THETA FUNCTIONS OF INDEFINITE BINARY QUADRATIC FORMS

(BELİRSİZ İKİLİ KUADRATİK FORMLARIN THETA FONKSİYONLARI)

by

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Thesis

Submitted in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

in

MATHEMATICS

in the

GRADUATE SCHOOL OF SCIENCE AND ENGINEERING

of

GALATASARAY UNIVERSITY

OCTOBER 2017

Approval of the thesis:

THETA FUNCTIONS OF INDEFINITE BINARY QUADRATIC FORMS

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ACKNOWLEDGMENTS

I would like to express my gratitude to my supervisor Ayberk Zeytin for the useful comments, remarks and engagement through the learning process of this master thesis. The door to his office was always open whenever I had a question about my research and writing.

I particularly send my special thanks to my sister Şahika İzgi, within that period she labored not only her clients' but also my changeable psychology. I appreciate her for stabilizing my psychology.

I am grateful to my fiancee Melendiz Dalyan, who have provided me through moral and emotional support in my life. I would never finish this thesis without her supports.

Last but not the least, I will be always greatful to other members of my family and my friends who have supported me along the way.

This thesis supported by TUBITAK with project number 114R073. I would like to acknowledge the financial support, I am honoured to be a part of this project.

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LIST OF SYMBOLS

$\mathbb{O}(2)$: Set of orthogonal matrices with n dimensional vector space

$\mathbb{L}(2)$: Set of lattices in $\mathbb{R}^2$

Q : Quadratic form

B : Bilinear form

$M_2(\mathbb{R})$: Set of all 2×2 matrices with real coefficient

$Sym(2)$: Set of 2×2 symmetric matrices with left-up, right-down entries and determinant are positives

$\mathfrak{H}$: Upper half plane

$\mathcal{O}^*(\mathbb{C})$: Set of holomorphic invertible functions in $\mathbb{C}$

$\wp$: Weierstraß elliptic function

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ABSTRACT

The general theory encapsulating the relationship between classical theory of theta functions and representation of integers by positive definite quadratic forms is well established. Indeed, it is the finiteness of the set of representations of a given positive integer by a fixed quadratic form forces the convergence of the corresponding theta function. These theta functions also possess certain symmetries with respect to both of their variables.

In this thesis with the aim of generalizing the theory of theta functions to the indefinite case we consider the analogous question for the case of indefinite binary quadratic forms. By the classical theory of indefinite binary quadratic forms it is well known that the cardinality of the set of number of representations of any integer is infinite unless empty. It was S. Zwegers who defined the corresponding theta functions. The convergence and symmetries of these new theta functions corresponding to indefinite binary quadratic forms are investigated.

The thesis is organized as follows: the first part discusses 1-1 correspondences between equivalence classes of rank 2 lattices and positive definite binary quadratic forms. Second chapter is devoted to recalling basic facts around theta functions (their convergence and symmetries) and then discusses two classical uses of theta functions : in obtaining elliptic functions and in obtaining modular forms. The final chapter of thesis is devoted to the definition of theta functions corresponding to indefinite binary quadratic forms and their symmetries.

Keywords : theta functions, lattices, indefinite binary quadratic forms

ÖZET

Bu çalışmada, pozitif belirli ikili kuadratik formların tamsayı temsilleri ile klasik theta fonksiyonları arasındaki ilişkiyi kapsayan genel teori açıklanmıştır. Bilindiği üzere, sabitlenmiş bir pozitif tamsayının verilen pozitif beliri ikili kuadratik form ile farklı temsillerinin sayısının sonlu oluşu, bu kuadratik forma karşılık gelen theta fonksiyonunun yakınsamasını zorunlu hale getirir. Aynı zamanda bu theta fonksiyonları, değişkenlerinin her ikisine göre belli bir simetriye sahiptir.

Theta fonksiyonları teorisini, belirsiz kuadratik formlarda genellemeyi amaçlayan bu tezde, belirsiz ikili kuadratik formlarda da benzer soruyu göz önünde bulunduruyoruz. Bilindiği üzere, klasik ikili kuadratik formlar teorisine göre, herhangi bir tamsayının temsil sayısı kümesinin eleman sayısı boş değilse sonsuzdur. S.Zwegers, belirsiz ikili kuadratik formlar için theta fonksiyonlarını tanımlamıştır. Bu çalışmada, belirsiz ikili kuadratik formlara karşılık gelen bu yeni theta fonksiyonlarının yakınsaklıkları ve simetrileri araştırılmıştır.

Bu tezde, birinci bölümde iki boyutlu kafeslerdeki denklik sınıflarıyla, pozitif belirli ikili kuadratik formlar arasındaki birebir ilişki tartışılmıştır. İkinci bölümde theta fonksiyonlarına ilişkin temel kurallar (bunların yakınsaklıkları ve simetrilerini) hatırlatılmış ve daha sonra theta fonksiyonlarının iki klasik uygulaması olan eliptik fonksiyonların elde edilmesi ve modüler formların elde edilmesi tartışılmıştır. Son bölümde ise, belirsiz ikili kuadratik formlar ve simetrilere karşılık gelen theta fonksiyonlarının tanımına değinilmiştir.

Anahtar Kelimeler : theta fonksiyonları, kafesler, belirisiz ikili kuadratik formlar

To my grandmother...

1 INTRODUCTION

In this chapter, we will define orthogonal transformations and give examples, then we will define lattices and finally we will mention quadratic forms which have very important role for theta functions.

1.1 Orthogonal Transformations

A linear transformation $T : V \to V$ on a real inner product space $(V, \langle \cdot, \cdot \rangle)$ is called **an orthogonal transformation**, if it preserves the inner product. In particular, for each pair (u, v) of elements of V, an orthogonal transformation preserves lengths of vectors and angles between vectors, that is we have ;

$$\langle u, v \rangle = \langle Tu, Tv \rangle.$$

In finite-dimensional spaces, the matrix of **an orthogonal transformation** is called an orthogonal matrix with columns being orthogonal vectors with unit norm. The determinant of any orthogonal matrix is either 1 or -1. The product of two orthogonal matrices and the inverse of an orthogonal matrix is also an orthogonal matrix. So, we have proved :

Proposition 1.1. *The set of orthogonal matrices form a group denoted by $\mathbb{O}(n)$, where n denotes the dimension of the ambient vector space V.*

In two dimensional vector spaces (which will be identified with $\mathbb{R}^2$ henceforth) we may classify orthogonal transformations into two groups : reflections and rotations. The image of a figure by a **reflection** is its mirror image in the axis of reflection. For example, the mirror image of the letter **b**, for a reflection with respect to a vertical axis would be **d** and the transformation matrix of this reflection is $R_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; for a horizontal axis would be **p** and the transformation matrix of this reflection is $R_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The reflection matrix has a determinant -1 and it fixes all the points on its axis of reflection.

Rotations are orthogonal transformations that rotate points of $\mathbb{R}^2$ about a fixed point by angle ϕ. In this case, the matrix $A_\phi = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$ rotates points in the vector space $\mathbb{R}^2$ counter-clockwise by ϕ about the origin. For instance, if we rotate the vector $u = (1,2)$ counter-clockwise by an angle $\frac{\pi}{2}$ about the origin, we get :

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$x = \cos\frac{\pi}{2} - 2\sin\frac{\pi}{2} = -2$$
$$y = \sin\frac{\pi}{2} + 2\cos\frac{\pi}{2} = 1.$$

So the rotation of u around origin by an angle $\frac{\pi}{2}$ is $v = (-2,1)$. Rotation matrices have a determinant 1 and they fix the origin.

Theorem 1.1. *All orthogonal transformations of a two dimensional vector space V can be written as a composition of reflections and rotations.*

Proof. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an arbitrary element of $\mathbb{O}(2)$. $T(0,0) = (0,0)$.
Say $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and $e_1 = (1,0)$, $e_2 = (0,1)$ are the unit vectors of $\mathbb{R}^2$. Then

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ r \end{pmatrix}$$
$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} q \\ s \end{pmatrix}.$$

Since orthogonal transformations preserve length ; $p^2 + r^2 = 1$ and $q^2 + s^2 = 1$. So (p,r) and (q,s) are the elements of the unit circle S^1. (i.e.$(p,r) = (\cos(\theta_1), \sin(\theta_1))$ and $(q,s) = (\cos(\theta_2), \sin(\theta_2))$ for some $\theta_1,\ \theta_2 \in (0, 2\pi]$.).
Orthogonal transformations preserve dot product, since $e_1 \perp e_2$ then $(p,r) \perp (q,s)$. Thus $pq+rs = \cos(\theta_1)\cos(\theta_2)+\sin(\theta_1)\sin(\theta_2) = \cos(\theta_1-\theta_2) = 0$. So $\theta_1-\theta_2 = k\pi+\dfrac{\pi}{2}$ for some $k \in \mathbb{Z}$. So

$$S = \begin{pmatrix} \cos(\theta_1) & \cos(\theta_2) \\ \sin(\theta_1) & \sin(\theta_2) \end{pmatrix} = \begin{pmatrix} \cos(\theta_1) & \cos(\theta_1 - (k\pi + \frac{\pi}{2})) \\ \sin(\theta_1) & \sin(\theta_1 - (k\pi + \frac{\pi}{2})) \end{pmatrix}.$$

There are two cases ;

— If k is odd, then the matrix $S = \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix}$. So, S is a rotation.

— If k is even, then the matrix $S = \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) \\ \sin(\theta_1) & -\cos(\theta_1) \end{pmatrix}$. If we compose S with the reflection matrix $R_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we get ; $(R_2 \circ S) = \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) \\ \sin(\theta_1) & -\cos(\theta_1) \end{pmatrix}$. This composition is also a rotation.

□

1.2 Lattices in $\mathbb{C}$

Definition 1.1. A lattice Λ of rank 2 is $\mathbb{Z}-$ submodule of $\mathbb{R}^2$ generated by two $\mathbb{R}-$linearly independent vectors :

$$\Lambda = \{a_1 u + a_2 v \mid (a_1, a_2) \in \mathbb{Z}^2\}$$

where $\{u, v\}$ is a basis for Λ. We write $\mathbb{L}(2)$ for the set of lattices in $\mathbb{R}^2$.

Example 1.2.1. The lattice obtained from unit vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ is $\mathbb{Z}^2$

Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two vectors of $\mathbb{R}^2$ and Λ be a lattice obtained from these vectors. The group $\mathbb{O}(2)$ acts on Λ producing another lattice defined as

$$\begin{aligned} \bullet : \mathbb{O}(2) \times \mathbb{L} &\rightarrow \mathbb{L} \\ (\gamma, \Lambda) &\mapsto \gamma \bullet \Lambda \end{aligned}$$

where ;

$$\begin{aligned} \gamma \bullet \Lambda &= \left(\begin{pmatrix} p & q \\ r & s \end{pmatrix} u \right) \mathbb{Z} + \left(\begin{pmatrix} p & q \\ r & s \end{pmatrix} v \right) \mathbb{Z} \\ &= \{\mathbb{Z}(p(u_1 + v_1) + q(u_2 + v_2), r(u_1 + v_1) + s(u_2 + v_2))\} \end{aligned}$$

Theorem 1.2. *The above map $\bullet$ defines an action of $\mathbb{O}(2)$ on $\mathbb{L}(2)$.*

Proof. $I \bullet \Lambda = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u \right) \mathbb{Z} + \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v \right) \mathbb{Z} = u\mathbb{Z} + v\mathbb{Z} = \Lambda$ for all u, v in Λ.

Let $\gamma_1 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \gamma_2 = \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}$ be two orthogonal transformations.

$$\begin{aligned}
\gamma_1 \bullet (\gamma_2 \bullet \Lambda) &= \gamma_1 \left\{ \left(\begin{pmatrix} p'u_1 + q'u_2 \\ r'u_1 + d'u_2 \end{pmatrix} \right) \mathbb{Z} + \left(\begin{pmatrix} p'v_1 + q'v_2 \\ r'v_1 + d'v_2 \end{pmatrix} \right) \mathbb{Z} \right\} \\
&= \left(\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} p'u_1 + q'u_2 \\ r'u_1 + d'u_2 \end{pmatrix} \right) \mathbb{Z} + \left(\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} p'v_1 + q'v_2 \\ r'v_1 + d'v_2 \end{pmatrix} \right) \mathbb{Z} \\
&= \begin{pmatrix} pp'u_1 + pq'u_2 + qr'u_1 + qs'u_2 \\ rp'u_1 + rq'u_2 + sr'u_1 + ss'u_2 \end{pmatrix} \mathbb{Z} + \begin{pmatrix} pp'v_1 + pq'v_2 + qr'v_1 + qs'v_2 \\ rp'v_1 + rq'v_2 + sr'v_1 + ss'v_2 \end{pmatrix} \mathbb{Z} \\
&= \begin{pmatrix} pp' + qr' & pq' + qs' \\ rp' + sr' & rq' + ss' \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mathbb{Z} + \begin{pmatrix} pp' + qr' & pq' + qs' \\ rp' + sr' & rq' + ss' \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mathbb{Z} \\
&= (\gamma_1 \bullet \gamma_2) \bullet \Lambda.
\end{aligned}$$

□

We obtain an equivalence relation on the set of lattices. Namely two lattices Λ and Λ' are equivalent if and only if there exists an element $\gamma \in \mathbb{O}(2)$ such that $\Lambda = \gamma \bullet \Lambda'$. For convenience we will denote the equivalance class of a lattice Λ in $\mathbb{C}$ again by Λ. That is, we will denote elements of the set $\mathbb{L}(2)$ by elements of $\mathbb{L}(2)/\mathbb{O}(2)$.

Definition 1.2. A homothety is a transformation of a space with respect to a point S that takes any point M in a one-to-one correspondence with a point M' on the straight line SM in accordance with the rule $SM' = kSM$ where k is constant real number, which is called **homothety ratio** and the point S is said to be **homothety center**. Homoteties are similarities that fix a point and either preserve (if $k > 0$) or reverse (if $k < 0$) the direction of all vectors.

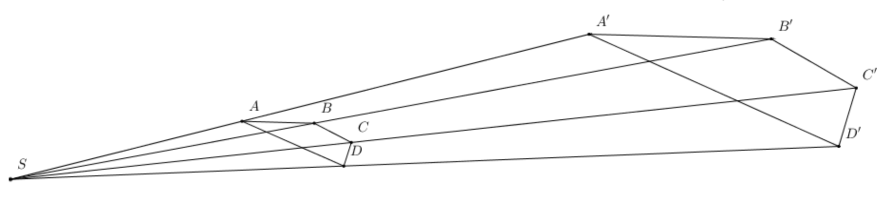

Figure 1.1: A Transformation with Homothety Center S

We fix the origin of $\mathbb{C}$ as the homothety center. Then a homothety in $\mathbb{C}$ is determined uniquely by a real non zero number x_0 in the sense that homotheties in this case are given by maps from $\mathbb{C}$ to $\mathbb{C}$ of the form $z \mapsto x_0\, z$. This homothety is denoted by m_{x_0}. With this notation, two lattices Λ and Λ' are called **homothetic** if there is a $x_0 \in \mathbb{R}^*$ so that $\Lambda = m_{x_0} \bullet \Lambda'$.

Theorem 1.3. *Homothety defines an equivalence relation on the set* $\mathbb{L}(2)$.

Proof. If we choose $x_0 = 1$ then we have $\Lambda \sim \Lambda$. For some m_{x_0}, we have $\Lambda \sim \Lambda'$ if and only if $\Lambda = m_{x_0} \bullet \Lambda'$. If we choose $\dfrac{1}{x_0}$ then we have $\Lambda' = m_{\frac{1}{x_0}} \bullet \Lambda$. Thus $\Lambda' \sim \Lambda$. For some x_0 and x_1 in $\mathbb{R}^*$ we have $\Lambda = m_{x_0} \bullet \Lambda'$ and $\Lambda' = m_{x_1} \bullet \Lambda''$. This implies $\Lambda = m_{x_0} m_{x_1} \bullet \Lambda''$. So $\Lambda \sim \Lambda''$ if we choose $m_{x_0 x_1}$. $\square$

1.3 Quadratic Forms

A quadratic form on $\mathbb{R}^n$ to $\mathbb{R}$ is a real-valued function of the form

$$Q(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} x_i x_j$$

where $X = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $a_{i,j} \in \mathbb{R}$. Note that each monomial in this finite set is of degree two. This formula may be rewritten by using the symmetric matrix

$$A = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,j} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,j} \\ \vdots & \vdots & \ddots & \vdots \\ b_{i,1} & b_{i,2} & \cdots & b_{i,j} \end{pmatrix}$$

such that,

$$Q(X) = X^T A X;$$

where $b_{i,j} = b_{j,i} = \dfrac{a_{i,j}}{2}$. Whenever $n = 2$ the quadratic form is called a **binary quadratic form**.

Example 1.3.1. Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a vector in $\mathbb{R}^2$ and $A = \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix}$ be a 2×2 symmetric matrix. So the quadratic form represented by A can be written as

$$Q(X) = (x_1 \;\; x_2) \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 - 2x_1 x_2 - 3x_2^2.$$

Definition 1.3. A quadratic form is called **positive definite** if $Q(X) \geq 0$, and **negative definite** if $Q(X) \leq 0$ for all $X \in \mathbb{R}^n$. $Q(X)$ is called **indefinite** if $Q(X)$ is attains both negative and positive values, that is $Q(X)$ is positive for some $X \in \mathbb{R}^n$ and negative for some $X \in \mathbb{R}^n$.

Proposition 1.2. *The binary quadratic form Q determined by the symmetric matrix $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{1,2} & a_{2,2} \end{pmatrix}$ is positive (respectively negative) definite if $\det(A) > 0$ and the upper left entry $a_{1,1} > 0$ (respectively $a_{1,1} < 0$). Q is indefinite if $\det(A) < 0$.*

Proof. Let $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{1,2} & a_{2,2} \end{pmatrix}$ be symmetric matrix. For any $X = (x_1, x_2) \in \mathbb{R}^2$, quadratic form associated with the matrix A is

$$\begin{aligned}
Q(X) &= a_{1,1}(x_1)^2 + 2a_{1,2}x_1x_2 + a_{1,2}(x_2)^2 \\
&= a_{1,1}\big((x_1)^2 + \frac{2a_{1,2}}{a_{1,1}}x_1x_2 + \frac{a_{2,2}}{a_{1,1}}(x_2)^2\big) \\
&= a_{1,1}\big((x_1)^2 + \frac{2a_{1,2}}{a_{1,1}}x_1x_2 + \frac{a_{2,2}}{a_{1,1}}(x_2)^2\big) - \frac{a_{1,2}^2}{a_{1,1}^2}((x_2)^2) + \frac{a_{2,2}}{a_{2,2}}(x_2)^2 \\
&= a_{1,1}\left((x_1 + \frac{a_{1,2}}{a_{1,1}}x_2)^2 + \frac{(x_2)^2}{a_{1,1}^2}(a_{2,2}a_{1,1} - a_{1,2}^2)\right).
\end{aligned}$$

Since $\frac{(x_2)^2}{a_{1,1}^2} \leq 0$ and $\frac{(x_2)^2}{a_{1,1}^2} \leq 0$, $Q(X)$ is positive definite when $a_{1,1} > 0$ and $(a_{2,2}a_{1,1} - a_{1,2}^2) > 0$, respectively $Q(X)$ is negative definite when $a_{1,1} < 0$ and $(a_{2,2}a_{1,1} - a_{1,2}^2) > 0$. On the other hand $Q(X)$ is indefinite if $(a_{2,2}a_{1,1} - a_{1,2}^2) < 0$. $\square$

Corollary 1.1. *The binary quadratic form Q determined by the symmetric matrix $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{1,2} & a_{2,2} \end{pmatrix}$ is definite (respectively indefinite) when $\det(A) = a_{2,2}a_{1,1} - a_{1,2}^2 < 0$ (respectively $a_{2,2}a_{1,1} - a_{1,2} > 0$)*

Example 1.3.2. A quadratic form $Q(X) = x_1^2 - 2x_1x_2 + 3x_2^2$ can be rewritten as $Q(X) = (x_1 - x_2)^2 + 4x_2^2$. So that the quadratic form $Q(X)$ is positive definite.

Example 1.3.3. The quadratic form in $\mathbb{R}^2$ defined by the matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is $Q(x) = -x_1^2 - x_2^2$. For any $x \in \mathbb{R}^2$, $Q(x)$ is always non-positive so the quadratic form $Q(x)$ is negative definite.

Definition 1.4. A **bilinear form** on $\mathbb{R}^2$ is a map

$$B \; : \; \mathbb{R}^2 \; \times \; \mathbb{R}^2 \quad \to \quad \mathbb{R}$$

such that

— $B(X_1 + X_2, y) = B(X_1, Y) + B(X_2, Y)$ for all $X_1, X_2, Y \in \mathbb{R}^2$

— $B(X, Y_1 + Y_2,) = B(X, Y_1) + B(X, Y_2)$ for all $X, Y_1, Y_2 \in \mathbb{R}^2$

— $B(aX, Y) = aB(X, Y)$ for all $X, Y \in \mathbb{R}^2$

— $B(X, aY) = aB(X, Y)$ for all $X, Y \in \mathbb{R}^2$

Remark 1.1. Given a quadratic form Q, consider the form

$$B_Q(X, Y) \;\; := \;\; \frac{1}{2}(Q(X + Y) - Q(X) - Q(Y)).$$

Then, for any $X_1, X_2, Y \in \mathbb{R}^2$, we have

$$B(X_1 + X_2, Y) \;\; = \;\; \frac{1}{2}(Q(X_1 + X_2 + Y) - Q(X_1 + X_2) - Q(Y)).$$

If we calculate value of Q by definition, then we get

$$\begin{aligned}
&= \frac{1}{2}\big((X_1 + X_2 + Y)^T A(X_1 + X_2 + Y) - (X_1 + X_2)^T A(X_1 + X_2) - Y^T AY\big) \\
&= \frac{1}{2}\big(X_1^T AX_1 + X_1^T AX_2 + X_1^T AY + X_2^T AX_1 + X_2^T AX_2 + X_2^T AY + Y^T AX_1 \\
&\quad Y^T AX_2 + Y^T AY\big) - X_1^T AX_1 - X_1^T AX_2 - X_2^T AX_1 - X_2^T AX_2 - Y^T AY \\
&= \frac{(Q(X_1 + Y) + Q(X_2 + Y) - Q(X_1) - Q(X_2) - 2Q(Y))}{2} \\
&= B(X_1, Y) + B(X_2, Y).
\end{aligned}$$

Second equation holds from same method.

For third equation $a \in \mathbb{R}$ and $X, Y \in \mathbb{R}^2$.

$$\begin{aligned}
B(aX, Y) &= \frac{1}{2}(Q(aX + Y) - Q(aX) - Q(Y)) \\
&= \frac{1}{2}\big((aX + Y)^T A(aX + Y) - (aX)^T(aX) - Y^T AY\big) \\
&= \frac{1}{2}\big((aX)^T A(aX) + aX^T AY + Y^T A(aX) + Y^T AY \\
&\quad -(aX)^T A(aX) - y^T AY\big)
\end{aligned}$$

$$= aB(X,Y)$$

.

Fourth equation holds from same method above. We say the B is a bilinear form associated to Q.

Example 1.3.4. The binary quadratic form which appareared in Example 1.3.1 is

$$\begin{aligned} B(X,Y) &= \frac{1}{2}\big((x_1+y_1)^2 - 2(x_1+y_1)(x_2+x_2) - 3(x_2+y_2)^2 - (x_1+x_2)^2 \\ &\quad -2(x_1+x_2) - 3(x_1+x_2)^2 - (y_1+y_2)^2 - 2(y_1y_2) - 3(y_1+y_2)^2\big) \\ &= x_1y_1 - x_1y_2 - x_2y_1 - 3x_2y_2. \end{aligned}$$

The **type** of Q is defined as the pair $(2-s,s)$ where s is the largest dimension of a linear subspace of R^2 on which Q is negative definite. If Q has type $(2,0)$ then by definition, the dimension of a linear subspace on which Q is negative definite is 0, hence is $\{0\}$. This means simply that Q is positive definite. Similarly, if Q has type $(0,2)$ then by definition, the dimension of a linear subspace on which Q is negative definite is 2, hence is $\mathbb{R}^2$. This means simply that Q is negative definite.

2 THETA FUNCTIONS AND QUADRATIC FORMS : DEFINITE CASE

In this chapter, we will mention quadratic forms associated to lattices, theta functions, elliptic funtions and modular forms. First, we will show how to obtain definite quadratic forms from lattices and then we will define theta functions associated to these quadratic forms. This will be followed by the definition of we will define elliptic funtions and their relation between theta functions. Finally, we will define modular forms and observe that Riemann theta function gives rise to a modular form of level 4.

2.1 Quadratic Forms Associated to Lattices

Let Λ be a lattice generated by the vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$. We have a new symmetric matrix A_Λ with the dot products of the vectors u, v such that

$$A_\Lambda = \begin{pmatrix} u \bullet u & u \bullet v \\ u \bullet v & v \bullet v \end{pmatrix} = \begin{pmatrix} (u_1)^2 + (u_2)^2 & (u_1)(v_1) + (u_2)(v_2) \\ (u_1)(v_1) + (u_2)(v_2) & (v_1)^2 + (v_2)^2 \end{pmatrix}.$$

The matrix A_Λ represents a quadratic form.

Proposition 2.1. *Given any element $\gamma \in M_2(\mathbb{R})$, we have $A_{\gamma\cdot\Lambda} = \gamma \cdot A_\Lambda \cdot \gamma^T$.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $M_2(\mathbb{R})$, $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ be bases of the lattice Λ. We denote $\gamma\Lambda$, the lattice obtained from γ transformation of Λ. So,

$$\gamma \cdot \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} a\vec{u} + b\vec{v} \\ c\vec{u} + d\vec{v} \end{pmatrix} = \begin{pmatrix} \left((au_1 + bv_1), (au_2 + bv_2)\right) \\ \left((cu_1 + dv_1), (cu_2 + dv_2)\right) \end{pmatrix}.$$

The quadratic form obtained from this lattice is

$$A_{\gamma\cdot\Lambda} = \begin{pmatrix} (au_1 + bv_1)^2 + (au_2 + bv_2)^2 & (au_1 + bv_1)(cu_1 + dv_1) + (au_2 + bv_2)(cu_2 + dv_2) \\ (au_1 + bv_1)(cu_1 + dv_1) + (au_2 + bv_2)(cu_2 + dv_2) & (cu_1 + dv_1)^2 + (cu_2 + dv_2)^2 \end{pmatrix}$$

Now we calculate $\gamma \cdot A_\Lambda \cdot \gamma^T$.

$$\begin{aligned}
\gamma \cdot A_\Lambda \cdot \gamma^T &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} (u_1)^2 + (u_2)^2 & (u_1)(v_1) + (u_2)(v_2) \\ (u_1)(v_1) + (u_2)(v_2) & (v_1)^2 + (v_2)^2 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \\
&= \begin{pmatrix} a(u_1)^2 + a(u_2)^2 + b(u_1)(v_1) + b(u_2)(v_2) & a(u_1)(v_1) + a(u_2)(v_2) + b(v_1)^2 + b(v_2)^2 \\ c(u_1)^2 + c(u_2)^2 + d(u_1)(v_1) + d(u_2)(v_2) & c(u_1)(v_1) + c(u_2)(v_2) + d(v_1)^2 + d(v_2)^2 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \\
&= \begin{pmatrix} (au_1 + bv_1)^2 + (au_2 + bv_2)^2 & (au_1 + bv_1)(cu_1 + dv_1) + (au_2 + bv_2)(cu_2 + dv_2) \\ (au_1 + bv_1)(cu_1 + dv_1) + (au_2 + bv_2)(cu_2 + dv_2) & (cu_1 + dv_1)^2 + (cu_2 + dv_2)^2 \end{pmatrix} \\
&= A_{\gamma \cdot \Lambda}
\end{aligned}$$

So, for any element γ of $M_2(\mathbb{R})$ we have $A_{\gamma\cdot\Lambda} = \gamma \cdot A_\Lambda \cdot \gamma^T$. □

Theorem 2.1. *Let Λ be a lattice in $\mathbb{C}$ and $\gamma \in M_2(\mathbb{R})$. Then $\gamma \cdot \Lambda = \Lambda$ if and only if $\gamma \in SL_2(\mathbb{Z})$.*

Proof. Let Λ be in lattice in $\mathbb{C}$ with the base vectors u and v. So we have $\Lambda = \langle u, v\rangle$. Suppose that γ is any element of $SL_2(\mathbb{Z})$ such that $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Base vectors of $\gamma \cdot \Lambda$ is

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} pu + qv \\ ru + sv \end{pmatrix} = \langle pu + qv, ru + sv\rangle$$

Since $p, q, r, s \in \mathbb{Z}$, $pu + qv \in \langle u, v\rangle$ and $ru + sv \in \langle u, v\rangle$. So $\langle pu + qv, ru + sv\rangle \subseteq \langle u, v\rangle$. So given any matrix γ, we have $\gamma \cdot \Lambda \subseteq \Lambda$. As $\gamma \in SL_2(\mathbb{Z})$, γ is an invertible matrix with its inverse $\gamma^{-1} \in SL_2(\mathbb{Z})$. By changing $\Lambda \to \gamma \cdot \Lambda$ and $\gamma \to \gamma^{-1}$, we get

$$\begin{aligned}
\Lambda &\to \gamma\Lambda \\
&= \gamma^{-1}(\gamma \cdot \Lambda) \subseteq \gamma \cdot \Lambda \subseteq \Lambda \\
& \Lambda \subseteq \gamma \cdot \Lambda \subseteq \Lambda.
\end{aligned}$$

So this relation forces Λ to be equal to $\gamma \cdot \Lambda$.

For the other direction, suppose $\gamma \cdot \Lambda = \Lambda$.

Suppose that $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ with components $p, q, r, s \in \mathbb{R}$. Let $\Lambda = \langle u, v\rangle$ for u and v two vectors of $\mathbb{C}$. So $\gamma \cdot \Lambda = \langle pu + qv, ru + sv\rangle$. Since $\gamma \cdot \Lambda = \Lambda$ then for $a, b, k, l \in \mathbb{Z}$, we have $a(pu + qv) + b(ru + sv) = ku + lv$.

This equations shows us $ap + br = k$ and $aq + bs = l$. If we translate this equation into the matrix equation we get for all $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$, there exists one and only one $\begin{pmatrix} k \\ l \end{pmatrix} \in \mathbb{R}^2$ such that

$$\begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} k \\ l \end{pmatrix}.$$

First, if $\det \begin{pmatrix} p & r \\ q & s \end{pmatrix} = 0$, the equation would have infinitly many solutions but we have unique solution. So $\det \begin{pmatrix} p & r \\ q & s \end{pmatrix} \neq 0$. This means the matrix is invertible.
Secondly, since the equation holds for all (a, b), we can choose $(a, b) = (1, 0)$. Then we get $(p, q) = (k, l)$. Due to the fact that k and l are integers, p and q should be integers, too. On the other hand, if we choose $(a, b) = (0, 1)$, we get $(r, s) = (k, l)$, as we saw in previous sentence, r and s should be integers. This is a contradiction and since $\det \begin{pmatrix} p & r \\ q & s \end{pmatrix} \neq 0$ and $p, q, r, s \in \mathbb{Z}$, $\det(M)$ must be ∓ 1. It means $\gamma \in SL_2(\mathbb{Z})$. $\square$

Λ does not have any canonical set of generators. This is to say we may choose many different elements (u, v) of Λ which generate and we don't have any particular preference. The association $\Lambda \mapsto A_\Lambda$ depends on the choice of generators of Λ. This forces us to define an equivalence relation on the set of symmetric matrices with elements $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $a, b, c \in \mathbb{R}$. To overcome this difficulty, we define two symmetric matrices M and M' to be equivalent if there is some $\gamma \in SL_2(\mathbb{Z})$ such that

$$\gamma \cdot M \cdot \gamma^T = M'.$$

Proposition 2.2. *This is an equivalence relation and we write $M \sim M'$.*

Proof. Let M, M' and M'' be symmetric matrices and let γ and δ be elements of $SL_2(\mathbb{Z})$. Since $\gamma, \delta \in SL_2(\mathbb{Z})$, their transposes γ^T, δ^T and their inverses γ^{-1}, δ^{-1} are also elements of $SL_2\mathbb{Z}$.
We know that identity matrix is an element of $SL_2(\mathbb{Z})$. So if we choose $\gamma = I_2$, we get

$$\gamma \cdot M \cdot \gamma^T = I_2 \cdot M \cdot I_2^T = M.$$

So this relation is reflexive. If $\gamma \cdot M \cdot \gamma^T = M'$, then by multiplying left hand side with γ^{-1} and right hand side $(\gamma^T)^{-1}$, we get

$$\begin{aligned} \gamma^{-1} \cdot \gamma \cdot M \cdot \gamma^T \cdot (\gamma^T)^{-1} &= \gamma^{-1} \cdot M' \cdot (\gamma^T)^{-1} \\ M &= \gamma^{-1} \cdot M' \cdot (\gamma^T)^{-1}. \end{aligned}$$

Since γ^{-1} and $(\gamma^T)^{-1}$ are elements of $SL_2(\mathbb{Z})$. So this relation is symmetric.
If $\gamma \cdot M \cdot \gamma^T = M'$ and $\delta \cdot M' \cdot \delta^T = M''$, then we get

$$\gamma \cdot \delta \cdot M \cdot \gamma^T \cdot \delta^T = M''.$$

As γ and δ are elements of $SL_2(\mathbb{Z})$, $\gamma \cdot \delta$ and $(\gamma \cdot \delta)^T$ are also elements of $SL_2(\mathbb{Z})$. Note $\gamma \cdot \delta = \mu$. Then we have

$$\mu \cdot M \cdot \mu^T = M''.$$

So this relation is transitive. $\square$

Thus, we obtain a map from set of lattices $\mathbb{L}(2)$ to the set of symmetric matrices with the equivalance relation in 2.1. Now given a symmetric matrix $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, we will find a lattice Λ_M (i.e. an element of $\mathbb{L}(2)$). So the symmetric matrix associated to Λ_M is M. We must have

$$\begin{aligned} u \bullet u &= a, \\ u \bullet v &= b, \\ v \bullet v &= c. \end{aligned} \tag{2.1}$$

Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$. We have equation system 2.1, this means

$$\begin{aligned} u_1^2 + u_2^2 &= a, \\ u_1 v_1 + u_2 v_2 &= b, \\ v_1^2 + v_2^2 &= c. \end{aligned} \tag{2.2}$$

Since u_1, u_2, v_1, v_2 are elements of $\mathbb{R}$, there are infinitely many solutions for 2.2. Let $R_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ be reflection matrix. By multiplying u and v with R_1 we get

$$\begin{aligned} R_1 \cdot u &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -u_1 \\ u_2 \end{pmatrix} \\ R_1 \cdot v &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ v_2 \end{pmatrix}. \end{aligned}$$

These new vectors are also a solution of 2.2.

$$\begin{aligned} (-u_1)^2 + u_2^2 = u_1^2 + u_2^2 &= a, \\ (-u_1)(-v_1) + u_2 v_2 = u_1 v_1 + u_2 v_2 &= b, \\ (-v_1)^2 + v_2^2 = v_1^2 + v_2^2 &= c. \end{aligned}$$

If we multiply two vectors with R_2, we'll find new vectors. It is obvious that these vectors are also a solution for 2.2. On the other hand, let $\gamma = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$ be rotation matrix with $\phi \in (0, 2\pi]$. By multiplying u and v with γ, we get we get

$$\gamma \cdot u = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos(\phi)u_1 - \sin(\phi)u_2 \\ \sin(\phi)u_1 + \cos(\phi)u_2 \end{pmatrix}$$

$$R_1 \cdot v = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos(\phi)v_1 - \sin(\phi)v_2 \\ \sin(\phi)v_1 + \cos(\phi)v_2 \end{pmatrix}.$$

These new vectors are also a solution of 2.2.

$$\begin{aligned} (\cos(\phi)u_1 - \sin(\phi)u_2)^2 + (\sin(\phi)u_1 + \cos(\phi)u_2)^2 = u_1^2 + u_2^2 &= a, \\ (\cos(\phi)u_1 - \sin(\phi)u_2)(\sin(\phi)u_1 + \cos(\phi)u_2) + (\cos(\phi)v_1 - \sin(\phi)v_2)(\sin(\phi)v_1 + \cos(\phi)v_2) = u_1 v_1 + u_2 v_2 &= b, \\ (\cos(\phi)v_1 - \sin(\phi)v_2)^2(\sin(\phi)v_1 + \cos(\phi)v_2)^2 = v_1^2 + v_2^2 &= c. \end{aligned}$$

This gives rise to the equivalence relation defined by action of $\mathbb{O}(2)$ on $\mathbb{L}(2)$.
On the other hand, for one of these solutions we can start by choosing $u = (\sqrt{a}, 0)$, with respect to this choise by using 2.2, we can see that $v = (\frac{b}{\sqrt{a}}, \sqrt{\frac{ac-b^2}{a}})$. This base vectors make us to see that, the system given in 2.2 does not have any solutions for any symmetric matrix $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. First, a must be positive and $ac - b^2$ must be non negative. Since b^2 is non negative, c must also be non negative. We see that $ac - b^2$ is determinant of M.
These properties are valid for one choice of bases vectors but as $\mathbb{O}(2)$ acts on $\mathbb{L}(2)$, the properties must be valid for all choices. So denote $Sym(2)$ for the symmetric matrices $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with a being positive, c being non negative and determinant of M being also non negative.

Theorem 2.2. *There is a one to one correspondence between $\mathbb{L}(2)/\mathbb{O}(2)$ and $Sym(2)\ /SL_2(\mathbb{Z})$.*

Proof. At the beginning of this chapter, for each basis of a lattice we find new symmetric matrix. Since a lattice can have infinitly many bases, by dividing these bases by the action of $\mathbb{O}(2)$, we can say that these two lattices are identic with respect to this division. On the other hand, we see that the group $SL_2(\mathbb{Z})$ acts on the set of symmetric matrices. So we can say that some symmetric matrices are identic with respect to this group effect.

So we have a map, say ϕ, from $\mathbb{L}(2)/\mathbb{O}(2)$ to $Sym(2)/SL_2(\mathbb{Z})$, which sends a lattice Λ to A_Λ, which is an element of $Sym(2)$. If $\Lambda = \langle u, v\rangle$ then $A_\Lambda = \begin{pmatrix} u \bullet u & u \bullet v \\ v \bullet u & v \bullet v \end{pmatrix}$.

That is to say $\phi(\Lambda) = A_\Lambda$.

On the other hand, we have also a map, say ψ, from $Sym(2)/\mathbb{O}(2)$ to $\mathbb{L}(2)/SL_2(\mathbb{Z})$, which sends M to the lattice denoted by Λ_M with the base vectors u_M and v_M, so we have $\Lambda_M = \langle u_M, v_M\rangle$. Thus $\psi(M) = \Lambda_M$ where $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in Sym(2)$ with $a = u \bullet u$, $b = u \bullet v$ and $c = v \bullet v$.

For any element $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ in $Sym(2)$, we have

$$(\phi \circ \psi)(M) = \phi(\psi(M)) = \phi(\Lambda_M) = \phi(\langle u_M, v_M\rangle)$$

where $u_M \bullet u_M = a$, $u_M \bullet v_M = b$ and $v_M \bullet v_M = c$.

$$\phi(\langle u_M, v_M\rangle) = A_{\Lambda_M} = \begin{pmatrix} u_M \bullet u_M & u_M \bullet v_M \\ v_M \bullet u_M & v_M \bullet v_M \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = M.$$

Thus, $\phi \circ \psi$ is identity, which means both *phi* and ψ are $1-1$ and onto. $\square$

2.2 Theta Functions Associated to Quadratic Forms

A holomorphic function $\theta : \mathbb{C} \to \mathbb{C}$ is called a **theta function** with respect to a lattice Λ (or corresponding quadratic form A_Λ) if, for any $\lambda \in \Lambda$, there exists a holomorphic function $e_\lambda(z)$ which is invertible such that

$$\theta(z + \lambda) = e_\lambda(z)\theta(z), \qquad \forall z \in \mathbb{C}.$$

Each element of the set of functions $\{e_\lambda(z)\}$ is called the **theta factor** of θ.

Example 2.2.1. Let $\Lambda = \mathbb{Z}\tau + \mathbb{Z}$, where $\tau \in \mathfrak{H}$. Set

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi(n^2\tau+2zn)}.$$

This function is holomorphic on $\mathbb{C}$. To check the convergence of this function, we calculate its absolute value :

$$|\theta(z,\tau)| = \left|\sum_{n\in\mathbb{Z}} e^{i\pi(n^2\tau+2zn)}\right| \leq \sum_{n\in\mathbb{Z}} \left(|e^{i\pi n^2(a+bi)}||e^{2\pi inz}|\right)$$

$$\leq \sum_{n\in\mathbb{Z}} \left(|e^{i\pi n^2 a}||e^{-\pi bn^2}||e^{2\pi inz}|\right).$$

Since a is a real number, $|e^{i\pi n^2 a}| = 1$. Thus we should only check the convergence of the series $\sum\limits_{n\in\mathbb{Z}} |e^{-\pi bn^2}||e^{2\pi inz}|$. If we fix $z \in \mathbb{C}$, then there exists a positive real number M such that $|e^{2\pi inz}| \leq M$. So we have

$$|\theta(z,\tau)| \leq M \sum_{n\in\mathbb{Z}} |e^{-\pi bn^2}|$$

$$\sum_{n\in\mathbb{Z}} |e^{-\pi bn^2}| = 1 + 2.\sum_{n=1}^{\infty} |e^{-\pi bn^2}|.$$

According to d'Alembert's ratio test,

$$\lim_{n\to\infty} \frac{e^{-\pi b(n+1)^2}}{e^{-\pi bn^2}} = \lim_{n\to\infty} e^{-\pi b(2n+1)}.$$

Since τ is an element of $\mathfrak{H}$, the integer b is positive. So $\lim_{n\to\infty} e^{-\pi b(2n+1)} = 0$. This calculation shows us $|\theta(z,\tau)| \leq \sum\limits_{n\in\mathbb{Z}} M|e^{-\pi bn^2}| < \infty$. Thus $\theta(z,\tau)$ is convergent, when z is fixed.

Now, let us fix $\tau \in \mathfrak{H}$. For any $z \in \mathbb{C}$, there exists a positive real number M such that $|z| \leq M$, as a result of this inequality, we have $|e^{2\pi inz}| \leq |e^{2\pi nM}|$. So $|\theta(z,\tau)| \leq \sum\limits_{n\in\mathbb{Z}} |e^{-\pi bn^2}||e^{2\pi nM}|$.

$$|\theta(z,\tau)| \leq \sum_{n\in\mathbb{Z}} |e^{-\pi bn^2}||e^{2\pi nM}| = 1 + \sum_{n=1}^{\infty} |2e^{-\pi bn^2}| \frac{|e^{2\pi nM}|^2+1}{|e^{2\pi nM}|}.$$

According to d'Alembert's ratio test,

$$\lim_{n\to\infty} \frac{|2e^{-\pi b(n+1)^2}|}{|2e^{-\pi bn^2}|} \frac{\frac{|e^{2\pi(n+1)M}|^2+1}{|e^{2\pi(n+1)M}|}}{\frac{|e^{2\pi(n)M}|^2+1}{|e^{2\pi(n)M}|}}.$$

If we simplify expression above, we get

$$\lim_{n\to\infty} \frac{|e^{-2\pi(n+1)b}|}{|e^{2\pi M}|} \frac{|e^{4\pi(n+1)M}|+1}{|e^{4\pi n M}|+1}.$$

It is obvious that $\lim_{n\to\infty} \dfrac{|e^{4\pi(n+1)M}|+1}{|e^{4\pi n M}|+1} = e^{4\pi M}$ and since b is positive, $\lim_{n\to\infty} |e^{-2\pi(n+1)b}| = 0$. Thus $\theta(z,\tau)$ is convergent, when τ is fixed. So $\theta(z,\tau)$ is convergent for all $z \in \mathbb{C}$ and $\tau \in \mathfrak{H}$.

Consider $\lambda \in \Lambda$, i.e. $\lambda = p\tau + q$ for $p, q \in \mathbb{Z}$

$$\theta(z+\lambda,\tau) = \theta(z+p\tau+q) = \sum_{n\in\mathbb{Z}} e^{(i\pi n^2\tau + 2\pi i n z + 2\pi i n p\tau + 2\pi i n q)}.$$

If we complete the square and rearrange the summands then

$$= \sum_{n\in\mathbb{Z}} e^{(\pi i n^2\tau + 2\pi i n p\tau + \pi i p^2\tau - \pi i p^2\tau + 2\pi i n z + 2\pi i p z - 2\pi i p z + 2\pi i n q)}$$

$$= e^{-\pi i p^2\tau - 2\pi i p z} \sum_{n\in\mathbb{Z}} e^{\pi i (n+p)^2\tau} e^{2\pi i (n+p) z} e^{2\pi i n q}$$

We know that $e^{2\pi i n q} = 1$ for all $n \in \mathbb{Z}$, and if we make a index shift $m = n + p$, then we get

$$= e^{-\pi i p^2\tau - 2\pi i p z} \sum_{m\in\mathbb{Z}} e^{(\pi i m^2\tau + 2\pi i m z)}.$$

For any $\lambda = p\tau + q \in \Lambda$, we have

$$\theta(z+\lambda,\tau) = e^{-\pi i p^2\tau - 2\pi i p z}\theta(z,\tau).$$

So $\theta(z,\tau)$ is a theta function with the theta factor

$$e_{p\tau+q}(z) = e^{-\pi i p^2\tau - 2\pi i p z}.$$

This theta function is called **the Riemann theta function**.

Riemann theta function can be also rewritten as :

$$\theta(z,\tau) = 1 + 2e^{\pi i\tau}\cos(2z) + 2e^{4\pi i\tau}\cos(4z) + 2e^{9\pi i\tau}\cos(6z) + \ldots + 2e^{n^2\pi i\tau}\cos(2nz) + \ldots$$

Example 2.2.2. $\nu(z,\tau) = 2e^{\frac{1}{4}}\cos(z) + 2e^{\frac{9}{4}}\cos(3z) + 2e^{\frac{25}{4}}\cos(5z) + \ldots + 2e^{\frac{(2n+1)^2}{4}}\cos((2n+1)z) + \ldots.$

ν is also a theta function. We invite the reader to check that the conditions.

Lemma 2.1. There are two relations between functions θ and ν.

$$\theta(z,\tau) = e^{\frac{\pi i\tau}{4}} e^{iz}\nu(z + \frac{\pi\tau}{2}, \tau),$$

$$\nu(z,\tau) = e^{\frac{\pi i\tau}{4}} e^{iz}\theta(z + \frac{\pi\tau}{2}, \tau). \tag{2.3}$$

Proof. For the equation (2.3), calculate the $\nu(z+\dfrac{\pi\tau}{2},\tau)$

$$\begin{aligned}
\nu(z+\frac{\pi\tau}{2},\tau) &= 2e^{\frac{1}{4}}\cos(z+\frac{\pi\tau}{2}) + 2e^{\frac{9}{4}}\cos(3z+\frac{3\pi\tau}{2}) + 2e^{\frac{25}{4}}\cos(5z+\frac{5\pi\tau}{2}) + ... \\
&= e^{\frac{\pi i\tau}{4}}(e^{iz+\frac{\pi i\tau}{2}} + e^{-iz-\frac{\pi i\tau}{2}}) + e^{\frac{9\pi i\tau}{4}}(e^{3iz+\frac{3\pi i\tau}{2}} + e^{-3iz-\frac{3\pi i\tau}{2}}) + ...
\end{aligned}$$

By multiplying both sides with $e^{\frac{\pi i\tau}{4}}$, we get

$$\begin{aligned}
&= e^{\frac{\pi i\tau}{2}}(e^{iz+\frac{\pi i\tau}{2}} + e^{-iz-\frac{\pi i\tau}{2}}) + e^{\frac{5\pi i\tau}{2}}(e^{3iz+\frac{3\pi i\tau}{2}} + e^{-3iz-\frac{3\pi i\tau}{2}}) \\
&\quad + e^{\frac{13\pi i\tau}{2}}(e^{3\ iz+\frac{3\pi i\tau}{2}} + e^{-3iz-\frac{3\pi i\tau}{2}}) + ... \\
&= (e^{iz+\pi i\tau} + e^{-iz}) + (e^{3iz+4\pi i\tau} + e^{-3iz+\pi i\tau}) + (e^{5iz+9\pi i\tau} + e^{-5iz+4pii\tau}) + ...
\end{aligned}$$

Now mutliply both sides with e^{iz}, then finally we get

$$\begin{aligned}
&= 1 + e^{2iz+\pi i\tau} + e^{4iz+4\pi i\tau} + e^{-2iz+\pi i\tau} + e^{6iz+9\pi i\tau} + e^{-4iz+4\pi i\tau} \\
&= 1 + e^{\pi i\tau}(e^{2iz} + e^{-2iz}) + e^{4\pi i\tau}(e^{4iz} + e^{-4iz}) + ... \\
&= 1 + 2e^{\pi i\tau}\cos(2z) + 2e^{4\pi i\tau}\cos(4z) + ... \\
e^{\frac{\pi i\tau}{4}}e^{iz}\nu(z+\frac{\pi\tau}{2},\tau) &= \theta(z,\tau).
\end{aligned}$$

□

Proposition 2.3. *Riemann theta function have some transformations.*

$$\begin{aligned}
&\textbf{\textit{i)}} \quad \theta(z+1,\tau) = \theta(z,\tau), \\
&\textbf{\textit{ii)}} \quad \theta(z+\pi i,\tau) = \theta(z,\tau), \qquad (2.4) \\
&\textbf{\textit{iii)}} \quad \theta(z+\pi\tau,\tau) = e^{-\pi i\tau}e^{-2iz}\theta(z,\tau).
\end{aligned}$$

Proof. It is easy to check first equation.

$$\theta(z+1,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i(n^2\tau+2n(z+1))} = \sum_{n\in\mathbb{Z}} e^{\pi i(n^2\tau+2nz)}e^{2n\pi i} = \theta(z,\tau).$$

For the second equation, we have

$$\theta(z,\tau)=\sum_{n\in\mathbb{Z}}e^{\pi i(n^2\tau+2nz)}=1+2e^{\pi i\tau}\cos(2z)+2e^{4\pi i\tau}\cos(4z)+2e^{9\pi i\tau}\cos(6z)+...+2e^{n^2\pi i\tau}\cos(2nz)+...$$

From this equality we get

$$\begin{aligned}\theta(z+\pi,\tau) &= 1+2e^{\pi i\tau}\cos(2z+2\pi)+2e^{4\pi i\tau}\cos(4z+4\pi)+...+2e^{n^2\pi i\tau}\cos(2nz+2n\pi)+...\\ &= 1+2e^{\pi i\tau}\big(\cos(2z)\cos(2\pi)-\sin(2z)\sin(2\pi)\big)+2e^{4\pi i\tau}\big(\cos(4z)\cos(4\pi)-\sin(4z)\sin(4\pi)\big)\\ &\quad +2e^{n^2\pi i\tau}\big(\cos(2nz)\cos(2n\pi)-\sin(2nz)\sin(2n\pi)\big)+...\\ &= \theta(z,\tau).\end{aligned}$$

For the third equation, first we should start with the first equation of the equation system 2.3. We have

$$\theta(z,\tau)=e^{\frac{\pi i\tau}{4}}e^{iz}\nu(z+\frac{\pi\tau}{2}).$$

By changing z with $z+\frac{\pi\tau}{2}$, then we get

$$\nu(z+\frac{\pi\tau}{2},\tau)=e^{\frac{\pi i\tau}{4}}e^{i(z+\frac{\pi\tau}{2})}\theta(z+\pi\tau,\tau).$$

Finally multiplying both sides with $e^{\frac{\pi i\tau}{4}}e^{iz}$, we get

$$e^{\frac{\pi i\tau}{4}}e^{iz}\nu(z+\frac{\pi\tau}{2},\tau) = e^{\frac{\pi i\tau}{4}}e^{iz}e^{\frac{\pi i\tau}{4}}e^{i(z+\frac{\pi\tau}{2})}\theta(z+\pi\tau,\tau).$$

Finally using the second equation of the equation system 2.4, we have

$$\theta(z,\tau) = e^{\pi i\tau+2iz}\theta(z+\pi\tau,\tau).$$

□

Example 2.2.3. Let $\Lambda=\mathbb{Z}\pi+\mathbb{Z}\pi\tau$, where $\tau\in\mathfrak{H}$. Set

$$\theta(z,\tau)=\frac{1}{2}\sum_{n\in\mathbb{Z}}e^{i\pi n^2\tau}(e^{2inz}+e^{-2inz}).$$

This function is holomorphic on $\mathbb{C}$. If we calculate $|\theta(z,\tau)|$, we get

$$\begin{aligned}\left|\frac{1}{2}\sum_{n\in\mathbb{Z}}e^{i\pi n^2\tau}(e^{2inz}+e^{-2inz})\right| &\leq \frac{1}{2}\sum_{n\in\mathbb{Z}}\big(|e^{i\pi n^2(a+bi)}||e^{2\pi inz}+e^{-2inz}|\big)\\ &\leq \frac{1}{2}\sum_{n\in\mathbb{Z}}\big(|e^{i\pi n^2a}||e^{-\pi bn^2}||\frac{e^{4inz}+1}{e^{2inz}}|\big).\end{aligned}$$

Since a is a real number, $|e^{i\pi n^2a}|=1$.

Thus we should only check the convergence of the serie $\sum_{n\in\mathbb{Z}} |e^{-\pi bn^2}||\frac{e^{4inz}+1}{e^{2inz}}|$. If we fix $z \in \mathbb{C}$, then there exists a positive real number M such that $|\frac{e^{4inz}+1}{e^{2inz}}| \leq M$. So we have

$$|\theta(z,\tau)| \leq \frac{M}{2}\sum_{n\in\mathbb{Z}} |e^{-\pi bn^2}|$$

$$\sum_{n\in\mathbb{Z}} |e^{-\pi bn^2}| = 1 + 2.\sum_{n=1}^{\infty} |e^{-\pi bn^2}|.$$

If we apply the d'Alembert's ratio test, same as example (3.4.1), we'll see that $\theta(z,\tau)$ converges, when z is fixed.

On the other hand, if we fix $\tau \in \mathfrak{H}$, for any $z \in \mathbb{C}$, there exists a positive real number M such that $|z| \leq M$, as a result of this inequality, we have $|\frac{e^{4inz}+1}{e^{2inz}}| \leq |\frac{e^{4inM}+1}{e^{2inM}}|$. So $|\theta(z,\tau)| \leq \sum_{n\in\mathbb{Z}} |e^{-\pi bn^2}||\frac{e^{4inM}+1}{e^{2inM}}|$.

$$|\theta(z,\tau)| \leq \sum_{n\in\mathbb{Z}} |e^{-\pi bn^2}||\frac{e^{4inz}+1}{e^{2inz}}| = 1 + 2\sum_{n=1}^{\infty} |e^{-\pi bn^2}|\frac{e^{4inM}+1}{e^{2inM}}.$$

Same as exapmple (3.4.1), if we apply the d'Alembert's ratio test, we get $\theta(z,\tau)$ converges when τ is fixed. Thus $\theta(z,\tau)$ is convergent.

To find theta factor of this function, first we should calculate $\theta(z + a\pi, \tau)$ where $a\pi$ is an element of Λ.

$$\begin{aligned}\theta(z + a\pi, \tau) &= \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau}(e^{2\pi in(z+a\pi)} + e^{-2in(z+a\pi)}) \\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau}(e^{2inz}e^{2ina\pi} + e^{-2inz}e^{-2ina\pi}).\end{aligned}$$

Since n and a are integers, $e^{2ina\pi}$ and $e^{-2ina\pi}$ are equal to $(-1)^a$. So we have

$$\begin{aligned}\theta(z + a\pi, \tau) &= \frac{1}{2}\sum_{n\in\mathbb{Z}} (-1)^a e^{i\pi n^2\tau}(e^{2inz} + e^{-2inz}) \\ &= (-1)^a\theta(z,\tau). \qquad (2.5)\end{aligned}$$

Now we calculate $\theta(z + \pi\tau)$ where $\pi\tau$ is an element of Λ.

$$\begin{aligned}\theta(z + \pi\tau, \tau) &= \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau}(e^{2\pi in(z+\pi\tau)} + e^{-2in(z+\pi\tau)}) \\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau}(e^{2inz}e^{2in\pi\tau} + e^{-2inz}e^{-2in\pi\tau}).\end{aligned}$$

If we multiply both sides with $(e^{\pi i\tau})e^{2iz}$, we get

$$
\begin{aligned}
e^{\pi i\tau}e^{2iz}\theta(z+\pi\tau,\tau) &= \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau}e^{\pi i\tau}e^{2inz}(e^{2inz}e^{2in\pi\tau}+e^{-2inz}e^{-2in\pi\tau}) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau}(e^{2inz+2in\pi\tau+\pi i\tau+2iz}+e^{-2inz-2in\pi\tau+\pi i\tau+2iz}e^{-2in\pi\tau}).
\end{aligned}
$$

If we simplify the equation we get

$$
\theta(z+\pi\tau,\tau) = e^{-\pi i\tau}e^{-2iz}\theta(z,\tau). \tag{2.6}
$$

Finally, to find the theta factor, for any element $\lambda = a\pi + b\pi\tau$ of Λ, we calculate $\theta(z+a\pi+b\pi\tau,\tau)$. Using equation 2.5, we can see that

$$
\begin{aligned}
\theta(z+a\pi+b\pi\tau,\tau) &= (-1)^a\theta(z+b\pi\tau,\tau) \\
&= (-1)^a\theta(z+(b-1)\pi\tau+\pi\tau,\tau).
\end{aligned}
$$

Using the equation 2.6 we get

$$
\begin{aligned}
\theta(z+a\pi+b\pi\tau,\tau) &= (-1)^a e^{-\pi i\tau}e^{-2iz}\theta(z+(b-1)\pi\tau) \\
&= (-1)^a e^{-\pi i\tau}e^{-2iz}\theta(z+(b-2)\pi\tau+\pi\tau,\tau) \\
&= (-1)^{a+1}(e^{-\pi i\tau}e^{-2iz})^2\theta(z+(b-2)\pi\tau,\tau).
\end{aligned}
$$

If we simplify this equation b times with the same method, we get

$$
\theta(z+a\pi+b\pi\tau,\tau) = (-1)^{a+b}(e^{-\pi i\tau}e^{-2iz})^b\theta(z,\tau).
$$

So theta factor of this theta function is $(-1)^{a+b}(e^{-\pi i\tau}e^{-2iz})^b$.

We invite the reader to check that for $\Lambda = \mathbb{Z}\pi + \mathbb{Z}\pi\tau$, where $\tau \in \mathfrak{H}$.

$$
\begin{aligned}
\theta_1(z,\tau) &= \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{(n+\frac{1}{2})^2\pi\tau}(e^{(2n+1)iz}+e^{-(2n+1)iz}) \\
\theta_2(z,\tau) &= \frac{1}{2}\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi in^2\tau}(e^{2inz}+e^{-2inz}).
\end{aligned}
$$

are also theta functions with theta factors $e_{\lambda_1} = (-1)^a(e^{-\pi i\tau}e^{-2iz})^b$ and $e_{\lambda_2} = (-e^{-\pi i\tau}e^{-2iz})^b$, respectively, see (Bellman, 1961).

Remark 2.1. Theta factor $e_\lambda(z)$ of Riemann theta function satisfies the following condition :

$$e_{\lambda+\lambda'}(z) = e_\lambda(z+\lambda')e_{\lambda'}(z).$$

Indeed ; let $\lambda = p\tau + q$ and $\lambda' = p'\tau + q'$ be two elements of Λ. We have

$$\begin{aligned} e_{\lambda+\lambda'}(z) = e_{(p+p')\tau+(q+q')}(z) &= e^{-\pi i(p+p')^2\tau-2\pi i(p+p')z} \\ &= e^{-\pi ip^2\tau-2\pi ipp'\tau-\pi i(p')^2\tau-2\pi ipz-2\pi ip'z} \\ &= e^{-\pi ip^2\tau-2\pi ipz-2\pi ipp'\tau}.e^{-\pi i(p')^2\tau-2\pi ip'z}. \end{aligned}$$

Since $p, q \in \mathbb{Z}$ then $e^{-2\pi ipq} = 1$. If we multiply both sides of equation with $e^{-2\pi ipq}$, then we get

$$\begin{aligned} &= e^{-\pi ip^2\tau-2\pi ipz-2\pi ipp'\tau-2\pi ipq}.e^{-\pi i(p')^2\tau-2\pi ip'z} \\ &= e^{-\pi ip^2\tau-2\pi ip(z+p'\tau+q)}.e^{-\pi i(p')^2\tau-2\pi ip'z} \\ &= e_\lambda(z+\lambda').e_{\lambda'}(z). \end{aligned}$$

More generally, we have the following :

Lemma 2.2. All theta factors with respect to the lattice Λ, satisfies the following condition

$$e_{\lambda+\lambda'}(z) = e_\lambda(z+\lambda')e_\lambda(z).$$

Proof. Let θ be theta function with theta factor e_λ. So $\theta(z+\lambda) = e_\lambda(z)\theta(z)$ for any $\lambda \in \Lambda$.

For any $\lambda' \in \Lambda$, we can find $\theta(z+(\lambda+\lambda')) = e_{\lambda+\lambda'}(z)\theta(z)$ and also $\theta((z+\lambda)+\lambda') = e_{\lambda'}(z+\lambda)\theta(z+\lambda)$. If we equalize two equations then we have

$$e_{\lambda+\lambda'}(z)\theta(z) = e_{\lambda'}(z+\lambda)\theta(z+\lambda)$$

$$e_{\lambda+\lambda'}(z)\theta(z) = e_{\lambda'}(z+\lambda)e_\lambda(z)\theta(z)$$

$$e_{\lambda+\lambda'}(z) = e_{\lambda'}(z+\lambda)e_\lambda(z).$$

□

Theorem 2.3. *Let $\phi(z) \in \mathcal{O}^*(\mathbb{C})$ be holomorphic invertible function on $\mathbb{C}$. For any theta function $\theta(z)$ with theta factor $e_\lambda(z)$, the function $f(z)\phi(z)$ is also theta func-*

tion with theta factor $\dfrac{e_\lambda(z)\phi(z+\lambda)}{\phi(z)}$.

Proof. Denote $g(z) = f(z)\theta(z)$. Then for any $\lambda \in \Lambda$ we have

$$\begin{aligned} g(z+\lambda) &= f(z+\lambda)\theta(z+\lambda) \\ &= e_\lambda(z)f(z)\theta(z+\lambda) \\ &= e_\lambda(z)\theta(z+\lambda)\frac{f(z)\phi(z)}{\phi(z)} \\ &= \frac{e_\lambda(z)\phi(z+\lambda)g(z)}{\phi(z)}. \end{aligned}$$

Thus $g(z) = f(z)\phi(z)$ is a theta function with the theta factor $\dfrac{e_\lambda(z)\phi(z+\lambda)}{\phi(z)}$. □

Definition 2.1. Two theta functions of $\{e_\lambda\}$ are called equivalent if they either related by some holomorphic invertible function $\phi(z)$ or are obtained from each other by translation of the argument $z \to z + \lambda$.

Theorem 2.4. *This is an equivalance relation.*

Proof. Let $e_\lambda(z)$ and $\tilde{e}_\lambda(z)$ two theta factors of $\{e_\lambda\}$. We say $e_\lambda \sim \tilde{e}_\lambda(z)$, if either

$$\tilde{e}_\lambda(z) = \frac{\phi(z+\lambda)}{\phi(z)} \tag{2.7}$$

or

$$\tilde{e}_\lambda(z) = e_\lambda(z+a). \tag{2.8}$$

To show this is an equivalence relation, we must check three conditions. In 2.7 pick $\phi(z) = c$ for all $c \in \mathbb{C}^*$ then we have $\tilde{e}_\lambda(z) = e_\lambda(z)$.

In 2.8 pick $a = 0$, then we have $\tilde{e}_\lambda(z) = e_\lambda(z)$. So this relation is reflexive.

In 2.7, if we choose a holomorphic invertible function $\psi(z) = \dfrac{1}{\phi(z)}$, then we have

$$\tilde{e}_\lambda(z) = \frac{\frac{1}{\psi(z+\lambda)}}{\frac{1}{\psi(z)}}.e_\lambda = \frac{\psi(z)}{\psi(z+\lambda)}.e_\lambda$$

$$e_\lambda = \frac{\psi(z+\lambda)}{\psi(z)}.\tilde{e}_\lambda(z).$$

In 2.8, $\tilde{e}_\lambda(z + (-a)) = e_\lambda(z + (-a) + a) = e_\lambda(z)$. If we denote $(-a) = b$, then we have $\tilde{e}_\lambda(z+b) = e_\lambda(z)$.

To check transitivity, we'll investigate four different cases.

If $\tilde{e}_\lambda(z) = e_\lambda(z+a)$ and $e'_\lambda(z) = \tilde{e}_\lambda(z+b)$, then we have

$$\tilde{e}_\lambda(z+b) = e_\lambda(z+a+b) = e'_\lambda(z).$$

If $\tilde{e}_\lambda(z) = e_\lambda(z+a)$ and $e'_\lambda(z) = \dfrac{\phi(z+\lambda)}{\phi(z)}\tilde{e}_\lambda(z)$, then $e'_\lambda(z) = \dfrac{\phi(z+\lambda)}{\phi(z)}e_\lambda(z+a)$. Choose $a = 0$, then we have

$$e'_\lambda(z) = \frac{\phi(z+\lambda)}{\phi(z)}e_\lambda(z).$$

If $\tilde{e}_\lambda(z) = e'_\lambda(z+a)$ and $\tilde{e}_\lambda(z) = \dfrac{\phi(z+\lambda)}{\phi(z)}e_\lambda(z)$, then $e'_\lambda(z+a) = \dfrac{\phi(z+\lambda)}{\phi(z)}e_\lambda(z+a)$. Also choose $a = 0$, then we have

$$e'_\lambda(z) = \frac{\phi(z+\lambda)}{\phi(z)}e_\lambda(z).$$

For the last case, if $\tilde{e}_{(}z) = \dfrac{\phi(z+\lambda)}{\phi(z)}e_\lambda(z)$ and $e'_{(}z) = \dfrac{\overline{\phi}(z+\lambda)}{\overline{\phi}(z)}\tilde{e}_\lambda(z)$. If we use two equations, we have $e'_{(}z) = \dfrac{\overline{\phi}(z+\lambda)}{\overline{\phi}(z)}\dfrac{\phi(z+\lambda)}{\phi(z)}$.
Set $\delta(z) = \phi(z)\overline{\phi}(z)$ a holomorphic invertible function, then

$$e'_\lambda(z) = \frac{\delta(z+\lambda)}{\delta(z)}e_\lambda(z).$$

$\square$

2.3 Elliptic Functions via Theta Function

Definition 2.2. Let f be a meromorphic function on $\mathbb{C}$. We say f is **elliptic** if it is periodic in two $\mathbb{R}$-linearly independent directions. Formally, f is an elliptic function on $\mathbb{C}$, if there exist two complex numbers u and v with $\dfrac{u}{v} \notin \mathbb{R}$, such that $f(z+u) = f(z)$ and $f(z+v) = f(z)$ for all $z \in \mathbb{C}$. If u and v are periods of f, then all other periods of f, say w, can be written as $mu+nv$ where m and n are integers. Herein u and v are called fundamental periods of f and all elliptic functions have fundamental periods, but they are not unique. Thus we can denote lattice of periods of f by $\Lambda = \{mu+mv \mid (m,n) \in \mathbb{Z}^2\}$, it follows that $f(z) = f(z+w)$ for all $w \in \Lambda$.
Example 2.3.1. Let $\Lambda = \langle 1, \tau\rangle$ be lattice with $\tau \in \mathfrak{H}$. For all $z \in \mathbb{C}$, the Weierstraß's $\wp$ function written as

$$\wp(z;1,\tau) = \frac{1}{z^2} + \sum_{(m,n)\in\Lambda\setminus(0,0)} \left\{\frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}\right\}$$

is an elliptic function.

Example 2.3.2. Derivative of Weierstraß's elliptic function

$$\wp'(z;1,\tau) = \frac{-2}{z^3} - \sum_{(m,n)\in\Lambda\setminus(0,0)} \frac{2}{(z+m+n\tau)^3}$$

is also an elliptic function.

Theorem 2.5. *(Jones and Singerman, 1987) Given any elliptic function f on $\mathbb{C}$ periodic with respect to the lattice $\Lambda = \langle 1, \tau\rangle$ (i.e. given any meromorphic function f periodic with respect to 1 and τ), there are rational functions R_1 and R_2 in $\mathbb{C}(t)$ such that f can be written as*

$$f = R_1(\wp) + \wp' R_2(\wp).$$

Remark 2.2. Suppose F and G are two theta functions with respect to lattice $\Lambda = \langle 1, \tau\rangle$, with the same theta factors. Then by definition for any $\lambda \in \Lambda$ we have

$$\begin{aligned} F(z+\lambda,\tau) &= e_\lambda(z)F(z,\tau) \\ G(z+\lambda,\tau) &= e_\lambda(z)G(z,\tau). \end{aligned}$$

By dividing two equations we get

$$\frac{F}{G}(z+\lambda,\tau) = \frac{F}{G}(z,\tau) \quad \text{for all} \quad z\in\mathbb{C} \quad \text{and} \quad \lambda\in\Lambda.$$

Denote $f = \dfrac{F}{G}$, so f is periodic with respect to Λ. By the theorem 2.5, there must exist some rational functions R_1 and R_2 in $\mathbb{C}$ such that

$$\frac{F}{G} = R_1(\wp) + \wp' R_2(\wp).$$

Theorem 2.6. *We define new function $\theta^{(x)} = \theta(z - (\frac{1}{2}) - (\frac{\tau}{2}) - x)$ where θ is Riemann theta function. Fix a positive integer r and choose any two sets of r complex numbers $\{x_i\}$ and $\{y_j\}$ such that $\sum_i x_i - \sum_j y_j$ is an integer. Then the ratio of translated theta functions*

$$R(z) = \frac{\prod_i x_i}{\prod_j y_j}$$

is a meromorphic Λ-periodic function on $\mathbb{C}$.

Proof. The function $\theta^{(x)}(z)$ has zeroes at points $x+\Lambda$. Note that

$$\theta^{(x)}(z+1) = \theta(z) \quad \textit{and} \quad \theta^{(x)}(z+\tau) = -e^{2\pi i(z-x)}\theta(z)$$

Define a new function $R(z)$ of the ratio

$$R(z) \;=\; \frac{\prod_i \theta^{(x_i)}(z)}{\prod_j \theta^{(y_j)}(z)}.$$

It is clear that this function is meromorphic and periodic on $\mathbb{C}$, i.e. $R(z+1) = R(z)$. So it must be Λ-periodic if and only if $R(z+\tau) = R(z)$. But

$$
\begin{aligned}
R(z+\tau) &= \frac{\prod_{i}^{m} \theta^{(x_i)}(z+\tau)}{\prod_{j}^{n} \theta^{(y_j)}(z+\tau)} \\
&= (-1)^{m-n} \frac{\prod_{i=1}^{m} e^{-2\pi i(z-x_i)} \theta^{(x_i)}(z)}{\prod_{j=1}^{n} e^{-2\pi i(z-y_j)} \theta^{(y_j)}(z)} \\
&= (-1)^{m-n} e^{-2\pi i[(m-n)z + \sum_j y_j - \sum_i x_i]} R(z).
\end{aligned}
$$

Thus extra factor $(-1)^{m-n} e^{-2\pi i[(m-n)z + \sum_j y_j - \sum_i x_i]}$ have to be 1 for all z in $\mathbb{C}$ so m must be equal to n and if so, this factor is 1 if and only if

$$
\sum_i x_i - \sum_j y_j \in \mathbb{Z}.
$$

$\square$

2.4 Modular Forms and Theta Functions

Definition 2.3. A **modular form** of weight k for the modular group $\mathrm{SL}_2(\mathbb{Z})$ is a complex function f on $\mathfrak{H}$ such that :

1) f is holomorphic on $\mathfrak{H}$.

2) For any z in $\mathfrak{H}$ and any matrix $\gamma = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$, the equation $f(\frac{az+b}{cz+d}) = (cz+d)^k\, f(z)$ is required to hold.

3) f is holomorphic at $z = \infty$.

To define holomorphicity at ∞, we introduce $q = e^{2\pi i z}$, and note that $q = e^{2\pi i z} = e^{2\pi i(x+iy)} = e^{2\pi i x} e^{-2\pi y}$. And if we compute $|q| = |e^{2\pi i x}||e^{-2\pi y}| = e^{-2\pi y}$. If $y \to \infty$ (i.e. $Im(z) \to \infty$), then $q \to 0$. This means $Im(z) \to \infty$. We say that a function $f\ :\ \mathfrak{H} \to \mathbb{C}$ is holomorphic at ∞, if $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ is holomorphic at $q = 0$.

Example 2.4.1. For any $c \in \mathbb{C}$, the constant function $f(z) = c$ is a modular form.

Indeed :

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= 0, \\ f(\frac{az+b}{cz+d}) &= c = f(z), \\ \lim_{Im(z)\to\infty} f(z) &= c < \infty. \end{aligned}$$

So f is a modular form of weight 0.

Remark 2.3. Let f is a modular form of weight k.
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$, so we have $f(z) = f(z+1)$. We denote T for this matrix.
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$, so we have $f(\frac{-1}{z}) = z^k f(z)$. We denote S for this matrix.
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$, so we have $f(z) = (-1)^k f(z)$. This equality shows uS if weight of modular form is odd, then it is equal to 0.

Theorem 2.7. *The group $SL_2(\mathbb{Z})$ is generated by;*
$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Example 2.4.2. Consider Eisenstein Series $G_n(z) = \sum\limits_{(a,b)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(az+b)^n}$.

For $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{Z})$

$$\begin{aligned} G_n(\frac{pz+q}{rz+s}) &= \sum_{(a,b)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(a(\frac{pz+q}{rz+s})+b)^n} \\ &= \sum_{(a,b)\in\mathbb{Z}^2\setminus(0,0)} \frac{(rz+s)^n}{(a(pz+q)+b(rz+s))^n} \\ &= (rz+s)^n \sum_{(a,b)\in\mathbb{Z}^n\setminus(0,0)} \frac{1}{(a(pz+q)+b(rz+s))^n} \end{aligned}.$$

Set $pa + rb = A$ and $qa + sb = B$. Now we define a function ϕ such that :

$$\begin{aligned} \phi \quad : \mathbb{Z}^2/\{(0.0)\} &\to \mathbb{Z}^2/\{(0.0)\} \\ (c,d) &\mapsto (pa+rb, qa+sb) \end{aligned}.$$

We know $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Suppose that gcd(p,r)=g. It means g/p and g/r and

$ps - qr = gp's - gr'q = g(p's - r'q) = 1$. So $g/1 \Rightarrow g = 1$. It is same for q and s gcd(q,s)=1. So ϕ is surjective.

Let (x, y) and (x', y') two elements of $\mathbb{Z}^2/\{(0.0)\}$. Suppose $\phi(x, y) = \phi(x', y')$

$$\begin{array}{lcl} (px + ry, qx + sy) & = & (px' + ry', qx' + sy'), \\ A' = px + ry = px' + ry' & \& & B' = qx + sy = qx' + sy', \\ p(x - x') = r(y' - y) & \& & q(x - x') = s(y' - y), \\ gcd(p, r) = 1 & \& & gcd(q, s) = 1, \\ p/y - y' & \& & r/x - x', \\ q/y - y' & \& & s/x - x', \end{array}$$

Thus $rl = x - x', sn = x - x', pk = y - y', qm = y - y'$ for $k, l, m, n \in \mathbb{R}$

$A' = p(x' + rl) + r(pk + y') \quad \Rightarrow px' + ry' + prl + prk = px' + ry'$. So $rp(k + l) = 0$.

$B' = q(sn + x') + s(qm + y') \Rightarrow qsn + qx' + sqm + sy' = qx' + sy'$. So $qs(m + n) = 0$.

If $l + k = 0$, then $l = -k$ and $pk = y - y'$, $-rk = x - x'$. It means p and r are divisible by k. It is contradiction, because gcd(p,r)=1. So $l + k \neq 0$. Similar method can be applied to m and n gives $m + n \neq 0$. These results show that $pr = 0$ and $qs = 0$.

If $p = 0$, then $y = y'$ and $q \neq 0 \Rightarrow s = 0 \Rightarrow x = x'$.

If $r = 0$, then $x = x'$ and $s \neq 0 \Rightarrow q = 0 \Rightarrow y = y'$.

Thus ϕ is injective.

$G_n(\dfrac{pz + q}{rz + s}) = (rz + s)^2 \sum\limits_{(A,B) \in \mathbb{Z}^2} \dfrac{1}{(Az + B)^2}$.

So $G_n(z)$ is modular form of weight n.

Remark 2.4. To check $f : \mathfrak{H} \to \mathbb{C}$ is modular form of weight k, it suffices to check $f(z + 1) = f(z)$ and $f(\frac{-1}{z}) = z^k f(z)$.

In example 2.4.2, to show that Eisenstein Series is modular form, it would be enough to check two transformations by using 2.4.

First check $G_n(z + 1) = \sum\limits_{(a,b) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(az + a + b)^n}$. Since $(a, b) \in \mathbb{Z}^2$, we can change the variables $(k, l) \mapsto (a, a + b)$. Then we get;

$$G_n(z + 1) = \sum_{(k,l) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(kz + l)^n} = G_n(z).$$

Secondly, we check $G_n(\frac{-1}{z})$.

$$\begin{aligned} G_n(\frac{-1}{z}) &= \sum_{(a,b)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(a(\frac{-1}{z})+b)^n} \\ &= \sum_{(a,b)\in\mathbb{Z}^2\setminus(0,0)} \frac{z^n}{(-a+bz)^n} \\ &= z^n \sum_{(a,b)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(bz-a)^n}. \end{aligned}$$

Also changing variables $(k,l)\mapsto(b,-a)$ then we get

$$\begin{aligned} G_n(\frac{-1}{z}) &= z^n \sum_{(k,l)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(kz+l)^n} \\ &= z^n G_n(z) \end{aligned}$$

Theorem 2.8. *Eisenstein series $G_n(z) = \sum_{(a,b)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(az+b)^n}$ converge absolutely and uniformly on upper half plane, when $k>2$.*

Proof. Let $z = x+yi$ be an element of $\mathfrak{H}$, such that $|x|\le p, y\ge r$. Then we have the inequality

$$|(az+b)^2| = (ax+b)^2 + a^2y^2 \ge (ax+b)^2 + a^2q^2.$$

If we fix a and b, we'll analyse two cases; the first case is $|b|\le 2p|a|$and the second is $|b|\ge 2p|a|$. From the first case, we have

$$|az+b|^2 \ge a^2q^2 \ge \frac{q^2}{2}a^2 + \frac{q^2}{2(2p)^2} \ge min\{\frac{q^2}{2},\frac{q^2}{8p^2}\}(a^2+b^2).$$

From the second case, if we apply triangle inequality, then we have

$$|az+b|^2 \ge (|ax|-|b|)^2 + a^2q^2 \ge \big(\frac{|b|}{2}\big)^2 + a^2q^2 \ge min\{\frac{1}{4},q^2\}(a^2+b^2).$$

Associating both cases and putting

$$c = min\{\frac{q^2}{2},\frac{q^2}{8p^2},\frac{1}{4},q^2\},$$

we get the inequality

$$|az+b| \ge c(a^2+b^2)^{\frac{1}{2}} \qquad for\ all\ a,b\in\mathbb{Z},\ z\in\mathfrak{H}.$$

This inequality implies that for any $z\in\mathfrak{H}$, we have

$$|G_n(z)| \leq \frac{1}{c^n} \sum_{(a,b)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(a^2+b^2)^{1/2}}.$$

We rearrange the sum by grouping together for each fixed $k = 1, 2, 3, ...$ all pairs (a, b) with $max = \{|a|, |b|\} = k$. Note that for each k pairs, there are $8k$ pairs (a, b), each of which satisfies

$$k^2 \leq (a^2 + b^2)^2.$$

From this inequality we get

$$\begin{aligned} |G_n(z)| &\leq \sum_{k=1}^{\infty} \frac{1}{c^n} \\ &= \frac{8}{c^n} \sum_{k=1}^{\infty} \frac{1}{k^{n-1}} \end{aligned}$$

which is finite and independent of z. $\square$

So far, in this chapter, we have been occupied with the behaviour of $\theta(z, \tau)$ for its first component. But, it has also a behaviour as a function of τ. θ is periodic up to a factor for a group acting z and τ. If we have two generators of $\Lambda_\tau = \langle 1, \tau \rangle$, written as $a\tau + b$ and $c\tau + d$, we could have made theta functions. They were periodic with respect to $z \mapsto z + c\tau + d$ and up to a factor for $z \mapsto z + a\tau + b$ which is exponential. Then the new theta functions don't be very different from originals. Fix any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and assume that multiplication of each columns are even. That is to say ab and cd are even.

Now, we consider the function $\theta((c\tau + d)w, \tau)$. When we replace w by $w + 1$, the function does not change except for a factor which is exponential. It is easy to find out an factor e^{Aw^2}. This factor corrects $\tau((c\tau + d)w, \tau)$ to a periodic function $w \mapsto w + 1$. Actually, let

$$\upsilon(w, \tau) = e^{\pi i c(c\tau+d)w^2} \theta((c\tau + d)w, \tau).$$

Then if we make some simple calculations, we would have

$$\upsilon(w + 1, \tau) = \upsilon(w, \tau).$$

On the other hand, the periodic behaviour of θ for $z \mapsto z + \tau$ gives the second period for υ,

$$\upsilon(w+\frac{a\tau+b}{c\tau+d},\tau)=e^{-\pi i\frac{a\tau+b}{c\tau+d}-2\pi iw}\upsilon(w,\tau)$$

This time we make some calculations, by definition we have

$$\frac{\upsilon(w+\frac{a\tau+b}{c\tau+d},\tau)}{\theta((c\tau+d)w+a\tau+b,\tau)} = e^{\pi ic(c\tau+d)w^2+2\pi icw(a\tau+b)+\pi ic\frac{(a\tau+b)^2}{c\tau+d}}$$

and

$$\begin{aligned}\frac{\theta((c\tau+d)w+a\tau+b,\tau)}{\upsilon(w,\tau)} &= \frac{e^{-\pi ia^2\tau-2\pi iaw(c\tau+d)}\theta((c\tau+d)w,\tau)}{e^{\pi ic(c\tau+d)^2}\theta((c\tau+d)w,\tau)}\\ &= e^{-\pi ia^2\tau-2\pi iaw(c\tau+d)-\pi ic(c\tau+d)^2}.\end{aligned}$$

We know that determinant of a matrix in $SL_2(\mathbb{Z})$ is 1, so we have $ad-bc=1$, by multiplying two equations above, we get ;

$$\begin{aligned}\frac{\upsilon(w+\frac{a\tau+b}{c\tau+d},\tau)}{\upsilon(w,\tau)} &= e^{-2\pi iw(ad-bc)+\pi ic\frac{(a\tau+b)^2}{c\tau+d}-\pi ia^2\tau}\\ &= e^{-2\pi iw-\frac{\pi i}{c\tau+d}(a^2\tau(c\tau+d)-c(a\tau+b)^2)}\\ &= e^{-\pi iw-\frac{\pi i}{c\tau+d}(a^2d\tau-2abc\tau-b^2c)}.\end{aligned}$$

If we factorise the exponent, we get

$$\begin{aligned}a^2d\tau-2abc\tau-b^2c &= a(ad-bc)\tau-ab(c\tau+d)+b(ad-bc)\\ &= (a\tau+b)-ab(c\tau+d).\end{aligned}$$

We assumed that ab is even so we had what we want. On the other hand, for $\tau'=\dfrac{a\tau+b}{c\tau+d}$, the unique function invariant under $\Lambda_{\tau'}$ is $\theta(w,\tau')$. Thus, we have

$$\upsilon(w,\tau)=\psi(\tau)\theta(w,(\frac{a\tau+b}{c\tau+d}))$$

for some function $\psi(\tau)$. Say it differently, if $w=\dfrac{z}{c\tau+d}$, then

$$\theta(z,\tau)=\psi(\tau)e^{\frac{-\pi icz^2}{(c\tau+d)}}\theta(\frac{z}{(c\tau+d)},\frac{(a\tau+b)}{(c\tau+d)}).$$

For evaluating $\psi(\tau)$, note that the property "zeroth term in Fourier expansion of $\theta(z,\tau)$ is 1" is normalised $\theta(z,\tau)$. That is to say

$$\int_0^1 \theta(w,\tau)\,\mathrm{d}y = 1.$$

From here

$$\psi(\tau) = \int_0^1 \upsilon(w,\tau)\,\mathrm{d}y = \int_0^1 e^{\pi ic(c\tau+d)w^2}\theta((c\tau+d)y,\tau)\,\mathrm{d}y.$$

If we continue calculating, finally we will find that

$$\psi(\tau) = \frac{1}{c(\frac{(\tau+d)/c}{i})^{\frac{1}{2}}} \sum_{1\leq n\leq c} e^{\frac{\pi in^2 d}{c}}.$$

Theorem 2.9. *(Mumford, 2007) For all $\tau \in \mathfrak{H}$, given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with ab and cd are even, there exists a function η which is eighth root of 1, then we have*

$$(E_1)\ :\ \theta(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}) \quad = \quad \eta(c\tau+d)^{\frac{1}{2}} e^{\frac{\pi icz^2}{c\tau+d}}\theta(z,\tau).$$

By assuming $c \geq 0$ and $d > 0$, we get $Im(c\tau+d) \geq 0$ and choose $(c\tau+d)^{\frac{1}{2}}$. We evaluate two cases, to fix η, $c > 0$

— *If d is odd and c is even, then $\eta = i^{\frac{1}{2}(d+1)}(\frac{c}{|d|})$.*

— *If d is even and c is odd, then $\eta = e^{\frac{-\pi ic}{4}}(\frac{d}{c})$.*

Definition 2.4. We call **level** of a an elliptic modular form f to an integer N such that

$$\Gamma_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ \mid\ b,c \equiv 0(mod\ N), \quad a,d \equiv 1(modN) \right\}$$

here the set of Γ_N is a subgroup of $SL_2(\mathbb{Z})$.

Theorem 2.10. *(Mumford, 2007) The square of Riemann Theta function $\theta(z,\tau)$ is a modular form with weight 1 and level 4.*

3 THETA FUNCTIONS ASSOCIATED TO QUADRATIC FORMS : INDEFINITE CASE

In previous chapter, we worked on definite theta functions and their properties. In addition there are indefinite theta functions. To obtain these functions, we should define new sets. At the first section, we'll define these new sets and obtain indefinite theta functions, and then we'll mention their properties.

3.1 Basic Definitions

Let A be symmetric 2×2 matrix with integer coefficients. We consider the quadratic form $Q : \mathbb{R}^2 \rightarrow \mathbb{R}$, and the associated bilinear form $B(x,y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$ with $Q(x) = \frac{1}{2}\langle x, Ax\rangle$. Now we assume that Q has type $(1,1)$, i.e., Q is negative definite on 1 dimensional linear subspace of $\mathbb{R}^2$. Then the set of vectors $c \in \mathbb{R}^2$ with $Q(c) < 0$ has two components. If $B(c_1, c_2) < 0$, then c_1 and c_2 belong to the same component, if not then c_1 and c_2 belong to opposite components. Let C_Q be one of the two components. If c_0 is a vector in that component, then C_Q is given by : $C_Q = \{c \in \mathbb{R}^2 \ : \ Q(c) < 0,\ B(c, c_0) < 0\}$. We further define
$S_Q = \{c = (c_1, c_2) \in \mathbb{Z}^2 \ : pgcd(c_1, c_2) = 1,\ Q(c) = 0, B(c, c_0) < 0\}$.

Example 3.1.1. In previous example, we see that the bilinear form associated to the quadratic form Q which is defined by the matrix A in the Example 1.3.1 is $B(X,Y) = x_1y_1 - x_1y_2 - x_2y_1 - 3x_2y_2$.

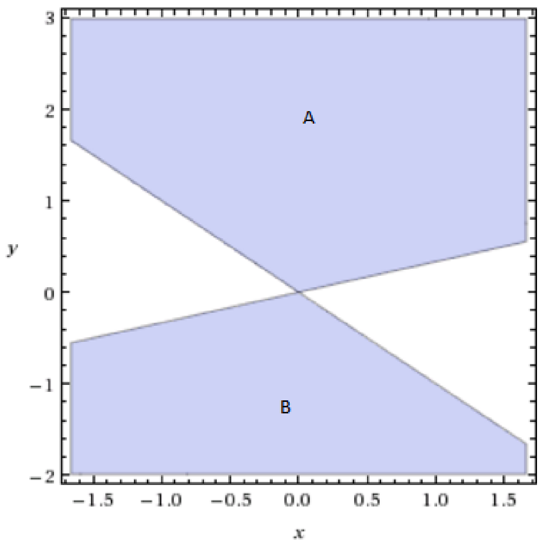

Figure 3.1: Negative Definite Areas of $Q(c)$

$Q(c)$ is negative definite for $c \in A$ or $c \in B$ where A is the set of following system of inequalities

$$\{x + y > 0,\ x - 3y < 0\}$$

and B is the set of solutions of the

$$\{x + y < 0,\ x - 3y > 0\}.$$

Set A and set B to be the shaded areas in Figure 3.1. Let $c_1 = (0, 2)$ and $c_2 = (0, -1)$, then we get $B(c_1, c_2) = 6 > 0$. It is clear to see in the picture, $c_1 \in A$ and $c_2 \in B$ so they belong to opposite components. And if we choose $c_1 = (0, 2)$ and $c_2 = (0, 3)$, then we get $B(c_1, c_2) = -18 < 0$. Both c_1 and c_2 belong to same component A. Thus we get $C_Q = A \cup B$.

Let $c_0 \in C_Q$ and $Q(c) = 0$, so c is the boundary point. For this example there are only four boundary vectors which are primitive and satisfy $B(c, c_0) < 0$. So $S_Q = \{(1, -1), (-1, 1), (3, 1), (-3, -1)\}$.

For some cases S_Q can be empty, for example $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Further we put $\overline{C}_Q :=$ $C_Q \cup S_Q$ and for $c \in \overline{C}_Q$ we define

$$R(c) = \begin{cases} \mathbb{R}^2 & \text{if } c \in C_Q; \\ \{a \in \mathbb{R}^2 : B(c, a) \notin \mathbb{Z}\} & \text{if } c \in S_Q. \end{cases}$$

and

$$D(c) := \{(z, \tau) \in \mathbb{C}^2 \times \mathcal{H} \ : \ \left(\frac{Im(z_1)}{Im(\tau)}, \frac{Im(z_2)}{Im(\tau)}\right) \in R(c)\},$$

where $z = (z_1, z_2)$ is an element of $\mathbb{C}^2$.

Example 3.1.2. In example 3.1.1, if we choose $c \in A \cup B$, then we have $R(c) = \mathbb{R}^2$. On the other hand, if we choose $c \in S_Q$, for instance $c = (3, 1)$, then

$$\begin{aligned} R((3,1)) &= \{a = (a_1, a_2) \in \mathbb{R}^2 : B((3,1), (a_1, a_2)) \notin \mathbb{Z}\} \\ &= \{a = (a_1, a_2) \in \mathbb{R}^2 : B((3,1), (a_1, a_2)) = (2a_1 - 6a_2) \notin \mathbb{Z}\}. \end{aligned}$$

So, all elements in $\mathbb{Z}^2$ is not an element of $R((3, 1))$, however $(3, \frac{2}{7})$ is an element of $R((3, 1))$.

Definition 3.1. *Let $c_1, c_2 \in \overline{C}_Q$. For $(z,\tau) \in D(c_1) \cap D(c_2)$, we say $\theta_{a,b}(z,\tau)$ theta function of the indefenite quadratic form Q, with respect to (c_1, c_2), such that*

$$\begin{aligned} \theta(z,\tau) &= \theta_{a,b}^{c_1,c_2}(z,\tau) := e^{-2\pi i Q(a)\tau - 2\pi i B(a,b)}\theta_{a,b} \\ &= \sum_{n\in\mathbb{Z}^2} \rho(n+a,\tau) e^{2\pi i Q(n)\tau + 2\pi i B(n,z)}, \end{aligned}$$

where $\rho(\nu,\tau)$ is defined by

$$\rho(\nu,\tau) = \rho^{c_1,c_2}(\nu,\tau) := \rho^{c_1}(\nu,\tau) - \rho^{c_2}(\nu,\tau)$$

with

$$\rho^c(\nu,\tau) = \begin{cases} E(\frac{B(\nu,\tau)\sqrt{Im(\tau)}}{\sqrt{-Q(c)}}) & if\ c \in C_Q; \\ sgn(B(c,\nu)) & if\ c \in S_Q. \end{cases}$$

where $E(z) = 2\int_0^z \mathrm{e}^{-\pi u^2} du$.
And with $a, b \in \mathbb{R}^2$ defined by $z = a\tau + b$, so $a = \frac{Im(z)}{Im(\tau)}$ and $b = \frac{Im(\bar{z}\tau)}{Im(\tau)}$

Example 3.1.3. For all $X = (x_1, x_2) \in \mathbb{Z}^2$, consider the indefinite quadratic form $Q(X) = x_1^2 - x_1x_2 - x_2^2$. For all $Y = (y_1, y_2) \in \mathbb{Z}^2$, the bilinear form associated to this quadratic form is $B(X,Y) = 2x_1y_1 - 2x_2y_2 - x_1y_2 - x_2y_1$. Let $c_1 = (0,2)$ and $c_2 = (3,0)$ be two elements of C_Q. By definition

$$\begin{aligned} \rho^{c_1,c_2}(n+a,\tau) &= \rho^{c_1}(n+a,\tau) - \rho^{c_2}(n+a,\tau) \\ \rho^{c_1}(n+a,\tau) &= E(\frac{B(c_1,n+a)\sqrt{Im(\tau)}}{\sqrt{-Q(c_1)}}) \\ \rho^{c_2}(n+a,\tau) &= E(\frac{B(c_2,n+a)\sqrt{Im(\tau)}}{\sqrt{-Q(c_2)}}). \end{aligned}$$

Since the function E is defined as an integral, ρ^{c_i} are real valued functions with respect to n and c_i with $i = 1,2$. So say $\rho^{c_1}(n+a,\tau) = A_{n,c_1}$ and $\rho^{c_2}(n+a,\tau) = A_{n,c_2}$. Since c_1 and c_2 are elements of C_Q, $D(c) = \mathbb{C}^2 \times \mathfrak{H}$. Let $(z,\tau) = (z_1, z_2; \tau)$ be an element of $D(c)$, then theta function of quadratic form Q with respect to c_1 and c_2 is

$$\theta_{a,b}^{c_1,c_2} = \sum_{n\in\mathbb{Z}^2} (A_{n,c_1} - A_{n,c_2}) e^{2\pi i Q(n)\tau + 2\pi i B(n,z)}.$$

If we calculate Q and B we get

$$\theta_{a,b}^{c_1,c_2} = \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} (A_{n,c_1} - A_{n,c_2}) e^{2\pi i(n_1^2 - n_1 n_2 - n_2^2)\tau + 2\pi i(2n_1 z_1 - 2n_2 z_2 - n_1 x_2 - n_2 z_1)}.$$

3.2 Properties of Indefinite Theta Function

In this section, we will give some properties and some transformations of indefinite theta functions. To be able to prove them, first we will give a lemma.

Lemma 3.1. Let f be any holomorphic function. $\int_0^z f(u)du = -\int_0^{-z} f(u)du$ if f is even.

Proof. Denote $F(z) = \int_0^z f(u).du$ so $F(-z) = \int_0^{-z} f(u)du$. By replacing u with $-t$, we get

$$F(-z) = \int_0^z f(-t)(-dt) = \int_0^z -f(t)dt = -F(z).$$

So since $F(z) = -F(-z)$, we have proved the lemma. $\square$

Proposition 3.1. *The indefinite theta functions satisfy :*

1) For $(z,\tau) \in D(c_1) \cap D(c_2) \cap D(c_3)$ and c_1, c_2, c_3 are elements of $\overline{C}_Q$, we have $\theta^{c_1,c_2} + \theta^{c_2,c_1} = 0$ and $\theta^{c_1,c_2} + \theta^{c_2,c_3} + \theta^{c_3,c_1} = 0$.

2)For all $\nu \in A^{-1}\mathbb{Z}^2$ and $\lambda \in \mathbb{Z}^2$, we have $\theta(z+\lambda\tau+\nu,\tau) = e^{-2\pi i Q(\lambda)\tau - 2\pi i B(z,\lambda)}\theta(z,\tau)$.

3)$\theta(-z,\tau) = -\theta(z,\tau)$.

4) θ^{c_1,c_2} is continuous function on $C_Q \times C_Q$.

5) Let $c_1, c_2 \in C_Q$, $c_3 \in S_Q$ and $(z,\tau) \in D(c_3)$. Set $c(t) = c_3 + tc_2$. Then $c(t) =\in C_Q$.

6) Let $D'(c) := \{(z,\tau) \in D(c) | \ (\frac{z}{\tau}, -\frac{1}{\tau}) \in D(c)\} = \{(a\tau + b, \tau) | \ \tau \in \mathfrak{H}, \ a,b \in \mathbb{R}^2, \ B(c,a) \notin \mathbb{Z}, \ B(c,b) \notin \mathbb{Z}\}$. If $(z,\tau) \in D'(c_1) \cap D'(c_2)$ then

$$\theta(\frac{z}{\tau}, -\frac{1}{\tau}) = \frac{i}{\sqrt{-\det A}}(i\tau)^{r/2} \sum_{p\in A^{-1}\mathbb{Z}^2/\mathbb{Z}^2} e^{2\pi i Q(z+p\tau)/\tau}\theta(z+p\tau,\tau).$$

Proof. Proof of (1) follows from relations of ρ^{c_1,c_2}.

For (2), it is easy to see that $\theta(z+\nu,\tau) = \theta(z,\tau)$ for $\nu \in A^1\mathbb{Z}$ and by replacing n by $n+\lambda$ we get $\theta(z+\lambda\tau,\tau) = e^{-2\pi i Q(\lambda)\tau - 2\pi i B(z,\lambda)}\theta(z,\tau)$.

For the proof of (3), the function $f(u) = e^{-u^2}$ is even we can clearly see this in power series of $f(u)$

$$f(u) = e^{-u^2} = \sum_{n=0}^{\infty}(-1)^n \frac{\pi^n u^{2n}}{n!}$$

is even function. By using Lemma 3.1, we know that $E(z)$ is an odd function. On the other hand, it is obvious that sgn function is odd function. Thus, as ρ is also an odd function.

$$\theta(-z,\tau) = \sum_{n\in\mathbb{Z}^2} \rho(n-a,\tau)e^{2\pi iQ(n)\tau+2\pi iB(n,-z)}$$

by replacing n to $-n$ we get

$$\theta(-z,\tau) = \sum_{-n\in\mathbb{Z}^2} \rho(-n-a,\tau)e^{2\pi iQ(-n)\tau+2\pi iB(-n,-z)}.$$

Since $Q(-n) = \frac{1}{2}\langle -n, -An\rangle = Q(n)$, $B(-n,-z) = B(n,z)$ and we showed that ρ is odd function so we have

$$\begin{aligned}
\theta(-z,\tau) &= \sum_{-n\in\mathbb{Z}^2} \rho(-n-a,\tau)e^{2\pi iQ(-n)\tau+2\pi iB(-n,-z)} \\
&= \sum_{n\in\mathbb{Z}^2} -\rho(n+a,\tau)e^{2\pi iQ(n)\tau+2\pi iB(n,z)} \\
&= \theta(z,\tau).
\end{aligned}$$

□

Corollary 3.1. *(Zwegers, 2012) The indefinite theta function $\theta_{a,b}$ satisfies following transformation properties ;*

1) $\theta_{a+\lambda,b} = \theta_{a,b}$ for all $\lambda \in \mathbb{R}^2$.

2) $\theta_{a,b+\nu} = e^{2\pi iB(a,\nu)}\theta_{a,b}$ for all $\nu \in A^{-1}\mathbb{Z}^2$.

3) $\theta_{-a,-b} = -\theta_{a,b}$.

4) $\theta_{a,b}(\tau+1) = e^{-2\pi iQ(a)-\pi iB(A^{-1}A^*,A)}\theta_{a,a+b+\frac{1}{2}A^{-1}A^*(\tau)}$ with A^* the vector of diagonal elements of A.

5) If $a,b \in R(c_1)\cap R(c_2)$ then

$$\theta_{a,b}(-\frac{1}{\tau}) = \frac{i}{\sqrt{-\det A}}(-i\tau)^{r/2}e^{2\pi iB(a,b)} \sum_{p\in A^{-1}\mathbb{Z}^2 mod\mathbb{Z}^2} \theta_{b+p,-a}(\tau).$$

4 CONCLUSION

In this thesis, theta functions corresponding to positive definite binary quadratic forms is investigated. As is well known, these theta functions are convergent with respect to both of its variables z and t, whose convergence is a consequence of the finiteness of the number of representations of a given positive integer by the given positive definite binary quadratic form. Due to the symmetry properties of these functions, these theta functions play a key role in both producing elliptic functions and modular forms.

Whenever the binary quadratic form is indefinite however a word for word definition does lead to a divergent theta function. We observe that the factor (introduced by S.Zwegers) forces the infinite sum to converge yet this new series does not converge absolutely. Therefore, we obtain a function of two complex variables and another variable in the upper half space. The symmetries of these new theta functions with respect to both variables are studied.

In the light of the results obtained concerning the symmetry relations of these new theta functions we believe that it should be possible to use these new family of theta functions in producing :

- functions defined on abelian surfaces, and
- modular forms.

REFERENCES

Bellman, R. (1961). *A brief introduction to theta functions*, Athena Series : Selected Topics in Mathematics, Holt, Rinehart and Winston, New York.

Deitmar, A. (2013). *Automorphic forms*, Universitext, Springer, London. Translated from the 2010 German original.

Jones, G. A. and Singerman, D. (1987). *Complex functions*, Cambridge University Press, Cambridge. An algebraic and geometric viewpoint.
URL: *http ://dx.doi.org/10.1017/CBO9781139171915*

Mumford, D. (2007). *Tata lectures on theta. I*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA. With the collaboration of C. Musili, M. Nori, E. Previato and M. Stillman, Reprint of the 1983 edition.
URL: *http ://dx.doi.org/10.1007/978-0-8176-4578-6*

Zwegers, S. P. (2012). Mock Maass theta functions, *Q. J. Math.* **63**(3) : 753–770.
URL: *http ://dx.doi.org/10.1093/qmath/har020*

APPENDIX A UPPER HALF-PLANE

The upper half plane is defined as the set of all complex numbers with positive imaginary part, and denoted $\mathfrak{H}$:

$$\mathfrak{H} \;=\; \{z \in \mathbb{C} \;\mid\; Im(z) > 0\}.$$

The set $\mathbb{D} = \{z \in \mathbb{C} \;\mid\; |z| < 1\}$ is called **unit disk**.
Let U and V be two subsets of $\mathbb{C}$ and $f : U \to V$ be a holomorphic function. We call f is **biholomorphic**, if f is bijective and its inverse omoromorphic.

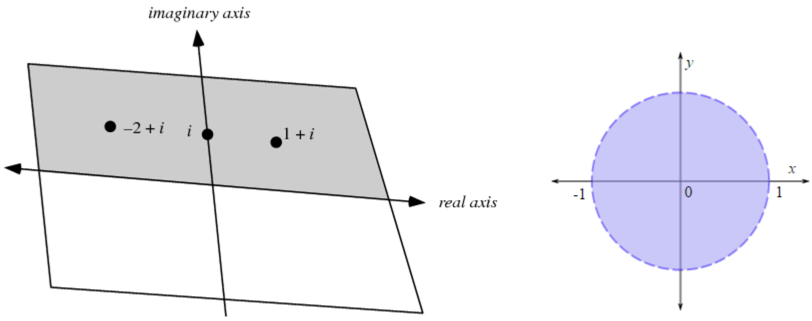

Figure APPENDIX A.1: Upper Half Plane and Unit Disk

The Cayley Transformation is the mapping of the upper half plane to the unit disk, given by

$$\kappa \;:\; \mathfrak{H} \;\to\; \mathbb{D}$$

$$z \;\mapsto\; \frac{z-i}{z+i}.$$

It is clear to see that $\dfrac{\partial}{\partial \bar{z}}(\kappa(z)) = 0$. So κ is a holomorphic. κ is surjective because any element y of $\mathbb{D}$ can be written $y = \dfrac{z-i}{z+i}$. Indeed, we get $z = \dfrac{i(y+1)}{1-y}$. This is an element of $\mathfrak{H}$ because $\left|\dfrac{i(y+1)}{1-y}\right| = |i| \left|\dfrac{y+1}{1-y}\right| = \dfrac{|y+1|}{|1-y|}$, since $y \in \mathbb{D}$, $|y+1|$ never vanishes, so $\dfrac{|y+1|}{|1-y|} > 0$. κ is surjective.

To show κ is bijective function, we should find its inverse.

$$\begin{aligned} \kappa^{-1} : \mathbb{D} &\rightarrow \mathfrak{H} \\ z &\mapsto \frac{i(z+1)}{-z+1}. \end{aligned}$$

Now we compute $\kappa \circ \kappa^{-1}(z)$.

$$\begin{aligned} \kappa \circ \kappa^{-1}(z) &= \kappa(\frac{i(z+1)}{-z+1}) \\ &= \frac{\frac{i(z+1)}{-z+1} - i}{\frac{i(z+1)}{-z+1} + i} \\ &= \frac{iz + i + iz - i}{iz + i - iz + i} \\ &= z. \end{aligned}$$

Thus $\mathfrak{H}$ and $\mathbb{D}$ are biholomorphic.

Lemma(Schwarz Lemma). If $F : \mathbb{D} \rightarrow \mathbb{D}$ is a holomorphic function and if $F(0) = 0$, then

$$|F(z)| \leq |z| \quad \text{ } and \text{ } \quad |F'(0)| \leq 1.$$

If either $|F(z)| = |z|$ for some $z \neq 0$ or $|F'(0)| = 1$, then F is a rotation. (i.e $F(z) = e^{i\theta}z$ for some $\theta \in [0.2\pi]$.) We'll use Schwarz's Lemma to determine all automorphisme of $\mathbb{D}$. First note that $\rho_\theta : \mathbb{D} \mapsto \mathbb{D}$ sends z to $e^{i\theta}z$ is an automorphism of the unit disk. In addition, $a \in \mathbb{C}$ with $|a| < 1$, we define a function ϕ_a, from $\mathbb{D}$ to itself, given by;

$$\begin{aligned} \phi_a : \mathbb{D} &\rightarrow \mathbb{D} \\ z &\mapsto \frac{z-a}{1-\bar{a}z}. \end{aligned}$$

Let $z_0 \in \partial\mathbb{D}$, then $|z_o| = 1$ and $z_0 = e^{i\theta_0}$

$$|\phi_a(z_0) = \left| \frac{e^{i\theta_0} - a}{1 - \bar{a}e^{i\theta_0}} \right| = \frac{|e^{i\theta_0} - a|}{|e^{i\theta_0}||e^{-i\theta_0} - \bar{a}|}.$$

Set $e^{i\theta_0} - a = w$, so we get $\bar{w} = e^{-i\theta_0} - \bar{a}$

$$|\phi_a(z_0)| = \frac{|w|}{|\bar{w}|} = 1.$$

According to maximum modulus principle, $|\phi|$ attains its maximum on the boundary. So $|\theta_a(z_0)| < 1$ in $\mathbb{D}$.

The inverse of ϕ_a is $(\phi_a)^{-1}(z) : \mathbb{D} \to \mathbb{D}$ such that $(\phi_a)^{-1}(z) = \dfrac{z+a}{1+\bar{a}z}$.

$$\begin{aligned} \phi_a(z) \circ (\phi_a)^{-1}(z) &= \phi_a\left(\frac{\frac{z+a}{1+\bar{a}z} - a}{1 - \bar{a}\frac{z+a}{1+\bar{a}z}}\right) \\ &= \frac{z + a - a - \bar{a}az}{1 + \bar{a}z - \bar{a}z - \bar{a}a} \\ &= \frac{z(1-|a|)}{1-|a|}. \end{aligned}$$

According to definiton $|a| < 1$, $(1 - |a|) \neq 0$ thus $\phi_a(z) \circ (\phi_a)^{-1}(z) = z$. This shows that ϕ_a is bijective and since $\dfrac{\partial}{\partial \bar{z}}(\phi_a(z)) = 0$, ϕ_a is also holomorphic function.

So $\phi_a : \mathbb{D} \to \mathbb{D}$ is a holomorphic automorphism.

If $F : \mathbb{D} \to \mathbb{D}$ is any holomorphic automorphism, then there exist $a \in \mathbb{D}$ and $\theta \in [0, 2\pi)$ such that $F = \phi_a \circ \rho_\theta$.

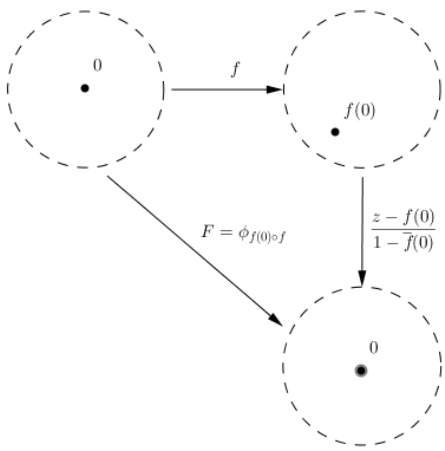

Figure APPENDIX A.2: Automorphism of the Unit Disk

Since ϕ_a and f are biholomorphic functions $F = \phi_a \circ f$ is also biholomorphic, thus $F^{-1} : \mathbb{D} \to \mathbb{D}$ is biholomorphic function too. $F^{-1}(0) = 0$, according to Schwarz Lemma $|(F^{-1})'(0)| \leq 1$

$$F \ : \mathbb{D} \to \mathbb{D} \quad , \quad F(0) = 0 \Rightarrow |F'(0)| \leq 1,$$

$$F^{-1} \ : \mathbb{D} \to \mathbb{D} \quad , \quad F^{-1} = 0 \Rightarrow |(F^{-1})'(0)| \leq 1,$$

$$|(F^{-1})'(0)| = \frac{1}{|(F)(0)|}| \leq 1.$$

If we use the equations above, it is easy to see that $|F'(0)| = 1$, according to Schwarz Lemma, F is a rotation. Thus $F(z) = e^{i\theta_0}z = \phi_{f(0)} \circ f$ i.e. $(\theta_{f(o)})^{-1} = \phi_{-f(0)} \Rightarrow \phi_{-f(0)} \circ \rho_\theta = f$. This shows us any biholomorphic function of $\mathbb{D}$ is a composition of ϕ_a and ρ_θ. Conversely, any composition of ϕ_a and ρ_θ is a biholomorphism of $\mathbb{D}$. So we proved:

Theorem APPENDIX A.1. *For any biholomorphism $f : \mathfrak{H} \to \mathfrak{H}$, there are some $a \in \mathbb{D}$ and some $\theta \in [0.2\pi]$ such that $f = \kappa^{-1} \circ (\phi_a \circ f_\theta) \circ \kappa$.*

The special linear group $\mathrm{SL}_2(\mathbb{Z})$ is the group of all integer 2×2 matrices with determinant one.

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc| = 1 \right\}.$$

More specially, we denote $\mathrm{PSL}_2(\mathbb{R})$, the group of $\mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ $\mathrm{PSL}_2(\mathbb{Z})$ is a subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

Two particular elements of $\mathrm{PSL}_2(\mathbb{Z})$ are

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The matrix $S^2 = I_2$, $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $(ST)^3 = I_2$ with $ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. S and T generates $SL_2(\mathbb{Z})$.

BIOGRAPHICAL SKETCH

Şahan İzgi was born in Mersin on Sep 3, 1990. He was graduated from 19 Mayıs Anatolian High School in 2007. He received university diploma form Galatasaray University, Istanbul, Mathematics in 2014.