MARKOV THEORY AND OUTER AUTOMORPHISM OF PGL$(\mathbf{2}, \mathbb{Z})$

(MARKOV TEORİSİ VE
PGL$(2, \mathbb{Z})$ GRUBUNUN DIŞ OTOMORFİZMİ)

by

B u k e t E R E N, B.S.

Thesis

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

MASTER OF SCIENCE

in

MATHEMATICS

in the

GRADUATE SCHOOL OF SCIENCE AND ENGINEERING

of

GALATASARAY UNIVERSITY

June 2018

Acknowledgements

I would like to express my sincere gratitude to my advisor Prof. Dr. A. Muhammed ULUDAĞ, who provided me with the tools that I needed to choose the right direction, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. It has been a period of intense learning for me, not only in the scientific arena, but also on a personal level thanks to him.

I would also like to thank professor Hakan AYRAL for the help on my all technical problems.

My thanks and appreciations also go to the people who are directly or indirectly helped me out in developing this thesis.

Thanks to the Turkish Scientific and Technical Research Institute (TÜBİTAK) for supporting this work under project grant 115F412.

18 May 2018
Buket EREN

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List of Symbols

$\mathrm{PGL}(2,\mathbb{Z})$	: Projective general linear group of determinant ± 1 integral 2×2 matrices
$\mathrm{SL}(2,\mathbb{Z})$	: Special linear group of unimodular integral 2×2 matrices
$\Gamma(N)$	: Congruence modular group of level N
$\mathcal{L}$	: Lagrange spectrum
$\mathcal{M}$	: Markov spectrum
$\mathcal{M}$	: Set of Markov numbers
$f_m(x,y)$	: Markov form
$\mathbb{H}$	: Upper half plane
γ_m	: Markov irrational associated Markov number
γ_t	: Markov irrational associated Farey index
$T_{\mathcal{F}}$	: Farey tree
$T_{\mathcal{M}}$	: Markov tree
T_C	: Cohn tree
con	: Conumerator function
num	: Numerator function

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Abstract

The Markov numbers are positive integers that arise from the solutions of a special Diophantine equation, called Markov equation. These numbers appear in the context of continued fractions and so Diophantine approximation.The Lagrange number of the real number α is defined as the supremum of real numbers ℓ such that $|\alpha - p/q| < 1/\ell q^2$ for an infinite number of rationals p/q. The Lagrange spectrum is defined as the set of Lagrange numbers of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Further, the solutions are related with the Lagrange-Markov spectrum which consists of those quadratic numbers which are badly approximable by rational numbers. We focus here on the set of irrationals α, called Markov irrationals, with $L(\alpha) < 3$ such that continued fraction expansion of such numbers is in the form $[..., a_1, a_2, a_3, ...]$ with $a_i \in \{1, 2\}$. However, there is a fundamental involution Jimm of real line induced by the outer automorphism of the extended modular group $\mathrm{PGL}(2, \mathbb{Z})$. Its action on the real line, explicitly on continued fraction expansion, recently discovered.

In this dissertation, with the aim of finding relations on various parts of this work, we consider to ask whether there is a classification related to Markov number or a special property the set of image of Markov irrationals under the involution Jimm. Markov numbers also appear in many parts of mathematics such as binary quadratic forms, hyperbolic geometry and the combinatorial of words and we investigated analogous problem in different topics.

Keywords: Continued fraction, Lagrange number, Diophantine approximation, Markov equation, Markov number.

Özet

Markov sayıları, Markov denklemi adı verilen özel bir diyofant denklemin çözümlerinden ortaya çıkan pozitif tamsayılardır. Bu sayıların sürekli kesirler ve dolayısıyla diyofant yaklaşımı ile ilgisi vardır. Bir α reel sayısı için Lagrange sayısı, $|\alpha - p/q| < 1/\ell q^2$ eşitsizliğini sonsuz sayıda rasyonel p/q sayı için sağlayan reel ℓ değerlerinin supremumu olarak tanımlanır. Ayrıca tüm α reel sayılarına ait Lagrange sayılarının kümesine Lagrange spektrumu denir.

Dahası, Markov sayıları bazı kuadratik irrasyonellere ait Lagrange sayılarını içeren Lagrange-Markov spektrumu ile ilgilidir. Bu çalışmada Lagrange sayıları 3'ten küçük öyle ki sürekli kesir açılımları $[\ldots, a_1, a_2, a_3, \ldots]$, $a_i \in \{1,2\}$ formunda olan Markov kuadratik irrasyonelleri kümesi üzerinde durulacaktır. Öte yandan, genişletilmiş modüler grubun dış otomorfizminden kaynaklanan, reel sayıların Jimm adında temel bir envolüsyonu vardır. Bu envolüsyonun reel sayılara diğer bir deyişle sürekli kesir açılımlarına etkisi son zamanlarda keşfedilmiştir.

Bu tezde, çalışmanın çeşitli bölümleri arasında bağlantılar bulmayı amaçlayarak, Markov irrasyonellerinin yukarıda bahsedilen Jimm envolüsyonu altında görüntü kümesinin Markov sayılarına bağlı olarak bir sınıflandırmasının veya özgün bir özelliğinin olup olmadığı araştırılmıştır. Markov sayıları ikili kuadratik formlar, hiperbolik geometri ve kombinatuar gibi matematiğin bir çok alanında karşımıza çıkar. Bu sebeple benzer sorular farklı alanlara taşınarak muhtemel sonuçlar tartışılmıştır.

Anahtar Sözcükler : Sürekli kesirler, Lagrange sayıları, diyofant yaklaşma, Markov denklemi, Markov sayıları.

Chapter 1

Introduction

1.1 A Brief History of Rational Approximation

Every irrational number is the limit of a sequence of rational numbers since the density of rational numbers in real numbers. Let us consider an example :

$$\sqrt{2} = 1,4142356237309504....$$

Since $\sqrt{2}$ is irrational, the decimal expansion does not stop so a sequence of rational numbers that converges to $\sqrt{2}$ can be choosen as follows:

$$(1) \quad (1,4) \quad (1,41) \quad (1,414) \quad (1,4142) \quad (1,41423) \quad (1,414235) \quad \ldots \tag{1.1}$$

As decimal representation of all the term of sequence terminates, we have a sequence of rational numbers which converges to $\sqrt{2}$. Note that the distance between $\sqrt{2}$ and n-th rational approximation is smaller than 10^{1-n}. If we want to find a good approximation, we should choose n sufficiently large. But in this case, we may have:

$$\sqrt{2} \approx \frac{14142356237309504}{10000000000000000}$$

means denominators grow very fast. Thus, a natural question is: Can we find a good approximation with smaller denominator ? Let us take another example,

$$\sqrt{2} - \frac{17}{12} \approx 0,0024$$

On the other hand,

$$\sqrt{2} - \frac{141}{100} \approx 0,0042$$

This says that $17/12$ is a better approximation than $141/100$. In fact, we could choose infinitely many rational numbers instead of $17/12$ with comparatively small denominator than rational numbers listed in 1.1.

How to find the good approximations? What about numbers like π or $\sqrt[3]{2}$? We will try to answer all the questions by general theory in this part. Before to give a classical theorem in approximation theory, let us introduce a measure of approximation. Let α be a real number and p/q with $(p,q) = 1$ be a rational then define:

$$\mu := q|\alpha - \frac{p}{q}| = |q\alpha - p|$$

Note that if p/q is a good approximation, then $|\alpha - p/q|$ is small and q is not so large. Then quantity μ is small. Conversely, if μ is small then $|\alpha - p/q|$ is also small. So the quantity μ gives us information about our approximation.

Theorem 1.1.1. *(Dirichlet's Theorem)* [1, Chapter 1] Let α be an irrational and $N \in \mathbb{N}$. Then there exists infinitely many p/q rational with $q \leq N$ such that

$$|q\alpha - p| < \frac{1}{N}$$

In particular,

$$|\alpha - \frac{p}{q}| \leq \frac{1}{q^2}$$

Proof. Let α be an irrational and N be an integer. Consider $N+1$ numbers as:

$$\alpha, 2\alpha, 3\alpha, 4\alpha, ..., (N+1)\alpha$$

Choose integer part of $k\alpha$ as $p_k = \lfloor k\alpha \rfloor$ hence we have $0 \leq |k\alpha - p_k| \leq 1$ for $k \in \{1, 2, ..., N+1\}$. Let us distribute numbers between 0 and 1 in N distinct containers :

Put numbers between 0 and $\frac{1}{N}$, in $1st$ container.

Put numbers between $\frac{1}{N}$ and $\frac{2}{N}$, in $2nd$ container.

$\vdots$

Put numbers between $\frac{N-1}{N}$ and $\frac{N}{N}$, in Nth container.

We have $N+1$ numbers between 0 and 1 and put them in N container. In this case, at least one of these containers must contain at least two of these numbers by pigeonhole principle. Suppose that $|k\alpha - p_k|$ and $|l\alpha - p_l|$ are in same container then we have :

$$0 < |k\alpha - p_k - (l\alpha - p_l)| < \frac{1}{N}$$

Let $p = p_k - p_l$ and $q = k - l$, then $|q\alpha - p| < \frac{1}{N}$ which means $|\alpha - \frac{p}{q}| < \frac{1}{qN}$. Since k and l are inferior to $N+1$, we have $q \leq N$ so $|\alpha - \frac{p}{q}| \leq \frac{1}{q^2}$. As N is arbitrary, the inequality $|\alpha - \frac{p}{q}| < \frac{1}{qN}$ guarantees the existence of infinite number of distinct fractions $\frac{p}{q}$.

□

On the other hand, we say that a real number α can be approximated to order t if there exist infinitely many p/q and a constant c_α depending on α such that

$$|\alpha - \frac{p}{q}| \leq \frac{c_\alpha}{q^t}$$

It is clear that, for any α of order t, this inequality holds also for values smaller than t. Hence the remarkable question : What is the possible highest order ? Before to give an amazing result recall that a real number that is a root of a polynomial with integer coefficients is called algebraic of degree d where d is smallest degree for which such a polynomial exists. Now, we will see a relation between algebraic degree and order of approximation with theorem proved by Liouville:

Theorem 1.1.2. *(Liouville's Theorem)* [1, Chapter 1] Let $\alpha \in \mathbb{R}$ be an algebraic of degree d. Then there is a constant $c > 0$ such that

$$|\alpha - \frac{p}{q}| > \frac{c}{q^d} \quad \text{for all} \quad \frac{p}{q} \in \mathbb{Q}.$$

Proof. Let $\alpha \in \mathbb{R}$ be an algebraic number of degree d and $p/q \in \mathbb{Q}$, $p/q \neq \alpha$. Then observe two cases :

- For $d = 1$, α is rational so $\alpha = \frac{x}{y}$ with $x \in \mathbb{Z}$, $y \in \mathbb{N}^*$. Then,

$$|\alpha - \frac{p}{q}| = |\frac{x}{y} - \frac{p}{q}| = \frac{|qx - py|}{qy} \geq \frac{1}{qy}$$

  since $qx - py \neq 0$, $c = \frac{1}{y}$ convient.

- For $d>1$: Let α be root of polynomial $P(x)=a_0+a_1x+a_2x^2+....+a_dx^d$ where $P(\alpha)=0$. Let β be real root different than α of $P(x)$ which is closest to α with $|\alpha-\beta|=\delta$. Consider interval $(\alpha-\delta,\alpha+\delta)$. Let us now $\frac{p}{q}\in\mathbb{Q}$ arbitrary. Then we have two cases:

✓ If $|\alpha-\frac{p}{q}|\geq\delta$, then we have $|\alpha-\frac{p}{q}|>\frac{M}{q^d}$ with $M=\frac{\delta}{2}$.

✓ If $|\alpha-\frac{p}{q}|<\delta$, then compute $|P(\frac{p}{q})|$:
($P(\frac{p}{q})\neq 0$ because we supposed that β is closest to α that is root of $P(x)$ such that $|\alpha-\beta|=\delta$.)

$$P(\frac{p}{q})|=\frac{|a_0q^d+a_1pq^{d-1}+....+a_dp^d|}{q^d}\geq\frac{1}{q^d} \tag{1.2}$$

By intermediate value theorem, there exist $\xi\in(\alpha-\delta,\alpha+\delta)$ such that

$$\frac{P(\frac{p}{q})-P(\alpha)}{\frac{p}{q}-\alpha}=\frac{P(\frac{p}{q})}{\frac{p}{q}-\alpha}=P'(\xi)\neq 0$$

where $\xi\in(\alpha,\frac{p}{q})$. Hence, we obtain:

$$|\alpha-\frac{p}{q}|.|P'(\xi)|=|P(\frac{p}{q})|\geq\frac{1}{q^d}$$

If we take K such that $K>|P'(x)|$, $\forall x\in(\alpha-\delta,\alpha+\delta)$, then

$$|\alpha-\frac{p}{q}|>\frac{1}{Kq^d}$$

In conclusion, $c=\min(M,\frac{1}{K})$ satisfies two cases.

□

It means that an algebraic number of degree d can be approximated to order at most d. For instance, we can deduce by Dirichlet 's and Liouville's theorem that

i. If α can be approximated to every order n, it must be transcendental. In this case, α is called well-approximable.

ii. If α can be approximated to order 1, then it must be rational.

iii. If α can be approximated to order greater than 1, then it must be irrational.

The first observation allows us to construct many examples of transcendental numbers. The second is already clear. For the last one, let us return to algebraic numbers of degree d. Then the natural question is whether there is an order of approximation smaller than d. The answer is found by Klaus Roth who was awarded Fields Medal in 1958. Roth's theorem states that every algebraic number α of any degree has approximation order 2.

Theorem 1.1.3. *(Roth's Theorem)* [10, part D] Let α be an algebraic number. For any $\varepsilon > 0$, there are only finitely many rational numbers p/q satisfying

$$|\alpha - \frac{p}{q}| \leq \frac{1}{q^{2+\varepsilon}}$$

Proof. See [10, part D, p. 300].

It shows that the exponent 2 can not be improved when α is an algebraic number. Hence we focus on all real numbers $L > 0$ such that

$$|\alpha - \frac{p}{q}| \leq \frac{1}{Lq^2}$$

for infinitely many rational p/q.

Our aim is to determine the best constant L depending on α which is called *Lagrange number of* α. Indeed, the main question is : Is there a connection between any given α and its approximation constant ? The answer is found by A. A. Markov in 1879 and this result connects very surprisingly approximation theory and a Diophantine equation we will study in detail.

Chapter 2

Literature Review

Diophantine Analysis name is due to Greek mathematician Diophant of Alexandria in the third century, is very old subject in mathematics; it relates the theory of Diophantine approximations with the theory of Diophantine equations. Markov equation is one of the interesting Diophantine equation and it occurred initially in Markov's doctoral studies about minima of quadratic forms in [11]. The solutions are related with the Lagrange-Markov spectrum, which consists of those quadratic numbers which are badly approximable by rational numbers. It can be find general Markov theory in [7].

The main idea of the Markov's Theorem 3.1.3 comes from Dirichlet's Theorem 1.1.1 proved by using pigeonhole principle. In fact, it is the first concrete application; so it is also called Dirichlet principle. After some years, Liouville gives us a method to construct the first concrete transcendental numbers by Liouville's Theorem 1.1.2 in 1844. In the same years, another important result was given by Hurwitz's Theorem 2.1.4 which said that $\sqrt{5}$ is best possible approximation constant. In 1873, A. Korkine and E.I. Zolotarev proved that the first two Lagrange numbers are $\sqrt{5}$ and $\sqrt{8}$. All these works inspired Frobenius and he introduced a problem which is open for more than 100 years, known as the uniqueness conjecture for Markov numbers in 1913. Anyway the Conjecture has been proved for certain classes of Markov numbers by A. Baragar (1996), P. Schmutz (1996), J.O. Button (1998), M.L. Lang, S.P. Tan (2005), Y. Zhang (2006) [19].

Markov numbers occur also in other parts of mathematics, in particular free groups, Fuchsian groups and hyperbolic Riemann surfaces and they were investigated by many mathematicians as Ford, Lehner, Cohn, Rankin, Conway, Coxeter, Hirzebruch.

2.1 Continued Fractions

History of continued fractions goes back to antiquity since it is closely related to Euclid algorithm; the process of finding the continued fraction expansion of a real number is exactly same with Euclid algorithm. More precisely, let us consider $p, q \in \mathbb{Z}_{>0}$:

$$\begin{aligned}
p &= a_0 q + r_0 && 0 < r_0 < q \\
q &= a_1 r_0 + r_1 && 0 < r_1 < r_0 \\
r_0 &= a_2 r_1 + r_2 && 0 < r_2 < r_1 \\
&\vdots \\
r_{k-2} &= a_k r_{k-1} && 0 = r_k < r_{k-1} < \cdots < r_0 < q
\end{aligned}$$

These equalities give :

$$\begin{aligned}
\frac{p}{q} &= a_0 + \frac{r_0}{q} && 0 < \frac{r_0}{q} < 1 \\
&= a_0 + \frac{1}{q/r_0} && 1 < \frac{q}{r_0} < a_1 + \frac{r_1}{r_0} \\
&= a_0 + \frac{1}{a_1 + \frac{r_1}{r_0}} && 0 < \frac{r_1}{r_0} < 1 \\
&\vdots \\
&= a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}
\end{aligned}$$

This leads us to define a representation of a real number.

Definition 2.1.1. A real number α can be written in the form:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} \tag{2.1}$$

where $a_i \in \mathbb{Z}$, $a_i > 0$ for $i > 0$ and this form is called *simple continued fraction expansion* of α and denoted by $\alpha = [a_0, a_1, a_2, \dots]$. and the following continued fraction expansion

$$\frac{p_n}{q_n} = [a_0, a_1, a_2, \dots, a_n]$$

is called *n-th convergent* of α.

Proposition 2.1.1. Let $a_0, a_1, a_2, ...$ be a sequence of positive real numbers. Let p_n and q_n be defined as follows:

$$p_n = a_n p_{n-1} + p_{n-2}, \quad p_0 = a_0, \; p_{-1} = 1 \tag{2.2}$$

$$q_n = a_n q_{n-1} + q_{n-2}, \quad q_0 = 1, \quad q_{-1} = 0 \tag{2.3}$$

Then we have $p_n/q_n = [a_0, a_1, ..., a_n]$ for all n.

Proof. We prove by induction:

- ✓ $\dfrac{p_0}{q_0} = [a_0]$
- ✓ $\dfrac{p_1}{q_1} = \dfrac{a_1 p_0 + p_{-1}}{a_1 q_0 + q_{-1}} = a_0 + \dfrac{1}{a_1} = [a_0, a_1]$
- ✓ Suppose that it is true for $n-1$ and consider :

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

Put $a'_{n-1} = a_{n-1} + \dfrac{1}{a_n}$ and obtain:

$$[a_0, a_1, ..., a_n] = [a_0, a_1, ..., a_{n-2}, a'_{n-1}] = [a_0, a_1, ..., a_{n-2}, a_{n-1} + \frac{1}{a_n}]$$

$$= \frac{(a_{n-1} + \frac{1}{a_n})p_{n-2} + p_{n-3}}{(a_{n-1} + \frac{1}{a_n})q_{n-2} + q_{n-3}} = \frac{a_n(a_{n-1}p_{n-2} + p_{n-3}) + p_{n-2}}{a_n(a_{n-1}q_{n-2} + q_{n-3}) + q_{n-2}}$$

$$= \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_n}{q_n}$$

□

Now we prove some useful facts about convergence.

Lemma 2.1.1. Let $[a_0, a_1, \dots]$ be a continued fraction expansion of a real number and $p_n/q_n = [a_0, a_1, \dots, a_n]$ be the n-th convergent. Then we have:

i. $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$ where $n \geq 0$.

ii. $gcd(p_n, q_n) = 1$ where $n \geq 0$.

Proof. To see the first one, we use induction on n for the recurrence formula from previous Proposition 2.1.1 then it is easily seen that we get

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

If we take determinants in two side we get desired result. The second can be deduced from this by Bezout's theorem .

□

It is clear that corresponding continued fraction expansion terminates if and only if representing number is a rational number. And also rationals are not determined unique way however every irrational has a uniquely determined continued fraction expansion. On the other hand, a natural question is which irrational numbers have an eventually periodic expansion. The answer is given by Lagrange:

Theorem 2.1.2. *(Lagrange's Theorem)* An irrational number have an eventually periodic continued fraction expansion if and only if it is a quadratic irrational.

Proof. [1, chapter 1, p. 15].

One of the famous theorem in Diophantine approximation theory gives us a connection between continued fraction and rational approximation :

Theorem 2.1.3. *(Legendre's Theorem)* [1, Chapter 1] Let α be an irrational number and p_n/q_n be a convergent of α. If there exist p/q satisfying

$$|\alpha - \frac{p}{q}| \leq \frac{1}{2q^2}$$

then $p/q = p_n/q_n$ for some n.

Proof. Suppose, by contradiction, that p/q is not a convergent of α with $q_n \leq q < q_{n+1}$. Then we have from Fact 1 in Appendix A :

$$q_n \leq q \Rightarrow |q_n\alpha - p_n| \leq |q\alpha - p| = |\alpha - \frac{p}{q}|q < \frac{1}{2q}$$

$$|\alpha - \frac{p_n}{q_n}| < \frac{1}{2qq_n}$$

But, since $\frac{p}{q} \neq \frac{p_n}{q_n}$, we have $|pq_n - qp_n| \geq 1$ which brings us to:

$$\frac{1}{qq_n} \leq \frac{|pq_n - qp_n|}{qq_n} = |\frac{p}{q} - \frac{p_n}{q_n}| \leq |\alpha - \frac{p}{q}| + |\alpha - \frac{p_n}{q_n}| < \frac{1}{2q^2} + \frac{1}{2qq_n}$$

Then we obtain

$$\frac{1}{qq_n} < \frac{1}{q^2} \Longleftrightarrow q_n > q$$

which contradicts our hypothesis.

□

Our next theorem will show that we can do better :

Theorem 2.1.4. *(Hurwitz's Theorem)*[1, Chapter 1] Let α be an irrational number. There are infinitely many rational p/q such that

$$|\alpha - \frac{p}{q}| \leq \frac{1}{\sqrt{5}q^2}$$

Proof. Let p_n/q_n be a convergent of α. It suffices to show that for all k positive, at least one

$$\frac{p}{q} \in \{\frac{p_{k-1}}{q_{k-1}}, \frac{p_k}{q_k}, \frac{p_{k+1}}{q_{k+1}}\}$$

satisfies

$$|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$$

Assume that p_{k-1}/q_{k-1} et p_k/q_k does not satisfy this inequality and since α must be between two consecutives convergents from Fact 2 in Appendix A, we have:

$$|\alpha - \frac{p_{k-1}}{q_{k-1}}| + |\alpha - \frac{p_k}{q_k}| = |\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}| = \frac{1}{q_k q_{k-1}} \geq \frac{1}{\sqrt{5}q_{k-1}^2} + \frac{1}{\sqrt{5}q_k^2}$$

Then,

$$1 \geq \frac{q_k}{\sqrt{5}q_{k-1}} + \frac{q_{k-1}}{\sqrt{5}q_k} \Leftrightarrow \frac{q_k}{q_{k-1}} + \frac{q_{k-1}}{q_k} \leq \sqrt{5}$$

from Lemma 1 in Appendix A we get

$$\frac{q_k}{q_{k-1}} < \frac{\sqrt{5}+1}{2}$$

If $|\alpha - \frac{p_{k+1}}{q_{k+1}}| \geq \frac{1}{\sqrt{5}q_{k+1}^2}$ is also true, then

$$\frac{q_{k+1}}{q_k} < \frac{\sqrt{5}+1}{2}$$

with $q_{k+1} = a_{k+1}q_k + q_{k-1}$, we conclude that:

$$\frac{\sqrt{5}+1}{2} > \frac{q_{k+1}}{q_k} = a_{k+1} + \frac{q_{k-1}}{q_k} \geq 1 + \frac{q_{k-1}}{q_k} > 1 + \frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}+1}{2},$$

which is contradiction.

□

2.2 Lagrange Spectrum

Recall that our aim is to find the best interval for a given irrational number. In other words we want to determine the largest L for infinitely many p/q:

Definition 2.2.1. For a given $\alpha \in \mathbb{R}$, $L(\alpha) = \sup L$ over all L that satisfy for infinitely many $p/q \in \mathbb{Q}$,

$$|\alpha - \frac{p}{q}| \leq \frac{1}{Lq^2}$$

is said to be Lagrange number of α. The set of Lagrange number as α varies is called *Lagrange spectrum*.

Legendre's [1] and Hurwitz's Theorem [1] guarante us all good approximations are convergents

$$\mid \alpha - \frac{p_n}{q_n} \mid < \frac{1}{\sqrt{5}q^2}$$

which means it may be possible to compute Lagrange number of α bigger than $\sqrt{5}$ using continued fraction expansion of α:

Assume that $\alpha = [a_0, a_1, a_2, \dots]$ with $\alpha = [a_0, a_1, a_2, \dots, a_n, \alpha_{n+1}]$. Proposition 2.1.1 tells us that

$$\begin{aligned}|\alpha - \frac{p_n}{q_n}| &= \frac{1}{q_n(\alpha_{n+1}q_n + q_{n-1})} \\ &= \frac{1}{q_n^2(\alpha_{n+1} + q_{n-1}/q_n)} \\ &= \frac{1}{(\alpha_{n+1} + 1/\beta_n)q_n^2}\end{aligned}$$

where $\alpha_{n+1} = [a_{n+1}, a_{n+2}, \dots]$ and $\beta_n = q_n/q_{n-1}$. Let us show that

$$\beta_n = \frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, \dots, a_1]$$

for all $n \geq 1$. We prove by induction on n. For $n = 1$, it is clear that $\beta_n = q_1/q_0 = a_1$. Suppose it is true for $n-1$ then

$$\begin{aligned}\beta_n = \frac{q_n}{q_{n-1}} = \frac{a_n q_{n-1} + q_{n-2}}{q_{n-1}} &= a_n + \frac{1}{\beta_{n-1}} && (2.4) \\ &= a_n + [0, a_{n-1}, \dots, a_1] && (2.5) \\ &= [a_n, a_{n-1}, \dots, a_1] && (2.6)\end{aligned}$$

as desired. Therefore we obtain

$$|\alpha - \frac{p_n}{q_n}| = \frac{1}{\mu_n(\alpha)q_n^2} \tag{2.7}$$

where

$$\mu_n(\alpha) := \alpha_{n+1} + \frac{1}{\beta_n} = [a_{n+1}, a_{n+2}, \dots] + [0, a_{n-1}, \dots, a_1] \tag{2.8}$$

Consequently, one of the method to compute this number is following:

Proposition 2.2.1. With preceding notation, Lagrange number of α is

$$L(\alpha) = \lim_{n\to\infty} \sup([a_{n+1}, a_{n+2}, \dots] + [0, a_n, \dots, a_1])$$

Proof. Assume that $K(\alpha) := \lim\limits_{n\to\infty} \sup \lambda_n(\alpha)$. We want to show that $L(\alpha) = K(\alpha)$. Let us consider $L > \sqrt{5}$ such that $|\alpha - p_n/q_n| < 1/Lq_n^2$ for all $n \in \mathbb{N}$.

$$|\alpha - \frac{p_n}{q_n}| = \frac{1}{\mu_n(\alpha)q_{n^2}} < \frac{1}{Lq_n^2} \Longleftrightarrow \mu_n(\alpha) > L$$

Hence $K(\alpha) = \lim\limits_{n\to\infty} \sup \lambda_n(\alpha) \geq L(\alpha)$. On the other hand, for all $\varepsilon > 0$, there exist infinitely n such that

$$\mu_n(\alpha) > K(\alpha) - \varepsilon$$

which means

$$\frac{1}{(K(\alpha)-\varepsilon)q_n^2} > |\alpha - \frac{p_n}{q_n}| \implies K(\alpha) - \varepsilon \leq L(\alpha) \qquad \text{for all } \varepsilon > 0$$

thus $K(\alpha) \leq L(\alpha)$. In conclusion, we have $K(\alpha) = L(\alpha)$.

□

Let us compute Lagrange numbers for some real numbers:

Example 2.2.1. *Let* $\alpha = [1,1,1,1,\ldots]$.

$$\begin{aligned} L(\alpha) &= \limsup_{n\to\infty}([1,1,1,\ldots]+[0,1,1,\ldots,1]) \\ &= \frac{1+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}} \\ &= \sqrt{5} \end{aligned}$$

Thus,

$$L(\frac{1+\sqrt{5}}{2}) = \sqrt{5}$$

which means

$$|\frac{1+\sqrt{5}}{2} - \frac{p}{q}| \leq \frac{1}{\sqrt{5}q^2}$$

holds for infinitely many rationals p/q and it is the best approximation.

Before to complete this part, we need to define an equivalence relation on irrational numbers :

Definition 2.2.2. We say that α and β are equivalent if their continued fraction expansions eventually coincide, that is

$$\alpha = [a_0, a_1, a_2, \ldots, \gamma] \quad \sim \quad \beta = [b_0, b_1, b_2, \ldots, \gamma]$$

Remark. If two continued fraction eventually coincide, then their Lagrange numbers must be equal.

Proposition 2.2.2. Equivalent numbers have the same Lagrange numbers:

$$\alpha \sim \beta \Longrightarrow L(\alpha) = L(\beta)$$

Proof. We will see in the next chapter.

Attention. The converse is not always true. However, it is conjectured to be true under some hypothesis.

Chapter 3

Markov's Approximation Theory

A Diophantine equation is an equation with integers coefficients having only integer solutions. The most famous example is of course Fermat's equation:

$$x^n + y^n = z^n \tag{3.1}$$

This Diophantine equation has integer solutions known as pythagorean triples, for $n = 2$ by the work of Andrew Wiles in 1995, it has no integer solutions (x, y, z) with $xyz \neq 0$ for $n \geq 3$. On the other hand, we know, by undecidability of Hilbert's 10th problem, there is no algorithm which for a given arbitrary Diophantine equation would tell whether the equation has a solution or not.

3.1 Markov Numbers

In this part, we are interested in the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz \tag{3.2}$$

called *Markov equation*. This equation is the leading part the Markov theory and we will find all solutions by a simple algorithm. For a moment, let us consider this Diophantine equation as an algebraic variety V

$$V := \left\{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^2 - 3xyz = 0\right\}$$

and the surface defined by Markov equation has many automorphisms. It is not so difficult to see that permutation of variables or change of signs is an automorphism of the variety defined by this equation. Furthermore there is another automorphism which produces all the integral solutions except $(0, 0, 0)$. Let us see how we can find all possible positive integer solutions of Markov equation. Assume that (m, m_1, m_2) be a solution of this equation:

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$$

We get a quadratic equation in the third coordinate m of which we know a solution by fixing two coordinates of this solution, say m_1 and m_2 so equation becomes

$$x^2 - 3m_1m_2x + m_1^2 + m_2^2 = 0$$

has two solutions : m and m' with

$$m + m' = 3m_1m_2 \quad \& \quad mm' = m_1^2 + m_2^2$$

We have to verify that m' is also a positive integer. The first equality shows us that m' is an integer because (m, m_1, m_2) is already a solution triple so that they are integers furthermore m' is positive by the second equality. Thus $m' = 3m_1m_2 - m$ gives us another solution triple

$$(3m_1m_2 - m, m_1, m_2)$$

In conclusion, if we start with a solution (m, m_1, m_2) of the equation, we can extract three other solutions

$$(m' = 3m_1m_2 - m, m_1, m_2) \qquad (m, m_1' = 3mm_2 - m_1, m_2) \qquad (m, m_1, m_2' = 3mm_1 - m_2)$$

of Markov equation by changing the role of m_i above.

Definition 3.1.1. The solution $(m_1, m_2, m_3) \in \mathbb{N}^3$ of the equation with $m_i \neq 0$ for $i \in \{1, 2, 3\}$ are called *Markov triples* and each number m_i is called *Markov number.*

It is easy to see that $(1, 1, 1)$ is a Markov triple. All solutions can be generated from the first triple by using previous algorithm, we will say two solutions are neighbors if they share two components:

First solution $(1, 1, 1)$ has only one neighbor : $(1, 2, 1)$.

Second solution $(1, 2, 1)$ has two neighbors: $(1, 1, 1)$ and $(1, 5, 2)$.

After $(1, 5, 2)$ any other solution has exactly three neighbors.

The Markov triples are :

$$(1,1,1), \quad (1,1,2), \quad (1,2,5), \quad (1,5,13), \quad (2,5,29), \ldots$$

Hence, the set of Markov numbers is union of the solutions and denoted by $\mathcal{M}$ as follows:

$$\mathcal{M} = \{1, 2, 5, 13, 29, \dots\} \tag{3.3}$$

To order all solutions derived from the first, take any Markov triple and write the maximum in the middle

$$(m_1, m, m_2)$$

Then we can see easily

$$m_2' = 3m_1m - m_2 > m \quad \text{and} \quad m_1' = 3m_2m - m_1 > m \tag{3.4}$$

It means that two of neighbors have larger maximum than the initial solution and the third is smaller, namely m', since it comes from the previous one:

$$m_2' > m_1' > m > m' \quad \text{or} \quad m_1' > m_2' > m > m' \tag{3.5}$$

depending on which of m_1 or m_2 is larger. The recursive rule is following:

$$\begin{array}{ccc} & a, b, c & \\ \swarrow & & \searrow \\ a, 3ab - c, b & & b, 3bc - a, c \end{array}$$

Hence we can see all the solutions as a tree, namely Markov Tree,

$$\begin{array}{c}
(1,1,1) \\
| \\
(1,2,1) \\
| \\
(1,5,2) \\
\swarrow \qquad \searrow \\
(1,13,5) \qquad\qquad (5,29,2) \\
(1,34,13) \quad (13,194,5) \quad (5,433,29) \quad (29,169,2) \\
(1,89,34) \quad (34,1325,13) \quad \dots \quad (29,14701,169) \quad (169,985,2)
\end{array}$$

Figure 3.1: Markov Tree

Now we can show some properties of Markov numbers.

Proposition 3.1.1. There is no other solution (m_1, m_2, m_3) containing repeated numbers other than $(1,1,1)$ and $(1,2,1)$.

Proof. Suppose that $m_2 = m_3$ and Markov equation tells us that:

$$m_1^2 + 2m_2^2 = 3m_1m_2^2$$

which means

$$m_2 \mid m_1^2 \Rightarrow m_1 = bm_2$$

where b is positive integer. We put it into Markov equation then we obtain:

$$(bm_2)^2 + 2m_2^2 = 3bm_2m_2^2 \Longleftrightarrow b^2 + 2 = 3bm_2$$

implies that

$$b \mid 2 \Longleftrightarrow b = 2 \quad \text{or} \quad b = 1$$

Therefore, if $b = 1$ we have $m_1 = m_2 = m_3 = 1$ so $(1,1,1)$ and otherwise if $b = 2$ then it gives us triple $(1,2,1)$.

□

$(1,1,1)$ and $(1,2,1)$ are called singular solutions and others are called nonsingular solutions. In the above case, the smallest nonsingular solution is $(1,5,2)$.

Theorem 3.1.1. Every Markov number appears as maximum for some Markov triple.

Proof. It is clear for singular solution. Suppose that $m \geq 5$ is one of the Markov number in a nonsingular solution (m_1, m_2, m_3) with $m_1 > m_2 > m_3$.

- ✓ If $m = m_1$, we are done .
- ✓ If $m = m_2$, then m_2 is the maximum of the smaller neighbor of (m_1, m_2, m_3).
- ✓ If $m = m_3$, going back in the tree, m_3 eventually becomes largest after two next step or m stays the smallest all the way. But in this case, if we continue up to $(1,5,2)$, then we conclude that $m = 1$ which contradicts with $m \geq 5$.

□

Lemma 3.1.2. Any two Markov numbers in a Markov triple are relatively prime.

Proof. Obviously, it is true for $(1,1,1),(1,2,1)$ and $(1,5,2)$. If d is a divisor of, without loss of generality , m_1 and m_2 then d must be divide also the third coordinate m_3 since

$$m_1^2+m_2^2+m_3^2=3m_1m_2m_3 \tag{3.6}$$

Going back in the tree, we will see that d divides $1,5$ and 2 which are number of the smallest nonsingular solution. It is equivalent to say $d=1$. Thus any two Markov numbers in a solution triple are relatively prime.

□

In the next section, we will see the main result in Markov's 1880 paper [12]. For this, we first define two characteristic numbers for a Markov triple:

Definition 3.1.2. Suppose (m,m_2,m_3) be a Markov triple with Markov number m such that $m\geq m_1$, $m\geq m_2$.

i. Let u be the least positive residue of $\pm m_1/m_2 \mod m$. It is said to be the *characteristic number* of (m,m_2,m_3).

ii. We define v by the following equation

$$u^2+1=mv \tag{3.7}$$

Let us compute characteristic number of some Markov triples.

Example 3.1.1. For (2,29,5), Markov number is 29 so the values u and v are as follows:

$$2x\equiv\pm 5 \mod 29 \Rightarrow x=12 \text{ or } x=17$$

Since $12<29/2$ the characteristic number u of 29 is 12. Thus:

$$12^2=-1+29v\Rightarrow v=5$$

Let us see characteristic numbers and Markov numbers for the first fifteen Markov triples in following table:

Markov number	u	v
1	0	1
2	1	1
5	2	1
13	5	2
29	12	5
34	13	5
89	34	13
69	70	29
194	75	29
233	89	34
433	179	74
610	233	89
985	408	169
1325	507	194
1597	610	233

Table 3.1: Characteristic numbers of some Markov triples

3.1.1 From Markov Forms to Markov Theorem

A *real quadratic form* is a homogenous polynomial of degree 2 in two integer variables as the form

$$f(x,y) = ax^2 + bxy + cy^2 \qquad \text{with } \ a,b,c \in \mathbb{R} \tag{3.8}$$

If we complete the square to obtain:

$$\begin{aligned} 4af(x,y) &= 4a^2x^2 + 4abxy + 4acy^2 && (3.9)\\ &= (2ax+by)^2 + 4acy^2 - b^2y^2 && (3.10)\\ &= (2ax+by)^2 - Dy^2 && (3.11) \end{aligned}$$

where $D = b^2 - 4ac$ discriminant of quadratic form. This value plays an important role to determine the type of quadratic form:

- If $D < 0$, the right hand side is always positive, so the sign of $f(x,y)$ depends only the sign of a so that this type is called *definite form*. In this case, when $a > 0$ we say *positive definite form* and otherwise we say *negative definite form*.
- If $D > 0$, the quadratic form takes positive or negative values so we call *indefinite form*.

Suppose that f is indefinite form with discriminant Δ. Let us define:

$$m(f) := \inf\{\mid f(x,y) \mid : f(x,y) \neq 0, (x,y) \in \mathbb{Z}^2\} \tag{3.12}$$

Set

$$M(f) = \frac{\sqrt{\Delta}}{m(f)} \tag{3.13}$$

which we call *Markov constant*.

Definition 3.1.3. The set

$$\mathcal{M} = \{M(f) : f \text{ indefinite form } \}$$

is called *Markov spectrum*.

Definition 3.1.4. Let (m, m_2, m_3) be a Markov triple with $m \geq m_2, m_3$ and u be the characteristic number of m. Then *Markov form* $f_m(x,y)$ is defined as follows:

$$f_m(x,y) = mx^2 + (3m - 2u)xy + (v - 3u)y^2$$

The discriminant of f_m is:

$$\Delta(f_m) = 9m^2 - 4$$

By the theory of quadratic forms, we get also :

$$\inf(\mid f_m(x,y) \mid) = m$$

Hence, by previous construction, we get :

$$M(f_m) = \sqrt{9 - \frac{4}{m^2}}$$

As we see, $M(f_m)$ is always smaller than 3 and it is the limit point in the Markov spectrum. These constants will play an important role in the next part and we denote it by $\mathcal{M} \cap [0,3)$. See [11] for more details.

Markov Triple	Quadratic form $f_m(x,y)$	Markov constant $M(f_m)$	
$(1,1,1)$	x^2+xy-y^2	$\sqrt{5}$	$=2,23606797\ldots$
$(1,2,1)$	$2x^2+4xy-2y^2$	$\sqrt{8}$	$=2,82842712\ldots$
$(1,5,2)$	$5x^2+11xy-5y^2$	$\sqrt{221}/5$	$=2,97321374\ldots$
$(1,13,5)$	$13x^2+29xy-13y^2$	$\sqrt{1517}/13$	$=2,99605262\ldots$
$(2,29,5)$	$29x^2+63xy-31y^2$	$\sqrt{7565}/29$	$=2,99920718\ldots$
$(1,34,13)$	$34x^2+76xy-34y^2$	$\sqrt{2600}/17$	$=2,99942324\ldots$
$(1,89,34)$	$89x^2+199xy-89y^2$	$\sqrt{71285}/89$	$=2,99991583\ldots$
$(2,169,29)$	$169x^2+367xy-181y^2$	$\sqrt{257045}/169$	$=2,99997665\ldots$
$(5,194,13)$	$194x^2+432xy-196y^2$	$\sqrt{338720}/194$	$=2,99998228\ldots$
$(1,233,89)$	$233x^2+521xy-233y^2$	$\sqrt{488597}/233$	$=2,99998772\ldots$

Table 3.2: Markov Triples, corresponding quadratic forms and Markov constants

Even if Markov spectrum and Lagrange spectrum seem to be very irrelavant each other, they are closely related. We can see this relation by an analogy in two spectrum:

Lagrange Spectrum	Markov Spectrum
Let α, β and be two irrationals. Then $\alpha \sim \beta$ if there is a matrix	Let f and g be two quadratic forms. Then $f \sim g$ if there is a matrix
$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$
such that $\alpha = A.\beta$	such that $g = A.f$
i.e. $\alpha = \dfrac{a\beta + b}{c\beta + d}$	i.e. $g(x,y) = f(ax+by, cx+dy)$
Equivalent numbers have the same Lagrange numbers:	Equivalent forms have the same $m(f)$ and Δ so:
$\alpha \sim \beta \implies L(\alpha) = L(\beta)$	$f \sim g \implies M(f) = M(g)$

Markov proved in his famous paper [12] that every quadratic form f with $M(f) < 3$ is equivalent to a Markov form $f_m(x,y)$ and then he proved that Lagrange and Markov spectrum coincide up to 3 that is

$$\mathcal{M} \cap [0,3) = \mathcal{L} \cap [0,3)$$

as stated in the main theorem of Markov theory:

Theorem 3.1.3. *(Markov's Theorem)* [1, Chapter 1]
Suppose $\mathcal{M} = \{1,2,5,13,29,34,\dots\}$ be the set of Markov numbers.

i. The Lagrange spectrum below 3 is given by the set

$$\mathcal{L}_{<3} = \left\{ \sqrt{9 - \frac{4}{m^2}} : m \in \mathcal{M} \right\}$$

Also, there is an inequivalent sequence of quadratic irrationals such as

$$\gamma_m = \frac{m + 2u + \sqrt{9m^2 - 4}}{2m}$$

where u is the characteristic number of m and whose Lagrange numbers are

$$L(\gamma_m) = \sqrt{9 - \frac{4}{m^2}}$$

ii. Conversely every $\alpha \notin \mathbb{Q}$ with $L(\alpha) < 3$ is of this form which means that α is equivalent to exactly one γ_m with $L(\gamma_m)$ as described above.

Proof. [1, chapter 9, p. 185].

To be more clear, take an example:

Example 3.1.2. Let us take $(13,1,5)$ Markov triple. The characteristic numbers u and v are 5 and 2 respectively. Thus, Markov form $f_{13}(x,y)$ is:

$$f_{13}(x,y) = 13x^2 + 29xy - 13y^2$$

with discriminant $\Delta = 1517$. Hence

$$M(f_{13}) = \frac{\sqrt{1517}}{13}$$

On the other hand, Markov theorem says that this number is the Lagrange number of quadratic irrational γ_{13} :

$$L(\gamma_{13}) = \frac{\sqrt{9.13^2 - 4}}{13} = \frac{\sqrt{1517}}{13}$$

In fact this number shows that for any given quadratic irrational α, there is infinitely many rational p/q such that

$$|\alpha - \frac{p}{q}| \leq \frac{13}{\sqrt{1517}q^2}$$

Furthermore, there is no L superior to $\sqrt{1517}/13$ when α is equivalent to β as in Proposition 2.2.2 where:

$$\beta = \frac{23 + \sqrt{1517}}{26}$$

3.1.2 Uniqueness Conjecture

Uniqueness of the Markov triples was a little hidden in Markov's works [11], [12] and it was the subject of a question asked from Frobenius in his treatise 1913 [9] and it is known today as the uniqueness conjecture for Markov numbers. Despite the simplicity of the statement, Cassels showed the depth and difficulty of this conjecture in [3], which made it famous. Several mathematicians [7], [6] have shown that this conjecture branched out into several problems in number theory and Diophantine approximation.

Uniqueness Conjecture I. [1, Chapter 1] For a Markov number $m > 2$, there exists exactly one pair (m_1, m_2) of positive integers with $m > m_1 > m_2$ such that

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$$

In other words, every Markov number appears exactly once as the maximum in a Markov triple.

Uniqueness Conjecture II. [1, Chapter 1] Suppose that α and β are two irrational numbers with $L(\alpha) < 3$ and $L(\beta) < 3$. Then

$$L(\alpha) = L(\beta) \Longrightarrow \alpha \sim \beta$$

Let us see these two conjectures are equivalent.

Proposition 3.1.2. Uniqueness Conjecture *I* and *II* are equivalent.

Proof. Assume that uniqueness of Markov numbers holds and let α and β be two irrationals with $L(\alpha) = L(\beta) < 3$. Markov theorem 3.1.3 says that there exists γ_r and γ_s such that

$$\alpha \sim \gamma_r \quad \text{and} \quad \beta \sim \gamma_s \tag{3.14}$$

where r and s are two Markov numbers so they are maximum in the corresponding triple. Since equivalent numbers have the same Lagrange number, we obtain

$$L(\alpha) = L(\gamma_r) \quad \text{and} \quad L(\beta) = L(\gamma_s) \tag{3.15}$$

By hypothesis, we know that $L(\alpha) = L(\beta)$ so we have also $L(\gamma_r) = L(\gamma_s)$ that is

$$\frac{\sqrt{9r^2-4}}{r} = \frac{\sqrt{9s^2-4}}{s} \Longrightarrow r = s$$

Recall that γ_m depends on Markov number m and characteristic number u. Since uniqueness property holds, there exist exactly one pair that determines characteristic number of $r = s$. Therefore $\gamma_r = \gamma_s$ and then we have

$$\alpha \sim \gamma_r = \gamma_s \sim \beta$$

which implies that $\alpha \sim \beta$.

Conversely, assume that uniqueness property does not hold. We have to show that there exist two inequivalent numbers α and β with $L(\alpha) = L(\beta) < 3$. Since uniqueness assumption is false, there exist a Markov number $r = s$ which is the maximum in two different triples so there exist two different characteristic numbers u_r and u_s corresponding same Markov number. Markov theorem implies that

$$L(\gamma_r) = L(\gamma_s) < 3$$

and we can take $\alpha = \gamma_r$ and $\beta = \gamma_s$ which are inequivalent.

$\square$

Even if the uniqueness hypothesis can not be verified for almost 100 years, there is a lot of work that shows the uniqueness of a certain Markov number.

3.2 Combinatorics of Markov Numbers

3.2.1 Farey index

It is well known that there is a method to produce all rationals between 0 and 1 by an operation which is called *Farey sum* or the *mediant* of two rational numbers:

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$$

Now let us present the process of construction. We start by two rationals 0 and 1 so write them in the first row

$$\frac{0}{1} \qquad\qquad\qquad\qquad \frac{1}{1}$$

then copy the first row in the second by replacing their Farey sum between 0 and 1

$$\begin{array}{ccc} \frac{0}{1} & & \frac{1}{1} \\ \frac{0}{1} & \frac{1}{2} & \frac{1}{1} \end{array}$$

and continue by the same rule, we rewrite $(n-1)$-th row in the n-th line by replacing their Farey sum between two consecutive rationals. Thus, we get a piece of Farey table as follows:

$\frac{0}{1}$				$\frac{1}{2}$				$\frac{1}{1}$
$\frac{0}{1}$		$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$		$\frac{1}{1}$
$\frac{0}{1}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{1}{1}$

Table 3.3: Farey Table

We will see that this table consists all rational numbers between 0 and 1. Let us now prove some useful lemmas to see this fact.

Lemma 3.2.1. Let p/q be rational between 0 and 1.

i. Every row in Farey table arises a sequence strictly increasing from 0 to 1.

ii. Every fraction p/q is reduced in Farey table.

Proof. We prove the first assertion by induction. It is clear for the first row so assume that is true for two consecutive numbers in some row $p/q < r/s$ but then

$$\begin{aligned} \frac{p}{q} < \frac{r}{s} \Leftrightarrow ps < qr \Leftrightarrow pq + ps < pq + qr \\ \Leftrightarrow p(q+s) < q(p+r) \\ \Leftrightarrow \frac{p}{q} < \frac{p+r}{q+s} \end{aligned}$$

on the other hand

$$\begin{aligned} \frac{p}{q} < \frac{r}{s} \Leftrightarrow ps < qr \Leftrightarrow rs + ps < rs + qr \\ \Leftrightarrow s(p+r) < r(q+s) \\ \Leftrightarrow \frac{p+r}{q+s} < \frac{r}{s} \end{aligned}$$

hence we must have

$$\frac{p}{q} < \frac{p+q}{r+s} < \frac{r}{s}$$

in the next row, as desired.
To prove second assertion, we claim that $ps - qr = 1$ where p/q and r/s be two consecutive rationals in a Farey row. It is true for the first row so assume for n th row and check for the next:

$$1 = ps - qr = pq + ps - pq - qr = p(q+s) - q(p+r)$$

It means that $gcd(p+r, q+s) = 1$ and do similar for the right hand side which completes the proof.

$\square$

In fact, Farey table can be seen as a binary tree produced by the following recursive rule

$$\begin{array}{ccc} & \frac{p}{q}, \frac{x}{y}, \frac{r}{s} & \\ \swarrow & & \searrow \\ \frac{p}{q}, \frac{p+x}{q+y}, \frac{x}{y} & & \frac{x}{y}, \frac{r+x}{s+y}, \frac{r}{s} \end{array}$$

with starting triple $(\frac{0}{1}, \frac{1}{2}, \frac{1}{1})$ and every triple in a row is called the *Farey triple*. The following binary tree is called *Farey tree* and denote it by $T_{\mathcal{F}}$:

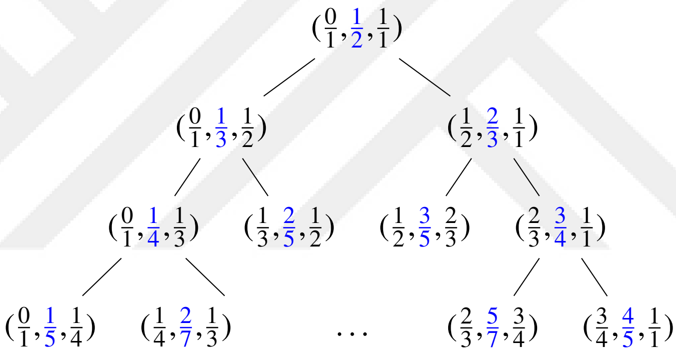

Figure 3.2: Farey Tree $T_{\mathcal{F}}$

Theorem 3.2.2. [1, Chapter 2] Farey tree contains all rational numbers $t \in [0,1]$ and every $t \neq 0, 1$ is generated by a Farey sum exactly once.

Proof. See [1, chapter 2].

Now we have two combinatorically identical binary trees $T_{\mathcal{F}}$ and $T_{\mathcal{M}}$. Hence each Markov number can be indexed by a Farey rational that is

$$\begin{aligned} \iota : \mathbb{Q} \cap [0,1] &\to \mathcal{M} \\ \frac{p}{q} &\mapsto m_{\frac{p}{q}} \end{aligned}$$

where the place of Markov number m in Markov tree is the same place corresponding rational number p/q in Farey tree. Therefore uniqueness conjecture can be reformulated as follows:

Uniqueness Conjecture III. [1, Chapter 2] The map $\iota : \mathbb{Q} \cap [0,1] \to \mathcal{M}$ is injective.

3.2.2 Cohn Matrices

Theorem 3.2.3. *(Fricke Identity)* [1, Chapter 2] Let A and B be matrices in $\mathrm{SL}(2,\mathbb{Z})$, then

$$\operatorname{tr}(A)^2+\operatorname{tr}(B)^2+\operatorname{tr}(AB)^2=\operatorname{tr}(A)\operatorname{tr}(B)\operatorname{tr}(AB)+\operatorname{tr}(ABA^{-1}B^{-1})+2$$

Proof. See [1, chapter 2].

Recall that a triple $(m_1/3, m_2/3, m_3/3)$ is a solution of Markov-like equation

$$x^2+y^2+z^2=xyz \tag{3.16}$$

if and only if (m_1, m_2, m_3) is a Markov triple. We would like to draw your attention to the similarity between Fricke identity and Markov equation. Indeed there is one to one correspondance between solutions of them when the last two terms of identity are omitted so we can obtain Markov numbers from the trace of unimodular integral 2×2 matrices satisfying an elementary identity of Fricke. Harwey Cohn [5] , what he did is exactly point out this coincidence of the solutions of two equations.

Corollary 3.2.4. Let $A, B \in \mathrm{SL}(2,\mathbb{Z})$ such that $\operatorname{tr}(A), \operatorname{tr}(B)$ and $\operatorname{tr}(AB)$ are positive. Then

$$\left(\frac{\operatorname{tr}(A)}{3}, \frac{\operatorname{tr}(B)}{3}, \frac{\operatorname{tr}(AB)}{3}\right)$$

is a solution of Markov equation if and only if $\operatorname{tr}(ABA^{-1}B^{-1}) = -2$.

Definition 3.2.1. Let m_t be a Markov number and a_t, c_t be integers. A matrix in the following term

$$C_t=\begin{pmatrix} a_t & m_t \\ c_t & 3m_t-a_t \end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z})$$

is called *Cohn matrix*. A matrix triple (R, T, S) is said to be a *Cohn triple* if R, T and S are Cohn matrices with $T = RS$ and (m_r, m_t, m_s) Markov numbers where $r,t,s \in \mathbb{Q}\cap[0,1]$, $t = r\oplus s$ associated respectively .

Theorem 3.2.5. [1, Chapter 2] Let (M, MN, N) be a Cohn triple associated with the Markov triple (m_r, m_t, m_s) where $t = r\oplus s$. Then we have

$$\begin{tikzcd}[column sep=tiny] & M, MN, N \arrow[dl, dash] \arrow[dr, dash] & \\ M, MN^2, N & & MN, MN^2, N \end{tikzcd}$$

where (M, M^2N, MN) and (MN, MN^2, N) is a Cohn triple associated with Markov triples $(m_r, m_{r\oplus t}, m_t)$ and $(m_t, m_{t\oplus s}, m_s)$ respectively.

Proof. See [1, chapter 2].

This result gives the recursive rule for the construction of the Cohn tree denoted by T_C. In fact, there are infinitely many Cohn trees with starting triple depending on a_t and c_t which are integers. The natural question here is : Is it possible to classify all starting Cohn triples ? The answer is as follows:

Theorem 3.2.6. [1, Chapter 2] All starting triples of Cohn matrices $C_{0/1}$, $C_{1/2}$ and $C_{1/1}$ for $m_{0/1} = 1$, $m_{1/2} = 5$ and $m_{1/1} = 2$ are given by

$$C_{0/1} = \begin{pmatrix} a & 1 \\ 3a - a^2 - 1 & 3 - a \end{pmatrix}$$

,

$$C_{1/2} = \begin{pmatrix} 5a + 2 & 5 \\ -5a^2 + 11a + 5 & 13 - 5a \end{pmatrix} \quad \text{and} \quad C_{1/1} = \begin{pmatrix} 2a + 1 & 2 \\ -2a^2 + 4a + 2 & 5 - 2a \end{pmatrix}$$

Proof. See [1, chapter 2].

Note that the trace of Cohn matrices C_t, indexed by $t \in \mathbb{Q} \cap [0, 1]$ according their place in Farey tree, depends on Markov number. Therefore the correspondance between Farey tree and Markov tree allows us to say another version of uniqueness conjecture in this context.

Uniqueness Conjecture IV. [1, Chapter 2] The matrices in the Cohn tree that arise from any starting Cohn triple have different traces.

3.3 Geometry of Markov Numbers in Hyperbloic Plane

3.3.1 Uniqueness Conjecture in Hyperbolic Plane

Let us begin to define a special group which will play an important role in this part.

Definition 3.3.1. The set of invertible integral two-by-two matrices

$$\mathrm{GL}(2, \mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = \pm 1 \right\}$$

form a group under the matrix product and it is called *general linear group*.

Proposition 3.3.1. The general linear group $\mathrm{GL}(2,\mathbb{Z})$ is generated by two matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.17}$$

This group is closely related with continued fraction algorithm. Our interest is the action on the left of this group on the complex plane $\mathbb{C}$ defined by

$$\begin{gathered} \hat{T} : \mathrm{GL}(2,\mathbb{Z}) \times \mathbb{C} \longrightarrow \mathbb{C} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \longmapsto \hat{T}(z) = \frac{az+b}{cz+d} \end{gathered}$$

Here, the maps $\hat{T}$ is called *fractional linear map* and they form a group called *projective linear group* denoted by $\mathrm{PGL}(2,\mathbb{Z})$ with composition as operation. We can deduce immediately that T and $-T$ represente the same transformation from the following result:

Lemma 3.3.1. The map $\varphi : \mathrm{GL}(2,\mathbb{Z}) \to \mathrm{PGL}(2,\mathbb{Z})$ where $\varphi(T) = \hat{T}$ is a group homomorphism with kernel $\{\pm I\}$. Therefore, $\mathrm{PGL}(2,\mathbb{Z}) \cong \mathrm{GL}(2,\mathbb{Z})/\{\pm I\}$.

Recall from Section 2.2. that two irrational numbers are equivalent if their continued fraction expansions eventually coincide. The previous lemma allows us to give an alternative equivalence relation on irrational numbers.

Proposition 3.3.2. Let α and β be two irrational numbers. They are equivalent if and only if $\hat{T}(\alpha) = \beta$ for some $T \in \mathrm{GL}(2,\mathbb{Z})$.

Proof. See [7, chapter 2].

It is possible to classify all matrices in the linear group hence in projective linear group according to their trace. Look at the fixed points of the action on $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that is find the solutions of the equation

$$\frac{az+b}{cz+d} = z$$

we obtain two solutions as fixed points

$$\zeta_1, \zeta_2 = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c} = \frac{(a-d) \pm \sqrt{(\mathrm{tr}\,T)^2 - 4}}{2c} \tag{3.18}$$

Remark that T and $-T$ represente same transformation so we may assume that $\mathrm{tr}(T) \geq 0$.

- If $\mathrm{tr}(T) < 2$, then ζ_1 and ζ_2 are conjugate complex numbers and T is called *elliptic*.
- If $\mathrm{tr}(T) = 2$, then the map has only one fixed point which is rational $(a-d)/2c$ and T is called *parabolic*.
- If $\mathrm{tr}(T) > 2$, then ζ_1 and ζ_2 are conjugate quadratic irrationals and T is called *hyperbolic*.

The last is the interesting case for our purposes. Let (m_r, m_t, m_s) be a Markov triple with Markov number m_t. Recall that, for $a \in \mathbb{Z}$, $t \in \mathbb{Q} \cap [0,1]$, we have :

$$C_t(a) = \begin{pmatrix} am_t + u_t & m_t \\ (3a-a^2)m_t - (2a-3)u_t - v_t & (3-a)m_t - u_t \end{pmatrix}$$

where u_t and v_t characteristic numbers of m_t. For any given $a \in \mathbb{Z}$, there is a starting triple and therefore a Cohn tree $T_C(a)$ arising from this triple. Note that the matrices $C_t(a)$ are distinct in the $T_C(a)$ for every $a \in \mathbb{Z}$. Let us consider B_t as transpose of $C_t(2)$ that is

$$B_t = \begin{pmatrix} 2m_t + u_t & 2m_t - u_t - v_t \\ m_t & m_t - u_t \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

Note that $\mathrm{tr}(T) = 3m_t > 2$ since $m_t \in \{1, 2, 5, 13, 29, \dots\}$ and fixed points can be obtain from quadratic equation

$$\frac{(2m_t + u_t)x + 2m_t - u_t - v_t}{m_t x + m_t - u_t} = x$$

with discriminant $\Delta = 9m_t^2 - 4$ hence fixed points of the map B_t

$$\gamma_t, \gamma_t' = \frac{m_t + 2u_t \pm \sqrt{9m_t^2 - 4}}{2m_t}$$

Proposition 3.3.3. Let γ_t, γ_t' be Markov quadratic irrational as defined above. They are $\mathrm{SL}(2,\mathbb{Z})$-equivalent; there is a matrix $T \in \mathrm{SL}(2,\mathbb{Z})$ such that $T(\gamma_t) = \gamma_t'$. Thus, we have $L(\gamma_t) = L(\gamma_t')$.

Proof. See [1, chapter 2].

Our aim is to introduce a connection between geodesics and the Lagrange spectrum. Now we come to the the main result:

Theorem 3.3.2. [1, Chapter 3] Let $T \in \Gamma(3)$ be hyperbolic with fixed points ζ_T and $\zeta_T^{'}$. The projection $\pi(A_T)$ is simple if and only if the Lagrange numbers $L(\zeta_T) < 3$ and $L(\zeta_T^{'}) < 3$.

Proof. See [1, chapter 2].

Markov's Theorem says that every irrational $\alpha \notin \mathbb{Q}$ with $L(\alpha) < 3$ is equivalent to some γ_t hence α' to $\gamma_t^{'}$. Hence we get

$$L(\alpha) = L(\gamma_t) \quad \text{and} \quad L(\alpha') = L(\gamma_t^{'})$$

Consequently the result in the previous Proposition 3.3.3 becomes the following:

Corollary 3.3.3. Let $T \in \Gamma(3)$ be hyperbolic with fixed points ζ_T and $\zeta_T^{'}$. The projection $\pi(A_T)$ is simple if and only if the Lagrange numbers $L(\zeta_T) = L(\zeta_T^{'}) < 3$.

Proposition 3.3.4. Let A be axis of a hyperbolic map $T \in \Gamma(3)$ whose projection $\Pi = \pi(A) \in \mathbb{H}/\Gamma(3)$ is a simple closed geodesic. The length of Π is given by

$$\ell(\Pi) = 2\log\frac{t+\sqrt{t^2-4}}{2} \tag{3.19}$$

where $t = \mathrm{tr}(B)$ is the trace of a primitive matrix B with axis A.

Note that primitve matrices have the same trace so the length of simple closed geodesic is well defined. Now return to the group $\Gamma(3)$; we denote the axis of B_t by A_t.

Lemma 3.3.4. B_t is primitive for the stabilizer of the axis A_t.

Proof. See [1, chapter 2].

Remark that B_t is not in $\Gamma(3)$ because of $3 \nmid m$ however B_t^2 is in $\Gamma(3)$

$$B_t(3) = B_t^2 = \begin{pmatrix} 6m_t^2 + 3m_t u_t - 1 & 6m_t^2 - 3m_t u_t - 3m_t v_t \\ \\ 3m_t^2 & 3m_t^2 - 3m_t u_t - 1 \end{pmatrix} \in \Gamma(3) \tag{3.20}$$

thus $B_t(3)$ is a primitive map with axis A_t and

$$\mathrm{tr}(B_t(3)) = 9m_t^2 - 2 \tag{3.21}$$

by previous Proposition 3.3.4 the length of $\Pi_t = \pi(A_t)$ is given by

$$\ell(\Pi_t) = 2\log\frac{9m_t^2 - 2 + 3m_t\sqrt{(9m_t^2-4)}}{2} \tag{3.22}$$

In conclusion, we notice that the length of Π_t surprisingly depends only on the Markov number m_t. Let us define an equivalence relation on two simple closed geodesic.

Definition 3.3.2. Let Π and $\hat{\Pi}$ be two simple closed geodesics. We say that Π and $\hat{\Pi}$ are equivalent if there is an automorphism σ such that $\sigma(\Pi) = \hat{\Pi}$.

It is clear that equivalent geodesics have the same length but the converse is not always true.

Proposition 3.3.5. Two simple closed geodesics Π and $\hat{\Pi}$ on $\mathbb{H}/\Gamma(3)$ of the same length are equivalent if and only if the uniqueness conjecture is true.

Proof. Let us take two simple closed geodesics Π and $\hat{\Pi}$ on $\mathbb{H}/\Gamma(3)$ of the same length

$$\ell(\Pi) = \ell(\hat{\Pi}) \tag{3.23}$$

Then we have $\Pi = \pi(A)$ with fixed points ζ and ζ' such that $L(\zeta) = L(\zeta') < 3$. Markov's Theorem tells us that there exists quadratic irrationals γ_t and its conjugate $\gamma_t^{'}$ such that $\zeta \sim \gamma_t$ and $\zeta' \sim \gamma_t^{'}$ for some $t \in \mathbb{Q} \cap [0,1]$. It follows that $R(A) = A_t$ for some $R \in \mathrm{GL}(2,\mathbb{Z})$ which means $\Pi(A)$ and $\Pi(A_t)$ are equivalent. Furthermore, equivalent geodesic have the same length that is $\ell(\Pi) = \ell(\Pi_t)$. Similarly, we can deduce also $\ell(\hat{\Pi}) = \ell(\Pi_{\hat{t}})$. By hypothesis, we conclude that

$$\ell(\Pi_t) = \ell(\Pi_{\hat{t}}) \tag{3.24}$$

which depend only on the Markov numbers. Hence $m_t = m_{\hat{t}}$ as indicated in the conjecture. Suppose now the uniqueness conjecture is true, then A and $\hat{A}$ and so their projection Π and $\hat{\Pi}$ on $\mathbb{H}/\Gamma(3)$ are connected by an automorphism. Because if not, there are two Markov numbers m_t and $m_{\hat{t}}$ with $t \neq \hat{t}$. In this case, γ_t and $\gamma_{\hat{t}}$ are not equivalents and it is equivalent to say there is no map carrying Π onto $\hat{\Pi}$.

□

It is a new reformulation of the uniqueness conjecture.

Uniqueness Conjecture V. [1, Chapter 3] Two simple closed geodesics Π and $\hat{\Pi}$ on $\mathbb{H}/\Gamma(3)$ of the same length are equivalent.

Chapter 4

Outer Automorphism of $\mathbf{PGL}(\mathbf{2},\mathbb{Z})$

4.1 Functional Equations

In this part, we will introduce two functional equations and our aim is to find two functions that arise from these equations. The first one is as follows:

$$\begin{cases} g(x+1) = g(x) + g(\frac{1}{x}) \\ g(x) = g(\frac{x}{x+1}) \end{cases} \tag{4.1}$$

Can we find such a function g ? Let us take some numerical examples for natural numbers, $x \in \mathbb{N}^*$:

$$\begin{aligned} g(2) &= g(1) + g(1) = 2g(1) \\ g(3) &= g(2) + g(1/2) = 2g(1) + g(1/2) \\ g(4) &= g(3) + g(1/3) = 2g(1) + g(1/2) + g(1/3) \\ &\vdots \\ g(n+1) &= 2g(1) + g(1/2) + g(1/3) + \cdots + g(1/n) \end{aligned}$$

Remark that we have $g(1/n) = g(1/(n+1))$ for every $n \in \mathbb{N}^*$ by the second equality of the functional equation. So we obtain

$$g(2) = 2g(1)$$
$$g(3) = 3g(1)$$
$$g(4) = 4g(1)$$
$$\vdots$$
$$g(n+1) = (n+1)g(1)$$

It means that it suffices to say $g(1)$ to determine image of all the values of $\mathbb{N}^*$. For example, choosing $g(1) = 1$, we obtain identity map of $\mathbb{N}^*$:

$$g : \mathbb{N}^* \to \mathbb{N}^*$$
$$n \mapsto n$$

Now, we ask same question to find a function which is defined on positive rational numbers satisfying the same functional equation, let us choose $x = p/q \in \mathbb{Q}_{>0}$ and by the second equality, observe that

$$g(p/q) = g(p/(p+q)) = g(p/(2p+q)) = \cdots = g(p/(np+q)) = \ldots$$

On the other hand, we have

$$g(\frac{p+q}{p+mq}) = g(\frac{p}{np+q}) + g(\frac{q}{p+rq}) \tag{4.2}$$

where $m,n,r \in \mathbb{N}^*$. Hence, we infer that the value $g(p/q)$ does not depend on the denominator. For any $p/q \in \mathbb{Q}_{>0}$ and $a,b,c \in \mathbb{N}^*$,

$$g(\frac{p+q}{a}) = g(\frac{p}{b}) + g(\frac{q}{c}) \tag{4.3}$$

Thus, we can see easily following function which we call *numerator* function is one of the functions satisfiying functional equation of g:

$$\text{num} : \mathbb{Q}_{>0} \to \mathbb{N}^*$$
$$p/q \mapsto p$$

Further if we suppose $\text{num}(-p/q) = -\text{num}(p/q)$ where $p \in \mathbb{Z}$, $q \in \mathbb{Z}_{>0}$, we can extend it to all rational numbers, that is

$$\begin{aligned} \text{num} : \mathbb{Q} &\to \mathbb{Z} \\ p/q &\mapsto p \end{aligned}$$

In a similar way, we try to determine a function f such that

$$\begin{cases} f(x+1) = f(x) + f(\frac{1}{x}) \\ f(x) = f(\frac{1}{x+1}) \end{cases} \tag{4.4}$$

Take $x \in \mathbb{N}^*$,

$$\begin{aligned} f(2) &= f(1) + f(1) = 2f(1) \\ f(3) &= f(2) + f(1/2) = f(2) + f(1) = 3f(1) \\ f(4) &= f(3) + f(1/3) = f(3) + f(2) = 5f(1) \\ f(5) &= f(4) + f(1/4) = f(4) + f(3) = 8f(1) \\ &\vdots \\ f(n) &= F_{n+1} f(1) \end{aligned}$$

where F_n is nth Fibonacci number. Hence, the function f from natural numbers depends only on the choice of $f(1)$. If we choose $f(1) = 1$, we get

$$\begin{aligned} f : \mathbb{N}^* &\to \mathbb{N}^* \\ n &\mapsto F_{n+1} \end{aligned}$$

Is it possible to extend this function to rational numbers ? The answer is given by the following lemma:

Lemma 4.1.1. For all $x \in \mathbb{Q}_{>0}$ and $n \in \mathbb{N}$, we have

$$f(n+x) = F_{n+1} f(x) + F_n f(1/x)$$

where F_n is nth Fibonacci number.

Proof. We prove by induction. It is obvious for $n = 0$ and $n = 1$. Assume that it is true for n then we prove for $n+1$

$$\begin{aligned} f(n+1+x) &= F_{n+1}f(1+x)+F_nf(1/1+x) \\ &= F_{n+1}(f(x)+f(1/x))+F_nf(x) \\ &= (F_{n+1}+F_n)f(x)+F_{n+1}f(1/x) \\ &= F_{n+2}f(x)+F_{n+1}f(1/x) \end{aligned}$$

as desired.

□

It means that for any positive rational number x, we can reduce the value of $f(x)$ into some multiple of $f(1)$ which is determined only by the Fibonnacci numbers by applying this Lemma 4.1.1. For example, let us find $f(5/3)$:

$$\begin{aligned} f(5/3) &= f(1+2/3) \\ &= F_1f(2/3)+F_2f(3/2) \\ &= F_1f(1/2)+F_2f(1+1/2) \\ &= F_1f(1)+F_2(F_1f(1/2)+F_2f(2)) \\ &= F_1f(1)+F_2(F_1f(1)+F_2F_3f(1)) \\ &= (F_1+F_2F_1+F_2^2F_3)f(1) \\ &= 4f(1) \end{aligned}$$

We call this extension *conumerator* function and denoted by $\mathrm{con}(x)$. Now we will see interesting relations between these two functions satisfying functional equations:

Proposition 4.1.1. Let f and g be satisfied two functional equations as defined above. Then

i. $f\left(\dfrac{f(x)}{f(1/x)}\right) = g(x)$

ii. $g\left(\dfrac{f(x)}{f(1/x)}\right) = f(x)$

Proof. To see these, it suffices to show that functions left hand side satisfy functional equations of numerator function and conumerator function respectively. We can see eaisly by using functional equations of f and g:

i.

$$\begin{aligned}
g(x+1) = f\left(\frac{f(x+1)}{f(1/(x+1))}\right) &= f\left(\frac{f(x)+f(1/x)}{f(x)}\right) \\
&= f\left(1+\frac{f(1/x)}{f(x)}\right) \\
&= f\left(\frac{f(1/x)}{f(x)}\right) + f\left(\frac{f(x)}{f(1/x)}\right) \\
&= g(x)+g(1/x)
\end{aligned}$$

And also,

$$\begin{aligned}
g(x/(x+1)) = f\left(\frac{f(x/(x+1))}{f((x+1)/x)}\right) &= f\left(\frac{f(x/(x+1))}{f(1+1/x)}\right) \\
&= f\left(\frac{f(1/x)}{f(1/x)+f(x)}\right) \\
&= f\left(\frac{1}{\frac{f(1/x)+f(x)}{f(1/x)}}\right) \\
&= f\left(\frac{1}{1+\frac{f(x)}{f(1/x)}}\right) \\
&= f\left(\frac{f(x)}{f(1/x)}\right) = g(x)
\end{aligned}$$

ii.

$$\begin{aligned}
f(x+1) = g\left(\frac{f(x+1)}{f(1/(x+1))}\right) &= g\left(\frac{f(x)+f(1/x)}{f(x)}\right) \\
&= g\left(1+\frac{f(1/x)}{f(x)}\right) \\
&= g\left(\frac{f(1/x)}{f(x)}\right) + g\left(\frac{f(x)}{f(1/x)}\right) \\
&= f(x)+f(1/x)
\end{aligned}$$

And also,

$$\begin{aligned} f(1/(x+1)) = g\left(\frac{f(1/(x+1))}{f(x+1)}\right) &= g\left(\frac{f(x)}{f(x)+f(1/x)}\right) \\ &= g\left(\frac{1}{\frac{f(1/x)+f(x)}{f(x)}}\right) \\ &= g\left(\frac{1}{1+\frac{f(1/x)}{f(x)}}\right) \\ &= g\left(\frac{f(x)}{f(1/x)}\right) = f(x) \end{aligned}$$

$\square$

Indeed, we have

$$\operatorname{con}\left(\frac{\operatorname{con}(x)}{\operatorname{con}(1/x)}\right) = \operatorname{num}(x)$$

and

$$\operatorname{num}\left(\frac{\operatorname{con}(x)}{\operatorname{con}(1/x)}\right) = \operatorname{con}(x)$$

Because of the last functional relation between numerator and conumerator, we can extend conumerator function to all rational numbers by

$$\operatorname{con}(-x) = -\operatorname{con}(1/x)$$

Now define a new function which we call Jimm and denote it by $J(x)$:

$$J(x) := \operatorname{con}(x)/\operatorname{con}(1/x)$$

In fact this function can be defined on real numbers, we will see how it is possible in the next part.

4.2 An involution Jimm on the Real Line

We will see some properties of Jimm such as continuity on irrationals and certain equations by the following lemma:

Lemma 4.2.1. Let $J(x) = \mathrm{con}(x)/\mathrm{con}(1/x)$. Then we have,

i. $J(J(x)) = x$

ii. $J(1/x) = 1/J(x)$

iii. $J(1+x) = 1+1/J(x)$

for all $x \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. Recall that conumerator function satisfies functional equation 4.4

i. To show first property,

$$J(J(x)) = \frac{\mathrm{con}\left(\dfrac{\mathrm{con}(x)}{\mathrm{con}(1/x)}\right)}{\mathrm{con}\left(\dfrac{\mathrm{con}(1/x)}{\mathrm{con}(x)}\right)} = \frac{\mathrm{num}(x)}{\mathrm{num}(1/x)} = x$$

Hence, we can say Jimm is an involution.

ii. The second is immediately obvious,

$$J(1/x) = \mathrm{con}(1/x)/\mathrm{con}(x) = 1/(\mathrm{con}(x)/\mathrm{con}(1/x)) = 1/J(x)$$

iii. By using functional equation of conumerator 4.4 we get,

$$\begin{aligned} J(1+x) = \mathrm{con}(x+1)/\mathrm{con}(1/x+1) &= (\mathrm{con}(x)+\mathrm{con}(1/x))/\mathrm{con}(x) \\ &= 1+\mathrm{con}(1/x)/\mathrm{con}(x) \\ &= 1+1/J(x) \end{aligned}$$

□

Even if Jimm is derived from conumerator function, it is more interesting because it can be defined on real numbers. Now, by using these two functional equations of Jimm:

$$J(1+x) = 1+1/J(x) \quad \& \quad J(1/x) = 1/J(x) \tag{4.5}$$

Let us compute the image of a continued fraction expansion of a real number $[a_0, a_1, a_2, \ldots]$ under Jimm involution :

$$\begin{aligned}
J([a_0, a_1, a_2, a_3, \ldots]) &= J(1 + [a_0 - 1, a_1, a_2, a_3, \ldots]) \\
&= 1 + \cfrac{1}{J([a_0 - 1, a_1, a_2, a_3, \ldots])} \\
&= 1 + \cfrac{1}{1 + \cfrac{1}{J([a_0 - 2, a_1, a_3, \ldots])}} \\
&= 1 + \cfrac{1}{1 + \cfrac{1}{\ddots 1 + \cfrac{1}{J([0, a_1, a_3, \ldots])}}} \\
&= 1 + \cfrac{1}{1 + \cfrac{1}{\ddots 1 + \cfrac{1}{1 + J([a_1, a_2, a_3, \ldots])}}} \\
&= 1 + \cfrac{1}{1 + \cfrac{1}{\ddots 1 + \cfrac{1}{2 + \cfrac{1}{J([a_1 - 1, a_2, a_3, \ldots])}}}} \\
&= [\underbrace{1, 1, 1, \ldots, 1}_{a_0 - 1 \text{ times}}, 2, J([a_1 - 1, a_2, a_3, \ldots])] \\
&\vdots \\
&= [\underbrace{1, 1, 1, \ldots, 1}_{a_0 - 1 \text{ times}}, 2, \underbrace{1, 1, 1, \ldots, 1}_{a_1 - 2 \text{ times}}, 2, \underbrace{1, 1, 1, \ldots, 1}_{a_2 - 2 \text{ times}}, 2, 1, 1, \ldots]
\end{aligned}$$

Definition 4.2.1. Let $[a_0, a_1, a_2, a_3, \ldots]$ be a continued fraction expansion of a real number. Then Jimm is defined on real numbers as follows:

$$J : [a_0, a_1, a_2, a_3, \ldots] \longrightarrow [1_{a_0 - 1}, 2, 1_{a_1 - 2}, 2, 1_{a_2 - 2}, 2, \ldots]$$

with these two rules:

$$[\dots,a,1_0,b,\dots] := [\dots,a,b,\dots]$$
$$[\dots,a,1_{-1},b,\dots] := [\dots,a+b-1,\dots]$$

where 1_k represents sequence $1,1,\dots,1$ of length k.

Example 4.2.1. Let us compute Jimm of some continued fractions :

1. $J([1,1,1,\dots]) = \infty$
2. $J([2,2,2,\dots])) = [1,2,2,2,\dots]$
3. $J([n,n,n,\dots]) = [1_{n-1},\overline{2,1_{n-2}}]$
4. $J([\overline{2,2,1,1}]) = [1,\overline{2,4}]$
5. $J([\overline{2,2,1,1,1,1}]) = [1,\overline{2,6}]$
6. $J([\overline{2,2,2,2,1,1}]) = [1,\overline{2,2,2,4}]$

Proposition 4.2.1. Jimm sends the set of quadratic irrational numbers to itself.

Proof. An infinite continued fraction is eventually periodic if and only if it represents quadratic irrational. It is clear that Jimm preserves the set of irrationals having periodic continued fraction expansion, so we are done.

□

Since all Markov irrationals are quadratic, their images under Jimm are also quadratic by this proposition. Let us find images of continued fraction expansion of some Markov irrationals:

Markov number	Continued Fraction of γ_m	Continued Fraction of $J(\gamma_m)$
1	$[\bar{1}]$	∞
2	$[\bar{2}]$	$[1,\bar{2}]$
5	$[\overline{2_2,1_2}]$	$[1,\overline{2,4}]$
13	$[\overline{2_2,1_4}]$	$[1,\overline{2,6}]$
29	$[\overline{2_4,1_2}]$	$[1,\overline{2_3,4}]$
34	$[\overline{2_2,1_6}]$	$[1,\overline{2,8}]$
89	$[\overline{2_2,1_8}]$	$[1,\overline{2,10}]$
169	$[\overline{2_6,1_2}]$	$[1,\overline{2_5,4}]$
194	$[\overline{2_2,1_2,2_2,1_4}]$	$[1,\overline{2,4,2,6}]$
233	$[\overline{2_2,1_{10}}]$	$[1,\overline{2,12}]$
433	$[\overline{2_3,1_2,2_2,1_2}]$	$[1,\overline{2_2,4,2,4}]$
610	$[\overline{2_4,1_2,2_2,1_2}]$	$[1,\overline{2_3,4,2,4}]$
985	$[\overline{2_8,1_2}]$	$[1,\overline{2_7,4}]$
1325	$[\overline{2_2,1_4,2_2,1_6}]$	$[1,\overline{2,6,2,8}]$
1597	$[\overline{2_2,1_{14}}]$	$[1,\overline{2,16}]$

Table 4.1: Continued fraction expansion of image some γ_m under Jimm.

It may be interesting also to see Jimm of Markov irrationals as quadratic form.We know that it can be obtained of course from the limit of convergents of contiued fraction expansion but we will present a nice way to find them in next section.

Chapter 5

Conclusion

This thesis consists of two main parts. The first is about Diophantine approximation theory via Markov equation and the second part is about a fundamental involution of real line induced by the outer automorphism of the extended modular group $\mathrm{PGL}(2,\mathbb{Z})$. Our aim was to find possible relation between them. In the first part, we introduced the general theory of Markov including Markov numbers arised from Markov Diophantine equation and presented the five different reformulation of Uniqueness conjecture in different context such as hyperbolic geometry and combinatorics.

During this period, we studied several questions in relation with different topics. For example, since conumerator function is defined on rational numbers, we thought that it may be interesting to study image of the some branches of Farey tree under conumerator function because Farey numbers are in relation with uniqueness of Markov numbers. And also a fundamental involution of real line called Jimm induced by the outer automorphism of the extended modular group $\mathrm{PGL}(2,\mathbb{Z})$ which is recently discovered inspired us that the effect of Jimm on the Markov quadratic irrationals suggests that this involution must play a role in Markov's theory. Finally, we found a method to find directly the quadratic form of the image of Markov quadratics and then we proved an interesting result on the subset of the image of Markov quadratic irrationals in relation with Fibonacci numbers under Jimm by using a method coming from geometry of Markov numbers.

5.1 Jimm of Markov Irrationals

General functional equation is

$$J(Mx) = J(M)J(x) \quad \text{for all} \;\; M \in PGL_2(\mathbb{Z}) \tag{5.1}$$

Here, $J(M)$ is the image of M under Dyer's automorphism of $PGL_2(\mathbb{Z})$:

$$J : PGL_2(\mathbb{Z}) \to PGL_2(\mathbb{Z})$$
$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \mapsto J(M) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If there is a matrix fixing an irrational then the following is true:

$$x = Mx \Longrightarrow J(x) = J(Mx) = J(M)J(x) \tag{5.2}$$

which means $J(x)$ is fixed also by $J(M)$. Hence, to find the image of a quadratic irrational under Jimm, it suffices to find the image of matrix associated.

Question. For a given matrix $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, how to find $J(M)$?

We know that $J(1) = 1$ and $J(2) = 2$,

$$J(M1) = J(M)J(1) \Longrightarrow J\left(\frac{p+q}{r+s}\right) = \frac{a+b}{c+d} \tag{5.3}$$
$$J(M2) = J(M)J(2) \Longrightarrow J\left(\frac{2p+q}{2r+s}\right) = \frac{2a+b}{2c+d} \tag{5.4}$$

It means that

$$\text{con}\left(\frac{p+q}{r+s}\right) = a+b \tag{5.5}$$
$$\text{con}\left(\frac{r+s}{p+q}\right) = c+d \tag{5.6}$$
$$\text{con}\left(\frac{2p+q}{2r+s}\right) = 2a+b \tag{5.7}$$
$$\text{con}\left(\frac{2r+s}{2p+q}\right) = 2c+d \tag{5.8}$$

Find a, b, c, d and replace in the matrix :

$$J\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} \operatorname{con}\left(\frac{2p+q}{2r+s}\right) - \operatorname{con}\left(\frac{p+q}{r+s}\right) & 2\operatorname{con}\left(\frac{p+q}{r+s}\right) - \operatorname{con}\left(\frac{2p+q}{2r+s}\right) \\ \operatorname{con}\left(\frac{2r+s}{2p+q}\right) - \operatorname{con}\left(\frac{r+s}{p+q}\right) & 2\operatorname{con}\left(\frac{r+s}{p+q}\right) - \operatorname{con}\left(\frac{2r+s}{2p+q}\right) \end{pmatrix} \tag{5.9}$$

Recall.

1. Suppose that (m, m_2, m_3) is a Markov triple with Markov number m such that $m \geq m_1$, $m \geq m_2$.

   i. Let u be the least positive residue of $\pm m_1/m_2 \mod m$. It's said to be the characteristic number of m.

   ii. We define v by the following equation

$$u^2 + 1 = mv \tag{5.10}$$

2. For all $m \in \mathcal{M} = \{1, 2, 5, 13, 29, 34, \dots\}$, Markov quadratic irrational is of the form

$$\gamma_m = \frac{m + 2u + \sqrt{9m^2 - 4}}{2m}$$

   and it is fixed by the matrix

$$M = \begin{pmatrix} 2m+u & 2m-u-v \\ m & m-u \end{pmatrix}$$

   which is hyperbolic with trace $3m$.

Example 5.1.1. Let us compute the image of some Markov quadratics under Jimm by using previous method.

- Let $m = 1$, $u = 0$, $v = 1$ and quadratic irrational assocaited is

$$\gamma_1 = (1 + \sqrt{5})/2$$

  and it is fixed by the matrix

$$M_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

If we compute image of this matrix under Jimm by using formula, we obtain:

$$J(M_1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

whose fixed point is ∞ .

- Let $m = 2$, $u = 1$, $v = 1$ and quadratic irrational associated is

$$\gamma_2 = 1 + \sqrt{2}$$

and it is fixed by the matrix

$$M_2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

If we compute image of this matrix under Jimm by using previous formula, we obtain:

$$J(M_2) = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

whose fixed points are $\pm\sqrt{2}$.

- Let $m = 5$, $u = 2$, $v = 1$ and quadratic irrational assocaited is

$$\gamma_5 = (9 + \sqrt{221})/10$$

and it is fixed by the matrix

$$M_5 = \begin{pmatrix} 12 & 7 \\ 5 & 3 \end{pmatrix}$$

If we compute image of this matrix under Jimm by using formula, we obtain:

$$J(M_5) = \begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$$

whose fixed points are $-1 \pm \sqrt{6}$.

- Let $m = 13$, $u = 5$, $v = 2$ and quadratic irrational assocaited is

$$\gamma_{13} = (23 + \sqrt{1517})/26$$

and it is fixed by the matrix

$$M_{13} = \begin{pmatrix} 31 & 19 \\ 13 & 8 \end{pmatrix}$$

If we compute image of this matrix under Jimm by using formula, we obtain:

$$J(M_{13}) = \begin{pmatrix} 3 & 16 \\ 2 & 11 \end{pmatrix}$$

whose fixed points are $-2 \pm \sqrt{12}$.

Let us see Jimm of Markov irrationals in the following table:

m	$\gamma_m = (m+2u+\sqrt{9m^2-4})/2m$	$J(\gamma_m)$
1	$(1+\sqrt{5})/2$	∞
2	$1+\sqrt{2}$	$\sqrt{2}$
5	$(9+\sqrt{221})/10$	$\sqrt{6}-1$
13	$(23+\sqrt{1517})/26$	$\sqrt{12}-2$
29	$(53+\sqrt{7565})/58$	$(\sqrt{210}-6)/6$
34	$(15+\sqrt{650})/17$	$\sqrt{20}-3$
89	$(157+\sqrt{71285})/178$	$\sqrt{30}-4$
169	$(309+\sqrt{257045})/338$	$(\sqrt{7140}-35)/35$
194	$(344+\sqrt{338720})/388$	$\sqrt{119}-2$
233	$(411+\sqrt{488597})/466$	$\sqrt{42}-5$
433	$(791+\sqrt{1687397})/866$	$(12\sqrt{143}-60)/59$
610	$(1076+\sqrt{3348896})/1220$	$\sqrt{56}-6$
985	$(1801+\sqrt{8732021})/1970$	$(\sqrt{60639}-102)/102$
1325	$(2339+\sqrt{15800621})/2650$	$(\sqrt{3906}-42)/14$
1597	$(2817+\sqrt{22953677})/3194$	$\sqrt{72}-7$

Table 5.1: Image of some Markov irrationals under Jimm as quadratic form

It seems that some of these images in the form $\sqrt{a}-b$ with $a,b\in\mathbb{N}$. To prove this observation, we require the followings:

Lemma 5.1.1. Let F_n be n th Fibonacci number. Then the following hold:

i. $\mathrm{con}(\dfrac{F_n}{F_{n+1}}) = 1$

ii. $\mathrm{con}(\dfrac{F_{n+1}}{F_n}) = n$

Proof. For the first one is easy to see, since we have

$$\operatorname{con}(\frac{F_n}{F_{n+1}}) = \operatorname{con}(\frac{1}{1+F_n/F_{n+1}}) = \operatorname{con}(\frac{F_{n+1}}{F_{n+2}})$$

for all $n \in \mathbb{N}$, it is clear that

$$\operatorname{con}(\frac{F_n}{F_{n+1}}) = \operatorname{con}(\frac{F_1}{F_2}) = \operatorname{con}(1) = 1$$

The second can be proved by induction, obviously we have for $n = 1$ and $n = 2$:

$$\operatorname{con}(\frac{F_2}{F_1}) = 1$$

and

$$\operatorname{con}(\frac{F_3}{F_2}) = \operatorname{con}(2) = 2$$

Suppose that it is true for n then show that for $n+1$ by using previous assertion:

$$\operatorname{con}(\frac{F_{n+2}}{F_{n+1}}) = \operatorname{con}(1+\frac{F_n}{F_{n+1}}) = \operatorname{con}(\frac{F_n}{F_{n+1}}) + \operatorname{con}(\frac{F_{n+1}}{F_n}) = n+1$$

which completes the proof.

□

Corollary 5.1.2. Let F_n be the n th Fibonacci number. Then the following hold:

i. $\operatorname{con}(1+\frac{F_{2n}}{F_{2n+2}}) = 4n+1$

ii. $\operatorname{con}(1+\frac{F_{2n+1}}{F_{2n+3}}) = 4n+3$

iii. $\operatorname{con}(2+\frac{F_{2n}}{F_{2n+2}}) = 6n+1$

iv. $\operatorname{con}(2+\frac{F_{2n+1}}{F_{2n+3}}) = 6n+4$

Proof.

i. Let us show the first one, we compute

$$\operatorname{con}(1+\frac{F_{2n}}{F_{2n+2}}) = \operatorname{con}(\frac{F_{2n}}{F_{2n+2}}) + \operatorname{con}(\frac{F_{2n+2}}{F_{2n}})$$

Since F_{2n}/F_{2n+2} and F_{2n+1}/F_{2n} are in the same $< 1/(1+x) >$-orbits and by Lemma 5.1.1 ii.

$$\mathrm{con}(\frac{F_{2n}}{F_{2n+2}}) = \mathrm{con}(\frac{1}{1+F_{2n+1}/F_{2n}}) = \mathrm{con}(\frac{F_{2n+1}}{F_{2n}}) = 2n$$

similary we have

$$\begin{aligned}\mathrm{con}(\frac{F_{2n+2}}{F_{2n}}) = &\ \mathrm{con}(1+\frac{F_{2n+1}}{F_{2n}}) \\ &= \mathrm{con}(\frac{F_{2n+1}}{F_{2n}}) + \mathrm{con}(\frac{F_{2n}}{F_{2n+1}}) \\ &= 2n+1\end{aligned}$$

Hence $\mathrm{con}(1+F_{2n}/F_{2n+2}) = 4n+1$.

ii. To prove the second, we have

$$\mathrm{con}(1+\frac{F_{2n+1}}{F_{2n+3}}) = \mathrm{con}(\frac{F_{2n+1}}{F_{2n+3}}) + \mathrm{con}(\frac{F_{2n+3}}{F_{2n+1}})$$

Since F_{2n+1}/F_{2n+2} and F_{2n+1}/F_{2n+3} are in the same $< 1/(1+x) >$-orbits and by Lemma 5.1.1 ii.

$$\mathrm{con}(\frac{F_{2n+1}}{F_{2n+3}}) = \mathrm{con}(\frac{1}{1+F_{2n+2}/F_{2n+1}}) = \mathrm{con}(\frac{F_{2n+2}}{F_{2n+1}}) = 2n+1$$

and also we have

$$\begin{aligned}\mathrm{con}(\frac{F_{2n+3}}{F_{2n+1}}) = &\ \mathrm{con}(1+\frac{F_{2n+2}}{F_{2n+1}}) \\ &= \mathrm{con}(\frac{F_{2n+2}}{F_{2n+1}}) + \mathrm{con}(\frac{F_{2n+1}}{F_{2n+2}}) \\ &= 2n+2\end{aligned}$$

Hence $\mathrm{con}(1+F_{2n+1}/F_{2n+3}) = 4n+3$.

iii. By using Lemma 4.1.1, we obtain

$$\mathrm{con}(2+\frac{F_{2n}}{F_{2n+2}}) = F_3\,\mathrm{con}(\frac{F_{2n}}{F_{2n+2}}) + F_2\,\mathrm{con}(\frac{F_{2n+2}}{F_{2n}})$$

since it is already known from previous part, we obtain

$$\mathrm{con}(2+\frac{F_{2n}}{F_{2n+2}}) = 2(2n)+2n+1 = 6n+1$$

iv. By using Lemma 4.1.1, we obtain

$$\mathrm{con}(2+\frac{F_{2n+1}}{F_{2n+3}}) = F_3\,\mathrm{con}(\frac{F_{2n+1}}{F_{2n+3}}) + F_2\,\mathrm{con}(\frac{F_{2n+3}}{F_{2n+1}})$$

since it is already known from the previous part, we obtain

$$\mathrm{con}(2+\frac{F_{2n+1}}{F_{2n+3}}) = 2(2n+1)+2n+2 = 6n+4$$

$\square$

Proposition 5.1.1. Let γ_m be a Markov quadratic irrational with $m = F_{2n+1}$ as follows

$$\gamma_{F_{2n+1}} = \frac{F_{2n+1}+F_{2n-1}+\sqrt{9F_{2n+1}^2-4}}{2F_{2n+1}}$$

fixed by the hyperbolic matrix

$$M = \begin{pmatrix} 2F_{2n+1}+F_{2n-1} & F_{2n+2}-F_{2n-3} \\ F_{2n+1} & F_{2n} \end{pmatrix}$$

Then image of M under Jimm is

$$J(M) = \begin{pmatrix} 3 & 6n-2 \\ 2 & 4n-1 \end{pmatrix}$$

and positive fixed point of this matrix which is equal to $J(\gamma_m)$ is

$$\sqrt{n^2+n}-n+1$$

Proof. Let us show that the matrix above is equal to the matrix $J(M)$ which comes from the formula 5.9 that is,

$$J(M) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

$$a_{11} = \mathrm{con}(\frac{4F_{2n+1}+2F_{2n-1}+F_{2n+2}-F_{2n-3}}{F_{2n+3}}) - \mathrm{con}(\frac{2F_{2n+1}+F_{2n-1}+F_{2n+2}-F_{2n-3}}{F_{2n+2}})$$
$$a_{12} = 2\,\mathrm{con}(\frac{2F_{2n+1}+F_{2n-1}+F_{2n+2}-F_{2n-3}}{F_{2n+2}}) - \mathrm{con}(\frac{4F_{2n+1}+2F_{2n-1}+F_{2n+2}-F_{2n-3}}{F_{2n+3}})$$
$$a_{21} = \mathrm{con}(\frac{F_{2n+3}}{4F_{2n+1}+2F_{2n-1}+F_{2n+2}-F_{2n-3}}) - \mathrm{con}(\frac{F_{2n+2}}{2F_{2n+1}+F_{2n-1}+F_{2n+2}-F_{2n-3}})$$
$$a_{22} = 2\,\mathrm{con}(\frac{F_{2n+2}}{2F_{2n+1}+F_{2n-1}+F_{2n+2}-F_{2n-3}}) - \mathrm{con}(\frac{F_{2n+3}}{4F_{2n+1}+2F_{2n-1}+F_{2n+2}-F_{2n-3}})$$

By using Corollary 5.1.2 iii. and 5.1.2 iv, we find first a_{11} and a_{21}

$$\begin{aligned}\mathrm{con}(\frac{4F_{2n+1}+2F_{2n-1}+F_{2n+2}-F_{2n-3}}{F_{2n+3}}) &= \mathrm{con}(2+\frac{F_{2n+1}}{F_{2n+3}}) \\ &= 6n+4\end{aligned}$$

and also

$$\begin{aligned}\mathrm{con}(\frac{2F_{2n+1}+F_{2n-1}+F_{2n+2}-F_{2n-3}}{F_{2n+2}}) &= \mathrm{con}(2+\frac{F_{2n}}{F_{2n+2}}) \\ &= 6n+1\end{aligned}$$

implies that $6n+4-(6n+1)=3$. Similary, we have also by Corollary 5.1.2 i. and 5.1.2 ii.

$$\begin{aligned}\mathrm{con}(\frac{F_{2n+3}}{4F_{2n+1}+2F_{2n-1}+F_{2n+2}-F_{2n-3}}) &= \mathrm{con}(1+\frac{F_{2n+1}}{F_{2n+3}}) \\ &= 4n+3\end{aligned}$$

and

$$\begin{aligned}\mathrm{con}(\frac{F_{2n+2}}{2F_{2n+1}+F_{2n-1}+F_{2n+2}-F_{2n-3}}) &= \mathrm{con}(1+\frac{F_{2n}}{F_{2n+2}}) \\ &= 4n+1\end{aligned}$$

we obtain $4n+3-(4n+1)=2$ so the first column of the image is constant as follows

$$J(M) = \begin{pmatrix} 3 & a_{12} \\ \\ 2 & a_{22} \end{pmatrix}$$

Now, we must determine the values a_{12} and a_{22}. But we know

$$\mathrm{con}(\frac{F_{2n+3}}{4F_{2n+1}+2F_{2n-1}+F_{2n+2}-F_{2n-3}}) - \mathrm{con}(\frac{F_{2n+2}}{2F_{2n+1}+F_{2n-1}+F_{2n+2}-F_{2n-3}}) = 2$$

so it suffices to show that the last entry is

$$\mathrm{con}(\frac{F_{2n+2}}{2F_{2n+1}+F_{2n-1}+F_{2n+2}-F_{2n-3}}) - 2 = 4n-1$$

which is already true by Corollary 5.1.2 i. Finally, we have

$$J(M) = \begin{pmatrix} 3 & a_{12} \\ \\ 2 & 4n-1 \end{pmatrix}$$

Since $J(M)$ is a matrix in $\mathrm{PGL}(2,\mathbb{Z})$, the determinant is 1. After a short computation we see that a_{12} is equal to $6n-2$ so we can conclude that the matrix is as the form

$$J(M) = \begin{pmatrix} 3 & 6n-2 \\ \\ 2 & 4n-1 \end{pmatrix}$$

and fixed point can be calculated from the quadratic equation

$$x^2 + (2n-2)x - 3n + 1 = 0$$

with discriminant $\Delta = 4(n^2+n)$ then positive root of the polynomial

$$\sqrt{n^2+n} - n + 1$$

as desired.

$\square$

n	$m = F_{2n+1}$	$u = F_{2n-1}$	$v = F_{2n-3}$	M	$J(M)$	$J(\gamma_m)$
1	2	1	1	$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$	$\sqrt{2}$
2	5	2	1	$\begin{pmatrix} 12 & 7 \\ 5 & 3 \end{pmatrix}$	$\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$	$\sqrt{6}-1$
3	13	5	2	$\begin{pmatrix} 31 & 19 \\ 13 & 8 \end{pmatrix}$	$\begin{pmatrix} 3 & 16 \\ 2 & 11 \end{pmatrix}$	$\sqrt{12}-2$
4	34	13	5	$\begin{pmatrix} 81 & 50 \\ 34 & 21 \end{pmatrix}$	$\begin{pmatrix} 3 & 22 \\ 2 & 15 \end{pmatrix}$	$\sqrt{20}-3$
5	89	34	13	$\begin{pmatrix} 212 & 131 \\ 89 & 55 \end{pmatrix}$	$\begin{pmatrix} 3 & 28 \\ 2 & 19 \end{pmatrix}$	$\sqrt{30}-4$
6	233	89	34	$\begin{pmatrix} 555 & 343 \\ 233 & 144 \end{pmatrix}$	$\begin{pmatrix} 3 & 34 \\ 2 & 23 \end{pmatrix}$	$\sqrt{42}-5$
7	610	233	89	$\begin{pmatrix} 1453 & 898 \\ 610 & 377 \end{pmatrix}$	$\begin{pmatrix} 3 & 40 \\ 2 & 27 \end{pmatrix}$	$\sqrt{56}-6$
8	1597	610	233	$\begin{pmatrix} 3804 & 2351 \\ 1597 & 987 \end{pmatrix}$	$\begin{pmatrix} 3 & 46 \\ 2 & 31 \end{pmatrix}$	$\sqrt{72}-7$

Table 5.2: Quadratic form of $J(\gamma_{F_{2n+1}})$.

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Appendix

Some facts about continued fraction as follows:

Fact 1. Let p_n/q_n be n-th convergent of a real number α. Then we have

i. $|\alpha - \dfrac{p}{q}| < |\alpha - \dfrac{p_n}{q_n}| \Rightarrow q > q_n$ for all $n \geq 1$.

ii. $|q\alpha - p| < |q_n\alpha - p_n| \Rightarrow q \geq q_{n+1}$ for all $n \geq 1$.

Fact 2. Let $\alpha = [a_0, a_1, a_2, \dots]$ be a real number and let $r_n = p_n/q_n = [a_0, a_1, a_2, \dots, a_n]$ be n-th convergent of α. Then

i. $r_0 < r_2 < r_4 \dots, \dots < r_5 < r_3 < r_1$,

ii. We have $r_{2i} < \alpha < r_{2j+1}$ for all $i, j \geq 0$ and $\lim_{n\to\infty} r_n = \alpha$ exists.

Lemma 1. Assume $\alpha \geq 1$ is a real number with $\alpha + \alpha^{-1} \leq \sqrt{5}$. Then

$$\alpha \leq \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \alpha^{-1} \geq \frac{\sqrt{5}-1}{2}$$

and equality $\alpha + \alpha^{-1} = \sqrt{5}$ is satisfied for $\alpha = \dfrac{1+\sqrt{5}}{2}$.

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