INTERPRETABLE FIELDS IN ACF
(CEBİRSEL KAPALI CİSİMLERDE YORUMLANABİLİR CİSİMLER)

by

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LIST OF SYMBOLS

ACF : The theory of algebraically closed fields

$\mathbf{ACF_p}$: The theory of algebraically closed fields of characteristic p where p is either zero or a prime number

$\mathcal{L}$: A language

$\mathcal{M}$: An $\mathcal{L}$-structure whose universe is set M

$\neg\phi$: The negation of formula ϕ

$\wedge$: And

DCC : Descending chain condition

$\mathbf{K}^+$: The additive subgroup of a field K

$\mathbf{K}^*$: The multiplicative subgroup of a field K

$\mathbf{K}^{\mathbf{alg}}$: The algebraic closure of a field K

$\mathbf{K}^{\mathbf{sep}}$: The seperable closure of a field K

$\mathbf{Gal(K/k)}$: The Galois group of a field extension K over a field k

$\mathcal{L}_\mathcal{A}$: The set of $\mathcal{L} \cup A$ formulae where $\mathcal{L}$ is a language and A is a subset of the universe of the $\mathcal{L}$-structure.

Aff(K) : The group of affine transformations from a field K to itself

RM : Morley rank

$\mathbf{Diag_{el}}(\mathcal{M})$: The elementary diagram of a structure $\mathcal{M}$

ABSTRACT

This thesis focuses on some results regarding the interaction between algebraic geometry and model theory. It concerns the characterization of definable fields in an algebraically closed field which is provided by Bruno Poizat. In this work, we present the basic model theoretic notions with strong theorems, the background about groups of finite Morley rank and linear algebraic groups. Such a presentation is not only essential to understand this characterization (Poizat, 2001) but also enlightening to see the aforementioned interaction. Then to conclude that an infinite field K which is definable in an algebraically closed field F is definably isomorphic to F, we will introduce the Poizat's idea which provides two different methods depending on characteristic of F.

Keywords : MODEL THEORY OF ALGEBRAICALLY CLOSED FIELDS, GROUPS OF FINITE MORLEY RANK

ÖZET

Bu tez cebirsel geometri ve model teorenin etkileşimlerini göz önüne alan bazı sonuçlara odaklanmıştır. Uğraşı, cebirsel kapalı cisimlerde tanımlanabilir cisimlerin Bruno Poizat tarafından verilen karakterizasyonudur. Bu çalışmada, modeller teorisinin temel kavramlarını güçlü teoremlerle birlikte sunarken Morley rankı sonlu gruplar ve doğrusal cebirsel gruplardan hakkında gerekli altyapıyı vermekteyiz. Böyle bir sunum yalnızca bahsedilen karakterizasyonun (Poizat, 2001) anlaşılması için değil, aynı zamada da bahsi geçen etkileşimin de aydınlatılması için önemlidir. Ardından cebirsel kapalı bir F cisminde tanımlanabilir olan her sonsuz K cisminin F'e tanımlabilir şekilde izomorfik olduğu sonucuna varmak için Poizat'ın F'in karakteristiğine bağlı olarak iki yöntem ortaya koyduğu fikrini tanıtacağız.

Anahtar Kelimeler : CEBİRSEL KAPALI CİSİMLERİN MODELLER KURAMI, MORLEY RANKI SONLU OLAN GRUPLAR

1 INTRODUCTION

One of the main interests of model theory is to specify the definable sets, that are given by a formula, in a given structure. In this aspect, the theory of algebraically closed fields ACF is quite rich. It is well-known that, by quantifier elimination, definable sets in algebraically closed fields are exactly the constructible sets in algebraic geometric sense. This is not the only interaction between model theory and algebraic geometry. At that point, I would like to share the following sentences of Wilfrid Hodges, " According to Zil'ber's programme, if the history of mathematics had been crazily different and we had discovered model theory before algebraic geometry, the natural development of model theory would have forced us to invent algebraic geometry as a canonical example." (Rabinovich, 1993)

In this work, we focus on some results regarding the aforementioned interaction. We present that the definable groups in an algebraically closed fields are algebraic groups. Furthermore, we give the Bruno Poizat's proof of the fact that an infinite field K which is definable in an algebraically closed field F is definably isomorphic to F.

The outline of this work can be given as follows.

In Chapter 2, we give a summarized version of our literature survey.

In Chapter 3, the fundamental model theoretic background is given such as Compactness Theorem and the notions of completeness and model-completeness. Also we state the fact that ACF eliminates imaginary elements which allows us to identify any quotient of a model of ACF by a definable equivalance relation with an image of a definable function.

In Chapter 4, we introduce two rank notions, namely abstract rank and Morley rank and see how their equivalance in our setting is beneficial. In addition, we give proofs of basic properties of those ranks.

In Chapter 5, the groups of finite Morley rank will be our main interest. We develop some tools to explore the subgroups of finite index of a group of finite Morley rank G regarding the generic types over G. Moreover, we see that one has the notion of connectedness of G which is analogous to the connectedness concept in algebraic geometry.

In Chapter 6, we define new concepts which will be crucial to conclude the characterization of definable fields in an algebraically closed field and bring the necessary facts as well.

In Appendix A, B, C, we bring some essential information from the various disciplines of mathematics such as algebra, linear algebra and linear algebraic groups.

This work is an attempt to explore and clarify the relevant parts of (Poizat, 2001). One could have difficulties while studying on (Poizat, 2001), so our work may be useful to comprehend the details.

2 LITERATURE REVIEW

A group of finite Morley Rank is a group G endowed with a rank function that assignes an integer to a definable subset of G with nice properties. Morley rank was introduced by Michael Morley in 1965. Alexandre Borovik discovered the concept of ranked groups (Borovik and Nesin, 1994) and Bruno Poizat showed that a group is ranked if and only if it is a group of finite Morley rank (Poizat, 2001).

Angus Macintyre showed that a group of finite Morley has DCC on its definable subgroups (Macintyre, 1971a). This result yields to the fact that the connected component of a group of finite Morley rank is a definable subgroup of finite index (Borovik and Nesin, 1994). Also he proved that an infinite field K is algebraically closed if and if K is a field of finite Morley rank (Macintyre, 1971b).

Bruno Poizat gives a proof of the fact that every group which is definable in an algebraically closed field is definably isomorphic to an algebraic group (Poizat, 2001) and attributes to Ehud Hrushovski. This proof is inspried by a result of André Weil (Weil, 1948), (Weil, 1955) and then Bruno Poizat named this theorem as Weil - Hrushovski.

Bruno Poizat proves that $G/Z(G)$ is definably isomorphic to a linear group for any connected, definable group G in an algebraically closed field K with center $Z(G)$ by extending a result of Maxwell Rosenlicht which can be found (Rosenlicht, 1956). Moreover, he proves that an infinite field K which is definable in a pure algebraically closed field F is definably isomorphic to F in 1987 (Poizat, 2001).

3 PRELIMINARIES

This chapter encloses basic but required model theoretic concepts with relevant theorems. In the first section, there is a brief introduction to fundamental concepts with some examples.

3.1 Basic Concepts

Definition 3.1. *A theory T has* ***quantifier elimination*** *if for every formula φ, there is a quantifier-free formula ψ such that*

$$T \models \varphi \leftrightarrow \psi.$$

That is equivalent to say that $\varphi \leftrightarrow \psi$ is satisfied in every model of T.

Theorem 3.2. *ACF admits quantifier elimination.*

Definition 3.3. *A theory T is called* ***strongly minimal*** *if for any model $\mathcal{M}$ of T, every definable subset of M is either finite or cofinite where M is the universe of $\mathcal{M}$.*

Example 3.4. *ACF is a strongly minimal theory. Indeed, it is a consequence of quantifier elimination on ACF. Let K be an algebraically closed field with a definable subset X of K^n. Then, by quantifier elimination, X is a finite Boolean combination of Zariski closed sets. It follows from the fact that Zariski closed sets are given by the zeros of the polynomials and polynomials have finitely many roots.*

Theorem 3.5. *(**Compactness Theorem**) A theory T is satisfiable if and only if every finite subset of T is satisfiable.*

Definition 3.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures where M and N are their universes respectively. An injection $j : M \to N$ is called an **$\mathcal{L}$-embedding** from $\mathcal{M}$ to $\mathcal{N}$, if the interpretation of all of the symbols of $\mathcal{L}$ is preserved under j.*

Definition 3.7. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures. An $\mathcal{L}$-embedding $j : \mathcal{M} \to \mathcal{N}$ is called **elementary** if*

$$\mathcal{M} \models \varphi(a_1, \dots, a_n) \Leftrightarrow \mathcal{N} \models \varphi(j(a_1), \dots, j(a_n))$$

for all $\mathcal{L}$-formulas and each $a_1, \dots, a_n \in M$.

*If there is an elementary inclusion map from $\mathcal{M}$ to $\mathcal{N}$, then $\mathcal{M}$ is said to be **elementary substructure** of $\mathcal{N}$ and it is denoted by $\mathcal{M} \prec \mathcal{N}$.*

Definition 3.8. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\mathcal{L}_{\mathcal{M}}$ be the language where each element of M is added to $\mathcal{L}$ as a constant symbol. The **elementary diagram** of $\mathcal{M}$ is defined as*

$$\{\varphi(m_1, \dots, m_n) : \mathcal{M} \models \varphi(m_1, \dots, m_n), \varphi(\bar{x}) \textit{ is an } \mathcal{L} - \textit{formula and } n \in \mathbb{N}\}.$$

We will denote it by $Diag_{el}(\mathcal{M})$.

Proposition 3.9. *If $\mathcal{N} \models Diag_{el}(\mathcal{M})$, then there is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. In other words $\mathcal{N}$ is an elementary extension of $\mathcal{M}$*

Definition 3.10. *Let $\mathcal{M} = (M; \dots)$ be an $\mathcal{L}$-structure for fixed language $\mathcal{L}$ with $A \subseteq M$.*

*An element $a \in M$ is called **definable** over A, if there is an element $\bar{b} \in A$ and a $\mathcal{L}$-formula $\varphi(u, \bar{b})$ such that the set of x in M that satisfies $\varphi(x, \bar{b})$ is $\{a\}$. The **definable closure** of A is then defined as*

$$dcl(A) = \{x \in M : x \textit{ is definable over } A\}$$

*An element $a \in M$ is called **algebraic** over A, if there is an element $\bar{b} \in A$ and a $\mathcal{L}$-formula $\varphi(u, \bar{b})$ such that the set of x in M that satisfies $\varphi(u, \bar{b})$ is finite and $\varphi(a, \bar{b})$ holds. Then **algebraic closure** of A is defined as*

$$acl(A) = \{x \in M : x \textit{ is algebraic over } A\}.$$

Definition 3.11. *A theory T is **model-complete** if every embedding of any model of T is elementary.*

Definition 3.12. *A theory T is said to be* ***complete****, for any sentence φ from its language, if either $T \models \varphi$ or $T \models \neg\varphi$.*

Example 3.13. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Then the full theory*

$$Th(\mathcal{M}) = \{\phi : \mathcal{M} \models \phi \text{ where } \phi \text{ is an } \mathcal{L}\text{-sentence}\}$$

is complete.

Example 3.14. *The following examples show that there is no implication between the notions of model-complete and complete. In other words, model-complete theories do not have to be complete or vice versa.*

- *ACF is a model-complete but not a complete theory. Model-completeness is a consequence of quantifier elimination. To see that ACF is not complete, consider the characteristic of an arbitrary algebraically closed field. Since neither of the sentence that states the characteristic of this field nor its negation needs to be a logical consequence of ACF, there is at least one sentence φ in the language of rings such that neither $T \models \varphi$ nor $T \models \neg\varphi$.*
- *Consider the theory of dense linear orders with maximal and minimal elements, call this theory T. One can see that T is complete by using a back and forth argument. Let $([0,1];<), ([0,2];<)$ be two models of T. When the latter model satisfies the formula that states "there exists $x, x > 1$", the first one does not. So the inclusion map is not elemantary. Thus T is not model-complete.*

Definition 3.15. *Let T be a theory with models of size κ where κ is an infinite cardinal. Then T is said to be κ-****categorical*** *if any two models of T of cardinality of κ are isomorphic.*

Example 3.16. *(Marker, 2000) Let κ be an uncountable cardinal. Then the theory of algebraically closed fields of characteristic p where p is zero or a prime number is κ-categorical.*

Theorem 3.17. *(****Categoricity Theorem****) If T is κ-categorical for some uncountable cardinal κ, then T is λ-categorical for every uncountable cardinal λ.*

Theorem 3.18. *(****Łoś -Vaught Test****) Let T be a satisfiable, κ-categorical, for some infinite cardinal $\kappa \geqslant |\mathcal{L}|$, $\mathcal{L}$-theory with no finite models. Then T is complete.*

Theorem 3.19. *ACF_p is complete, where p is either zero or a prime number.*

Proof Let F and K be two algebraically closed fields of the characteristic p. Then $\mathcal{F}$ and $\mathcal{K}$ satisfy ACF_p where $\mathcal{F} = (F; \mathcal{L}_{ring})$ and $\mathcal{K} = (K; \mathcal{L}_{ring})$. Let φ be a formula from $\mathcal{L}_{ring}$. Since ACF has quantifier elimination, we can find a quantifier free formula ψ that is equivalent to φ. By using the fact that $\mathcal{F}$ and $\mathcal{K}$ are extensions of either $\mathbb{Q}$ or $\mathbb{F}_p$, we have

$$\mathcal{F} \models \psi \Leftrightarrow \mathbb{Q} \models \psi \Leftrightarrow \mathcal{K} \models \psi$$

or

$$\mathcal{F} \models \psi \Leftrightarrow \mathbb{F}_p \models \psi \Leftrightarrow \mathcal{K} \models \psi$$

depending on the characteristic. Thus,

$$\mathcal{F} \models \varphi \Leftrightarrow \mathcal{F} \models \psi \Leftrightarrow \mathcal{K} \models \psi \Leftrightarrow \mathcal{K} \models \varphi.$$

That means $\mathcal{F}$ and $\mathcal{K}$ are elemantarily equivalent. Since ACF_p is κ-categorical for all uncountable cardinals κ and any algebraically closed field has infinitely many elements, by using *Łoś -Vaught Test* we can conclude that ACF_p is complete.

□

Definition 3.20. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure with universe M. Let $A \subseteq M$. An **n-type** $p(\bar{x}) = p$ of $\mathcal{M}$ over A is a set of $\mathcal{L}_{\mathcal{A}}$-formulae in n free variables such that for each finite subset of $p_0(\bar{x})$ of $p(\bar{x})$, there is a tuple $(c_1, \ldots, c_n) \in M^n$ with $\mathcal{M} \models p_0(c_1, \ldots, c_n)$.*

*An n-type p over A is called **complete** if for any $\mathcal{L}_{\mathcal{A}}$-formula ϕ either $\phi \in p$ or $\neg\phi \in p$ holds. The set of complete n-types over A is denoted by $S_n^{\mathcal{M}}(A)$.*

*A complete n-type can be defined by using a tuple from M. Let $\bar{a} = (a_1, \ldots, a_n)$ and $A \subseteq M$, then the **complete type of $\bar{a}$ over A** , $tp(\bar{a}/A) = \{\psi(\bar{x}) \in \mathcal{L}_{\mathcal{A}} : \mathcal{M} \models \psi(\bar{a})\}$.*

Let ϕ be an $\mathcal{L}_{\mathcal{A}}$-formula in n-free variables. One can construct a set by collecting the complete types from $S_n(A)$ that contains ϕ, define

$$[\phi] = \{p \in S_n(A) : \phi \in p\}.$$

*One can see that there is a topology on $S_n(A)$ which is generated by the sets $[\phi]$ as principal open sets. This topology is called **Stone Topology**.*

*A type $p \in S_n(A)$ is called **isolated**, if the singleton $\{p\}$ is an open subset of $S_n(A)$.*

Proposition 3.21. *(Marker, 2000) Let $p \in S_n(A)$. Then the following are equivalent.*

1. *p is isolated.*
2. *$\{p\} = [\phi]$ for some $\mathcal{L}_A$-formula ϕ.*
3. *There is an $\mathcal{L}_A$-formula $\phi \in p$ such that , for all $\mathcal{L}_A$-formulas ψ, $\psi \in p$ if and only if*

$$Th_A(\mathcal{M}) \models \phi \to \psi.$$

Example 3.22. *Consider the set of complex numbers $\mathbb{C}$ in the field structure $\mathcal{C} = (\mathbb{C}; 0, 1, +, \cdot)$. Since $\sqrt{2}$ is algebraic over $\mathbb{Q}$ with minimal polynomial $x^2 - 2$ over $\mathbb{Q}$ and π is transcendental over $\mathbb{Q}$, we have the following types;*

$$tp(\sqrt{2}/\emptyset) = \{\phi : \mathcal{C} \models \forall x, (x^2 - 2 = 0 \to \phi(x))\}.$$

$$tp(\pi/\emptyset) = \{p(x) \neq 0 : p(x) \in \mathbb{Z}[X] \setminus \{0\}\} \cup \{\text{logical consequences}\}.$$

Hence, any transcendental element of $\mathbb{C}$ over $\mathbb{Q}$ has the same type over $\emptyset$. We will generalize this idea to any algebraically closed field in the next chapters.

Definition 3.23. *A type $p \in S_n(A)$ is called **definable over** B if for any $\mathcal{L}$-formula $\psi(\bar{x}, \bar{y})$, there exists an $\mathcal{L}_\mathcal{B}$-formula $d_p^\psi(\bar{y})$ such that*

$$\psi(\bar{x}, \bar{a}) \in p \Leftrightarrow d_p^\psi(\bar{a}) \text{ holds}$$

for each $\bar{a} \in A$.

As a final notion of this section, we will define a saturated model.

Definition 3.24. *Let κ be an infinite cardinal. A structure $\mathcal{M}$ is called κ-**saturated**, if for any $A \subseteq M$ with $|A| < \kappa$, for each type $p \in S_n(A)$, there is $\bar{c} \in M^n$ such that $\mathcal{M} \models \phi(\bar{c})$ for all $\phi \in p$.*

Example 3.25. *(Marker, 2000) ACF_p has a countable saturated model which is the algebraically closed field of characteristic p of transcendence degree $\aleph_0$.*

3.2 Elimination of Imaginaries

Fix a complete theory T which is constructed by $\mathcal{L}$-sentences where $\mathcal{L}$ is a countable language. We will work with the *monster model* of T. Roughly speaking, monster model of T is a saturated model of T with very big cardinality, say κ. The existence of such a cardinality is guaranteed by ZFC and it yields to a model theoretic fact that states there is a saturated model of size κ (Marker, 2000). Let $\mathfrak{M}$ be a monster model of T.

$\mathfrak{M}^{eq}$ as an $\mathcal{L}^{eq}$-structure is defined as follows; for each $\emptyset-$definable equivalance relation

$$E(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

on $\mathfrak{M}^n$, there will be a new sort $\mathfrak{M}^n/E$ that is added to $\mathfrak{M}$ and a new function symbol π_E that is added to L such that

$$\pi_E : \mathfrak{M}^n \to \mathfrak{M}^n/E$$

is the projection map. The elements which belong to those new sortes are called ***imaginary elements***. Since every model $\mathcal{M}$ of T can be embedded into $\mathfrak{M}$ elementarily , we can define $\mathcal{M}^{eq}$ by considering an elementary substructure of $\mathfrak{M}^{eq}$. The set of all definable subsets of M^{eq} with parameters will be denoted by $\mathrm{Def(M^{eq})}$.

Our main interest is about eliminating those new elements. A theory admits ***elimination of imaginaries*** if for every model $\mathcal{M}$ of the theory with universe M, for any A-definable equivalence relation E on M^n, where $A \subseteq M$, for some l, there is an A-definable function $f : M^n \to M^l$ such that

$$\bar{x}E\bar{y} \text{ if and only if } f(\bar{x}) = f(\bar{y})$$

This allows us to identify any quotient M^n/E with an image of a definable (with parameters) function where E is a definable (with parameters) equivalance relation on M^n.

Theorem 3.26. *ACF admits elimination of imaginaries*

Elimination of imaginaries on algebraically closed fields allows us to play with equivalance relations without disturbing definability. We will use this facility a lot.

Recall that a field K is called perfect if $char(K) = 0$ or if $char(K) = p$ and the Frobenius map $x \mapsto x^p$ is an automorphism of K. A perfect closure of a field K is the smallest perfect field containing K. In char p, define

$$K_{ins} = \{x \in (K)^{alg} : x^{p^n} \in k \text{ for some } n\}.$$

Since the Frobenius map $x \mapsto x^p$ is an injective endomorphism of K, K_{ins} is the perfect closure of K.

Theorem 3.27. *(Bouscaren, 1998) Suppose F is an algebraically closed field. Let k be a subfield generated by A where $A \subseteq F$. For $a \in F$,*

$$a \in dcl(A) \text{ if and only if } a \in k_{ins}.$$

Proof

Let $a \in dcl(A)$ be arbitrary. Then $a \in dcl(k_{ins})$ because $A \subseteq k \subseteq k_{ins}$. Let $P(X)$ be the minimal polynomial of a over k_{ins}. Then $tp(a/k_{ins})$ is isolated by the formula $P(X) = 0$ since any polynomial from $k_{ins}[X]$ that vanishes at a is a multiple of $P(X)$. If $degP > 1$, then there must be more than one root of $P(X)$ because of the fact that k_{ins} is perfect. On the other hand $a \in dcl(k_{ins})$ implies that there could be only one root of $P(X)$. So we have $degP = 1$, hence $a \in k_{ins}$.

Suppose $a \in k_{ins}$. Then $a^{p^n} \in k$ for some n. In other words $a^{p^n} = b$ for some $b \in k$. Let $\varphi(x)$ be a formula with parameter b that states

$$x^{p^n} = b.$$

Since Frobenius is an injective map, the set of realizations of $\varphi(x)$ is $\{a\}$. Then $a \in dcl(k)$, equivalently $a \in dcl(A)$.

□

Theorem 3.28. *Assume that F is an algebraically closed field with a subfield k. Let $X \subseteq F^n$ be k-definable and $f : X \to F$ be a k-definable function. Then X can be written as $X_1 \cup X_2 \cup \ldots \cup X_m$, for some m, where X_i's are k-definable sets such that for each i, there is some non-negative $j(i)$ such that $f \restriction X_i = (Fr^{-j(i)} \circ f_i) \restriction X_i$ for some rational f_i.*

Proof Consider the set of sentences

$$\sum(v_1, \ldots, v_n) = \{f(\bar{x}) \neq (Fr^{-t} \circ g)(\bar{x}) : g(\bar{x}) \in k(\bar{x}), t \in \mathbb{N} \setminus \{0\}\} \cup \{\bar{v} \in X\} \cup Diag_{el}(F).$$

We aim to show that $\sum$ is not a satisfiable theory. To obtain a contradiction, suppose that there exists a model $\mathcal{K}$ of $\sum$. In other words, $\mathcal{K} \models Diag_{el}(F)$ with $b_1, \ldots, b_n$ such that $\mathcal{K} \models \sum(\bar{b})$. $\mathcal{K} \models Diag_{el}(F)$ implies that $\mathcal{K}$ is an elementary extension of F by Proposition 3.9. Thus we are allowed to extend f to

$$f^{\mathcal{K}} : X^{\mathcal{K}} \longrightarrow K.$$

As mentioned above, each element of $(k(\bar{b}))_{ins}$ is in the form of $g(\bar{b})^{1/p^n}$ for some $g(\bar{x}) \in k(\bar{x})$ and for some $n \in \mathbb{N}$. So $\mathcal{K} \models \sum(\bar{b})$ implies that $f(\bar{b}) \notin (k(\bar{b}))_{ins}$. By above theorem we have $f(\bar{b}) \notin dcl(k(\bar{b}))$. This contradicts the fact that f is k-definable. So $\sum$ is not satisfiable. Thus, by Compactness Theorem 3.5, there exist $f_1, \ldots, f_n \in k(\bar{x})$ and $j(1), \ldots, j(m)$ such that for all $\bar{x} \in X$

$$f(\bar{x}) = Fr^{-j(i)} \circ f_i$$

for some $i \in \mathbb{N}$. Let

$$X_i = \{\bar{x} : f(\bar{x}) = Fr^{-j(i)} \circ f_i\}.$$

Then X_i is definable for each i.

$\square$

4 TWO RANK NOTIONS

In this chapter, we will introduce two ways of measuring in model theoretic sense. Throughout this section definable means definable with parameters.

Definition 4.1. *A function* $rk : \mathrm{Def}(\mathrm{M}^{\mathrm{eq}}) \to \mathbb{N}$ *is called* ***rank*** *on* M^{eq} *if the following axiom are satisfied for all* $A, B \in \mathrm{Def}(\mathrm{M}^{\mathrm{eq}})$.

1. *(Monotonicity of rank)* $rk(A) \geq n+1$ *if and only if there are infinitely many pairwise disjoint, non-empty, definable subsets of* A *each of rank at least* n.

2. *(Definability of rank) If* $f : A \to B$ *is a definable function, then the set* $\{b \in B : rk(f^{-1}(b) = n\}$ *is definable for each* n.

3. *(Additivity of rank) If* $f : A \to B$ *is a definable surjection and if for all* $b \in B$, $rk(f^{-1}(b)) = n$, *then* $rk(A) = rk(B) + n$.

4. *(Elimination of infinite quantifiers) For any definable function* $f : A \to B$, *there is an integer* m *such that for any* $b \in B$, *the pre-image of* b *under* f *is infinite when it contains at least* m *elements.*

Proposition 4.2. *For any sets* A *and* B *from* M^{eq}, *we have the following properties of rank function.*

1. A *is finite if and only if* $rk(A) = 0$.

2. $A \subseteq B$ *implies that* $rk(A) \leq rk(B)$.

3. $rk(A \cup B) = max\{rk(A), rk(B)\}$.

4. *$rk(A^n) = n.rk(A)$ for any n.*

5. *If there is a definable injection f between A and B, then $rk(A) \leq rk(B)$. In particular, $rk(A) = rk(B)$ if f is a definable bijection.*

Proof

1. Suppose that A is finite and $rk(A) \geq 1$. By the axiom of monotonicity there are infinitely many pairwise disjoint, non-empty, definable subsets of A each of rank at least 0, hence A contains infinitely many elements. Since A is finite, this is not possible.

   Conversely assume that $rk(A) = 0$ and A is infinite. Consider the definable family of subsets of A which consists of the singletons $\{a_i\}$, where $a_i \in A$. By above paragraph, we have $rk(\{a_i\}) = 0$, hence there is an infinite family of pairwise disjoint, non-empty, definable subsets of A each of rank at least 0. Thus $rk(A) \geq 1$ and it contradicts the fact that $rk(A) = 0$.

2. Suppose $rk(A) = n+1$, then by the axiom of monotonicity, A has infinitely many disjoint definable subsets A_i of rank at least n. Since $A \subseteq B$, for each i, $A_i \subseteq B$. By applying the same axiom, one can conclude that $rk(B) \geq n+1$. If $rk(A) = 0$, $rk(A) \leq rk(B)$ holds trivially.

3. Let $C = A \cup B$. By previous part, we know that $rk(C) \geq max\{rk(A), rk(B)\}$. Assume $rk(C) = n + 1$. Let C_i be an infinite family of disjoint definable subsets of C each of rank n. Now consider the infinite families $A \cap C_i$ and $B \cap C_i$. For each i, at least one of the following $rk(A \cap C_i) = n$, $rk(B \cap C_i) = n$ holds, hence $rk(A) \geq n + 1$, $rk(B) \geq n + 1$ or both. Thus $max\{rk(A), rk(B)\} \geq rk(C)$.

4. We will see that the equation holds by proceeding induction on n. $n = 0$ is the trivial case since $A^0 = \{\emptyset\}$. Suppose $rk(A^n) = n.rk(A)$ for some n. Consider the definable surjection $pr_1 : A^{n+1} \to A$. Then for each $a \in A$, $pr_1^{-1}(a) = A^n$, hence $rk(pr_1^{-1}(a)) = n.rk(A)$. Then, $rk(A^{n+1}) = rk(A) + n.rk(A) = (n + 1).rk(A)$ by axiom of additivity.

5. If A is a finite set, then the equality follows. If A is infinite, we will proceed by induction on $rk(A)$. Suppose that $rk(A) = n$ implies that $rk(A) \leq rk(B)$ for some n. Now assume that $rk(A) = n+1$. Let $(A_i)_i$ be an infinite family of pairwise disjoint definable subsets of A of rank n. Then $f(A_i)_i$ gives an infinite family of pairwise disjoint definable subsets of B, since f is an injection. By induction

hypotesis, each of element of this family is of rank n. Then, by monotonicity of rank, $rk(B) \geq n+1 = rk(A)$.

□

Remark 4.3. *One can ease the conditions that we introduced in the Axiom of Monotonicity of Rank. To say that $rk(A) \geq n+1$ it is enough to have infinitely many definable subsets A_i of A such that $rk(A_i) = n$ and $rk(A_i \cap A_j) < n$ for any $A_i \neq A_j$. It can be seen that by considering infinitely many pairwise disjoint, definable subsets B_i of A, $B_i := A_i \setminus ((A_i \cap A_0) \cup \ldots \cup (A_i \cap A_{i-1})$ with $B_0 = A_0$. Then, by Proposition 4.2 $n = rk(A_i) = max\{rk(B_i), rk((A_i \cap A_0) \cup \ldots \cup (A_i \cap A_{i-1}))\}$. It follows that $rk(B_i) = n$ since $rk(A_i \cap A_j) < n$ for all $j < i$.*

Definition 4.4. *Assume that $\mathcal{M}$ is an $\mathcal{L}$-structure and ϕ is an $\mathcal{L}_{\mathcal{M}}$-formula.The relation $RM^{\mathcal{M}}(\phi) \geq \alpha$, for some ordinal α, is defined inductively as follows*

1. *$RM^{\mathcal{M}}(\phi) \geq 0$ if and only if $\phi(\mathcal{M})$ is non-empty.*
2. *$RM^{\mathcal{M}}(\phi) \geq \alpha + 1$ if and only if there are $\mathcal{L}_{\mathcal{M}}$-formulae ψ_i such that $\psi_1(\mathcal{M})$, $\psi_2(\mathcal{M})$, ... is an infinite family of pairwise disjoint subsets of $\phi(\mathcal{M})$ and*

$$RM^{\mathcal{M}}(\psi_i) \geq \alpha$$

   for all i.
3. *$RM^{\mathcal{M}}(\phi) \geq \alpha$ if and only $RM^{\mathcal{M}}(\phi) \geq \beta$, for all $\beta < \alpha$, where α is a limit ordinal.*

Then we let $RM^{\mathcal{M}}(\phi) = \alpha$ if $RM^{\mathcal{M}}(\phi) \geq \alpha$ but $RM^{\mathcal{M}}(\phi) \not\geq \alpha + 1$. By the convention if $\phi(\mathcal{M}) = \emptyset$, then $RM^{\mathcal{M}}(\phi) = -1$.

One can check that $RM^{\mathcal{M}}(\phi) = RM^{\mathcal{N}}(\phi)$ where $\mathcal{M}$ and $\mathcal{N}$ are $\aleph_0$-saturated so that $\mathcal{M} \prec \mathcal{N}$ by induction on $RM^{\mathcal{M}}(\phi)$.

We want to define Morley rank of an $\mathcal{L}_{\mathcal{M}}$-formula ϕ which is independent from the model $\mathcal{M}$. Following this, we let ***Morley rank*** *of ϕ, $RM(\phi) = RM^{\mathcal{N}}(\phi)$ where $\mathcal{N}$ is an $\aleph_0$-saturated elementary extension of $\mathcal{M}$. Consequently Morley rank is preserved under elementary extension. By the convention if $\phi(\mathcal{M}) = \emptyset$, then $RM(\phi) = -1$.*

The ***Morley rank of a set*** *X which is defined by a $\mathcal{L}_{\mathcal{M}}$-formula ϕ, $RM(X) = RM(\phi)$.*

Let p be a complete type over $A \subseteq M$. Then the ***Morley rank of*** *p is defined as $RM(p) = min\{RM(\phi) : \phi \in p\}$.*

Our main interest will be groups which are definable in an algebraically closed field. To this end, we will introduce some facts about Morley rank in algebraically closed fields.

Example 4.5. *The Morley rank of an algebraically closed field F is 1. Since F is infinite, $RM(F) \geq 1$. In Example 3.4 we stated that ACF is strongly minimal, hence for any definable subset X of F is either finite or cofinite. So it is not possible to find two infinite definable subsets of F which are disjoint. That means $RM(F) = 1$.*

Theorem 4.6. *Let F be an algebraically closed field. For any set X which is definable in F, $RM(X)$ is equal to Krull dimension of Zariski closure of X.*

5 GROUPS OF FINITE MORLEY RANK

Theorem 5.1. *Let G be a group of finite Morley rank. Morley rank is the only rank that satisfies the conditions in Definition 4.1.*

Bruno Poizat proved the above theorem in (Poizat, 2001). Hence we are allowed to replace $rk(A)$ with $RM(A)$ for any definable A in a group G of finite Morley rank.

Theorem 5.2. *(Marker, 2000) Types over a group of finite Morley rank are definable.*

Proposition 5.3. *[**Lascar's equality**] Let G be a group of finite Morley rank with a definable subgroup $H \leq G$. Then $RM(G) = RM(G/H) + RM(H)$.*

Proof The left coset space $G/H = \{\bar{g} = g.H : g \in G\}$ is definable, since the map $\psi_g(h) := g.h$ is definable for all $g \in G$. On the other hand $\psi_g(h)$ is a bijection, hence $RM(gH) = RM(H)$ by Proposition 4.2. Consider the canonical surjective homomorphism

$$\begin{array}{rccc} \varphi : & G & \longrightarrow & G/H \\ & g & \mapsto & \bar{g} = gH \end{array}$$

For arbitrary $\bar{g} \in G/H$, $\varphi^{-1}(\bar{g}) = gH$. Then $RM(\varphi^{-1}(\bar{g})) = RM(gH) = RM(H)$. Hence, by additivity of rank, $RM(G) = RM(G/H) + RM(H)$.

□

Lemma 5.4. *Let G be a group of finite Morley rank and H and K be definable subgroups of G with $K \leq H$. Then*

1. $[H : K] = \infty$ *if and only if* $RM(K) < RM(H)$.
2. $[H : K] < \infty$ *if and only if* $RM(K) = RM(H)$.
   (Borovik and Nesin, 1994)

Proof Let n be the Morley rank of K. For any $h \in H$, consider the definable bijection

$$\begin{array}{cccc} \varphi_h : & K & \longrightarrow & hK \\ & k & \mapsto & hk \end{array}$$

Recall that every left coset of K in H has rank n.

If $[H : K] = \infty$, then there are infinitely many, disjoint, definable subsets of H each of rank n, namely the left cosets. By the axiom of monotonicity, $RM(H) \geq n + 1$, i.e. $RM(K) < RM(H)$.

If $[H : K] < \infty$, H can be written as a disjoint union of left cosets of K in H. Then by Proposition 4.2 $RM(H) = max\{RM(hK) : h \in H\} = n$.

Now assume $RM(K) = RM(H)$. If $[H : K] = \infty$ holds, then $\{hK : h \in K\}$ is an infinite family of disjoint, definable subsets of H each of rank n. This implies $RM(H) \geq n + 1$ which is impossible by the assumption $RM(K) = RM(H)$. Hence $[H : K] < \infty$. Since $K \leq H$, the sufficient condition of the first statement has been proven also. □

It is said that a group G satisfies ***descending chain condition for definable subgroups*** if any descending chain of definable subgroups H_i of G,

$$H_0 > H_1 > \ldots > H_n > \ldots$$

becomes stationary at a finite step, that is, there is an $n \in \mathbb{N}$ such that $H_m = H_n$ for all $n \leq m$. We abrreviate descending chain condition as DCC.

Theorem 5.5. ***[Macintyre]*** *A group of finite Morley rank G satisfies the descending chain condition for its definable subgroups.*

Proof Let $(H_i)_i$ be a descending chain of definable subgroups of G. Then $(RM(H_i))_i$ is a decreasing sequence by Proposition 4.2. This sequence needs to stabilize after finite terms. i.e. there is an n such that, for all $m \geq n$, $RM(H_n) = RM(H_m)$, say $RM(H_n) = k$.

Now suppose that $(H_i)_i$ does not become stationary. Then we have $H_n = \sqcup_{j=1}^{r} h_j.H_{n+1}$ for non-identity $h_j \in H_n$ since $RM(H_n) = RM(H_{n+1})$ implies that $[H_n : H_{n+1}] < \infty$ by Lemma 5.4. By a similar argument, for any $m \geq n$, $H_m = \sqcup_{j=1}^{t} \tilde{h}_j.H_{m+1}$ for some non-identity $\tilde{h}_j \in H_m$.

Since the left cosets of H_{m+1} compose an infinite family of disjoint definable subsets in H_{i_m} for any $m \geq n$, we have

$$H_n = t_n.H_{n+1} \sqcup t_{n+1}.H_{n+2} \sqcup \ldots \sqcup t_{n+s}.H_{n+s+1} \sqcup \ldots$$

for some $t_{n+s} \in H_{n+s}$ and $RM(t_{n+s}.H_{n+s+1}) = k$.

Then by the axiom of monotonicity $RM(H_n) = k + 1$. This contradicts the fact that $RM(H_n) = k$. Hence, $(H_i)_i$ becomes stationary at a finite step.

$\square$

5.1 Connected Groups

Definition 5.6. *A group G is* ***connected*** *if it has no definable proper subgroup of finite index.*

Assume that G is a group of finite Morley rank. The **connected component of** G is defined as the intersection of all definable subgroups of finite index of G and denoted by G^{o}.

$$G^{o} = \bigcap_{[G:H]<\infty} H$$

where H's are definable subgroups of G.

Since G has DCC on its definable subgroups, G^{o} becomes the intersection of finitely many definable subgroups of finite index of G.

$$G^{o} = \bigcap_{i=1}^{n} H_i$$

where each H_i is a definable subgroup of G with $[G : H_i] < \infty$.

Hence G^o is the smallest definable subgroup of G that has finite index in G. Note that G^o is normal subgroup in G.

Remark 5.7. *According to Definition 5.6, a group G of finite Morley rank is called* ***connected*** *if and only if* $G^o = G$.

It can be seen that a finite group G is connected if and only if G is trivial.

Proposition 5.8. *Let H be a definable, normal subgroup of a group G of finite Morley rank. Assume that H and G/H are connected. Then G is connected. (Borovik and Nesin, 1994, Exercise)*

Proof Let G be a group of finite Morley rank with a definable, normal subgroup H such that H and G/H are connected groups. Recall that G/H is a group of finite Morley rank by Proposition 5.3. Since a group of finite Morley rank satisfies the DCC on its definable subgroups, one can find $A_1, \ldots, A_m$ such that

$$(G/H)^o = \bigcap_{i=1}^{m} A_i$$

where each A_i is a definable subgroup of G/H with $[G/H : A_i] < \infty$.

Note that each A_i is in the form of B_i/H for some definable subgroup B_i of G with $H \subseteq B_i$. Thus

$$(G/H)^o = \bigcap_{i=1}^{m} B_i/H.$$

Consider the canonical homomorphism $\varphi : G \to G/H$ with $\varphi^{-1}(B_i/H) = B_i$. Since $[G/H : A_i] = [G/H : B_i/H]$ is finite, $[G : B_i]$ is finite. Thus $G^o \subseteq B_i$ for all i. So $G^o \subseteq \varphi^{-1}(B_i/H)$ for all i.

Apply φ to both sides to get

$$\varphi(G^o) \subseteq B_i/H$$

for all i. Then we have

$$\varphi(G^o) \subseteq \bigcap_{i=1}^{m} B_i/H = (G/H)^o$$

Since $\varphi(G^o) = G^oH/H$, we have $G^oH/H \subseteq (G/H)^o$. This yields to $G^oH/H = (G/H)^o$ by using the fact that image of a definable subgroup of finite index of G is a definable subgroup of finite index of G/H under φ.

G/H is connected means that $(G/H)^{\circ} = G/H$ and then we get $G^{\circ}H/H = G/H$ by above observations. On the other hand $H^{\circ} \subseteq G^{\circ}$ implies that $H \subseteq G^{\circ}$ by the connectedness of H. Then we have

$$G/H = G^{\circ}H/H = G^{\circ}/H.$$

Hence $G^{\circ} = G$, in other words G is connected.

□

Corollary 5.9. *A semidirect product of two connected groups is a connected group.*

Example 5.10. *The group of linear transformations from a field K of finite Morley rank to itself, namely* Aff(K) *is connected. We will prove that the additive and multiplicative subgroups of K are connected in 6.1. Note that* Aff(K) *is isomorphic to semidirect product of K^{+} by K^{*}. Then by Corollary 6.3,* Aff(K) *is connected.*

5.2 Generic Types

We introduced the notion of type in Chapter 3. Now we will examine a special kind of types, namely generic types, with more details. Throughout this section, we will work with $\mathcal{G} = (G; \cdot, 1, \ldots)$ where G is a group of finite Morley rank and $\mathbb{G}$ is a monster model with $\mathcal{G} \prec \mathbb{G}$. One should be careful about an abuse of notation, both of the universe and structure of the monster model of $\mathcal{G}$ will be denoted by $\mathbb{G}$. It is not required that $\mathcal{G}$ is a pure group structure, that is, there may be extra symbols.

Definition 5.11. *Let $p \in S_1(G)$. Then*

- *The complete type p is called **generic** if $RM(p) = RM(G)$*
- *An element a of $\mathbb{G}$ is called **generic** over G if $RM(tp(a/G)) = RM(G)$.*

Proposition 5.12. *Let G be a group of finite Morley rank. For any definable subset X of G, there are generic types over X.*

Proof Suppose that X be a definable subset of G. Let $\phi(x)$ be the formula that defines X and n be the Morley rank of $\phi(x)$, i.e. $RM(\phi) = RM(X) = n$. When $n = 0$, type of an each element of X is generic, hence assume $n \neq 0$. We may assume that ϕ is irreducible, that is, there is no partition of ϕ by finitely many formula of Morley rank

n. (If $\phi = \psi_1 \wedge \ldots \wedge \psi_m$ such that $RM(\psi_i) = n$ where m is the biggest number for such a partition, we are dealing with one of ψ_i's) Now we will find a complete type $p \in S_1(G)$ such that $\phi \in p$ and $RM(p) = RM(\phi) = n$.

Let $p = \{\psi : RM(\neg\psi \wedge \phi) < n\}$. Obviously $\phi \in p$. We will see that p is a complete type.

Let $\psi_1, \ldots, \psi_m \in p$. To show that $\psi_1 \wedge \ldots \wedge \psi_m$ is satisfiable, we will obtain that $RM(\psi_1 \wedge \ldots \wedge \psi_m) \neq 0$. By Proposition 4.2, we have the following equation.

$$RM(\phi) = max\{RM(\phi \wedge (\psi_1 \wedge \ldots \wedge \psi_m)), RM(\phi \wedge \neg(\psi_1 \wedge \ldots \wedge \psi_m))\}.$$

By considering $RM(\phi \wedge \neg(\psi_1 \wedge \ldots \wedge \psi_m)) = RM(\phi \wedge (\neg\psi_1 \vee \ldots \vee \neg\psi_m))$, we get

$$RM(\phi) = max\{RM(\phi \wedge (\psi_1 \wedge \ldots \wedge \psi_m)), RM((\phi \wedge \neg\psi_1) \vee \ldots \vee (\phi \wedge \neg\psi_m))\}.$$

Recall that $\psi_i \in p$, hence $RM(\phi \wedge \neg\psi_m) < n$ for each i. Since

$$RM((\phi \wedge \neg\psi_1) \vee \ldots \vee (\phi \wedge \neg\psi_m)) = max\{RM((\phi \wedge \neg\psi_1), \ldots, RM(\phi \wedge \neg\psi_m))\},$$

we have $RM((\phi \wedge \neg\psi_1) \vee \ldots \vee (\phi \wedge \neg\psi_m)) < n$. It follows that $RM(\phi \wedge (\psi_1 \wedge \ldots \wedge \psi_m)) = n$, since $RM(\phi) = n$

Let ψ be $\psi_1 \wedge \ldots \wedge \psi_m$. In order to show that ψ is consistent, consider the following equation

$$RM(\phi) = max\{RM(\phi \wedge \psi), RM(\phi \wedge \neg\psi)\}$$

Since $\psi \in p$, $RM(\phi \wedge \neg\psi) < n$. Hence $RM(\psi) \geq RM(\phi \wedge \psi) = RM(\phi) = n$. Then ψ is satisfiable.

Let θ be a formula. Assume that $\theta \notin p$. We will show that $\neg\theta \in p$. Since $\theta \notin p$, $RM(\phi \wedge \neg\theta) \geq n$. Then $RM(\phi \wedge \neg\theta) = n$. So $RM(\phi \wedge \theta) < n$ by irreducibility of ϕ. Hence $\neg\theta \in p$.

□

Proposition 5.13. *There are only finitely many generic types over G.*

Proof Let $RM(G) = n$ and ϕ define G. Suppose there are infinitely many generic types p_i's over G. For each i, there is an irreducible formula ψ_i such that $p_i \in [\psi_i]$ and $RM(\psi_i) = n$. We will see that ψ_i's are distinct. Let p_i, p_j be two distinct generic types.

To get a contradiction, assume that $[\psi_i] = [\psi_j]$. Then there is a formula ϕ_{ij} where $\phi_{ij} \in p_i \setminus p_j$. Since p_j is a complete type, $\neg\phi_{ij} \in p_j$. Then we have $RM(\psi_i \wedge \phi_{ij}) = n$, $RM(\psi_i \wedge \neg\phi_{ij}) = n$ because $\psi_i \wedge \phi_{ij} \in p_i$ and $\psi_i \wedge \neg\phi_{ij} \in p_j$. This contradicts with the fact that ψ_i is irreducible.

Now consider $(\psi_i \wedge \psi_j)(\mathbb{G}) \in \psi_i(\mathbb{G})$. Since ψ_i is irreducible, $RM(\psi_i \wedge \psi_j) < n$ for any $i \neq j$.

We get an infinite family of subsets $\psi_1(\mathbb{G}), \psi_2(\mathbb{G}), \ldots$ of $\phi(\mathbb{G})$ such that $RM(\psi_i \wedge \psi_j) < n$ and $RM(\psi_i) = n$ for all i. By Remark 4.3, we have $RM(G) > n$. This contradicts with the fact that $RM(G) = n$, hence there are only finitely many generics.

□

Definition 5.14. *Suppose $a, b \in \mathbb{G}$. Let $\mathcal{G}_b$ be an elementary extension of $\mathcal{G}$ that contains b with universe G_b. We say that a is* ***independent*** *from b over G if $RM(tp(a/G)) = RM(tp(a/G_b))$.*

There are lots of nice properties of this independence notion. We will introduce the one that we will use. For a detailed explanation, one can check (Marker, 2000).

Theorem 5.15. *(Marker, 2000) Independence has a symmetry property, that is, a is independent from b over G implies that b is independent from a over G.*

Proposition 5.16. *Let p be a generic type over G. There are two independent realizations of p over G.*

Proof Let $a \in \mathbb{G}$ be a realization of p. Then we have the following identities $RM(tp(a/G)) = RM(p) = RM(G)$.

Let $\phi \in p$ be such that $RM(p) = RM(\phi)$. Suppose that ψ is an irreducible $\mathcal{L}_{G_a}$-formula such that $\psi(\mathbb{G}) \subseteq \phi(\mathbb{G})$ of Morley rank n.

Let q be the set of $\mathcal{L}_{G_a}$-formulae so that, for any $\theta \in q$, $RM(\theta \wedge \psi) = n$. For any $\mathcal{L}_{G_a}$-formula φ, one has

$$RM(\psi) = max\{RM(\psi \wedge \varphi), RM(\psi \wedge \neg\varphi)\}.$$

If $\varphi \notin q$, then $RM(\psi \wedge \varphi) < n$. Hence $RM(\psi \wedge \neg\varphi) = n$ by above equality.

Let $\varphi_1, \varphi_2 \in q$. Then $RM(\varphi_1 \wedge \psi) = RM(\varphi_2 \wedge \psi) = n$. Since ψ is irreducible of Morley rank n, $RM(\varphi_1 \wedge \psi) = RM(\varphi_2 \wedge \psi) = n$ implies that $RM(\psi \wedge \neg\varphi_i) < n$ for $i = 1, 2$. Note that

$$RM(\psi \wedge \neg(\phi_1 \wedge \phi_2)) = RM(\psi \wedge (\neg\phi_1 \vee \neg\phi_2)) = RM((\psi \wedge \neg\varphi_1) \vee (\psi \wedge \neg\varphi_2))$$

Since $RM(\psi \wedge \neg\varphi_i) < n$ for $i = 1, 2$, then

$$RM((\psi \wedge \neg\varphi_1) \vee (\psi \wedge \neg\varphi_2)) = max\{RM(\psi \wedge \neg\varphi_1), RM(\psi \wedge \neg\varphi_2)\} < n.$$

But then $RM(\psi) = max\{RM(\psi \wedge (\phi_1 \wedge \phi_2), RM(\psi \wedge \neg(\phi_1 \wedge \phi_2))\}$ implies that

$$RM(\psi \wedge (\phi_1 \wedge \phi_2) = n.$$

In other words q is finitely realizable.

We have seen that q is a type of rank n and $p \subseteq q$. Now let b be a realization of q. Then $RM(tp(b/G_a)) = RM(q) = n$. Also b is a realization of p because $p \subseteq q$. Hence, $RM(tp(b/G)) = RM(p) = n$. It follows that

$$RM(tp(b/G_a)) = RM(tp(b/G)).$$

That means b independent from a over G. Since indepence over G is symmetric by Theorem 5.15 , we also know that a independent from b over G. Hence a and b are independent realizations of p.

□

Proposition 5.17. *Let $tp(b/G)$ be a generic type and $a \in G$. Then $tp(ab/G)$ and $tp(b^{-1}/G)$ are generic types.*

Proof Consider the definable bijections $f : \mathbb{G} \to \mathbb{G}$ where $f(x) = ax$ and $f' : \mathbb{G} \to \mathbb{G}$ where $f'(x) = x^{-1}$. Then $f(b) = ab$ and $f(b) = b^{-1}$. Hence, by Proposition 4.2, $RM(tp(ab/G)) = RM(tp(b/G))$ and $RM(tp(b^{-1}/G)) = RM(tp(b/G))$. Thus $tp(ab/G)$ and $tp(b^{-1}/G)$ are generic. □

Proposition 5.18. *Any element of G is a product of two generics.*

Proof Pick $g \in G$. Let $a \in \mathbb{G}$ be generic. Consider the definable bijection $f' : \mathbb{G} \to \mathbb{G}$ where $f(x) = gx^{-1}$. Then, by Proposition 4.2, $f(a) = ga^{-1}$ is generic over G. By letting $f(a) = b$, we have $g = a.b$.

$\square$

5.2.1 The action on one types

We will see that G acts on its complete one types $S_1(G)$. Define

$$\begin{array}{cccc} \phi : & G \times S_1(G) & \longrightarrow & S_1(G) \\ & (g, p) & \mapsto & gp \end{array}$$

where $gp = \{\varphi(x) : \varphi(gx) \in p\}$.

Consider $(g.h)p = \{\varphi(x) : \varphi(g.h.x) \in p\}$. We have

$$\varphi(x) \in (g.h)p \quad \Leftrightarrow \quad \varphi(gx) \in hp \quad \Leftrightarrow \quad \varphi(x) \in g(hp)$$

Hence $(g.h)p = g(hp)$. Also $1p = \{\varphi(x) : \varphi(1.x) \in p\}$, hence $1p = p$. So ϕ defines a left action of G on its complete 1-types $S_1(G)$.

Since we have an action ϕ, we may consider the stabilizer of a complete 1-type under this action. Let

$$stab_p := \{g \in G : gp = p\}.$$

Theorem 5.19. *$stab_p$ is a definable subgroup of G for any $p \in S_1(G)$.*

Proof Let $p \in S_1(G)$. Define

$$stab_p^{\varphi} = \{g \in G : \varphi(hx) \in p \Leftrightarrow \varphi(hgx) \in p \text{ for all } h \in G\}$$

where $\varphi \in p$.

Let $\varphi \in p$. Now we will observe that $stab_p^\varphi$ is a definable subgroup of G. Let $\psi(y, x)$ be the formula $\varphi(yx)$. Since types over a group of finite Morley rank are definable, there is a formula $d_p^\psi(y)$ with parameters from G such that

$$\psi(g, x) \in p \Leftrightarrow G \models d_p^\psi(g)$$

.Then $stab_p^\varphi = \{g \in G : d_p^\psi(h) \leftrightarrow d_p^\psi(hg)\}$ is a definable set in G.

Let g, g' be two elements from $stab_p^\varphi$. Then

$$\varphi(hx) \in p \Leftrightarrow \varphi(g'hx) \in p \Leftrightarrow \varphi(g'.g^{-1}hx) \in p$$

for all $h \in G$, hence $g'.g^{-1} \in stab_p^\varphi$. In other words, $stab_p^\varphi$ is a subgroup of G.

Now we will show that

$$stab_p = \bigcap_{\varphi \in p} stab_p^\varphi.$$

Let $g \in stab_p$. Then $gp = p$, that is,

$$\varphi(x) \in p \Leftrightarrow \varphi(gx) \in p$$

for any $\varphi \in p$. Let $h \in G$. By replacing x by hx, we have

$$\varphi(hx) \in p \Leftrightarrow \varphi(ghx) \in p$$

for each $\varphi \in p$. Thus $g \in stab_p^\varphi$.

Let $g \in \bigcap_{\varphi \in p} stab_p^\varphi$. Then $g \in stab_p^\varphi$ for all $\varphi \in p$. In other words, for each $\varphi \in p$

$$\varphi(hx) \in p \Leftrightarrow \varphi(hgx) \in p.$$

By letting $h = 1_G$, we get

$$\varphi(x) \in p \Leftrightarrow \varphi(gx) \in p$$

for all $\varphi \in p$. Thus $g \in stab_p$.

Thus, $stab_p$ is an intersection of definable subgroups of G. By Theorem 5.5, there are finitely many formulas φ_i in p such that $stab_p = \bigcap_{i=1}^n \varphi_i$. Therefore $stab_p$ is a definable subgroups of G. □

Proposition 5.20. *Let $p \in S_1(G)$. Then $stab_p$ is a subgroup of G^o.*

Proof We stated that G^o is a definable subgroup of G. So it is enough to show that G^o contains $stab_p$ as a set. Let $\psi(x)$ be the formula that defines G^o. Let G' be an elementary extension of G containing a realization a of p. Then $a \in h.(G')^o$ for some h, hence $h^{-1}.a \in (G')^o$. So $\psi(h^{-1}.a)$ holds, in other words, $\psi(h^{-1}.x) \in p$.

Let $g \in stab_p$. Then $\psi(h^{-1}.gx) \in p$. Consider an elementary extension G'' of G that contains a realization b of p. Since $\psi(x)$ defines G^o and G'' is an elemantary extension of G, we have $h^{-1}.g.b \in (G'')^o$ and $h^{-1}.b \in (G'')^o$. So $(h^{-1}.b)^{-1}.h^{-1}.g.b = b^{-1}.g.b \in (G'')^o$. Since $(G'')^o$ is normal, we have $g \in (G'')^o$. Then $G'' \models \psi(g)$ holds implies that $G \models \psi(g)$, hence $g \in G^o$

□

Those results hold for all complete 1-types over G but we will be interested in generic types which are enriched by strong properties.

Remark 5.21. *Let $a \in \mathbb{G}$. Recall that $\mathcal{G}_a$ is an elementary extension of $\mathcal{G}$ containing a with universe G_a. We noted that the Morley rank of a formula is preserved under an elementary extension. Since the formula "$v = v$" defines the universe, for any $a \in \mathbb{G}$, $RM(G) = RM(G_a)$.*

Suppose that G is a connected group. Then so is G_a. Otherwise there would be a subgroup H of G_a of finite index. One can find a formula that states there are only finitely many elements in G_a such that G_a can be written as a disjoint union of the translations of H by those elements but there are no such elements in G because G is connected. Since $\mathcal{G}_a$ is an elementary extension of $\mathcal{G}$, this gives a contradiction.

Proposition 5.22. *Let p be a generic type over G. Then $stab_p = G^o$.*

Proof Let p be a generic type over G. Then $RM(p) = RM(G)$. Consider the set

$$A = \{ap : a \in G\}.$$

By Proposition 5.17, hence A consists of generic types over G. Recall that there are only finitely many generic types. Thus, A is finite. Let $A = \{a_1p, \dots, a_mp\}$ for some m. Let $b \in G$ be arbitrary. Then $bp = a_ip$ for some $1 \leq i \leq m$. This implies that $a_i^{-1}.bp = p$. Thus $a_i^{-1}.b \in stab_p$. So we have $b \in a_i.stab_p$. Hence, $[G : stab_p] \leq m$, namely $stab_p$ is a definable subgroup of G of finite index. Then $G^o \leq stab_p$. By 5.20, $stab_p \leq G^o$. Hence, $stab_p = G^o$.

□

Proposition 5.23. *G is connected if and only if it has a unique generic type.*

Proof Suppose that G has a unique generic type, say p. To get a contradiction, assume $G \neq G^o$. Then there exists $g \in G$ such that $G^o \cap gG^o = \emptyset$. Let ψ_1 and ψ_2 be two formulas that define G^o and gG^o respectively. Let $p_1 \in [\psi_1]$ and $p_2 \in [\psi_2]$ be generic types. Since $G^o \cap gG^o = \emptyset$, $[\psi_1] \neq [\psi_2]$. Therefore $p_1 \neq p_2$.

Now assume that G is connected. Let p and q be two generic types over G and let a and b be two independent realizations of p and q over G. Then

$$RM(tp(b/G_a)) = RM(tp(b/G)) = RM(q)$$

by independence of a and b and $RM(q) = RM(G)$ because q is generic over G. So $RM(tp(b/G_a) = RM(G)$. By Remark 5.21, we have $RM(G) = RM(G_a)$, hence b is generic over G_a.

On the other hand G_a is connected by Remark 5.21. So we can use Proposition 5.22 to say that $stab_{tp(b/G_a)}$ is G_a. Hence,

$$a.tp(b/G_a) = tp(a.b/G_a) = tp(b/G_a).$$

In particular, $tp(ab/G) = tp(b/G) = q$.

Similarly a^{-1} is generic over G_b by 5.4 and $stab_{tp(a^{-1})/G_b}$. Then

$$b^{-1}.tp(a^{-1}/G_b) = tp(b^{-1}.a^{-1}/G_b) = tp(a^{-1}/G_b).$$

In particular, $tp(b^{-1}.a^{-1}/G) = tp(a^{-1}/G)$. Hence, $tp(ab/G) = tp(a/G) = p$. So we reached the result $p = tp(a.b/G) = q$.

As a summary, we would like to present the following relations that we observed ;

$$
\begin{array}{rclcrcl}
tp(b/G_a) & = & tp(a.b/G_a) & & & & \\
 & \Downarrow & & & & & \\
tp(b/G) & = & tp(a.b/G) & = & tp(a.b/G) & = & tp(a/G) \\
 & & & & & \Uparrow & \\
 & & & & tp(a/G_b) & = & tp(a.b/G_b)
\end{array}
$$

Hence $p = tp(a.b/G) = q$

$\square$

6 FIELDS IN FIELDS

6.1 Fields of Finite Morley Rank

Theorem 6.1. *[**Macintyre**] An infinite field K of finite Morley rank is algebraically closed.*

Proof Let K be an infinite field of finite Morley rank. We will show that its additive and multiplicative subgroups are connected. Let a be a non-zero element from K, then multiplication by a gives an automorphism of K^+. Since $(K^+)^o$ is a subgroup of K^+, $(K^+)^o$ is an ideal of K. Then either we have $(K^+)^o = \{0\}$ or $(K^+)^o = K$. Recall that $(K^+)^o$ has finite index in K^+. This yields to $(K^+)^o = K$ since K is infinite. In other words, K^+ is connected. By Proposition 5.23, K^+ has a unique generic type. Note that $S_1^{\mathcal{F}}(K^*) \subseteq S_1^{\mathcal{F}}(K^+)$ by considering $K^* = K^+ \setminus \{0\}$. Then for any generic type p over K^*, p is also a generic type over K^+ since $RM(p) = RM(K^+) = RM(K^*)$. Then p is unique, hence K^* is connected by Proposition 5.23.

Now consider, for a fixed n, the group homomorphism $\varphi : K^* \to K^*$ with $\phi(x) = x^n$. Notice that $ker\varphi$ is defined by the formula $x^n - 1 = 0$, hence it is finite. Then by Lascar's equality we have $RM(K^*) = RM(ker\varphi) + RM(\varphi(K^*))$, i.e. $RM(K^*) = RM(\varphi(K^*))$. By Lemma 5.4, $\varphi(K^*)$ has finite index in K^*. We know that K^* is connected, in other words K^* has no proper definable subgroup of finite index, hence $\varphi(K^*) = K^*$. Therefore every element of $K = K^* \cup \{0\}$ has n^{th} root in K. In particular, K is perfect.

If $charK = p > 0$, then $\psi : K^+ \to K^+$ that is defined as $\psi(x) = x^p - x$ is a group homomorphism. By Lascar's equality, we get $RM(K^+) = RM(ker\psi) + RM(\psi(K^+))$. Note that $ker\psi$ is defined by the formula $x^p - x = 0$. Thus $ker\psi$ is finite, i.e. $RM(ker\psi) = 0$. So we have $RM(K^+) = RM(\psi(K^+))$. In a similar fashion, $(K^+) = \psi(K^+)$ as a consequence of connectedness of K^+.

We argue with two claims to conclude that K has no proper Galois extension.

Claim 1 : Let K is a field of finite Morley rank containing all m^{th} roots of unity where $m \leq n$ for some n. Then K has no proper Galois extension of degree n.

Suppose that n is the smallest number such that there is a field of Morley rank K that contains all m^{th} roots of unity where $m \leq n$ and K has a proper Galois extension L of degree n with corresponding Galois group $G = Gal(L/K)$. Let q be a prime number that divides $n = |G|$. Then G has an element of order q, say σ. By the Fundamental Theorem of Galois Theory, there is an intermediate field F such that the extension L/F is Galois and $Gal(L/F)$ is the subgroup of G generated by σ. Thus the degree of the extension L/F is q. By considering F as a vector space over K, one can see that $dim(F/K) = n/q$, i.e. $rk(F) = (n/q).rk(K)$. Hence F is a field of finite Morley rank that contains all m^{th} roots of unity where $m \leq n$. Then by the minimality of $n = q$ and $F = K$. So $|G| = q$, i.e. G is cyclic.

If $charK$ is different than q, by Theorem APPENDIX A.10 there is an $\alpha \in K$ such that $L = K(\alpha)$ and $p(\alpha) = 0$ where $p(x) = x^q - a$ for some $a \in K$. Note that $p(x)$ is the minimal polynomial of α over K since $deg(p(x)) = deg(L/K)$. Since K contains the q^{th} root of a, $p(x)$ is reducible.

If $charK = q$, by Artin-Schreier's theorem there is an $\alpha \in K$ such that $L = K(\alpha)$ and $p(\alpha) = 0$ where $p(x) = x^q - x - a$ for some $a \in K$. Then $p(x)$ is the minimal polynomial of α over K. Since ψ is surjective, there is a β such that $\beta^q - \beta = a$, whence $p(x)$ is reducible.

As a result, there is no proper Galois extension of K of degree n, since we observed contradictions in both cases.

Claim 2 : A field K of finite Morley rank contains all roots of unity.

Let n be the smallest number such that K does not contain all n^{th} roots of unity. Let

ζ be the primitive n^{th} root of unity. Consider the extension $K(\zeta)$ over K. Since it is a splitting field of $x^n - 1$, $(K(\zeta)/K)$ is Galois of degree at most $n-1$, say k. Since $k < n$, by the minimality of n, K contains all m^{th} roots of unity where $m \leq k$. Then by Claim 1, K has no proper Galois extension of degree k. But we have stated that $(K(\zeta)/K)$ is Galois of degree k. Since this gives a contradiction, such a number n does not exist. That means K contains all roots of unity.

It follows that, by Claim 1, K has no proper Galois extension. Therefore the separable closure of K is itself by Theorem APPENDIX A.6.($K^{sep} = K$) We proved that K is perfect, by Theorem APPENDIX A.5 the separable closure of K is algebraic closure of K ($K^{alg} = K^{sep}$), whence K is algebraically closed.

□

Lemma 6.2. *A field of finite Morley rank has no infinite definable proper subring.*

Proof Let K be a field of finite Morley rank and k be an infinite definable proper subring of K. Then k is an integral domain. Since K is a field of finite Morley rank and k is definable in K, k is a ring of finite Morley rank. By Theorem 5.5, there is a minimal definable non-zero ideal of k. (Consider the non-zero definable ideals of k as additive subgroups.) Fix such a minimal ideal I. For any non-zero $a \in I$, $I = (a)$ by the minimality of I. Then by a similar argument, for arbitrary non-zero $x \in k$, one can have $(a) = (ax)$. So there is a y such that $a = axy$. Since k is an integral domain and a is non-zero, we can conclude that $xy = 1$ and following this, k is a field.

We know that k is of finite Morley rank, hence it is algebraically closed. Consider the field extension of k by K. It needs to have an infinite degree since an algebraically closed field has no non-trivial algebraic extension. For each n, K contains n-many copy of k as a vector space over k. Then $RM(k^n) = n.RM(k)$ holds for all n, this forces $RM(K)$ to be infinite. This yields to a contradiction. □

Corollary 6.3. *Let K be a field of characteristic zero and of finite Morley rank. Then*

1. *K has no non-trivial definable additive subgroup.*
2. *Every definable additive map from K^n to K^m is K-linear for any n and m.*
3. *Any (additive) subgroup of $K^{n \times n}$ is a K-vector subspace for any n.*

Proof

1. Let A be a definable additive subgroup of K. Let R be the following set

$$\{a \in K : aA \subseteq A\}.$$

   Since R contains 0,1 and is closed under addition and multiplication, R is a subring of K. Moreover R is a definable subring of K since multiplication is definable in K . Also R is infinite. Otherwise, the cardinality of R gives a positive characteristic to K. Then we have $R = K$ by above lemma. That means $kA \subseteq A$ for all $k \in K$, hence A is an ideal of K. Since K is a field, A is either 0 or K.

2. Let ϕ be an additive homomorphism from K^n to K^m. Consider $R' \subseteq K$ defined as

$$R' = \{a \in K : \phi(ax) = a\phi(x)\}.$$

   By a similar argument, R' is a subring of K. Then $R' = K$. So $\phi(ax) = a\phi(x)$ for all $a \in K$, hence ϕ is K-linear.

3. Let A be a subgroup of $K^{n\times n}$. Let R'' be defined as

$$R' = \{a \in K : aA \subseteq A\}.$$

   One sees that R'' is a subring of K as described above. Hence, $R'' = K$. Since A is closed under addition, $R'' = K$ implies that A is a K-vector subspace.

□

Remark 6.4. *We know that K^* acts K-linearly on K^+. Consider the action defined as*

$$\begin{array}{rccl} \varphi : & K^* & \longrightarrow & End(K^+) \\ & a & \mapsto & \varphi_a : K^+ \to K^+ \\ & & & \quad x \mapsto a.x \end{array}$$

Suppose that there is a field F so that $K^+ = F^+$. Then $charF = 0$. One can see that K^* acts on F^+ by replacing K^+ with F^+. Since every definable additive endomorphism of F is F-linear by , K^* acts F-linearly on F^+. Consequently, K^* acts F-linearly on K^+.

Let $\odot$ be the multiplication on K to distinguish from the operation F. Then the multiplication on K ;

$$\begin{array}{rccc} \phi_{\odot}: & K \times K & \longrightarrow & K \\ & (x, y) & \mapsto & x \odot y \end{array}$$

Let $a \in F$. Then $\phi_{\odot}(ax, y) = (ax) \odot y = \varphi_y(ax)$ for non-zero x and y. (Otherwise the following are trivial) Since the action of K^* is F-linear on K^+, then we have $\varphi_y(ax) = a.\varphi_y(x) = a.(x \odot y) = a(\phi_{\odot}(x, y))$. Similarly, $\phi_{\odot}(x, ay) = a(\phi_{\odot}(x, y))$. Hence, the multiplication on K is F-bilinear.

We want to see that K^* acts F-linearly on K^+ where char$K = p > 0$ and $K = F$. Since we do not have the Corollary 6.3 a priori in characteristic p, we need to produce a new method.

Let K be a pure field of positive characteristic p. Bruno Poizat introduces the fact that the definable endomorphisms of K^+ are in the form of $a_{-n}x^{p^{-n}} + \ldots + a_0 x + \ldots + a_n x^{p^n}$ and consequently the definable automorphisms of K^+ are in the form of $a_n x^{p^n}$ where $n \in \mathbb{Z} \setminus \{0\}$ in (Poizat, 2001, page 46). One can see this fact follows from a result of Lou van den Dries (van den Dries, 1990, page 136, theorem 3). To see this implication, it is enough to clarify the statement of the aforementioned theorem. In our setting, the constructible sets are definable and the definable endomorphisms are morphisms of perfect groups which are polynomial maps, possibly composed with a power of the inverse of the Frobenius.

Let G be a group of definable automorphisms of K^+. To describe the elements of G more precisely, we will see the set of degrees of elements of G is bounded. To obtain a contradiction, assume that there is no such a bound. In other words, there is a model $\mathcal{F}$ of $\sum(v)$ where

$$\sum(v) = \{\forall b \ \forall a_n \ \exists v \ f_b(v) \neq a_n v^{p^n} : n \in \mathbb{N}\} \cup Diag_{el}(K).$$

Then $\mathcal{F}$ is an elementary extension of $\mathcal{K}$ by Proposition 3.9. So we are allowed to extend $f_b(x)$ to a definable automorphism of F^+. Since $\mathcal{F} \models Diag_{el}(K)$, F is a field of characteristic p. Hence, the definable automorphisms of F^+ are in the form of $a_n x^{p^n}$. It follows that $\sum$ is not satisfiable. By Compactness Theorem 3.5, there are $f_{b_1}, \ldots, f_{b_m}$ such that, for all $x \in K^+$, $f_{b_i}(x) = a_{n_i} x^{p^{n_i}}$ for each i. Hence there is a bound of degrees of elements of G.

Notice that G consists of the maps which are in the form of ax. Otherwise G would *contain a map ax^n, for some $n > 1$, then there would be no bound of the degrees of the elements of the subgroup of G which is generated by ax^n. Hence G is definably isomorphic to a subgroup of K^*.*

Definition 6.5. *An infinite group G is **minimal** if all its proper definable subgroups are finite.*

Example 6.6. *Let K be a field of characteristic zero and of finite Morley rank. Then K^+ is minimal since its only proper subgroup is 0 by Corollary 6.3.*

Proposition 6.7. *A connected group of Morley rank 1 is minimal. (Borovik and Nesin, 1994, Exercise)*

Proof Let G be a connected group of Morley rank 1. Suppose that G has a proper infinite definable subgroup, say H. Since G is connected, H cannot have finite index in G. Then H has an infinite index in G and by Lemma 5.4, $RM(H) < RM(G)$. Then $rk(G) = 1$ implies that $rk(H) = 0$. This contradicts with the fact that H is infinite. Hence all proper definable subgroups of G are finite.

□

6.2 Algebraic Groups

Definition 6.8. *Let K be a sufficiently large algebraically closed field. By an **affine variety**, we mean a Zariski closed subset of K^n for some n. A **morphism** between affine varities $V \subseteq K^n$ and $W \subseteq K^m$ can be described as a map $f : V \to W$ where $f = (f_1, \ldots, f_m)$ such that f_i belongs to $K[V] = K[X_1, \ldots, X_n]/I(V)$ for each i.*

The notion of affine variety can be generalized as an abstract variety. Then one can define a morphism between two abstract varities.

*A **variety** V is a set that is covered by finitely many subsets $V = V_1 \cup V_2 \cup \ldots \cup V_n$ such that, for each $1 \leq i \leq n$, there is an affine variety $U_i \subseteq K^{n_i}$ with a corresponding bijection $f_i : V_i \to U_i$ that satisfies*

1. *$U_{i,j} = f_i(V_i \cap V_j)$ is open in U_i (with respect to Zariski topology of U_i).*
2. *$f_{i,j} = f_i \circ f_j^{-1}$ which is a bijective rational map between U_{ji} and U_{ij}*

One can define the Zariski topology of a variety (V, V_i, f_i) as follows; A subset A of V is open if $f_i(A \cap V_i)$ is open in U_i for all i.

*A **morphism** h between two varities (V, V_i, f_i) and (W, W_j, g_j) is a map where h is continuous and $g_j \circ h$ is a morphism on $h^{-1}(W_j) \cap V_i$ into $g_j(W_j)$ for any pair i, j. By a continuous map h, we mean the pre-image of an (Zariski) open set under h is an (Zariski) open set.*

Definition 6.9. *A group G is called an **algebraic group** over an algebraically closed field K where G is a variety and the group operation and inversion (with respect to the group operation) as maps are morphisms.*

Example 6.10. *Let K be an algebraically closed field. Then the general linear group $GL_n(K)$ of invertible $n \times n$ matrices over K is an algebraic group. To see that $GL_n(K)$ is a variety, by using the fact that invertible matrices have non-zero determinant, identify $GL_n(K)$ with the following set*

$$\{(\alpha, X) \in K \times K^{n \times n} = K^{n^2+1} : \alpha.det(X) = 1\}$$

Since determinant is a polynomial map, this set is Zariski closed. Recall that the matrix multiplication and the corresponding inversion can be given by polynomials, hence they are morphisms. It follows that $GL_n(K)$ is an algebraic group over K.

Theorem 6.11. *(Poizat, 2001) [**Weil - Hrushovski**] Every group which is definable in an algebraically closed field is definably isomorphic to an algebraic group.*

Definition 6.12. *A **linear group** is a definable subgroup of $GL_n(K)$ for some n.*

Theorem 6.13. *(Poizat, 2001) [**Rosenlicht**] Let K be an algebraically closed field. If G is a connected, definable group in K with center $Z(G)$, then $G/Z(G)$ is definably isomorphic to a linear group.*

Proof Let $G \subseteq K^m$ be a connected group, which is definable in K, with identity element e for some m. Then by Theorem 6.11, G is algebraic. Note that G is a group of finite Morley rank since it is definable in K and $RM(K) = 1$.

Let U be an open neighborhood of e. We will deal with rational functions defined on U. Let A be a ring of germs of rational functions which is defined as the set of

$$f : U_f \to K$$

where U_f is a Zariski open subset of U containing e and the couple (f, U_f) satisfies the condition (E) where a couple (g, Ω) satisfies (E) if and only if there are polynomials $p(\bar{x})$ and $q(\bar{x})$ such that

1. $\forall \bar{x} \in \Omega,\ q(f_i^{-1}(\bar{x})) \neq 0$ and
2. $\forall \bar{x} \in \Omega,\ g(\bar{x}) = p(f_i^{-1}(\bar{x}))/q(f_i^{-1}(\bar{x}))$ and
3. There is no bigger Zariski open set $\Omega' \supsetneq \Omega$ for which the condition of (1) are satisfied.

We will see the elements of A as $\frac{p_i(\bar{x})}{q_i(\bar{x})}$'s. One should notice that each coordinate map on U belongs to A. Note that the ideal generated by $\frac{p_i(\bar{x})}{q_i(\bar{x})}$'s is exactly the ideal generated by $p_i(\bar{x})$'s since $q_i(\bar{x})$'s are invertible. Thus $K[\bar{x}]$ is Noetherian implies that A is Noetherian. Moreover, any $f(\bar{x}) \in A$ which is non-zero at e, there is an open neighborhood of e given by the relation $f(\bar{x}) \neq 0$. Thus it is possible to define $1/f(\bar{x})$ in this neighborhood of e, hence $1/f(\bar{x}) \in A$. Now consider $f(\bar{x}) \in A$ such that $f(e) = 0$. Let

$$M = \{f(\bar{x}) \in A : f(e) = 0\}$$

Clearly M is an ideal of A and also it consists of all non-unit elements of A. Hence, A is a local ring with maximal ideal M by Theorem APPENDIX A.3. As in every Noetherian local ring, we have $\cap_n M^n = \{\bar{0}\}$.

Now we are interested in the elements of A/M^n. For arbitrary $\frac{p(\bar{x})}{q(\bar{x})} \in A$, we know that $q(e) \neq 0$ implies $q(\bar{x})$ is invertible in an open neighborhood of e. Let $q(e) = c$. Then there is a non-invertible $q'(\bar{x})$ such that $q(\bar{x}) = c - q'(\bar{x})$. By replacing $q(\bar{x})$ with $\frac{q(\bar{x})}{c}$ and $q'(\bar{x})$ with $\frac{q'(\bar{x})}{c}$, one has $q(\bar{x}) = 1 - q'(\bar{x})$. Following this, by replacing $p(\bar{x})$ with $\frac{p(\bar{x})}{c}$, we get

$$p(\bar{x}).\frac{1-(q'(\bar{x}))^n}{1-q'(\bar{x})} = p(\bar{x}).(1 + q'(\bar{x}) + \ldots + (q'(\bar{x}))^{n-1})$$

since $1 - (q'(\bar{x}))^n = (1 - q'(\bar{x}))(1 + q'(\bar{x}) + \ldots + (q'(\bar{x}))^{n-1})$. It follows that

$$\frac{p(\bar{x})}{1-q'(\bar{x})} = \frac{p(\bar{x})}{1-q'(\bar{x})}.(q'(\bar{x}))^n + p(\bar{x}).(1 + q'(\bar{x}) + \ldots + (q'(\bar{x}))^{n-1})$$

Since $q'(\bar{x}) \in M$, $\frac{p(\bar{x})}{q(\bar{x})} = \frac{p(\bar{x})}{1-q'(\bar{x})}$ is congruent to the polynomial

$$p(\bar{x}).(1 + q'(\bar{x}) + \ldots + (q'(\bar{x}))^{n-1})$$

modulo M^n locally.

We want to conclude that any element of A/M^n can be represented by a polynomial of degree less than n. To see that, one can identify e with $\bar{0} \in K^m$ or equivalently change the variables of the polynomials with $(x_1 - e_1, \ldots, x_m - e_m)$. Then A/M^n is a finite dimensional vector space over K. We will denote the elements of A/M^n by $[f(\bar{x})]$.

The group of inner automorphisms of G, namely $Inn(G)$, consists of

$$\begin{array}{rccc} Inn_g : & G & \longrightarrow & G \\ & \bar{x} & \mapsto & g.\bar{x}.g^{-1} \end{array}$$

for each $g \in G$. As a notational remark, one should be careful about the fact that both of g and $\bar{x}$ are tuples. Since we use $\bar{x}$ to denote an element of G and the tuple of variables at the same time, we prefer this notation.

Let $f(\bar{x}) \in A$. Then $f^g(\bar{x}) := f(Inn_g(\bar{x})) \in A$, since $Inn_g(e) = e$. Consequently, $f(Inn_g(\bar{x})) \in M^n$ for each $f(\bar{x}) \in M^n$. Those observations lead us to define an action of G on A/M^n.

$$\begin{array}{rccc} \varphi_n : & G \times A/M^n & \longrightarrow & A/M^n \\ & (g, [f(\bar{x})]) & \mapsto & [f^g(\bar{x})] \end{array}$$

Now we will see that φ_n is definable. Recall that G is an algebraic group. Hence G is a variety and the group operation on G is compatible with chart maps. We will be interested in generic elements of G. The motivation of this interest depends on the fact that, which is given in the Proposition 5.18, any element of G is a product of two generics. Hence if we manage to see that φ_n is definable on a generic element of G, we can generalize it to the all elements of G.

There is generic $V \subseteq U$. Since G is connected, by Proposition 5.23, V contains all generic elements of G. Let $x \in V$. Since the multiplication and corresponding inversion on G are morphisms, for each $g \in G$, there is a rational function that corresponds to $Inn_g(\bar{x})$. Following this, we have $[f^g(\bar{x})]$ is a composition of two rational functions on

U, since $f(\bar{x}) \in A$. Therefore one can get rid of the denominator of $[f^g(\bar{x})]$ in A/M^n by following the steps described above. Thus we have a polynomial representation of $\varphi_n(g, [f(\bar{x})])$ in A/M^n for each $g \in G$. Hence φ_n is definable.

Moreover, φ_n is K-linear. Let $[f_1(\bar{x})], [f_2(\bar{x})] \in A/M^n$.

$$\begin{aligned}
\varphi_n(g, [(f_1+f_2)(\bar{x})]) &= [(f_1+f_2)^g(\bar{x})] \\
&= [(f_1+f_2)(Inn_g(\bar{x}))]) \\
&= [(f_1)(Inn_g(\bar{x})] + [(f_2)(Inn_g(\bar{x}))] \\
&= [(f_1)^g(\bar{x})] + [(f_2)^g(\bar{x})] \\
&= \varphi_n(g, [f_1(\bar{x})]) + \varphi_n(g, [f_2(\bar{x})])
\end{aligned}$$

For any $\alpha \in K$, $\varphi_n(g, \alpha[f(\bar{x})]) = [(\alpha f)^g(\bar{x})] = \alpha[f^g(\bar{x})] = \alpha\varphi_n(g, [f(\bar{x})])$ as well. That means we have a definable homomorphism ϕ_n corresponding to the action φ_n.

$$\begin{aligned}
\phi_n : \quad G &\longrightarrow \quad GL(A/M^n) \\
g &\mapsto \quad \phi_{n,g} : A/M^n \to A/M^n \\
& \qquad\qquad [f(\bar{x})] \mapsto [f^g(\bar{x})]
\end{aligned}$$

If the definable action ϕ_n is faithful on $G/Z(G)$, that is, ϕ_n is injective, then $G/Z(G)$ is isomorphic to a subgroup $GL(A/M^n)$ and this finishes the proof. Now we will see why ϕ_n is faithful on $G/Z(G)$.

Let $G_n = ker\phi_n$. Then G_n is definable since ϕ_n is definable. Recall that

$$G_n = \{g \in G : \phi_n(g, [f(\bar{x})]) = [f(\bar{x})] \text{ for all } [f(\bar{x})] \in A/M^n\}.$$

So, for any $g \in G_{n+1}$, one has $f^g(\bar{x}) - f(\bar{x}) \in M^{n+1}$. On the other hand, for each n, $M^{n+1} \subseteq M^n$. Then, $f^g(\bar{x}) - f(\bar{x}) \in M^n$, hence G_n's constitute a decreasing sequence of definable subgroups of G.

We know that a group of finite Morley rank has DCC on its definable subgroups, thus the aforementioned sequence stabilizes at a finite step, say l. Let $g \in \cap_{i=1}^{l} G_i$, then

$g \in G_n$ for all n. In other words, for all $f(\bar{x}) \in A$ and for all n, $f^g(\bar{x}) - f(\bar{x}) \in M^n$. Since $\cap_n M^n = \{\bar{0}\}$, we have $f^g(\bar{x}) = f(\bar{x})$, for all $f(\bar{x}) \in A$. In particular, for any coordinate map f_i on U, we have $f_i^g(\bar{x}) = f_i(\bar{x})$. Hence, for any generic $\bar{x}$, $Inn_g(\bar{x}) = x$. In other words, any $g \in \cap_{i=1}^{l} G_i$, g commutes with generic elements. Since every element of G is a product of two generics by Proposition 5.18, then g belongs to $Z(G)$. So we get that, for sufficiently large n (in our setting, $n \geq l$), we can identify G_n with $Z(G)$. Hence, by taking quotient of G with $Z(G)$, we make definable ϕ_n is injective on $G/Z(G)$. Thus $G/Z(G)$ is definably isomorphic to a subgroup $GL(A/M^n)$.

□

6.3 The Main Theorem

Theorem 6.14. *(Poizat, 2001) Every infinite field K which is definable in the pure algebraically closed field F is definably isomorphic to it.*

Proof Let K be an infinite field which is definable in the pure algebraically closed field F. Then K is a field of finite Morley rank, since $RM(F) = 1$. We have seen that K^+ and K^* are connected in Theorem 6.1 and we noted that then $Aff(K) = K^+ \ltimes K^*$ is connected in Example 5.10. There is no $ax + b$ that commutes with $cx + d$ for all $c \in K^*$, $d \in K^+$ other than the identity map, hence the center of Aff(K) is trivial.

The group structure Aff(K) can be written as a $(K^+ \times K^*; \cdot_\varphi, (0, 1))$ where the operation $\cdot_\varphi$ depending on the action of K^* on K^+ which is defined as follows;

$$(b_1, a_1) \cdot_\varphi (b_2, a_2) = (b_1 + a_1 b_2, a_1 a_2).$$

Since K is definable in F as a field, addition and multiplication on K are definable in F. Note that the operation $\cdot_\varphi$ consists of addition and multiplication on K. Hence Aff(K) is definable in F. By Theorem 6.13, Aff(K) is linear in F.

One sees that $Aff(K)$ is a definable extension of K^+ and K^* by considering $K^+ = K^+ \times \{1\}$ and $K^* = \{0\} \times K^*$ in F. Then the linearity of Aff(K)) implies that K^+ and K^* are linear in F. Thus K^+ is a definable subset of $GL_n(F)$ for some n. Since K^+ is abelian, K^+ has a triangularizing basis by Theorem APPENDIX B.6.

homomorphism $\varphi_i : K^+ \to F^*$ defined by

$$\begin{bmatrix} a_{11} & a_{12} & & \cdots & a_{1n} \\ & a_{22} & & \cdots & a_{2n} \\ & & \ddots & & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & a_{nn} \end{bmatrix} \longmapsto a_{ii}$$

where $a \in K^+$ is represented by above upper triangular matrix.

To figure out the image of K^+ under φ_i, we will examine the cases $charK = 0$ and $charK = p > 0$.

Suppose $charK = 0$. As we noted in Corollary 6.3, K^+ has no non-trivial subgroup. So $ker\varphi_i$ is either 0 or is K^+ for all i. If $ker\varphi_i = \{0\}$, φ_i is injective. Note that φ_i could not be surjective. Otherwise K^+ and F^* would be isomorphic. It is not possible since the former one is torsion-free but the latter one has torsion elements. Then $\varphi_i(K^+)$ is a proper, definable subgroup of F^*. Recall that F^* is minimal by Proposition 6.7. Hence, $\varphi_i(K^+)$ is finite. Since φ_i is injective and K^+ is infinite, $\varphi_i(K^+)$ needs to be infinite. Thus it is not possible to have $ker\varphi_i = \{0\}$. Hence $ker\varphi_i = K^+$, consequently $\varphi_i(K^+) = \{1\}$ for all i.

Now assume $charK = p > 0$. Then K^+ is a group of exponent p, because $\underbrace{x + \ldots + x}_{\text{p-times}} = 0$ for any $x \in K^+$. Then $\varphi_i(K^+)$ has a finite exponent by Proposition APPENDIX A.2, hence it is finite. On the other hand, the connectedness is preserved under group homomorphism, hence $\varphi_i(K^+)$ is a connected and finite group. By Remark 5.7, $\varphi_i(K^+) = \{1\}$.

In both cases, each element of K^+ has an upper triangular representation that consists of 1 in the diagonal. Thus all eigenvalues of the corresponding matrix to $a \in K^+$ are 1 for all $a \in K^+$, hence K^+ is unipotent.

We will deal with F by considering its characteristic in order to explore a relation between K^+ and F^+. Since $K^+ \subseteq GL_n(F)$, we work with a multiplicative group. Now we will introduce a way to identify K^+ with an additive group and then we will observe

$K^+ = F^+$. Then we will produce various arguments regarding characteristic of F to observe the desired isomorphism.

Suppose $char F = 0$. Recall that a square matrix X is nilpotent if $X^m = 0$ for some m, and unipotent if it is in the form of $1 + X$ where X is nilpotent. We will obtain a one-to-one correspondence between the multiplicative group of unipotent matrices and the additive group of nilpotent matrices over F of the same size via the logarithm.

$$\begin{array}{ccc} \mathsf{U} & \longrightarrow & \mathsf{N} \\ 1+X & \mapsto & log(1+X) \end{array} \qquad \begin{array}{ccc} \mathsf{N} & \longrightarrow & \mathsf{U} \\ X & \mapsto & exp(X) \end{array}$$

where U and N are the group of unipotent and nilpotent matrices respectively.

Consider the power series representations of exp and log,

$$log(1+X) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{X^n}{n} \qquad exp(X) = \sum_{n=1}^{\infty} \frac{X^n}{n!}$$

For any $X \in \mathsf{N}$, $X^m = 0$ for some m, then $log(1+X)$ and $exp(X)$ are finite sums, namely polynomials. Hence, they are definable in F. Following this, a formal substitution shows that exp and log are inverses to each other and log constitutes a bijection between U and N. Moreover if one have commutativity on N, then $exp(X).exp(Y) = exp(X+Y)$ with $exp(0) = 1$. Since $K^+ \subseteq GL_n(F)$ is a commutative unipotent group, K^+ is definably isomorphic to a subgroup of $(F^+)^{m\times m}$ via log. By Proposition 6.3, K^+ is definably isomorphic to an F-vector subspace. Since K^+ is minimal by Example 6.6, the dimension of K^+ over F is 1. Thus, $K^+ = F^+$.

By Remark 6.4 K^* acts F-linearly on K^+. Now we will produce a field isomorphism between K and F by using this observation.

Let $\odot$ be the multiplication on K with the identity element $\mathbb{1}$. Consider K as an F-vector space. Then by F-bilinearity of $\odot$ on K^+ which is stated in 6.4, we have

$$x\mathbb{1} \odot y\mathbb{1} = xy(\mathbb{1} \odot \mathbb{1}) = xy\mathbb{1}$$

where $x, y \in F$. Then

$$\begin{array}{cccc} \phi : & F & \longrightarrow & K \\ & x & \mapsto & x\mathbb{1} \end{array}$$

is an isomorphism.

We obtained a definable field isomorphim between K and F where the characteristic of F is 0.

Now suppose $charF = p$. Recall that the Proposition APPENDIX B.4, that is, any matrix A is unipotent if and only if the order of A is a power of p. In particular, the matrix representation of the identity element of K^+ has order a power of p. Then $charK = p$.

Recall that K^* is linear in F. Also the definability of K^* in F implies that K^* is definably isomorphic to an algebraic group by Theorem 6.11. Since $charK = p$, none of the elements of K^* has order p. Hence, K^* has no unipotent element by the Proposition APPENDIX B.4. Then K^* is a connected, abelian, algebraic group containing no unipotent elements. By Theorem APPENDIX C.2, K^* is torus, that is, isomorphic to $(F^*)^m$ for some m. By considering torsion elements of K^* and $(F^*)^m$, m needs to be 1. Thus $K^* = F^*$. Then $RM(K^*) = RM(F^*)$. Since F is an algebraically closed field, $RM(F^*) = 1$ by Example 3.4. Hence, K^+ is a connected group of Morley rank 1.

Recall that, in an algebraically closed field, the Morley Rank corresponds to the Krull dimension by Theorem 4.6. On the other hand, K^+ is definably isomorphic to an algebraic group by Theorem 6.11 as above. Thus we know that K^+ is definably isomorphic to a connected, one-dimensional, unipotent algebraic group. Hence, $K^+ = F^+$ by Theorem APPENDIX C.3. Then one can define ϕ between K and F like in characteristic 0 case, since we know that K^* acts F-linearly on K^+ by the characteristic p part of the Remark 6.4.

□

7 CONCLUSION

David Marker and Anand Pillay proved that in a reduct $\mathcal{M}$ of an algebraically closed field $\mathcal{F} = (F; +, \cdot)$, which is non-locally modular and expanding the additive structure, an infinite field $\mathcal{K} = (K; \oplus, \odot)$ is interpretable and then the multiplication on F is definable in the reduct $\mathcal{M}$ (Marker and Pillay, 1990). In their work, they use the result of Bruno Poizat from (Poizat, 2001) which is presented in this thesis. Our interest about the aforementioned article of Marker and Pillay is the main motivation of this work. Moreover, the proof of the Poizat's result reveals the interaction between model theory and algebraic geometry. Henceforth, it is quite stimulating for young mathematicians. This work is an attempt to explore and clarify the relevant parts of (Poizat, 2001) by being precise about implications and giving the omitted details in (Poizat, 2001). One could have difficulties while studying on (Poizat, 2001), so our work may be useful to comprehend the details.

While studying the proof, we noted that Poizat introduces two very distinct proofs depending on whether the characteristic is 0 or a prime p. It is a well-known model theoretic consequence of the compactness theorem that any first order statement in the language of rings which is true in all models of ACF_p for large p needs also to be true for all models of ACF_0. By considering this fact, we are interested in an open problem that asks whether having such a result for char p case is enough. If we could make the statements which we used to prove char p case first order, then there is no need to separate proof in two cases.

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A APPENDIX A

A.1 Group Theory

Definition APPENDIX A.1. *The **exponent of a group** G is defined as the least common multiple of the orders of elements of G.*

Proposition APPENDIX A.2. *Let G, H be two groups where G has finite exponent. Suppose there is a group homomorphism ϕ from G into H. Then $\phi(G)$ has finite exponent.*

A.2 Ring Theory

Theorem APPENDIX A.3. *A ring R is local if and only if non-unit elements of R form an ideal.*

A.3 Field and Galois Theory

Theorem APPENDIX A.4. *An algebraically closed field K has no non-trivial algebraic extension. Hence, K has no non-trivial finite extension.*

Theorem APPENDIX A.5. *K is a perfect field if and only if the separable closure of K is the algebraic closure of K.*

Theorem APPENDIX A.6. *A separable closure of a field K is a Galois extension of K.*

Theorem APPENDIX A.7. *[**Fundamental Theorem of Galois Theory - Relevant Part**] Suppose that L/K is a finite Galois extension with the corresponding Galois group $Gal(L/K)$. Let H be a subgroup of G and F be an intermediate field between L and K. We will denote the subfield of L which is fixed by H by L^H and the set of automorphisms of L fixing F by $Aut(L/F)$*

1. *There is an inclusion reversing one-to-one correspondence ϕ between the subfields of L containing K and the subgroups of G given by $\phi(F) = Aut(L/F)$ and $\phi^{-1}(H) = L^H$*
2. *The extension L/F is normal, hence Galois.*
3. *$[F : K] = [G : H]$ and $[L : F] = |H|$.*

A.4 Cyclic Extensions

Definition APPENDIX A.8. *Let K be a Galois extension of k. For an element α of K, its **norm** and **trace** is defined as follows;*

$$N_k^K(\alpha) = \prod_{\sigma} \sigma(\alpha), \quad Tr_k^K(\alpha) = \sum_{\sigma} \sigma(\alpha).$$

where $\sigma \in Gal(K/k)$.

Theorem APPENDIX A.9. *(Lang, 2002) [**Hilbert's Theorem 90**] Let K be a cyclic extension of k of degree n. Let $G = Gal(K/k)$ with $G =< \sigma >$. For $\beta \in K$, $N_k^K(\beta) = 1$ if and only if there is a non-zero element α in K such that $\beta.\sigma(\alpha) = \alpha$.*

Theorem APPENDIX A.10. *(Lang, 2002) Let K/k be a cyclic extension of degree n, where n is a positive integer that is coprime to the characteristic of k. Suppose that there is a primitive n^{th} root of unity in k. Then there exists $\alpha \in K$ such that $K = k(\alpha)$, and α is a root of $x^n - a$ for some $a \in k$.*

Proof Let ζ be a primitive n^{th} root of unity in k. Let $G = Gal(K/k)$ with $G =< \sigma >$. Since $\zeta \in k$, every element of G fixes ζ^{-1}. Then $N_k^K(\zeta^{-1}) = \zeta^{-1^n} = 1$. By Hilbert's Theorem 90, there exists $\alpha \in K$ such that $\sigma(\alpha) = \zeta.\alpha$. Since $\zeta \in k$, we have

$$\sigma^2(\alpha) = \sigma(\zeta.\alpha) = \zeta.\sigma(\alpha) = \zeta^2.\alpha.$$

This equation implies $\sigma^i(\alpha) = \zeta^i.\alpha$ for all $1 \leq i \leq n$ inductively. Thus $\zeta^i.\alpha$ are n distinct conjugates of α over k. Following this, $[k(\alpha) : k]$ is at least n. Then $[K : k] = n$ implies that $K = k(\alpha)$. Moreover,

$$\sigma(\alpha^n) = \sigma(\alpha)^n = (\zeta.\alpha)^n = \zeta^n.\alpha^n = \alpha^n$$

.

i.e. α^n is fixed under G, since $G =< \sigma >$. Accordingly, α^n is an element of k. Since α is a root of $x^n - \alpha^n$, we get the desired result.

□

Theorem APPENDIX A.11. *(Lang, 2002) [**Hilbert's Theorem 90, Additive Form**] Let K be a cyclic extension of k of degree n. Let $G = Gal(K/k)$ with $G =< \sigma >$. For $\beta \in K$, $Tr_k^K(\beta) = 0$ if and only if there is an element α in K such that $\beta = \alpha - \sigma(\alpha)$.*

Theorem APPENDIX A.12. *(Lang, 2002) [**Artin - Schreier**] Let k be a field of characteristic $p > 0$. Let K/k be an extension of degree p. Then there exists $\alpha \in K$ such that $K = k(\alpha)$ and α is a root of $x^p - x - a$ for some $a \in k$.*

Proof Let K/k be an extension of degree p, hence K/k is cyclic. Let $G = Gal(K/k)$ with generator σ. Observe that $Tr_k^K(-1) = p.(-1) = 0$, since -1 is fixed under G. By the additive form of Hilbert's Theorem 90, there exists α in K such that $1 = \sigma(\alpha) - \alpha$, or equivalently $\sigma(\alpha) = 1 + \alpha$. Then

$$\sigma^2(\alpha) = \sigma(1 + \alpha) = 1 + \sigma(\alpha) = 2 + \alpha.$$

By proceeding inductively, one can get that $\sigma^i(\alpha) = i + \alpha$ for all $1 \leq i \leq n$ and $\sigma^i(\alpha)$ are p distinct conjugates of α, whence $[k(\alpha) : k] \geq p$. It follows that $K = k(\alpha)$, since$[K : k] = p$. Furthermore,

$$\sigma(\alpha^p - \alpha) = \sigma(\alpha)^p - \sigma(\alpha) = (1 + \alpha)^p - (1 + \alpha) = \alpha^p - \alpha.$$

Since $G =< \sigma >$, $\alpha^p - \alpha$ is fixed under G. Thence $\alpha^p - \alpha \in k$. Since $\alpha^p - \alpha$ is a root of $x^p - x - \alpha^p - \alpha$, the proof has been completed.

□

B APPENDIX B

Theorem APPENDIX B.1. *[**Cayley - Hamilton**] A square matrix A satisfies its characteristic polynomial which is $det(A - Id.x)$.*

Definition APPENDIX B.2. *Let K be an algebraically closed field. A square matrix X over K is called*

- ***nilpotent** if $X^n = 0$ for some n*
- ***unipotent** if $(X - 1)$ is nilpotent where 1 is identity matrix Id.*

Remark APPENDIX B.3. *Let A be a unipotent matrix. Then $A = X + Id$ for some X such that $X^m = 0$ for some m. To obtain an eigenvalue of X, one needs to consider the equation $Xv = \lambda v$ for all v. Then $X^m = 0$ implies that $\lambda = 0$. It implies that, $Av = 1v = v$ for all v, whence all eigenvalues of A is 1. Converse can be obtained by saying that a matrix is nilpotent if all of its eigenvalues are zero. But this is a consequence of Cayley - Hamilton.*

Proposition APPENDIX B.4. *Let K be an algebraically closed field of characteristic $p > 0$. Let G be a definable subgroup of $GL_n(K)$ for some n. Then $A \in G$ is unipotent if and only if the order of A in G is a power of p.*

Proof Let $A \in G$ be a unipotent matrix where $A = 1 + X$ where $X^m = 0$ for some m. Let d be given such that $p^d > m$. Then

$$0 = X^{p^d} = (A - 1)^{p^d} = A^{p^d} - 1.$$

So $A^{p^d} = 1$. Hence the order of A in G divides p^d. Since p is a prime, the order of A is a power of p.

Let $A \in G$ be any matrix of order p^r for some r. Then $A^{p^r} = 1$ implies $(A-1)^{p^r} = 0$. So A is unipotent.

$\square$

Definition APPENDIX B.5. *A* ***flag*** *is a sequence of subspaces of a finite dimensional vector space V so that the corresponding sequence that consists of the dimensions of those subspaces is increasing. Let V be an n-dimensional vector space, then the flag on V;*

$$0 = V_0 < V_1 < \ldots < V_{n-1} < V_n = V$$

such that $0 = d_0 < d_1 < \ldots < d_n = n$ where $d_i = dim(V_i)$.

A flag is called ***complete*** *if $d_i = i$ for any i.*

Theorem APPENDIX B.6. *Let V be an n dimensional vector space over an algebraically closed field K for some n. Let M be a commutative subset of $End(V)$. Then there exists a basis of V such that, for any $f \in M$, the matrix representation of f with respect to this basis is upper triangular.*

Proof It is enough to show that there is a complete flag

$$0 = V_0 < V_1 < \ldots < V_{n-1} < V_n = V$$

preserved by M. Consider a basis $\{v_1, \ldots, v_n\}$ of V where $\{v_1, \ldots, v_i\}$ is the basis of V_i. For any $f \in M$, $f(v_i) \in V_i$, since the flag is preserved under M. Then $f(v_i) = \lambda_{1i}.v_1 + \lambda_{2i}.v_2 + \ldots + \lambda_{ii}.v_i$ for some $\lambda_{ji} \in K$. Then the corresponding matrix to f;

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} & & \dots & \lambda_{1n} \\ & \lambda_{22} & & \dots & \lambda_{2n} \\ & & \ddots & & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & \lambda_{nn} \end{bmatrix}$$

First of all, let us have an observation about M. If M consists of maps that give scalar multiplication, then for any basis $\{v_1, \dots, v_n\}$ of V, we have the complete flag $0 = V_0 < V_1 < \dots < V_{n-1} < V_n = V$ where $\{v_1, \dots, v_i\}$ is the basis of V_i. And one sees that $f(V_i) \subseteq V_i$. So we may assume that M has non-scalar elements.

Now we will show that such a flag exists by induction on n. If $n = 1$, consider $\{0\} = V_0 < V_1 = V$ and for any $f \in M, f(V) \subseteq V$ trivially holds. Suppose that the statement holds for all $m < n$. Pick a non-scalar $f \in M$. Then there is an $a \in K$ such that the kernel of the map $\varphi(x) := f(x) - a.x$ is neither $\{0\}$ nor V. (The existence of a such that $ker\varphi \neq \{0\}$ is a consequence of Cayley - Hamilton Theorem since $f \in End(V)$ is not nilpotent and f is non-scalar implies that $ker\varphi \neq V$.) Then $ker\varphi$ has a positive dimension which is smaller than V as a K-vector space. Let $M' = \{g \restriction_{ker\varphi} : g \in M\}$. Note that for any $g \in M$ and $v \in ker\varphi$, $f(v) = a.v$ implies that

$$f(g(v)) = g(f(v)) = g.(a.v) = a.g(v).$$

That means $g(v) \in ker\varphi$, hence $g(ker\varphi) \subseteq ker\varphi$ for any $g \in M$. Then M' is a commutative subset of $End(ker\varphi)$.By induction hypothesis, there is a complete flag on $ker\varphi$ preserved by M', hence by M.

Let $\{0\} = W_0 < W_1 < \ldots < W_m = ker(\varphi(x))$ be the complete flag preserved by M where $\{w_1, \ldots, w_i\}$ be a basis of W_i. Then for all $f \in M$, $f(w_1) \in (w_1)$, where (w_1) is a subspace generated by w_1. Therefore w_1 is an eigenvector of all elements of M.

Consider the quotient space $\overline{V} = V/(w_1)$. Let

$$\overline{M} = \{\bar{f} : \overline{V} \to \overline{V} : \bar{f}(\bar{x}) = \overline{f(x)} \text{ where } f \in M\}.$$

To observe that any $\bar{f} \in \overline{M}$ is well-defined, consider the following identities $\bar{f}(\bar{v_1}) - \bar{f}(\bar{v_2}) = \overline{f(v_1)} - \overline{f(v_2)} = \overline{f(v_1) - f(v_2)} = \overline{f(v_1 - v_2)}$ where $\bar{v_1} = \bar{v_2}$. Since $\bar{v_1} = \bar{v_2}$, $v_1 - v_2 = \alpha.w_1$ for some $\alpha \in K$. So $\overline{f(v_1 - v_2)} = \overline{f(\alpha.w_1)} = \alpha.\overline{f(w_1)} = \alpha.\overline{(\lambda.w_1)}$ for some $\lambda \in K$ since w_1 is an eigenvector of f. Hence, $\bar{f}(\bar{v_1}) - \bar{f}(\bar{v_2}) = \bar{0}$. Then $\overline{M} \subseteq End(\overline{V})$. Since $dim(\tilde{V}) = n-1$, by induction hypothesis, there is a complete flag on $\overline{V}$, say $0 = \overline{V}_0 < \overline{V}_1 < \ldots < \overline{V}_{n-1} = \overline{V}$ preserved by $\tilde{M}$. Let $\{\bar{v_1}, \ldots, \bar{v_i}\}$ be a basis of $\overline{V}_i$. Pick representatives v_i from $\bar{v_i}$, then $dim(V_i) = i$ by letting $V_i = (w_1, v_2, \ldots, v_i)$. Therefore

$$0 = V_0 < V_1 = (w_1) < V_1 = (w_1, v_2) < \ldots < V_n = (w_1, v_2, \ldots, v_{n-1}) = V$$

is a complete flag on V.

Now we will see that this flag is preserved under M. Let $f \in M$. Then $f(w_1) \subseteq (w_1)$, since $(w_1) \in ker\varphi$. We have $\bar{f}(\bar{v_i}) \in \overline{V}_i$ for any v_i, then $\overline{f(v_i)} \in \overline{V}_i$. Hence, $f(v_i) \in V_i$. Therefore $f(V_i) \subseteq V_i$.

$\square$

C APPENDIX C

Throughout this chapter, K is an algebraically closed field.

Definition APPENDIX C.1. *An* ***abstract torus*** T *is an algebraic group over* K *that is isomorphic to the direct product of* m*-many copies of* K^* *for some* m.

Theorem APPENDIX C.2. *(Digne and Michel, 1991) A connected, solvable, algebraic group containing no unipotent elements is a torus.*

Theorem APPENDIX C.3. *(Springer, 2009) Let* G *be a connected one-dimensional unipotent algebraic group over* K*. Then* G *is isomorphic to the additive subgroup of* K.

BIOGRAPHICAL SKETCH

Zeynep Kısakürek was born in Şişli, İstanbul on May 19, 1990. She was graduated from Maltepe Anatolian High School (Maltepe Anadolu Lisesi) in 2008 and earned a Bachelor of Science degree in Mathematics from İstanbul Bilgi University in 2015. She performed as a part-time Teaching Assistant between October, 2015 and December, 2015 for İstanbul Bilgi University. She has been working as a Research Assistant at Mathematics Undergraduate Programme for İstanbul Bilgi University since December, 2015.