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Ercan BALCI

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LIST OF SYMBOLS

$V(f_1, \ldots, f_n)$	Vanishing set of polynomials $f_1, \ldots, f_n$.
$\mathcal{O}_{X,p}$	Localization of X at p
$Sing(X)$	Singular locus of X
$\overline{A}$	Integral clousure of A
$ht(P)$	Height of prime ideal p
$B_{(0,0)}(\mathbb{K}^n)$	Blow-up of $\mathbb{K}^n$ at $(0,0)$
$\varepsilon(C)$	Enriques tree associated to the consetellation C
$p_D(O_j)$	Direct predecessor of O_j
$p_I(O_j)$	Indirect predecessor of O_j
$B_I(X)$	Blow-up of X along I
$w(x_i)$	Weight of a vertex x_i
$v(x_i)$	Valence of a vertex x_i
$\overline{\{x_i\}}$	Set of vertices adjeceant to x_i
$M(\Gamma)$	Incidence matrix of Γ
$\varepsilon(y, x)$	Elementary modification of first kind
$\varepsilon(z, x, y)$	Elementary modification of second kind
$\Gamma < \Gamma'$	Γ' dominates Γ
$V(C)$	Multiple sails associated to C
$K(C)$	Kites associated to multiple sails $V(C)$
$p(Z)$	Arithmetic genus of cycle Z

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ABSTRACT

In this thesis, we study in details the article (Popescu-Pampu, 2009). We apply its results for plane curve singularities to the rational singularities of complex surfaces using the results given in (Lê & Tosun, 1990).

In Chapter 1 we review some basic definitions from algebra which will be useful in the sequel.

In Chapter 2, we define a singular point (or a singularity) of a variety and a desingularization of a singular point. It is well known that, H.Hironaka proved in (Hironaka, 1964) that any singularity of an algebraic variety S of any dimension defined on a field of characteristic 0 can be desingularized by a sequence of normalized blowing up. A desingularization of a singularity of S consists of finding a proper birational map $\pi : X \to S$ with X non-singular variety. The existence of a desingularization in the case of fields of characteristic p is still an open problem. Assume that S is a normal variety. Otherwise, we normalize it, that means to find the integral closure of the local rings $\mathcal{O}_{S,p}$ (see Section 1.2); the normalised variety may have a finite number of isolated singularities. Let us fix a singular point p of a normal algebraic variety S and blow it up (see Section 2.2). The blow up of a point consists of the blowing up of the maximal ideal of the point. If the blowed up surface $Bl_1(S)$ is not normal we again normalize it. The normalised blowed up surface $\overline{Bl_1(S)}$ contains a connected curve over the singular point p which is the projectivized tangent cone of the singular point. Of course, it is possible that the blowed up surface is non-singular (has no singularity). By Hironaka's theorem, after a finite blow ups and normalising process $\overline{Bl_k(S)}$ will be a non-singular variety. Here we are interested rather easy singularities of curves in $\mathbb{C}^2$ and, of surfaces in $\mathbb{C}^n$ which are absolutely isolated (means that each blowed up surface in the desingularisation process is normal). Following the literature, we associate a graph, called dual graph, to the inverse image of the singularity p by the desingularization map π, called exceptional curve. This correspondence is very important as we can obtain many properties of a singularity by using simply the combinatorial properties of its desingularization graph.

In Chapter 3, we study the singularities of plane curves and the effects of the blowing ups of a point q on the exceptional curve according to the fact that q is a smooth point or a singular point of the exceptional curve; these blow ups are called the elementary modifications.

In chapter 4, we introduce Enriques tree corresponding to a plane curve singularity. It helps to read the geometry of each blowing up in the desingularization process. As in (Popescu-Pampu, 2009), we construct sails corresponding to the elementary modifications leading to "the kite of a plane singularity"; it is a perfect configuration where we can read many information on the corresponding singularity, namely each

blow up process, self-intersection of the irreducible components of the exceptional fibre, so the dual graph and Enrique tree of the singularity. This is the only configuration having so many information about the plane curve singularity.

In Chapter 5, we try to understand all the concepts given in Chapter 4 in the case of singularities of complex surfaces. For this, we first introduce the minimal singularities of complex surfaces. The dual graphs of these singularities are very special, means easy to work in purely combinatorics way. Using (Spivakovsky, 1990), we define the distance of a vertex in a minimal tree and construct a tree which will be called depth tree of the minimal singularity. The word "depth" is used in (Lê & Tosun, 1990) where the authors give a generalisation of the distance of a vertex in a minimal tree to a rational tree. So we claim that the depth tree of a rational singularity serves as an Enriques tree of a plane curve singularity. Using this information, we define a new elementary modification and construct the sail for rational singularities. To distinguish the dimensional difference, we call it "royal sail"; also it doesn't contain only triangles as in the case of curves but also some rectangles, pentagons,... etc. The problem we have in the case of surface singularities, the sail doesn't say anything about the self intersections (or weights) of the irreducible components of the exceptional curve (of the vertices in the dual graph) of the singularity. We call this problem "weight problem". This is because we don't have control on the number neither on the self-intersection of an irreducible components of the exceptional fibre after each blow up map. We also know that we can obtain many different rational trees having the same shape with different weights. However, we can take the minimum weights of the vertices such that the tree is rational and draw the dual graph of the singularity from the corresponding the royal sail.

In the last section, we aimed to find a relation between the sails of the singularities of plane curves and of the rational singularities using sandwiched singularities as they are the singularities of surfaces obtained by the blow up of a complete ideal in $\mathbb{C}^2$ and are a subclass of the class of rational singularities. We hope to find a nice royal sail configuration for a sandwiched singularity having all the information as in the case of plane curve singularities.

ÖZET

Bu tezde (Popescu-Pampu, 2009) makalesini detaylı şekilde çalıştık. Bu makelenin düzlemsel eğri tekillikleri üzerine elde edilmiş sonuçlarını, (Lê & Tosun, 1990) makelesinde verilen sonuçları da kullanarak kompleks rasyonel tekilliklere uyguladık.

Birinci bölüme, tezin ilerleyen bölümlerinde kullanacağımız bazı cebirsel tanım ve özellikleri vererek başladık.

İkinci bölümde ise kompleks tekillikler ve onların çözümlenmesiyle ilgili bazı tanım ve özellikler verdik. Herhangi bir yüzey tekilliğinin normalleştirilmiş patlatma dizisi yolu ile çözümlenebileceğini (Hironaka, 1964) makalesinden biliyoruz. Karakteristiği 0 olan cisimlerde yaşayan tekilliklerin her zaman çözümlenebileceği yine (Hironaka, 1964) makalesinde gösterilmiştir. Bir V cebirsel kümesinin tekilliklerinin çözümlenmesi, W tekil olmayan bir cebirsel küme olmak üzere; birasyonel ve proper bir $f : W \to V$ tasviri bulmak demektir. Burada $dim(V) < dim(W)$'dir. Çözümlenmenin karakteristiği sıfır olmayan cisimlerde her zaman bulunup bulunamayacağı halen açık bir problemdir. Bir S cebirsel kümesini alalım ve S'in normal olduğunu varsayalım. Eğer değilse S'i normalleştirelim. Bu da $\mathcal{O}_{S,p}$'nin cebirsel kapanışını bulmak demektir (bknz. Bölüm 1.2); normalleştirilmiş bir cebirsel kümenin sonlu sayıda izole tekilliği olabilir. Normal S cebirsel kümesinden bir p izole tekilliği seçelim ve S'i bu noktada patlatalım (bknz. Bölüm 2.2). Bir nokta patlatması, o noktaya karşılık gelen maksimal ideali patlatmaktan ibarettir. Eğer patlatılmış yüzey $Bl_1(S)$ normal değil ise, tekrar normalleştirme uygularız. Normal patlatılmış yüzey $\overline{Bl_1(S)}$, p tekilliği üzerine bağlantılı bir eğriyi kapsar. Bu eğri de tekilliğin projectivize edilmiş tanjant konudur ve onun üzerinde patlatılmış yüzey tekillikleri vardır. Tabi ki patlatılmış yüzeyin tekilliğinin olmaması durumu da mevzu bahistir. Hironaka'nın teoremi, k kez patlatma ve normalleştirme işlemi uygulanarak tekil olmayan düzgün bir $\overline{Bl_k(S)}$ bulunulabileceğini göstermektedir.

Biz daha çok $\mathbb{C}^2$'de yaşayan eğri tekillikleri ve $\mathbb{C}^n$'de yaşayan tamamen izole yüzey tekillikleri ile ilgileneceğiz. Böylece her patlatmadan sonra normalleştirme yapmamıza gerek kalmayacak. Ardından, literatürü takip ederek p tekilliğinin çözümlenme tasviri altındaki ters görütüsüne dual diagram olarak adlandırılan bir graf ataması yaptık. Dual diagramlar ve çözümlenme arasındaki bu ilişki ve dual diagramların kombinatorik özellikleri bizim için tekilliklerin özelliklerini araştırmada önemli olacak.

3. bölümde, eğri tekilliklerini ve bir q noktasını patlatmanın dual diagramlar üzerindeki etkilerini anlamaya çalıştık. Bunun için iki temel modifikasyon tanımladık.

4. bölümde, eğri tekilliklerine karşılık gelen Enriques ağaçlarını tanıttık. Bu ağaçlar, çözümlenme işlemindeki patlatmaları geometrik olarak okumada bize yardım etti. (Popescu-Pampu, 2009) makalesindeki gibi temel dönüşümlere karşılık gelen yelkenleri oluşturduk ve bu da bize düzlem tekilliklerine atanan uçurtmayı verdi. Bu

söz konusu olan tekilliğin özelliklerini özellikle de patlatma sürecini anlamamız için önemli bi yapıdır.

5. bölümde, 4. bölümde verilen bütün konseptleri yüzey tekillikleri için anlamaya çalıştık. Öncelikle minimal yüzey tekilliklerini tanıttık. Bu tekillikler gayet özeldir, kombinatorik olarak anlaşılması kolaydır. (Spivakovsky, 1990) makalesinden yararlanarak minimal bir ağaçta uzaklık kavramını tanımladık ve minimal tekillikler için derinlik ağacı adını verdiğimiz bir ağaç tanımladık. "Derinlik" kelimesi (Lê & Tosun, 1990) makalesinde kullanılmıştır ve yazarlar minimal ağaçlar için tanımlanan uzaklık kavramını, rasyonel tekillikler için genelleştirmiştir. Biz rasyonel tekillikler için derinlik ağacının, eğri tekilliklerine atanan Enriques ağaçlarına karşılık geldiğini iddaa ediyoruz. Bu bilgiyi kullanarak, yeni bir temel dönüşüm tanımladık ve rasyonel tekillikler için yelken tanımladık. Boyutsal farklılığı ayırt etmek için bunlara royal yelkenler adını verdik. Ayrıca bu yelkenler eğri tekilliklerindeki gibi sadece üçgenlerden oluşmaz, dörtgen ve beşgenler de bulunabilir. Yüzey tekillikleri için oluşturduğumuz yelkenlerin, istisnai bölenlerin ağırlğı hakkında bilgi vermemesi gibi bir problemimiz var. Bu probleme ağırlık problemi adını verdik. Çünkü yüzeylerdeki patlatmalardan sonra istisnai bölenlerin kesişim sayılarını kontrol edemiyoruz. Ayrıca, dual diagramlar olarak aynı şekle ancak farklı ağırlıklara sahip olan farklı rasyonel tekillikler mevcuttur. Buna rağmen, ağırlıkları ağaçlar minimal kalmak şartıyla minimal alıp, tekilliğe karşılık gelen dual diagramlara yelkenlerden ulaşabiliyoruz.

Son bölümde, eğri tekilliklere ve rasyonel tekilliklere atanan yelkenler arasında bir ilişki bulmayı hedefledik ve rasyonel tekilliklerin bir alt sınıfı olan ve complete ideallerin patlatılması ile elde edilen sandiviç tekilliklerini kullandık. Eğri tekilliklerinde olduğu gibi rasyonel tekillikler ile ilgili bütün bilgilere ulaşabileceğimiz uygun bir royal yelken oluşturmayı hedefliyoruz.

1. INTRODUCTION

Let A be a ring. A zero divisor in A is a non-zero element x such that there exists an $y \in A$ with $y \neq 0$ and $xy = 0$. A ring A is said to be an integral domain if $A \neq \{0\}$ and it has no zero divisors. In other words, a nonzero ring A is an integral domain if, for all $x, y \in A$ with $x \neq 0$ and $y \neq 0$ we have $xy \neq 0$. It is well known that a finite integral domain A is a field.

1.1 Characteristic of a Field

Let A be an integral domain. Consider the function $f : \mathbb{Z} \to A$ defined by $f(n) = n \cdot 1$ is a ring homomorphism and the image by f is the cyclic subgroup $< 1 >$ of $(A, +)$ generated by 1. So either 1 has finite order n, hence $< 1 >= \mathbb{Z}/n\mathbb{Z}$ or, 1 has infinite order, hence $< 1 >= \mathbb{Z}$. In the first case, $n = p$ is a prime number and every nonzero element of A has order p; in the later case, every nonzero element of A has infinite order.

Definition 1.1.1. *Let A be an integral domain. If $1 \in A$ has infinite order, the characteristic of A is said to be zero. If $1 \in A$ has finite order, which is necessarily a prime p, we say that the characteristic of A is p. This is written as write $char(A) = 0$ or $char(A) = p$.*

1.2 Integral Closure

An element $x \in k$ is **integral over** A if one of the following equivalent condition holds:
(1) x verify an equation of the form

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0$$

where $a_i \in A$ for all i.
(2) $A[x] \subset k[x]$ is finitely generated A-module.

Definition 1.2.1. *The integral closure $\bar{A}$ of A is the set of $x \in k$ integral over A. The domain A is called integrally closed if $\bar{A} = A$.*

Exemple 1.2.1. The integral domain $\mathbb{C}[t^2, t^3]$ is not integrally closed since the rational function $t = \dfrac{t^3}{t^2}$ is a root of the monic polynomial $x^2 - t^2$. Then t is integral over $\mathbb{C}[t^2, t^3]$ but $t \notin \mathbb{C}[t^2, t^3]$. In fact $\overline{\mathbb{C}[t^2, t^3]} = C[t]$.

1.3 Normal Local Ring

Let $\mathbb{K}$ be a field. A **discrete valuation** on $\mathbb{K}$ is a mapping υ from $\mathbb{K} - \{0\}$ onto $\mathbb{Z}$ such that for all $x, y \in \mathbb{K} - \{0\}$ we have following properties:
(i) $\upsilon(xy) = \upsilon(x) + \upsilon(y)$
(ii) $\upsilon(x + y) \geq \min(\upsilon(x), \upsilon(y))$.

If υ is a discrete valuation, the subring

$$R = \{x \in \mathbb{K} | \upsilon(x) \geq 0\} \cup \{0\}$$

of $\mathbb{K}$ is called **discrete valuation ring** of υ. Moreover, R is a local ring with maximal ideal

$$m = \{x \in \mathbb{K} | \upsilon(x) > 0\} \cup \{0\}.$$

In general, an integral domain A which is equal to the valuation ring of some valuation of its quotient field is called discrete valuation ring.

Exemple 1.3.1. Let $K = \mathbb{K}(x)$ and $p \in \mathbb{K}[x]$ be an irreducible polynomial. Every $r \in K = \mathbb{K}(x)$ can be written uniquely as $p^{\alpha}q$ where $\alpha \in \mathbb{Z}$, numerator and denominator of q are both prime to p. We define valuation $\upsilon_p(r)$ to be α. The corresponding discrete valuation ring is the prime ideal (p).

Let A be a discrete valuation ring and $I \neq 0$ be an ideal of A. Then $\{\upsilon(y) | y \in I\}$ allows minumum value. So, for some $x \in I$, $\upsilon(x) \leq \upsilon(y)$ for all y in I. Suppose that $\upsilon(x) = d$. It follows that I contains every $y \in A$ with $\upsilon(y) \geq d$ and the only non-zero ideals in A are

$$m_k = \{y \in A | \upsilon(y) \geq k\}$$

So A is Noetherian. The dimension of Noetherian domain A defined as the length of longest possible chain of prime ideals. We say that A is a Noetherian local domain of dimension 1.

Proposition 1.3.1. (Atiyah & MacDonald, 1990) *Let A be a noetherian local domain of dimension one, m its maximal ideal, $k = A/m$ its resideue field. Then, the followings are equivalent:*

(i) A is a discrete valuation ring,
(ii) A is integrally closed,
(iii) m is a principal ideal,
(iv) $dim_k(m/m^2) = dim(A)$,
(v) Every non-zero ideal is a power of m.

If A be a Noetherian local ring with maximal ideal m and residue field k, then $dim_k(m/m^2) \geq dim(A)$. We say that A is **regular local ring** if

$$dim_k(m/m^2) = dim A.$$

Let A be an integral domain with quotient field $\mathbb{K}$. The **normalization** of A in $\mathbb{K}$ denoted by $\bar{A}$, is the unique largest subring $\bar{M} \subset \mathbb{K}$ such that evey homomorphism $\phi : M \longrightarrow R$ to a discrete valuation ring extends to a homomorphism $\overline{\phi} : \bar{M} \longrightarrow R$.

Theorem 1.3.2. (Atiyah & MacDonald, 1990) *The integral clousure $\bar{A}$ of A in K is equal to the intersection of all the valuation ring of K which contain A.*

Corallary 1.3.3. *The normalization of an integral domain A is equal to its integral clousure in its quotient field.*

Definition 1.3.4. *An integrally closed local ring is called normal.*

1.4 Krull Dimension

Let R be a ring and M be an R-module. The sequence $x_1, \ldots, x_r$ of elements of R is called **regular sequence** for M if x_1 is not zero divisor in M and for all $i = 2, \ldots, r$; x_i is not zero divisor in

$$M/_{(x_1,\ldots,x_{i-1})M}$$

If R is local ring with maximal ideal m, the depth of M is defined as the maximum length of a regular sequence $x_1, \ldots, x_r$ for M with all $x_i \in m$. We say that a local noetherian domain R is **Cohen-Macaulay** if $depthR = dimR$.

Theorem 1.4.1 *Let R be a local noetherian ring with maximal ideal m.*
(a) If R is regular, then it is Cohen-Macaulay.
(b) If R is Cohen-Macaulay, then any localization of R at prime ideal is also Cohen-Macaulay.
(c) If R is Cohen-Macaulay, then a set of elements $x_1, \ldots, x_r \in m$ forms a regular sequence for R if and only if $dimR/_{(x_1,\ldots,x_r)} = dimR - r$.
(d) If R is Cohen-Mcaulay and $x_1, \ldots, x_r$ is a regular sequence for R, then $R/_{x_1,\ldots,x_r}$ is also Cohen-Macaulay.

Definition 1.4.2. *Let R be a ring. The **Krull dimension** of R, denoted $dimR$, is the maximal length of an ascending chain of prime ideals in R:*

$$p_0 \subsetneqq p_1 \subsetneqq \cdots \subsetneqq p_d$$

*not counting the minimal prime p_0. If $p \in SpecR$, the **height** of p is defined by $ht(p) = dimR_p$ (which is the maximal length of a chain of prime ideals contained in p)*

Theorem 1.4.3. *A noetherian ring R is normal if and only if it satisfies the following two conditions:*
(i) for every prime ideal $p \subseteq R$ of height ≤ 1, A_p is regular (hence a field or a discrete valuaiton ring); and
(ii) for every prime ideal $p \subseteq R$ of height ≥ 2, we have depth $A_p \geq 2$.

Definition 1.4.4. *Let I be an ideal of a commutative ring R. An element $x \in R$ is called **integral over** I, if*

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} a_n = 0$$

*for some $a_i \in I^i$ for $i = 1, 2, \ldots, n$. The set consisting of elements which are integral over I is called integral closure of I and denoted by $\overline{I}$. We say that I is a **complete ideal** if $I = \overline{I}$.*

Following (Kollar, 2007):

Theorem 1.4.5. *Let R be a one-dimensional, normal, noetherian integral domain. Then R is regular. In other words, for every maximal ideal $m \subset R$, the quotient m/m^2 is one-dimensional over R/m.*

Proof. We can reduce the case where (R, m) is local ring by localizing R at m_p. Let $x \in m \backslash m^2$. If $m = (x)$, we are done. If $m \neq (x)$, $m/(x)$ is non-trivial. Since R local ring with dimension 1, $R/(x)$ is 0-dimensional and it means $m/(x)$ is killed by a power of m. Thus, there is a $y \in m\backslash(x)$ such that $my \in (x)$. Hence

$$\frac{y}{x}m \subset R.$$

If $\frac{y}{x}m$ contains an unit element, then $\frac{y}{x}z = u$ for some $z \in m$ and a unit u. That gives us $x = yzu^{-1} \in m^2$, which is impossible.

Thus $\frac{y}{x}m \subset m$. Now, we will use Nakayama's lemma :

Let $m = (x_1, \ldots, x_n)$. Then there are $r_{ij} \in R$ such that

$$\frac{y}{x}x_j = \sum_{i}^{max} r_{ij}x_j.$$

So, $(x_1, \ldots, x_n)$ is a null vector of the matrix

$$\frac{y}{x}\mathbf{1}_n - (r_{ij}),$$

and the determinant is zero. This determinant is a monic polynomial in $\frac{y}{x}$ with coefficients in R. Since R is normal that gives us $\frac{y}{x} \in R$ and $y \in (x)$ contrary to our choice of y.

■

2. DESINGULARIZATION MAP

An algebraic set $X \subseteq \mathbb{C}^n$ is the vanishing set of a finite set of polynomials $f_1, \ldots, f_k$ in $\mathbb{C}[x_1, \ldots, x_n]$; it is denoted by $X = \mathbb{V}(f_1, \ldots, f_k)$. When $k = 1$, the algebraic set $V(f_1)$ is called an hypersurface. An irreducible algebraic set is called variety. Here we are interested in varieties of dimension 1 in $\mathbb{C}^2$, called plane curves and, of dimension 2 in $\mathbb{C}^n$, called surfaces.

2.1 Singular Points

Definition 2.1.1. *Let C be a curve defined by $f(x, y) = 0$ in $\mathbb{C}^2$. A point $p = (a, b)$ is called singular point of C if*

$$f(a, b) = \frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0$$

If $f(a, b) = 0$ but $\frac{\partial f}{\partial x}(a, b) \neq 0$ or $\frac{\partial f}{\partial y}(a, b) \neq 0$ then p is called a regular (or smooth) point of C. A plane curve having only regular points is called a smooth curve.

Definition 2.1.2. *Let X be surface defined as the zero locus of*

$$f_1(x_1, x_2, \ldots, x_n) = \ldots = f_m(x_1, x_2, \ldots, x_n) = 0$$

in $\mathbb{C}^n$. A point $p \in X$ is said to be a regular point of X if the rank of the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})(p)$ attains maximal. Otherwise, we say that p is a singular point (or a singularity) of X.

If all points of X are regular we say that X is a smooth surface, means that the local ring $\mathcal{O}_{X,p}$ at each point p is a regular local ring. The set of singular points of X is called singular locus of X and denoted by $Sing(X)$; it is a proper closed subset of X. Note that $dimX = min_{p \in X}\{dimT_pX\}$. In general, $dimT_pX \geq dimX$.

Definition 2.1.3. *Let $(X, 0)$ be a normal surface with singularity at 0. This says that $\mathcal{O}_{X,0}$ is a normal ring. A map $\pi : X' \longrightarrow X$ is called a desingularization of $(X, 0)$ if X' is non-singular and the map π is birational and proper.*

Recall that a function $f : X \rightarrow Y$ between two topological spaces is called **proper** if the preimage of every compact set in Y is compact in X.

2.2 Blow-up

The blowing up of a point in a variety is a significative example of birational maps. Let us first look at blowing up of $\mathbb{C}^2$ at $(0, 0)$. The idea depends on the fact that every point of $\mathbb{C}^2$ except the origin lies on unique line through origin and set of all lines through the origin corresponds $\mathbb{P}^1$. Let us consider the set

$$B = \{(x, l) \in \mathbb{C}^2 \times \mathbb{P}^1 | x \in l\} \subset \mathbb{C}^2 \times \mathbb{P}^1$$

The blow-up of $(0,0)$ in $\mathbb{C}^2$ is the natural projection of B to the first factor

$$\Pi : B \longmapsto \mathbb{C}^2$$
$$(x,l) \longmapsto x$$

Hence we have:

$$B = \{(a,b,p_1 : p_2) \in \mathbb{C}^2 \times \mathbb{P}^1 | ap_2 = bp_1\}.$$

where $x = (a,b)$ and $l = (p_1 : p_2)$. More generally, we have:

$$\begin{aligned} B_{(0,\ldots,0)}(\mathbb{C}^n) &= \{(x,l) \in \mathbb{C}^n \times \mathbb{P}^{n-1} | x \in l\} \\ &= \{(x_1,\ldots,x_n; y_1 : \ldots : y_n) | x_i = \lambda y_i, \forall i, \lambda \in \mathbb{C}\} \\ &= \mathbb{V}(x_i y_j - x_j y_i | 0 \leq i < j \leq n) \end{aligned}$$

The blow up of origin in $\mathbb{C}^n$ is the projection Π of $B_{(0,\ldots,0)}(\mathbb{C}^n)$ into the first factor.

Let S be an algebraic variety in $\mathbb{C}^n$. The blow up of a point $p \in S$ in S, denoted by $B_p(S)$ is the closure of $\Pi^{-1}(S \setminus p)$ in $B_p(\mathbb{C}^n)$. The fibre $\Pi^{-1}(p)$ is called **exceptional divisor** of Π.

Consider the plane curve $X = \mathbb{V}(y^2 - x^2 - x^3) \subset \mathbb{C}^2$ which has a singularity at $(0,0)$. The blow-up $(0,0)$ in X is nothing but closure of $\Pi^{-1}(X \setminus (0,0))$ together with the restriction of the natural projection. Since Π is an isomorphism from $B \setminus \Pi^{-1}(0,0)$ to $\mathbb{K}^2 \setminus (0,0)$, restiriction of Π to $\Pi^{-1}(X \setminus (0,0))$ is an isomorphism onto $X \setminus (0,0)$. We have:

$$\Pi^{-1}(\mathbb{V}(y^2 - x^2 - x^3)) = \{(a,b,p_1 : p_2) | b^2 - a^2 - a^3, ap_2 = bp_1\}.$$

Let's look at first chart defined by $p_1 \neq 0$:

$$\begin{aligned} B_1(X) &= \{(a,b,1:t) | b^2 - a^2 - a^3, at = b\} \\ &\cong \{(a,b,t) | a^2 t^2 - a^2 - a^3\} \\ &\cong \{(a,t) | a^2(t^2 - 1 - a)\} \end{aligned}$$

The subvariety $\mathbb{V}(a^2)$ is called the exceptionnel divisor of Π. The subvariety $\mathbb{V}(t^2 - 1 - a)$ is called the strict transform of X.

Definition 2.2.1.(Blow-up of an ideal) *Let $X = \mathbb{V}(f_1,\ldots,f_r) \subset \mathbb{C}^n$ an affine variety. Let $I = (g_1,\ldots,g_l)$ be an ideal in $A = \mathbb{C}[x_1,\ldots,x_n]/(f_1,\ldots,f_l)$. The blow-up of X along I is the graph $B_I(X)$ of the rational map*

$$\begin{aligned} \varphi : \quad & X \longrightarrow \mathbb{P}^{l-1} \\ & x \longmapsto [g_1(x),\ldots,g_l(x)] \end{aligned}$$

together with the projection $\Pi : B_I(X) \subset X \times \mathbb{P}^{l-1} \longrightarrow X$. The restriction of the projection into $B_I(X) \setminus \Pi^{-1}(\mathbb{V}(I))$ give rise to an isomorphism onto $X \setminus \mathbb{V}(I)$.

Theorem 2.2.2. (Hironaka, 1964) *Let S be an algebraic variety over an algebrically closed field of char 0. There exists a desingularization of the singularities of S, means that there exists a map $\pi : \tilde{X} \longrightarrow S$ such that*

(i) $\tilde{X}$ is a smooth surface

(ii) π is a birational map

(iii) π is a proper map

Remark 2.2.3. *A desingularization of a variety is a sequence of finite number of blow-ups of points.*

By (Hironaka, 1964), there exists a unique desingularization which is dominated by all the other desingularizations of $(S, 0)$, called the minimal desingularization. Let X be a surface with singularity at 0 and $\pi : X' \to X$ be its resolution. The fibre $\pi^{-1}(0) := E$ is called exceptional fibre.

Theorem 2.2.4. *(Zariski's Main Theorem) If X is a normal surface then the exceptional fiber $\pi^{-1}(0) := E$ is connected and has dimension* 1.

If X is a normal surface, the exceptional fibre is in the form $E = E_1 \cup E_2 \cup \cdots \cup E_n$ with E_i's are irreducible components called exceptional divisors. The desingularization π is called strong resolution if E is normal crossing. Normal crossing means that all the exceptional divisors are smooth and intersect each other at most one point.

3. PLANE CURVES AND DESINGULARIZATION

Here we will introduce the Enrique tree of a plane curve singularity is an invariant of the curve describing the desingularization process of a plan curve singularity.

Let C be a curve with singularity at O and $\Pi_1 : \Sigma^{(1)} \longrightarrow C$ be the blow-up of C with center O. The pullback of C by Π_1 contains the strict transform $C^{(1)}$ together with the exceptional locus $E^{(1)}$. The singularities of $C^{(1)}$ lying above $O \in C$ are called infinitely near points in the first infinitesimal neighborhood of (C, O). Inductively, the singularities in the first infinitesimal neighborhood of $\Pi_1^{-1}(O) \subset C^{(1)}$ are called infinitely near points in the second infinitesimal neighborhood of (C, O). Now, let us take the strong desingularization map of (C, O) :

$$(\Sigma^{(N+1)}, E^{(N+1)}) \xrightarrow{\Pi_{N+1}} (\Sigma^N, E^N) \xrightarrow{\Pi_N} \dots \xrightarrow{\Pi_1} \Sigma^0 = (\mathbb{C}^2, O)$$

where Π_{i+1} denoting the blow-up of Σ^i with finite set of infinitely near singularities $\{O_j | j \in J(i)\} \subset E^{(i)}$ where $J(i)$ is set of indices depending on i.

Let $O \in \Sigma^{(k)}$ be an infinitely near singularity. We consider the map

$$\Pi_{k+1} : \Sigma^{(k+1)} \longrightarrow \Sigma^{(k)}$$

If $O_1 \in \Pi_{k+1}^{-1}(O)$, we say that O is the **direct predecessor** of O_1 and we write $p_D(O_1) = O$. We denote the set of infinitely near singularities on $\Sigma^{(i)}$ by $P^{(i)}$.

Definition 3.0.1 *The set $\bigcup_{i=0}^{N} P^{(i)}$ is called the **constellation** of desingularization. So, the constellation consists of centers of the blowing-up process of (C, O).*

Let $O_i \in P^{(j)} \subset \Sigma_j$. The point O_i is called **satellite point** if $O_i \in E_{k_1}^{(j)} \cap E_{k_2}^{(j)}$ for $k_1 \neq k_2$. When $p_D(O_i) = O_{k_1}^{(j-1)}$, then we set $p_I(O_i) = O_{k_2}^{(j-r)}$ where $r \leq j$ and $p_I(O_i)$ is called **indirect predecessor** of O_i. If O_i is not a satellite point (this means if O_i is smooth point of $E^{(j)}$, O_i is called a **free point**. The set of proximity points of O_i denoted by $P(O_i)$ defined as

$$P(O_i) = P_D(O_i) \cup P_I(O_i)$$

where $P_D(O_i) = \{O_l | p_D(O_l) = O_i\}$ $P_I(O_i) = \{O_l | p_I(O_l) = O_i\}$. So, these two sets contain direct and indirect predecessors of O_i.

Here is an exemple of the blowing-up process of (C, O) :

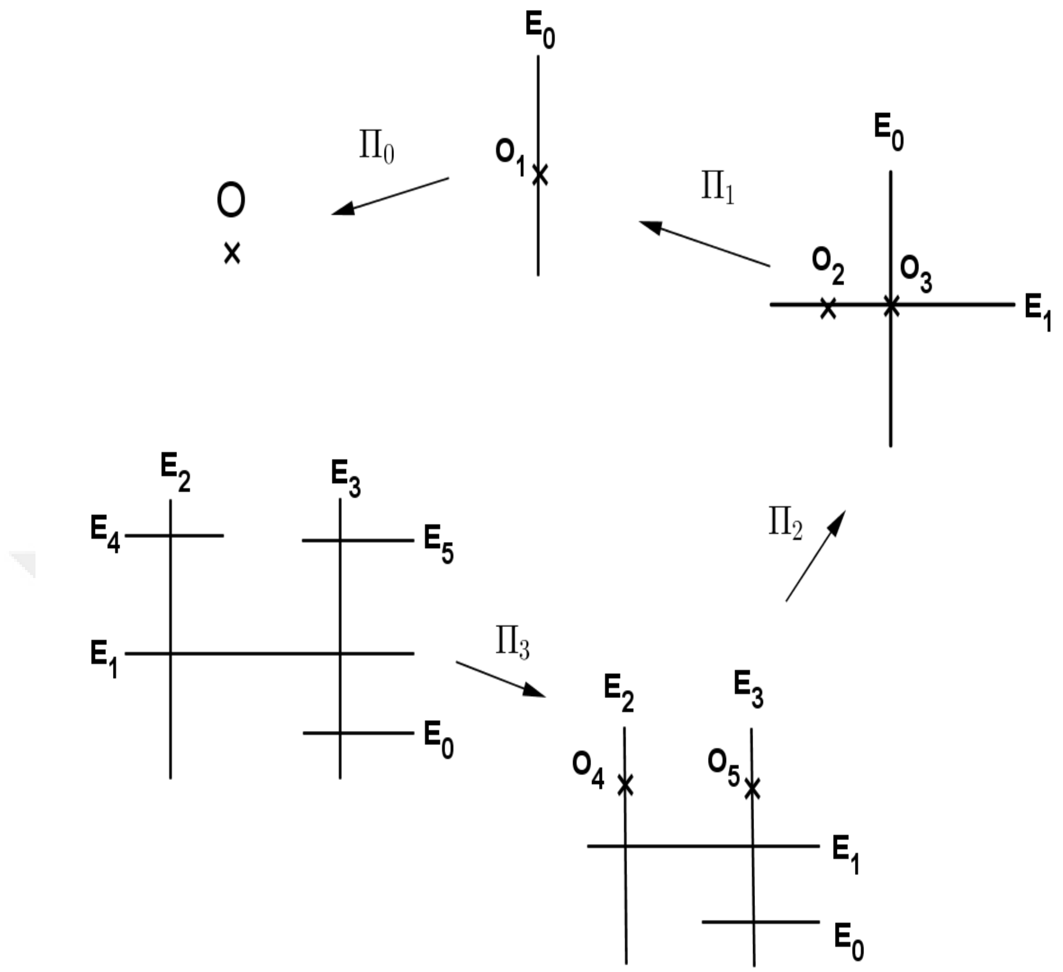


Figure 3.1: The constellation is $P = \{O, O_1, O_2, O_3, O_4, O_5\}$ of (C, O).

Let (C, O) be a curve singularity and let P be its constellation. The ***Enriques tree*** $\varepsilon(C)$ of (C, O) is defined as follows:

1. The vertices corresponds to the points O_i of the constellation,

2. There is an edge between two vertices O_i and O_j if one of them is direct predecessor of the other one. Let us say $p_D(O_j) = O_i$,

3. If O_j is a satellite point, the edge between them is a curvilinear line, if O_j is a free point then the edge between them is a segment,

4. The union of two consecutive segments is a straight line if the two starting points O_i and $p_D(O_i)$ have the same indirect predecessor, if indirect predessors of O_i and $p_D(O_i)$ differ, the union of two segments is broken line.

The Enriques tree of the constellation above is:

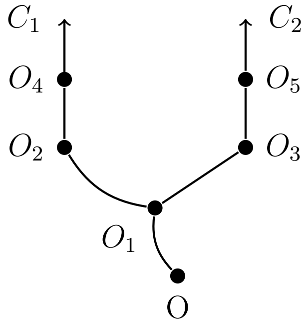


Figure 3.2: Enriques tree of (C, O) in Figure 3.1.

Corallary 3.0.2 (Castellini, 2015) *Let C_1 and C_2 be curve singularities. The following statements are equivalent:*

(i) C_1, C_2 have same topological type;

(ii) C_1, C_2 have isomorphic Enriques trees;

(iii) C_1, C_2 have isomorphic dual graphs.

3.1 Elementary Modifications

We saw that there are two kinds of point on an exceptional fibre of the blowing-up map to continue to desingularization process: Free point and satellite point. Here we will explain the effects of a blow-up of each these points: Let (C, O) be a curve singularity in $(\mathbb{C}^2, O)$ and let

$$X = \Sigma^{(k)} \xrightarrow{\Pi_k} \dots \xrightarrow{\Pi_2} \Sigma^{(1)} \xrightarrow{\Pi_1} \Sigma^0 = (\mathbb{C}^2, O)$$

be a strong desingularization of (C, O). Let us define $\Pi_1 \circ \Pi_2 \circ \cdots \circ \Pi_j := \Phi_j : \Sigma^j \to \Sigma^0$. Let us consider the exceptional fibre $\Phi_j^{-1}(O)$. Suppose that Φ_j is not giving the strong resolution. Then, we need a few blow-ups to get strong resolution. Consider the blow-up $\Pi_{j+1} : \Sigma_{j+1} \to \Sigma_j$ of Σ_j at some point of exceptional fibre $\Phi_j^{-1}(O)$. The type of that point gives us two elementary modification on dual graphs. Let us denote Γ the dual graph corresponding to $\Phi_j^{-1}(O)$ and Γ' the dual graph corresponding to $\Phi_{j+1}^{-1}(O)$.

(a) Let $x \in |\Gamma|$ and E_x corresponding exceptional divisor. If the point we blow-up is a free point $y \in E_x$ we get **elementary modification of the first kind** denoted by $\varepsilon(y, x)$ which turns Γ into Γ' obtained by adding a new vertex y to Γ such that:

$$\begin{array}{ll} (1) & \overline{\{y\}} = \{x, y\} \quad \text{in} \quad \Gamma', \\ (2) & w_{\Gamma'}(y) = 1 \\ & w_{\Gamma'}(x) = w_\Gamma(x) + 1 \\ & w_{\Gamma'}(z) = w_\Gamma(z) \quad for \quad z \neq x, y. \end{array}$$

We can describe this effect like this:

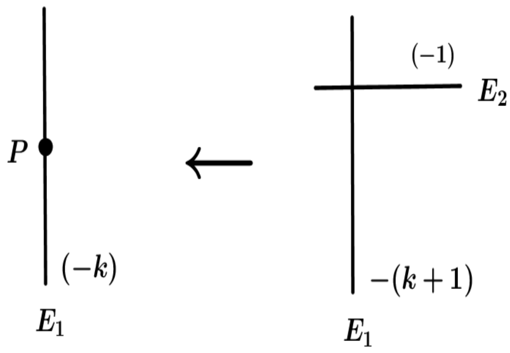


Figure 3.3. Effects of blow-up a free point on weights

(b) Let $x, y \in \Gamma$ be two adjacent vertices with corresponding exceptional divisor E_x and E_y. This adjaceny says $E_x \cap E_y \neq \emptyset$. If the point we blow-up is a satellite point $y \in E_x \cap E_y$ we get **elementary modification of the second kind**, $\varepsilon(z, x, y)$ which turns Γ into Γ' obtained by adding a new vertex z between x and y such that:

$$\begin{array}{ll} (1) & \overline{\{z\}} = \{x, y, z\} \quad \text{in} \quad \Gamma', \\ (2) & w_{\Gamma'}(z) = 1 \\ & w_{\Gamma'}(x) = w_{\Gamma}(x) + 1 \\ & w_{\Gamma'}(y) = w_{\Gamma}(y) + 1 \\ & w_{\Gamma'}(t) = w_{\Gamma}(t) \quad for \quad t \neq x, y.z \end{array}$$

For the case of satellite points, we can describe like this:

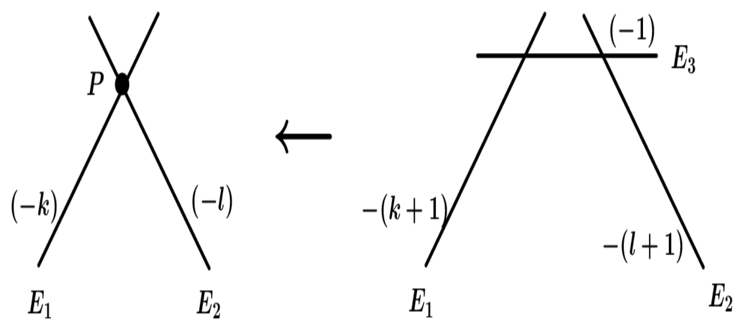


Figure 3.4. Effects of blow-up a satellite point on weights

(c) For $\Gamma = \emptyset$, we have elementary modification of first kind $\varepsilon(x)$ which is

$$\emptyset \longrightarrow \bullet^1_x$$

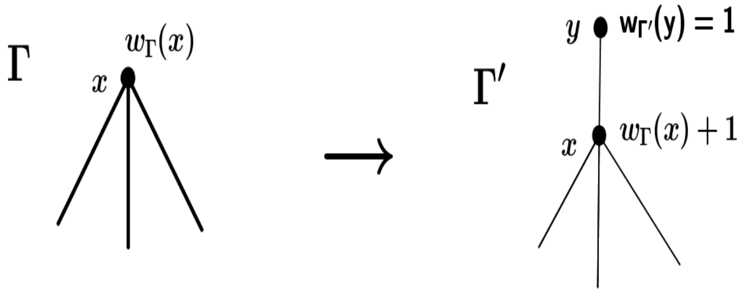


Figure 3.5. An elementary modification of first kind

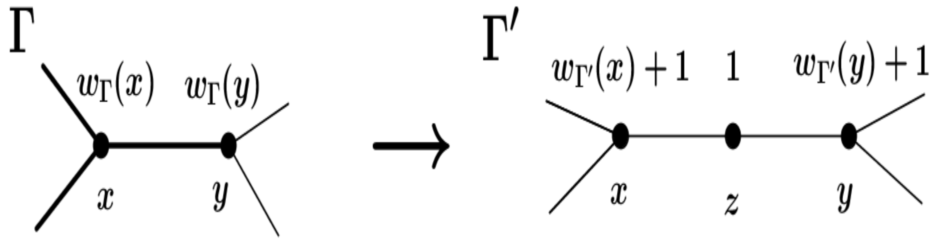


Figure 3.6. An elementary modification of second kind

Definition 3.1.1 *An **elementary sequence** is a sequence of elementary modifications. Let* Γ, Γ' *be two weighted graphs. We say that* Γ' *dominates* Γ, *denoted by* $\Gamma < \Gamma'$, *if* Γ' *can be obtained from* Γ *by an elementary sequence.*

4. SAILS AND KITES

Definition 4.0.1 *A* ***half sail*** *is a triangle with three edges named as the basis of triangle, simple edge and invisible edge. The corner point appearing as the intersection of the basis and invisible edge will be called the fancy point of the half-sail. The other two vertices will be called simple vertices; but they will differ as the basis vertex and the terminal vertex.*

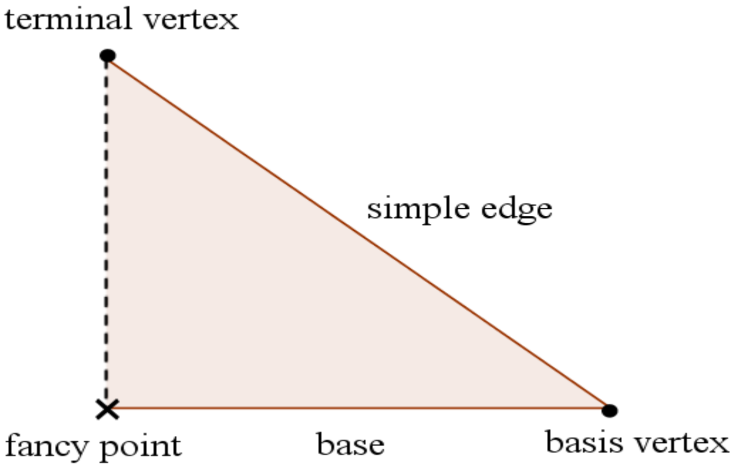


Figure 4.1: A half sail

A half sail will be oriented by ordering its vertices: The basis vertex, the terminal vertex and then the fancy point.

Definition 4.0.2 *A simple sail is the gluing of two half-sails in such a way. It has two simple edges, one base. All vertices of simple sail are simple vertices, one of them is terminal vertex and the other two are basis vertices.*

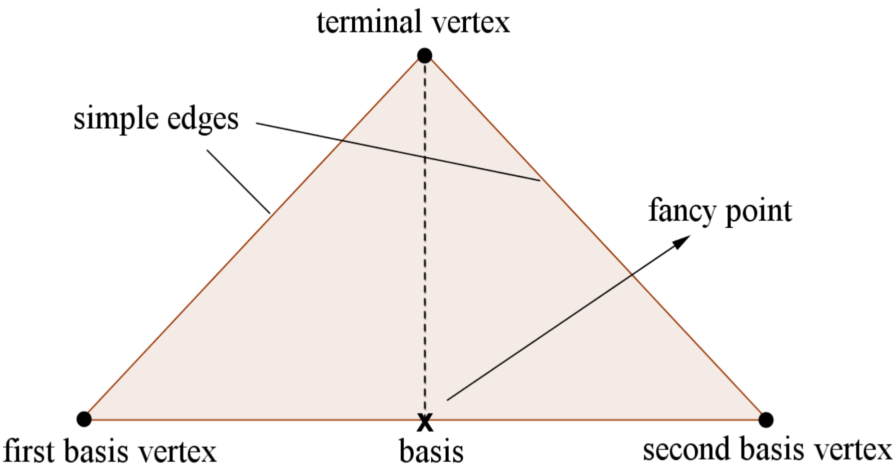


Figure 4.2: A simple sail

By gluing half sails and simple sails we will obtain multiple sails. The gluing conditions described as follows:

1. If we dont have a multiple sails already existed, we start with taking axis of the elementary sail.

2. Suppose that we have a multiple sails already existed, let us denote this multiple sails by $\mathcal{V}$. We glue a half sail or a simple sail to $\mathcal{V}$ with respecting following conditions:
(i) Assume that we will attach a half sail S. The basis vertex of S will be attached to a simple vertex of $\mathcal{V}$.

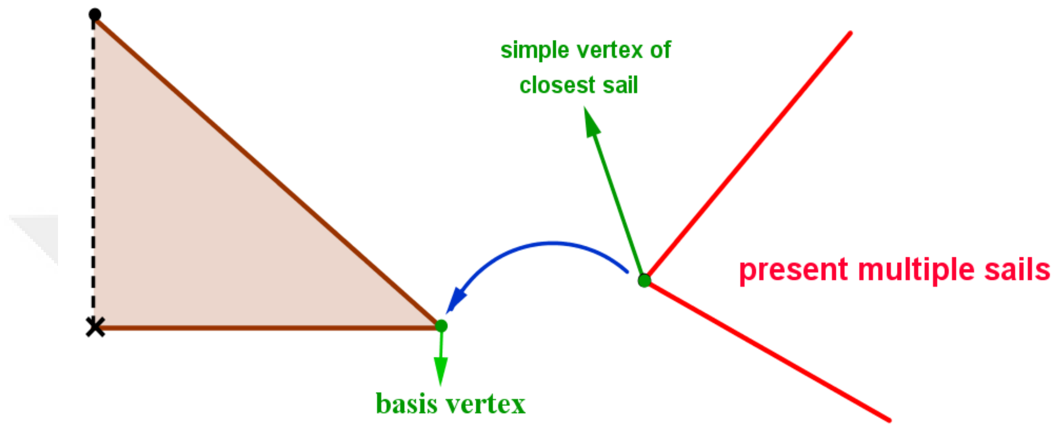


Figure 4.3: Gluing of a half sail

(ii) Assume that we will attach a simple sail S to $\mathcal{V}$. We glue the basis of S to the simple edge l of multiple sails $\mathcal{V}$ by defining an (unique) affine isomorphisme which identifies the second basis vertex of S to terminal vertex lying on l.

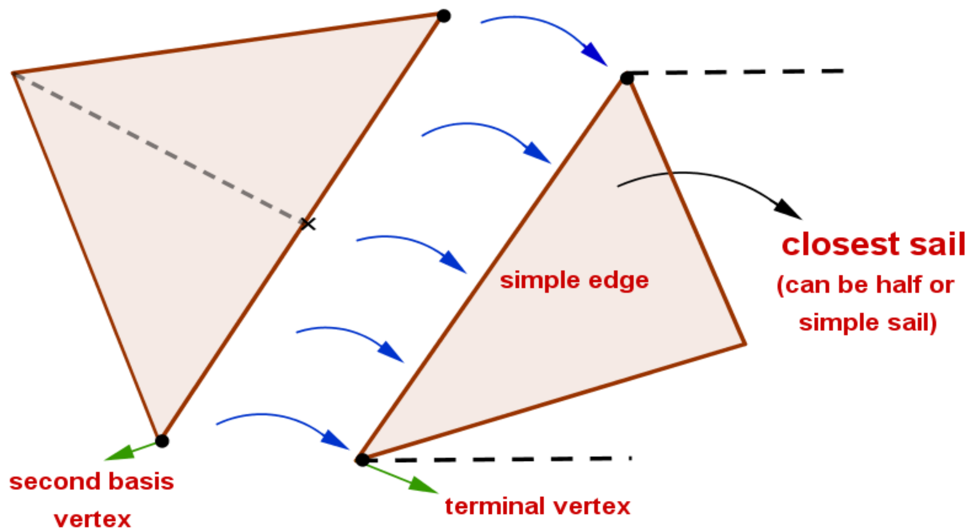
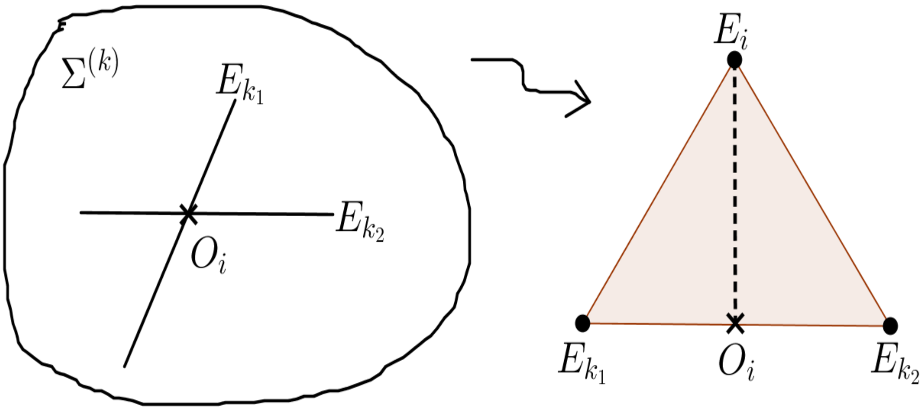


Figure 4.4: Gluing of a simple sail

We can construct multiple sails by adding elementary sails one by one. Now, we will build a relation between multiple sails and constellation. Briefly, we can say that half sails corresponding to free points and simple sails corresponding to satellite points. More precisely, let us consider the curve singularity (C, O) and its strong resolution

$$(\Sigma^{(N+1)}, E^{(N+1)}) \xrightarrow{\Pi_{N+1}} (\Sigma^N, E^N) \xrightarrow{\Pi_N} \dots \xrightarrow{\Pi_1} \Sigma^0 = (\mathbb{C}^2, O)$$

Let $\mathcal{C}$ be the associated constellation. Let O_i be the satellite point which lies on the intersection of E_{k_1} and E_{k_2} where $E_{k_1}, E_{k_2} \in E^{(k)}$. Let $E_i := \Pi_{k+1}^{-1}(O_i)$ be the divisor existed by blowing-up O_i. Then we will associate a simple sail. The vertices will corresponds bijectively to the divisors E_{k_1}, E_{k_2} and E_i. The segment $[E_{k_1}, E_{k_2}]$ will be the basis of simple sail and then E_{k_1} and E_{k_2} will be the basis vertices and E_i will be the terminal vertex of simple sail. The middle point of the basis so the fancy point corresponds to the infinitely near singularity O_i. Consequently, the invisible edge will be between the fancy point O_i and the terminal vertex E_i which is the exceptional divisor existed by blowing up O_i.



Now let us consider free point O_i lies on $E_{k_1} \in E^{(k)}$. Let $E_i := \Pi_{k+1}^{-1}(O_i)$ be the exceptional divisor existed by O_i. We will associate a half sail as follows. The vertices fancy vertex, basis vertex and terminal vertex corresponds bijectively (with respecting the order) to infinitely near singularity O_i, exceptional divisor E_{k_1} and the exceptional divisor E_i. The invisible edge will be between the infinitely near singularity O_i and the exceptional divisor E_i which is the exceptional divisor existed by blowing up O_i.

We want to divide a multiple sails into smaller pieces. Let $\mathcal{V}$ be a multiple sails. If we remove vertices of every sails in a multiple sails $\mathcal{V}$, we can regard $\mathcal{V}$ as a union of connexe components. Then, adherence of each of those components are callded **complete sails**. All of the complete sails except axis has exactly one half-sail due to the construction rules just described above.

We will now assign an orientation to the complete sails. Only half sail of complete sails will be the key to assign an orientation on complete sails. More precisely, we will assign an orientation on every complete sails by extending the orientation on the half sail belongs to them. After assigning an orientation on every complete sails, we will define new notion to distinguish two simple edges of a simple sail from each other.

Thus, the simple edge of a simple sail coming after the base with respecting to the orientation is called right simple edge and the other one is called left simple edge.

Gluing a simple sail to a multiple sails can be made on right simple edges or left simple edges of closest sails. But, if closest sail is a half sail, attachement is always made on right simple edges. Thus, consider the suite of simple sails $(\tau_1, \ldots, \tau_n)$ being glued each other with respecting the order. If we glue all of them on right (or left) simple edges, we say that these simple sails turning in the same direction.

We will now associate **kites** $\mathcal{K}$ to a given multiple sails $\mathcal{V}$: To do that, we will attach some vertices on the multiple sails by two types of cords, free cord or satellite cord. Each gluing of an elementary sail gives a cord of $\mathcal{K}$. The type of cord is depending on type of elementary sail we have been glued:

(a) If we glue a half-sail; we attach a **free cord** identfying its final point to fancy vertex of the half-sail, its initial point to fancy vertex or fancy point of the closest sail.

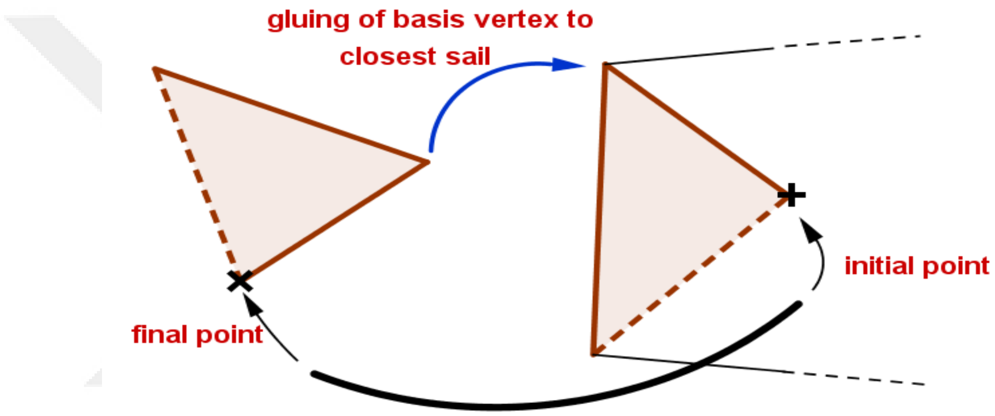


Figure 4.5: Gluing of a half sail and existence of a free cord

(b) Assume that we glue a simple sail. Let us denote B terminal vertex and C basis vertex of the closest sail such that BC is the simple edge on which we glue basis of the simple sail and M is fancy vertex of the closest sail. we attach a **satellite cord** identfying its final point to midpoint of BC and its initial point to M.

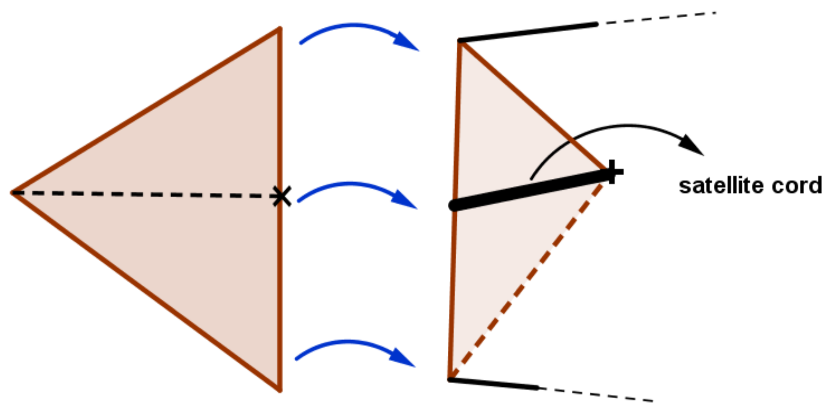


Figure 4.6: Gluing of a simple sail and existence of a satellite cord

In Figure 4.7, we have a kite associated to multiple sail. We always represent free cords by curved lines to distinguish them from satellite cords which are represented by straight lines.

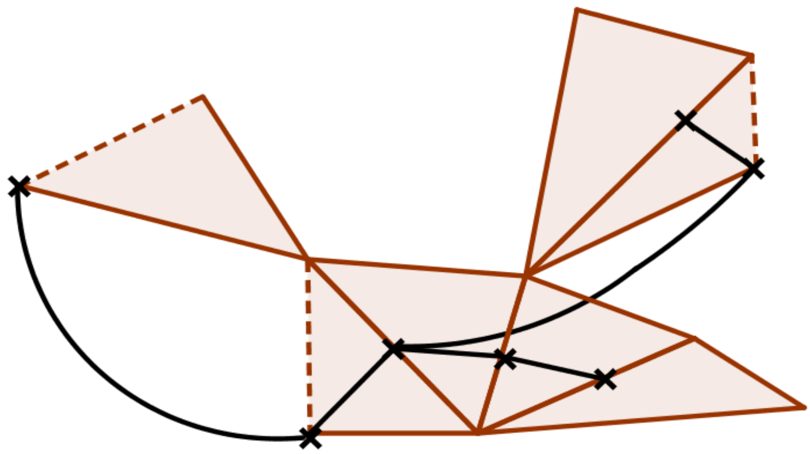


Figure 4.7: A multiple sails with associated kite

In the sequel of this section, we will interrelate sails with desingularization which is our main topic. Briefly, we associate a multiple sails to the constellation of desingularization of a curve singularity. Two kind of sails will corresponds to two elementary operation of desingularization which are blow-up a free point or blow-up a satellite point. In Chapter 3, we explain these two elementary operation by charecterization of their effects on dual graphs of desingularization. Now, we will present a new charecterization using elementary sails. Let's say C is a finite constellation and C' is another constellation which contains C and C' has exactly one more infinitely near singularity O_i. We will explain how to construct multiple sails $V(C)')$ associated to C' from $V(C)$ de C:

(i) If $C = \emptyset$, $V(\{O_i\})$ is just axis we represent with dotted line where initial vertex of axis noted by $I(O)$ and the terminal vertex by $T(O)$.

(ii) If $C \neq \emptyset$, we have two case to consider depending on the type of infinitely near singularity O_i:

Case 1: If O_i is free point, $p_D(O_i) = O_j$, we attach a half-sail $\upsilon(O_i)$ to $V(C)$ identifying its basis vertex with $T(O_j)$. Fancy point of $\upsilon(O_i)$ is denoted by $I(O_i)$ and its terminal vertex is denoted by $T(O_i)$.

Case 2: If O_i is satellite point with $p_D(O_i) = O_j$ and $p_I(O_i) = O_k$, we attach a simple sail $\upsilon(O_i)$ to $V(C)$ gluing its base to simple edge with simple vertices O_j, O_k of $V(C)$ identfying second basis vertex of $\upsilon(O_i)$ with O_j. Fancy vertex of new simple sail denoted by $I(O_i)$ and terminal vertex by $T(O_i)$.

Let us see how to construct the other configurations given one of the followings in purely combinatorics way:

- A multiple sail,
- A constellation C
- An Enriques tree
- Dual Graph
- $f(x, y) = 0$ in $\mathbb{C}^2$

From Enriques tree of (C, O) we can easily extract the configuration exceptional fibre. So, we can write associated dual graphs without weights. Moreover, next two proposition tells us about how we extract weights of vertices of dual graph from Enrique tree. With respecting the notation in Definition 3.1, let

$$\Pi_k : (\Sigma^{(k)}, E^{(k)}) \to (\Sigma^{(k-1)}, E^{(k-1)})$$

be the blow-up of $\Sigma^{(k-1)}$ along the point P of the exceptional divisor $D \subset E^{(k-1)}$.

Proposition 4.0.3 *Let D' be the strict transform of D. Then, we have*

$$(D')^2 = (D)^2 - 1.$$

Proof. Let $\Pi^*(D)$ be the full preimage of D in X'. Then, $\Pi^*(D) = D' + E$.

$$\begin{aligned} D^2 &= \Pi^*(D)\ \Pi^*(D) \\ &= (D + E')(D + E') \\ &= (D')^2 + 2D'E + E^2 \\ &= (D')^2 + 2 - 1 = (D')^2 + 1. \end{aligned}$$

■

Corallary 4.0.4 *Let $E_i^{(N)} \subset \sum^{(N)}$ be an irreducible component. Then,*

$$(E_i^{(N)})^2 = -1 - \#P(O_i).$$

Proof. When we blow-up an infinitely near point O_i, corresponding exceptional divisor exists with self intersection (-1). By the proposition above, each time we blow-up a point of E_i its self intersection drops by one. ■

Exemple 4.0.1 In Figure 4.8, we give an example of cosntellation C. At first, we will find its Enriques tree $\varepsilon(C)$ and dual graph $D(C)$ using usual way. See Figure 4.9 and Figure 4.10.

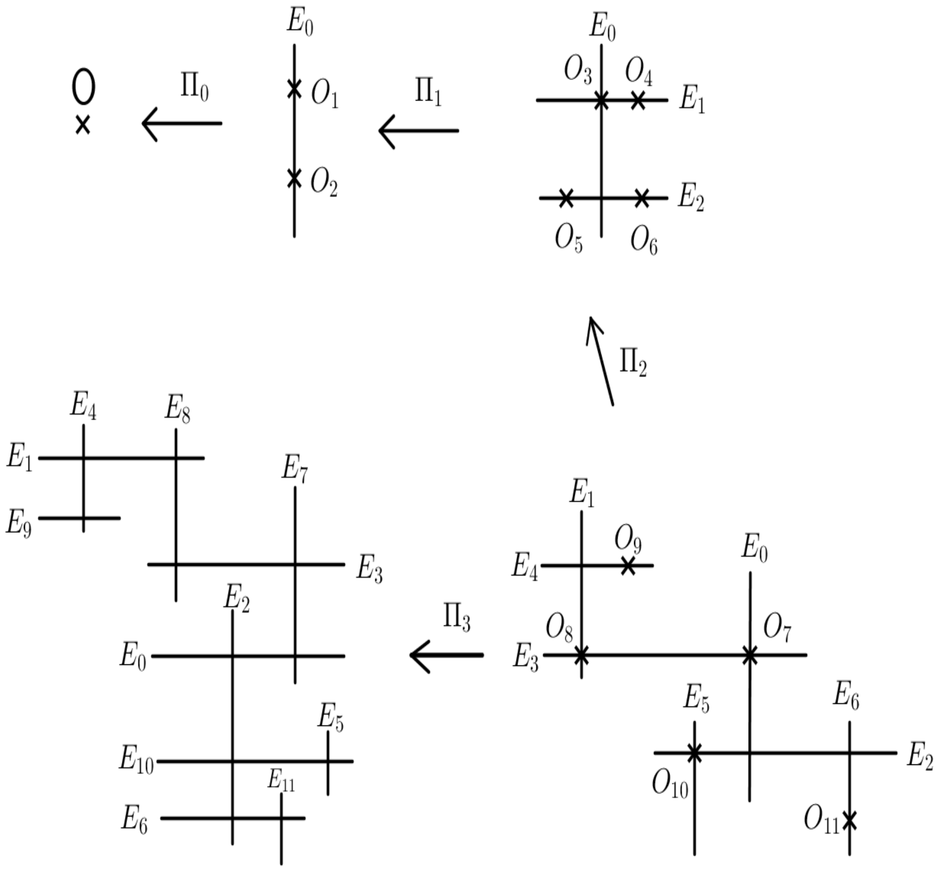


Figure 4.8 : A finite constellation $C = \{O, O_1, O_2, \ldots, O_{11}\}$.

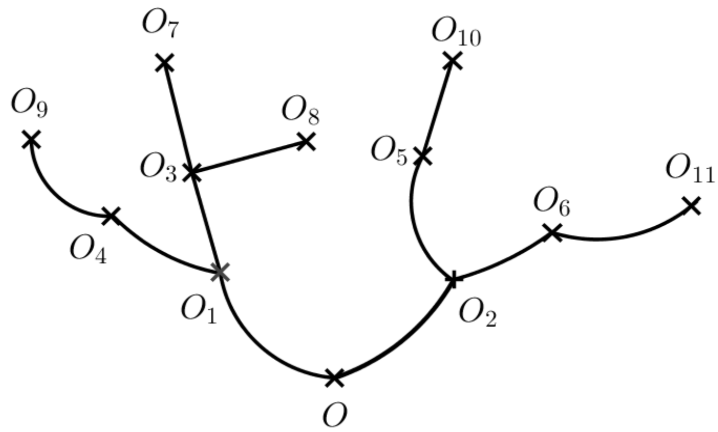


Figure 4.9 : Enriques tree of the constellation C above.

Now, in order to find dual graph of constellation above, we will use Corallary 4.0.4 which says weight of a vertex e_i with corresponding infinitely near singularity O_i is equal to

$$w(e_i) = -1 - \#P(O_i).$$

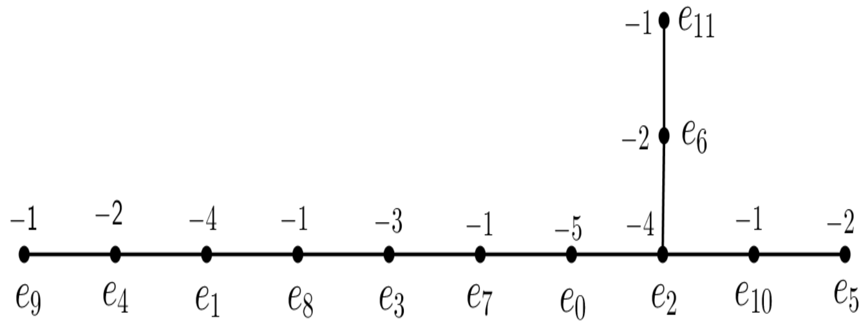


Figure 4.10 : Dual graph associated to the constellation C above.

The following theorem tells us how to extract a dual graph and Enriques tree of a constellation C from multiple sails $V(C)$ and kites $K(C)$ associated to C.

Following (Popescu-Pampu, 2009):

Theorem 4.0.5 *Let C be a finite constellation starting from O.*

(i) *Enriques tree $\varepsilon(C)$ is isomorphic to kites $K(C)$ of constellation by an isomorphism which sends infinitely near singularity O_i to the fancy vertex or fancy point of the sail corresponding to O_i. Curvilinear lines of $\varepsilon(C)$ corresponding to the free cords of $K(C)$ and segments of $\varepsilon(C)$ corresponding to the satellite cords.*

(ii) *Dual graph $D(C)$ of constellation is isomorphic to the graph obtained by union of simple edges of multiple sails $V(C)$ where the isomorphisme sends every infinitely near singularity O_i of C to the terminal vertex (which represents the exceptional divisor existed by blowing-up O_i) of the sail corresponding to O_i. Weight of an exceptional divisor which corresponds to a simple vertex v of $K(C)$ is equal to*

$$(-1) \quad \times \quad \# \{\textit{number of elementary sails has arrived at } v\}$$

where l'axis included.

Exemple 4.0.2 In Figure 4.11, we will associate multiple sail $V(C)$ to the constellation C in Figure 4.8. In Figure 4.12, we will extract kites $K(C)$ from $V(C)$.

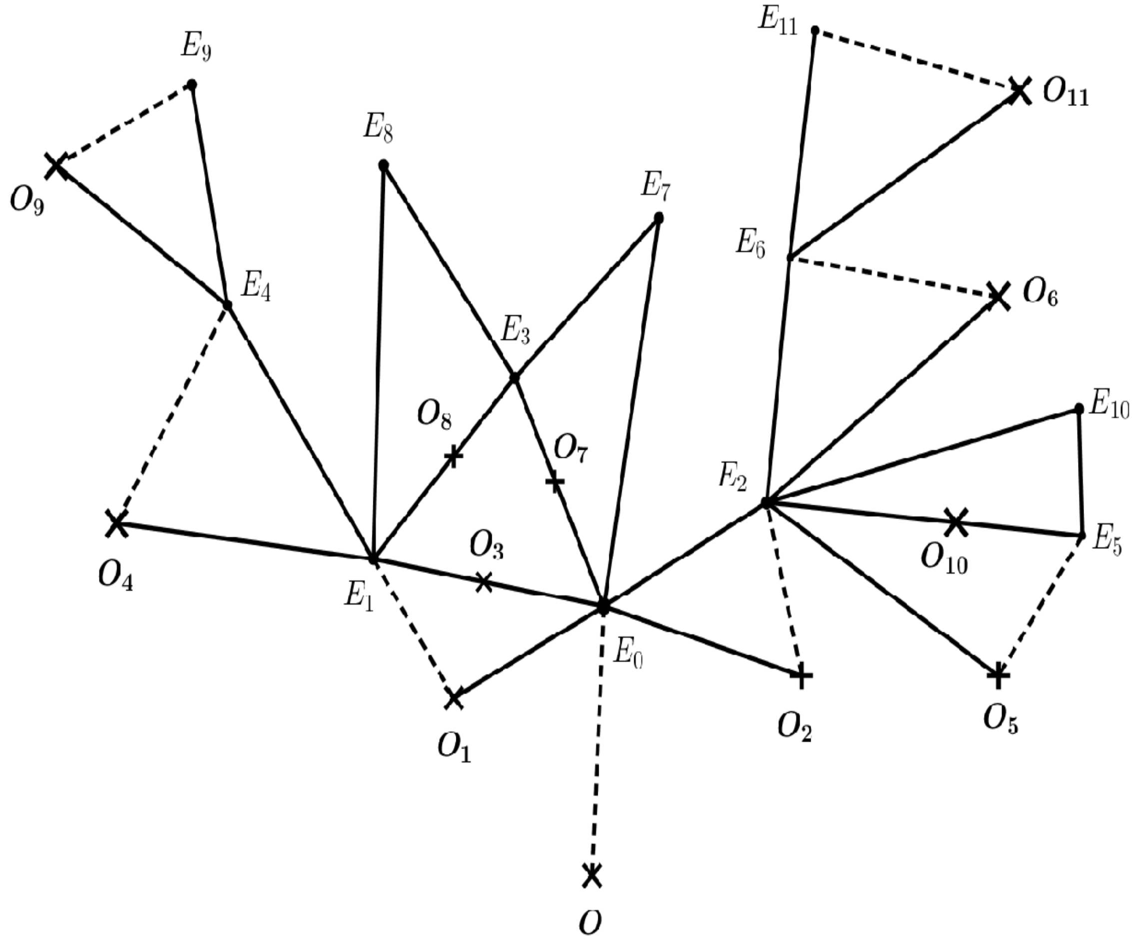


Figure 4.11: Multiple sails $V(C)$ asssociated to the constellation C in Figure 4.8.

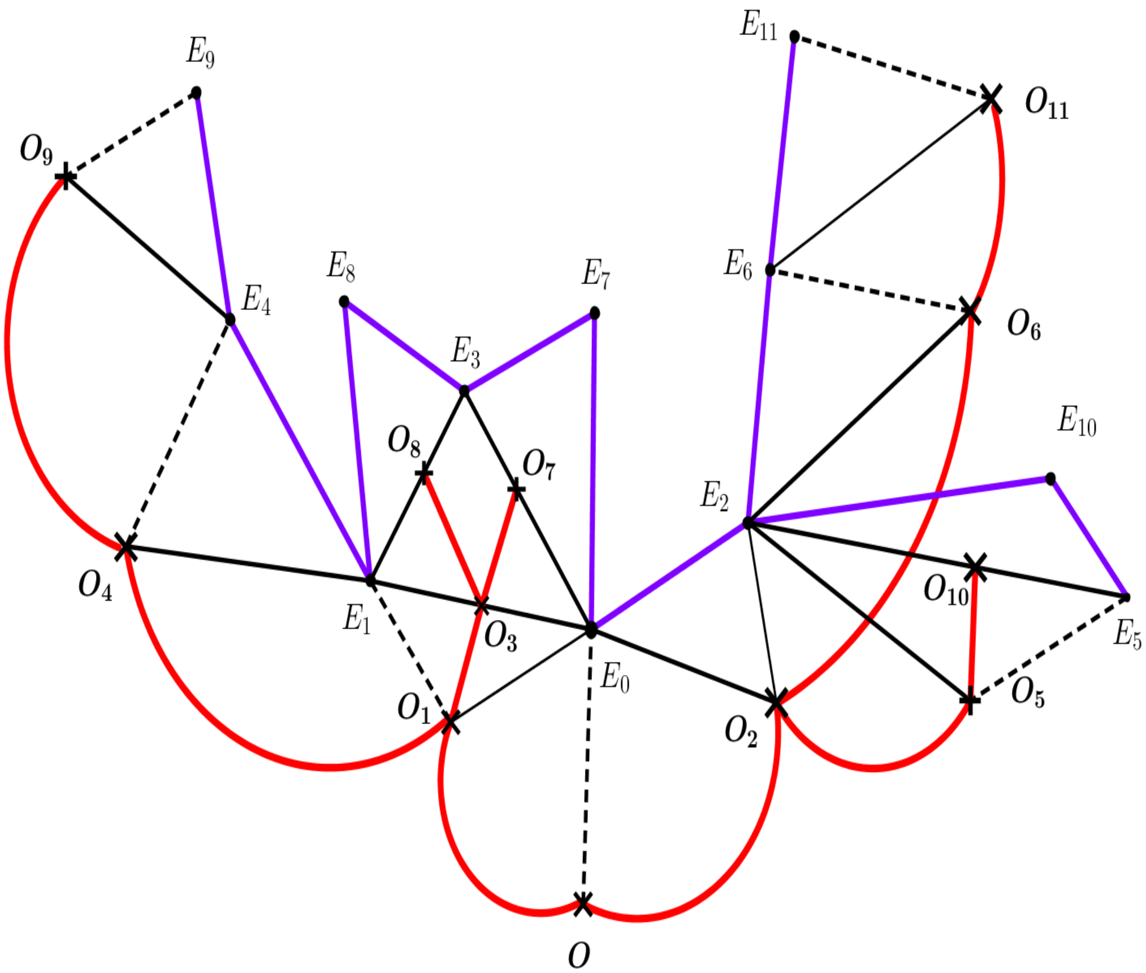


Figure 4.12: Kites $K(C)$ associated to multiple sails $V(C)$.

It is easy to check the tree $K(C)$ (in red) is isomorphic with the tree we have found in Figure 4.9; the tree (in purple) is isomorphic to the dual graph in Figure 4.10 where weight of a vertex E_i is equal to number of elementary sails arrived to the vertex E_i.

Remark 4.0.6 *A kite is the best configuration corresponding to a plane curve singularity in the sense that a kite collect many invariants of the singularity at once. Note that given any of these information we can construct the kite.*

5. SINGULARITIES OF SURFACES

5.1 Desingularization Graph

In this section, we will be interested in desingularization process of a surface singularity. We will define a special tree for a surface singularity giving the similar information as Enrique tree in the case of the curve singularities.

Let S denotes a normal surface in $\mathbb{C}^n$. As any normal surface has isolated singularities (Laufer, 1973), we assume that 0 is the only singularity of the surface S. In 1935, R. J. Walker proved the existence of desingularization of an analytic surface $(S,0)$ and, in 1939 O. Zariski proved it for algebraic surfaces. Let $\pi : (X,E) \longrightarrow (S,0)$ be a desingularization of $(S,0)$. By the Main Theorem of Zariski, the normality of the surface S implies that the exceptional fibre $E := \pi^{-1}(0)$ is a connected curve. Hence E is of dimension 1. The universal property of blow-up (Hartshorne, 1977) says that:

Theorem 5.1.1. *With preceding notation, let $\mathfrak{m}$ be the maximal ideal in the local ring $\mathcal{O}_{S,0}$. If $\mathfrak{m}\mathcal{O}_X$ is invertible in X then π can be factorized by the blow-up of 0 in S.*

Furthermore, we can find many desingularization of S and a desingularization which is convenient for our aim. For example, π is called a good desingularization if E is normal crossing and, for all i, E_i is non-singular. Let us assume that π is a good desingularization of $(S,0)$. We associate a matrix $M(E)$ to E using the intersection form of the curves E_i, called intersection matrix; the coefficients of $M(E)$ are defined by $(e_{ij})_{1\leq i,j\leq n}$ with $e_{ii} = -(E_i \cdot E_i)$ and $e_{ij} =$ is the number of intersection points of E_i and E_j.

Theorem 5.1.2. (Mumford, 1961) *The intersection matrix is negative definite.*

Now let us associate a weighted graph Γ to $\pi^{-1}(0)$. The vertices of the graph Γ are in one-to-one correspondence with the irreducible components of $\pi^{-1}(0)$. Two vertices are connected by an edge if the corresponding irreducible components intersect. If x_i is a vertex with corresponding irreducible component E_i, we define the weight of vertex x_i as

$$w_\Gamma(x_i) = -E_i^2$$

where E_i^2 is the self-intersection number of E_i. The weighted graph Γ is called desingularization graph (or dual graph) of π.

Theorem 5.1.3. (Artin, 1966) *A weighted graph is the dual graph of a surface singularity if and only if the intersection matrix of the corresponding curve configuration is negative definite.*

A prime divisor on X is an irreducible subvariety of codimension 1 in X. A divisor $D = \sum m_i E_i$ on a desingularization X is a formal linear combination of prime divisors $E_i \subset X$ with coefficients $m_i \in \mathbb{Z}$. The group of divisors $Div(X)$ is a free abelian group generated by prime divisors.

5.2 Minimal Trees

Let Γ be a weighted graph with vertices $x_1, \dots, x_k$ such that:

(i) For each $i \neq j$, there exists a unique path between the vertex x_i and the vertex x_j,
(ii) There is no cycle in Γ.

These two conditions together are equivalent to say that Γ is a tree. Form now on, Γ will represent a tree and $|\Gamma|$ will represent the set of vertices of Γ, means we have $|\Gamma| = \{x_1, \dots, x_k\}$. We say that the distance between the vertices x_i and x_j is 1 (or they are adjacent) if they are attached to each other by an edge. We denote it as $dist_\Gamma(x_i, x_j) = 1$. The number of all the vertices in Γ having distance 1 to a vertex x is called the valency of x in Γ. We will denote it by $\nu_\Gamma(x)$.

Definiton 5.2.1. *Let S be a normal complex surface and 0 a singular point of S. The singular point 0 of S is called minimal singularity if*
(i) $mult_0 S = emdim_0(S) - dim_0(S) + 1$
(ii)The tangent cone $C_{S,0}$ of S at 0 is reduced.

Definiton 5.2.2. *A weighted tree Γ is called minimal tree if $w(x) \geq \nu_\Gamma(x)$ for all $x \in |\Gamma|$.*

Proposition 5.2.3. (Artin, 1966) *If Γ is a minimal tree then the matrix $M(\Gamma)$ is negative definite.*

It is clear that we have:

Proposition 5.2.4. (Artin, 1966) *Any subtree of a minimal tree Γ is a minimal tree.*

Remark 5.2.5. For a normal surface singularity being minimal can be charecterized entirely by the dual graph. A normal surface singularity whose dual graph is a minimal tree Γ, is a minimal singularity. A minimal tree Γ is dual graph of some minimal normal surface singularity. In the next section, we will see that a normal minimal singularity is a rational singularity with reduced fundemental cycle.

Remark 5.2.6. *If $0 \in S$ is a minimal singularity then blow-up $B_0 S$ has only minimal singularities.*

Remark 5.2.7. *Let Γ be a minimal tree. So $w(x_i) \geq v(x_i)$ for all $i \in\mid \Gamma \mid$ and, this implies $(Z \cdot x_i) \leq 0$ for all i where $Z = \sum_{i=1}^{k} x_i$ is a divisor in X supported on E.*

Theorem 5.2.8. *Let $(S, 0)$ be a minimal singularity and*

$$\pi : (X, E) \to (S, 0)$$

its minimal desingularization. Let $\sigma : \bar{S} \to S$ the blow-up of 0 in S. Then there is a map $r : X' \to \bar{S}$ such that $\pi = \sigma \circ r$ and a component x_i of $E = \pi^{-1}(0)$ is contracted to a point by r if and only if $(Z \cdot x_i) = 0$ where $Z = \sum_{i=1}^{k} x_i$.

From now on, let us denote by $\mathcal{M}$ a minimal tree. Let E be the curve configuration corresponding to $\mathcal{M}$ with the irreducible components $E_i, \dots, E_k$. We take a unique irreducible component E_{i_0}. We want to compute the appearance level of E_{i_0} in the sequence of blowing ups giving π. This level is called **the depth of E_i (or** x_i in π, denoted by $depth_\pi(x_i)$. By theorem , a vertex $x \in |\mathcal{M}|$ has depth one if $w(x) > v(x)$, which means that these components of E appears after the first blow up of 0 in S. Let Γ_{TC} be the set of vertices $x_1, \ldots, x_l$ of depth one. We have:

$$\mathcal{M} - \{x_1, \ldots, x_l\} = \coprod T_j, \quad j = 1, \ldots, p$$

Each T_j is called a Tjurina subtree of $\mathcal{M}$. The $depth_\pi(x)$ of a vertex x in $\mathcal{M}$ is defined as:

$$depth_\pi(x) = d_x := min(dist(x, x_i))$$

among all $x_i \in \Gamma_{TC}$. This means that the component E_x of E corresponding to x appears in the d_x-th blow up of 0.

5.3 Royal Sails and Kites

Here we will try to find the sails and kites for a minimal normal surface singularity. In the case of surfaces, the irreducible components of the exceptional fibre don't appear one by one after the blow ups process of the singularity. By theorem , the blow up of the minimal singularity may produce many singularities of the blowed up surface and many irreducible components of the exceptional divisor. Before we proceed we need to add one more elementary modifications to three modifications given before in the case of the singularity of a plane curve.

(d) Let $E_1, \ldots, E_m$ be the irreducible components which intersect all at one point p which is singular for the blowed up surface $\bar{S}$. We call that point non-normal crossing point. Let us blow up p in $\bar{S}$; such a blow up will be called an **elementary modification of the third kind**, $\varepsilon(x_1, \ldots, x_m)$. We will get some new components separating all of them. When $m = 3$, geometrically this can be seen as:

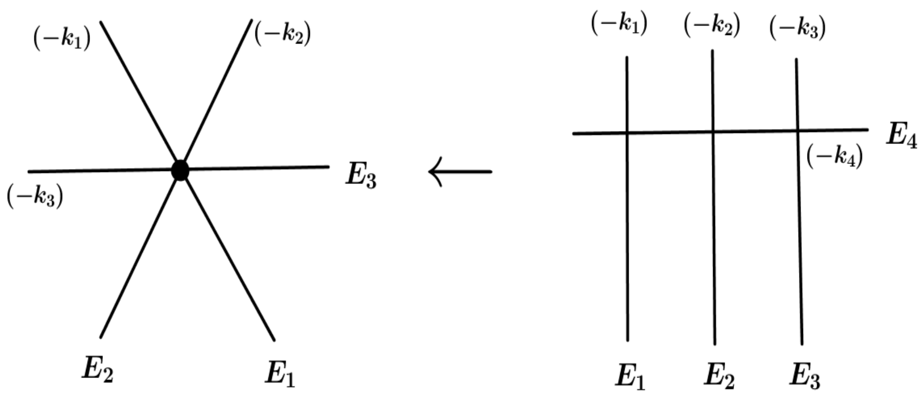
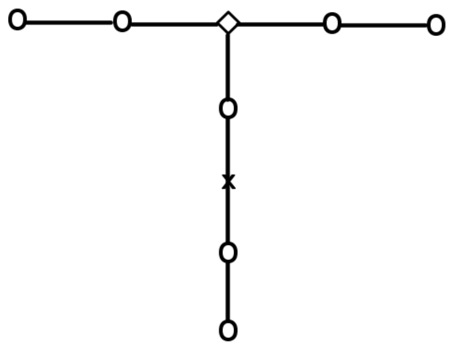
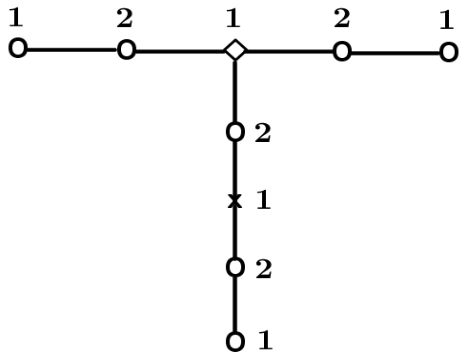


Figure 5.1: Elementary modification of third kind

Now let us consider the following minimal tree:



The vertices ∘ (resp. ×, and ◇) represents the weight 2 (reap. 3 and 4). It is easy to compute the depth of each vertex from the discussion above. In this special case, the depths are obtained as:



Definition 5.3.1. *The tree above is called the depth graph of the minimal tree.*

Remark 5.3.2. *The depth graph of a surface singularity gives the same information as the Enriques tree in the case of plane curve singularities. So, a depth graph can be seen as the generaized Enriques tree of the minimal surface singularity.*

Definition 5.3.3. (**Construction of royal sails**)
Now, we will attach royal sail to depth graph of a minimal normal surface singularity.

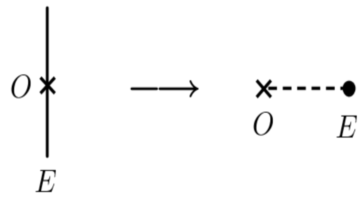
(i) First of all, we start to construct the royal sail by putting the singularity O at the bottom, we can think the place of O like ground floor,

(ii) We put divisors with depth 1 *to first floor of the royal sail, divisors with depth* 2 *to the second floor of the royal sail and goes like this ...*

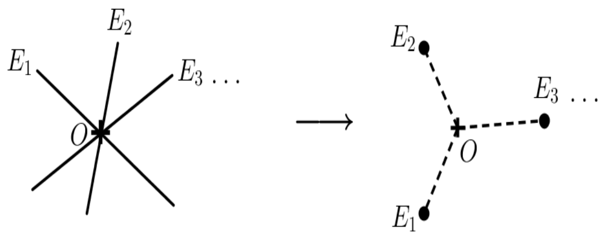
(iii) If a divisor E_i existed by blowing-up of a singularity lying above divisor E_j, there is a line between E_i and E_j. If E_i and E_j are not neighborhood in desingularization graph, the line between them is a dotted line.

(iv) We now give how to describe exceptional divisors and the singularities lying above them. We have two situation to describe:

(a) The singularity O lying above exactly one exceptional divisor



(b) The singularity O lying above more than one exceptional divisor



(v) If O_i is existed by blowing-up O_j, there is a dotted line between them.

Exemple 5.3.1. Let us consider the minimal graph of type A_n. The depth graph for A_n changes according to $n = 2k$ or $n = 2k + 1$ with $k \in \mathbb{N}_{\geq 0}$.

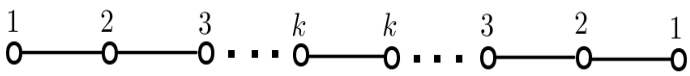


Figure 5.2: Depth graph of A_n when $n = 2k$, $k \in \mathbb{N}$.

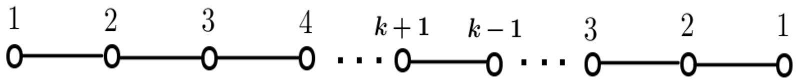


Figure 5.3: Depth graph of A_n when $n = 2k + 1$, $k \in \mathbb{N}$.

We will now give royal sails associated to A_n. We again have two cases:

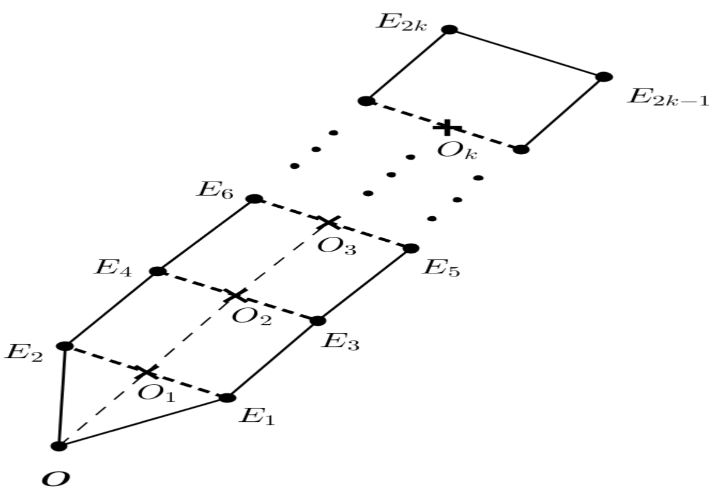


Figure 5.4: Royal sail associated to dual graph in Figure 5.2. ($n = 2k$)

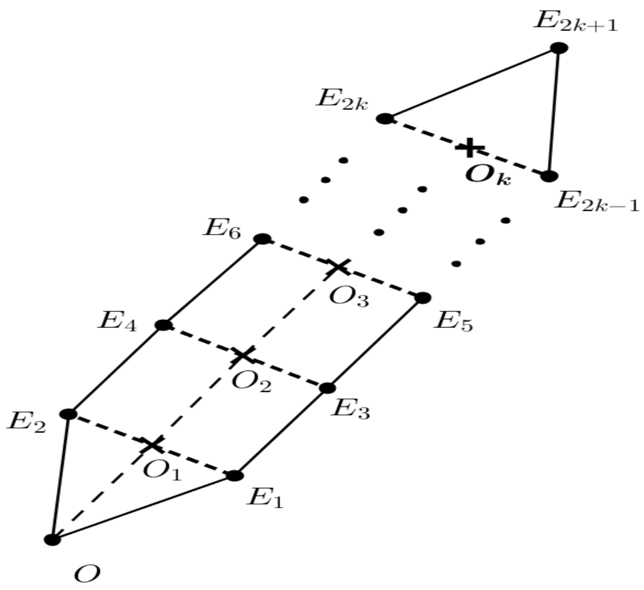


Figure 5.5: Royal sail associated to dual graph in Figure 5.3. ($n = 2k + 1$)

Exemple 5.3.2. Let us consider another depth graph Γ_1.

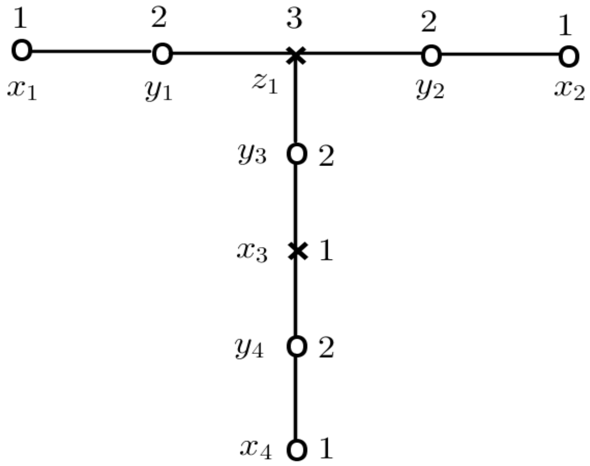


Figure 5.6: Depth graph Γ_1

We will now give blow-up process and royal sails associated to Γ_1.

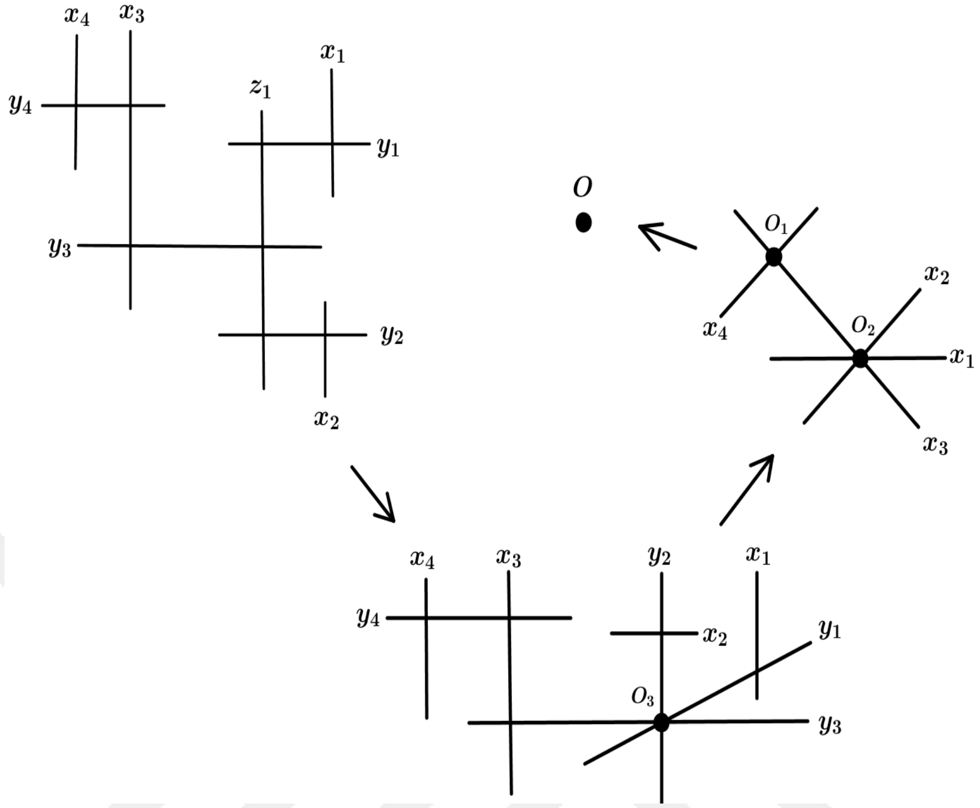


Figure 5.7: Blow-up process of Γ_1

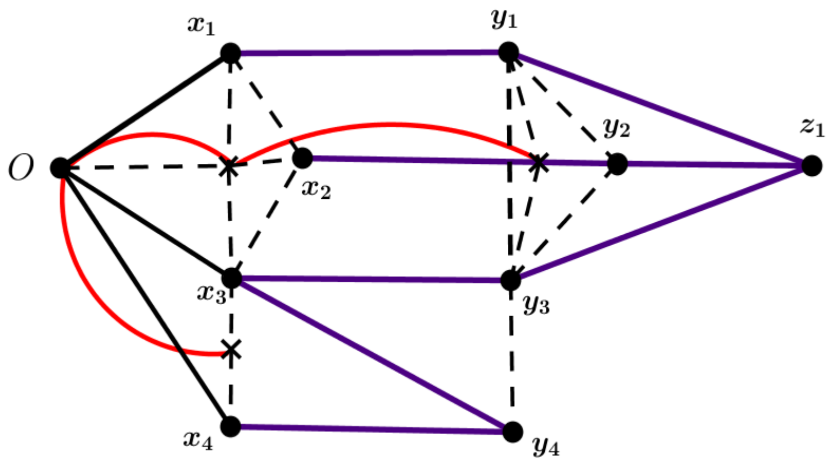


Figure 5.8: Royal sail associated to Γ_1

Remark 5.3.4. *The dual graph of a plane curve singularity can be obtained from Enriques tree as we have seen in Proposition 4.5 but that process doesn't work in the case of depth graph.*

5.4 Rational Trees

In this section, we will generalize the concept of depth to the case of rational singularities of surfaces given in (Lê & Tosun, 1999) Let Γ be a dual graph of a rational singularity, called rational tree. If we consider a weighted tree Γ with vertices $x_1, \ldots, x_n$ and weights w_i we can say whether Γ is a rational tree as follows: Let $D = \sum_{i=1}^{k} m_i E_i$ be a divisor supported on Γ with $m_i \geq 1$ for all i. Assume that $(D \cdot E_i) \leq 0$ for all i. Such divisors form a semigroups of divisors which admits a smallest element, called Artin divisor.

Definition 5.4.1. *Let $Z = \sum_{i=1}^{k} a_i E_i$ be the Artin divisor of Γ. If*

$$\frac{Z \cdot Z + \sum a_i(w_i - 2)}{2} + 1 = 0$$

then Γ is a rational tree.

Proposition 5.4.2. (Artin, 1966) *Any subtree of a rational tres Γ is a rational tree.*

Let Γ be a rational tree. A vertex of Γ is called non-Tjurina component of Γ with respect to D if $(D \cdot x_i) < 0$ (or equivalently $(D \cdot x_i) < 0$. Let us denote by $y_1, \ldots, y_k$ the vertices of Γ such that $(D \cdot x_i) < 0$. As before, we have:

$$\Gamma - \{y_1, \ldots, y_k\} = \coprod T_j, \quad j = 1, \ldots, k$$

where T_j is a rational tree. The subtrees T_j are called Tjurina subtrees of Γ with respect to D.

Consider a rational tree Γ and its Artin divisor $Z(\Gamma)$. Let us denote by $\mathcal{T}$ a Tjurina component with respect to Z. Let F be a vertex of Γ which is contained in $\mathcal{T}$. Put $\mathcal{T}_0 = \mathcal{T}$. Let $Z(\mathcal{T}_0)$ be the Artin divisor of $\mathcal{T}_0$. If $(Z(\mathcal{T}_0) \cdot F) < 0$ then the depth of F is said to be 0. If $(Z(\mathcal{T}_0) \cdot F) = 0$ let us denote by $\mathcal{T}_1$ the Tjurina component with respect to $Z(\mathcal{T}_0)$ containing F. A finite step in this way will give a finite sequence

$$\mathcal{T}_m \subset \mathcal{T}_{m-1} \subset \ldots \mathcal{T}_0 = \mathcal{T}$$

where $\mathcal{T}_i$ is the Tjurina component of the Artin divisor $Z(\mathcal{T}_{i-1})$. Note that each $\mathcal{T}_i$ contains F as non-Tjurina component, which means we have $(Z(\mathcal{T}_i) \cdot F) = 0$ for all i and $(Z(\mathcal{T}_m) \cdot F) < 0$.

Definition 5.4.3. *The length m of the chain above is called the depth of F in Γ.*

Exemple 5.4.1. Let us consider the rational tree E_6. The depth graph for E_6 and blow-up process is as below:

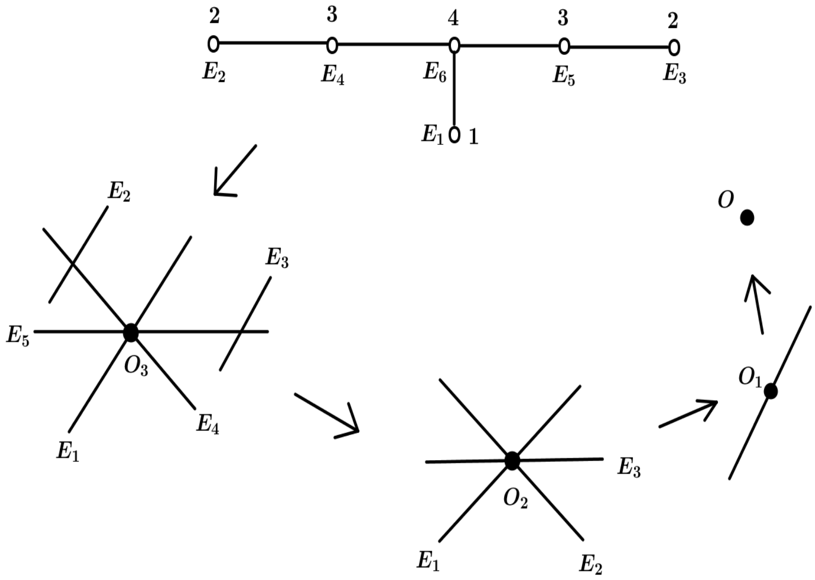


Figure 5.9: Blow-up process of E_6

The corresponding royal sail is:

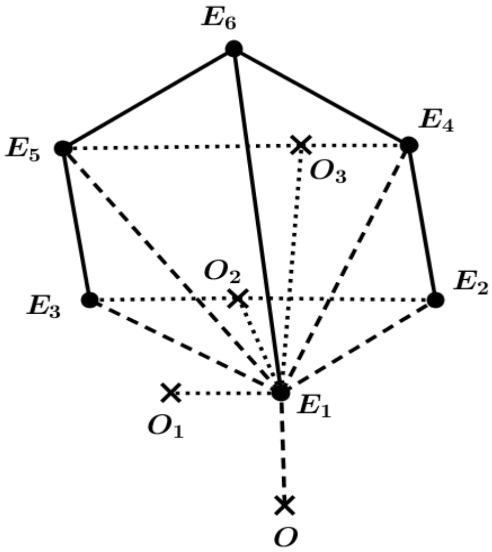


Figure 5.10: Royal sail of E_6

Corallary 5.4.4. Let Γ be a rational tree and let $(S, 0)$ be the corresponding singularity. The number of blowing-ups leading to the minimal resolution is $b := max_{i \in |\Gamma|} h_i$ where h_i is the depth of the vertex E_i in $|\Gamma|$.

Proposition 5.4.5. In a minimal tree, the number b is given(bounded) by the largest subtree of A_n.

5.5 Sandwiched Singularites

Let $\mathcal{O}$ be a normal 2-dimensional complex-analytic local ring. We say that $\mathcal{O}$ has a sandwiched singularity if there exists a non-singular algebraic surface X_0 over $\mathbb{C}$, an ideal sheaf $\mathcal{I}$ on X_0 and a point ξ in the blowing-up S of X_0 along $\mathcal{I}$ such that $(\mathcal{O}_{S,\xi})_{an} \cong \mathcal{O}$. A normal local ring $\mathcal{O}$ which is a localization of a finitely generated $\mathbb{C}$-algebra is said to have a sandwiched singulartiy if $\mathcal{O}_{an}$ has one. Let X_0 be a regular surface and η a point of X_0, I be a complete ideal in $\mathcal{O}_{X_0,\eta}$ and let $I = \prod_{i=1}^{r} p_i^{\alpha(i)}$ be its factorizationinto simple complete ideals. Let $\pi_0 : S \to X_0$ be blowing-up of I and $\pi : X \to S$ the minimal desingularization of S. Each p_i is a valuation ideal for some valuation ν of the function field K of X_0. We define A_i as set of all the simple ν-ideals p such that $p \subseteq p_i$. Let

$$A := \bigcup_{1 \le i \le r} A_i.$$

M.Spivakovsky (1990) proved that the map

$$\pi_0 \circ \pi : X \to X_0$$

is blowing-up of the ideal $\prod_{p \in A} p$. This is an important result, because blowing-up maps are birational dominant maps. Hence,

$$\pi_0 \circ \pi : X \to S \to X_0$$

is a birational dominant map between two non-singular surfaces. Birational dominant maps between non-singular surfaces can be decomposed into a sequence of point blowing-ups. Hence $\pi_0 \circ \pi$ is nothing but sequence of point blowing-ups.

Dual graph Γ associated to $\pi^{-1}(\xi)$ is the weighted subgraph of the dual graph Γ' associated to $(\pi_0 \circ \pi)^{-1}(\eta)$. Since X is a minimal desingularization of ξ, Γ contains no vertices of weight 1. It follows any connected subgraph of Γ' containing no vertices of weight 1 corresponding to some sandwiched singularity.

Definition 5.5.1. *We will define two kinds of dual graph:*

(1) Γ *is called non-singular if* Γ *dominates the empty graph.*

(2) Γ *is sandwiched if there exists a non-singular graph* Γ' *containing* Γ *as weighted subgraph.*

As expected, dual graph associated to the desingularization of some sandwiched singularity is a sandwiched graph. Next proposition will say the converse is also true. This proposition is analagous to the theorem proved in (Artin, 1966) and it says for a rational tree Γ, we can always find a desingularization of rational singularity with dual desingularization graph Γ.

Following (Spivakovsky, 1990):

Proposition 5.5.2. *The followings on* Γ *are equivalent:*

(1) *Any singularity having a desingularization with dual graph* Γ *is sandwiched.*

(2) *There exists a sanwiched singularity having a desingularization with dual graph* Γ.

(3) Γ *is sandwiched.*

We will classify the sandwiched singularities using dual graphs. Note that if there is a sandwiched graph Γ there is infinitely many non-singular graph Γ' containing Γ as weighted subgraph. It will be convenient to choose one type between them:

Proposition 5.5.3. (Spivakovsky, 1990) *Let* Γ *be sandwiched graph. The among the non-singular graphs containing* Γ *there exists a graph* Γ^* *such that for any* $x \in |\Gamma^*| \setminus |\Gamma|$

$$\mathrm{dist}_{\Gamma^*}(x, \Gamma) = w_{\Gamma^*}(x) = 1.$$

Proof. Let Γ^* be any non-singular graph containing Γ. By definition, there exists an elementary sequence

$$\emptyset \xrightarrow{\varepsilon_1} \Gamma_1 \xrightarrow{\varepsilon_2} \cdots \xrightarrow{\varepsilon_n} \Gamma_n$$

For $1 \leq i \leq n$, let x_i denote the unique vertex in $|\Gamma_i| \setminus |\Gamma_i - 1|$. We write

$$\varepsilon_i = \varepsilon(x_i, y_i) \qquad \text{if } \varepsilon_i \text{ is of the first kind,}$$
$$\varepsilon_i = \varepsilon(x_i, y_i, z_i) \qquad \text{if } \varepsilon_i \text{ is of the second kind,}$$

where $y_i, z_i \in |\Gamma_i - 1|$. Now, replace the elementary sequence above with

$$\emptyset \xrightarrow{\varepsilon_1'} \Gamma_1' \xrightarrow{\varepsilon_2'} \ldots \xrightarrow{\varepsilon_n'} \Gamma_n'$$

where

$$\begin{aligned} \varepsilon_i &= \varepsilon(x_i, \{y_i, z_i\} \cap |\Gamma|) \quad if \quad \{y_i, z_i\} \cap |\Gamma| \neq \emptyset \quad and \quad x_i \in |\Gamma|, \\ &= \varepsilon(x_i) \quad \text{if} \quad \Gamma_{i-1}' = \emptyset \quad \text{and} \quad x_i \in |\Gamma|, \\ \varepsilon_i' & \quad \text{is an isomorphisme otherwise .} \end{aligned}$$

In the first two cases we identify the unique vertex of $|\Gamma_i| \setminus |\Gamma_i - 1|$ with x_i. Replacing maps ε_i' s are also elementary modifications. So Γ_n' is non-singular as it is dominates empty graph. For any vertex $x \in |\Gamma|$, we have $x = x_i$ for some i, $1 \leq i \leq n$. Hence, x appears as a vertex in Γ_n', actually x is created by ε_i'. This costruction gives us some important results. At first,

$$\begin{aligned} \{ \quad j \quad | \quad i < j \leq n \quad & \text{and x appears as an argument in } \varepsilon_i \} \\ & \geq \{ \quad j \quad | \quad i < j \leq n \quad \text{and x appears as an argument in } \varepsilon_i' \}. \end{aligned}$$

Also by the construction of the ε_i' s, the set of arcs of Γ_n' is a subset of arcs of Γ. Hence, Γ_n' contains Γ as unweighted graph, with

$$w_\Gamma(x) \neq w_{\Gamma_n'}(x)$$

for all $x \in |\Gamma_n'|$. Take any $x \in |\Gamma_n'| \setminus |\Gamma|$. Then $x = x_i$ for some $1 \leq i \leq n$. Since $x \notin \Gamma$, by definition of the ε_j', x does not appear in any ε_j' such that $j > i$. And by definition of ε_j', there exists $y \in |\Gamma|$ such that

$$\mathrm{dist}_{\Gamma_i'}(x, y) = 1.$$

Hence

$$\begin{aligned}\mathrm{dist}_{\Gamma'_n}(x,\Gamma) \leq & \mathrm{dist}_{\Gamma'_n}(x,y) = \mathrm{dist}_{\Gamma'_i}(x,y) = 1,\\ & w_{\Gamma'_n}(x) = w_{\Gamma'_i}(x) = 1,\end{aligned}$$

as desired. Iterating the elementary modification $\varepsilon(z,x)$, $w_\Gamma(x) - w_{\Gamma'_n}(x)$ times for every $x \in |\Gamma|$, we obtain a weighted graph Γ^* with desired properties.

■

Now, let's concentrate on what proposition above actually says. Let Γ' be the non-singular graph with desired properties. The proposition says that all vertices $x \in |\Gamma'| \setminus |\Gamma|$ has weight 1 at least. So, the sequence of elementary modifications which transform Γ to Γ' are just elementary modifications of the first kind. Let's denote those modifications by $\varepsilon(x,y)$. And, the second equality $\mathrm{dist}_{\Gamma^*}(x,\Gamma) = 1$ says that if $\varepsilon(x,y)$ is one of those modification, x is always an element of $|\Gamma|$. Well, this choice of Γ' depending on those combinotorial properties leads us the following algebraic corollory:

Corallary 5.5.4. (Spivakovsky, 1990) *Let $\mathcal{O}$ has sandwiched singularity. With respecting notation in Section 5.5, let $\pi_0 : S \to X_0$ blowing up of $\mathcal{I}$ and $\pi : X \to S$ the minimal desingularization of S. Let $\eta = cosupp(\mathcal{I})$. Then X_0 and $\mathcal{I}$ can be chosen in such a way that*

(1) ξ is the only singularity of S.

(2) Every irreducible curve in $(\pi_0 \circ \pi)^{-1}(\eta) \setminus \pi^{-1}(\xi)$ is an exceptional curve of the first kind.

For sandwiched singularity, we shall always assume that $\mathcal{I}$ and X_0 are chosen as above.

6. CONCLUSION

In this thesis, we wanted understand in detail Sails and Kites theory on curve singularities given in (Popescu-Pampu, 2009). Then we tried to generalise it to some singularities of complex surfaces, namely to rational singularities of surfaces. In first two chapters, we studied the basic definitions and results on singularities and their resolutions. In chapter 3 and 4, we started to understand our main reference (Popescu-Pampu, 2009) and in chapter 4, we presented the sail and the kite associated to a curve singularities based on Enriques tree of the singularity. In the last chapter, we defined the notion of depth for a vertex in a minimal tree and in rational tree, which is equal to the minimum number of blow-ups to make appear the corresponding exceptional divisor in the resolution process of the singularity. We claim that the depth graph (the resolution graph with the depth assigned on each if its vertices) is the generalised Enriques tree for surface singularities with rational singularities. Following (Popescu-Pampu, 2009), we define one more elementary modification to construct a sail for a rational singularity. The corresponding sail is called royal sail to distinguish the cases of curve and surface singularities. Since we can't control the number of the irreducible components and their self-intersection as in the case of plane curve singularities, the kite is not a perfect configuration where we can read as many information as we did in the curve case. We just discovered two articles in the literature which can help us to develop our works.

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BIOGRAPHICAL SKETCH

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