**GALATASARAY UNIVERSITY**

**GRADUATE SCHOOL OF SCIENCE AND ENGINEERING**

# BIFURCATION OF REACTION DIFFUSION EQUATION

**Hatice Kübra PEKMEZ**

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# BIFURCATION OF REACTION DIFFUSION EQUATION

(REAKSİYON DİFÜZYON DENKLEMİNİN BİFÜRKASYONU)

by

**Hatice Kübra PEKMEZ, B.S.**

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This is to certify that the thesis entitled

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**LIST OF SYMBOLS**

- $R(X)$ — Resultant of the matrix $X$
- $D/\sim$ — Equivalence class
- $\mathbb{P}_1(\mathbb{C})$ — Projective Space with dimension 1
- $\mathbb{P}_2(\mathbb{C})$ — Projective Space with dimension 2
- $\wp$ — Weierstrass Elliptic Function
- $sn(t,k)$,$cn(t,k)$,$dn(t,k)$ — Jacobi Elliptic Functions

## LIST OF FIGURES

## ABSTRACT


First of all, we review important topological properties of elliptic curves in dependence of coefficient of defining equation of elliptic curves. Especially, we study singular points of elliptic curves by means of discriminant of defining equation of the curve. We see that the elliptic curve is non singular if it is defined by an equation whose coefficients will give non zero discriminant.

In Chapter 2, we review fundamental properties of elliptic functions, i.e. doubly periodic meromorphic functions on projective space $\mathbb{P}_1(\mathbb{C})$.

In Chapter 3, we define Jacobi elliptic functions as inverse function to elliptic integrals. Moreover, we see that Jacobi elliptic function solves mechanic problem on the periodicity of Galileo pendulum.

In Chapter 4, we study solution to non linear reaction diffusion equation. This equation is described by an quartic polynomial that depends on three parameters. We examine the bifurcation of the elliptic curves associated to this quartic equation. That is to say first we draw a bifurcation diagram in a 2-dimensional parameter space for a fixed energy level parameter. Then we shall achieve case studies for each connected component of the bifurcation diagram to analyze the behavior (e.g. the length of the period) of periodic solution to the reaction diffusion equation expressed by Jacobi elliptic function.

# ÖZET

İlk önce, tanımlanan eliptik eğri denklemlerinin kat sayılarına bağlı olarak, eliptik eğrilerin önemli topolojik özelliklerini inceliyoruz. Özellikle, tanımlanan bu eğri denkleminin diskriminantı vasıtasıyla bulunan tekil noktalar üzerinde çalışıyoruz. Eğer bu eğriler, kat sayılarının sıfırdan farklı diskriminant veren bir denklem tarafından tanımlanmışsa, bu eliptik eğrilerin tekil olmadığını görüyoruz.

2. ünitede eliptik fonksiyonların temel özellikleri, yani $\mathbb{P}_1(\mathbb{C})$ projektif uzayında bulunan iki kat periyodik meromorfik fonksiyonlar üzerinde çalışıyoruz.

3. ünitede, Jacobi eliptik fonksiyonları, eliptik integrallerin ters fonksiyonu olarak tanımlıyoruz. Ayrıca, Galileo sarkacının periyotluğu üzerinden, Jacobi eliptik fonksiyonların mekanik problemleri çözdüğünü görüyoruz.

4. ünitede, doğrusal olmayan reaksiyon difüzyon türevli denklemi üzerinde çalışıyoruz. Bu denklemi, 3 parametreye bağlı 4. dereceden polinomla belirtiyoruz. Bu denklemle ilişkili olan eliptik eğrilerin bifürkasyonunu inceliyoruz. Yani, sabitlenmiş enerji seviye parametresinin 2 boyutlu parametre uzayında bifürkasyon diyagramını çiziyoruz. Bunun sonucunda, Jacobi eliptik fonksiyonu ile ifade edebilen reaksiyon difüzyon denkleminin periyodik çözümünün davranışını analiz etmek için bifürkasyon diyagramının bütün bağlı unsurlarının durum incelemelerine ulaşıyoruz.

## 1. INTRODUCTION

We can see ellipses such as orbits of planets because of Kepler's laws. Also, we learn in analytic geometry that ellipses can be described as below.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Generally, people think that it is called an *elliptic curve*. But, they do not. So that we should analyze what elliptic curves are. In the plane $\mathbb{C}^2$, an elliptic curve is the set of solutions to an equation of the form $F(x, y) = 0$ where $F(x, y)$ is a cubic polynomial in $x$ and $y$. If we compare figure of an *ellipse* and an *elliptic curve*, we can see they look different.

If we want to search relation between *ellipse* and *elliptic curve*, we should consider history by analyzing first *the ellipses*, secondly *elliptic integrals*, next *elliptic functions* and finally *elliptic curves.*

The 4th century BCE mathematician **Menaechmus** studied on one of the three classical construction problems which is called the Duplication Problem. He showed that proportions, which are found by **Hyppocrates of Chios** (460-380 BCE), yield the curves which are two parabolas and a hyperbola. Next, **Menaechmus** described these as conic sections, discovering *the ellipses* in the process.

There was a question about how to find are length of an ellipse. Thanks to finding of the integral in calculus in the 1660's, new ways were found to answer this question and many mathematicians such as **Isaac Newton**, **Leonhard Euler** and **Colin Maclaurin** searched new processes. **Legendre** developed kinds of non elementary integrals and he found the integrals which are defined as *elliptic integrals.*

Moreover, **Abel** and **Jacobi** rewrote *elliptic integrals* by substituting of trigonometry. **Jacobi** had an idea which we can find *the inverse of elliptic integrals* which are called *elliptic functions.* After that, by using series, **Weierstrass** showed that the Weierstrass differential equation lies on a cubic curve.

In this thesis, we try to search the behavior of the quartic equation which is called *reaction diffusion equation* by using *the elliptic functions* and *the Jacobi elliptic functions.* Therefore, understanding their definition and applications is significant.

## 2. ELLIPTIC CURVES

### 2.1 Addition of Points

**Definition 2.1.1.** *Let $f$ be a non zero polynomial. The set of points $(a, b) \in \mathbb{R}^2$ is called* ***plane algebraic curve*** *satisfying $f(a, b) = 0$.*

To describe the addition of points, we need natural laws. For example, we can apply the laws on any straight line and on the unit circle $x^2 + y^2 = 1$. Let us think about the addition points of a line. If we take a fixed point $O$ on the line, we get that the sum of points $X$ and $Y$ is the point $Z$ where $\overrightarrow{OZ} = \overrightarrow{OX} + \overrightarrow{OY}$. Moreover, if we want to search the addition point on the circle, we find that the sum of the points $(\cos x, \sin x)$, $(\cos y, \sin y)$ is the point $(\cos(x + y), \sin(x + y))$.

We can interpret geometrically the addition of points. (Prasolov & Solovyev, 1997) Let $K$ be the point $(0, 1)$, $L$ and $M$ be two points of the unit circle. The sum of the points $L$ and $M$ is the point $P$ where the straight line through $K$ which is parallel to the straight line $LM$ intersects the circle at.

Now, we will define the addition of points on any conic i.e. on a second order curve.

**Definition 2.1.2.** *Let $K$ be a fixed point on a conic.* ***The sum of the points*** *$L$ and $M$ is the intersection of the straight line through $K$ parallel to the straight line $LM$ on the conic for second time.*

**Definition 2.1.3.** *A plane algebraic curve is called* **cubic**, *if $\sum a_i j x^i y^j = 0$ where the maximum value of the addition of $i$ and $j$ is equal to 3.*

Now, we will define the sum of the points on a cubic by using a new point on the cubic as different from a conic.

**Definition 2.1.4.** *For fixed point $K$, we suppose that $X$ which the straight line $LM$ intersect on a cubic is a point.* ***The sum of the points*** *$L$ and $M$ is the point of the intersection of the line $XK$ with the cubic.*

In addition, we will think about a new condition where $L$ is a fixed point on a curve and a point $M$ moves towards $L$ along the curve.

## 2.2 The Tangents and Inflection Points

We analyze a line $A_1A_2$ on a cubic which has a fixed point $A_1$ on the cubic and a point $A_2$ moves towards $A_1$ along the cubic. The line has the tendency to a fixed line and the tangent at $A_1$. We will search the addition of $A_1$ and $A_1$, so that we will find the tangent at $A_1$ rather than drawing the line $A_1A_2$.

Also, we can see that if $A_2$ coincides with $A_1$, the equation of the curve has a multiple root at $A_1$. Therefore, the restriction of the equation of the curve to the tangent has a multiple root at the tangent point. We will use this information to get the equation of the tangent of a curve on a cubic.

Let $F = 0$ be a curve, a point $A = (a_1, a_2, a_3)$ be on the curve and let $X = (x_1, x_2, x_3)$ be an arbitrary point. We can describe the points of the plane $TX$ is of the form $\lambda T + \mu X$. Now, we consider that the curve $F$ is restricted to the line $TX$ as a function $t$. As $F$ is a polynomial of degree 3, we can write that

$$F(T + tX) = F(T) + at + bt^2 + ct^3 = Q(t) \tag{2.2.1}$$

where $a = \sum F_i(T)x_i$ and $b = \frac{1}{2}\sum F_{ij}(T)x_ix_j$ where $F_i$ is the partial derivative of $F$ reference to $i$th variable.(Prasolov & Solovyev, 1997) We can see easily that when $a = 0$, $Q(t)$ has a multiple root at $t = 0$.

**Definition 2.2.1.** *We call* ***non singular point*** $T$ *of the curve* $F$ *in* (2.2.1)*, if at least one of the* $F_i(T)$ *is nonzero.*

So that the line $l$ which is determined by the equation $\sum F_i(T)x_i$ tangent to the curve at $T$ which is a non singular point.

**Definition 2.2.2.** *We call that* $F$ *is an* ***inflection point*** *of the curve* $F$ *if* $F = 0$ *and* $a = \sum F_i(T)x_i = 0$ *suggest that* $b = \frac{1}{2}\sum F_{ij}(T)x_ix_j = 0$

Moreover, we have that if the second degree polynomial $x^TAx$ which is in the matrix form is *divisible* by the linear function $x^Tl$ where $x^TAx = x^Tlk^Tx$ for some $k$. So that we can say that the matrix $A = lk^T$ is the product of a column by a row. Especially, $detA = 0$. Therefore, it can be concluded that if $T$ is an inflection point, then $det(F_{ij}(T)) = 0$. We remark that the converse condition also true for non singular points.

To sum up, all the inflection points of the curve $F = 0$ is included in the set of intersection points of the curves $F = 0$ and $H = 0$ where $H(X) = det(F_{ij}(X))$ which is called ***Hessian*** of the curve $F = 0$.

Let us consider the third degree curves. Let $F(x, y)$ and $H(x, y)$ be two third degree polynomials in the form

$$F(x, y) = a_0 y^3 + a_1(x)y^2 + a_2(x)y + a_3(x) \tag{2.2.2}$$

$$H(x, y) = b_0 y^3 + b_1(x)y^2 + b_2(x)y + b_3(x) \tag{2.2.3}$$

where $a_k(x)$ and $b_k(x)$ are two polynomials. We can say that if $F(x, y) = 0$ and $H(x, y) = 0$ have a common point $(x_0, y_0)$, then $y_0$ is a common root of the curves

$$f(y) = a_0 y^3 + a_1 y^2 + a_2 y + a_3 \tag{2.2.4}$$

$$h(y) = b_0 y^3 + b_1 y^2 + b_2 y + b_3 \tag{2.2.5}$$

such that $a_k = a_k(x)$ and $b_k = b_k(x)$. Moreover, we can say that if the polynomials have a common root $y_0$, then the point $(x_0, y_0)$ is the common point of the curves.

While analyzing over $\mathbb{C}$, we think that having a common root is the same meaning of having a common non constant *divisor*. Further, if $a_0 b_0 \neq 0$, then we say that the polynomials $f(y)$ and $h(y)$ have a common divisor in the case that we can find the polynomials $f_1$ and $h_1$ such that $fh_1 = hf_1$ where the degrees of $F(x, y)$ and $H(x, y)$ are higher than the degrees of $f_1$ and $h_1$ respectively.

Therefore, we can substitute $f_1 = fd^{-1}$ and $h_1 = hd^{-1}$. As $fh_1 = hf_1$ and $degf_1 < degf$, we can write all prime divisors of $f$ in the prime factorization of $hf_1$. But, we cannot do it in the factorization of $f_1$.

Let $h_1(y) = m_0y^2 + m_1y + m_2$ and $f_1(y) = n_0y^2 + n_1y + n_2$ be two polynomials. We can express the equality $fh_1 = hf_1$ as

$$\begin{aligned} a_0m_0 - b_0n_0 &= 0 \\ a_1m_0 + a_0m_1 - b_1n_0 - b_0n_1 &= 0 \\ a_2m_0 + a_1m_1 + a_0m_2 - b_2n_0 - b_1n_1 - b_0n_2 &= 0 \\ a_3m_0 + a_2m_1 + a_1m_2 - b_3n_0 - b_2n_1 - b_1n_2 &= 0 \\ a_3m_1 + a_2m_2 - b_3n_1 - b_2n_2 &= 0 \\ a_3m_2 - b_3n_2 &= 0 \end{aligned} \tag{2.2.6}$$

**Definition 2.2.3.** (Prasolov & Solovyev, 1997) *The determinant is called **resultant** which vanishes in the case that the system* (2.2.6) *of linear homogeneous equations related to $m$ and $n$ i.e.*

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \end{vmatrix} = 0 \tag{2.2.7}$$

*where $a_k$ and $b_k$ depend on $x$. The resultant noted by $R(X)$.*

That is to say that for every root $x_0$ of the polynomial $R(x)$, the curves $F(x, y) = 0$ and $H(x, y) = 0$ have a common point $(x_0, y_0)$.

Here, we try to find multiple root of the equation of a curve by using its derivation.

**Proposition 2.2.4.** *Let $H(w)$ a polynomial. If $H(w)$ has a multiple root if and only if $H(w)$ and $H'(w)$ has a common root.*

**Proof.** Let $H(w) = (w - w')^2D(w)$ such that $D(w')\neq0$. So that we find that $H'(w) = 2(w - w')D(w) + (w - w')^2D'(w)$, then we can see easily $w'$ is also a solution of $H'(w)$. □

We want to analyze *discriminant* as an example for resultant. Let take a curve $G(w)$ such that

$$H(w) = h_0 w^3 + h_1 w^2 + h_2 w + h_3 \tag{2.2.8}$$

So that the derivative of the equation of $H(w)$ is

$$H'(w) = 3h_0 w^2 + 2h_1 w + h_2 \tag{2.2.9}$$

**Definition 2.2.5.** *We call* ***the discriminant of H(w)*** (2.2.8) *is the determinant of the matrix A such that*

$$\begin{pmatrix} h_0 & h_1 & h_2 & h_3 & 0 \\ 0 & h_0 & h_1 & h_2 & h_3 \\ 3h_0 & 2h_1 & h_2 & 0 & 0 \\ 0 & 3h_0 & 2h_1 & h_2 & 0 \\ 0 & 0 & 3h_0 & 2h_1 & h_2 \end{pmatrix} \begin{pmatrix} {w_0}^5 \\ {w_0}^4 \\ {w_0}^3 \\ {w_0}^2 \\ w_0 \end{pmatrix} = 0 \tag{2.2.10}$$

*where* $A = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 & 0 \\ 0 & h_0 & h_1 & h_2 & h_3 \\ 3h_0 & 2h_1 & h_2 & 0 & 0 \\ 0 & 3h_0 & 2h_1 & h_2 & 0 \\ 0 & 0 & 3h_0 & 2h_1 & h_2 \end{pmatrix}$ *and* $w_0$ *is a solution of* $H(w) = H'(w) = 0$.

**Proposition 2.2.6.** *The following statements are equivalent.*

1. $detA = 0$ *where* $A$ *is the given matrix* (2.2.10) *in definition of discriminant.*

2. *The equation* $H(w) = 0$ (2.2.8) *has a multiple solution* $w = w_0$

3. *The system of the equations* (2.2.8) *and* (2.2.9) *has a common solution.*

We have an analogous result for the quartic equation

$$G(w) = h_0 w^4 + h_1 w^3 + h_2 w^2 + h_3 w + h_4 \tag{2.2.11}$$

So that the derivative of the equation of $G(w)$ is

$$G'(w) = 4h_0 w^3 + 3h_1 w^2 + 2h_2 w + h_3 \tag{2.2.12}$$

*discriminant* of $G(w)$ is determinant of the matrix $B$ given

$$B = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 & h_4 & 0 & 0 \\ 0 & h_0 & h_1 & h_2 & h_3 & h_4 & 0 \\ 0 & 0 & h_0 & h_1 & h_2 & h_3 & h_4 \\ 4h_0 & 3h_1 & 2h_2 & h_1 & 0 & 0 & 0 \\ 0 & 4h_0 & 3h_1 & 2h_2 & h_1 & 0 & 0 \\ 0 & 0 & 4h_0 & 3h_1 & 2h_2 & h_1 & 0 \\ 0 & 0 & 0 & 4h_0 & 3h_1 & 2h_2 & h_1 \end{pmatrix} \tag{2.2.13}$$

## 2.3 Singular Points

We defined non singular points in the previous section. Also, we call ***non singular curve*** (elliptic curve) which its all points are non singular.

We try to construct the equation of a non singular cubic curve as below (Prasolov & Solovyev, 1997)

$$y^2 = (x - a_1)(x - a_2)(x - a_3) \tag{2.3.1}$$

where $a_1$, $a_2$, and $a_3$ are different.

Let us suppose the inequality $a_1 < a_2 < a_3$.

If $a_1$ and $a_2$ are equal, we can find a curve example which can be constructed by following equation

$$y^2 = x^2(x-1) \tag{2.3.2}$$

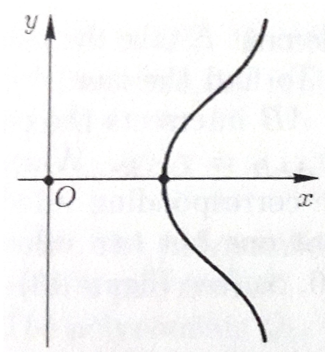


Figure 2.1: Graph of $y^2 = x^2(x - 1)$

Also, if $a_2$ and $a_3$ are equal, we can find similarly a curve example which can be constructed by following equation

$$y^2 = x^2(x+1) \tag{2.3.3}$$

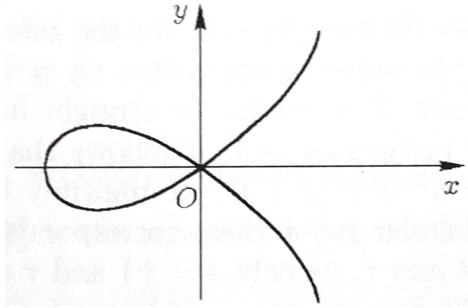


Figure 2.2: Graph of $y^2 = x^2(x+1)$

Over $\mathbb{R}$, the difference between the curves can be seen clearly. On the other hand, we cannot see this difference over $\mathbb{C}$.

If we suppose that all roots are same, we can see easily that the equation of the curve (2.3.1) becomes

$$y^2 = x^3 \tag{2.3.4}$$

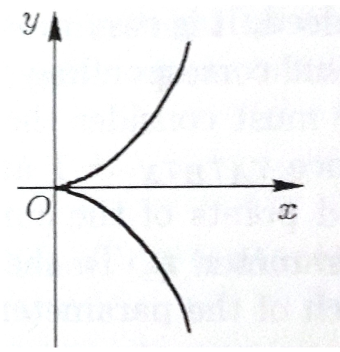


Figure 2.3: Graph of $y^2 = x^3$

Now, we will define two notions which are *the multiplicity* and *singular point.*

**Definition 2.3.1.** *We call* ***the multiplicity*** *which describes the number of time that a curve passes through a point on the curve.*

**Definition 2.3.2.** *If the multiplicity of a point $X$ is at least two, the point $X$ is called **singular point**.*

If we search singular points these three curves, we get that the point $(0, 0)$ is a singular point. So that any straight lines which pass through the origin intersect the curves $r^2y^2 = x^2(x \pm 1)$ and $r^2y^2 = x^3$ at the singular point with multiplicity two or more than two for all $r \in \mathbb{R}$. If we analyze that, $r^2y^2 = x^2(x \pm 1)$ and $r^2y^2 = x^3$ have root at $x = 0$ where the curves pass through this point at least 2 times. Moreover, the sum of the singular points is the other singular point which any straight lines which pass through the origin intersect the curves at.

Let us analyze the curve (2.3.4). We can make two substitutions as $x = k^{-2}$ and $y = k^{-3}$ on the curve. We will find out the intersection points of this curve with the line $t_1x + t_2y + t_3 = 0$. With the parametrization, the line transforms to $t_3k^3 + t_1k + t_2 = 0$. By definition, if the line does not pass through the singular point, then we can say that $t_3 \neq 0$, so $k_1 + k_2 + k_3 = 0$ where $k_1$, $k_2$, $k_3$ are the roots of the equation since the coefficient of $k^2$ is equal to 0.

Now, we assume a *infinite point* for the zero element $P$ corresponding to the parameter $k_P = 0$ and let $k_L$ and $k_M$ be parameters refers to the points $L$ and $M$ of the curve. If the straight line $LM$ passes through a point $X$ on the cubic, then we get $k_L + k_M + k_X = 0$ and the straight line $PX$ passes through $L + M$ on the curve, so $k_P + k_X + k_{L+M} = 0$. Therefore, we find $-k_X = k_{L+M} = k_L + k_M$ and it can be concluded that we must add the corresponding values of the parameter $k$ for the addition points on the curve (2.3.4).

Secondly, let us apply a similar method on the curve (2.3.3). If we suppose the equality $y = kx$, we get $x = k^2 - 1$ and $y = k^3 - k$ since $k^2x^2 = x^2(x + 1)$. In the same way, we will find out the intersection points of this curve with the line $t_1x + t_2y + t_3 = 0$. With the parametrization, the line transforms to $t_1(k^2 - 1) + t_2(k^3 - k) + t_3 = 0$. We have $k_1k_2 + k_2k_3 + k_1k_3 = -1$ in the case $t_2 \neq 0$ where $k_1$, $k_2$, $k_3$ are roots of the line as above. After the substitution $k = (1 + \kappa)(1 - \kappa)^{-1}$, we get $\kappa_1\kappa_2\kappa_3 = 1$ where $\kappa_1$, $\kappa_2$, $\kappa_3$ are roots of the line.

Now, we assume a *infinite point* for the zero element $P'$ corresponding to the parameter $\kappa_{P'} = 1$ and let $\kappa_{L'}$ and $\kappa_{M'}$ be parameters refers to the points $L'$ and $M'$ of the curve. If the straight line $L'M'$ passes through a point $X'$ on the cubic, then we get

$\kappa_{L'}\kappa_{M'}\kappa_{X'} = 1$ and the straight line $P'X'$ passes through $L' + M'$ on the curve, so $\kappa_{P'}\kappa_{X'}\kappa_{L'+M'} = 1$. Therefore, we find $\kappa_{L'+M'} = \kappa_{L'} + \kappa_{M'}$ and it can be concluded that we add points by multiplying the corresponding values of the parameter $\kappa$ for the addition points on the curve (2.3.3). However,there exist two values which correspond of the parameters $k$ and $\kappa$. They are $k = -1, 1$ and $\kappa = 0, \infty$.

### 2.4 Projective Spaces

**Definition 2.4.1.** *Let $D$ be a set and $\sim$ be a binary relation on the set $D$. $D/\!\sim$ is called **equivalence class** which provides the following statements*

- *$a \sim a$ for all $a \in D$;*
- *if $a \sim b$, then $b \sim a$ for all a,b $\in D$;*
- *if $a \sim b$ and $b \sim c$, then $a \sim c$ for all a,b,c $\in D$.*

**Definition 2.4.2.** (Brieskorn & Knörrer, 1986) *Let $(x_0, x_1)$, $(x_0', x_1')$,$(x_0'', x_1'')$ be in $\mathbb{C}^2$-$\{(0,0)\}$ and $\lambda$ and $\mu$ be in $\mathbb{C}$-$\{0\}$. We call that $\mathbb{P}_1(\mathbb{C})$ is **projective space with dimension 1** such that $\mathbb{P}_1(\mathbb{C})$=$\mathbb{C}^2$-$\{(0,0)\}/\sim$, i.e. $\mathbb{P}_1(\mathbb{C})$ is an equivalence class where $\sim$ is a binary relation on $\mathbb{C}^2$-$\{(0,0)\}$ such that*

- *$(x_0, x_1) \sim (x_0, x_1)$ i.e. $x_0$=$1.x_0$ and $x_1$=$1.x_1$;*
- *if $(x_0, x_1) \sim (x_0', x_1')$, i.e. $x_0' = \lambda x_0$ and $x_1' = \lambda x_1$, then $(x_0', x_1') \sim (x_0, x_1)$, i.e. $x_0 = \lambda^{-1}x_0'$ and $x_1 = \lambda^{-1}x_1'$;*
- *if $(x_0, x_1) \sim (x_0', x_1')$ and $(x_0', x_1') \sim (x_0'', x_1'')$, i.e. $x_0' = \lambda x_0$ and $x_1' = \lambda x_1$ and $x_0'' = \mu x_0'$ and $x_1'' = \mu x_1'$, then $(x_0, x_1) \sim (x_0'', x_1'')$, i.e. $x_0'' = \mu\lambda x_0$ and $x_1'' = \mu\lambda x_1$.*

Let us consider $\mathbb{P}_1(\mathbb{C})$=$A_0 \cup A_1$ where

$$A_0 = \{(a_0, a_1) \in \mathbb{P}_1(\mathbb{C}) : a_0 \neq 0\} \qquad A_1 = \{(a_0, a_1) \in \mathbb{P}_1(\mathbb{C}) : a_1 \neq 0\} \tag{2.4.1}$$

Now, let us take a point $(a_0, a_1) \in A_0$ and let us consider the complex number $z_0 = \frac{x_0}{x_1}$. In the similar way, let us take a point $(a_0, a_1) \in A_1$ and let us consider the complex number $z_1 = \frac{x_1}{x_0}$. It is enough taking only one point in a equivalence class to analyze.

In addition let $f$, $g$ be two bijective applications such that

$$f : A_0 \to \mathbb{C} \qquad g : A_1 \to \mathbb{C} \tag{2.4.2}$$

$$(a_0, a_1) \mapsto z_0 \quad (a_0, a_1) \mapsto z_1$$

If we look at the intersection $A_0 \cap A_1$, we get the equality $z_1 = \frac{1}{z_0}$. We should find out what happened to the point $(0, 1)$. To obtain $\mathbb{P}_1(\mathbb{C})$ on $A_0$, $\frac{x_1}{x_0}$ tends to infinity as $x_0 \to 0$. (Brieskorn & Knörrer, 1986). Let us explain this statement with an example.

On the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z-1)^2 = 0\} \in S^2$, we can see easily that the set $A_0 = \left\{(\frac{2x}{2-z}, \frac{2y}{2-z}, 0)\right\}$, which is in $\mathbb{R}^2$, is well defined for $z \neq 2$.

Since $\mathbb{R}^2 \cong \mathbb{C}$, $\mathbb{C}$ is homeomorphic to $A_0$, i.e. $S^2 - \{(0, 0, 2)\}$ is homeomorphic to $A_0$ since the following application is a homeomorphism.

$$d : S^2 - \{(0, 0, 2)\} \to \mathbb{R}^2 \tag{2.4.3}$$

$$(x, y, z) \mapsto (\frac{2x}{2-z}, \frac{2y}{2-z}, 0)$$

In the same way, On the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z-1)^2 = 0\} \in S^2$, we can see easily that the set $A_1 = \left\{(\frac{2x}{2+z}, \frac{2y}{2+z}, 0)\right\}$, which is in $\mathbb{R}^2$, is well defined for $z \neq -2$.

So that $S^2 - \{(0, 0, -2)\}$ is homeomorphic to $A_1$ since the following application is a homeomorphism.

$$h : S^2 - \{(0, 0, -2)\} \to \mathbb{R}^2 \tag{2.4.4}$$

$$(x, y, z) \mapsto (\frac{2x}{2+z}, \frac{2y}{2+z}, 0)$$

We have that $\mathbb{P}_1(\mathbb{C}) = A_0 \cup A_1$. So that

$$\mathbb{P}_1(\mathbb{C}) = A_0 \cup A_1 = (S^2 - \{(0, 0, 2)\}) \cup (S^2 - \{(0, 0, -2)\}) = S^2$$

**Proposition 2.4.3.** $\mathbb{P}_1(\mathbb{C})$ *is homeomorphic to sphere* $S^2$.

Now, we will study the application on $A_0 = \{(a_0, a_1) \in \mathbb{P}_1(\mathbb{C}) : a_0 \neq 0\}$

$$f : \mathbb{P}_1(\mathbb{C}) \to \mathbb{P}_1(\mathbb{C}) \tag{2.4.5}$$

$$(x_0, x_1) \mapsto ({x_0}^2, {x_1}^2)$$

We can transform the application $f$ to

$$z_0 = \frac{x_0}{x_1} \mapsto (1, \frac{{x_0}^2}{x_1}) = (1, {z_0}^2)$$

where $z_0 \in A_0$. But, the application

$$g : A_0 \to A_0 \qquad (2.4.6)$$
$$z_0 \mapsto {z_0}^2$$

is not injective since $f^{-1}((1, {z_0}^2)) = (1, z_0) \cup (1, -z_0)$ except $z_0 = 0$. Therefore, we get $0 \leq arg z_0 = t < \pi$ ou $\pi \leq arg z_0 = t' < 2\pi$ where $z_0 = r.e^{it}$ ou $z_0 = r.e^{it'}$.

It can be concluded that the image of the application (2.4.5) turns around the origin. Let us analyze the application (2.4.5) on the following sets

$$H^+ = \{z_0 \in \mathbb{C} : Im(z_0) > 0\} \qquad H^- = \{z_0 \in \mathbb{C} : Im(z_0) < 0\}$$

Then we get the applications $f'$ and $f''$ by restricting $f$ to $A_0$

$$f' : H^+ \to \mathbb{C} - \left\{\mathbb{R}^+\right\} \qquad (2.4.7)$$
$$r.e^{it} \mapsto r^2 e^{2it}$$

where $0 < t < 2\pi$ and

$$f'' : H^- \to \mathbb{C} - \left\{\mathbb{R}^+\right\} \qquad (2.4.8)$$
$$r.e^{it} \mapsto r^2 e^{2it}$$

where $-\pi < t < 0$.

# 3. ELLIPTIC FUNCTIONS

## 3.1 Topological Interpretation of Non Singular Curves in $\mathbb{P}_2(\mathbb{C})$

If we analyze the addition of points on the circle, it is related to its parametrization by the functions *sine* and *cosine*. Let $f{:}\mathbb{R} \to S^1$ a map such that $f(\phi) = (\cos\phi, \sin\phi)$. We draw a parametrization of the circle by using real numbers such that the addition of points on the circle accords to the addition of real numbers.

We will study a similar parametrization for cubics. We defined the addition of the points on the cubic in the first chapter.

In this chapter, we will define new functions which are called *elliptic functions*, and we will demonstrate how one can parametrize a non singular cubic by using elliptic functions.

Let us describe the topological structure of non singular cubics in $\mathbb{P}_2(\mathbb{C})$. We can define $\mathbb{P}_2(\mathbb{C}){=}\mathbb{C}^3 - \{(0,0,0)\}/{\sim}$ where $(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$ for $\lambda{\in}\mathbb{C} - \{0\}$ similarly $\mathbb{P}_1(\mathbb{C})$. We can write the equation of any non singular cubic in $\mathbb{P}_2(\mathbb{C})$ as below (Prasolov & Solovyev, 1997)

$$y^2 z = (x{-}t_1 z)(x{-}t_2 z)(x{-}t_3 z) \tag{3.1.1}$$

where $t_k$'s are different. The equation describes a complex curve in $\mathbb{P}_2(\mathbb{C})$. Let us consider the projection map

$$\begin{aligned}\pi : \mathbb{P}_2(\mathbb{C}){-}\{(0,1,0)\} &\to \mathbb{P}_1(\mathbb{C}) \\ (x, y, z) &\mapsto (x, z)\end{aligned} \tag{3.1.2}$$

We have that the complex projection line $\mathbb{P}_1(\mathbb{C})$ is homoemorphic to the 2-dimensional sphere $S^2$ (see proposition (2.4.3)).

We know that $y^2 = b$ has two different solutions for $b \neq 0$.

If we suppose that $z \neq 0$ and $x - t_k z \neq 0$, then there exists a point $(x, z) \in \mathbb{P}_1(\mathbb{C})$ which has two preimages of $\pi$ that belong to the curve $y^2 z = (x - t_1 z)(x - t_2 z)(x - t_3 z)$.

If we suppose that $z \neq 0$ and $\frac{x}{z}$ is equal to one of $t_k$, then we can find only one preimage. If we suppose that $z = 0$, the equation becomes to $x^3 = 0$. Then it is argued from that there exists only one preimage of the point $\infty$=$(1, 0)$, that is to say $(0, 1, 0)$.

Also, we can say more generally that the preimage of the point $(1, z)$ converge to $(0, 1, 0)$ as $z$ tends to 0.

Let us add the points $t_k$ and $\infty$ of $\mathbb{P}_1(\mathbb{C})$. So we can find that all points have two preimages. We want to study more detail on the structure of the projection map $\pi$ in neighborhood of the points $t_k$ and $\infty$.

Let us suppose that $t_1$=0. If we consider affine coordinates by setting $z = 1$, the projection map of the curve is described as $(x, y)\mapsto x$. Therefore, we write that in the form

$$y^2 = x(x - t_2)(x - t_3) \tag{3.1.3}$$

where $t_2t_3 \neq 0$. We have nearly the equation $y^2$=$cx$ for points $x$ that its value is near to zero. So we find solutions which are in the form $x$=$c\lambda^2e^{2i\theta}$, $x$=$c\lambda e^{i\theta}$. While $\theta$ is changing from 0 to $\pi$, a revolution happens around the point $(0, 1)$ on $\mathbb{P}_1(\mathbb{C})$. In this situation, $y$ alters sign. While raising the revolution around $(0, 1)$ to the curve $y^2z = (x - t_1z)(x - t_2z)(x - t_3z)$, it does not get back the initial point. However, it can get back the initial point as $y_0$ changing the sign.

In the same way, we can construct the structure of the projection of the curve on $\mathbb{P}_1(\mathbb{C})$ in neighborhood of the point $\infty$. To see it more clearly, let $x = 1$. So we have almost the equation $y^2 = 1/z$ in a neighborhood of $z = 0$. Then we can say that under a revolution about the point $z = 0$, $y$ changes sign.

To sum up, *the ramification points* of the curve are $(x, y, z)$=$(t_1, 0, 1)$,$(t_2, 0, 1)$,$(t_3, 0, 1)$, $(0, 1, 0)$. If we consider the following cubic curve

$$y^2z = (x{-}t_1z)(x{-}t_2z)(x{-}t_3z), \tag{3.1.4}$$

it also has 4 ramification points $(t_k, 0, 1) \in \mathbb{P}_2(\mathbb{C})$ where $k = 1, 2, 3, 4$.

Now, we will cut $\mathbb{P}_1(\mathbb{C})$ from $t_1$ to $t_2$ and from $t_3$ to $\infty$.

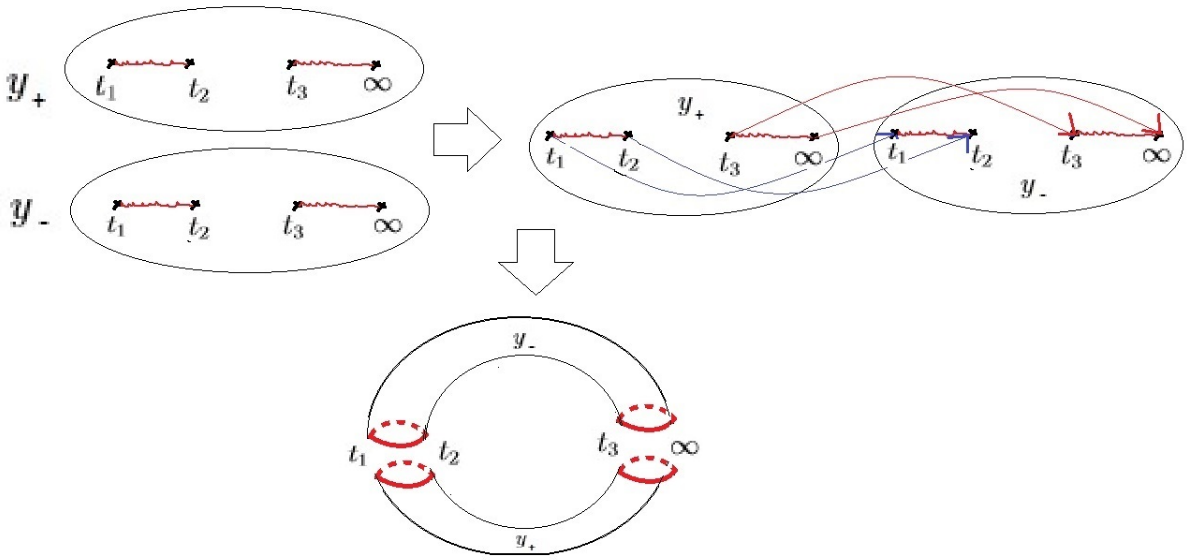


Figure 3.1: Steps of construction of a torus

Then we see above that the part of the curve that lies this plane which consists of two pieces. Now, we will glue this pieces. To apply this, we match same roots which are in the planes $y_+$ and $y_-$ . Therefore, we construct *a torus.*

### 3.2 Elliptic Functions

We parametrize a cubic in $\mathbb{P}_2(\mathbb{C})$ by using a map $h$:$\mathbb{C}^1 \rightarrow \mathbb{P}_2(\mathbb{C})$ where $h(z) = (H_1(z), H_2(z), 1)$. Then, we reach that its image is a torus. Therefore, we identify points of the form $z + aw_1 + bw_2$. Indeed, we say that $w_1$ and $w_2$ are periods of the functions $H_1$ and $H_2$, respectively. (Brieskorn & Knörrer, 1986).

Before we begin the section, we should learn a few definition to use there.

**Definition 3.2.1.** *$L$ is called **lattice**, if $L = \{aw_1 + bw_2; (a, b) \in \mathbb{Z}^2\}$ where $w_1$,$w_2 \in \mathbb{C} - \{0\}$.*

**Definition 3.2.2.** *A function $f$ is called **meromorphic**, if $f$ has no singular points other than poles in a finite domain of $\mathbb{C}$ where $f$ is an analytic function.*

**Definition 3.2.3.** *Let $f$ be a function. We call **doubly periodic**, if $f(z + aw_1 + bw_2) = f(z)$ for any $(a, b) \in \mathbb{Z}^2$ where $Im(w_1/w_2) > 0$.*

**Definition 3.2.4.** *A function $f$ is called **elliptic**, if $f$ is a meromorphic function and doubly periodic.*

**Theorem 3.2.5.** (Golubev, 1960) *An elliptic function is constant where it has no poles.*

**Proof.** Firstly, we assume that a function $f$ which is elliptic has no poles. Then, we can say that $|f|$ is continuous on $\mathbb{C}$. Also, $f$ is bounded since the lattice is compact. By using *Liouville's theorem*, we can say that $f$ is constant. $\square$

**Theorem 3.2.6.** *For any non-constant function $f$, we have*

$$\sum v_a(f) = \#\{zeros\} - \#\{poles\} = 0 \tag{3.2.1}$$

*i.e. the difference the number of zeros of $f$ and the number of poles of $f$ is equal to* 0.

**Corollary 3.2.7.** *There exist no elliptic function which has only one simple pole.*

Now, let us analyze *Weierstrass's function* as an example of elliptic functions.

### 3.3 Weierstrass's Function

**Definition 3.3.1.** (Armitage, 2008) *A meromorphic function $\wp$ is called **Weierstrass's function** such that*

$$\wp(z) = \frac{1}{z^2} + \sum \frac{1}{(z + n_1w_1 + n_2w_2)^2} - \frac{1}{(n_1w_1 + n_2w_2)^2} \tag{3.3.1}$$

*where $(n_1, n_2) \in \mathbb{Z}^2 - \{0\}$.*

**Theorem 3.3.2.** *The function $\wp$ is an elliptic function, i.e.*

1. *$\wp(z)$ is convergent for $z \neq L$ where $L$ is a lattice, i.e. $\wp(z)$ is holomorphic on $\mathbb{C} - L$.*
2. *$\wp(z + n_1w_1 + n_2w_2) = \wp(z)$ for any $z \in \mathbb{C}$ where $(n_1, n_2) \in \mathbb{Z}^2$*

**Proof.** 1.We can see easily that

$$\begin{aligned}
&\frac{1}{(z + n_1w_1 + n_2w_2)^2} - \frac{1}{(n_1w_1 + n_2w_2)^2} \\
&= \frac{(n_1w_1 + n_2w_2)^2 - (z^2 + 2z(n_1w_1 + n_2w_2)) + (n_1w_1 + n_2w_2)^2}{(z + n_1w_1 + n_2w_2)^2(n_1w_1 + n_2w_2)^2} \\
&= \frac{-(z^2 + 2z(n_1w_1 + n_2w_2))}{(z + n_1w_1 + n_2w_2)^2(n_1w_1 + n_2w_2)^2}
\end{aligned}$$

Let us suppose that $w = n_1w_1 + n_2w_2$. Then, we can draw a comparison as below

$$|z||\frac{z + 2w}{(z + w)^2w^2}| < \frac{C|z|}{|w|^3}$$

and we see that $C$ is positive number on the condition $|w|{>}2|z|$.

Now, let us study on the case $L' = \{w \in L; |w| > 2|z|\}$. Then we draw a comparison as below

$$\sum_{w \in L'} |\frac{1}{(z+w)^2} - \frac{1}{w^2}| < C \sum_{|w|>2|z|} \frac{|z|}{|w^3|}$$

Also, we can write that

$$|n_1 w_1 + n_2 w_2| = |w_1||n_1 + n_2 \frac{w_2}{w_1}| = |w_1||n_1 + (\alpha + \beta i)n_2| = |w_1|\sqrt{(n_1 + \alpha n_2)^2 + \beta^2 {n_2}^2}$$

where $\alpha + \beta i = \frac{w_2}{w_1}$ such that $\beta > 0$.

Moreover, we can draw another a comparison as below

$$\frac{1}{|n_1 w_1 + n_2 w_2|} < \frac{C_1}{\sqrt{{n_1}^2 + {n_2}^2}}$$

and an integral $C \int \frac{dxdy}{\sqrt{(x^2+\beta^2 y^2)^3}}$ greater than $\sum_{|w|>2|z|} \frac{1}{|w^3|}$. We know that $\int \frac{dxdy}{\sqrt{(x^2+\beta^2 y^2)^3}}$ is convergent. So that it be concluded that Weierstrass's function is convergent.

We have that $\wp(z) = \frac{1}{z^2} + \sum \frac{1}{(z+w)^2} - \frac{1}{w^2}$ where $w{\in}L - \{0\}$.

Let us add a fixed number $w'$ to $x$. Then,

$$\begin{aligned} \wp(z + w') &= \frac{1}{(z+w')^2} + \sum \frac{1}{(z+w+w')^2} - \frac{1}{w^2} \\ &= \frac{1}{z^2} + \sum_{w \neq w'} \frac{1}{(z+w)^2} + \frac{1}{z^2} - \frac{1}{w^2} \end{aligned}$$

Therefore, if we have a parametrization $\tilde{w} = w + w'$, we find that

$$\wp(z + w') = \frac{1}{(z+\tilde{w})^2} + \sum \frac{1}{(z+w+w')^2} - \frac{1}{w^2} = \wp(z)$$

where $\tilde{w}, w{\in}L - \{0\}$. In conclusion, $\wp(z + w') = \wp(z)$ □

Now, let us analyze Weierstrass's function as differential equation and find differential equation satisfying the function $\wp(z)$. If we regard a new function

$$\zeta(z) = \frac{1}{z} + \sum_{w \in L - \{0\}} \frac{1}{z-w} + \frac{1}{w} - \frac{z}{w^2}, \tag{3.3.2}$$

We can see easily that $-\zeta'(z) = \wp(z)$. (Brieskorn & Knörrer, 1986) Also, we would like to rewrite $\frac{1}{z-w}$ as a serie.

$$\frac{1}{z-w} = -\frac{1}{w}\frac{1}{(1-\frac{z}{w})} = -\frac{1}{w}\sum_{k=0}^{\infty}(\frac{z}{w})^k = -\frac{1}{w}(1+\frac{z}{w}+\frac{z^2}{w^2}+\ldots)$$

Therefore, we see that

$$\frac{1}{z-w}+\frac{1}{w}+\frac{z}{w^2} = (-\frac{1}{w}-\frac{z}{w^2}-\frac{z^2}{w^3}+\ldots)+\frac{1}{w}+\frac{z}{w^2}$$
$$= \sum_{k=2}^{\infty} -\frac{z^k}{w^{k+1}}$$

where $w \in L - \{0\}$. If we simply this serie, we find a more clear serie which is

$$\sum_{k=2}^{\infty}\sigma_{2k}z^{2k-1}$$

where $\sigma_{2k} = \sum \frac{1}{w^{2k}}$. So that

$$\zeta(z) = \frac{1}{z} + \sum_{k=2}^{\infty}\sigma_{2k}z^{2k-1}$$

$$\Rightarrow \wp(z) = \frac{1}{z^2} - \sum_{k=2}^{\infty}(2k-1)\sigma_{2k}z^{2k-2} \tag{3.3.3}$$

Let us search derivative of $\wp(z)$.

$$\wp'(z) = -\frac{2}{z^3} - \sum_{k=2}^{\infty}(2k-1)(2k-2)\sigma_{2k}z^{2k-3} \tag{3.3.3a}$$

$$\Rightarrow (\wp'(z))^2 = \frac{4}{z^6} - -\frac{24\sigma_4}{z^2} - 80\sigma_6 \tag{3.3.3b}$$

$$4(\wp(z))^3 = \frac{4}{z^6} - -\frac{36\sigma_4}{z^2} + 60\sigma_6 \tag{3.3.3c}$$

It follows that

$$(\wp'(z))^2 - 4(\wp(z))^3 + 60\sigma_4\wp(z) + 140\sigma_6 = o(z) \tag{3.3.4}$$

is an elliptic function without pole. Therefore, the right side $o(z) = 0$ by theorem (3.2.5). In addition, we can say that the following differential equation is valid.

$$(\frac{d\wp(z)}{dz})^2 = 4(\wp(z))^3 - 60\sigma_4\wp(z) - 140\sigma_6$$

$$= 4(\wp(z))^3 - g_2\wp(z) - g_3 \tag{3.3.5}$$

where $g_2 = 60\sigma_4$, $g_3 = 140\sigma_6$. Then, we find that $\wp(z)$ is the solution of the equation

$$\frac{dw}{dz} = \sqrt{4w^3 - g_2w - g_3}$$

by parametrizing $\wp(z) = w$. Therefore, it can be concluded that

$$\int_{\wp(z_0)}^{\wp(z)} \frac{dw}{\sqrt{4w^3 - g_2w - g_3}} = z - z_0 \tag{3.3.6}$$

To say that this integral is well defined, we have to claim the path of the integration. So that let us consider the curve

$$\{(y, w) \in \mathbb{C}^2; y^2 - (4w^3 - g_2w - g_3) = 0\}$$

and let us suppose that the equation $4w^3 - g_2w - g_3$ has three distinct solutions which are $e_1$,$e_2$,$e_3$. So that we rewrite the equation of the curve as below

$$y^2(w) = 4(w - e_1)(w - e_2)(w - e_3)$$

$$\Rightarrow y(w) = \pm\sqrt{4(w - e_1)(w - e_1 + e_1 - e_2)(w - e_1 + e_1 - e_3)}$$

Moreover, $y(w)$ behaves like $\sqrt{4(e_1 - e_2)(e_1 - e_3)}.\sqrt{(w - e_1)}(1+o(w-e_1))$ as $w-e_1\rightarrow 0$. Then, we say that $\sqrt{w - e_1}$ can be expressed by $\sqrt{r}e^{i\theta/2}$. So that $w = e_1 + \sqrt{r}e^{i\theta/2}$ where $0\leq\theta\leq 2\pi$, and we can see easily that the behavior of $\sqrt{w - e_1}$ is nearly to the point $w = e_1$.

Since the equation $\sqrt{4(w - e_1)(w - e_2)(w - e_3)}$ has two different roots in the way plus and minus, values of the integral follow two different paths as below

Therefore, we have to make clear paths of which follows the function $y(w)$. Let us parametrize $w = \frac{1}{w'}$. If we substitute it on the equation, we write

$$\frac{d(\frac{1}{w'})}{\sqrt{4\frac{1}{w'}^3 - g_2\frac{1}{w'} - g_3}} = \frac{\sqrt{(w')^3}}{-(w')^2} \frac{dw'}{\sqrt{4 - g_2(w')^2 - g_3(w')^3}}$$

If we replace $w' = 0$, we find $w = \infty$ the point of ramification, since $y(w)$ have two different values except at points of ramification. More clearly, the function $\sqrt{4(w-e_1)(w-e_2)(w-e_3)}$ change sign during that $w$ turn around points $e_1$,$e_2$,$e_3$,$\infty$ in other words ramification points.

# 4. JACOBI ELLIPTIC FUNCTIONS

## 4.1 Jacobi Elliptic Functions

We have from trigonometry

$$sin^{-1}(x) = \arcsin(x) = \int_0^{\varphi} \frac{dx}{\sqrt{1-x^2}} \tag{4.1.1}$$

We will focus on $\sin(x)$ and we will form *Jacobi elliptic functions* by using ellipse. We know the general ellipse equation as below

$$\frac{x^2}{m^2} + \frac{y^2}{n^2} = 1 \tag{4.1.2}$$

Now, we will suppose that $n = 1$, so that the equation becomes

$$\frac{x^2}{m^2} + y^2 = 1 \tag{4.1.3}$$

Also, it provides that $x^2 + y^2 = r^2$.

**Definition 4.1.1.** ***The eccentricity*** *of an ellipse is $\epsilon$ where the equation of the conic is $\frac{x^2}{m^2} + \frac{y^2}{n^2} = 1$ such that*

$$\epsilon = \sqrt{1 - \frac{n^2}{m^2}} \tag{4.1.4}$$

Since we suppose that $n = 1$, the eccentricity of (4.1.2) is

$$\epsilon = \sqrt{1 - \frac{1}{m^2}} = k \tag{4.1.5}$$

where $k$ is called ***modulus*** of the elliptic function (4.1.3) That's why, $0 < k < 1$.

In addition, we know the argument of a trigonometric function is an angle $\varphi$. But, we will describe it differently for elliptic function.

**Definition 4.1.2.** *We call that $u$ is* ***argument*** *of the elliptic function* (4.1.3) *such that*

$$u = \int_B^A r d\varphi \tag{4.1.6}$$

Here, we do not talk about arc length or area. However, in the condition $m \to 1$ or $k \to 0$, $u$ becomes an angle since the ellipse turn to circle in this case.

By using the argument and the modulus of the elliptic function (4.1.3), we construct a new function system as below

$$sn(u,k) = y \qquad cn(u,k) = x/m \qquad dn(u,k) = r/m \tag{4.1.7}$$

where $x$, $y$ describe $\cos\varphi$, $\sin\varphi$ respectively and $r$ is not constant on an ellipse. Moreover, when $m = 1$, these functions tend to $y$, $x$, $1$ respectively since $r \to 1$. Let us define these functions.

**Definition 4.1.3.** (Meyer, 2001) *Let $k$ be in the interval $(0,1)$ and let $t$ be real variable which describes time. The functions $sn(t,k)$, $cn(t,k)$, and $dn(t,k)$ are called* ***Jacobi elliptic functions*** *as the solutions of the system of differential equations*

$$\begin{aligned} m^2 y' &= xr \\ x' &= -ry \\ r' &= -k^2 yx \end{aligned} \tag{4.1.8}$$

*satisfying the initial conditions $y(0) = 0$, $x(0) = 1$, and $r(0) = 1$ where the variables $x$, $y$, and $r$ are given in* (4.1.7).

We know from trigonometry that $\varphi = \arctan(\frac{y}{x}) = \tan^{-1}(\frac{y}{x})$.

$$\Rightarrow d\varphi = \frac{xdy - ydx}{x^2(1 + \frac{y^2}{x^2})} = \frac{xdy - ydx}{x^2 + y^2} = \frac{xdy - ydx}{r^2}. \tag{4.1.9}$$

Also, we have $du = rd\varphi$ by definition of the argument, so that

$$du = rd\varphi = \frac{xdy - ydx}{r} \tag{4.1.10}$$

Moreover, we have $\frac{x}{m^2}dx + ydy = 0$ from (4.1.3). Then, by using this, we can rewrite as below

$$dy = -\frac{x}{m^2 y}dx$$

$$dx = -\frac{m^2 y}{x}dy \tag{4.1.11}$$

So that we substitute these equalities to $du$. Therefore, we obtain

$$du = \frac{1}{r}(-\frac{x^2}{m^2 y} - y)dx$$

$$du = \frac{1}{r}(x + \frac{m^2 y}{x})dy \tag{4.1.12}$$

By using the differential equation system as above, we reach the equalities which are

$$du = \frac{dy}{\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}} \tag{4.1.13}$$

$$du = \frac{dx}{\sqrt{(1-x^2)}\sqrt{(k'^2+k^2x^2)}} \tag{4.1.14}$$

where $k^2 = 1 - \frac{1}{m^2}$. Anymore we will use the time variable $t$ instead of the variable $u$ since we will use the Jacobi elliptic functions. Because of the equalities (4.1.12), we get a system of differential system

$$\begin{aligned}
\frac{d}{dt}sn(t,k) &= cn(t,k)dn(t,k) \\
\frac{d}{dt}cn(t,k) &= -dn(t,k)sn(t,k) \\
\frac{d}{dt}dn(t,k) &= -k^2 sn(t,k)cn(t,k)
\end{aligned} \tag{4.1.15}$$

Now, let us search the solutions of $t$ by using the system of differential system as above. We have $cn(t,k) = \sqrt{1 - sn^2(t,k)}$ and $dn(t,k) = \sqrt{1 - k^2 sn^2(t,k)}$. If we replace these on the differential equation $\frac{d}{dt}sn(t,k) = cn(t,k)dn(t,k)$, we get

$$\frac{d}{dt}sn(t,k) = cn(t,k)dn(t,k) = \sqrt{1 - sn^2(t,k)}\sqrt{1 - k^2 sn^2(t,k)}. \tag{4.1.16}$$

Since $y = sn(t,k)$, we obtain

$$(\frac{dy}{dt})^2 = (1-y^2)(1-k^2y^2) \tag{4.1.17}$$

$$\Rightarrow t = c_1 + \int_{y(0)}^{y(t)} \frac{dy}{\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}} \tag{4.1.18}$$

In the similar way, we get also

$$t = c_2 + \int_{x(0)}^{x(t)} \frac{dx}{\sqrt{(1-x^2)}\sqrt{(k'^2+k^2x^2)}} \tag{4.1.19}$$

from $\frac{d}{dt}cn(t,k) = -dn(t,k)sn(t,k)$; and

$$t = c_3 + \int_{r(0)}^{r(t)} \frac{dx}{\sqrt{(1-r^2)}\sqrt{(r^2-k'^2)}} \tag{4.1.20}$$

from $\frac{d}{dt}dn(t,k) = -k^2sn(t,k)cn(t,k)$.

**Proposition 4.1.4.** *$(-sn(-t,k), cn(-t,k), dn(-t,k))$, $(sn(-t,k), -cn(-t,k), dn(-t,k))$ and $(sn(-t,k), cn(-t,k), -dn(-t,k))$ are solutions of the system (4.1.15) in the case that $(sn(t,k), cn(t,k), dn(t,k))$ is a solution of the system (4.1.15).*

**Proof.** Let us assume $(p(t), q(t), s(t)) = (-sn(-t,k), cn(-t,k), dn(-t,k))$. If we take derivatives of $(p(t), q(t), s(t))$, we get

$$\begin{aligned} p'(t) &= \frac{d}{dt}sn'(-t,k) = cn(-t,k)dn(-t,k) = q(t)s(t) \\ q'(t) &= -\frac{d}{dt}cn'(-t,k) = dn(-t,k)sn(-t,k) = -s(t)p(t) \\ s'(t) &= -\frac{d}{dt}dn'(-t,k) = k^2sn(-t,k)cn(-t,k) = -k^2p(t)q(t) \end{aligned} \tag{4.1.21}$$

It can be concluded that $(p(t), q(t), s(t))$ is a solution also. For other solutions, we follow the same way. □

That's why, this proposition says us that when we take solution with reversing time, it gives us another solution. We call ***reversing symmetries*** for such symmetries.

Moreover, by the proposition, we get thet $sn(t,k)$ is an odd function of $t$; $cn(t,k)$ and $dn(t,k)$ are even functions of $t$ for fixed $k\in(0,1)$.

Now, let us interpret geometrically these solutions by using the differential equations

$$sn^2(t,k) + cn^2(t,k) = 1 \tag{4.1.22}$$

$$k^2 sn^2(t,k) + dn^2(t,k) = 1 \tag{4.1.23}$$

The first equation gives us a circular cylinder, and the second equation gives us a right elliptic cylinder. Since we study both cylinder in the same case, we look at the intersection of two cylinder. Therefore, we get the inequalities

$$\begin{aligned} -1 &\leq sn(t,k) \leq 1 \\ -1 &\leq cn(t,k) \leq 1 \\ k' &\leq dn(t,k) \leq 1 \end{aligned} \tag{4.1.24}$$

The solution $(sn(t,k), cn(t,k), dn(t,k))$ starts in this intersection region, and it rests in this area for all $t$.

**Theorem 4.1.5. (Theorem of Continuous Dependence of Solutions)** (Hirsch & Smale, 2004) *Let $w' = f(w)$ be a differential equation such that $f$:$\mathbb{R}^n \to \mathbb{R}^n$ is derivable, and let $w(t)$ be a solution of $f(w)$ on the interval $[t_0, t']$ such that $x(t_0) = x_0$. Therefore, there exist a neighborhood $D \subset \mathbb{R}$ of $x_0$ and a constant $M$ in the case that there exists a unique solution $y(t)$ on the interval $[t_0, t']$ such that $y_0 \in D$ and $y(t_0)$=$y_0$. In addition, the given inequality occurs*

$$|y(t) - x(t)| \leq M|y_0 - x_0|e^{M(t-t_0)} \tag{4.1.25}$$

*for all $t \in [t_0, t']$.*

So that the theorem say us that the solutions $x(t)$ and $y(t)$ reste close for $t$ nearby $t_0$ in the case that they move on approximately.

**Proposition 4.1.6.** *If $k$ tends to 0 from the right, we get*

$$sn(t,k) \to \sin(t), \qquad cn(t,k) \to \cos(t), \qquad dn(t,k) \to 1 \tag{4.1.26}$$

and if $k$ tends to 1 from the left, we get

$$sn(t,k) \to \tanh(t), \qquad cn(t,k) \to (t), \qquad dn(t,k) \to (t). \tag{4.1.27}$$

*Moreover, they convergent uniformly on compact sets.*

**Proof.** By using the formulas (4.1.18), (4.1.19), and (4.1.20), we can found these limits. Also we can see easily that this is the solution of the system (4.1.8). □

### 4.2 Periodicity of Jacobi Elliptic Functions

Now, we will try to find the periodicity of the Jacobi elliptic functions. Let us begin by analyzing the function $sn(t,k)$. We have the integral from (4.1.18)

$$\Rightarrow t = c_1 + \int_{z(0)}^{z(t)} \frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}} \tag{4.2.1}$$

where $k < 1$. Let $c_1$ equal to 0. If we analyze the integral (4.2.1), we see that there exist four ramification points which are $z = 1, -1, 1/k, -1/k$. If paths of the integration (4.2.1) circulate around the ramification points or circulate around a pole of the integral (4.2.1), then it is possible that the integral (4.2.1) has more than one valued function in z. Therefore, the Riemann surface is formed by cutting on the projective space along two intervals which are bounded by the ramification points.

Let us assume that the intervals are $[-1, 1/k]$ and $[1, 1/k]$. As we made the ramification process in *Chapter 2, Section 1*, by using an other copy of this space, we glue these two copies along the cuts. Therefore, we obtain a torus. Let us analyze the following torus and define the paths $\lambda_1$ and $\lambda_2$ on the torus

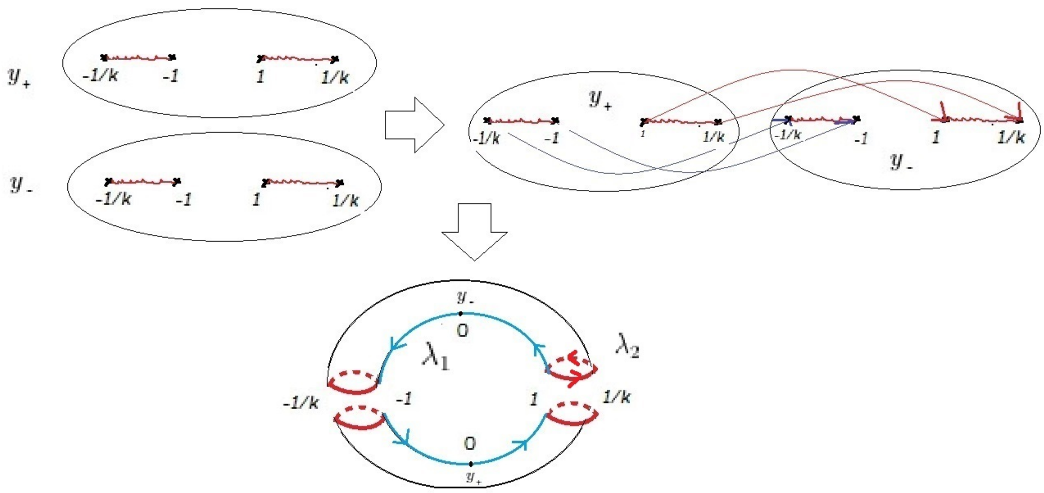


Figure 4.1: Ramification of Jacobi elliptic function $sn$

$$K=\int_0^1 \frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}} \tag{4.2.2}$$

$$K'=\int_1^{1/k} \frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}} \tag{4.2.3}$$

First, we want to calculate the value of the path $\lambda_1$ by using the integrals $K$ and $K'$. While seeing the figure as above, we can make calculation as below.

$$\int_{\lambda_1} \frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}}$$

$$=\int_0^1 \frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}}+\int_1^{-1} -\frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}}+\int_1^0 \frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}}$$

$$=4\int_0^1 \frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}}=4K \tag{4.2.4}$$

Let us apply the similar calculation for the path $\lambda_2$ on the torus.

$$\int_{\lambda_2} \frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}}$$

$$=\int_1^{1/k} \frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}}-\int_{1/k}^{1} -\frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}} \tag{4.2.5}$$

$$=2\int_1^{1/k} \frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}}=2K'$$

Briefly, if there is a path $\mu$ which have initial point in $\mu(0)$ and ending point in $\mu(z)$, it can be found a simple path $\alpha$ which have same initial and ending point. As we obtain the periodicity of the path as above, we describe like

$$\int_{\mu(0)}^{\mu(z)} \frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}} = \int_{\alpha} \frac{dz}{\sqrt{(1-z^2)}\sqrt{(1-k^2z^2)}} + 4p_1K + 2p_2K' \tag{4.2.6}$$

where $p_1$ and $p_2$ are integers.

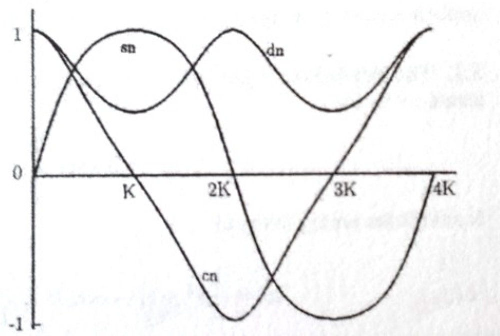


Figure 4.2: Periodicity of Jacobi elliptic functions

The figure as above show periodicity Jacobi elliptic function $sn$, $cn$ and $dn$ in one graph.

### 4.3 Galileo Pendulum

Now, we will apply *Jacobi elliptic functions* to analyze the differential equation of *Galileo pendulum* which is as below

$$ml\frac{d^2}{dt^2}\varphi(t) = -mg\sin\varphi(t) \tag{4.3.1}$$

where the angle $\varphi(t)$ describes the angle depending on the time $t$ which results from the matter, $m$ describes the mass of the matter, and $l$ describes the length of the span. Let us analyze the given system of differential equation as below

$$\gamma' = -\frac{g}{l}\sin\varphi \qquad \varphi' = \gamma \tag{4.3.2}$$

In addition, $\gamma$ describes the angular velocity in the system as above.

**Definition 4.3.1.** (Hirsch & Smale, 2004) *Let $H$:$\mathbb{R}^2 \to \mathbb{R}$ be a $C^\infty$ function. We call that $H$ is* ***Hamiltonian function*** *such that*

$$a' = \frac{\partial H}{\partial b}(a,b)$$

$$b' = -\frac{\partial H}{\partial a}(a,b)$$

*where $(a,b) \in \mathbb{R}^2$. Moreover, such system of differential equation as above is called* ***Hamiltonian system****.*

So that we say that the system (4.3.2) is an *Hamiltonian system*. Therefore,

$$H(\varphi, \gamma) = \frac{1}{2}\gamma^2 - \frac{g}{l}\cos\varphi \tag{4.3.3}$$

$$\Rightarrow \quad H = \frac{1}{2}\varphi'^2 - \frac{g}{l}\cos\varphi \tag{4.3.4}$$

Let us suppose $\varphi(0) = 0$. If we apply the multiplication of $\varphi'$ on both sides of the equation (4.3.1)

$$ml\varphi''\varphi' = -mg\sin\varphi\varphi', \tag{4.3.5}$$

we get

$$\frac{ml}{2}\frac{d}{dt}(\varphi')^2 = mg\frac{d}{dt}\cos\varphi \tag{4.3.6}$$

$$\Rightarrow \quad (\varphi')^2 = \frac{4g}{l}(k^2 - \sin^2(\frac{\varphi}{2})) \tag{4.3.7}$$

where $k^2 = \frac{l(\varphi')^2(0)}{4g}$. Therefore, we find out three situation on the equation (4.3.7) regarding to the value of $k$.

**Situation 1** : If $k = 1$, we have $(\varphi')^2 = \frac{4g}{l}\cos^2(\frac{\varphi}{2})$ and so that $\varphi' = 2\sqrt{\frac{g}{l}}\cos(\frac{\varphi}{2})$.Then we reach that

$$\varphi(t) = \int_0^t \varphi' dt = \int_{\varphi(0)}^{\varphi(t)} 2\sqrt{\frac{g}{l}}\cos(\frac{\varphi}{2})d\varphi \tag{4.3.8}$$

$$t = \sqrt{\frac{l}{g}}\log\tan(\frac{\varphi + \pi}{4}) \tag{4.3.9}$$

**Situation 2** : If $k < 1$, let us define a new variable $z_1$ such that $z_1 = \frac{1}{k}\sin\frac{\varphi}{2}$. When we replace the variable $z_1$ in the equation (4.3.7), we obtain

$$(\frac{dz_1}{dt})^2 = \frac{g}{l}(1-{z_1}^2)(1-k^2{z_1}^2). \tag{4.3.10}$$

So that we can integrate the equation as above

$$t = \int_0^t dt = \int_0^{z_1} \frac{dz_1}{\sqrt{\frac{g}{l}(1-{z_1}^2)(1-k^2{z_1}^2)}} \tag{4.3.11}$$

By previous section, this equation is in the form of *the Jacobi elliptic function sn.*

**Situation 3** : If $k > 1$, in the similar way with situation 2, let us define a new variable $z_2$ such that $z_2 = \sin\frac{\varphi}{2}$. When we replace the variable $z_2$ in the equation (4.3.7), we obtain

$$(\frac{dz_2}{dt})^2 = k^2\frac{g}{l}(1-{z_2}^2)(1-\frac{{z_2}^2}{k^2}). \tag{4.3.12}$$

So that we can integrate the equation as above

$$t = \int_0^t dt = \int_0^{z_2} \frac{dz_2}{\sqrt{\frac{g}{l}(1-{z_2}^2)(1-\frac{{z_2}^2}{k^2})}} \tag{4.3.13}$$

By previous section, this equation is in the form of *the Jacobi elliptic function sn.*

# 5. BIFURCATION OF REACTION DIFFUSION EQUATION

## 5.1 Dissipative Structures

Let us analyze the chemical system which is imagined by *Schlögl* (Lefever, 1978). The following equation describes the kinetic equation by considering the diffusion of $K$

$$\frac{\partial K}{\partial t_1} = a_1 A K^2 - a_2 K^3 - a_3 K + a_4 B + D\frac{\partial^2 K}{\partial r_1^2} \tag{5.1.1}$$

satisfying the conditions $K(0) = K(l) = \epsilon$ where $l$ is length and $\epsilon$ is a solution of (5.1.1) by choosing parameters as below

$$\delta = \frac{a_3}{a_2} \quad \chi = \frac{a_1 A}{a_2} = \frac{\delta}{\epsilon} + \epsilon \quad \epsilon(\delta + \epsilon^2 - \chi\epsilon) = \frac{a_4 B}{a_2}. \tag{5.1.2}$$

If we apply the transformation which are $t = a_2 t_1$, $r = (\frac{a_2}{D})^1/2r_1$, and $K = x + \epsilon$, we can rewrite the equation (5.1.1) as

$$\frac{\partial x}{\partial t} = -x^3 + (\frac{\delta}{\epsilon} - 2\epsilon)x^2 + (\delta - \epsilon^2)x + \frac{\partial^2 x}{\partial r^2} = f(x) + \frac{\partial^2 x}{\partial r^2} \tag{5.1.3}$$

where $f(x) = -x^3 + (\frac{\delta}{\epsilon} - 2\epsilon)x^2 + (\delta - \epsilon^2)x$.

## 5.2 A Solution Analysis on The Reaction Diffusion Equation

### 5.2.1 Dissipative Wave Solution

We shall look for the dissipative wave solution $x(x_1, t) = x(x_1 - ct)$ to the reaction diffusion equation

$$\frac{\partial x}{\partial t} = f(x) + \frac{\partial^2 x}{\partial {x_1}^2} \tag{5.2.1}$$

where $f(x)$ is polynomial. In the case $f(x) = rx(1 - \frac{x}{N})$ the logistic $x(x_1, t)$. As we look for the dissipative wave solution, we assume that $x = x(x_1 - ct)$. Let $z = x_1 - ct$.

So that $\frac{\partial x}{\partial t} = -c\frac{\partial x}{\partial z}$ and $\frac{\partial x^2}{\partial {x_1}^2} = \frac{\partial x^2}{\partial z^2}$. Thus (5.2.1) transformed into a second order non linear equation

$$x'' + cx' + f(x) = 0 \tag{5.2.2}$$

where $x' = \frac{\partial x}{\partial z}$. If we consider the Lyapunov function (Hirsch & Smale, 2004) $\frac{1}{2}(x')^2 + F(x)$ where $\frac{d}{dx}(F(x)) = f(x)$ ,

$$\frac{d}{dt}(\frac{1}{2}(x')^2 + F(x)) = -c(x')^2 \leq 0 \tag{5.2.3}$$

provided the dissipative wave speed $c$ is positive.

Let us assume that 3 roots of $p_1, p_2, p_3$ of $f(x)$ are located in such away that $\eta(-a) = p_1$, $\eta(0) = p_2$, $\eta(a) = p_3$ for some linear function $\eta$ real value $a \neq 0$. Then in making the scale transform $x = Av(\beta z) + B$ choosing suitable $c$, the equation (5.2.2) is transformed into

$$v'' + 3av' - 2v^3 + 2a^2 v = 0 \tag{5.2.4}$$

In other words, the linear equation $\begin{pmatrix} -a & 1 & p_1 \\ 0 & 1 & p_2 \\ a & 1 & p_3 \end{pmatrix} \begin{pmatrix} A \\ B \\ 1 \end{pmatrix} = 0$ has non zero solution $\begin{pmatrix} A \\ B \\ 1 \end{pmatrix}$.

**Proposition 5.2.1.** *The solution to* (5.2.4) *is expressed by Jacobi elliptic function* $sn_{k^2=-1}(u)$

$$v(w) = Me^{-aw} sn_{k^2=-1}(Me^{-aw} + M_1) \tag{5.2.5}$$

*where $M$ and $M_1$ are constants.*

**Proof.** Let $v(w) = Me^{-aw}V(Me^{-aw} + M_1)$ and $u = Me^{-aw} + M_1$. We look for the function $V(u)$.

$$\frac{\partial v}{\partial w}(w) = -Mae^{-aw}V(u) - V'(u)M^2ae^{-2aw} \tag{5.2.6}$$

$$\frac{\partial^2 v}{\partial w^2}(u) = M^3 a e^{-3aw} V''(u) - 2M^2 a^2 e^{-2aw} V'(u) + M a^2 e^{-aw} V(u) \tag{5.2.7}$$

When we replace (5.2.5) and (5.2.6) into (5.2.4), we get

$$\begin{aligned} M^3 a e^{-3aw} V''(u) + 3M^2 a^2 e^{-2aw} V'(u) + M a^2 e^{-aw} V(u) + 3a(-M a e^{-aw} V(u) \\ -M^2 a e^{-2aw} V'(u)) + 2M^3 e^{-3aw} V^3(u) + 2a^2 M e^{-aw} V(u) \\ = -M^3 e^{-3aw}(a^2 V''(u) - 2V^3(u)) \end{aligned} \tag{5.2.8}$$

Now $a^2 V''(u) - 2V^3(u)$ can be integrated by

$$\frac{\partial}{\partial u}((aV'(u))^2 - V^4(u) + 1) = 0$$

$$\Rightarrow a^2 (\frac{\partial V(u)}{\partial u})^2 = V^4(u) - 1$$

$$\Rightarrow u = a \int_{V(0)}^{V(u)} \frac{dV(u)}{\sqrt{(V^2(u)-1)(V^2(u)+1)}} \tag{5.2.9}$$

$$\Rightarrow V(u) = sn_{k^2=-1}(\frac{u}{a}) \tag{5.2.10}$$

by definition, i.e. inverse function to Jacobi elliptic integral. Jacobi elliptic integral for general $k$

$$u = \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}}$$

$$\Rightarrow z = sn(u)$$

$\square$

Now, we aim to transform the integral

$$\frac{u}{a} = \int_{\alpha_i}^{\alpha_j} \frac{dx}{\sqrt{(x^2-1)(x^2+1)}} \tag{5.2.11}$$

to the integral

$$\frac{u}{a} = \int_{\beta_i}^{\beta_j} \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}} \tag{5.2.12}$$

where k is in the domain $(0,1)$. For this purpose, we consider the projective transformation $w=\frac{Az+B}{Cz+D}$ where $AD-BC\neq 0$, especially $AD-BC=1$.

**Definition 5.2.2.** ***The projective transformation*** *is the mapping given by*

$$T(z)=\frac{Az+B}{Cz+D} \tag{5.2.13}$$

*where $T$: $\mathbb{P}_1(\mathbb{C})\rightarrow\mathbb{P}_1(\mathbb{C})$ and $A$, $B$, $C$, and $D$ are complex numbers such that $AD-BC\neq 0$.*

Let us analyze the above integral by using projective transformation and write

$$\begin{aligned}(V^4(z)-1)^{\frac{1}{2}} &= (\prod_{i=1}^{4}(\frac{Az+B}{Cz+D}-\alpha_i))^{\frac{1}{2}}=\frac{(\prod_{i=1}^{4}(Az+B-\alpha_i(Cz+D)))^{\frac{1}{2}}}{(Cz+D)^{\frac{1}{2}}}\\ &= \frac{1}{(Cz+D)^{\frac{1}{2}}}(\prod_{i=1}^{4}(Az+B-\alpha_i(Cz+D)))^{\frac{1}{2}} \end{aligned} \tag{5.2.14}$$

Let $w=\frac{Az+B}{Cz+D}$. Then we can see easily that $dw=\frac{(A(Cz+D)-(Az+D)C)dz}{(Cz+D)^{\frac{1}{2}}}$. Also, we can rewrite such as

$$\prod_{i=1}^{4}(A-\alpha_iC)z+(B-\alpha_iD)=\prod_{i=1}^{4}(A-\alpha_iC)(z-\frac{D\alpha_i-B}{-C\alpha_i+A}).$$

In addition, we can find out that the inverse of T(z) is $T^{-1}(z)=\frac{Dz-B}{-Cz+A}$. The integral (5.2.11) transforms to an integral as below. We choose $\alpha_i$, $\alpha_j$ from the set of the points $\{+1,-1,+i,-i\}$.

$$\int_{\beta_i}^{\beta_j}\frac{\pm dz}{\sqrt{\prod_{i=1}^{4}(A-\alpha_iC)(z-\beta_i)}}=\frac{1}{\sqrt{A^4-C^4}}\int_{\beta_i}^{\beta_j}\frac{\pm dz}{\sqrt{\prod_{i=1}^{4}(z-\beta_i)}} \tag{5.2.15}$$

where $\beta_j=\frac{D\alpha_j-B}{-C\alpha_j+A}=T^{-1}(\alpha_j)$, i.e $T(\beta_j)=\alpha_j$ and $A^4-C^4=k^2$. Therefore,

$$\int_{\alpha_i}^{\alpha_j}\frac{dx}{\sqrt{(x^2-1)(x^2+1)}}=\int_{\beta_i}^{\beta_j}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \tag{5.2.16}$$

We look for transformation $T(z)$ such that $\{\alpha_1=-1,\alpha_2=i,\alpha_3=-i,\alpha_4=1\}$ $\{\beta_1=-\frac{1}{k},\beta_2=\frac{1}{k},\beta_3=-1,\beta_4=1\}$ for some $k\in(0,1)$. This relation gives rise to the following linear equation

$$\begin{pmatrix} -1 & -k & 1 & -k \\ 1 & k & -i & -ki \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.2.17}$$

that has non trivial solution $(A, B, C, D) \neq (0, 0, 0, 0)$ if and only if the determinant of the matrix $\begin{pmatrix} -1 & -k & 1 & -k \\ 1 & k & -i & -ki \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \end{pmatrix}$ equals to $-\frac{2(1-6k+k^2)}{k^2}$. Therefore, the equation (5.2.17) has non trivial solution $k = 3 \pm 2\sqrt{2}$. We choose $k = 3 - 2\sqrt{2}$ since $k \in (0, 1)$. In conclusion, the elliptic integral (5.2.11) is reduced to the elliptic integral (5.2.12) with $k = 3 - 2\sqrt{2}$ with period 34.9958.

### 5.2.2 Analysis on The Bifurcation Sets of The Stationary Reaction Diffusion Equation

Especially, the stationary state of the system $\frac{\partial x}{\partial t} = 0$ (5.1.3) is given by the following non linear differential equation

$$(\frac{dx}{dr})^2 + F(x; d) = 0. \tag{5.2.18}$$

for $F(x; d)$ such that $\frac{1}{2}F'(x; d) = f(x)$ (5.2.1). Let us consider the following fourth order polynomial for $\frac{1}{2}F'(x; d) = f(x)$

$$F(x; d) = -\frac{x^4}{2} + \frac{2d_3x^3}{3} + d_2x^2 + d_0, \tag{5.2.19}$$

here the parameter $d = (d_0, d_2, d_3)$, especially the constant term $d_0$ corresponds to the energy level.

The discriminant of the quartic equation $F(x; d) = 0$ is given by

$$\Delta(d) = d_0(4(4d_2 - {d_3}^2){d_2}^3 + 256{d_0}^2 + d_0(-27{d_3}^4 + 144d_2{d_3}^2 - 128{d_2}^2)). \tag{5.2.20}$$

**Definition 5.2.3.** *We say that the point $(d_2, d_3)$ is on the* ***bifurcation set*** *where $d_2$,$d_3$ are coefficients of the function (5.2.19), if the function $F(x;d) = 0$ (5.2.19) has less than three different critical values.*

Its bifurcation loci i.e. the discriminant of $\Delta(d)$ as a polynomial in $d_0$ is equal to

$$Bif(d_2, d_3) = {d_2}^6 {d_3}^2 ({d_3}^2 + 4d_2)^3 (2{d_3}^2 + 9d_2)^2. \tag{5.2.21}$$

If we replace $d_2$ by $(\delta - \epsilon^2)$ and $d_3$ by $(\delta/\epsilon - 2\epsilon)$, then we get

$$B(\delta, \epsilon) = \frac{\delta^6(\delta - 2\epsilon^2)^2(\delta - \epsilon^2)^6(-\epsilon^4 + \delta\epsilon^2 + 2\delta^2)^2}{\epsilon^{12}}. \tag{5.2.22}$$

Also, if we replace $d_2$ by $(\delta - \epsilon^2)$ and $d_3$ by $(\delta/\epsilon - 2\epsilon)$ on $F(x;d)$, we get

$$H(x;d) = -\frac{x^4}{2} + \frac{2}{3}(\delta/\epsilon - 2\epsilon)x^3 + (\delta - \epsilon^2)x^2 + d_0 \tag{5.2.23}$$

In (5.2.21), it can be concluded that the curves $A_1 : \delta = 2\epsilon^2$, $A_2 : \delta = \epsilon^2$, $A_3 : \delta = \epsilon^2/2$ $A_4 : \delta = 0$, and $A_5 : \delta = -\epsilon^2$ are the bifurcation sets of the reaction diffusion equation. Let us analyze the solutions of the equation on the bifurcation sets.

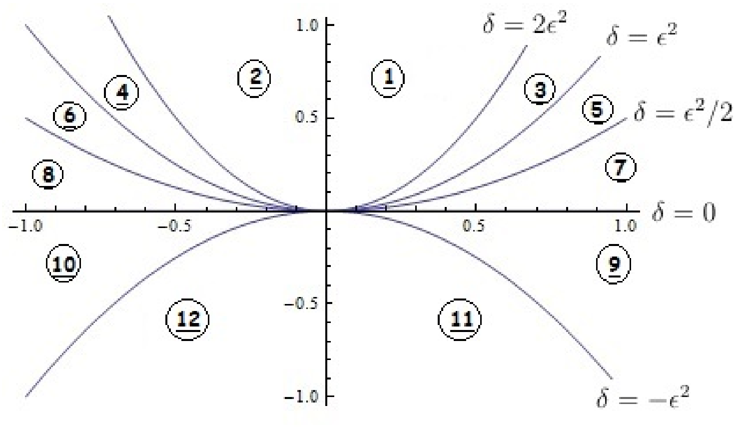


Figure 5.1: Bifurcation diagram of $H(x;d)$

When we solve $H(x;d)$ either on bifurcation sets or between bifurcation sets, we see that there are 3 types root which are 4 different real roots, 2 different real and 2 imaginary complex roots, and 2 double imaginary complex roots. We will analyze real roots in the cases. Let us look at the case of 2 imaginary complex roots. Let $x_1 = m - ni$, $x_2 = m + ni$, $x_3 = p$ and $x_4 = r$ be roots of $H(x; d)$ where $m$, $n$, $p$, $r$ are real numbers. If we apply projective transformation, we reach that the periodicity of solution is

$$4 \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1+k^2z^2)}}$$

.

Moreover, we see that there is no real periodicity if $H(x; d)$ has 2 double imaginary complex roots since the integral is divergent.

**Case 1:** Let $(\epsilon, \delta) = (1, 2)$ be a point on $A_1$ of the bifurcation set. So that in the case $d_0 = -1/2$ the equation transforms to

$$F_{11} = -\frac{x^4}{2} + x^2 - \frac{1}{2} \tag{5.2.24}$$

We have choices the parameters $\epsilon = 1$ or $\epsilon = -1$ and $\delta = 2$ boundary corresponds to a point on the bifurcation set that is the boundary between the domains (1) and (3) or (2) and (4), i.e. on the curve $\delta = 2\epsilon^2$. If we choose $d_0 = 0$, then we get 2 real double roots $x = -1$ and $x = 1$. We study the critical points and critical values of the function $F_{11}$. The calculation above shows that the polynomial $F_{11}$ has three critical points $(-1, 0)$, $(0, -1/2)$, $(1, 0)$. But the critical values are $-1/2$ and 0. That is to say the function has only two instead of three critical values. Therefore, we can say that the point $(\epsilon, \delta) = (\pm 1, 2)$ is on the bifurcation set.

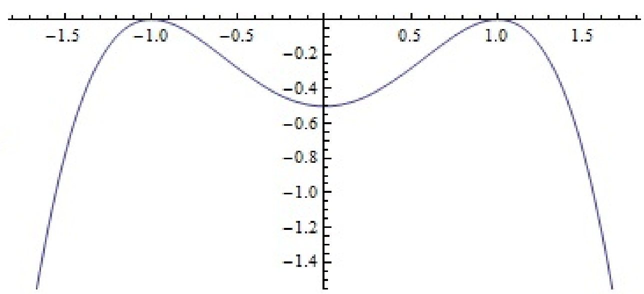


Figure 5.2: Graph of $H(x; (1, 2, 0))$

In other hand, when we take $-1/2 < d_0 < 0$, we get 4 distinct real roots. Therefore, we can apply transformation method in this interval to find solution. For example, let $d_0 = -1/3$. So that we get the function

$$F_{11a} = -\frac{x^4}{2} + x^2 - \frac{1}{3} \tag{5.2.25}$$

We see that the roots of $F_{11a}$ are $x_1 = -\sqrt{\frac{1}{3}(3-\sqrt{3})}$, $x_2 = \sqrt{\frac{1}{3}(3-\sqrt{3})}$, $x_3 = -\sqrt{\frac{1}{3}(3+\sqrt{3})}$, $x_4 = \sqrt{\frac{1}{3}(3+\sqrt{3})}$.

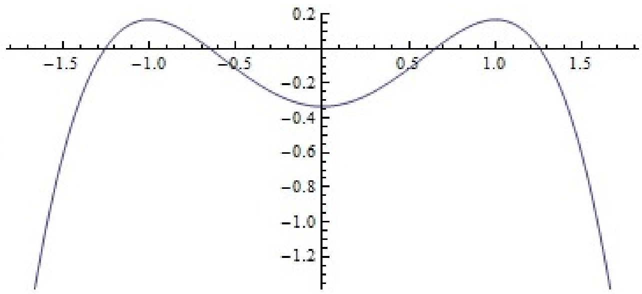


Figure 5.3: Graph of $H(x;(1,2,-\frac{1}{3})$

In the symmetric case as the above example, we will study on an easier method. Let $\pm\alpha$ and $\pm\beta$ the roots of the equation $F(x;d) = 0$ where $\beta > \alpha > 0$. We will try to transform of the Jacobi elliptic function $sn(r,k)$ form. We have periods

$$t = \int_{\alpha}^{\beta} \frac{dx}{\sqrt{1/2(\alpha^2 - x^2)(\beta^2 - x^2)}}. \tag{5.2.26}$$

If we substitute $x = \alpha z$, we find out

$$\sqrt{2}\int_{1}^{\frac{\beta}{\alpha}} \frac{\alpha dz}{\sqrt{(\alpha^2 - \alpha^2 z^2)(\beta^2 - \alpha^2 z^2)}}. \tag{5.2.27}$$

$$= \sqrt{2}\int_{1}^{\frac{\beta}{\alpha}} \frac{\alpha}{\alpha\beta} \frac{dz}{\sqrt{(1-\alpha^2 z^2)(1 - \frac{\alpha^2}{\beta^2} z^2)}} \tag{5.2.28}$$

If we establish the equality $k = \frac{\alpha}{\beta}$, we have

$$= \frac{\sqrt{2}}{\beta}\int_{1}^{\frac{\beta}{\alpha}} \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}} \tag{5.2.29}$$

Hence we have a solution $x(r) = \alpha sn(\frac{\beta r}{\sqrt{2}}, \frac{\alpha}{\beta})$ with periods $\frac{2\sqrt{2}}{\beta}K'(\frac{\alpha}{\beta})$ and $\frac{4\sqrt{2}}{\beta}K(\frac{\alpha}{\beta})$. Therefore, we can see easily that the solution of $F_{11a}$ is

$$x(r) = \sqrt{\frac{1}{3}(3-\sqrt{3})}sn(\frac{\sqrt{\frac{1}{3}(3+\sqrt{3})}r}{\sqrt{2}}, \frac{\sqrt{(3-\sqrt{3})}}{\sqrt{(3+\sqrt{3})}}) \tag{5.2.30}$$

with periods $\frac{2\sqrt{2}}{\sqrt{\frac{1}{3}(3+\sqrt{3})}}K'(\frac{\sqrt{(3-\sqrt{3})}}{\sqrt{(3+\sqrt{3})}})$ and $\frac{4\sqrt{2}}{\sqrt{\frac{1}{3}(3+\sqrt{3})}}K(\frac{\sqrt{(3-\sqrt{3})}}{\sqrt{(3+\sqrt{3})}})$.

**Case 2:** Let $(\epsilon, \delta) = (-1, 1)$ be a point on $A_2$ of the bifurcation set. So that in the case $d_0 = 0$ the equation transforms to

$$F_{12} = \frac{2x^3}{3} - \frac{x^4}{2} \tag{5.2.31}$$

We have choices the parameters $\epsilon = -1$ and $\delta = 1$ boundary corresponds to a point on the bifurcation set that is the boundary between the domains (3) and (5) or (4) and (6), i.e. on the curve $\delta = \epsilon^2$. If we choose $d_0 = 0$, then we get 2 real roots $x = 4/3$ and $x = 0$ where $x = 0$ is triple root, i.e. the inflection point. We study the critical points and critical values of the function $F_{12}$. The calculation above shows that the polynomial $F_{12}$ has two critical points $(0, 0)$, $(1, 1/6)$.

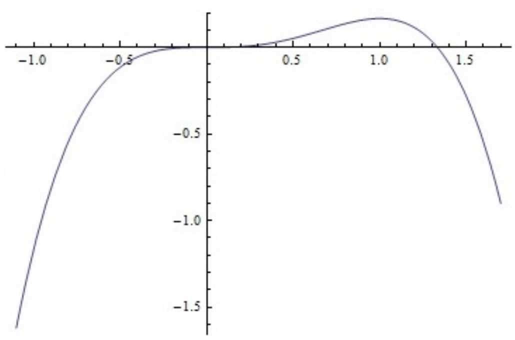


Figure 5.4: Graph of $H(x; (-1, 1, 0))$

But the critical values are 0 and 1/6. That is to say the function has only two instead of three critical values. Therefore, we can say that the point $(\epsilon, \delta) = (-1, 1)$ is on the bifurcation set.

However, if we suppose $d_0 \neq 0$, we see that there exist two double different conjugate complex roots of $F_{12}$.

**Case 3:** Let $(\epsilon, \delta) = (-1, 1/2)$ be a point on $A_3$ of the bifurcation set. In the case $d_0 = 0$ the equation transforms to

$$F_{13} = -\frac{x^4}{2} + x^3 - \frac{x^2}{2} \tag{5.2.32}$$

We have choices the parameters $\epsilon = -1$ and $\delta = 1/2$ boundary corresponds to a point on the bifurcation set that is the boundary between the domains (6) and (8), i.e. on the curve $\delta = 2\epsilon^2/2$. If we choose $d_0 = 0$, then we get 2 real double roots $x = 0$ and $x = 1$. We study the critical points and critical values of the function $F_{13}$. The calculation above shows that the polynomial $F_{13}$ has three critical points $(0, 0)$, $(1/2, -1/32)$, $(1, 0)$. But the critical values are $-1/32$ and $0$. That is to say the function has only two instead of three critical values. Therefore, we can say that the point $(\epsilon, \delta) = (-1, 1/2)$ is on the bifurcation set.

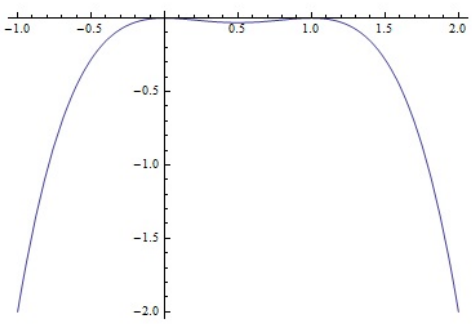


Figure 5.5: Graph of $H(x; (-1, \frac{1}{2}, 0))$

If we assume that $d_0 \neq 0$, we get three different types root of $F(x; d) = 0$ as below

**Situation 1** : If $0 < d_0 < 1/32$, there are four distinct real roots. For example; if $d_0 = 1/64$, the roots of $H(x; d) = 0$ are $x_1 = \frac{1}{4}(2 - \sqrt{(2 - \sqrt{2})})$, $x_2 = \frac{1}{4}(2 + \sqrt{(2 - \sqrt{2})})$, $x_3 = \frac{1}{4}(2 - \sqrt{(2 + \sqrt{2})})$ and $x_4 = \frac{1}{4}(2 + \sqrt{(2 + \sqrt{2})})$

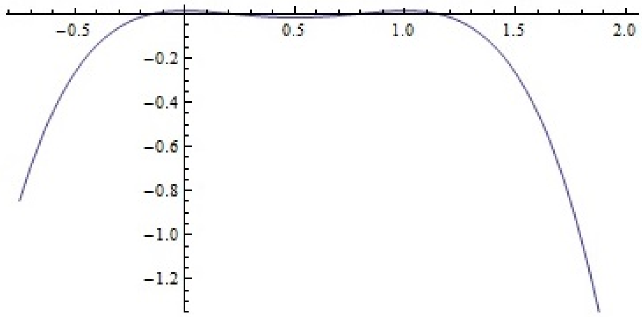


Figure 5.6: Graph of $H(x;(-1,\frac{1}{2},\frac{1}{64}))$

If we apply the transformation $x' = x + \frac{1}{2}$, we get

$$F_{13a} = \frac{1}{64} - \frac{1}{2}(\frac{1}{2} + x')^2 + (\frac{1}{2} + x')^3 - \frac{1}{2}(\frac{1}{2} + x')^4 \tag{5.2.33}$$

Therefore, we can see that the roots of $H(x;d) = 0$ are $x_1 = -\sqrt{\frac{1}{4} + \frac{1}{4\sqrt{2}}}$, $x_2 = \sqrt{\frac{1}{4} + \frac{1}{4\sqrt{2}}}$, $x_3 = -\frac{1}{2}\sqrt{\frac{1}{2}(2-\sqrt{2})}$ and $x_4 = \frac{1}{2}\sqrt{\frac{1}{2}(2-\sqrt{2})}$. Since the roots are symmetric, we can apply directly the method in (5.2.29)

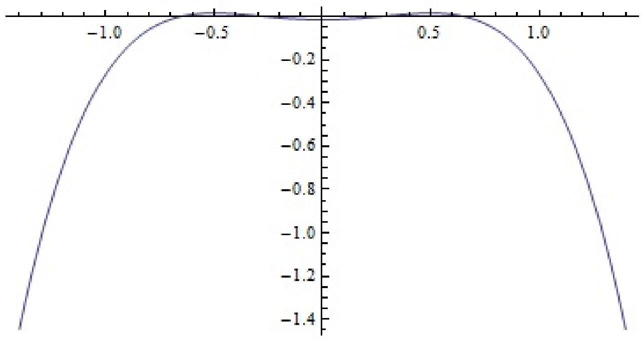


Figure 5.7: Graph of Graph of $F_{13a}(x';d)$

So that in this case, the solution of $H(x;d) = 0$ is

$$x'(t) = \sqrt{\frac{1}{4} + \frac{1}{4\sqrt{2}}}sn(\frac{\sqrt{(2-\sqrt{2})}t}{4}, \frac{1}{\sqrt{2}}) \tag{5.2.34}$$

Because of the transformation $x' = x + \frac{1}{2}$, we get the solution

$$x(t) = \sqrt{\frac{1}{4} + \frac{1}{4\sqrt{2}} sn(\frac{\sqrt{(2-\sqrt{2})}t}{4}, \frac{1}{\sqrt{2}})} - \frac{1}{2} \tag{5.2.35}$$

with periods $4\sqrt{2-\sqrt{2}}K'(\frac{1}{\sqrt{2}})$ and $8\sqrt{2-\sqrt{2}}K(\frac{1}{\sqrt{2}})$.

**Situation 2** : If $d_0 < 0$, there are two double distinct complex conjugate roots. For example; if $d_0 = -1/16$, the roots of $F(x; d) = 0$ are $x_1 = 1/2(1 - \sqrt{1 - i\sqrt{2}})$, $x_2 = 1/2(1 + \sqrt{1 - i\sqrt{2}})$, $x_3 = 1/2(1 - \sqrt{1 + i\sqrt{2}})$ and $x_4 = 1/2(1 + \sqrt{1 + i\sqrt{2}})$.

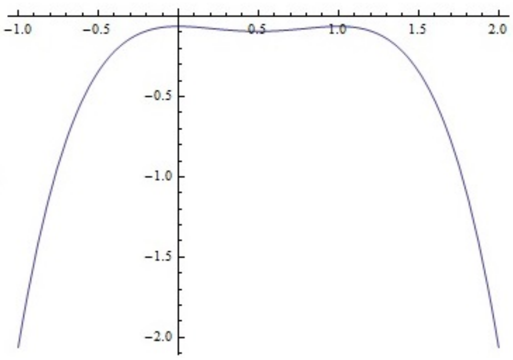


Figure 5.8: Graph of $H(x; (-1, \frac{1}{2}, -\frac{1}{64}))$

**Situation 3** : If $1/32 < d_0$, there are one double complex conjugate roots and two distinct real roots. For example; if $d_0 = 1/16$, the roots of $H(x; d) = 0$ are $x_1 = 1/2(1 - i\sqrt{-1 + \sqrt{2}})$, $x_2 = 1/2(1 + i\sqrt{-1 + \sqrt{2}})$, $x_3 = 1/2(1 - \sqrt{1 + \sqrt{2}})$ and $x_4 = 1/2(1 + \sqrt{1 + \sqrt{2}})$.

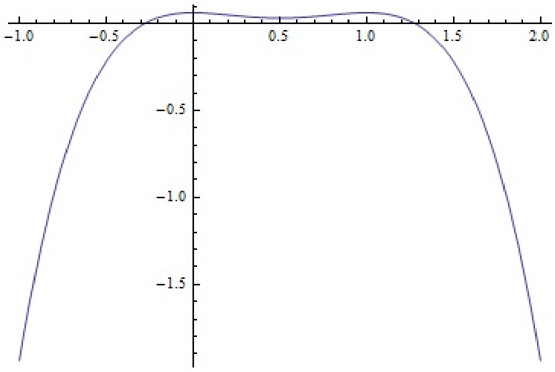


Figure 5.9: Graph of $H(x; (-1, \frac{1}{2}, \frac{1}{16}))$

**Case 4:** Let $(\epsilon, \delta) = (-1, -1)$ be a point on $A_5$ of the bifurcation set. So that in the case $d_0 = 0$ the equation transforms to

$$F_{14} = -\frac{x^4}{2}+2x^3-2x^2 \tag{5.2.36}$$

We have choices the parameters $\epsilon = -1$ and $\delta = -1$ boundary corresponds to a point on the bifurcation set that is the boundary between the domains (10) and (12), i.e. on the curve $\delta = -\epsilon^2$. If we choose $d_0 = 0$, then we get 2 real double roots $x = 0$ and $x = 2$. We study the critical points and critical values of the function $F_{14}$. The calculation above shows that the polynomial $F_{14}$ has three critical points $(0, 0)$, $(1, -1/2)$, $(2, 0)$. But the critical values are $-1/2$ and $0$. That is to say the function has only two instead of three critical values. Therefore, we can say that the point $(\epsilon, \delta) = (-1, 2)$ is on the bifurcation set.

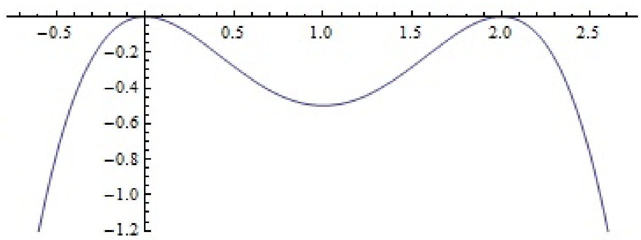


Figure 5.10: Graph of $H(x; (-1, -1, 0)$

**Situation 1** : If $0 < d_0 < \frac{1}{2}$, there are four distinct real roots. For example; if $d_0 = \frac{1}{3}$, the roots of $H(x; d) = 0$ are $x_1 = \frac{1}{3}(3 - \sqrt{3(3-\sqrt{6})})$, $x_2 = \frac{1}{3}(3 + \sqrt{3(3-\sqrt{6})})$, $x_3 = \frac{1}{3}(3 - \sqrt{(3+\sqrt{6})})$ and $x_4 = \frac{1}{3}(3 + \sqrt{(3+\sqrt{6})})$.

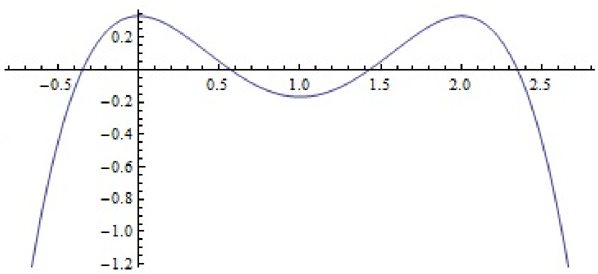


Figure 5.11: Graph of $H(x; (-1, -1, \frac{1}{3})$

If we apply the transformation $x' = x + 1$, we get

$$F_{14a} = \frac{1}{3} - 2(1 + x')^2 + 2(1 + x')^3 - \frac{1}{2}(1 + x')^4 \quad (5.2.37)$$

Therefore, we can see that the roots of $H(x; d) = 0$ are $x_1 = -\sqrt{\frac{1}{3}(3 - \sqrt{6})}$, $x_2 = -\sqrt{\frac{1}{3}(3 + \sqrt{6})}$, $x_3 = \sqrt{\frac{1}{3}(3 - \sqrt{6})}$ and $x_4 = \sqrt{\frac{1}{3}(3 + \sqrt{6})}$. Since the roots are symmetric, we can apply directly the method in (5.2.29).

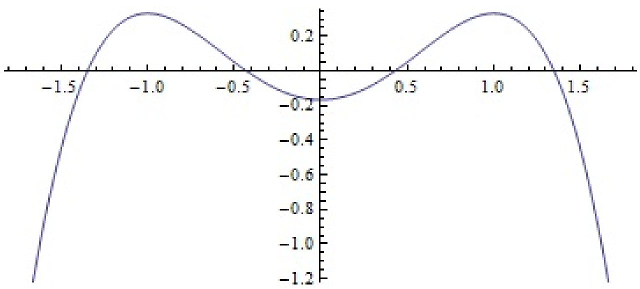


Figure 5.12: Graph of $F_{14a}$

So that in this case, the solution of $H(x; d) = 0$ is

$$x'(t) = \sqrt{\frac{1}{3}(3 - \sqrt{6})}sn(\sqrt{\frac{1}{2} + \frac{1}{\sqrt{6}}}t, \sqrt{5 - 2\sqrt{6}}) \quad (5.2.38)$$

Because of the transformation $x' = x + 1$, we get the solution

$$x(t) = \sqrt{\frac{1}{3}(3 - \sqrt{6})}sn(\sqrt{\frac{1}{2} + \frac{1}{\sqrt{6}}}t, \sqrt{5 - 2\sqrt{6}}) - 1 \quad (5.2.39)$$

with periods $2\sqrt{6 - 2\sqrt{6}}K'(\sqrt{5 - 2\sqrt{6}})$ and $4\sqrt{6 - 2\sqrt{6}}K(\sqrt{5 - 2\sqrt{6}})$.

**Situation 2** : If $d_0 < 0$, there are two double distinct complex conjugate roots. For example; if $d_0 = -1$, the roots of $H(x; d) = 0$ are $x_1 = 1 - \sqrt{1 - i\sqrt{2}}$, $x_2 = 1 + \sqrt{1 - i\sqrt{2}}$, $x_3 = 1 - \sqrt{1 + i\sqrt{2}}$ and $x_4 = 1 + \sqrt{1 + i\sqrt{2}}$.

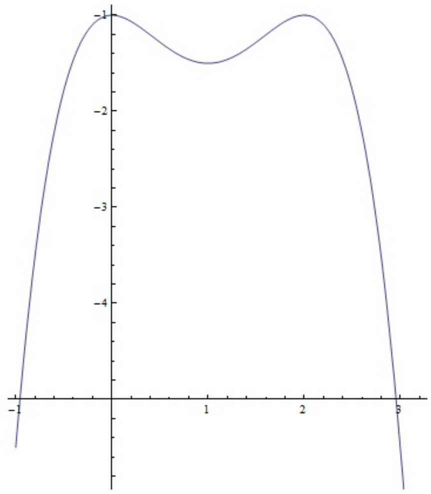

Figure 5.13: Graph of $H(x;(-1,-1,-1))$

**Situation 3** : If $\frac{1}{2} < d_0$, there are one double complex conjugate roots and two distinct real roots. For example; if $d_0 = 1$, the roots of $H(x;d) = 0$ are $x_1 = 1 - i\sqrt{-1+\sqrt{2}}$, $x_2 = 1 + i\sqrt{-1+\sqrt{2}}$, $x_3 = 1 - \sqrt{1+\sqrt{2}}$ and $x_4 = 1 + \sqrt{1+\sqrt{2}}$.

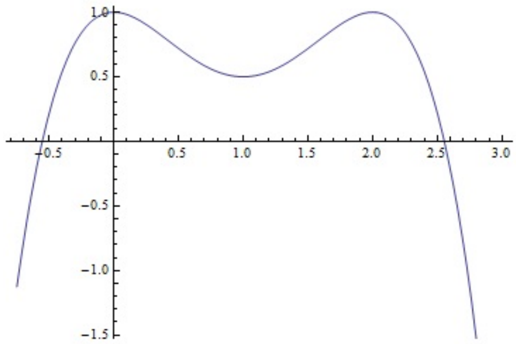

Figure 5.14: Graph of $H(x;(-1,-1,1))$

### 5.2.3 Analysis Between The Bifurcation Sets of The Reaction Diffusion Equation

**Proposition 5.2.4.** *Let $x_1$, $x_2$, $x_3$ and $x_4$ be 4 different real roots of $H(x;d)$ such that $x_1 < x_2 < x_3 < x_4$. Therefore, $H(x;d)$ has solutions with period*

$$sn_k^{-1} 1 \qquad where \quad k = \sqrt{\frac{(x_3 - x_2)}{(x_4 - x_2)} \cdot \frac{(x_4 - x_1)}{(x_3 - x_1)}}$$

**Case 1:** Let $(\epsilon, \delta) = (1, 3)$ be a point between on the region (1). In the case $d_0 = 0$ the equation transforms to

$$G_{11} = 4x + 2x^2 - 2x^3 \tag{5.2.40}$$

We have choices the parameters $\epsilon = 1$ and $\delta = 3$ boundary corresponds to a point out of the bifurcation set that is in the domain (1), i.e. on the region $\delta > 2\epsilon^2$. When we choose $d_0 = 0$, then we get 3 real roots $x_1 = 0$, $x_2 = \frac{2}{3}(1 - \sqrt{10})$ and $x_3 = \frac{2}{3}(1 + \sqrt{10})$. We study the critical points and critical values of the function $G_{11}$. The calculation above shows that the polynomial $G_{11}$ has three critical points $(-1, \frac{5}{6})$, $(0, 0)$, $(2, \frac{16}{3})$ and the critical values are $\frac{5}{6}$, 0 and $\frac{16}{3}$. Therefore, we can say that the point $(\epsilon, \delta) = (1, 3)$ is not on the bifurcation set.

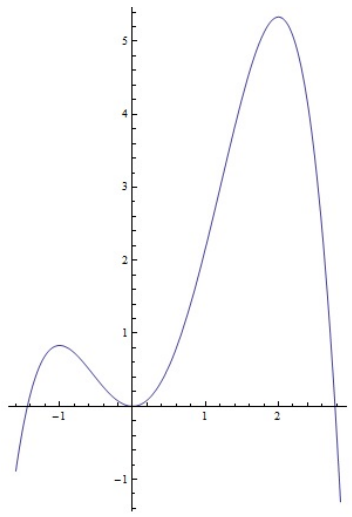


Figure 5.15: Graph of $G_{11}$

If we assume that $d_0 \neq 0$, we get three different types root of $H(x; d) = 0$ as below

**Situation 1** : If $-\frac{5}{6} < d_0 < 0$, there are four distinct real roots. For example; if $d_0 = -\frac{44649}{64220}$, the roots of $H(x; d) = 0$ are $x_1 = -1.2$, $x_2 = 2.75$, $x_3 = -0.755582$ and $x_4 = 0.557668$.

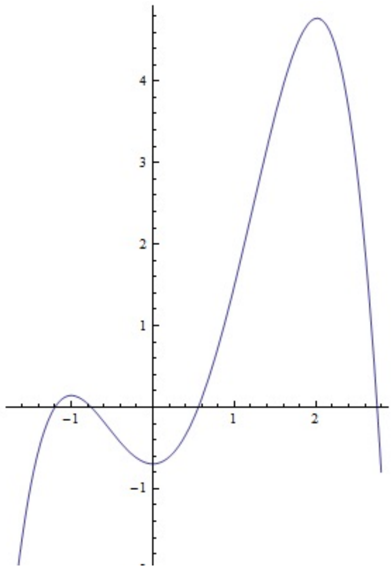


Figure 5.16: Graph of $H(x;(1,3,-\frac{44649}{64220})$

Since all roots of $H(x;d)=0$ is real in this situation, we can apply the projective transformation and we find $k=0.435126$ in the integral of Jacobi elliptic function $sn(r,k)$. So that by using the value $k$, we find the solution of $H(x;d)$ as the following graphic with period 0.0802142.

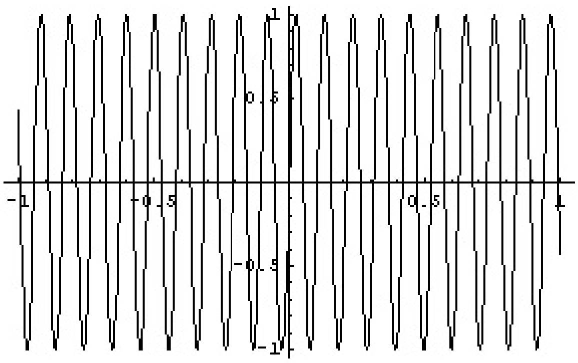


Figure 5.17: Graph of solution of $H(x;(1,3,-\frac{44649}{64220}))$

**Situation 2** : If $0<d_0$ or $-\frac{16}{3}<d_0<-\frac{5}{6}$, there are one double complex conjugate roots and two distinct real roots. For example; if $d_0=\frac{1}{2}$, the roots of $H(x;d)=0$ are $x_1=0.0339426-0.480065i$, $x_2=0.0339426+0.480065i$, $x_3=-1.53934$ and $x_4=2.80479$.

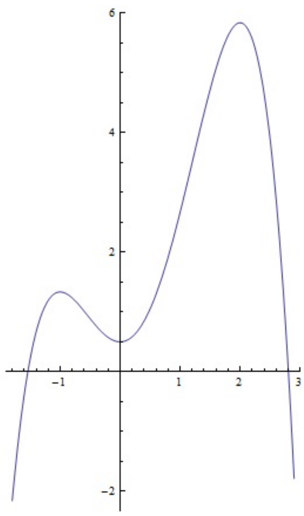


Figure 5.18: Graph of $H(x;(1,3,\frac{1}{2}))$

**Situation 3** : If $d_0 < -\frac{16}{3}$, there are two double distinct complex conjugate roots. For example; if $d_0 = -6$, the roots of $H(x;d) = 0$ are $x_1 = -1.36302 + 0.990422i$, $x_2 = -1.36302 - 0.990422i$, $x_3 = 2.02968 - 0.328026i$ and $x_4 = 2.02968 + 0.328026i$

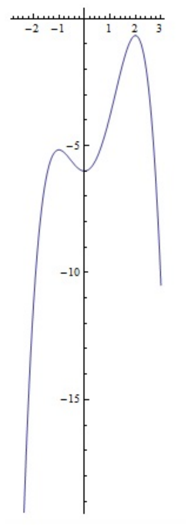


Figure 5.19: Graph of $H(x;(1,3,-6))$

**Case 2:** Let $(\epsilon,\delta) = (1,\frac{3}{2})$ be a point on the region (3). So that in the case $d_0 = 0$ the equation transforms to

$$G_{12} = \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2} \tag{5.2.41}$$

We have choices the parameters $\epsilon = 1$ and $\delta = \frac{3}{2}$ boundary corresponds to a point out of the bifurcation set that is between the domains (1) and (3), i.e. on the region $\epsilon^2 < \delta < 2\epsilon^2$. When we choose $d_0 = 0$, then we get 3 real roots $x_1 = 0$, $x_2 = \frac{2}{3}(1-\sqrt{10})$ and $x_3 = \frac{2}{3}(1+\sqrt{10})$. We study the critical points and critical values of the function $G_{12}$. The calculation above shows that the polynomial $G_{12}$ has three critical points $(-1, \frac{1}{3})$, $(0, 0)$, $(\frac{1}{2}, \frac{5}{96})$ and the critical values are $\frac{1}{3}$, 0 and $\frac{5}{96}$. Therefore, we can prove that the point $(\epsilon, \delta) = (1, \frac{3}{2})$ is not on the bifurcation set.

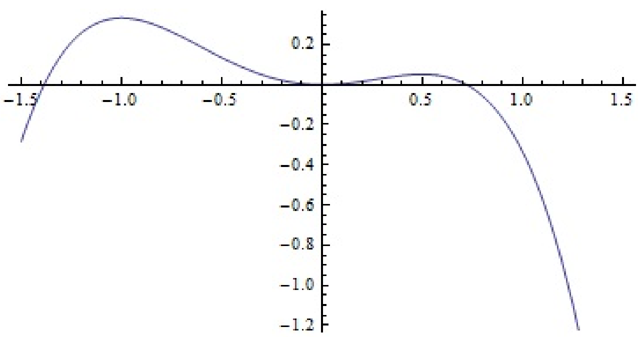


Figure 5.20: Graph of $G_{12}$

If we assume that $d_0 \neq 0$, we get three different types root of $H(x; d) = 0$ as below

**Situation 1** : If $-\frac{5}{96} < d_0 < 0$, there are four distinct real roots. For example; if $d_0 = -\frac{2028807}{23360000}$, the roots of $H(x; d) = 0$ are $x_1 = -1.35$, $x_2 = 0.55$, $x_3 = -0.373794$ and $x_4 = 0.625849$.

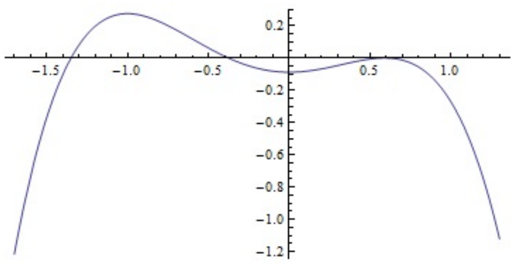


Figure 5.21: Graph of $H(x; (1, \frac{3}{2}, -\frac{2028807}{23360000}))$

Since all roots of $H(x; d) = 0$ is real in this situation, we can apply the projective transformation and we find $k = 0.670223$ in the integral of Jacobi elliptic function

$sn(t,k)$. So that by using the value $k$, we find the solution of $H(x;d)$ as the following graphic with period 0.940988.

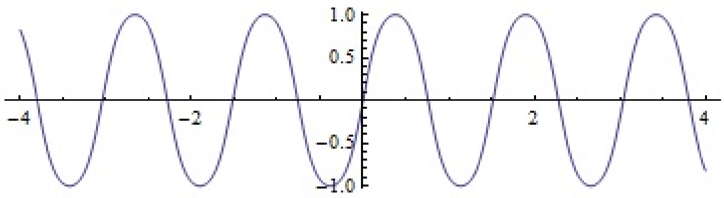


Figure 5.22: Graph of solution of $H(x;(1,\frac{3}{2},-\frac{2028807}{23360000}))$

**Situation 2** : If $0 < d_0$ or $-\frac{1}{3} < d_0 < -\frac{5}{96}$, there are one double complex conjugate roots and two distinct real roots. For example; if $d_0 = -\frac{1}{4}$, the roots of $H(x;d) = 0$ are $x_1 = 0.632878 - 0.414422i$, $x_2 = 0.632878 + 0.414422i$, $x_3 = -1.21089$ and $x_4 = -0.721533$.

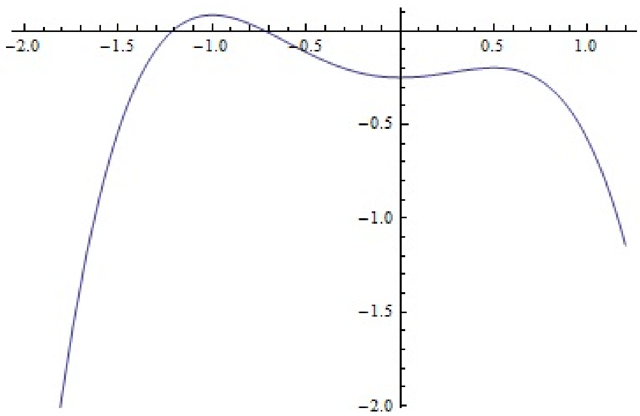


Figure 5.23: Graph of $H(x;(1,\frac{3}{2},-\frac{1}{4}))$

**Situation 3** : If $d_0 < -\frac{1}{3}$, there are two double distinct complex conjugate roots. For example; if $d_0 = -\frac{1}{2}$, the roots of $H(x;d) = 0$ are $x_1 = 0.720376 - 0.554882i$, $x_2 = 0.720376 + 0.554882i$, $x_3 = -1.05371 - 0.314845i$ and $x_4 = -1.05371 + 0.314845i$.

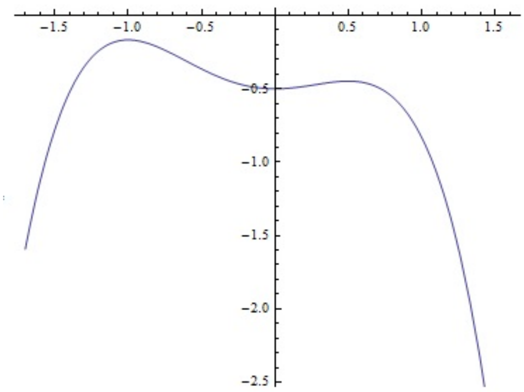


Figure 5.24: Graph of $H(x;(1,\frac{3}{2},-\frac{1}{2}))$

**Case 3:** Let $(\epsilon,\delta)=(1,\frac{3}{4})$ be a point on the region (5). In the case $d_0=0$ the equation transforms to

$$G_{13}=-\frac{x^2}{4}-\frac{5x^3}{6}-\frac{x^4}{2} \tag{5.2.42}$$

We have choices the parameters $\epsilon=1$ and $\delta=\frac{3}{4}$ boundary corresponds to a point out of the bifurcation set that is between the domains (3) and (5), i.e. on the region $\frac{1}{2}\epsilon^2<\delta<\epsilon^2$. When we choose $d_0=0$, then we get 3 real roots $x_1=0$, $x_2=\frac{1}{6}(-5-\sqrt{7})$ and $x_3=\frac{1}{6}(-5+\sqrt{7})$. We study the critical points and critical values of the function $G_{13}$. The calculation above shows that the polynomial $G_{13}$ has three critical points $(-1,\frac{1}{12})$, $(0,0)$, $(-\frac{1}{4},-\frac{7}{1536})$ and the critical values are $\frac{1}{12}$, 0 and $-\frac{7}{1536}$. Therefore, we prove that the point $(\epsilon,\delta)=(1,\frac{3}{4})$ is not on the bifurcation set.

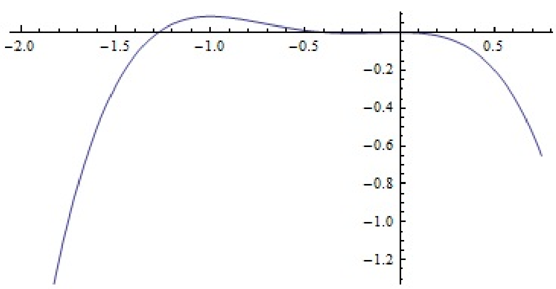


Figure 5.25: Graph of $G_{13}$

If we assume that $d_0\neq 0$, we get three different types root of $H(x;d)=0$ as below

**Situation 1** : If $0 < d_0 < \frac{7}{1536}$, there are four distinct real roots. For example; if $d_0 = \frac{136}{40625}$, the roots of $H(x;d) = 0$ are $x_1 = -1.28$, $x_2 = 0.1$, $x_3 = -0.317423$ and $x_4 = -0.164789$.

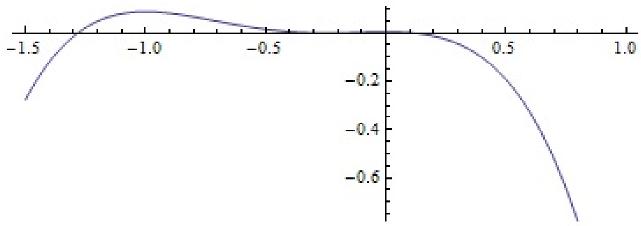


Figure 5.26: Graph of $H(x;(1,\frac{3}{4},\frac{136}{40625})$

Since all roots of $H(x;d) = 0$ is real in this situation, we can apply the projective transformation and we find $k = 0.14946$ in the integral of Jacobi elliptic function $sn(t,k)$. So that by using the value $k$, we find the solution of $H(x;d)$ as the following graphic with period 3.76741.

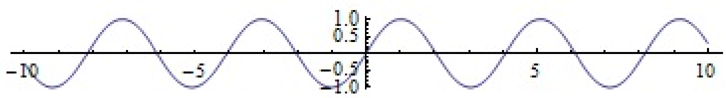


Figure 5.27: Graph of solution of $H(x;(1,\frac{3}{4},\frac{136}{40625})$

**Situation 2** : If $-\frac{1}{12} < d_0 < 0$ or $\frac{7}{1536} < d_0$, there are one double complex conjugate roots and two distinct real roots. For example; if $d_0 = \frac{1}{2}$, the roots of $H(x;d) = 0$ are $x_1 = -0.377602 - 0.876168i$, $x_2 = -0.377602 + 0.876168i$, $x_3 = -1.59866$ and $x_4 = 0.687197$.

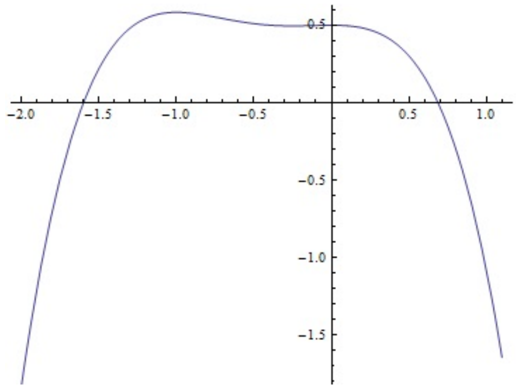


Figure 5.28: Graph of $H(x;(1,\frac{3}{4},\frac{1}{2}))$

**Situation 3** : If $d_0 < -\frac{1}{12}$, there are two double distinct complex conjugate roots. For example; if $d_0 = -\frac{1}{8}$, the roots of $H(x;d) = 0$ are $x_1 = 0.204409 + 0.424413i$, $x_2 = 0.204409 - 0.424413i$, $x_3 = -1.03774 - 0.222884i$ and $x_4 = -1.03774 + 0.222884i$

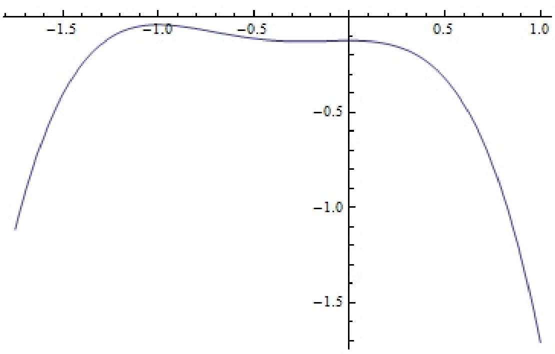


Figure 5.29: Graph of $H(x;(1,\frac{3}{4},-\frac{1}{12}))$

**Case 4:** Let $(\epsilon,\delta) = (1,\frac{1}{3})$ be a point between on the region (7). So that in the case $d_0 = 0$ the equation transforms to

$$G_{14} = -\frac{2x^2}{3} - \frac{10x^3}{9} - \frac{x^4}{2} \tag{5.2.43}$$

We have choices the parameters $\epsilon = 1$ and $\delta = \frac{1}{3}$ boundary corresponds to a point out of the bifurcation set that is in the domain (5), i.e. on the region $0 < \delta < \frac{1}{2}\epsilon^2$. When we choose $d_0 = 0$, then we get 3 real roots $x_1 = 0$, $x_2 = \frac{2}{9}(-5 - i\sqrt{2})$ and $x_3 = \frac{2}{9}(-5 + i\sqrt{2})$. We study the critical points and critical values of the function $G_{14}$.

The calculation above shows that the polynomial $G_{14}$ has three critical points $(-1, \frac{1}{18})$, $(-\frac{2}{3}, -\frac{16}{243})$, $(0,0)$ and the critical values are $-\frac{16}{243}$, 0 and $\frac{1}{18}$. Therefore, we can say that the point $(\epsilon, \delta) = (1, \frac{1}{3})$ is not on the bifurcation set.

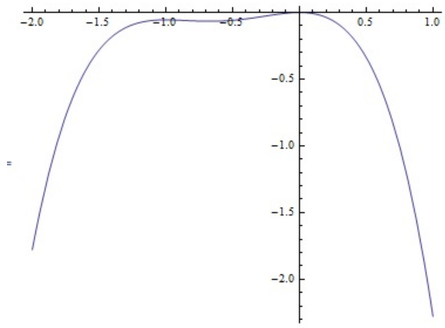


Figure 5.30: Graph of $G_{14}$

If we assume that $d_0 \neq 0$, we get three different types root of $H(x; d) = 0$ as below

**Situation 1** : If $\frac{1}{18} < d_0 < \frac{16}{243}$, there are four distinct real roots. For example; if $d_0 = \frac{876258}{16015625}$, the roots of $H(x; d) = 0$ are $x_1 = -1.8$, $x_2 = 0.24$, $x_3 = -0.898026$ and $x_4 = -0.470104$.

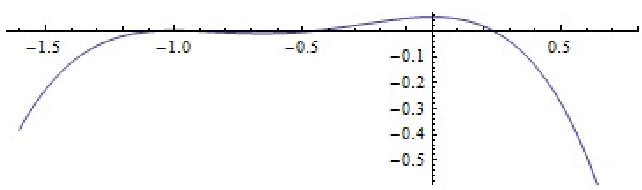


Figure 5.31: Graph of $H(x; (1, \frac{1}{3}, \frac{876258}{16015625}))$

Since all roots of $H(x; d) = 0$ is real in this situation, we can apply the projective transformation and we find $k = 0.397154$ in the integral of Jacobi elliptic function $sn(t, k)$. So that by using the value $k$, we find the solution of $H(x; d)$ as the following graphic with period 6.75411.

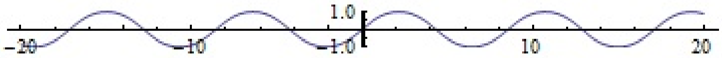


Figure 5.32: Graph of solution of $H(x;(1,\frac{1}{3},\frac{876258}{16015625}))$

**Situation 2** : If $\frac{16}{243} < d_0$ or $0 < d_0 < \frac{1}{18}$, there are one double complex conjugate roots and two distinct real roots. For example; if $d_0 = \frac{1}{20}$, the roots of $H(x;d) = 0$ are $x_1 = -1.01938 - 0.12256i$, $x_2 = -1.01938 + 0.12256i$, $x_3 = -0.413103$ and $x_4 = 0.229635$.

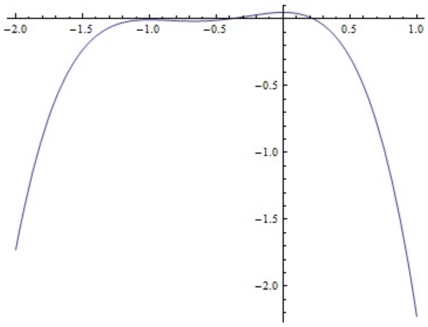


Figure 5.33: Graph of $H(x;(1,\frac{1}{3},\frac{1}{20}))$

**Situation 3** : If $d_0 < 0$, there are two double distinct complex conjugate roots. For example; if $d_0 = -\frac{1}{3}$, the roots of $H(x;d) = 0$ are $x_1 = -1.3139 - 0.593298i$, $x_2 = -1.3139 + 0.593298i$, $x_3 = 0.202787 - 0.528817i$ and $x_4 = 0.202787 + 0.528817i$.

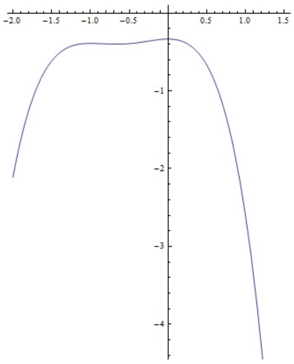

Figure 5.34: Graph of $H(x;(1,\frac{1}{3},-\frac{1}{3}))$

**Case 5:** Let $(\epsilon,\delta) = (1,-\frac{1}{2})$ be a point between on the region (9). In the case $d_0 = 0$ the equation transforms to

$$G_{15} = -\frac{3x^2}{2} - \frac{5x^3}{3} - \frac{x^4}{2} \tag{5.2.44}$$

We have choices the parameters $\epsilon = 1$ and $\delta = -\frac{1}{2}$ boundary corresponds to a point out of the bifurcation set that is in the domain (9), i.e. on the region $-\epsilon^2 < \delta < 0$. When we choose $d_0 = 0$, then we get 3 real roots $x_1 = 0$, $x_2 = \frac{1}{3}(-5 - i\sqrt{2})$ and $x_3 = \frac{1}{3}(-5 + i\sqrt{2})$. We study the critical points and critical values of the function $G_{15}$. The calculation above shows that the polynomial $G_{15}$ has three critical points $(-\frac{3}{2}, -\frac{9}{32})$, $(-1, -\frac{1}{3})$, $(0, 0)$ and the critical values are $-\frac{9}{32}$, $-\frac{1}{3}$ and 0. Therefore, we can say that the point $(\epsilon, \delta) = (1, -\frac{1}{2})$ is not on the bifurcation set.

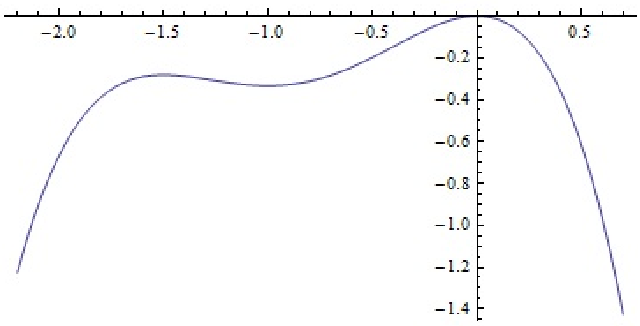


Figure 5.35: Graph of $G_{15}$

If we assume that $d_0 \neq 0$, we get three different types root of $H(x; d) = 0$ as below

**Situation 1** : If $\frac{9}{32} < d_0 < \frac{1}{3}$, there are four distinct real roots. For example; if $d_0 = \frac{43435359}{150781250}$, the roots of $H(x; d) = 0$ are $x_1 = -1.64$, $x_2 = 0.36$, $x_3 = -1.43617$ and $x_4 = -0.679474$.

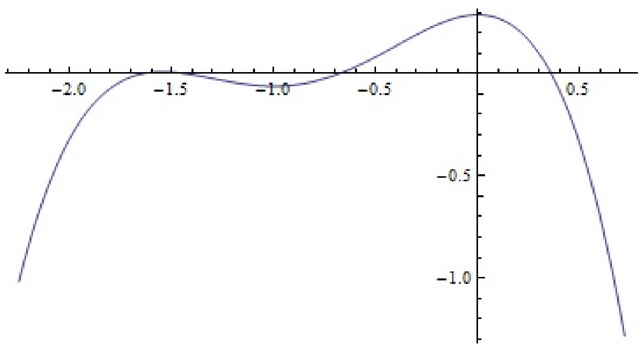


Figure 5.36: Graph of $H(x; (1, -\frac{1}{2}, \frac{43435359}{150781250})$

Since all roots of $H(x; d) = 0$ is real in this situation, we can apply the projective transformation and we find $k = 0.481004$ in the integral of Jacobi elliptic function $sn(t, k)$ . So that by using the value $k$, we find the solution of $H(x; d)$ as the following graphic with period 4.61404.

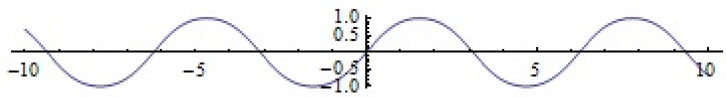


Figure 5.37: Graph of solution of $H(x; (1, -\frac{1}{2}, \frac{43435359}{150781250}))$

**Situation 2** : If $\frac{1}{3} < d_0$ or $0 < d_0 < \frac{1}{3}$, there are one double complex conjugate roots and two distinct real roots. For example; if $d_0 = \frac{1}{2}$, the roots of $H(x; d) = 0$ are $x_1 = -0.946059 - 0.500774i$, $x_2 = -0.946059 + 0.500774i$, $x_3 = -1.90045$ and $x_4 = 0.459234$.

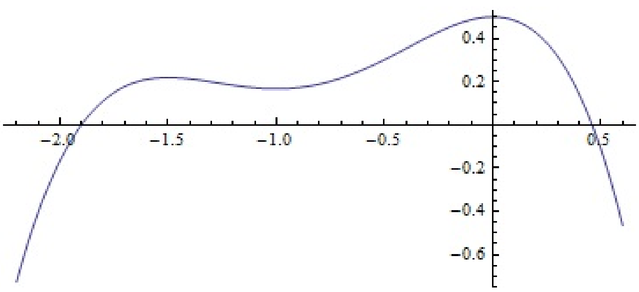


Figure 5.38: Graph of $H(x; (1, -\frac{1}{2}, \frac{1}{2}))$

**Situation 3** : If $d_0 < 0$, there are two double distinct complex conjugate roots. For example; if $d_0 = -\frac{1}{3}$, the roots of $H(x; d) = 0$ are $x_1 = -1.74281 - 0.587893i$, $x_2 = -1.74281 + 0.587893i$, $x_3 = 0.0761406 - 0.37683i$ and $x_4 = 0.0761406 + 0.37683i$.

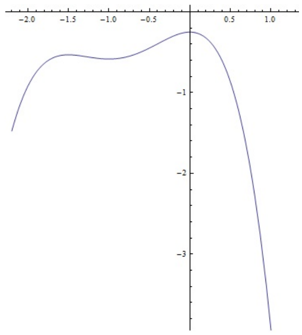


Figure 5.39: Graph of $H(x; (1, -\frac{1}{2}, -\frac{1}{3}))$

**Case 6:** Let $(\epsilon, \delta) = (1, -\frac{3}{2})$ be a point between on the region (11). So that in the case $d_0 = 0$ the equation transforms to

$$G_{16} = -\frac{5x^2}{2} - \frac{7x^3}{3} - \frac{x^4}{2} \tag{5.2.45}$$

We have choices the parameters $\epsilon = 1$ and $\delta = -\frac{3}{2}$ boundary corresponds to a point out of the bifurcation set that is in the domain (11), i.e. on the region $\delta < -\epsilon^2$. When we choose $d_0 = 0$, then we get 3 real roots $x_1 = -3$, $x_2 = -\frac{5}{3}$ and $x_3 = 0$. We study the critical points and critical values of the function $G_{16}$. The calculation above shows that the polynomial $G_{16}$ has three critical points $(-\frac{5}{2}, \frac{125}{96})$, $(-1, -\frac{2}{3})$, $(0, 0)$ and the critical values are $-\frac{2}{3}$, 0 and $\frac{125}{96}$. Therefore, we can say that the point $(\epsilon, \delta) = (1, -\frac{2}{3})$ is not on the bifurcation set.

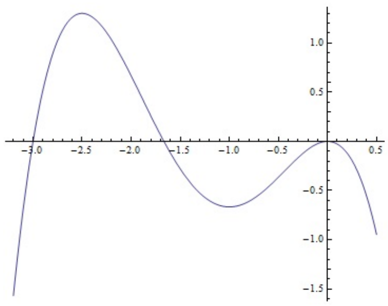


Figure 5.40: Graph of $G_{16}$

If we assume that $d_0 \neq 0$, we get three different types root of $H(x;d) = 0$ as below

**Situation 1** : If $0 < d_0 < \frac{2}{3}$, there are four distinct real roots. For example; if $d_0 = \frac{636291}{2176250}$, the roots of $H(x;d) = 0$ are $x_1 = -3.05$, $x_2 = 0.3$, $x_3 = -1.49256$ and $x_4 = -0.428179$.

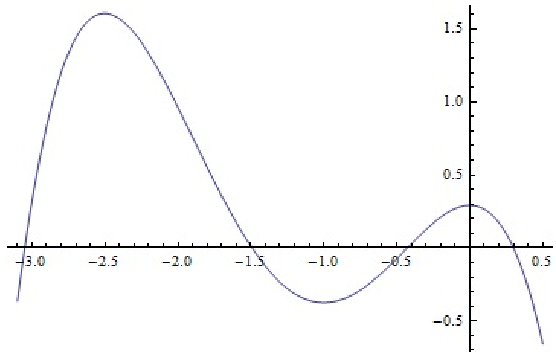


Figure 5.41: Graph of $H(x;(1,-\frac{3}{2},\frac{636291}{2176250}))$

Since all roots of $H(x;d) = 0$ is real in this situation, we can apply the projective transformation and we find $k = 0.341172$ in the integral of Jacobi elliptic function $sn(t,k)$. So that by using the value $k$, we find the solution of $H(x;d)$ as the following graphic with period 1.74795.

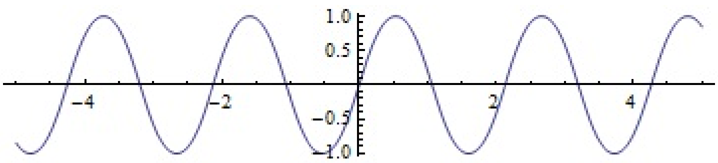


Figure 5.42: Graph of solution of $H(x;(1,-\frac{3}{2},\frac{636291}{2176250}))$

**Situation 2** : If $-\frac{125}{96} < d_0 < 0$ or $\frac{2}{3} < d_0$, there are one double complex conjugate roots and two distinct real roots. For example; if $d_0 = 1$, the roots of $H(x;d) = 0$ are $x_1 = -1.02 - 0.453773i$, $x_2 = -1.02 + 0.453773i$, $x_3 = -3.13804$ and $x_4 = 0.511379$.

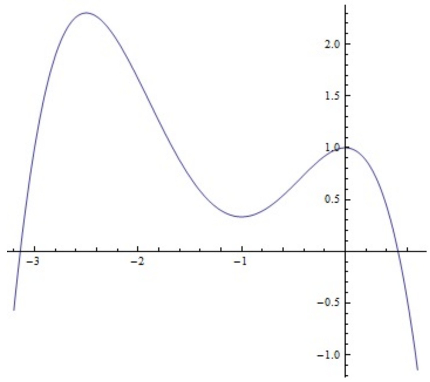


Figure 5.43: Graph of $H(x;(1,-\frac{3}{2},1))$

**Situation 3** : If $d_0 < -\frac{125}{96}$, there are two double distinct complex conjugate roots. For example; if $d_0 = -\frac{130}{90}$, the roots of $H(x;d) = 0$ are $x_1 = 0.179863 - 0.649883i$, $x_2 = 0.179863 + 0.649883i$, $x_3 = -2.5132 - 0.193053i$ and $x_4 = -2.5132 + 0.193053i$.

## 6. CONCLUSION

We draw a comparison between real solutions of the reaction diffusion equation different points of the bifurcation diagram.

In the cases where at least two roots are real, we can analyze the behavior of the solution as a real solution with a real period calculated by real elliptic integral. We find that

- If there are **four real roots** of the quartic equation, the period is given by $4\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$ for some $k$ $(0 < k < 1)$ calculated by means of a projective transformation.
- If there are only **two real roots** of the quartic equation, the period is given by $4\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1+k^2z^2)}}$ for some $k$ $(0 < k)$ calculated by means of a projective transformation.

In case all four complex roots, we cannot see the behavior of the solution as a real solution. In fact we see that there is no real periodic solution. Both of two periods are real.

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## BIOGRAPHICAL SKETCH

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