# MEASURES ON THE BOUNDARY OF THE FAREY TREE

(FAREY AĞACININ SINIRI
ÜZERİNDEKİ ÖLÇÜLER)

by

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Hamide KURU

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# LIST OF SYMBOLS

| | |
|---|---|
| $\mathrm{PGL}(2,\mathbb{Z})$ | : Projective general linear group of determinant $\pm 1$ integral $2 \times 2$ matrices |
| $G(V,E)$ : | : Graph with the set of vertices $V$ and the set of edges $E$. |
| $\mathcal{T}$ | : Tree |
| $\partial\mathcal{T}$ | : The boundary of the tree. |
| $\partial\mathcal{T}_e$ | : The boundary of the tree $\mathcal{T}$ with fixed starting edge $e$. |
| $\mathcal{F}$ | :Farey Tree with ribbon structure |
| $\mathfrak{J}$ | :Jimm Function |
| $\lvert\mathcal{F}\rvert$ | :Abstract Farey Tree without ribbon structure |
| $\sigma_\nu$ | : twist automorphism which twists every vertices in the set $\nu$. |
| $\tau_\mu$ | : shuffle automorphism. |
| $\mathbb{T}_\alpha$ | : Continued Fraction Map |

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# Abstract

In this work, we consider trees, their boundaries and some special Borel measures on the boundary of trees. Especially we work on Farey Tree. Farey Tree is a rooted binary planar tree whose vertices are described by all rational numbers in the unit interval. It is the first left branch of the Stern-Brocot Tree.

The boundary of a tree is the set of all infinite non back-tracking paths which is called end of the tree. If we fixed an edge or vertex on the tree we can define a metric on the bondary of the tree. A topology on the boundary of the tree is induced by this metric, it is generated by open balls. Some natural Borel measures appear on the boundary of the Farey tree. The Minkowski measure is the unique measure on the boundary which is invariant under the full action of the automorphism group of the tree on its boundary. Denjoy's measures are straightforward generalization of the Minkowski measure. Moreover, The Lebesgue's measure can be presented as a Boundary measure in a nice way.

Continued fraction map is defined on the unit interval which is a generalization of the Gauss map. The formal definition of this map is given in the Chapter 5. In this thesis we consider an even larger generalization of the continued fraction maps defined by using the tree structure. Our aim in this chapter is to find the invariant measures under these special continued fraction maps. Our claim is $f(y) = \frac{1}{y}$ satisfies the density of an invariant measure under this special members of maps and we proved it. At final chapter, we have a study that is not very compatible with the overall thesis. We associate a power series to continued fractions such that this series has some proprieties, such as convergence and continuity.

**Keywords:** Continued fraction, Farey Tree, Invariant measure, Minkowski measure, Denjoy's measure, Continued fraction map, Gauss map.

# Özet

Bu çalışmada, ağaçları, sınırlarını ve ağaçların sınırları üzerindeki bazı özel Borel ölçülerini ele alıyoruz. Özellikle Farey ağacı üzerinde çalışıyoruz. Farey ağacı, köşeleri birim aralığındaki rasyonel sayılarla tanımlanan köklü bir ikili ağaçtır. Stern Brocot ağacının ilk sol koludur.

Ağaçların sınırları son diye adlandırılan, tüm sonsuz, geri dönülmeyen yolların oluşturduğu kümedir. Eğer bir kenar ya da köşe sabitlersek ağaçların sınırları üzerinde metrik tanımlayabiliriz. Ağaçların sınırları üzerindeki topoloji bu metrik aracılığı ile tanımlanır, topoloji açık toplar tarafından gerilir. Ağacın sınırları üzerinde bazı özel Borel ölçüleri vardır. Minkowski ölçüsü ağacın otomorfizma gruplarının aksiyonu altında değişmez kalan tek ölçüdür. Denjoy ölçüsü ise Minkowski ölçüsünün genelleştirilmiş halidir. Ayrıca Lebesgue ölçüsü de ağacın sınırları üzerinde hoş bir yol ile tanımlanabilir.

Sürekli kesir fonksiyonu birim aralıktan tanımlanır ve Gauss fonksiyonunun bir genelleştirilmesidir. Bu fonksiyonun formal tanımı 5. bölümde verilir. Bu tezde, ağaç yapısı kullanılarak tanımlanan sürekli kesir fonksiyonlarının daha geniş bir genellemesini ele alıyoruz. Bu bölümdeki amacımız, özel sürekli kesir fonksiyonu altında değişmeyen ölçüleri bulmaktır. Iddiamız $f(y) = \frac{1}{y}$ bu özel fonksiyon üyeleri altındaki değişmez bir ölçünün yoğunluğunu sağlar ve bu iddiamızı ispatlıyoruz. Son bölümde tezin geneli ile çok uyumlu olmayan bir çalışmamız var. Her sürekli kesin tasvirine bir kuvvet serisi iliştiriyoruz öyle ki bu kuvvet serisi yakınsaklık ve süreklilik gibi bazı güzel özelliklere sahip olsun.

**Anahtar Sözcükler :** Sürekli kesirler, Farey Ağacı, Değişmez ölçü, Minkowski ölçüsü, Denjoy ölçüsü, Sürekli kesir fonksiyonu, Gauss fonksiyonu.

# 1 INTRODUCTION

## 1.1 Basic Definitions of The Graph Theory

Our aim is to study trees and their boundaries so we will give some basic definitions and examples from Graph Theory.
In general, a graph consists of vertices and edges that connects the vertices. There are various ways of formalising a graph. We will use the following.

**Definition 1.1.1.** A graph $G = (V,E)$ consists of a set of vertices $V$ together with a subset $E \subseteq \binom{V}{2}$ elements of which are called the edges of $G$ where $\binom{V}{2}$ is the set of subsets of $V$ having two elements.

**Example 1.1.1.** Set $G = (V,E)$ with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, e_3 = \{v_1, v_3\}\}$. Then we can draw this graph in $\mathbb{R}^2$ as follows:

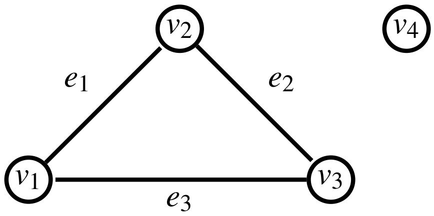



Figure 1.1: Drawing of the graph

*Remark.* In our definition multi-edges and loops are not allowed but actually such graphs are considered in the literature. Also note that our graphs are not directed.

Let $G = (V,E)$ be a graph.

**Definition 1.1.2.** Let $u, v \in V$ and suppose $e = \{u, v\} \in E$. Then $e$ is said to be **incident** to $u$ or to $v$. Two edges are called **incident** if they have a common vertex. Two vertices are called **adjacent** if they are connected by an edge. Furthermore, the set of all edges incident to $u \in V$ is called **the star** of $u$ and the number of edges in the star of $u$ is called **the degree** of $u$. A graph $G = (V,E)$ is said to be **locally finite** if $deg(u) < \infty$ for all $u \in V$.

From now on we shall assume that our graphs are locally finite.

**Example 1.1.2.** Let $V = \{v, v_1, v_2, v_3, v_4\}$ and
$E = \{e_1 = \{v, v_1\},\ e_2 = \{v, v_2\},\ e_3 = \{v, v_3\},\ e_4 = \{v, v_4\}\}$. Then the star of $v$ is $\{e_1, e_2, e_3, e_4\}$ so $deg(v) = 4$. Moreover all edges are incident in this graph. This graph is called the star tree and it is denoted by $S_4$.

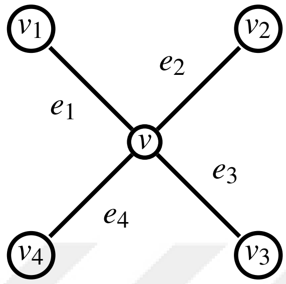



Figure 1.2: $S_4$

**Definition 1.1.3.** A **path** in $G$ is a finite or infinite sequence of edges $(e_1, e_2, e_3 \cdots)$ such that the consecutive edges are incident and if $v$ is a vertex in this path then the star of $v$ has at most 2 edges in this path. Let $e_1 = \{u, v\}$ and $e_2 = \{v, t\}$ then $u$ is called the **initial vertex** of the path. If the path is finite we have the **terminal vertex** such that the last edge $e_n$ is incident to this vertex but $e_{n-1}$ is not. If the terminal and initial vertex are the same such a path is called **circuit**.

**Example 1.1.3.** In the graph $S_4$ the set of paths is $\{(e_i, e_j) | i \neq j$ and $i, j \in \{1, 2, 3, 4\}\}$.

**Definition 1.1.4.** A graph $G = (V, E)$ is called **connected** if for any pair of vertices $\{u, v\}$ there exists a path from $u$ to $v$. Moreover, if a connected graph does not contain any loops or circuits then it is called **tree.**

**Example 1.1.4.** The graph in Figure 1.2 is a tree whereas in Figure 1.1 is not.

**Definition 1.1.5.** A graph is said to be **d-regular** if all of the vertices have the same degree $d$.

**Definition 1.1.6.** A graph is said to be **perfect** if all vertices are of degree $> 2$.

**Example 1.1.5.** The following tree is $3-$ regular.

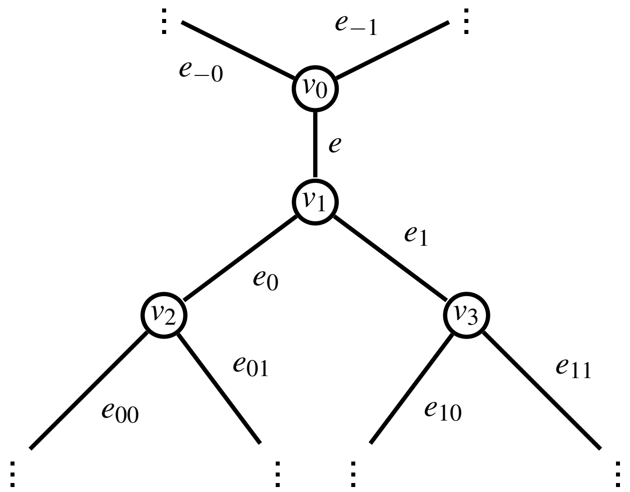



Figure 1.3: 3-regular tree

**Definition 1.1.7.** Two graphs $G = (V,E)$, $G' = (V',E')$ are said to be **isomorphic** if there exists a bijection $\phi : V \longrightarrow V'$ which preserves the incidence. That is, $\{u,v\} \in E$ iff $\{\phi(u),\phi(v)\} \in E'$. If $\phi$ is not a bijection but it preserves the incidence then it is called just a **morphism**.

**Example 1.1.6.** Let $V = \{v_1, v_2, v_3, v_4, v_5\}$ and
$E = \{e_{12} = \{v_1, v_2\}, e_{13} = \{v_1, v_3\}, e_{14} = \{v_1, v_4\}, e_{24} = \{v_2, v_4\}, e_{25} = \{v_2, v_5\},$
$e_{35} = \{v_3, v_5\}\}$. *Let* $V' = \{1,2,3,4,5\}$ *and*
$E' = \{f_{12} = \{1,2\}, f_{13} = \{1,3\}, f_{14} = \{1,4\}, f_{24} = \{2,4\}, f_{25} = \{2,5\},$
$f_{35} = \{3,5\}\}$. Then $\phi(v_i) := i$ for any $i \in \{1,2,\cdots,5\}$ is a bijective map from $V$ to $V'$ which preserves the incidence.

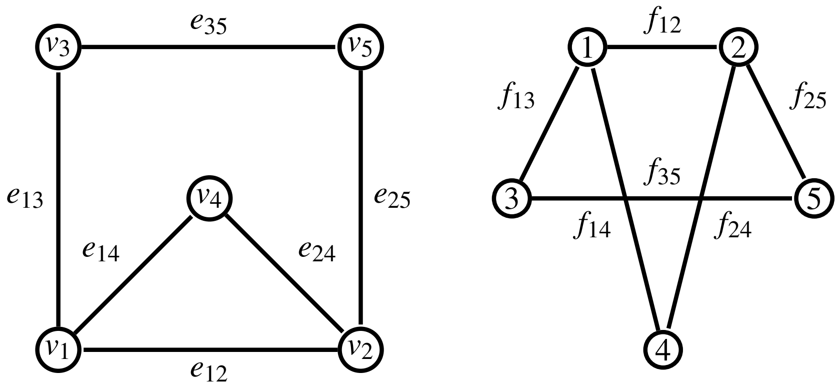



Figure 1.4: Isomorphic Graphs

**Proposition 1.1.1.** Up to isomorphism there exists a unique d-regular tree for any $d \geq 2$.

*Proof.* All of the vertices are d-regular and there doesn't exist any circuits and the tree is infinite. Choose a base vertex from both of them, $u$ and $u'$ where $\phi(u) = u'$. The image of other vertices are determined accordingly to preserve incidence. For $u$ and $u'$ infinitely different choice can be made. So we can find infinitely many isomorphisms.

**Definition 1.1.8.**

- A **planar graph** is a graph that can be drawn in the plane without any edge crossings.
- A **plane graph** is a graph that has been drawn in the plane without any edge crossings.
- A **planar tree** is a tree given by a drawing in the plane. Such a drawing endows $\mathcal{T}$ with an extra structure: one has a cyclic ordering of all stars of $\mathcal{T}$. Conversely, if we are given a cyclic ordering of stars, then there is a unique way of compatibly drawing the tree.

**Example 1.1.7.** $K_4$ is the complete graph with 4 vertices i.e. all vertices are connected to others. In the following figure, the first one contains one crossing edge so it is not a plane graph whereas the second one does not contain so it is a plane graph. But we say that $K_4$ is a planar graph due to the second graph.

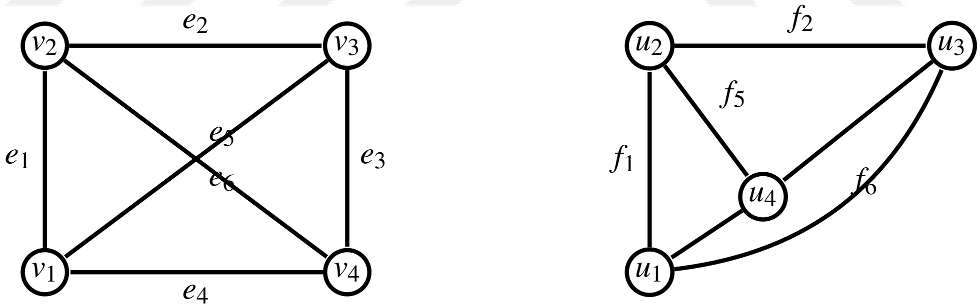



Figure 1.5: Drawing of the graph K4 with 2 different ways

# 2 LITERATURE REVIEW

Trees are very basic combinatorial objects and there is a vast literature on them. A basic reference is the now-classical book [16] which studies groups via their actions on trees. See also [3] and [6] for some recent work. Automorphism groups of trees have been studied in [11], [5] and boundary measures in relation to the tree automorphism groups have been studied in [15]. There are also the works on the measures on the boundary of some special trees for example the book [20] is one of them.

The Farey tree and the Strern-Brocot trees are binary rooted planar trees presented in a very special way: their vertices enumerate the rationals in a very natural fashion [1]. These are very fundamental objects and there is a growing literature about them. Our approach to the continued fraction maps can be traced back to [2].

The Gauss map is a very basic and elementary dynamical map with a very rich structure. We refer the reader to [14] for a classical study of continued fractions and the Gauss Map, and to [13] for a modern treatment. For the related ergodic theory, our reference is [8].Invariant measure is an important subject of dynamical systems. Since if we have two different invariant measures the two dynamical systems shows different proprieties of each others. In this thesis we are showing some invariant measures under continued fraction maps. To our knowledge, a more generalization of the continued fraction maps introduced in this thesis have not been studied elsewhere. We study the dynamics of these maps.

## 2.1 Boundary of A Tree

In this chapter some definitions and proofs are taken by an unpublished manuscript [23].

In what follows let $\mathcal{T}$ be a tree.

**Definition 2.1.1.** Two paths $\gamma = (e_1, e_2, \cdots)$ and $\gamma' = (f_1, f_2, \cdots)$ in the tree $\mathcal{T}$ are said to **eventually coincide** if there exists $n, m \in \mathbb{N}$ such that $e_{n+i} = f_{m+i}$ for any $i \in \mathbb{N}$.

**Definition 2.1.2.** An **end** of $\mathcal{T}$ is an equivalence class of one-sided, infinite, non-backtracking paths in $\mathcal{T}$ under the equivalence relation:

$$\gamma \sim \gamma' \iff \gamma \text{ and } \gamma' \text{ eventually coincide.}$$

The set of all ends of $\mathcal{T}$ is called the **boundary** of $\mathcal{T}$. We denote this set by $\partial\mathcal{T}$.

**Example 2.1.1.** Let $\mathcal{T}$ be the linear tree consisting of edges $e_i$ for all $i \in \mathbb{Z}$ such that the consecutive edges are incident. Then this tree has only two ends. Since let us choose an edge $e_n$ in the tree. Then $\gamma_+ = \{e_n, e_{n+1}, e_{n+2}, \cdots\}$ and $\gamma_- = \{e_n, e_{n-1}, e_{n-2}, \cdots\}$ are two paths of the tree representing the two ends. If we chose another edge, $e_m$ then the tree would have the paths $\{e_m, e_{m+1}, e_{m+2}, \cdots\}$ and $\{e_m, e_{m-1}, e_{m-2}, \cdots\}$ but they are equivalent to the first two paths. So $\partial\mathcal{T} = \{\gamma_+, \gamma_-\}$.

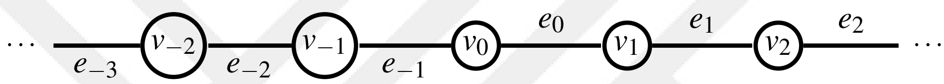



Figure 2.1: A linear tree

**Example 2.1.2.** In the tree which is described in the Figure 2.1, we can say that $(e_0, e_{01}, e_{011}, e_{011n_1}, e_{011n_1n_2}, \cdots) \sim (e_{011}, e_{011n_1}, e_{011n_1n_2}, \cdots)$, for any $n_i \in \{0,1\}$ but for any path which starts with $e_{00}$ is not equivalent to these paths.

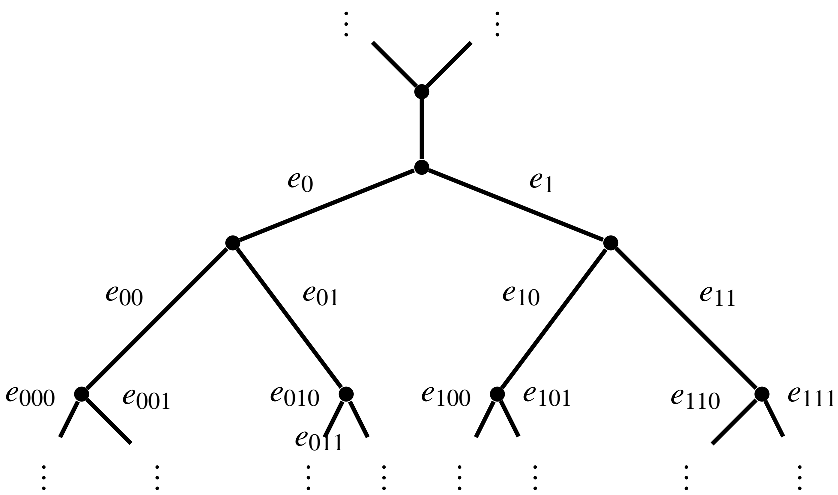



Figure 2.2: Binary Tree

**Proposition 2.1.1.** Let $e$ be a fixed edge of $\mathcal{T}$. Let $\gamma = (e_1, e_2, \cdots)$ be a one-sided non-backtracking path. Then there exists a unique path in the equivalence class of $\gamma$, which starts at $e$. This path is denoted by $\gamma_e$.

*Proof.* If $e = e_i$ for some $i \in \mathbb{N}$, then $\gamma' = (e, e_{i+1}, e_{i+2}, \cdots) \sim \gamma$, so we won. Otherwise, since $\mathcal{T}$ is connected, we can find a path $\gamma' = (e, f_1, \cdots, f_n, e_i, e_{i+1}, \cdots)$ for some $i \in \mathbb{N}$. Thus $\gamma \sim \gamma'$.
Assume there exist two paths which start at $e$ and are equivalent to $\gamma$. We call $\gamma' = (e, g_2, g_3, \cdots, e_i, e_{i+1}, \cdots)$ and $\gamma'' = (e, h_2, h_3, \cdots, e_j, e_{j+1}, \cdots)$ for some $i, j \in \mathbb{N} - \{0\}$. Without loss of generality assume that $i < j$. Then we have two paths in the tree $\mathcal{T}$, $(f, g_2, g_3, \cdots, e_i, \cdots, e_j)$ and $(f, h_2, h_3, \cdots, e_j)$. Then we obtained a circuit so which contradicts with the definition of a tree. Then $h_i = g_i$ for all $i$. Hence $\gamma' = \gamma''$. Therefore this path is unique.

## 2.2 The Topology on The Boundary of A Tree

The set of all ends in $\mathcal{T}$ which starts at $e$ is denoted by $\partial\mathcal{T}_e$. In this situation $e$ is called the **base edge**.

**Proposition 2.2.1.** Let e be an edge in $\mathcal{T}$. There is a canonical bijection between $\partial\mathcal{T}$ and $\partial\mathcal{T}_e$.

*Proof.* Let $\gamma \in \partial\mathcal{T}$. We define $f : \partial\mathcal{T} \longrightarrow \partial\mathcal{T}_e$ be a map such that $f(\gamma) \in \partial\mathcal{T}_e$ to be the path which starts at $e$ and is equivalent to $\gamma$. We know by Proposition 2.1.1 such an end exists and it is unique. Therefore $f$ is injective. Let $\gamma' \in \partial\mathcal{T}_e$ then it is obvious that $\gamma' \in \partial\mathcal{T}$ and $f(\gamma') = \gamma'$. So the map is surjective.

**Proposition 2.2.2.** Let $\gamma, \gamma' \in \partial\mathcal{T}_e$ with $\gamma = (e, e_1, e_2, \cdots)$, $\gamma' = (e, e'_1, e'_2, \cdots)$. Then $d(\gamma, \gamma') = 2^{-n}$ is a metric over the set $\partial\mathcal{T}_e$ with $n = \max\{i,\ e_j = e'_j, \forall j \leq i\}$. Hence $(\partial\mathcal{T}_e, d)$ is a metric space.

*Proof.* Let $\gamma \sim \gamma' \in \partial\mathcal{T}_e$. By Proposition 2.1.1 there exists a unique path which starts at $e$ and equivalence class of $\gamma$. So $\gamma = \gamma'$. Then $n = \infty$. Then $d(\gamma, \gamma') = \frac{1}{2}^{\infty} = 0$. Furthermore, if $d(\gamma, \gamma') = 0$ then $e_i = e'_i$ for all $i$. Then $\gamma = \gamma'$.
Due to the $e$, $1 \leq n$ for any $\gamma, \gamma' \in \partial\mathcal{T}_e$. Then $d(\gamma, \gamma') \leq \frac{1}{2}$. And also we have showed that if $\gamma \sim \gamma'$ then $d(\gamma, \gamma') = 0$. So for all $\gamma, \gamma' \in \delta\mathcal{T}_e$, $d(\gamma, \gamma') \in [0, \frac{1}{2}]$ since $2^{-n}$ is a monotone decreasing function.
It is clear that $d(\gamma, \gamma') = d(\gamma', \gamma)$ since $n$ does not change.
Let $\gamma, \gamma', \gamma'' \in \partial\mathcal{T}_e$ with $d(\gamma, \gamma') = 2^{-n}$ and $d(\gamma', \gamma'') = 2^{-m}$ then $d(\gamma, \gamma'') = 2^{-min\{m,n\}}$. Hence the triangle inequality satisfies.

This metric induces a topology on the set $\partial\mathcal{T}_e$ which is generated by open balls $B(\gamma, r) = \{\gamma' \mid d(\gamma, \gamma') < r\}$ for any $r > 0$.

**Example 2.2.1.** Let us choose an edge $e_0$ in the linear tree which is described in Figure 2.1. Then $\partial\mathcal{T}_e = \{\gamma = (e, e_1, \cdots), \gamma' = (e, e_{-1}, \cdots\}$. The topology over this set is generated by the open balls ; $B(\gamma, 2^{-n}), B(\gamma', 2^{-n})$, for any $n \in \mathbb{N}$. So $B(\gamma, 1) = \{\gamma, \gamma'\}$, $B(\gamma, 1/2) = \{\gamma\}$ and $B(\gamma', 1/2) = \{\gamma'\}$. So the topology on this set is the discrete topology : $\{\{\gamma, \gamma'\}, \{\gamma\}, \{\gamma'\}, \varnothing\}$.

**Theorem 2.2.1.** *If $\mathcal{T}$ is a perfect tree then $\partial\mathcal{T}_e$ is canonically homeomorphic to $\partial\mathcal{T}_{e'}$, for any edges $e$ and $e'$.*

*Proof.* We must find a bijective and continuous map between these two topological spaces. Let $\gamma_e \in \partial\mathcal{T}_e$. We define a map $h : \partial\mathcal{T}_e \longrightarrow \partial\mathcal{T}_{e'}$ such that $h(\gamma_e)$ is an end which starts at $e'$ and equivalent to $\gamma_e$ by the Proposition 2.1.1 such an end exists and unique so $h$ is an injection. This map is surjective. Since for any end in $\gamma_{e'} \in \partial\mathcal{T}_{e'}$, by the same Proposition there exists unique $\gamma_e$ such that $h(\gamma_e) = \gamma_{e'}$. So $h$ is a bijection.
Let $\omega \in \mathcal{T}_{e'}$. Let $B(\omega, r)$ be an open ball in the topological space $\partial\mathcal{T}_{e'}$ then $h^{-1}(B(\omega, r)) = B(h(\omega), r)$ so the inverse image of open balls are open ball in the topological space $\partial\mathcal{T}_e$. So this map is continuous. By the same reasoning the $h^{-1}$ is also continuous. Hence $\partial\mathcal{T}_e$ and $\partial\mathcal{T}_{e'}$ are homeomorphic.

**Definition 2.2.1.** Let $X$ be a topological space and $x \in X$. $x$ is said to be an ***isolated point*** if it has an open neighbourhood $O$ such that $X \cap O = \{x\}$.

**Example 2.2.2.** Let $X$ be a set and $(X, 2^X)$ be a topological space with the discrete topology on $X$. Then every point is isolated.
The topological space $(X, \{\emptyset, X\})$ has no isolated points.
In the boundary of $2-$regular tree both of the ends are isolated.

**Definition 2.2.2.** Let $T$ be a topological space. A subset $P \subset T$ is **perfect** if $P$ is closed and has no isolated points.

**Example 2.2.3.** Let $\mathcal{T}$ be a $2-$regular tree. Then $\partial\mathcal{T}_e$ is not perfect.

**Proposition 2.2.3.** If $\mathcal{T}$ is a perfect tree then $\partial\mathcal{T}_e$ is perfect for any base edge $e$.

*Proof.* Assume that the tree is perfect but $\partial\mathcal{T}_f$ is not a perfect. Then it has at least one isolated point, we say $\gamma$. Then there exists $n \in \mathbb{N}$ such that $B(\gamma, 2^{-n}) = \{\gamma\}$. Let $\gamma = (e, e_1, \cdots, e_n, e_{n+1} \cdots)$. Since $\mathcal{T}$ is a perfect tree there exists at least one edge which is adjacent to $e_n$ and $e_{n+1}$ at the same time, we call it $g_{n+1}$. So all ends which starts with the path $(e, e_1, \cdots, e_n, g_{n+1})$, is also an element of open ball $B(\gamma, 2^{-n})$. Hence $\gamma$ is not an isolated point, so we have a contradiction.

**Proposition 2.2.4.** If $\mathcal{T}$ is a perfect tree then the singletons are closed in $\partial\mathcal{T}_e$.

*Proof.* Let $\gamma = (e, e_1, e_2, \cdots)$ be an end in $\partial\mathcal{T}_e$. Let $\{e_1, e_{01}, e_{02}, \cdots, e_{0k_0}\}$ be the set of edges which are adjacent to $e$. Let $\gamma_{0i}$ be any path which starts at $e$ and passes $e_{0i}$ for all $i \in \{1, \cdots, k_0\}$. Then $d(\gamma, \gamma_{0i}) = \frac{1}{2}$. So $\{\gamma\} \not\subseteq \cup_{i=1}^{k_0} B(\gamma_{0i}, 2^{-1})$ Then we can say $\cup_{i=1}^{k_0} B(\gamma_{0i}, 2^{-1}) \subset \{\gamma\}^c$. Let define the set of all adjacent edges to $e_1$ which is $\{e, e_2, e_{11}, e_{12}, \cdots, e_{1k_1}\}$ and let $\gamma_{1i}$ be any path which starts at $e$ and passes $e_{1j}$ for any $j \in \{1, \cdots, k_1\}$. Then smilarly we can say that $\cup_{j=1}^{k_1} B(\gamma_{1j}, 2^{-2}) \subset \{\gamma\}^c$.
If we continue with this way we find $\bigcup_{n=0} \cup_{i=1}^{k_n} B(\gamma_{ni}, 2^{-n}) = \{\gamma\}^c$. $\{\gamma\}^c$ is open then $\{\gamma\}$ is closed.

**Proposition 2.2.5.** Let $\mathcal{T}$ be a $3-$ regular tree and $e$ be an edge in $\mathcal{T}$. Then $\partial\mathcal{T}_e$ is homeomorhic to $\{0,1\}^{\mathbb{N}}$ which is equipped with the product topology.

*Proof.* We can label the edges of the tree with the numbers 0 and 1:

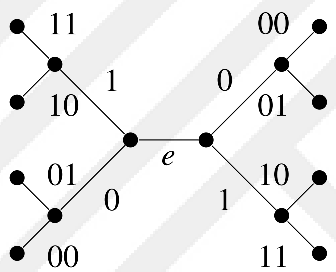



Figure 2.3: The Binary Tree Represented by the Numbers 0 and 1

So $\partial\mathcal{T}_e \cong \{0,1\}^{\mathbb{N}} \cup \{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$.

From now on, we assume that $\mathcal{T}$ is a perfect tree. The following proofs were made by us using ideas from standard proofs in topology on $\mathbb{R}^n$.

**Proposition 2.2.6.** $\partial\mathcal{T}_e$ is Hausdorff.

*Proof.* Let $\gamma = (e, e_1, e_2, \cdots), \gamma' = (e, e'_1, e'_2, \cdots)$ be two disjoint ends in $\partial\mathcal{T}_e$. Then there exist $n \in \mathbb{N}$ such that $e_{n+i} \neq e'_{n+i}$ for any $i \in \mathbb{N}$. Then $B(\gamma, 2^{-n})$ and $B(\gamma', 2^{-n})$ are the neighbourhoods of $\gamma$ and $\gamma'$ respectively and they are disjoints. So $\partial\mathcal{T}_e$ is Hausdorf.

Distance of between two edges is defined as the number of edges between these edges. It is denoted by $\mathbf{d}(e, e')$ for any $e, e' \in E(\mathcal{T})$.

**Proposition 2.2.7.** $\partial\mathcal{T}_e$ is compact.

*Proof.* Let $O$ be an open cover of the topological space $\partial\mathcal{T}_e$. We suppose that there exists a number $N \in \mathbb{N}$ for any $e' \in E(\mathcal{T})$ with $\mathbf{d}(e, e') = N$ such that there exists an open $O(e') \subset O$ with $O(e')$ is a cover of $B(\gamma', 2^{-N})$ where $\gamma'$ is the end which is starting at $e$

and passing at $e'$. So if our claim is true we can find finite $O(e')$ such that their unions cover $\partial\mathcal{T}_e$.
Assume that such an $N$ does not exist. So there are the infinitely many edges $e_1, e_2, \cdots$ such that we can not find such a subcover $O(e_i)$ for any $i$. Let $\{f_1, f_2, \cdots, f_n\}$ be the the set of all edges incident to $e$. Let $A_j$ be the set of all ends which starts with the path $(e, f_j)$ for any $j \in \{1, \cdots, n\}$. Then at least one of these $A'_j s$ must contain infinitely many $e'_i s$. We call it $A_k$. Let $\{e, g_1, g_2, \cdots, g_m\}$ be the the set of all edges incident to $f_k$. And again we define the set $B_j$ which starts with the path $(e, f_k, g_i)$. Then we will choose one $B_j$ which contains infinitely many $e_i$'s. So proceeding in this manner gives us a path which has infinitely many $e_i$'s. So this path can not be covered by any element of $O$, contradiction.

*Remark.* $\partial\mathcal{T}_e$ is disconnected.

*Proof.* Let $\{e_1, e_2, \cdots, e_n\}$ be the set of incident edges of $e$. Let $\gamma_i$ be any edge which starts at the edges $e, e_i$ for any $i$. Then $\partial\mathcal{T}_e = \cup_{i=1}^{n} B(\gamma_i, \frac{1}{2})$. And it is obvious that $B(i', \frac{1}{2}) \cap B(i'', \frac{1}{2}) = \varnothing$. So $\partial\mathcal{T}_e$ is disconnected.

**Proposition 2.2.8.** *$\partial\mathcal{T}_e$ is totally disconnected.*

*Proof.* Let $\gamma = (e, e_1, e_2, \cdots e_n, \cdots)$ be an end in $\partial\mathcal{T}_e$. Let $n \in \mathbb{N}$ and $B(\gamma, 2^{-n})$ be an open neighbourhood of $\gamma$. Then $B(\gamma, 2^{-n})$ contains all ends which starts with the path $(e, e_1, e_2, \cdots e_n)$. We define the set of the incident edges of $e$; $\{e_1, f_1, f_2, \cdots, f_n\}$. Consider $\gamma_{f_i}$ an end which starts with the path $(e, f_i)$ for any $i \in \{1, \cdots, n\}$. Let $u = e_{n-1} \cap e_n$. Let the set $U = \{e_{n-1}, e_n, g_1, \cdots, g_m\}$ be the star of the $u$. Let $\gamma_{g_j}$ be an end which starts with the path $(e, e_1, e_2, \cdots, e_{n-1}, g_j)$. Then $B(\gamma, 2^{-n})^c = \bigcup_{n}^{i=1} B(\gamma_{f_i}, 2^{-1}) \cup \bigcup_{n}^{j=1} B(\gamma_{g_j}, 2^{-n})$. So the complement of any neighbourhood of any end is open. Hence $\partial\mathcal{T}_e$ is totally disconnected.

**Theorem 2.2.2.** *(Brouwer) Every nonempty, totally disconnected, compact, metrizable space without isolated points is homeomorphic to the Cantor set.*

*Proof.* see [4].

**Theorem 2.2.3.** *If $\mathcal{T}$ is a perfect tree then $\partial\mathcal{T}_e$ is homeomorphic to the Cantor Set.*

*Proof.* We proved in the previous propositions that if $\mathcal{T}$ is a perfect tree then $\partial\mathcal{T}_e$ is a perfect, totally disconnected and compact topological space. These gives us the topological caracterisation of Cantor Set.

## 2.3 Ordered Boundary of the Tree

Let $\mathcal{T}$ be a planar tree. In this case there is a naturally defined cyclic order relation on $\partial\mathcal{T}_e$ induced by the planar structure of $\mathcal{T}$.

**Definition 2.3.1.** Let the edge $e$ has an orientation clockwise or counterclockwise. Let $\gamma, \gamma'$ be different two ends induce the same orientation on $e_0$ in $\partial\mathcal{T}_e$. If $\gamma$ differentiates from $\gamma'$ from the right of the $\gamma'$, we say that $\gamma < \gamma'$. Due to the this relation $\partial\mathcal{T}_e$ is an **ordered set**.

This order is compatible with the topology over the $\partial\mathcal{T}_e$. An interval can be defined as follows

$$[\gamma, \gamma'] = \{\alpha \mid \gamma \leq \alpha \leq \gamma'\} \text{ for any } \gamma, \gamma' \in \partial\mathcal{T}_e \text{ with } \gamma < \gamma'\}$$

**Example 2.3.1.** If we look the following tree we have 5 ends in $\partial\mathcal{T}_e$. Assume the cyclic ordering of the edge $e$ is clockwise so the direction of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are the same with the direction of $e$ whereas the direction of $\gamma_5$ is inverse. So we say that
$\gamma_5 < \gamma_4 < \gamma_3 < \gamma_2 < \gamma_1$.

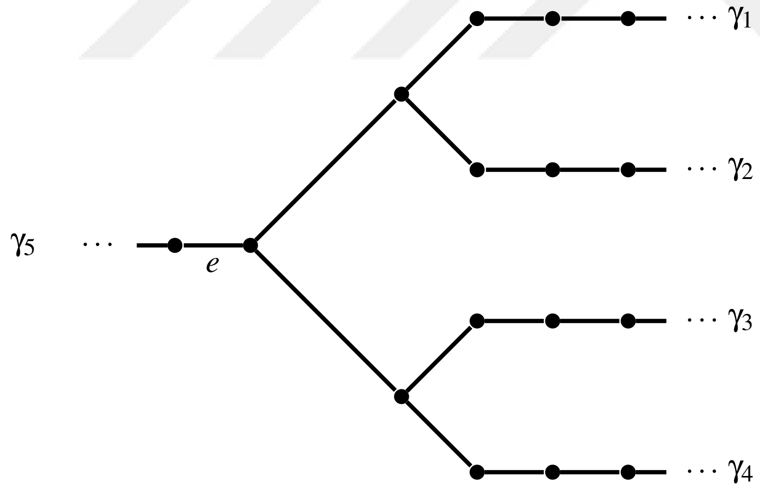



Figure 2.4: the Tree with base edge $e$

**Definition 2.3.2.** Since $\partial\mathcal{T}_e$ is an ordered set we can define a relation over this set. Let $\gamma_1, \gamma_2$ be two ends in $\partial\mathcal{T}_e$. We say $\gamma_1 \sim \gamma_2$ if $\gamma_1 \leq \alpha \leq \gamma_2$ or $\gamma_2 \leq \alpha \leq \gamma_1$ then $\gamma_1 = \alpha$ or $\gamma_2 = \alpha$ for any $\alpha \in \partial\mathcal{T}_e$. We denote the equivalence class of an end $\gamma$ by $\bar{\gamma}$.

**Example 2.3.2.** In the tree at Figure 2.4, we say that $\gamma_4 \sim \gamma_3$, $\gamma_3 \sim \gamma_2$, $\gamma_2 \sim \gamma_1$ *but* $\gamma_4 \not\sim \gamma_1$ *or* $\gamma_3 \not\sim \gamma_1$.

**Proposition 2.3.1.** This relation over the set $\partial\mathcal{T}_e$ is symmetric and reflexive but it is not always transitive.

*Proof.* It is obvious that the relation is reflexive and symmetric. Since $\gamma \sim \gamma$ by definition and if $\gamma_1 \sim \gamma_2$ then there is no end between these paths so $\gamma_2 \sim \gamma_1$ also satisfies. Moreover, we can see in the example 2.3.2 that this relation is not always transitive i.e. $\gamma_3 \sim \gamma_2, \gamma_2 \sim \gamma_1$ but $\gamma_3 \nsim \gamma_1$.

**Definition 2.3.3.** The transitive closure of a relation $R$ on a set $A$, is the smallest transitive relation which contains the relation $R$.

We know the relation over $\partial\mathcal{T}_e$ is not transitive. Let take the transitive closure of the relation on this tree and from now on we will say that $\sim$ is an equivalence relation. We consider the quotient topological space $\partial\mathcal{T}_e/\sim$ via the map

$$\phi : \partial\mathcal{T}_e \longrightarrow \partial\mathcal{T}_e/\sim$$

All proofs from here to the end of the chapter were made by us using ideas from standard proofs in topology on $\mathbb{R}^n$.

**Lemma 2.3.1.** Let $\mathcal{T}$ be a perfect tree and $\gamma \in \partial\mathcal{T}_e$. Then $\bar{\gamma}$ equivalents to either two ends or one end which is itself.

*Proof.* Assume that $\gamma \in \partial\mathcal{T}_e$ without loss of generality with the same orientation of $e$ such that after n-many edges it turns always right (left). Then we can find an end $\gamma'$ whose first $n-1$-many edges are the same with $\gamma$ and then it turns once the first right (left) of the $\gamma$ then it turns always left(right). So since $\gamma'$ turns right (left) before than $\gamma$ by definition $\gamma' < \gamma$ $(\gamma < \gamma')$. If it was $\gamma''$ such that $\gamma' < \gamma'' < \gamma$ $(\gamma < \gamma'' < \gamma')$ then $\gamma''$ must have the same common edges with $\gamma$ and $\gamma'$ and it must turn right before than $\gamma$ $(\gamma')$ and it must turns left before then $\gamma'$ $(\gamma)$. Then it must be equal to $\gamma'$ $(\gamma)$ or $\gamma$ $(\gamma')$ respectively. So we say that $\gamma \sim \gamma'$. And there is no another edges which is equivalent to $\gamma$. On the other hand there exists the ends which turns randomly left or right.
Claim :Let $\alpha$ be such an end then the class of $\alpha$ by the relation $\sim$ has just itself.
Proof: For any $\alpha' \in \partial\mathcal{T}_e$ such that $\alpha < \alpha'$ we can always find $\beta$ such that $\alpha < \beta < \alpha'$. If $\alpha < \alpha'$ $\alpha$ turns right before than $\alpha'$, assume at the $n^{th}$ edge. Let $n+k^{th}$ edge of $\alpha$ turns left then the end which of first $n+k-1$ edges are the same with $\alpha$ and $(n+k)^{th}$ edge turns the right of $\alpha$ is smaller than $\alpha$. It proves our claim.

**Proposition 2.3.2.** If $\mathcal{T}$ is a perfect tree the relation $\sim$ is an equivalence relation. i.e. it is transitive.

*Proof.* By the previous lemma if $\gamma \in \partial\mathcal{T}_e$ then the class of $\gamma$ has one or two edges. So if $\beta \sim \gamma$ then there does not exist another path which is equivalent to $\gamma$. That is $\sim$ is a transitive relation so it is an equivalence relation in the case of perfect trees.

*Notation.* We notice the set of all ends whose class has 2 elements by $\partial\mathcal{T}_e^2$ and the set of ends whose class has only itself by $\partial\mathcal{T}_e^1$.

**Lemma 2.3.2.** Let $\alpha, \alpha'$ be an end of $\partial\mathcal{T}_e$. An interval $[\alpha, \alpha']$ is not an open interval for the topological space $\partial\mathcal{T}_e$ if $\alpha$ or $\alpha'$ is an element of $\partial\mathcal{T}_e^1$.

*Proof.* Assume that $\alpha \in \partial\mathcal{T}_e^1$ and $[\alpha, \alpha']$ be an open interval. Then there exists $n \in \mathbb{N}$ such that $\alpha \in B(\alpha, 2^{-n})$ and $B(\alpha, 2^{-n}) \subseteq [\alpha, \alpha']$. We know $\alpha$ turns randomly left and right. Let $n + k^{th}$ edge of $\alpha$ turns right for $k > 1$. Then there exists an end $\alpha''$ whose both the first $n + k - 1$ edges are same with $\alpha$ and $n + k^{th}$ edge turns left. Then $\alpha'' \in B(\gamma, 2^{-n})$ but $\alpha'' > \alpha$ so $\alpha'' \notin [\alpha, \alpha']$, contradiction with our hypothesis. By the similar reason if $\alpha' \in \partial\mathcal{T}_e^1$ we can reach the same contradiction.

*Corollary* 2.3.1. By the proof of the Lemma 2.3.2 $[\alpha', \alpha)$, $(\alpha, \alpha']$ are not open intervals in $\partial\mathcal{T}_e$ if $\alpha' \in \partial\mathcal{T}_e^1$.

**Lemma 2.3.3.** For appropriately selected $\alpha < \alpha' \in \partial\mathcal{T}_e$ the all possible open intervals of the topological space $\partial\mathcal{T}_e$ are given by
*1.*$(\alpha, \alpha')$ for any $\alpha, \alpha' \in \partial\mathcal{T}_e$.
*2.*$[\alpha, \alpha')$ for $\alpha \in \partial\mathcal{T}_e^2$.
*3.*$(\alpha, \alpha']$ for $\alpha' \in \partial\mathcal{T}_e^2$.
*4.*$[\alpha, \alpha']$ for $\alpha, \alpha' \in \partial\mathcal{T}_e^2$.

*Proof.* Let $\alpha$ and $\alpha'$ have exactly $n+1-$many common edges and after that $\alpha$ turns always right and $\alpha'$ turns always left. Then $B(\alpha, 2^{-n}) = B(\alpha', 2^{-n}) = [\alpha, \alpha']$. Since by definition $B(\alpha, 2^{-n})$ and $B(\alpha', 2^{-n})$ contain all ends which has at least $n+1$- many common edges with $\alpha$ or $\alpha'$ respectively. And it is obvious that $\alpha$ is the minimal element of these sets whereas $\alpha'$ is the maximal element.
Hence all open balls can be described by an interval $[\alpha, \alpha']$ with $\alpha, \alpha' \in \partial\mathcal{T}_e^2$.

**Claim**: Any union of non disjoint intervals is interval.
Let $\{S_i | i \in \mathbb{N}\}$ be the set of intervals and $a \in \bigcap_{i\in\mathbb{N}} S_i$. Now assume that $\gamma < \gamma' < \gamma''$ and $\gamma, \gamma'' \in \bigcup_{i\in\mathbb{N}} S_i$. We have three cases: $\gamma' < a, \gamma' = a$, and $\gamma' > a$. If $y = a$ then $y \in \bigcap_{i\in\mathbb{N}} S_i \subset \bigcup_{i\in\mathbb{N}} S_i$. If $\gamma' < a$, let take $S_j$ for some $j \in \mathbb{N}$ such that $\gamma \in S_j$. Now $\gamma < \gamma' < a$ with $x, a \in S_j$, so $\gamma' \in S_j$ by definition of interval. Then $\gamma' \in \bigcup_{i\in\mathbb{N}} S_i$ If $\gamma' > a$ let $S_k$ be an interval such that $\gamma'' \in S_k$ I. Now $a < \gamma' < \gamma''$ with $a, \gamma'' \in S_k$, so $\gamma' \in S_k$ then $\gamma' \in \bigcup_{i\in\mathbb{N}} S_i$. Hence $\bigcup_{i\in\mathbb{N}} S_i$ is an interval.

Let's take an interval $[\alpha, \alpha')$ for $\alpha \in \partial\mathcal{T}_e^2$ and $\alpha' \in \partial\mathcal{T}_e^1$.

**Claim**: There exist $\alpha_i \in \partial\mathcal{T}_e^2$ for any $i \in \mathbb{N}$ such that

$$[\alpha, \alpha') = [\alpha, \alpha_1] \cup \bigcup_{i \in \mathbb{N}-\{0\}} [\alpha_i, \alpha_{i+1}].$$

Let $\alpha$ and $\alpha'$ have $n$ common edges and they have the same orientation with $e$. Assume that $\alpha$ always turns right after first $n$ edges. Assume that after first $n$ edges $\alpha'$ is a zigzag path that is if one edge of $\alpha$ turns left (right) then the consecutive edges turns right (left). Assume $\alpha = (e, e_1, \cdots, e_n, e_{n+1}, \cdots)$ and $\alpha' = (e, e_1, \cdots, e_n, f_1, f_2, \cdots)$ and we say that $f_1$ turns left. If we choose $\alpha_i$ the path which passes the finite path
$(e, e_1, e_2, \cdots, e_n, f_1, f_2, \cdots, f_{2i})$ and then it always turns right. By this choice $[\alpha_i, \alpha_{i+1}]$ is open interval for any $i \in \mathbb{N}$ since $[\alpha_i, \alpha_{i+1}] = B(\alpha_i, 2^{-(n+2i)})$. And $[\alpha, \alpha_1] = B(\alpha, 2^{-n})$. By previous claim the union of non disjoint intervals is interval and by the definition of topology the union of arbitrary opens is open. So $[\alpha, \alpha')$ is an open interval for appropriately selected $\alpha$ and $\alpha'$.
By the similar reason we will say that $(\alpha, \alpha']$ can be an open interval for $\alpha' \in \partial\mathcal{T}_e^2$.
Let $\alpha, \alpha', \alpha_1, \alpha_2$ have exactly $n$ many common edges. Assume that $\alpha_1$ always turns left, $\alpha_2$ always turns left and $\alpha$ and $\alpha'$ are zigzag path after $n$ edges such that $\alpha$ first left and $\alpha'$ turns right. Then $\alpha_1 > \alpha > \alpha' > \alpha_2$. And also $(\alpha, \alpha') = (\alpha, \alpha_2] \cap [\alpha_1, \alpha')$. Since $(\alpha, \alpha_2], [\alpha_1, \alpha')$ are opens then $(\alpha, \alpha')$ is an open interval for $\alpha, \alpha' \in \partial\mathcal{T}_e^2$.
For the other cases we must use the similar arguments.

From now on assume that $\mathcal{T}$ is a perfect tree.

**Proposition 2.3.3.** Any open subset of $\partial\mathcal{T}_e$ is the union of the disjoint open intervals.

*Proof.* If $U$ is an open subset of $\partial\mathcal{T}_e$ with $\gamma \in U$ then there exists an open interval $I \in U$ such that $x \in I$. (At least since U is an open subset then we know for any element of $x \in U$ we can find an open ball and all open balls can be written an open interval.) If there exists one such interval there exists one 'largest' interval which contains $x$. Denote by the set $\{I_\alpha\}$ all such maximal intervals. So $I_\alpha$ are pairwise disjoint otherwise they wouldn't be maximal. Then $U$ can be written as the union of the disjoint open intervals $\{I_\alpha\}$.

**Definition 2.3.4.** Since $\sim$ is an equivalence relation over $\partial\mathcal{T}_e$ then we can define the quotient topology on $\partial\mathcal{T}_e/\sim$. Let

$$\phi : \partial\mathcal{T}_e \longrightarrow \partial\mathcal{T}_e/\sim$$

This map is continuous and it determines the opens of the topological space $\partial\mathcal{T}_e/\sim$. We say that $U$ is an open subset of $\partial\mathcal{T}_e/\sim$ if its inverse image $\phi^{-1}(\partial\mathcal{T}_e/\sim)$ is an open subset of $\partial\mathcal{T}_e$.

*Remark.* Due to the continuity of the previous map $\phi$ and the compactness of $\partial\mathcal{T}_e$, the topological space $\partial\mathcal{T}_e/\sim$ is compact. Since continuous functions preserves compactness.

**Proposition 2.3.4.** The topological space $\partial\mathcal{T}_e/\sim$ is connected.

*Proof.* Let's define $X := \partial\mathcal{T}_e$ and $Y := \partial\mathcal{T}_e/\sim$. Assume that there exists open subsets $U,V$ of $Y$ such that $Y = U \cup V$ and $U \cap V = \varnothing$ and $U$ and $V$ are not empty. i.e. $Y$ is disconnected. Then $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are open since $\phi$ is continuous and $\phi^{-1}(U) \cup \phi^{-1}(V) = X$, $\phi^{-1}(U) \cap \phi^{-1}(V) = \varnothing$. So $\phi^{-1}(U)^c = \phi^{-1}(V)$. By the Proposition 2.3.3 $\phi^{-1}(U)$ and $\phi^{-1}(V)$ and can be written as the union of the open intervals. And we know all of the open intervals by the Lemma 2.3.3. We consider $\phi^{-1}(U) = \bigcup_{s\in\mathbb{N}}(I_s)$ and $\phi^{-1}(V) = \bigcup_{r\in\mathbb{N}}(J_r)$.
Let's we look all possible cases for the open intervals $I_r$ and $J_s$:
Assume that for some $\alpha,\alpha' \in \partial\mathcal{T}_e^2$ $[\alpha,\alpha'] = I_s$ for some $s \in \mathbb{N}$ such that $(\alpha',\beta) = J_r$ for some $r \in \mathbb{N}$. Claim: $U \cap V \neq \varnothing$. We know $\alpha' \in \partial\mathcal{T}_e^2$ and $\alpha < \alpha'$ so $\alpha'$ is a path which always turns left after finitely many edges. So there exists $\alpha'' > \alpha'$ and $\alpha'' \sim \alpha'$. So $\alpha'' \in U$ since $X$ is a quotient topology. But in the same time $\alpha'' \in V$. So this claim gives us a contradiction.
Assume that for some $r \in \mathbb{N}$ $I_r = (\alpha,\alpha']$ with $\alpha' \in \partial\mathcal{T}_e^2$ such that $J_s = (\alpha',\beta)$. Then by the same reason with the previous case there exists $\alpha'' \in U \cap V$.
Assume that for some $r \in \mathbb{N}$ $I_r = (\alpha,\alpha')$ with $\alpha,\alpha' \in \partial\mathcal{T}_e^1$ with $[\alpha',\beta]$ or $[\alpha',\beta)$ are not open for any $\beta$ so there is not such a case.
Assume that $r \in \mathbb{N}$ $I_r = (\alpha,\alpha')$ with $\alpha,\alpha' \in \partial\mathcal{T}_e^2$ with $[\alpha',\beta] = J_s$ for some $s \in \mathbb{N}$. So in this case we have $\alpha'' \sim \alpha'$ such that $\alpha'' \in U \cap V$ like the first case. So we reach the contradiction
Hence $Y$ is a connected topological space.

**Lemma 2.3.4.** $Y \setminus \{\alpha,\beta\}$ is disconnected for any $\{\alpha,\beta\} \subset Y$.

*Proof.* We can say that $Y - \{\alpha,\beta\} = (\alpha,\beta) \cup (\beta,\alpha)$. Then $Y$ is the union of two disjoint open intervals.

**Theorem 2.3.5.** *The topological space $\partial\mathcal{T}_e/\sim$ is homeomorphic to $\mathbb{S}^1$.*

*Proof.* The topological caracterisation of $\mathbb{S}^1$ is given as follows:

- Connected

- Compact
- For any $a, b \in \mathbb{S}^1$, $\mathbb{S}^1 - \{a, b\}$ is totally disconnected.

  So all topological spaces which has these properties are homeomorphic to $\mathbb{S}^1$, see [12].

# 3 TREE AUTOMORPHISMS

In this section, we study the automorphism groups of trees. This group may change dramatically if we consider trees with ribbon structure and if we require that the automorphisms must preserve the ribbon structure. For example, the automorphism group of a regular planar tree is countable, whereas if we consider it as an abstract graph (i.e. without the ribbon structure), then the automorphism group becomes an uncountable group.

## 3.1 Automorphisms of Planar Trees

If $\mathcal{T}$ has a ribbon structure we say $\mathcal{T}$ is a planar tree and in this case it has countably many automorphisms. We denote the automorphisms group of $\mathcal{T}$ by $Aut(\mathcal{T})$. $Aut(\mathcal{T})$ is generated by rotations around a vertex (or edge).

**Example 3.1.1.** Let $\mathcal{T}$ be a 3- regular planar tree with the set of edges ;
$E(\mathcal{T}) = \{\{0,1\}^* \cup \{0',1'\}^*\}$ and the set of vertices $V(\mathcal{T}) = (u, u0, u1)$. We can define countably many automorphisms on this tree but all of them can be generalized by the rotations.
We will choose 2 edges and we will define a function from the first to the second one. And the others are determined by this two edges.
*1.* $\varphi_1 : E(\mathcal{T}) \longmapsto E(\mathcal{T})$
$00 \longrightarrow 0'$ is an automorphism of $\mathcal{T}$. It just dislocates the edges and it preserves the planar structure. So $\varphi(000) = 1'$, $,\varphi(001) = e$, $\varphi(0) = 0'1'$ etc.
2. $\varphi_2 := (01e)$ is a rotation around the vertex $(e, 0, 1)$.. So it is an automorphism of $\mathcal{T}$. It rotates the tree around the edges $e$. So it preserves also the planar structure.
*3.* $\varphi_3 := (00')(11')$ is the rotation around the edge $e$ and also $\varphi_3 \in Aut(\mathcal{T})$.

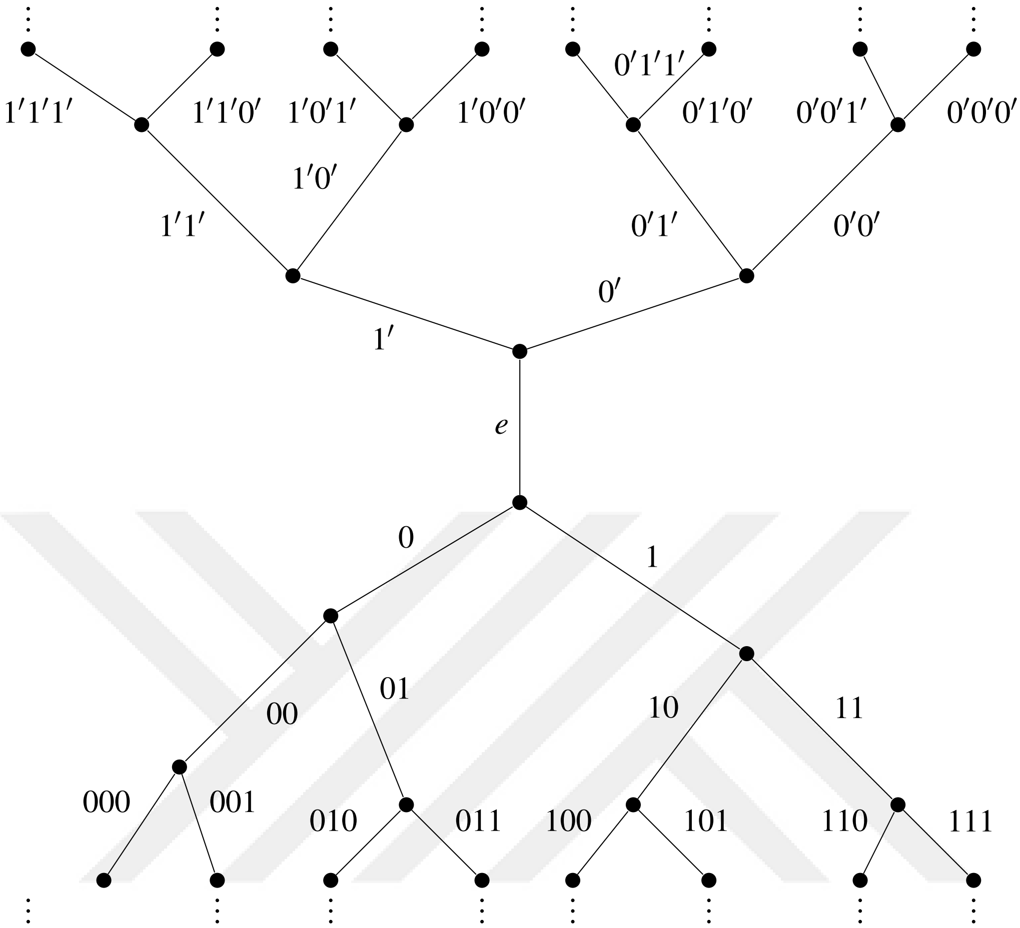



Figure 3.1: Binary Tree

Automorphism groups of non regular trees are less interesting than regular trees. Moreover, the automorphism group of a non regular tree can be trivial. For example, the tree $\mathcal{T}$ which is given in Figure 3.2 is non regular and $Aut(\mathcal{T}) = \{Id\}$. But this is not to say that the automorphism groups of non regular trees are always trivial or finite.

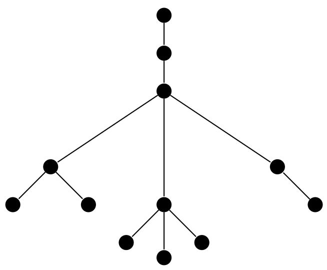


Figure 3.2: An example of non regular tree

### 3.1.1 Presentations of The Three-Regular Tree

In the Example 3.1.1 we saw a representation of the edges of $3-$ regular tree by using $\{0,1,0',1'\}$. But this representation is not the natural one. If the tree was a rooted tree we would use $\{0,1\}$ and it could be sufficient for the representation. But in our tree we must use $\{0',1'\}$ also for defining all of the edges and it breaks the naturalness. Via this representation we can describe easily the group $Aut(\mathcal{T})$.

#### The Natural Representation of The $3-$ Regular Trees

This representation is given in the article [17]. Consider the group
$\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/3\mathbb{Z} = \langle a,b \mid a^2 = b^3 = e := 1\rangle$ which is isomorphic the group PSL($2\mathbb{Z}$).
Consider the graph $\mathcal{F}$ with the set of edges $E(\mathcal{F}) = \mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/3\mathbb{Z}$ and the set of vertices $V(\mathcal{F}) = V_{\otimes}(\mathcal{F}) \cup V_{\bullet}(\mathcal{F})$ where $V_{\otimes}(\mathcal{F}) = \{\{w,wa\} | w \in \mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/3\mathbb{Z}\}$ and
$V_{\bullet}(\mathcal{F}) = \{\{w,wb,wb^2\} | w \in \mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/3\mathbb{Z}\}$.
This gives a tree since $\{a,b\}$ generates the group $\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/3\mathbb{Z}$ so the graph is connected. Moreover, it does not contain circuit or loop. Since there are no relations other than the relation $a^2 = b^3 = e$ between $a$ and $b$,

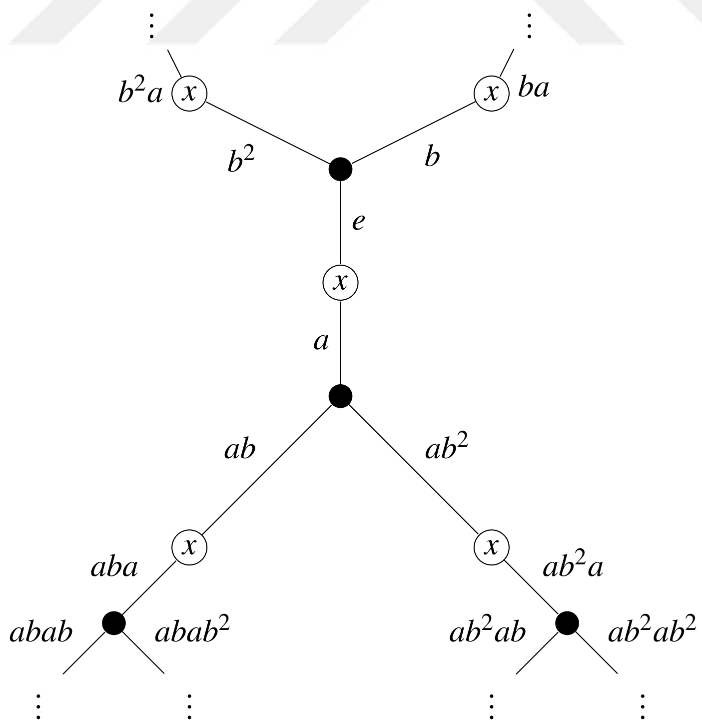



Figure 3.3: Binary Tree which is described by $\mathrm{PSL}_2(\mathbb{Z})$

*Remark.* In fact, via this representation we define a ribbon structure also over the tree and it is counterclockwise i.e. the vertices have an orientation $(m,mb,mb^2)$ and $(m,ma)$, we can see at the Figure 3.5. So the tree $\mathcal{F}$ is planar.

### 3.1.2 The Automorphism Group of Planar Farey Tree

Thanks to this representation defining the automorphisms is much easier.

**Proposition 3.1.1.** Let $M \in \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. Then the map $\varphi : E(\mathcal{F}) \longrightarrow E(\mathcal{F}$ sending the edge $w$ to the edge $Mw$ and mapping the vertices as

$$V_{\otimes}(\mathcal{F}) \longrightarrow V_{\otimes}(\mathcal{F})$$
$$\{w, wa\} \longmapsto \{Mw, Mwa\}$$
$$V_{\bullet}(\mathcal{F}) \longrightarrow V_{\bullet}(\mathcal{F})$$
$$\{w, wb, wb^2\} \longmapsto \{Mw, Mwb, , Mwb^2\}$$

is a planar automorphism of the tree $\mathcal{T}$. Every planar automorphism of the tree can be expressed this way so that $Aut(\mathcal{F}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

*Proof.* The multiplication map $\varphi$ preserves the incidence of edges since the image of a vertex $\{w, wb, wb^2\}$ or $\{w, wa\}$ is also a vertex. So if the edges are incident the image of the edges are incident. So $\varphi$ is a homomorphisme.
There is no relation between $a$ and $b$ such that i.e. $a^{k_1} b^{k_2} a^{k_3} \cdots \neq a^{m_1} b^{m_2} a^{m_3} \cdots$ for any $k_i, m_j \in \mathbb{Z}$ and $k_{2i+1}, m_{2i+1} \neq 2$, $k_{2i}, m_{2i} \neq 3$ for any $i, j \in \mathbb{N}$. So the map is a bijection. Hence it is an automorphism.
If we multiply all vertices with $M$ we can not change the orientation. So this automorphism preserves the ribbon sturcture. So it is an automorphism of $\mathcal{F}$.

### 3.1.3 The Automorphism Group of Finite Abstract Farey Tree

We can define the automorphism group of the abstract Farey tree $\mathcal{F}$ and since there is no ribbon structure of this tree, it has more automorphisms than $\mathcal{F}$. We will use the same representation of Farey tree to define the automorphisms of $|\mathcal{F}|$ since it is the most appropriate representation. But we don't have an expectation that the automorphisms preserve the ribbon structure.

**Proposition 3.1.2.** One has a proper inclusion Aut($\mathcal{F}$)<Aut($|\mathcal{F}|$), but Aut($\mathcal{F}$) is not a normal subgroup inside Aut($|\mathcal{F}|$).

*Proof.* Let $\varphi_b \in Aut(\mathcal{F})$. i.e. $\varphi_b(w) = w$. Let $\psi \in \mathrm{Aut}_u(|\mathcal{F}|)$ be the twist of the vertex $\{e, b, b^2\}$.i.e. $\psi\{\{e, b, b^2\}\} = \{e, b^2, b\}$. Then $\psi\varphi_b\psi^{-1}(b) = \psi\varphi_b(b^2) = \psi(e) = e$ $\psi\varphi_b\psi^{-1}(b^2) = \psi\varphi_b(b) = \psi(b^2) = b$ $\psi\varphi_b\psi^{-1}(e) = \psi\varphi_b(e) = \psi(b) = b^2$. Hence $\psi\varphi_b\psi^{-1} \notin Aut(\mathcal{F})$. Then $Aut(|\mathcal{F}|)$ is not normal.

Now let $\varphi \in Aut(|\mathcal{F}|)$ and $w \in E(|\mathcal{F}|)$ with $\varphi(w) = u$ where $u$ is an edge in $|\mathcal{F}|$. Then there is an automorphism $\psi \in Aut(|\mathcal{F}|)$ such that $\psi(u) = w$. Then $\varphi\psi(u) = u$. Hence $\varphi\psi$ fixes the edge u. All of the automorphisms which fixes the edge $u$ constitutes a subgroup of $Aut(|\mathcal{F}|)$ which we denote as $\mathrm{Aut}_u(|\mathcal{F}|)$.

**Proposition 3.1.3.** $\mathrm{Aut}_u(|\mathcal{F}|)$ is the subgroup of $Aut(|\mathcal{F}|)$ but it is not normal.

*Proof.* Let $\varphi_1, \varphi_2 \in \mathrm{Aut}_u(|\mathcal{F}|)$. Then $\varphi_1((\varphi_2)^{-1}(u)) = \varphi_1(u) = u$. Then it is closed under the composition operation. And also $\mathrm{Id} \in \mathrm{Aut}_u(|\mathcal{F}|)$ is obvious. Let $\varphi \in \mathrm{Aut}_u(|\mathcal{F}|)$ and let $\psi \in Aut(|\mathcal{F}|)$ with $\psi(u) = w$. Then $(\psi\varphi\psi^{-1})(w) = \psi(\varphi(u)) = \psi(u) = w$. Then it fixes $w$. So $\psi \mathrm{Aut}_u(|\mathcal{F}|)\, \psi^{-1} = \mathrm{Aut}_w(|\mathcal{F}|)$. Thus the subgroup is not normal.

Denote by $\mathcal{F}_n$ is the subtree of $\mathcal{F}$ consisting of vertices of distance $\leq n$ to the fixed edge $e$.

**Example 3.1.2.** The finite trees $\mathcal{F}_n$ are given in the following figures for $n = 1,2,3$. Let's give the $\mathrm{Aut}_e(|\mathcal{F}_n|)$ for each $n$.
Since they fix $e$ and they are not perfect we have finitely many automorphisms. Let define $\varphi : E(|\mathcal{F}_n|) \longrightarrow E(|\mathcal{F}_n|)$ with $\varphi(b) = b^2$, $\varphi(b^2) = b$, $\varphi(e) = e$. Then $\varphi \in \mathrm{Aut}_e(|\mathcal{F}_1|)$ is the unique non trivial automorphism. And $\varphi^2 \equiv Id$. Therefore, $\mathrm{Aut}_e(|\mathcal{F}_1|) \cong \mathbb{Z}/2\mathbb{Z}$.
*In* $|\mathcal{F}_2|$ *we can generate all automorphisms by two automorphisms which is given by* $\varphi_1, \varphi_2 : E(|\mathcal{F}_n|) \longrightarrow E(|\mathcal{F}_n|)$ *where* $\varphi_1(e) = e, \varphi_1(b) = b^2, \varphi_1(b^2) = b$ *so it must be* $\varphi_1(b^2a) = ba$ *and* $\varphi_1(ba) = b^2a$ *and it fixes the others,*
$\varphi_2(e) = e, \varphi_2(a) = a, \varphi_2(ab^2) = ab$, $\varphi_2(ab) = ab^2$ *and it fixes the other edges. So* $\mathrm{Aut}_e(|\mathcal{F}_2|) = \langle \varphi_1, \varphi_2 \rangle$ *and it is obvious that* ${\varphi_2}^2 = {\varphi_1}^2 := Id$. *Then* $|\mathrm{Aut}_e(|\mathcal{F}_2|)| = 4$.
*So* $\mathrm{Aut}_e(|\mathcal{F}_2|) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
*Since we want to find the automorphisms which preserves the edge e we can give the automorphisms of the finite tree which is the left of the edge e and the other finite tree which is the right of e. The automorphism group of the left side of the tree* $|\mathcal{F}_3|$ *( by the edge e) is isomorphic to the automorphism group of the right side of* $|\mathcal{F}_2|$.

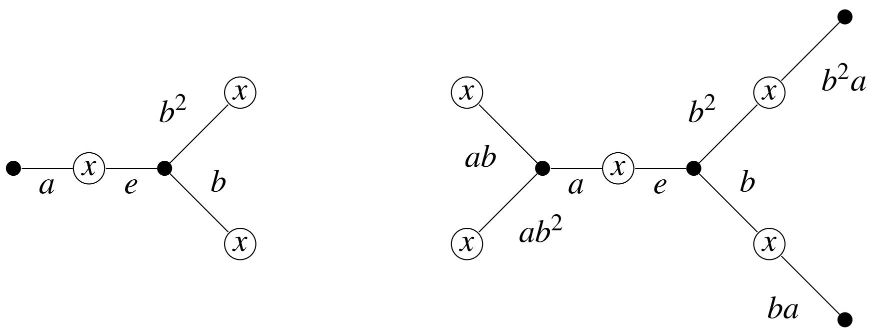



Figure 3.4: $\mathcal{F}_1$ and $\mathcal{F}_2$

*In the right side of the tree* $|\mathcal{F}_3|$ *by the edge e we can twist the tree vertices;* $v_1 := \{e, b, b^2\}, v_2 := \{ba, bab, bab^2\}, v_3 := \{b^2a, b^2ab, b^2ab^2\}$. *Let define* $G := \langle v_1, v_2, v_3 \rangle$. *So* $\mathrm{Aut}_e(|\mathcal{F}_3|) \cong \mathbb{Z}/2\mathbb{Z} \times G$.

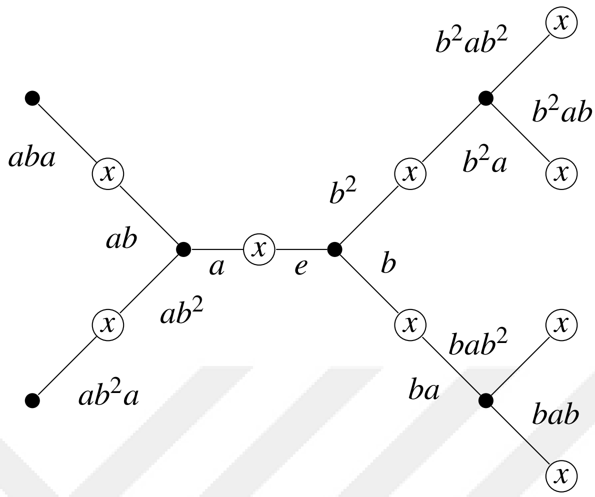
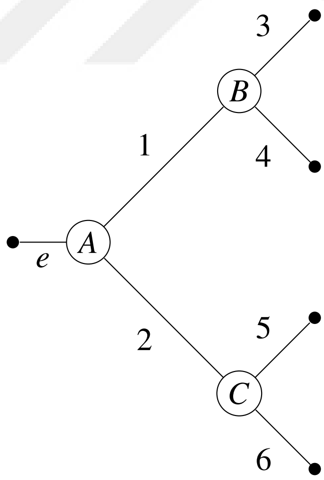



Figure 3.5: $\mathcal{F}_3$

*To facilitate the notation we will show the automorphisms on the following tree:*




*We can give all of the twists by permutation notation such as :* $\sigma_B := (34)$, $\sigma_C := (56)$, $\sigma_B\sigma_C := (34)(56)$, $\sigma_A := (12)(36)(45)$, $\sigma_B\sigma_A := (12)(3546)$, $\sigma_C\sigma_A := (12)(3645)$, $\sigma_C\sigma_B\sigma_A := (12)(35)(46)$. *That is, if we twist the vertex B,* $\sigma_B(3) = 4$ *and* $\sigma_B(4) = 3$.

*We can see that these automorphisms are the symmetries and rotations of a rectangle whose vertices are named with the numbers* $3, 4, 5, 6$. *That is G is isomorphic to the Dihedral Group* $D_8$. *As a result,* $\mathrm{Aut}_e(|\mathcal{F}_3|) \cong \mathbb{Z}/2\mathbb{Z} \times D_8$. *Hence* $|\mathrm{Aut}_e(|\mathcal{F}_3|)| = 16$.

More generally, we have

$$|\operatorname{Aut}_e(|\mathcal{F}_n|)| = \begin{cases} 2^{n-1} \times 2^n & \text{if } n \text{ is odd.} \\ 2^n \times 2^n & \text{if } n \text{ is even.} \end{cases}$$

### 3.1.4 Profinite Group

The following restriction map is surjective:

$$f := \operatorname{Aut}_e(|\mathcal{F}|) \longrightarrow \operatorname{Aut}_e(|\mathcal{F}_n|)$$

$$\varphi \longmapsto \varphi_{|\mathcal{F}_n}.$$

It is obvious that it is not injective.
Furthermore, we have a system of surjective maps given by:

$$\cdots \operatorname{Aut}_e(|\mathcal{F}_3|) \longrightarrow \operatorname{Aut}_e(|\mathcal{F}_2|) \longrightarrow \operatorname{Aut}_e(|\mathcal{F}_1|) \longrightarrow \operatorname{Aut}_e(|\mathcal{F}_0|)$$

where each group is finite.

**Definition 3.1.1.** A profinite group is a topological group that is isomorphic to the inverse limit of an inverse system of discrete finite groups. In this case the inverse limit is called the profinite limit. Let

$$\cdots \underset{f_3}{\longrightarrow} G_3 \underset{f_2}{\longrightarrow} G_2 \underset{f_1}{\longrightarrow} G_1 \underset{f_0}{\longrightarrow} G_0$$

be a system of finite groups. Then their profinite limit is given by:

$$\varprojlim G_i := \{x = (g_i)_{i\in I} \in \prod_{i\in I} G_i | f_i(x) = g_i\}$$

In our case we have a system of fine groups *:*

$$\cdots \underset{f_3}{\longrightarrow} \operatorname{Aut}_e(|\mathcal{F}_3|) \underset{f_2}{\longrightarrow} \operatorname{Aut}_e(|\mathcal{F}_2|) \underset{f_1}{\longrightarrow} \operatorname{Aut}_e(|\mathcal{F}_1|) \underset{f_0}{\longrightarrow} \operatorname{Aut}_e(|\mathcal{F}_0|)$$

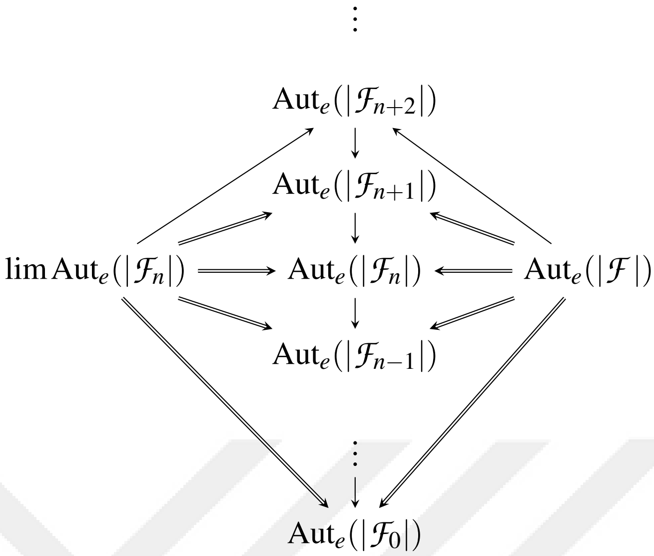
**Theorem 3.1.1.**

$$\operatorname{Aut}_e(|\mathcal{F}|) \cong \varprojlim \operatorname{Aut}_e(|\mathcal{F}_n|)$$

*Proof.* The following map is restriction and it is surjective:

$$\varphi : \varprojlim \operatorname{Aut}_e(|\mathcal{F}_n|) \longrightarrow \operatorname{Aut}_e(|\mathcal{F}_n|)$$

So we have the following diagram:




By this diagram we say that $\mathrm{Aut}_e(|\mathcal{F}|) \cong \lim_{\leftarrow} \mathrm{Aut}_e(|\mathcal{F}_n|)$.

## 3.2 Automorphism Group of An Abstract Tree

### 3.2.1 The Group $\mathrm{Aut}_e(|\mathcal{F}|)$

$\mathrm{Aut}_e(|\mathcal{F}|)$ is the group of automorphisms of the abstract tree $\mathcal{F}$ which fixes the edge $e$. We have two different type of automorphism.

**1.Twist**

**Definition 3.2.1.** Let $w \in E(\mathcal{F})$ and $v = \{w, wb, wb^2\} \in V_\bullet(\mathcal{F})$. Then the twist automorphism by the vertex $v$ is given by $\sigma_v$ as follows;

$$\sigma_v(x) = \begin{cases} x & \textit{if } x \textit{ does not start with } w \\ wb^{-k_1}ab^{-k_2}a\cdots & \textit{if } x = wb^{k_1}ab^{k_2}a\cdots \textit{ with } k_i \in \{1,2\} \textit{ for any } i. \end{cases}$$

*We know $b^3 = e$ so $b^{-1} = b^2$ and $b^{-2} = b$. So this automorphisms transforms all $b$ to $b^2$ whereas all $b^2$ to $b$.*

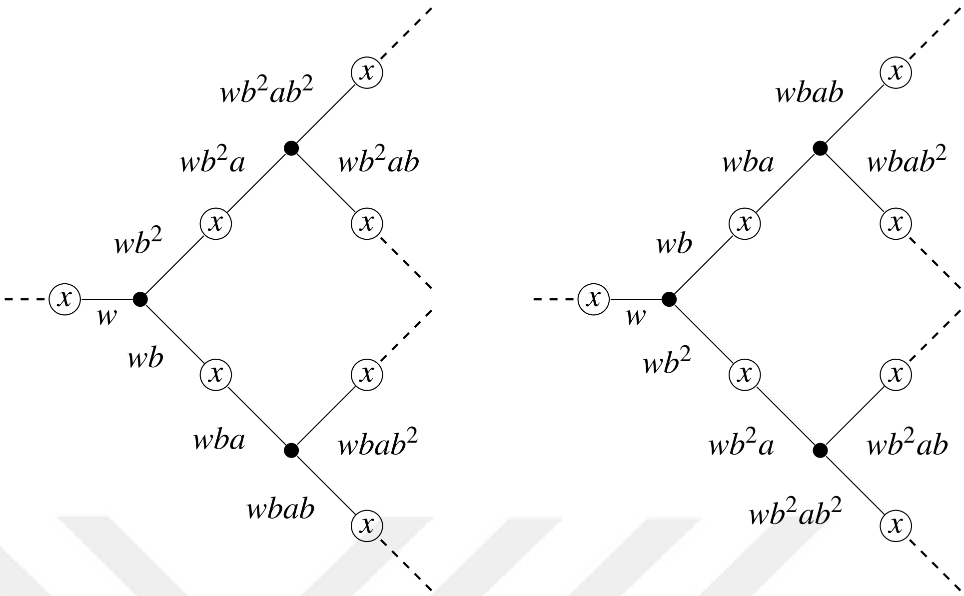



Figure 3.6: $\mathcal{F}$ and $\sigma_v(\mathcal{F})$

In the Figure 3.7 we can see that $v = \{w, wb, wb^2\}$ whereas $\sigma_v(v) = \{w, wb^2, wb\}$. So it changed the orientation i.e. it does not preserve the ribbon structure.
Let $\nu = \{v_1, v_2, v_3, \cdots\}$ be an arbitrary set of vertices of the tree $\mathcal{F}$. Let us sort vertices according to their distance to the fixed edge $e$ where
$d(v_i, e) :=$ the number of the edges between $v_i$, $e$ for any $i \in \mathbb{N}$ . So let
$\nu^{(1)} = \{{v_1}^{(1)}, {v_2}^{(1)}, {v_3}^{(1)}, \cdots\}$ be the current ordered set by the distance. That is
$d({v_i}^{(1)}, e) \leq d({v_{i+1}}^{(1)}, e)$.
$\sigma_\nu :=$ is the twist automorphism which twists every vertices in the set $\nu$ in order and appropriately.
Then the twist $\sigma_\nu$ is defined as follows: First, apply $\sigma_{{v_1}^{(1)}}$ to $\mathcal{F}$. Let call $\sigma_{\nu^{(1)}} = \sigma_{{v_1}^{(1)}}$. Then update the list, that is if we twist $v_1$ the places of other vertices may change so we must write it to the list by their new places. But it is obvious that the order does not change. So let $\nu_2 := \{{v_2}^{(2)}, {v_3}^{(2)}, \cdots\}$. Then apply $\sigma_{{v_2}^{(2)}}$ to the new tree. Let call $\sigma_{\nu^{(2)}} = \sigma_{{v_1}^{(1)}} \circ \sigma_{{v_2}^{(2)}}$. Then we will update again the set of vertices $\nu_2$, and we continue by this way. Therefore,

$$\sigma_\nu := \cdots \circ \sigma_{{v_3}^{(3)}} \circ \sigma_{{v_2}^{(2)}} \circ \sigma_{{v_1}^{(1)}}.$$

Let $\nu = \{v_1, v_2, v_3, \cdots\}$ be an infinite set. We know $\mathcal{F}_1$ has only one vertex to twist, $v := \{e, b, b^2\}$. Let ${v_1}^{(1)} = v$. So $\sigma_{{v_1}^{(1)}} \in V_\bullet(\mathcal{F}_1)$ but $\sigma_{{v_i}^{(i)}} \notin V_\bullet(\mathcal{F}_1)$ for any $i > 1$. Since $\mathcal{F}_2$ has just 2 vertices to twist, $\sigma_{{v_i}^{(i)}} \notin V_\bullet(\mathcal{F}_2)$ for any $i > 2$. Continuing this way we can observe that $(\sigma_{\nu^{(i)}})_{i=1}^{\infty}$ is a convergent sequence such that $\sigma_\nu = \lim_{\leftarrow} \sigma_{\nu^{(i)}}$.

**Shuffle**

**Definition 3.2.2.** Let $m \in E(\mathcal{F})$ and $u = \{m, mb, mb^2\} \in V_{\bullet}(\mathcal{F})$. Then the shuffle automorphism by the vertex $u$ is given by $\tau_u$ as follows;

$$\tau_u(x) = \begin{cases} x & \text{if } x \text{ does not start with } m \\ mby & \text{if } x = mb^2y \\ mb^2y & \text{if } x = mby \end{cases}$$

Shuffle of the edge $u$ is given by the following figure:

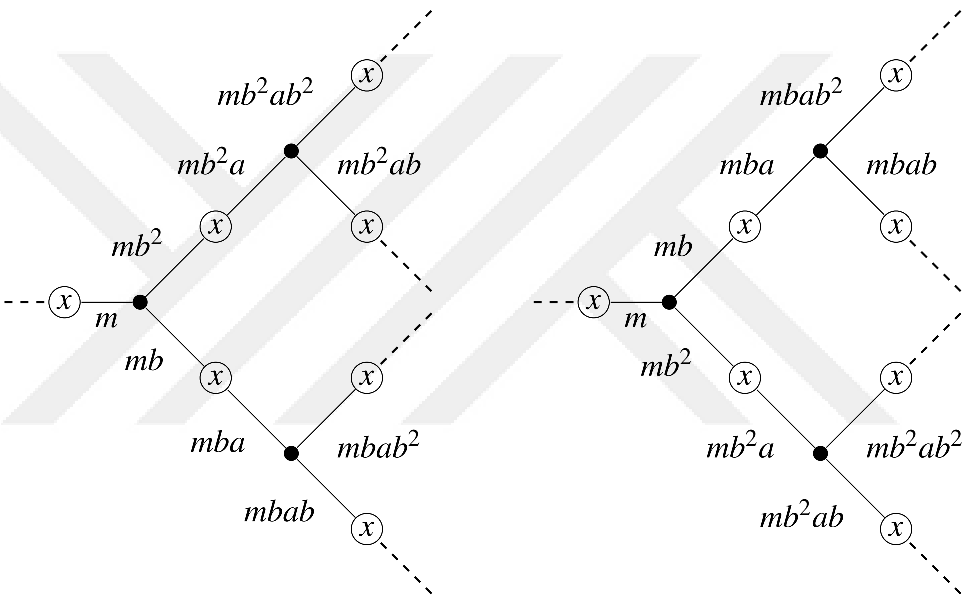



Figure 3.7: $\mathcal{F}$ and $\tau_u(\mathcal{F})$

*Remark.* It is clear that it does not preserve the orientation of the vertex $u$ whereas it preserves the others.

*Let $\mu = \{u_1, u_2, u_3, \cdots\}$ be an arbitrary set of vertices of the tree $\mathcal{F}$.Now $\tau_\mu$ is the shuffle automorphism which shuffles every vertices in the set $\mu$ in order. By the same steps with twist automorphism we will find $\tau_\mu$. First we sort vertices according to their distance to the fixed edge e. So let $\mu^{(1)} = \{{u_1}^{(1)}, {u_2}^{(1)}, {u_3}^{(1)}, \cdots\}$ be the current ordered set by the distance.*

*First, apply $\tau_{{u_1}^{(1)}}$ to $\mathcal{F}$. Then update the list $\mu_2 := \{{u_2}^{(2)}, {u_3}^{(3)}, \cdots\}$. Then apply $\tau_{{u_2}^{(2)}}$ to the new tree. So we will continue by the same way. Therefore,*

$$\tau_\mu := \cdots \circ \tau_{{u_3}^{(3)}} \circ \tau_{{u_2}^{(2)}} \circ \tau_{{u_1}^{(1)}}.$$

*And by the same explication,* $(\tau_{\mu^{(i)}})_{i=1}^{\infty}$ *is a convergent sequence such that* $\tau_\mu = \lim \leftarrow \tau_{\mu^{(i)}}$.

**Lemma 3.2.1.** Let $v = \{w, wb, wb^2\}$ and $\sigma_v$ be twist automorphism of the $\mathcal{F}$. Then there exists $\mu = \{u_1, u_2, \cdots\} \subset V_\bullet(\mathcal{F})$ such that the shuffle $\tau_\mu \equiv \sigma_v$.

*Proof.* $\sigma_v$ is given by:

$$\sigma_v(x) = \begin{cases} x & \text{if } x \text{ does not start with } w \\ wb^{-k_1}ab^{-k_2}a\cdots & \text{if } x = wb^{k_1}ab^{k_2}a\cdots \text{ with } k_i \in \{1,2\} \text{ for any } i. \end{cases}$$

$\tau_u(x) = \sigma_v(x)$ for any $x$ which does not start with $w$ and for any $u = \{wz, wzb, wzb^2\}$ with $z \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. So for such $x$'s we won.
Claim: $\mu := \{\{wz, wzb, wzb^2\} |$ for all z $\in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}\} \subset V_\bullet(\mathcal{F})$.i.e.$\mu := \{u_1 := \{w, wb, wb^2\}, u_2 := \{w\}\}$
Assume that $x = wb^{k_1}ab^{k_2}ab^{k_3}\cdots$. Then $\tau_{u_1^{(1)}}(x) = wb^{-k_1}ab^{k_2}ab^{k_3}\cdots$.
$\tau_{u^{(3)}}(x) = wb^{-k_1}ab^{-k_2}ab^{k_3}\cdots$ since there exist two vertices whose distance to the edge $e$ are 2. So the shuffle of one of them does not change the value of $x$. By the similar reason $\tau_{u^{(7)}}(x) = wb^{-k_1}ab^{-k_2}ab^{-k_3}\cdots$. So continue with this way we reach $\tau_\mu := \sigma_v$.

**Lemma 3.2.2.** Let $u = \{m, mb, mb^2\}$ and $\tau_u$ be a shuffle automorphism of the $\mathcal{F}$. Then there exists $\nu = \{v_1, v_2, \cdots\} \subset V_\bullet(\mathcal{F})$ such that the twist $\sigma_\nu := \tau_u$.

*Proof.* Let $x \in E(\mathcal{F})$ with $\tau_u(x) \neq x$. Since in this case the result is obvious. Let $x = mb^{k_1}ab^{k_2}ab^{k_3}\cdots$ then $\tau_u(x) = wb^{-k_1}ab^{k_2}ab^{k_3}\cdots$. Let take $\nu = \{v_1 = \{m, mb, mb^2\}, v_2 = \{mb^{k_1}a, mb^{k_1}ab, mb^{k_1}ab^2\}\} \subset V_\bullet(\mathcal{F})$. Then $\sigma_{v_1^{(1)}}(x) = wb^{-k_1}ab^{-k_2}ab^{-k_3}\cdots$ and then $\sigma_{\nu^{(2)}}(x) = wb^{-k_1}ab^{k_2}ab^{k_3}\cdots = \tau_u(x)$.

**Theorem 3.2.3.** *Every automorphisms of the abstract Farey tree* $\mathcal{F}$ *which preserves the fixe edge e can be written by twists (or shuffles).*

*Proof.* We know the automorphisms are shuffles and twists. And we showed that by the Lemma 3.2.2 every shuffles can be written by twists. So every automorphisms can be written by twists or vice versa.

**Proposition 3.2.1.** Let $\varphi \in \mathrm{Aut}_e(|\mathcal{F}|)$ and $\gamma, \gamma' \in \partial_e(\mathcal{F})$. Then $d(\gamma, \gamma') = d(\varphi(\gamma), \varphi(\gamma'))$.

**Graps of Automorphisms**

In fact, the vertices of $\mathcal{F}$ can be presented by the rational numbers:
By this tree $\sigma_{\frac{1}{1}}(x) = \frac{1}{x}$ for every $x \in \mathbb{Q}^+$. So the graphs is given by following figure:

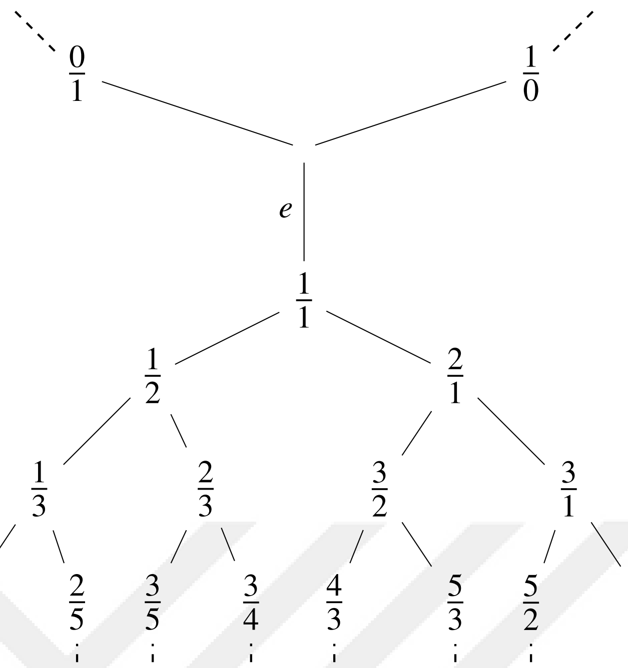



Figure 3.8: Farey Tree

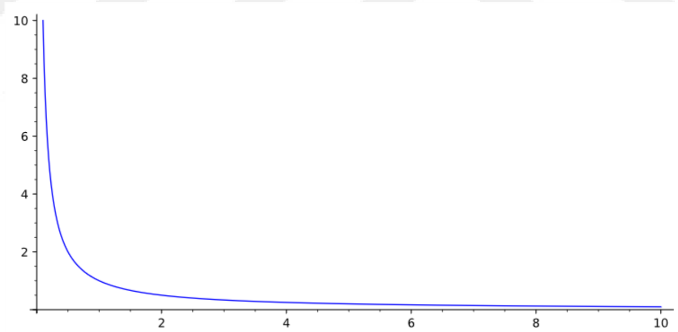



Figure 3.9: The graphs of $\sigma_1(x)$

Let's look at the graphs of some elements of $\mathrm{Aut}_e(|\mathcal{F}|)$:
$\sigma_{\nu_1}(x)$ with $\nu_1 = \{1, \frac{1}{2}\}$. So

$$\sigma_{\nu_1}(x) = \begin{cases} \frac{1}{x} & \text{, if } x \in (0,1) \\ 1 - \frac{1}{x} & \text{,otherwise} \end{cases}$$

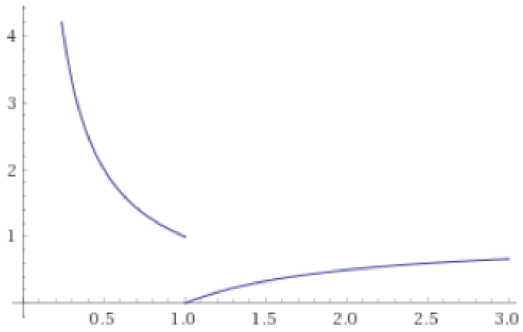



Figure 3.10: The graphs of $\sigma_{\nu_1}(x)$

$\sigma_{\nu_2}(x)$ with $\nu_2 = \{1, \frac{1}{2}, 2\}$. So $\sigma_{\nu_2}(x) = \tau_1(x)$ for any $x \in \mathbb{Q}^+$.

$$\sigma_{\nu_2}(x) = \begin{cases} \frac{1}{1-x} & \text{, if } x \in (0,1) \\ 1 - \frac{1}{x} & \text{,otherwise} \end{cases}$$

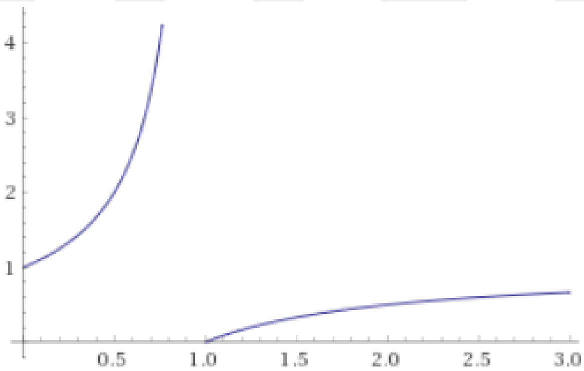



Figure 3.11: The graphs of $\sigma_{\nu_2}(x)$

Let's look at the automorphism $\sigma_{\frac{3}{5}}$ : We are just twisting the vertex $\frac{3}{5}$ so

$$\sigma_{\frac{3}{5}}(x) = \begin{cases} \frac{4x-3}{5x-4} & \text{, if } x \in (\frac{1}{2}, \frac{2}{3}) \\ x & \text{,otherwise} \end{cases}$$

Since the subtree of the Farey tree whose root is $\frac{3}{5}$ consists all rational numbers between $\frac{1}{2}$ and $\frac{2}{3}$. So the others does not change after the twist of $\frac{3}{5}$.
So the graphs of $\sigma_{\frac{3}{5}}$ is given by :

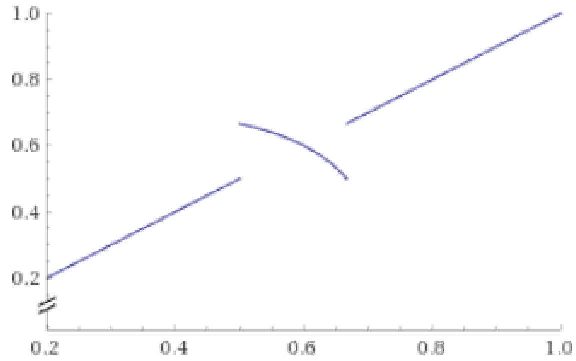



Figure 3.12: The graphs of $\sigma_{\frac{3}{5}}(x)$

# 4 MEASURES ON THE BOUNDARY OF THE FAREY TREE

## 4.1 Stern-Brocot Tree

Stern-Brocot tree is a binary tree which contains all positive rationals numbers once as a vertex. This particular tree was discovered by a mathematician Stern and a clockmarker Brocot. We construct this tree by taking the mediants.

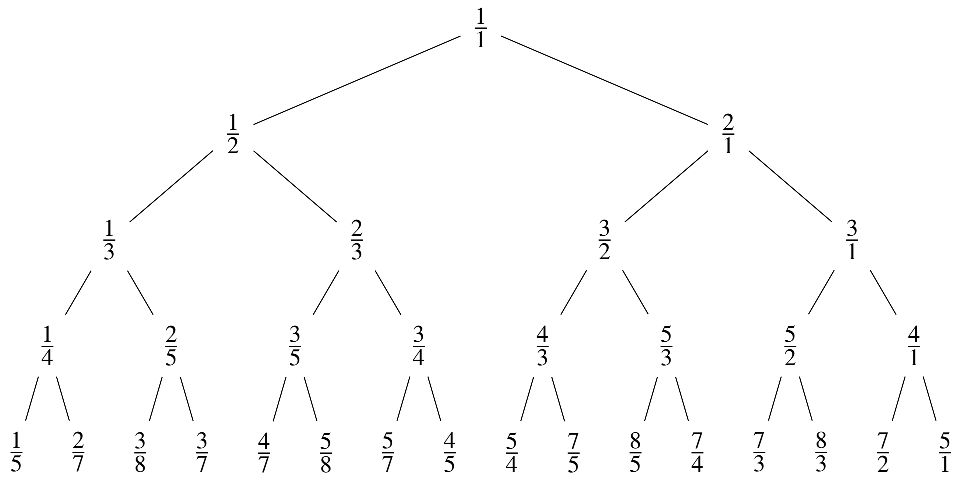
**Definition 4.1.1.** Let $\frac{a_1}{b_1}, \frac{a_2}{b_2} \in \mathbb{Q}^+$. The mediant of these two fractions is $\frac{a_1+a_2}{b_1+b_2} \in \mathbb{Q}$. For the mediant we use the notation $\frac{a_1}{b_1} \oplus \frac{a_2}{b_2} = \frac{a_1+a_2}{b_1+b_2}$.

*Remark.* Let $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}^+$ such that $\frac{p}{q} < \frac{r}{s}$. Then

$$\frac{p}{q} < \frac{p}{q} \oplus \frac{r}{s} < \frac{r}{s}. \tag{4.1}$$

*Proof.* We have $\frac{p}{q} < \frac{r}{s}$ then $p.s < q.r$. So
$p.s + p.q < q.r + p.q \iff p(q+s) < q(r+p) \iff \frac{p}{q} < \frac{r+p}{q+s}$. On the other hand
$p.s < q.r \iff p.s + r.s < q.r + r.s \iff s(p+r) < r(q+s) \iff \frac{r+p}{q+s} < \frac{r}{s}$. Thus
$\frac{p}{q} < \frac{p}{q} \oplus \frac{r}{s} < \frac{r}{s}$.

To construct the Stern- Brocot Tree we start with the fractions $\frac{0}{1}$ and $\frac{1}{0}$. The mediant of these given fractions is $\frac{1}{1}$. Then for finding the right child and left child of $\frac{1}{1}$ we will take the mediant of $\frac{0}{1}$ and $\frac{1}{1}$, we get $\frac{1}{2}$. This is the left child of the fraction $\frac{1}{1}$ and we will take the mediant of $\frac{1}{0}$ and $\frac{1}{1}$, we get $\frac{2}{1}$ which is the right child of $\frac{1}{1}$. Continuing this way we obtain a tree which is called Stern-Brocot Tree. On the other words, let's take a rational number $\frac{p}{q}$ in the tree such that $\frac{p}{q} = \frac{a+b}{c+d}$ where $\frac{a}{b}$ is the nearest left ancestor of $\frac{p}{q}$ and $\frac{c}{d}$ is the nearest right ancestor of $\frac{c}{d}$. We will use the notation $\mathcal{S}$ for the Stern Brocot tree. Furthermore $\mathcal{S}^n$ represents the set of vertices of $(n+1)^{th}$ line of the tree. For example $\mathcal{S}^0 = \{\frac{1}{0}, \frac{0}{1}\}, \mathcal{S}^1 = \{\frac{1}{1}\}, \mathcal{S}^2 = \{\frac{1}{2}, \frac{2}{1}\}$...You can see at the following figure which is described Stern-Brocot Tree.




**Lemma 4.1.1.** Let $\frac{a}{b} = \frac{p}{q} \oplus \frac{r}{s}$ be in the tree. Then the children of $\frac{a}{b}$ are $\frac{a}{b} \oplus \frac{p}{q}$ and $\frac{a}{b} \oplus \frac{r}{s}$.

*Proof.* Without lost of generality let $\frac{p}{q}$ be the nearest left ancestor of $\frac{a}{b}$ whereas $\frac{r}{s}$ is the right one. The right child of $\frac{a}{b}$ is the mediant of the its nearest left ancestor, $\frac{a}{b}$ and its nearest right ancestor is the same as $\frac{a}{b}'s$. So the right child of $\frac{a}{b}$ is $\frac{a}{b} \oplus \frac{r}{s}$. By the same thing the left child of $\frac{a}{b}$ is the mediant of the its nearest right ancestor, $\frac{a}{b}$ and its nearest left ancestor $\frac{p}{q}$.

**Proposition 4.1.1.** Let $\frac{a}{b} = \frac{p}{q} \oplus \frac{r}{s}$ then $|aq - bp| = |as - br| = 1$.

*Proof.* First, it is true for $\mathcal{S}^2 : |1.2 - 1.1| = |0.2 - 1.1| = 1$. Asumme that the hypothesis is true at $\mathcal{S}^i$ for all $i < n$. Let $\frac{a}{b} = \frac{p}{q} \oplus \frac{r}{s} \in \mathcal{S}^{n-1}$. Let $\frac{m}{n}$ be a child of $\frac{a}{b}$. Then it is equal to $\frac{a}{b} \oplus \frac{p}{q}$ or $\frac{a}{b} \oplus \frac{r}{s}$ by Lemma 4.1.1. Without lost of generality we assume that $\frac{m}{n} = \frac{a}{b} \oplus \frac{p}{q}$.

$$|(a+p)q - (b+q)p| = |aq - bp| = 1 \text{ by assumption.} \tag{4.2}$$

$$\text{And also } |(a+p)b - (b+q)a| = |aq - bp| = 1. \tag{4.3}$$

So the equation satisfies for $\mathcal{S}^n$.

*Corollary* 4.1.1. Let $\frac{p}{q}$ and $\frac{r}{s}$ be consecutive rational numbers in the tree such that $\frac{p}{q} < \frac{r}{s}$. Then

$$q.r - p.s = 1 \tag{4.4}$$

**Proposition 4.1.2.** All rational numbers in Stern-Brocot Tree is irreducible. On the other words, if $\frac{m}{n} \in \mathcal{S}$ then $(m,n) = 1$.

*Proof.* Let $\frac{p}{q}$ and $\frac{r}{s}$ be consecutive rational number in $\mathcal{S}$ such that $\frac{p}{q} < \frac{r}{s}$. Then $q.r - p.s = 1$ by Corollary 4.1.1. Hence $(p,q) = (r,s) = 1$.

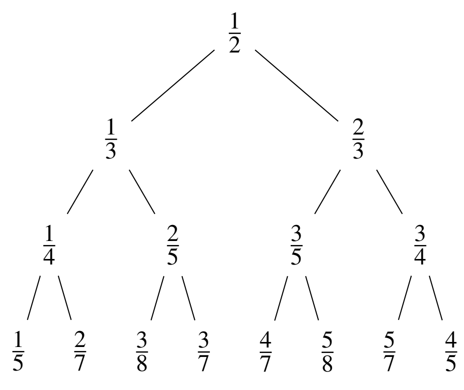
**Proposition 4.1.3.** Stern Brocot tree contains every positive rational numbers.

*Proof.* Let $\frac{a}{b} \in \mathbb{Q} \cap (0,1)$. Then we want to show that a vertex of the Stern Brocot tree is labeled with this rational number. $\frac{m}{n}, \frac{m_1}{n_1}$ be the rationals in $\mathcal{S}$ such that $\frac{m}{n} < \frac{a}{b} < \frac{m_1}{n_1}$. Now there are three cases:
First one, $\frac{m}{n} \oplus \frac{m_1}{n_1} = \frac{a}{b}$ so $\frac{a}{b} \in \mathcal{S}$, we win. Second, $\frac{m}{n} \oplus \frac{m_1}{n_1} < \frac{a}{b} < \frac{m_1}{n_1}$ and the third case is $\frac{m}{n} < \frac{a}{b} < \frac{m}{n} \oplus \frac{m_1}{n_1}$. Without lost of generality we assume that the second one satisfies. Then we write $\frac{m_2}{n_2} = \frac{m}{n} \oplus \frac{m_1}{n_1}$. Then we obtain a new inegality $\frac{m_2}{n_2} < \frac{a}{b} < \frac{m_1}{n_1}$. We will do the same things. Let $\frac{m_3}{n_3} = \frac{m_2}{n_2} \oplus \frac{m_1}{n_1}$. If $\frac{m_3}{n_3} = \frac{a}{b}$ then we win. If it is not true then we have two cases again: $\frac{m_3}{n_3} < \frac{a}{b}$ or $\frac{m_3}{n_3} > \frac{a}{b}$. Assume that the first one. Then we have $\frac{m_3}{n_3} < \frac{a}{b} < \frac{m_1}{n_1}$. And again we will contine by the same way. After finitely step we will obtain $\frac{m_k}{n_k} = \frac{a}{b}, thenwewin$. Hence all the rational numbers in $\mathbb{Q} \cap (0,1)$ is contained in Stern Brocot tree. But we can generalise this by takingthe rational in $\mathbb{Q} \cap (1,\infty)$. In this case the rational number is in first right branch of the Stern Brocot tree.

**Proposition 4.1.4.** [*10*] Stern Brocot tree contains a rational numbers at most once.

Some nice proprieties of Stern-Brocot tree can be found at [10] $pp. : 117 - 120$.

**Definition 4.1.2.** The Farey Tree is the first left branch of Stern Brocot Tree. So it contains all rational numbers in $(0,1) \cap \mathbb{Q}$. For Farey Tree we use the notation $\mathcal{F}$.




**Definition 4.1.3.** The top vertex of the tree is called the root. The root of the Stern-Brocot Tree is $\frac{1}{1}$ whereas the root of the Farey Tree is $\frac{1}{2}$.

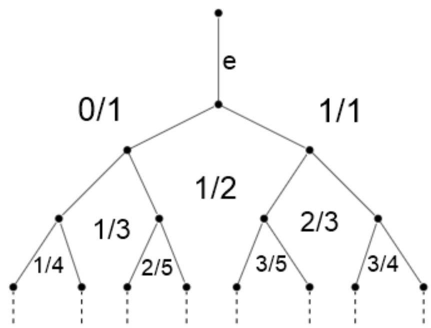



Figure 4.1: Continued Fraction expansion of The Paths of Farey Tree

If we have an end $\gamma$ the open ball $B(\gamma, 2^{-n})$ contains all ends which has at least $n+1$ common edges with $\gamma$. In other words, if $\gamma = (e, e_1, e_2, \cdots, e_n, \cdots)$ then $B(\gamma, 2^{-n})$ contains all ends which starts with the edges $e, e_1, \cdots, e_n$. Then we will use an other notation $O(e_n)$ which denotes the set of ends starts at $e$ and passes at $e_n$. For the Farey tree we will use the vertices instead of edges. We fix the root vertex $\frac{1}{2}$. So $O(\frac{a}{b})$ contains all ends which starts at $\frac{1}{2}$ and passes through $\frac{a}{b}$. Then $O(\frac{a}{b}) = [\frac{p}{q}, \frac{r}{s}]$ with $\frac{p}{q}, \frac{r}{s}$ are the ancestors of $\frac{a}{b}$. These intervals are called **Farey Interval**. We will denote them by $I(\frac{a}{b}) = [\frac{p}{q}, \frac{r}{s}]$.

## 4.2 The Monoid Structure on the Set of Vertices of The Tree

Now we will define a natural operation over the vertices of $\mathcal{F}$. For each vertex of $\mathcal{F}$ there is a unique path which starts at the root $\frac{1}{2}$ and ends at this vertex. The operation between the vertices can be defined as concatenating the paths which are corresponding to these vertices. To be more precise we will give this operation via defining the set $X$ and the $\star$ operation over this set and they will give us a monoid structure.
Now, we define the set $X := \{(n_1, n_2, n_3, \cdots, n_k | n_i \in \mathbb{N} \backslash \{0\}$ for any i$\}$.

**Proposition 4.2.1.** The following map is a bijection:

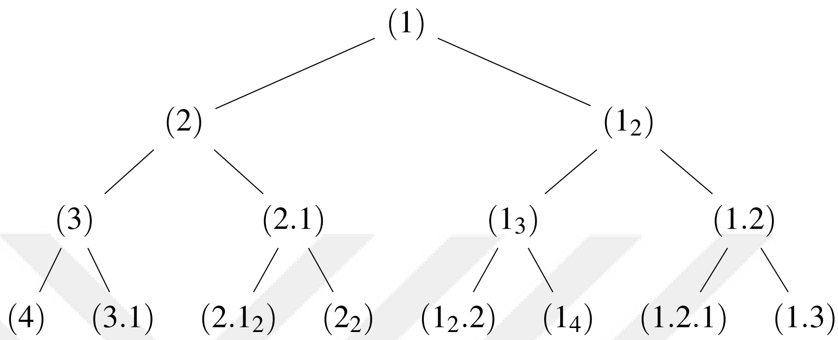
$$\theta \colon X \longrightarrow \mathbb{Q} \cap (0,1)$$

$$(n_1, n_2, \ldots, n_k) \mapsto [0, n_1, n_2, \ldots n_k, 1]$$

*Proof.* Let $\frac{p}{q} \in \mathbb{Q} \cap (0,1)$ then it can be write such a continued fraction. Let $[0, x_1, x_2, \cdots x_k]$ be the continued fraction representation of $\frac{p}{q}$. Since $[0, x_1, x_2, \cdots x_k] = [0, x_1, x_2, \cdots x_k - 1, 1]$, $\theta((x_1, x_2, \cdots, x_k - 1)) = a$.Let $\theta(x) = \theta((y)) = [0, n_1, n_2 \cdots, n_l, 1]$ then $x = y = (n_1, n_2 \cdots, n_l)$. Hence $\theta$ is a bijection.

**Example 4.2.1.** $\frac{1}{2} = [0,2] = [0,1,1] = \theta((1))$, $\frac{1}{3} = [0,3] = [0,2,1] = \theta((2))$, $\frac{2}{3} = [0,1,2] = [0,1,1,1] = \theta((1,1))$ are some examples of the values of $\theta$.

Since $\theta$ is a bijection map between $X$ and $\mathbb{Q} \cap (0,1)$, we can write all the rational numbers in $\mathbb{Q} \cap (0,1)$ by using the elements of the set $X$. So all vertices of the Farey Tree can be represented by a tuple in $X$.




**Definition 4.2.1.** Let $x = (n_1, n_2, \cdots, n_k) \in X$. *The depth of x is* $||x|| := n_1 + n_2 + \cdots + n_k - 1$ and the length of $x$ is $\ell(x) := k$.

**Example 4.2.2.** Let $x = \theta^{-1}(\frac{1}{2})$ then $x = (1)$, $||x|| = 0$ and $\ell(x) = 1$. For $x = \theta^{-1}(\frac{2}{3})$, $x = (1,1)$, $||x|| = 1+1-1 = 1$ and $\ell(x) = 2$.

*Remark.* [18] $x \in X$ is a right child if $\ell(x)$ is even, $x$ is a left child otherwise.

We define an operation over $X$ which is called star and described with $\star$ symbol, let $x = (n_1, n_2, \cdots, n_k), y = (m_1, m_2, \cdots, m_l) \in X$,

$$x \star y := \begin{cases} (n_1, n_2, \cdots, n_k, m_1 - 1, m_2, \cdots, m_l) & \text{if } x \text{ is a right child} \\ (n_1, n_2, \cdots, n_k + m_1 - 1, m_2, \cdots, m_l) & \text{if } x \text{ is a left child} \end{cases}$$

We assume that $(\cdots, m, 0, k, \cdots) = (\cdots, m+k, \cdots)$ and $(\cdots, m, 0) = (\cdots, m)$.

**Example 4.2.3.** $(1,1) \star (1,1) = (1,1,1-1,1) = (1,2)$.
$(2) \star (2,1) = (2+2-1,1) = (3,1)$

**Proposition 4.2.2.** The set $X$ is a monoid under this $\star$ operation.

*Proof.* First, $\star$ is associative.
Let $x = (n_1, n_2, \cdots, n_s)$, $y = (m_1, m_2, \cdots, m_l)$, $z = (k_1, k_2, \cdots k_r) \in X$. By definition of $\star$ operation

$$x \star y := \begin{cases} (n_1, n_2, \cdots, n_s, m_1 - 1, m_2, \cdots, m_l) & \text{if } x \text{ is a right child} \\ (n_1, n_2, \cdots, n_s + m_1 - 1, m_2, \cdots, m_l) & \text{if } x \text{ is a left child} \end{cases}$$

$$(x\star y)\star z := \begin{cases} (n_1,\cdots,n_s,m_1-1,m_2,\cdots,m_l,k_1-1,k_2\cdots,k_r) & \text{x and y right} \\ (n_1,\cdots,n_s,m_1-1,m_2,\cdots,m_l+k_1-1,k_2\cdots k_r) & \text{x right, y left} \\ (n_1,\cdots,n_s+m_1-1,m_2,\cdots,m_l,k_1-1,k_2\cdots,k_r) & \text{x left, y right} \\ (n_1,\cdots,n_k+m_1-1,m_2,\cdots,m_l+k_1-1,k_2\cdots k_r) & \text{x and y left} \end{cases}$$

Then

$$(x\star y)\star z := \begin{cases} (n_1,\cdots,n_s)\star(m_1,m_2,\cdots,m_l,k_1-1,k_2\cdots,k_r) & \text{y right} \\ (n_1,\cdots,n_s)\star(m_1,m_2,\cdots,m_l+k_1-1,k_2\cdots k_r) & \text{y left} \end{cases}$$

And we know

$$y\star z := \begin{cases} (m_1,m_2,\cdots,m_l,k_1-1,k_2\cdots,k_r) & \text{if y is a right child} \\ (m_1,m_2,\cdots,m_l+k_1-1,k_2\cdots k_r) & \text{if y is a left child} \end{cases}$$

So, $(x\star y)\star z = x\star(y\star z)$ that is, associativity satisfies.

$$x\star(1) := \begin{cases} (n_1,n_2,\cdots,n_s,0) = (n_1,n_2,\cdots,n_s) & \text{if } x \text{ is a right child} \\ (n_1,n_2,\cdots,n_s+1-1) = (n_1,n_2,\cdots,n_s) & \text{if } x \text{ is a left child} \end{cases}$$

$(1)\star x = (1+m_1-1,m_2,\cdots,m_l) = x$. So, $(1)$ is the neutral element of $X$. Besides, it corresponds to the root of the Stern-Brocot Tree, $\frac{1}{2}$. Then $(X,\star)$ is a monoid.

**Proposition 4.2.3.** We have the following equalities

$$\underbrace{(1,1)\star(1,1)\star\cdots\star(1,1)}_{n\ times} = (1,n)$$

$$\underbrace{(2)\star(2)\star\cdots\star(2)}_{n\ times} = (n+1)$$

*Proof.* We know that $(1,1)\star(1,1) = (1,2)$. Assume that $\underbrace{(1,1)\star(1,1)\star\cdots\star(1,1)}_{n-1\text{ times}} = (1,n-1)$. $(1,n-1)\star(1,1) = (1,n-1,0,1) = (1,n)$, by the induction hypothesis. $(2)\star(2) = (3)$.Assume that $\underbrace{(2)\star(2)\star\cdots\star(2)}_{n-1\text{ times}} = (n)$.

$(n)\star(2) = (n+1)$, by the induction hypothesis.

**Proposition 4.2.4.** Let $L := (2)$ and $R := (1,1)$. We can generate all elements of $X$ by $L$ and $R$ such that $(n_1,n_2,n_3,\ldots) = L^{n_1-1}\star R^{n_2}\star L^{n_3}\star R^{n_4}\star\cdots$.

*Proof.* Let $\ell(x)=1$ such that $x=(n_1)$ then $x=L^{n_1-1}$. If $\ell(x)=2$ such that $x=(n_1,n_2)$ then $x=(n_1)\star(1,n_2)=L^{n_1-1}\star R^{n_2}$. Assume that the hypothesis is true for $\ell(x)=2k-1$ such that $x=(n_1,n_2,\dots n_{2k-1})\in X$. Then we can write $x=L^{n_1-1}R^{n_2}\dots L^{n_{2k-1}}$ (if $\ell(x)$ had be an even number, $x$ would end with $R^{n_{2k-1}}$). Let $\ell(x)=2k$ such that

$$\begin{aligned} x &= (n_1)\star(1,n_2,n_3,\dots n_{2k}) \\ &= L^{n_1-1}\star(1,n_2)\star(n_3+1,n_4,\dots n_{2k}) \\ &= L^{n_1-1}\star R^{n_2}\star L^{n_3}\star\cdots\star R^{n_{2k}} \text{ by assumption.} \end{aligned} \tag{4.5}$$

Moreover if $\ell(x)=2k+1$ then,

$$\begin{aligned} x &= (n_1)\star(1,n_2,n_3,\dots n_{2k+1}) \\ &= L^{n_1-1}\star(1,n_2)\star(n_3+1,n_4,\dots n_{2k+1}) \\ &= L^{n_1-1}\star R^{n_2}\star L^{n_3}\star\cdots\star L^{n_{2k+1}} \text{ by assumption.} \end{aligned} \tag{4.6}$$

*Remark.* By using the bijection map $\theta$ we can transform $\star$ operation to an operation on $\mathbb{Q}\cap(0,1)$. And then we obtain $\underbrace{\frac{2}{3}\star\frac{2}{3}\star\cdots\star\frac{2}{3}}_{n\text{ times}}=\frac{n+1}{n+2}$ since $\theta((1,1))=\frac{2}{3}$ and $\theta((1,n))=[0,1,n,1]=\frac{n+1}{n+2}$. And also by this transformation we obtain $\underbrace{\frac{1}{3}\star\frac{1}{3}\star\cdots\star\frac{1}{3}}_{n\text{ times}}=\frac{1}{n+2}$. By conclusion, we can say that the Stern-Brocot three can be generate by $\frac{2}{3}$ and $\frac{1}{3}$.

Actually, this result is obvious seeing as $L$ means to go left wheras $R$ means to go right on the Farey Tree. So, we can walk on the whole tree via $L$ and $R$. Meanwhile, we can generate all elements of the $\mathbb{Q}\cap X$ by $L$ and $R$.

## 4.3 Some Special Automorphisms of The Farey Tree

### 4.3.1 The Automorphism K

We define the map:

$$\begin{aligned} K\colon X &\longrightarrow X \\ L &\longmapsto R \\ R &\longmapsto L \end{aligned}$$

In fact, the map $K$ finds the symmetry of the element on the tree according to the perpendicular line which passes the middle of the tree. To illustrate, the symmetry of $\frac{2}{5}$ is

$\frac{3}{5}$, the symmetry of $\frac{1}{3}$ is $\frac{2}{3}$ etc. So $K := \sigma_{\frac{1}{2}}$ which is the twist automorphism by the root vertex $\frac{1}{2}$. So $K(x) = 1 - x$ for any $x \in (0,1)$. We can show this result on the set $X$.

**Proposition 4.3.1.** *$K$ is an automorphism of $X$.*

*Proof.* Let $x = (n_1, n_2, \ldots, n_s)$ and $y = (m_1, m_2, \ldots, m_l)$ be two elements of $X$, without loss of generality $l(y)$ is an even number. Then

$$K(x \star y) = \begin{cases} K(n_1, n_2, \cdots, n_s, m_1 - 1, \ldots, m_l) = & \\ K(L^{n_1-1} \star R^{n_2} \star \cdots \star R^{n_s} \star L^{m_1-1} \star R^{m_2} \star \cdots \star R^{m_l}) = & \\ R^{n_1-1} \star L^{n_2} \star \cdots \star L^{n_s} \star R^{m_1-1} \star L^{m_2} \star \cdots \star L^{m_l} = & \\ K(n_1, n_2, \cdots, n_s) \star K(m_1, m_2, \ldots, m_l) = & \\ K(x) \star K(y) & \text{if } x \text{ is right} \\ K(n_1, n_2, \cdots, n_s + m_1 - 1, \ldots, m_l) = & \\ K(L^{n_1-1} \star R^{n_2} \star \cdots \star L^{n_s+m_1-1} \star R^{m_2} \star \cdots \star R^{m_l}) = & \\ R^{n_1-1} \star L^{n_2} \star \cdots \star R^{n_s} \star R^{m_1-1} \star L^{m_2} \star \cdots \star L^{m_l} = & \\ K(n_1, n_2, \cdots, n_s) \star K(m_1, m_2, \ldots, m_l) = & \\ K(x) \star K(y) & \text{if } x \text{ is left} \end{cases}$$

Hence, K is a homomorphism.

*Remark.* $K$ can be defined such a homomorphism of $\mathbb{Q} \cap (0,1)$ via the bijection $\theta$.

*Proof.* Let $x = L^{n_1-1} \star R^{n_2} \star L^{n_3} \star \cdots$. Then

$$\begin{aligned} Kx &= R^{n_1-1} \star L^{n_2} \star R^{n_3} \star \cdots \text{ since } K \text{ is a homomorphism} \\ &= (1, n_1 - 1) \star (n_2 + 1) \star (1, n_3) \star \cdots \\ &= (1, n_1 - 1, n_2, n_3, \ldots, n_k). \end{aligned} \tag{4.7}$$

**Lemma 4.3.1.** *Let $x = (n_1, n_2, \ldots, n_k) \in X$ then $Kx = 1 - x$.*

*Proof.* We showed that we can write $x$ via $L$ and $R$ such that $x = L^{n_1-1} \star R^{n_2} \star L^{n_3} \star \cdots$. Then

$$\begin{aligned} Kx &= R^{n_1-1} \star L^{n_2} \star R^{n_3} \star \cdots \text{ since } K \text{ is a homomorphism} \\ &= (1, n_1 - 1) \star (n_2 + 1) \star (1, n_3) \star \cdots \\ &= (1, n_1 - 1, n_2, n_3, \ldots, n_k). \end{aligned} \tag{4.8}$$

$$\begin{aligned}\theta(Kx) &= \theta(1, n_1-1, n_2, n_3, \ldots, n_k)\\ &= [0, 1, n_1-1, n_2, n_3, \ldots, n_k, 1]\\ &= \cfrac{1}{1+\cfrac{1}{\frac{1}{\theta(x)}-1}}\\ &= 1-\theta(x)\end{aligned}$$

If $K(x) = K(y)$ for any $x, y \in X$ then $x = y$ and also for any $x \in X$ we can write $K(1-x) = x$. So, $K$ is an automorphism. Via the bijection map $\theta$, $K$ determines an automorphism of the Farey Tree.

### 4.3.2 The Flip

The Flip map $\phi$ is defined on the set $X$ as follows:

$$\phi \colon X \longrightarrow X$$

$$(n_1, n_2, \ldots, n_k) \longmapsto (n_k, n_{k-1}, \ldots, n_1)$$

It is clear that $\phi(\phi(x)) = x$. Such maps are called involution. Via $\theta$, $\phi$ can be defined on the set $\mathbb{Q} \cap (0,1)$ as follows:

$$\phi([0, n_1, n_2, \ldots, n_k]) = [0, n_k - 1, n_{k-1}, \ldots n_2, n_1 + 1]$$

### 4.3.3 The Jimm Function

We define another map on $X$ which is called Jimm:

$$\mathfrak{J} \colon X \longrightarrow X$$
$$(n_1, n_2, \ldots, n_k) \longmapsto (1_{n_1-1}, 2, 1_{n_2-2}, 2, 1_{n_3-2}, 2, \ldots, 1_{n_k-1})$$

And we assume that $k > 1$ and we will eliminate $1_0$'s as $[\ldots, a, 1_0, b, \ldots] = [\ldots, a, b, \ldots]$ and $[1_0, a, \ldots] = [a, \ldots]$. If we have $1_{-1}$ then we will suppose $[\ldots, a, 1_{-1}, b, \ldots] = [\ldots, a+b-1, \ldots]$. Furthermore, by convention $\mathfrak{J}((n_1)) = (1_{n_1})$.

*Remark.* Jimm is an involution.i.e. $\mathfrak{J}(\mathfrak{J}(x)) = x$

It is coming with the simple calculation and the rules $1_0$ and $1_{-1}$.

**Proposition 4.3.2.** Jimm is an automorphism of the monoid $X$.

*Proof.* Let $x=(n_1,n_2,\ldots,n_s), y=(m_1,m_2,\ldots,m_l)\in X$.

$$\mathfrak{J}(x\star y)=\begin{cases}\mathfrak{J}(n_1,n_2,\cdots,n_s,m_1-1,\ldots,m_l)= \\ (1_{n_1-1},2,1_{n_2-2},2,\ldots,1_{n_s-2},2,1_{m_1-3},2,1_{m_2-2}\ldots,1_{m_l-1})= \\ (1_{n_1-1},2,1_{n_2-2},2,\ldots,1_{n_s-1})\star(1_{m_1-1},2,1_{m_2-2}\ldots,1_{m_l-1})= \\ \mathfrak{J}(x)\star\mathfrak{J}(y) & x \text{ is right} \\ \mathfrak{J}(n_1,n_2,\cdots,n_s+m_1-1,\ldots,m_l)= \\ (1_{n_1-1},2,1_{n_2-2},2,\ldots,1_{n_s+m_1-3},2,1_{m_2-2},\ldots,1_{m_l-1})= \\ (1_{n_1-1},2,1_{n_2-2},2,\ldots,1_{n_s-1})\star(1_{m_1-1},2,1_{m_2-2}\ldots,1_{m_l-1})= \\ \mathfrak{J}(x)\star\mathfrak{J}(y) & x \text{ is left}\end{cases}$$

Then Jimm is a homomorphism of $X$. Since Jimm is an involution for any $x\in X$ $\mathfrak{J}(\mathfrak{J}(\mathfrak{x}))=x$ so it is a surjective map. And also if $\mathfrak{J}(x)=\mathfrak{J}(y)$ then $\mathfrak{J}(\mathfrak{J}(x))=\mathfrak{J}(\mathfrak{J}(y))$ so $x=y$, then it is a bijection. Hence $\mathfrak{J}$ is an automorphism of $X$.

*Remark.* By $\theta$ we can transfer Jimm funtion to $\mathbb{Q}\cap(0,1)$ such that $\mathfrak{J}([0,n_1,n_2,\ldots,n_k])=[0,1_{n_1-1},2,1_{n_2-2},2,\ldots,1_{n_k-1}]$.

Let $x=[0,n_1,n_2,\ldots,n_k]$ then $\frac{x}{1+x}=\dfrac{1}{1+\frac{1}{x}}=\dfrac{1}{1+n_1+\dfrac{1}{n_2+\ldots}}=[0,n_1+1,n_2,\ldots,n_k]$.

Then

$$\begin{aligned}\mathfrak{J}(\tfrac{x}{x+1}) &= [0,1_{n_1},2,1_{n_2-2},2,\ldots,1_{n_k-1}] \\ &= [0,1,1_{n_1-1},2,1_{n_2-2},2,\ldots,1_{n_k-1}] \\ &= \tfrac{1}{1+\mathfrak{J}(x)}\end{aligned} \tag{4.9}$$

$\frac{1}{1+x}=[0,1,n_1,n_2,\ldots,n_k]$ for same $x$. Then we observe that

$$\begin{aligned}\mathfrak{J}(\tfrac{1}{x+1}) &= [0,1_0,2,1_{n_1-2},2,\ldots,1_{n_k-1}] \\ &= [0,2,1_{n_1-2},2,\ldots,1_{n_k-1}] \\ &= [0,1+1,1_{n_1-2},2,\ldots,1_{n_k-1}] \\ &= \dfrac{1}{1+\frac{1}{\mathfrak{J}(x)}} \\ &= \tfrac{\mathfrak{J}(x)}{\mathfrak{J}(x)+1}\end{aligned} \tag{4.10}$$

**Proposition 4.3.3.** The involution $\mathfrak{J}$ commutes with $K$.

*Proof.* Let $x = (n_1, n_2, \dots, n_k)$. We need to show that $\mathfrak{J}(Kx) = K\mathfrak{J}(x)$ for any $x$.

$$\begin{aligned} K(\mathfrak{J}(x)) &= K(1_{n_1-1}, 2, 1_{n_2-2}, 2, \cdots, 1_{n_k-1}) \\ &= (1, 0, 1_{n_1-2}, 2, 1_{n_2-2}, 2, \cdots, 1_{n_k-1}) \\ &= (2, 1_{n_1-3}, 2, 1_{n_2-2}, 2, \cdots, 1_{n_k-1}) \\ &= (1_0, 2, 1_{n_1-3}, 2, 1_{n_2-2}, 2, \cdots, 1_{n_k-1}) \\ &= \mathfrak{J}(1, n_1 - 1, n_2, n_3, \cdots, n_k) \\ &= \mathfrak{J}(K(x)) \end{aligned} \tag{4.11}$$

**Proposition 4.3.4.** The involution Jimm commutes with $\phi$.

*Proof.* Let $x = (n_1, n_2, \cdots, n_k) \in X$.

$$\begin{aligned} \mathfrak{J}(\phi(n_1, n_2, \cdots, n_k)) = \mathfrak{J}(n_k, n_{k-1}, \cdots, n_2, n_1) &= (1_{n_k-1}, 2, 1_{n_{k-1}-2}, 2, \cdots, 1_{n_1-1}) \\ &= \phi(1_{n_1-1}, 2, 1_{n_2-2}, 2, 1_{n_3-2}, \cdots, 1_{n_k-1}) = \phi\mathfrak{J}(x). \end{aligned}$$

## 4.4 Measures on The Boundary of The Farey Tree

In the first section we look at the topology on the boundary of a tree. Now we will work on the special tree which is Farey Tree. The topology on te boundary of a tree is generated by the open intervals $B(\gamma, 2^{-n})$ where $\gamma$ is an end which starts at the fixed edge (or vertex) and it is infinite and non-back tracking path. $n \in \mathbb{N}$ and the open ball is defined as $B(\gamma, 2^{-n}) = \{\gamma' | d(\gamma, \gamma') < n\}$. That is, it contains all ends which starts at the fixed edge (or vertex) and has at least $n+1$ common edges with $\gamma$. On the other words, let $e'$ be the $n^{th}$ edge of $\gamma$. Then $B(\gamma, 2^{-n})$ contains all ends which starts at fixed edge or vertex and passes the edge $e'$. And we label this set by $O_{e'}$.

Now on the Farey tree we don't choose a fixed edge we choose a fixed vertex which is root vertex $\frac{1}{2}$. So we suppose $\partial\mathcal{F}$ contains all ends which starts at $\frac{1}{2}$. Let I $\left[\frac{p}{q}, \frac{r}{s}\right]$ be a Farey interval. Actually the topology on the boundary of the Farey tree can be generated by $O_{\frac{a}{b}}$ with $\frac{a}{b} := \frac{p}{q} \oplus \frac{r}{s}$. Since there is a topology on the boundary of the Farey tree, there is a Borel algebra on $\partial\mathcal{F}_e$ which can be generated by opens of the topology on $\partial\mathcal{F}_e$ basicly. So we can put a probabilty measure on the Borel algebra.

**Definition 4.4.1.** Let $p, q \in \mathcal{F}$ such that they are siblings. Then we define a map $\pi : \mathcal{F} \mapsto \mathbb{Q} \cap (0,1)$ such that $\pi(p) + \pi(q) = 1$. All functions which provides this property is called **transition function**.

Actually, the map $\pi$ gives the probability of arriving to a choosen child from its parent.

Let's imagine that there is a man over the root vertex $\frac{1}{2}$ who will walk randomly non-backtracking on the tree. We want to calculate the probabilty of this man being at the interval $I[\frac{p}{q},\frac{r}{s}]$ at the end of the walk. Then we want to calculate the probabilty of the man reaching to $\frac{a}{b}=\frac{p}{q}\oplus\frac{r}{s}$. So far we know just the probability of coming $\frac{a}{b}$ from the parent of $\frac{a}{b}$ and it is $\pi(\frac{a}{b})$. We can find the probabilty of coming the parent of $\frac{a}{b}$ from the parent of the parent of the $\frac{a}{b}$. And we we will continue with this method until we reach the root and finally we will multiply the probability of all.
Let's define the map:

$$T_{\phi F}: X \longrightarrow X$$
$$(n_1,n_2,\ldots,n_{k-1},n_k) \longmapsto (n_1,n_2,\ldots,n_{k-1},n_k-1)$$

If $n_k-1=0$ we will ignore. The map gives the parent of the $(n_1,n_2,\ldots,n_{k-1},n_k)$. Since we know that an element of $X$ can be generated by $(1,1)$ and $(2)$. If we have a vertex $(n_1,n_2,\ldots,n_{k-1},n_k)$ the children of this vertex are $(n_1,n_2,\ldots,n_{k-1},n_k,1)$ and $(n_1,n_2,\ldots,n_{k-1},n_k+1)$. Then we give the probability measure on $\partial\mathcal{F}$ as follows

$$\mu_\pi(I(n_1,\ldots,n_k)) = \prod_{i=0}^{d-1}\pi\left(T^i_{\varphi F}(n_1,n_2,\ldots,n_k)\right) \tag{4.12}$$

where $d$ is the depth of the vertex i.e. it is just the number of edges between the vertex $(n_1,n_2,\cdots,n_k)$ and the root vertex $\frac{1}{2}$. see [18], $pg:9$.
So the cumulative distribution function is given as follows:

$$\begin{aligned}\mathbf{F}_\pi(x) := \mu_\pi([0,x]) &= \textstyle\sum_{k=1}^{\infty}(-1)^k\mu_\pi\{W\in I(n_1,n_2,\ldots,n_k)\}\\ &= \textstyle\sum_{k=1}^{\infty}(-1)^{1+k}\prod_{i=0}^{d-1}\pi\left(T^i_{\varphi F}(n_1,n_2,\ldots,n_k)\right)\end{aligned}$$

see [18]. There are special probability measures which is defined on the boundary of the Farey Tree. For example; **Minkowski Measure** is a probability measure on the boundary of the Farey tree which is defined as the previous it is just the special case of the measure $\mu_\pi$. It takes all transition function is equal except the root vertex. So it takes $\pi(x)=\frac{1}{2}$ for any $x\in X$. Then

$$\mu_\pi\{W\in I(n_1,\ldots,n_k)\} = 2^{1-n_1-n_2\cdots-n_k}.$$

where $W$ is a random walker.

**Proposition 4.4.1.** Its cumulative distribution function of Minkowski measure is

$$\mathbf{F}_\pi(x) = \sum_{k=1}^{\infty}(-1)^{1+k}2^{1-n_1-n_2\cdots n_k}.$$

It is called **Minkowski ? function**.

*Proof.* see [22] pg: 5-6.

The more general probability measure on the boundary of $\mathcal{F}$ is **Denjoy's measure**. Let $p$ and $q$ be two siblings such that $p$ is the right and $q$ is the left one. If we take $\pi(p) = a$ and $\pi(q) = 1 - a$ for any two siblings we obtain **Denjoy's measure**.
The third measure on the bondary of Farey Tree is Lebesgue's Measure. We denote the Lebesgue's Measure is a measure on the real numbers such that the measure of an interval is the length of this interval. Now we have Farey intervals so we can talk about the Lebesgue's measure on $\partial\mathcal{F}$. Let $\lambda$ be the Lebesgue's measure. Let $\pi_\lambda$ be the $\pi$ function corresponding to this measure. Then

$$\mu_\pi(I(x)) := \lambda(I(x)) = \prod_{i=0}^{||x||} \pi_\lambda\left(T^i_{\varphi F}(x)\right)$$

gives us the Lebesgue's measure where $x = (n_1, n_2, \cdots, n_k) \in X$ . Let $y$ be the parent of $x$ so $y = (n_1, n_2, \cdots, n_k - 1)$. And we know $\pi_\lambda(x)\mu_\pi(I(y)) = \mu_\pi(I(x))$. So

$$\pi_\lambda(x) = \frac{\mu_\pi(I(x))}{\mu_\pi(I(y))} \tag{4.13}$$

The end points of the interval $I(x)$ are $[0, n_1, n_2, \cdots, n_k]$ and $[0, n_1, n_2, \cdots, n_{k-1}]$ as a continued fraction expansion. We obtain it via the bijection $\theta$. Moreover, The end points of the interval $I(y)$ are given by $[0, n_1, n_2, \cdots, n_k - 1]$ and $[0, n_1, n_2, \cdots, n_{k-1}]$ as a continued fraction expansion. Let $\frac{p_k}{q_k} := [0, n_1, n_2, \cdots, n_k] = y$ be the canonical representation of the continued fraction. Then $\frac{p_{k-1}}{q_{k-1}} := [0, n_1, n_2, \cdots, n_{k-1}]$ is the $(k-1)^{th}$ convergent of $y$. Then by the corollary 7.0.1 we have the following equality:

$$\begin{aligned} \lambda(I(x)) &= \left| \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} \right| \\ &= \left| \frac{1}{q_k q_{k-1}} \right| \end{aligned} \tag{4.14}$$

Since the end points of $I(y)$ are $[0, n_1, n_2, \cdots, n_{k-1} n_k - 1]$ and $[0, n_1, n_2, \cdots, n_{k-1}]$, assume that $n'_k := n_k - 1$ so the $k^{th}$ term of the continued fraction $[0, n_1, n_2, \cdots, n_{k-1} n_k - 1]$.

So

$$\lambda(I(y)) = \left|\frac{p_{k'}}{q_{k'}} - \frac{p_{k-1}}{q_{k-1}}\right| = \left|\frac{1}{q_{k'}q_{k-1}}\right| \tag{4.15}$$

by corollary 7.0.1.

We denote $q_k := \langle n_1, \ldots, n_{k-1}, n_k \rangle$ for any $k \in \mathbb{N}$. So

$\lambda(I(x)) = \dfrac{1}{\langle n_1, \ldots, n_{k-1} \rangle \langle n_1, \ldots, n_{k-1}, n_k \rangle}$, $\lambda(I(y)) = \dfrac{1}{\langle n_1, \ldots, n_{k-1} \rangle \langle n_1, \ldots, n_{k-1}, n_k - 1 \rangle}$

Then we have

$$\begin{aligned} \pi_\lambda(x) &= \frac{\lambda(I(x))}{\lambda(I(y))} \\ &= \frac{\langle n_1, \ldots, n_{k-1}, n_k - 1 \rangle}{\langle n_1, \ldots, n_k \rangle} \\ &= \frac{\langle n_1, \ldots, n_k \rangle - \langle n_1, \ldots, n_{k-1} \rangle}{\langle n_1, \ldots, n_k \rangle} \\ &= 1 - \frac{\langle n_1, \ldots, n_{k-1} \rangle}{\langle n_1, \ldots, n_{k-1}, n_k \rangle} = 1 - [0, n_k, n_{k-1}, \ldots, n_1] \\ &= K\varphi\mathbb{T}_F(x). \end{aligned} \tag{4.16}$$

where $\mathbb{T}_F$ is the Farey map which is given by:

$$T_F : (n_1, n_2, \ldots, n_{k-1}, n_k) \to (n_1 - 1, n_2, \ldots, n_{k-1}, n_k).$$

If we return the set $X$,

$$\pi_\lambda (n_1, \ldots, n_k) = (1, n_k - 1, n_{k-1}, \ldots, n_1 - 1)$$

So we can say that

$$\pi_\lambda (n_1, \ldots, n_k) = \pi_\lambda (1, n_1 - 1, \ldots, n_k)$$

Then this equality gives us

$$\pi_\lambda(x) = \pi_\lambda(Kx).$$

That is, Lebesgue's measure is invariant under the automorphism $K$.

So the symmetry of the Lebesgue's measure is given by :

$$\pi_\lambda \mathfrak{J}(x) = \mathfrak{J}\pi_\lambda(x).$$

Since $\mathfrak{J}$ commutes with $K$, $\phi$ and $\mathbb{T}_F$ which was proved by the propositions 4.3.3 and 4.3.4. So we obtain the symmetry of Lebesgue's measure.

# 5 CONTINUED FRACTION MAPS

## 5.1 Continued Fraction Expansion of Real Numbers

An expression of the form

$$\alpha = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\vdots}}}$$

is called simple continued fraction expansion where $n_i \in \mathbb{N} - \{0\}$ for any $i$. We denote this continued fraction by $\alpha = [0, n_1, n_2, \ldots]$. Every real numbers can be presented by a continued fraction. If a continued fraction is finite it represents a unique rational number we showed it in the Proposition 7.0.1. Otherwise, if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then its continued fraction expansion must be infinite.

**Proposition 5.1.1.** Let $\alpha \in \mathbb{Q} \cap [0,1]$ then $\alpha$ has two different continued fraction representations such that if $\alpha = [0, n_1, n_2, \cdots, n_{k-1}, n_k, \infty]$ where $n_i \in \mathbb{Z}$ and $n_k > 1$ then $\alpha = [0, n_1, n_2, \cdots n_k, \infty] = [0, n_1, n_2, \cdots, n_{k-1}, n_k - 1, 1, \infty]$.

*Proof.* The first $k-1$ terms of two continued fractions are equals. So we will just show the equality of last terms.

$$\alpha = \cfrac{1}{n_1 + \cfrac{1}{\ddots\, n_{k-1} + \cfrac{1}{n_k + \cfrac{1}{\infty}}}} = \cfrac{1}{n_1 + \cfrac{1}{\ddots\, n_{k-1} + \cfrac{1}{n_k - 1 + \cfrac{1}{1 + \cfrac{1}{\infty}}}}}$$

We deonte these representations by $\alpha^+ := [0, n_1, n_2, \cdots, n_{k-1}, n_k - 1, 1, \infty]$ and $\alpha^- = [0, n_1, n_2, \cdots, n_{k-1}, n_k, \infty]$.

**Example 5.1.1.** The continued fraction representations of some numbers:
$\sqrt{2} = [1,2,2,2,\cdots] = [1,\overline{2}]$, $\frac{1}{2}^{+} = [0,1,1,\infty]$, $\frac{1}{2}^{-} = [0,2,\infty]$, The golden ratio; $\frac{1+\sqrt{5}}{2} = [1,1,1,\cdots] = [\overline{1}]$.

More information about continued fractions is available in Appendix.

We said that the edges of Farey tree can be presented by the elements of the free group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, or the vertices can be presented by the rational numbers. Now we will presented the paths of Farey tree which of their initial vertices are the same by continued fractions. Actually it is compatible with the presentation of vertices by rational numbers.

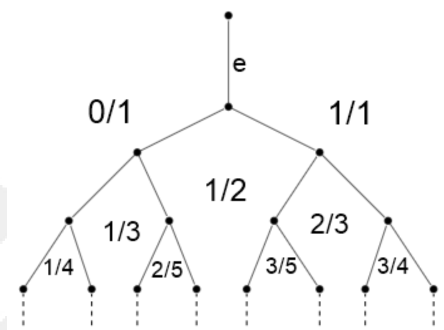



Figure 5.1: Continued Fraction expansion of The Paths of Farey Tree

For example, in the Figure 5.1 the path which starts the edge $e$ after turns always left is labeled with the rational numbers $\frac{0}{1}$ and otherwise the path wihch starts the edge $e$ and turns always right is labeled with the rational number $\frac{1}{1}$. It is very natural way to represent the path. Since the continued fraction expansion of $\frac{0}{1}$ is $[0,\infty] = [0,1+1+1+\cdots]$. Turning the same direction means adding 1 to the last number of the continued fraction. Otherwise, turning another direction means write 1 the end of the continued fraction expansion. The path which starts $e$ after always turns left presented by $[0,1,1+1+1+\cdots] = [0,1,\infty] = \frac{1}{1}$.

## 5.2 Dynamics of Continued Fraction Maps

**Definition 5.2.1.** A family of continued fraction maps $\mathbb{T}_\alpha$ is given by;

$$\mathbb{T}_\alpha(x) = \begin{cases} [0,m_{k+1},m_{k+2},m_{k+3},\cdots] & \text{if } n_k > m_k \\ [0,m_k - n_k,m_{k+1},m_{k+2},\cdots] & \text{if } n_k < m_k \end{cases}$$

where $\alpha = [0,n_1,n_2,n_3\cdots] \in [0,1]$ and $x = [0,m_1,m_2,m_3,\cdots] \in [0,1]$ and $k$ is the least positive integer such that $m_k \neq n_k$.

That is, this map cuts the common edges with the path $\alpha$ of the path $x$.

**Example 5.2.1.** Let $\alpha = [0,1,1,1,\cdots]$ and $x_1 \in [0,1,1,2,3,4,\cdots]$
$x_2 = [0,1,4,4,5,6,\cdots]$. Then $\mathbb{T}_\alpha(x_1) = [0,1,3,4,5,\cdots]$ and $\mathbb{T}_\alpha(x_2) = [0,3,4,5,\cdots]$.

Actually, we must define $\mathbb{T}_\alpha$ by hand for some special values:
The map $\mathbb{T}_\alpha$ is not continuous in the variable $\alpha$ for any $\alpha \in [0,1]$ i.e. $\mathbb{T}_{\alpha^+}(x)$ and $\mathbb{T}_{\alpha^-}(x)$ are two distinct maps with distinct dynamical proprieties. Let
$\alpha^+ := [0,n_1,n_2,\cdots,n_{k-1},n_k-1,1,\infty]$ and $\alpha^- = [0,n_1,n_2,\cdots,n_{k-1},n_k,\infty]$. Then
$\mathbb{T}_{\alpha^+}(x) \neq \mathbb{T}_{\alpha^-}(x)$ for $x = [0,n_1,n_2,\cdots,n_{k-1},n_k,m_1,m_2,\cdots,\infty]$.
Since $\mathbb{T}_{\alpha^+}(x) \neq \mathbb{T}_{\alpha^-}(x)$ for some $x \in [0,1]$ we always take $\alpha = \alpha^+$. i.e. if
$\alpha = [0,n_1,\cdots,n_k,\infty]$ then $n_k > 1$.
Furthermore, $\mathbb{T}_\alpha(\alpha)$ is not defined. By convention we say $\mathbb{T}_\alpha(\alpha) = 0$.
Finallly, $\mathbb{T}_\alpha(x)$ is not defined for some x where $x \in \mathbb{Q}$ and its continued fraction expansion is the same with the inital part of $\alpha$. i.e. if $x = [0,n_1,\cdots,n_k,\infty]$ and
$\alpha = [0,n_1,\cdots,n_k,n_{k+1},n_{k+2},\cdots]$. Actually in this case we accept that $\mathbb{T}_\alpha(x) = [0,\infty] = 0$.

**Proposition 5.2.1.** The following equation satisfies for any $\alpha, x \in [0,1]$

$$\mathfrak{J}\mathbb{T}_\alpha(x) = \mathbb{T}_{\mathfrak{J}(\alpha)}(\mathfrak{J}x)$$

where $\mathfrak{J}$ is the involution Jimm function which is defined in the previous chapter.

*Proof.* Let $\alpha = [0,n_1,n_2,n_3\cdots] \in [0,1]$ and $x = [0,m_1,m_2,m_3,\cdots] \in [0,1]$. Then

$$\mathfrak{J}\mathbb{T}_\alpha(x) = \begin{cases} [0,1_{m_{k+1}-1},2,1_{m_{k+2}-2},\cdots] & \text{if } n_k > m_k \\ [0,1_{m_k-n_k-1},2,1_{m_{k+1}-2},2,1_{m_{k+2}-2},\cdots] & \text{if } n_k < m_k \end{cases}$$

where $n_i = m_i$ for any $i \in \{1,\cdots k-1\}$. $\mathfrak{J}(\alpha) = [0,1_{n_1-1},2,1_{n_2-2},2,1_{n_3-2},2,\cdots]$ and $\mathfrak{J}(x) = [0,1_{m_1-1},2,1_{m_2-2},2,1_{m_3-2},2,\cdots]$. So if $m_i = n_i$ for any $i \in \{1,\cdots k-1\}$ then $T_{\mathfrak{J}(\alpha)}(\mathfrak{J}x) = [0,2-1,1_{m_{k+1}-2},2,1_{m_{k+2}-2},\cdots] = [0,1_{m_{k+1}-1},2,1_{m_{k+2}-2},\cdots]$ in the case $n_k > m_k$. If $n_k < m_k$ then $T_{\mathfrak{J}(\alpha)}(\mathfrak{J}x) = [0,1_{m_k-2-(n_k-2)-1},2,1_{m_{k+1}-2},2,\cdots] =$
$[0,1_{m_k-n_k-1},2,1_{m_{k+1}-2},2,1_{m_{k+2}-2},\cdots]$. So $\mathfrak{J}\mathbb{T}_\alpha(x) = \mathbb{T}_{\mathfrak{J}(\alpha)}(\mathfrak{J}x)$.

**Example 5.2.2.** *(Gauss Map)* The Gauss continued fraction map $h$ is given by:

$$h\colon [0,1] \to [0,1]$$
$$x \mapsto \frac{1}{x} - \lfloor \frac{1}{x} \rfloor.$$

where $\lfloor \frac{1}{x} \rfloor$ denotes floor function.

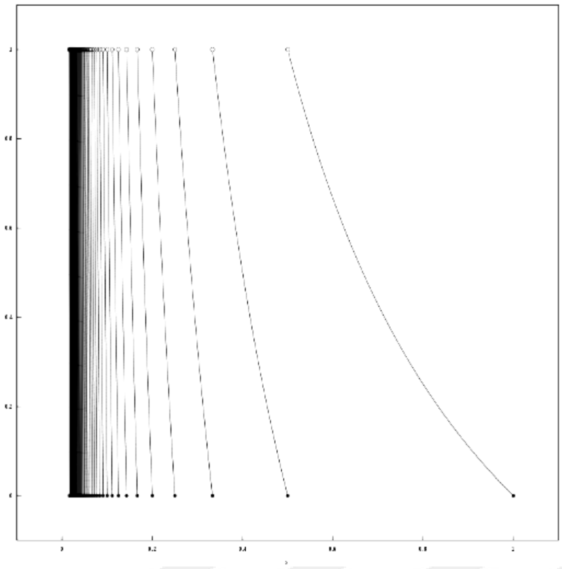


Figure 5.2: Gauss Map

*As we can see at Figure* 5.2 *it has an infinite number of jump discontinues.*
*Let* $\alpha := [0,\infty] = 0$ *and* $x = [0,n_1,n_2,\cdots] \in [0,1]$ *for any* $n_i \in \mathbb{Z}$*,* $i \in \{1,2,\cdots\}$ *which is rational or irrational number. Then by definition of continued fraction map* $\mathbb{T}_0(x) = [0,n_2,n_3,\cdots] = \frac{1}{x} - n_1 = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor = h(x)$*. That is, the map* $\mathbb{T}_0$ *is the Gauss map and it forgets just the first partial quotient of x for any* $x \in [0,1]$*.*

**Example 5.2.3.** *(The Fibonacci Map)* Let's take $\alpha = [0,1,1,\cdots] = \frac{\sqrt{5}-1}{2}$. Then $\mathbb{T}_\alpha(x) = [0,m_{k+1}-1,m_{k+2},\cdots]$ where $x = [0,1_k,m_{k+1},m_{k+2}\cdots]$ with $m_{k+1} > 1$ is called the Fibonacci Map.

**Proposition 5.2.2.** *[19]* The Gauss continued fraction map and Fibonacci map are conjugate under the involution Jimm. That is,

$$\mathfrak{J}\mathbb{T}_0(\mathfrak{J}(x)) = \mathbb{T}_{[0,1,1,\cdots]}(x)$$

for any $x \in [0,1]$.

*Proof.* $\mathfrak{J}([0,1,1,\cdots]) = 0$. By Proposition 5.2.1 $\mathfrak{J}\mathbb{T}_{[0,1,1,\cdots]}(x) = \mathbb{T}_{\mathfrak{J}([0,1,1,\cdots])}(\mathfrak{J}x)$ for any $x \in [0,1]$. Then $\mathfrak{J}\mathbb{T}_{[0,1,1,\cdots]}(x) = \mathbb{T}_0(\mathfrak{J}x)$. Since $\mathfrak{J}$ is an involution $\mathfrak{J}\mathfrak{J}\mathbb{T}_{[0,1,1,\cdots]}(x) = \mathbb{T}_{[0,1,1,\cdots]}(x)$. Then $\mathbb{T}_{[0,1,1,\cdots]}(x) = \mathfrak{J}\mathbb{T}_0(\mathfrak{J}x)$ for any $x \in [0,1]$.

Note that $\mathbb{T}_\alpha$ is piecewise $\mathrm{PSL}_2(\mathbb{Z})$ and that its inverse branches are given by

$$\mathbb{T}_\alpha{}^{-1}(y) = \begin{cases} [0,n_1,n_2,\cdots n_{k-1},i+y] & \text{if } 1 \leq k;\ 1 \leq i \leq n_k \\ [0,n_1,n_2,\cdots n_{k-1},n_k,y] & \text{if } 1 \leq k \end{cases} \tag{5.1}$$

So the inverse of Gauss Map:

$$\mathbb{T}_0{}^{-1}(y) = \frac{1}{y+n_1} = [0, n_1 + y].$$

Let $\psi$ be the cumulative distribution function of a probability measure on $[0,1]$. The operator $\mathbb{L}_\alpha$ is just the push back of $\mathbb{T}_\alpha$ and the cumulative distribution funtion of $y$ is equal to $\mathbb{L}_\alpha \Psi(y)$.

$$\begin{aligned}(\mathbb{L}_\alpha \Psi)(y) = &\sum_{k=1}^{\infty} (-1)^{1+k} \left(\Psi[0, n_1, \ldots, n_{k-1}, n_k, y] - \Psi[0, n_1, \ldots, n_{k-1}, n_k, 0]\right) \\ &+ \sum_{k=1}^{\infty} (-1)^k \sum_{i=1}^{n_k - 1} \left(\Psi[0, n_1, \ldots, n_{k-1}, i+y] - \Psi[0, n_1, \ldots, n_{k-1}, i]\right)\end{aligned} \tag{5.2}$$

We assume that $\Psi$ is differentiable. So we find the density function by taking the derivative of equation 5.2 term by term, we get the equation:

$$\mathcal{L}_\alpha \psi = \textstyle\sum_{k=1}^{\infty} (-1)^{1+k} \left\{ \frac{\mathrm{d}}{\mathrm{dy}} [0, n_1, \ldots, n_{k-1}, n_k, y] \right\} \psi[0, n_1, \ldots, n_{k-1}, n_k, y] + \\ \textstyle\sum_{k=1}^{\infty} (-1)^k \sum_{i=1}^{n_k - 1} \left\{ \frac{\mathrm{d}}{\mathrm{dy}} [0, n_1, \ldots, n_{k-1}, i+y] \right\} \psi[0, n_1, \ldots, n_{k-1}, i+y]$$

where $\psi$ is the derivative of $\Psi$.
This operator is called Gauss Kuzmin Wirsing operator of $\mathbb{T}_\alpha$. Now for finding the derivatives of continued fraction expansions of a function we will write them in the following form:

$$[0, n_1, \ldots, n_{k-1}, n_k, y] =: \frac{A_k y + B_k}{C_k y + D_k}.$$

with $A_k, B_k, C_k, D_k \in \mathbb{Z}$ and $A_k D_k - B_k C_k = (-1)^{1+k}$.
Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dy}} [0, n_1, \ldots, n_{k-1}, n_k, y] &= \frac{A_k}{(C_k y + D_k)} - \frac{C_k (A_k y + B_k)}{(C_k y + D_k)^2} \\ &= \frac{A_k D_k - B_k C_k}{(C_k y + D_k)^2} \\ &= \frac{(-1)^{k+1}}{(C_k y + D_k)^2}.\end{aligned} \tag{5.3}$$

So we obtain:

$$\begin{aligned}
\mathcal{L}_\alpha\psi &= \sum_{k=1}^{\infty} \frac{1}{(C_k y + D_k)^2}\psi[0,n_1,\ldots,n_{k-1},n_k,y] \\
&+ \sum_{k=1}^{\infty}\sum_{i=1}^{n_k-1} \frac{1}{(C_{k-1}(y+i)+D_{k-1})^2}\psi[0,n_1,\ldots,n_{k-1},i+y] \\
&= \sum_{k=1}^{\infty}\left|\frac{\mathrm{d}}{\mathrm{d}y}[0,n_1,\ldots,n_{k-1},n_k,y]\right|\psi[0,n_1,\ldots,n_{k-1},n_k,y] \\
&+ \sum_{k=1}^{\infty}\sum_{i=1}^{n_k-1}\left|\frac{\mathrm{d}}{\mathrm{d}y}[0,n_1,\ldots,n_{k-1},i+y]\right|\psi[0,n_1,\ldots,n_{k-1},i+y]
\end{aligned}$$

And we have the following equality:

$$\sum_{k=1}^{\infty}\left|\frac{\mathrm{d}}{\mathrm{d}y}[0,n_1,\ldots,n_{k-1},n_k,y]\right|\psi[0,n_1,\ldots,n_{k-1},n_k,y] =$$
$$\sum_{k=1}^{\infty}\left|\frac{\mathrm{d}}{\mathrm{d}y}[0,n_1,\ldots,n_{k-1},y]\right|^s\psi[0,n_1,\ldots,n_{k-1},y] - \left|\frac{\mathrm{d}}{\mathrm{d}y}[0,y]\right|\psi[0,y]$$

When we take the derivative the last term we obtain:

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}[0,y]\right|\psi[0,y] = -\frac{1}{y^{2s}}\psi\left(\frac{1}{y}\right).$$

Finally we have the density function :

$$(\mathcal{L}_\alpha\psi)(y) = -\tfrac{1}{y^2}\psi\left(\tfrac{1}{y}\right) + \textstyle\sum_{k=1}^{\infty}\sum_{i=0}^{n_k-1}\left|\tfrac{\mathrm{d}}{\mathrm{d}y}[0,n_1,\ldots,n_{k-1},i+y]\right|\psi[0,n_1,\ldots,n_{k-1},i+y]$$

The operation below

$$(\psi|M) := \frac{1}{|cx+d|^2}\psi\left(\frac{ax+b}{cx+d}\right)$$

defines an action of the group $\mathrm{PGL}_2(\mathbb{Z})$ on the set of functions on $\mathbb{R}$ with $M(x) = \frac{ax+b}{cx+d}$. It is called the **slash operator.**

*Proof.* Indeed, let $M(x) = x = \frac{1x+0}{0x+1}$ be the identity. Then
$(\psi|M)(x) = \frac{1}{|0x+1|^2}\psi\left(\frac{x}{1}\right) = \psi(x)$.
Now let $M(x) = \frac{ax+b}{cx+d}$, $N(x) = \frac{ex+f}{gx+h} \in \mathrm{PGL}_2(\mathbb{Z})$. We will show $(\psi|MN) = ((\psi|M)|N)$.
We know that

$$(\psi|M)(x) = \frac{1}{|cx+d|^2}\psi\left(\frac{ax+b}{cx+d}\right).$$

So,

$$
\begin{aligned}
((\psi|M)|N)(x) &= \frac{1}{(gx+h)^2}\frac{1}{\left(c\frac{ex+f}{gx+h}+d\right)^2}\psi\left(\frac{a(\frac{ex+f}{gx+h})+b}{c(\frac{ex+f}{gx+h})+d}\right) \\
&= \frac{1}{(c(ex+f)+d(gx+h))^2}\psi\left(\frac{a(ex+f)+b(gx+h)}{c(ex+f)+d(gx+h)}\right) \\
&= \frac{1}{(ce+dg)x+cf+dh)^2}\psi\left(\frac{(ae+bg)x+af+bh)}{(ce+dg)x+cf+dh)}\right) \\
&= (\psi|MN)(x)
\end{aligned}
$$

where $MN = \dfrac{(ae+bg)x+af+bh)}{(ce+dg)x+cf+dh)}$. Hence this is indeed a group action.

The modular group $\mathrm{PSL}_2(\mathbb{Z})$ is generated by the following transformations :

$$
\begin{array}{l} S : z \mapsto -1/z \\ T : z \mapsto z+1 \end{array} .
$$

Let $U : z \mapsto 1/z$, $K : z \mapsto 1-z$. So $T^n : z \mapsto z+n$ and $UT^n : z \mapsto 1/(z+n)$. Then we can described a continued fraction via these transformations:
$[0,n_0,n_1,n_2,\cdots,n_{k-1},i+y] = UT^{n_1}U\ldots UT^{n_{k-1}}UT^i(y)$. And by Equation 5.3 we obtain the derivative of a continued fraction. So we have the following equality:

$$
\left|\frac{\mathrm{d}}{\mathrm{d}y}[0,n_1,\ldots,n_{k-1},i+y]\right|\psi[0,n_1,\ldots,n_{k-1},i+y] = \left(\psi|UT^{n_1}U\ldots UT^{n_{k-1}}UT^i\right)(y)
$$

And also we have

$$
-\frac{1}{y^2}\psi\left(\frac{1}{y}\right) = -(\psi|U).
$$

Then we reach to the following equality

$$
\begin{aligned}
(\mathcal{L}_\alpha\psi)(y) &= -\frac{1}{y^2}\psi\left(\frac{1}{y}\right) + \sum_{k=1}^{\infty}\sum_{i=0}^{n_k-1}\left|\frac{\mathrm{d}}{\mathrm{d}y}[0,n_1,\ldots,n_{k-1},i+y]\right|\psi[0,n_1,\ldots,n_{k-1},i+y] \\
&= -(\psi|U) + \sum_{k=1}^{\infty}\sum_{i=0}^{n_k-1}\left(\psi|UT^{n_1}U\ldots UT^{n_{k-1}}UT^i\right)
\end{aligned}
\tag{5.4}
$$

Then

$$
(\mathcal{L}_\alpha\psi|T) = -(\psi|UT) + \sum_{k=1}^{\infty}\sum_{i=0}^{n_k-1}\left(\psi|UT^{n_1}U\ldots UT^{n_{k-1}}UT^{i+1}\right).
$$

*Remark.* The following equality satisfies

$$\sum_{k=1}^{\infty}(\psi|M_k)=\left(\psi|\sum_{k=1}^{\infty}M_k\right)$$

So by this remark we get

$$\mathcal{L}_\alpha\psi=\left(\psi|-U+\sum_{k=1}^{\infty}\sum_{i=0}^{n_k-1}UT^{n_1}U\dots UT^{n_{k-1}}UT^i\right)$$

So we get

$$\mathcal{L}_\alpha\psi=\left(\psi|-U+\sum_{k=1}^{\infty}UT^{n_1}U\dots UT^{n_{k-1}}U\sum_{i=0}^{n_{k-1}}T^i\right)$$

$$\sum_{i=0}^{n_k-1}T^i=(I-T)^{-1}(I-T^{n_k}).$$

Then we obtain

$$\mathcal{L}_\alpha\psi=\left(\psi|-U+\sum_{k=1}^{\infty}UT^{n_1}U\dots UT^{n_{k-1}}U(I-T)^{-1}(I-T^{n_k})\right)$$

By using the equation 5.4 we can compute $(\mathcal{L}_\alpha\psi|T)$;

$$(\mathcal{L}_\alpha\psi|T)=-(\psi|UT)+\sum_{k=1}^{\infty}\sum_{i=0}^{n_k-1}\left(\psi|UT^{n_1}U\dots UT^{n_{k-1}}UT^{i+1}\right)$$

$$\begin{aligned}(\mathcal{L}_\alpha\psi|I-T)&=(\mathcal{L}_\alpha\psi)-(\mathcal{L}_\alpha\psi|T)\\&=-(\psi|U(I-T))+\sum_{k=1}^{\infty}[(\psi|UT^{n_1}\dots T^{n_{k-1}}U)-(\psi|UT^{n_1}\dots T^{n_{k-1}}UT^{n_k})]\end{aligned}$$

$$((\mathcal{L}_\alpha\psi|I-T)|U)=-(\psi|U(I-T)U)+\sum_{k=1}^{\infty}[(\psi|UT^{n_1}\dots T^{n_{k-1}})-(\psi|UT^{n_1}\dots UT^{n_k}U)]$$

So

$$(\mathcal{L}_\alpha\psi|I-T)+((\mathcal{L}_\alpha\psi|I-T)|U)=((\mathcal{L}_\alpha\psi|I-T)|(I+U))$$

is given by

$$-(\psi|U-UT+I-UTU)+(\psi|I+U)=(\psi|UT+UTU)$$

So we obtain very simple equation by these computations. Assume that $\psi$ is an eigenfunction of $\mathcal{L}_\alpha$ with eigenvalue $\beta$, i.e. $\mathcal{L}_\alpha\psi = \beta\psi$. Then we obtain

$$(\psi|(I-T)(I+U)) = \frac{1}{\beta}(\psi|UTU+UT).$$

$(\psi|I-T-TU) = 0$ is called the Lewis three term and $(\star|UTU+UT)$ is Isola's transfer operator of the Farey map.

$$(\psi) - (\psi|T) + (\psi|U) - (\psi|UT) = \frac{1}{\beta}(\psi|UTU) + (\psi|UT).$$

So if we write explicitly we obtain the following functional equation;

$$\psi(y) - \psi(1+y) + \tfrac{1}{y^2}\left\{\psi\left(\tfrac{1}{y}\right) - \psi\left(\tfrac{y+1}{y}\right)\right\} = \\ \tfrac{1}{\beta(1+y)^2}\left\{\psi\left(\tfrac{y}{1+y}\right) + \psi\left(\tfrac{1}{1+y}\right)\right\}$$

**Definition 5.2.2.** Let $\mathcal{A}$ be the σ-algebra of $[0,1]$ and $\mu$ a measure on $[0,1]$. A map $\mathbb{T} : [0,1] \to [0,1]$ is called measurable if $\mathbb{T}^{-1}(A) \in \mathcal{A}$ for any $A \in \mathcal{A}$ and is called ***measure preserving*** if $\mu(\mathbb{T}^{-1}(A)) = \mu(A)$. In this case $\mu$ is said to be ***invariant measure*** under the map $\mathbb{T}$.

Now we will try to find the eigenfunction $\Psi$ for the Gauss Map which its inverse branch is given by $\mathbb{T}_0^{-1}(y) = \frac{1}{y+n_1} = [0, n_1 + y]$. Then

$$\mathcal{L}_0\psi(y) = \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}y}[0, n+y]\,\psi[0, n+y].$$

$\mathcal{L}_0\psi = \sum_{n=1}^{\infty}(\psi|UT^{n_1})$ And then
$(\mathcal{L}_0\psi|T) = \sum_{k=1}^{\infty}\left(\psi|UT^{n+1}\right) = (\mathcal{L}_0\psi|T) = \sum_{n=2}^{\infty}(\psi|UT^n)$. Assume $\psi$ is an eigenfunction of $(\mathcal{L}_0\psi)$ with eigenvalue $\lambda$ we obtain :

$$\begin{aligned} &\lambda(\psi - (\psi|T)) = (\psi|UT) \\ &\Leftrightarrow \psi - (\psi|T) = \tfrac{1}{\lambda}(\psi|UT) \\ &\Leftrightarrow \psi = (\psi|T) + \tfrac{1}{\lambda}(\psi|UT) \end{aligned}$$

If we write explicitly we find the following equality:

$$\psi(y) = \psi(1+y) + \frac{1}{\lambda}\frac{1}{(1+y)^2}\psi\left(\frac{1}{1+y}\right) \tag{5.5}$$

$\psi_0(y) := \frac{1}{\log 2}\frac{1}{1+y}$ is a solution of this equality. This equality is called the Gauss density. Moreover, it is straight forward to check that this measure satisfies the equation 5.5.

*Remark.* An invariant measure does not need to be unique. A map may have ∞-many invariant measure.

**Proposition 5.2.3.** *[19]* Minkowski measure is an invariant measure of $\mathbb{T}_\alpha$ for any $\alpha \in [0,1]$.

*Proof.* The cumulative distribution function of Minkowski measure is Minkowski ? function. The inverse branches of $\mathbb{T}_\alpha$ for any $\alpha \in [0,1]$ in given at 5.1. Let $\varphi_\gamma(y) := [0, n_1, n_2, \ldots, n_{k-1}, i+y]$ where $i \geq 0$. such that $\{\varphi_\gamma\}_{\gamma=1,2,\ldots}$ are given the set of inverse branches of $\mathbb{T}_\alpha$ for any $\gamma$. So we know by the equality 5.2 the cumulative distribution function for any $\alpha$ is the following

$$\mathbb{L}_\alpha ?(y) = \sum_\gamma (\Psi[0, n_1, \ldots, n_{k-1}, i+y] - \Psi[0, n_1, \ldots, n_{k-1}, i])$$

We will check that the Minkowski ? function is an eigenfunction of $\mathbb{L}_\alpha$ i.e. $\mathbb{L}_\alpha ?(y) = \lambda ?(y)$ where $\lambda$ is an eigenvalue. We assume that $\lambda = 1$.

$$\mathbb{L}_\alpha ?(y) = \sum_\gamma (?[0, n_1, \ldots, n_{k-1}, i+y] - ?[0, n_1, \ldots, n_{k-1}, i]) \quad = \sum_\gamma ?(y) 2^{-(n_1 + \cdots + n_{k-1} + i)} \tag{5.6}$$

So

$$\mathbb{L}_\alpha ?(y) = ?(y) \sum_\gamma 2^{-(n_1 + \cdots + n_{k-1} + i)}$$

$\sum_\gamma 2^{-(n_1 + \cdots + n_{k-1} + i)} = 1$. Hence it proves $\mathbb{L}_\alpha ?(y) = ?(y)$.

Let's take $\alpha = 1 = [0, 1, \infty]$. So the continued fraction map is given by ;

$$\mathbb{T}_1(x) = \begin{cases} [0, m_1 - 1, m_2, m_3, \cdots] & \text{if } m_1 > 1 \\ [0, m_3, m_4 \cdots] & \text{if } m_1 = 1 \end{cases}$$

where $x = [0, m_1, m_2, m_3, \cdots] \in [0,1]$.

$$\mathbb{T}_1(x) = \begin{cases} \left(\frac{1}{x} - 1\right)^{-1} \text{ if } 0 \leq x < \frac{1}{2}. \\ \mathbb{T}_0\left(\frac{1}{x} - 1\right) \text{ if } \frac{1}{2} \leq x < 1. \end{cases}$$

$$\begin{aligned} \mathcal{L}_1 \psi &= (\psi | UTU) + \sum_{i=1}^{\infty} \left(\psi | UTUT^i\right) \\ &= \sum_{i=0}^{\infty} \left(\psi | UTUT^i\right) \end{aligned}$$

Assume $\psi$ is an eigenfunction for $\mathcal{L}_1$. And also we assume that the eigenvalue is 1. So we get the following equality:

$$\begin{aligned}\psi &= \sum_{i=0}^{\infty}\left(\psi|UTUT^i\right)\\ &= \left(\psi|UTU\sum_{i=0}^{\infty}T^i\right)\\ &= \left(\psi|UTU(I-T)^{-1}\right)\\ &\Leftrightarrow (\psi|I-T) = (\psi|UTU)\end{aligned}$$

When we write explicitly we obtain

$$\psi(y) - \psi(1+y) = \frac{1}{(y+1)^2}\psi\left(\frac{y}{y+1}\right).$$

So $\psi(y) = \frac{1}{y}$ satisfies this equality which gives us an eigenfunction of $\mathcal{L}_1$. So it is the density function of an invariant measure on $T_1(x)$.

**Example 5.2.4.** $\mathcal{F}_1$ is the subtree of $\mathcal{F}$ consisting of vertices of distance $\leq 1$ to the fixed edge $e$. So $\mathbb{T}_{\mathcal{F}_1}$ is just cutting the first edges of the paths of the tree. That is,

$$\mathbb{T}_{\mathcal{F}_1}(x) = \begin{cases}[0, m_1-1, m_2, m_3, \cdots] & if\ \ m_1 > 1\\ [0, m_2, m_3, \cdots] & if\ \ m_1 = 1\end{cases}$$

*where* $x = [0, m_1, m_2, m_3, \cdots] \in [0,1]$. *If we write the function by the value x;*

$$\mathbb{T}_{\mathcal{F}_1}(x) = \begin{cases}\frac{x}{1-x} & if\ \ 0 \leq x < \frac{1}{2}.\\ \frac{1}{x} - 1 & if\ \ \frac{1}{2} \leq x < 1.\end{cases}$$

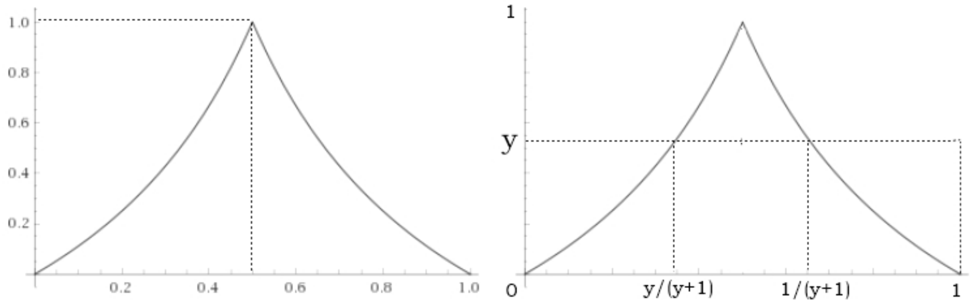



Figure 5.3: $\mathbb{T}_{\mathcal{F}_1}$

If $\mu$ is an invariant measure under the function $\mathbb{T}_{\mathcal{F}_1}$ then the cumulative distribution function satisfies the following equation:

$$(\mathbb{L}_{\mathcal{F}_1}\Psi)(y) = (\Psi[0,1,y] - \Psi[0,\infty]) + (\Psi[0,1+y] - \Psi[0,1,\infty])$$

Assuming that $(\mathbb{L}_{\mathcal{F}_1}\Psi)(y)$ is differentiable the desity function which is derivative of this function is given by:

$$\begin{aligned}(\mathcal{L}_\alpha\psi)(y) &= \left|\frac{\mathrm{d}}{\mathrm{d}y}[0,1,y]\right|\psi[0,1,y] + \left|\frac{\mathrm{d}}{\mathrm{d}y}[0,1+y]\right|\psi[0,1+y] \\ (\mathcal{L}_\alpha\psi) &= (\psi|UTU) + (\psi|UT)\end{aligned}$$

Now suppose that $\psi$ is an eigenfunction of $\mathcal{L}_\alpha$ then we get:

$$\begin{aligned}&\psi = (\psi|UTU) + (\psi|UT) \\ &\Leftrightarrow (\psi|U) = \left(\psi|UTU^2\right) + (\psi|UTU) \\ &\Leftrightarrow (\psi|U) = (\psi|UT) + (\psi|UTU) \\ &\Leftrightarrow (\psi - (\psi|U)) = 0\end{aligned}.$$

If we write explicitly the previous equation we obtain:

$$\psi(y) - \frac{1}{y^2}\psi\left(\frac{1}{y}\right) = 0.$$

$\psi(y) := \frac{1}{y}$ satisfies this equality for any $y$.

*Remark.* $\mathbb{T}_{\mathcal{F}_n} = (\mathbb{T}_{\mathcal{F}_1})^n$ so the invariant measure under the function $\mathbb{T}_{\mathcal{F}_n}$ is the same with $\mathbb{T}_{\mathcal{F}_1}$.

*Proof.* The map $\mathbb{T}_{\mathcal{F}_1}$ just cuts the first starting edge $e$. And $\mathbb{T}_{\mathcal{F}_2}$ cuts the sub tree $\mathcal{F}_2$. That is, it cuts only the edge in the first line; $e$ and the edges in the second line. So if we apply $\mathbb{T}_{\mathcal{F}_1}$ we cut the edge $e$ in the first line. And again we apply this map we cut the edges in the first line again but now these are the edges in the second line of the tree. So $\mathbb{T}^2_{\mathcal{F}_1} \equiv \mathbb{T}_{\mathcal{F}_2}$. Let $n \in \mathbb{N}$. Assume $\mathbb{T}_{\mathcal{F}_{n-1}} = (\mathbb{T}_{\mathcal{F}_1})^{n-1}$. The map $\mathbb{T}_{\mathcal{F}_n}$ cuts the sub tree $\mathcal{F}_n$. The map $\mathbb{T}_{\mathcal{F}_{n-1}}$ cuts the first $n-1$ lines of the tree. And if we apply the map $\mathbb{T}_{\mathcal{F}_1}$ we obtain the map $(\mathbb{T}_{\mathcal{F}_n})$. Then by induction $\mathbb{T}_{\mathcal{F}_n} = (\mathbb{T}_{\mathcal{F}_1})^n$.

The graphs of the maps $\mathbb{T}_{\mathcal{F}_2}$ and $\mathbb{T}_{\mathcal{F}_3}$ are as follows:

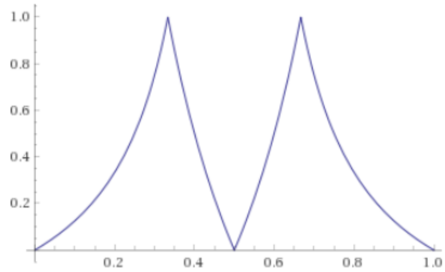



Figure 5.4: $\mathbb{T}_{\mathcal{F}_2}$

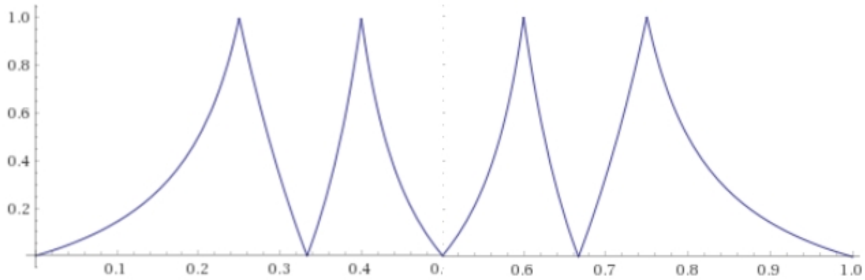



Figure 5.5: $\mathbb{T}_{\mathcal{F}_3}$

### 5.2.1 A More Generalisation of The Continued Fraction Maps

We know some ends of Farey tree can be labeled by rational numbers and the meaning of a continued fraction map $\mathbb{T}_\alpha$ can be thought as just cutting the end $\alpha$. So if we cut more than one end we obtain a generalization of these maps. This generalization is given by us.

Let us see this generalization over some examples:

Let's take the paths $\frac{1}{2}^+$ and $\frac{1}{2}^-$. We know the continued fraction expansion of them are not same so $\mathbb{T}_{\frac{1}{2}^+} \neq \mathbb{T}_{\frac{1}{2}^-} \neq \mathbb{T}_{\frac{1}{2}^+ \vee \frac{1}{2}^-}$.

$$\mathbb{T}_{\frac{1}{2}^+ \vee \frac{1}{2}^-}(x) = \begin{cases} [0, m_1 - 2, m_2, \cdots] & \text{if } m_1 > 2 \\ [0, m_3, m_4, \cdots] & \text{if } m_1 = 2 \\ [0, m_4, m_5, \cdots] & \text{if } m_1 = 1,\ m_2 = 1 \\ [0, m_2 - 1, m_3, \cdots] & \text{if } m_1 = 1,\ m_2 > 1. \end{cases}$$

where $x = [0, m_1, m_2, m_3, \cdots] \in [0,1]$.

$$\mathbb{T}_{\frac{1}{2}^+ \vee \frac{1}{2}^-}(x) = \begin{cases} \frac{x}{1-2x} & \text{if } 0 < x \leq \frac{1}{3} \\ \frac{x-m_2+2m_2x}{1-2x} & \text{if } \frac{1}{3} \leq x < \frac{1}{2} \\ \frac{1-x-2m_3x+m_3}{2x-1} & \text{if } \frac{1}{2} < x \leq \frac{2}{3} \\ \frac{1-x}{2x-1} & \text{if } \frac{2}{3} \leq x < 1. \end{cases}$$

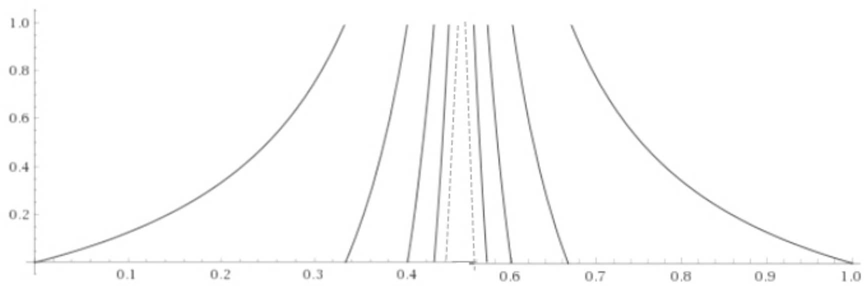



Figure 5.6: $\mathbb{T}_{\frac{1}{2}^+ \vee \frac{1}{2}^-}(x)$

The density function of an invariant measure $\mu$ is given by as follows:

$$f_\mu(y) = \sum_{m=0}^{\infty} \frac{1}{(2y+2m+1)^2}\Big(f_\mu\big(\frac{y+m}{2y+2m+1}\big) + f_\mu\big(\frac{y+m+1}{2y+2m+1}\big)\Big)$$

$$f_\mu(y) - f_\mu(y+1) = \frac{1}{(2y+1)^2}\Big(f_\mu\big(\frac{y}{2y+1}\big) + f_\mu\big(\frac{y+1}{2y+1}\big)\Big)$$

$f_\mu(y) = \frac{1}{y}$ satisfies the previous equality. So it can the density of an invariant measure $\mu$ under the function $\mathbb{T}_{\frac{1}{2}^+ \vee \frac{1}{2}^-}$. Let us to write $f_\mu(y) = \frac{1}{y}$ :

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{1}{(2y+2m+1)^2}\Big(f_\mu\big(\frac{y+m}{2y+2m+1}\big) + f_\mu\big(\frac{y+m+1}{2y+2m+1}\big)\Big) \\ &= \sum_{m=0}^{\infty} \frac{1}{(y+m)(y+m+1)} \\ &= \sum_{m=0}^{\infty} \frac{1}{y+m} - \frac{1}{y+m+1} = \frac{1}{y} \end{aligned}$$

Then $f_\mu(y) = \frac{1}{y}$ is the density of an invariant measure.

$$\mathbb{T}_{\frac{1}{3}^+\vee\frac{1}{3}^-\vee\frac{2}{3}^+\vee\frac{2}{3}^-}(x)=\begin{cases}[0,m_1-3,m_2,\cdots] & \text{if } m_1>3\\ [0,m_3,m_4,\cdots] & \text{if } m_1=3\\ [0,m_4,m_5,\cdots] & \text{if } m_1=2,\ m_2=1\\ [0,m_2-1,m_3,\cdots] & \text{if } m_1=2,\ m_2>1.\\ [0,m_3-1,m_4,\cdots] & \text{if } m_1=1,m_2=1,m_3>1\\ [0,m_5,m_6,\cdots] & \text{if } m_1=1,m_2=1,m_3=1\\ [0,m_4,m_5,\cdots] & \text{if } m_1=1,m_2=2\\ [0,m_2-2,m_3,\cdots] & \text{if } m_1=1,m_2>2\end{cases}$$

where $x=[0,m_1,m_2,m_3\cdots]\in[0,1]$. By the value $x$ function is given as:

$$\mathbb{T}_{\frac{1}{3}^+\vee\frac{1}{3}^-\vee\frac{2}{3}^+\vee\frac{2}{3}^-}(x)=\begin{cases}\frac{x}{1-3x} & \text{if } 0<x\le\frac{1}{4}\\ \frac{x-m_2+3m_2x}{1-3x} & \text{if } \frac{1}{4}\le x<\frac{1}{3}\\ \frac{1-2x-3m_3x+m_3}{3x-1} & \text{if } \frac{1}{3}<x\le\frac{2}{5}\\ \frac{1-2x}{3x-1} & \text{if } \frac{2}{5}\le x<\frac{1}{2}\\ \frac{2x-1}{2-3x} & \text{if } \frac{1}{2}<x\le\frac{3}{5}\\ \frac{2x-1-2m_4+3m_4x}{2-3x} & \text{if } \frac{3}{5}\le x<\frac{2}{3}\\ \frac{1-x-3m_3x+2m_3}{3x-2} & \text{if } \frac{2}{3}<x\le\frac{3}{4}\\ \frac{1-x}{3x-2} & \text{if } \frac{3}{4}\le x<1\end{cases}$$

So the density of an invariant measure under the previous function satisfies the equation:

$$\begin{aligned} f_\mu(y)=\sum_{m=0}^{\infty}\frac{1}{(3y+3m+1)^2}&\Big(f_\mu\big(\frac{y+m}{3y+3m+1}\big)+f_\mu\big(\frac{2y+2m+1}{3y+3m+1}\big)\Big)\\ &+\frac{1}{(3y+3m+2)^2}\Big(f_\mu\big(\frac{y+m+1}{3y+3m+2}\big)+f_\mu\big(\frac{2y+2m+1}{3y+3m+2}\big)\Big).\end{aligned}\tag{5.7}$$

Assuming that the density function is analytic we can write the following equality, finding a solution for this equality is simpler than the previous one.

$$\begin{aligned} f_\mu(y)-f_\mu(y+1)=\frac{1}{(3y+1)^2}&\Big(f_\mu\big(\frac{y}{3y+1}\big)+f_\mu\big(\frac{2y+1}{3y+1}\big)\Big)\\ &+\frac{1}{(3y+2)^2}\Big(f_\mu\big(\frac{y+1}{3y+2}\big)+f_\mu\big(\frac{2y+1}{3y+2}\big)\Big).\end{aligned}$$

$f(y)=\frac{1}{y}$ satisfies this equality. If we check it for the equation 5.7, it satisfies also. So it is the density of the invariant measure under the map $\mathbb{T}_{\frac{1}{3}^{+}\vee\frac{1}{3}^{-}\vee\frac{2}{3}^{+}\vee\frac{2}{3}^{-}}(x)$.

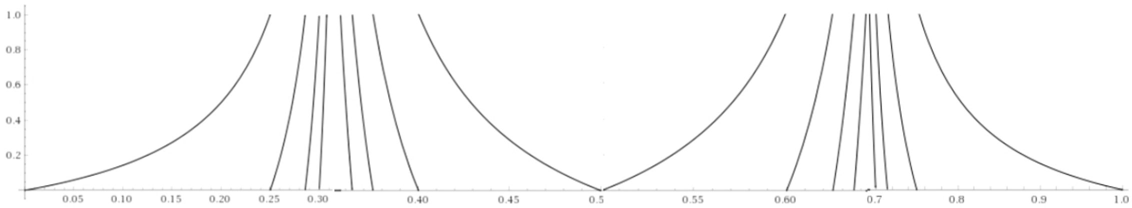



Figure 5.7: $\mathbb{T}_{\frac{1}{3}^{+}\vee\frac{1}{3}^{-}\vee\frac{2}{3}^{+}\vee\frac{2}{3}^{-}}(x)$

### 5.2.2 Invariant Measures for A Special Case of A Generalisation of The Continued Fraction Maps

Denote that $\mathcal{F}_n$ is a subtree of $\mathcal{F}$ consisting of vertices of distance $\leq n$ to the root vertex $\frac{1}{2}$. By convention, $V(\mathcal{F}_0) := \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$. Then $V(\mathcal{F}_1) := \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$, $V(\mathcal{F}_2) := \{\frac{0}{1}, \frac{1}{4}\frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}$ etc. Since it is a binary tree we can observe that $|V(\mathcal{F}_n)| = 2^{n+1} + 1$. So let

$$V(\mathcal{F}_n) = \left\{\frac{0}{1}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \cdots, \frac{a_k}{b_k}, \cdots, \frac{a_{2^{n+1}-1}}{b_{2^{n+1}-1}}, \frac{a_{2^{n+1}}}{b_{2^{n+1}}}\right\}$$

Then $V(\mathcal{F}_{n-1})$ and $V(\mathcal{F}_{n+1})$ is given by

$$V(\mathcal{F}_{n-1}) = \left\{\frac{0}{1}, \frac{a_2}{b_2}, \frac{a_4}{b_4}, \cdots, \frac{a_{2^{n+1}-2}}{b_{2^{n+1}-2}}, \frac{a_{2^{n+1}}}{b_{2^{n+1}}}\right\}$$

$$V(\mathcal{F}_{n+1}) = \{\frac{0}{1}, \frac{a_1}{1+b_1}, \frac{a_1}{b_1}, \frac{a_1+a_2}{b_1+b_2}, \frac{a_2}{b_2}, \frac{a_2+a_3}{b_2+b_3}, \frac{a_3}{b_3}, \cdots,$$
$$\frac{a_k}{b_k}, \frac{a_k+a_{k+1}}{b_k+b_{k+1}}, \frac{a_{k+1}}{b_{k+1}} \cdots, \frac{a_{2^{n+1}-1}}{b_{2^{n+1}-1}}, \frac{a_{2^{n+1}-1}+a_{2^{n+1}}}{b_{2^{n+1}-1}+b_{2^{n+1}}}, \frac{a_{2^{n+1}}}{b_{2^{n+1}}}\}$$

**Lemma 5.2.1.** Let $V(\mathcal{F}_n) = \left\{\frac{0}{1}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \cdots, \frac{a_k}{b_k}, \cdots, \frac{1}{2}, \cdots, \frac{c_k}{d_k}, \cdots, \frac{c_2}{d_2}, \frac{c_1}{d_1}, \frac{1}{1}\right\}$. Then $d_i = b_i$ and $c_i = b_i - a_i$ for any $i \in \{1, 2, \cdots, k, \cdots\}$. Indeed, $\frac{a_i}{b_i} + \frac{c_i}{d_i} = 1$ for any $i \in \mathbb{N}$.

*Proof.* It satisfies for $V(\mathcal{F}_0) := \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$. Assume that the propriety satisfies for $V(\mathcal{F}_n)$. We will show it satisfies for $V(\mathcal{F}_{n+1})$.

$$V(\mathcal{F}_{n+1}) = \{\frac{0}{1}, \frac{a_1}{1+b_1}, \frac{a_1}{b_1}, \frac{a_1+a_2}{b_1+b_2}, \frac{a_2}{b_2}, \frac{a_2+a_3}{b_2+b_3}, \frac{a_3}{b_3}, \cdots,$$
$$\frac{a_{k-1}}{b_{k-1}}, \frac{a_{k-1}+a_k}{b_{k-1}+b_k}, \frac{a_k}{b_k}, \cdots, \frac{1}{2}, \cdots, \frac{c_k}{d_k}, \frac{c_{k-1}+c_k}{d_{k-1}+d_k}, \frac{c_{k-1}}{d_{k-1}},$$
$$\cdots, \frac{c_3}{d_3}, \frac{c_3+c_2}{d_3+d_2}, \frac{c_2}{d_2}, \frac{c_2+c_1}{d_2+d_1}, \frac{c_1}{d_1}, \frac{c_1+1}{d_1+1}, \frac{1}{1}\}$$

Let $\frac{a_k}{b_k}, \frac{c_k}{d_k} \in V(\mathcal{F}_n)$. Then by assumption $b_{k-1} = d_{k-1}$ and $b_k = d_k$ then $b_{k-1} + b_k = d_{k-1} + d_k$. Moreover, by assumption $c_{k-1} = b_{k-1} - a_{k-1}$ and $c_k = b_k - a_k$. Then $c_{k-1} + c_k = b_{k-1} + b_k - (a_k + a_{k-1})$.

The following proprieties of Farey tree (following 2 lemmas) were showed by us.

**Lemma 5.2.2.** Assume that

$$V(\mathcal{F}_n) = \left\{ \frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots, \frac{a_k}{b_k}, \cdots, \frac{a_{2^n-1}}{b_{2^n-1}}, \frac{1}{2}, \frac{c_{2^n-1}}{b_{2^n-1}}, \cdots, \frac{c_k}{b_k}, \cdots, \frac{c_2}{b_2}, \frac{c_1}{b_1}, \frac{c_0}{b_0} \right\}.$$

Then

$$V(\mathcal{F}_{n+1}) = \{ \frac{a_0}{b_0}, \frac{a_0 + a_1}{b_1 + b_0}, \frac{a_1}{b_1}, \frac{a_1 + a_2}{b_1 + b_2}, \cdots, \frac{a_k + a_{k-1}}{b_k + b_{k-1}}, \frac{a_k}{b_k}, \cdots, \frac{a_{2^n-1} + 1}{b_{2^n-1} + 2} , \frac{1}{2}, \frac{b_1}{b_{2^n-1} + 2}, \frac{b_2}{b_{2^n-1}}, \cdots, \frac{b_4}{b_2}, \frac{b_3}{b_1 + b_2}, \frac{b_2}{b_1}, \frac{b_1}{b_1 + 1}, \frac{b_0}{b_0} \}.$$

*Proof.* We know $V(\mathcal{F}_0) = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$ and $V(\mathcal{F}_1) = \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$. So the hypothesis is true for this case. Assume that it is true for $V(\mathcal{F}_n)$. We will show it for $V(\mathcal{F}_{n+1})$. Let

$$V(\mathcal{F}_{n-1}) = \{ \frac{a_0}{b_0}, \frac{a_2}{b_2}, \frac{a_4}{b_4}, \cdots, \frac{1}{2}, \frac{c_{2^n-2}}{b_{2^n-2}} \cdots, \frac{c_4}{b_4}, \frac{c_2}{b_2}, \frac{c_0}{b_0} \}$$

If

$$V(\mathcal{F}_n) = \left\{ \frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \cdots, \frac{a_k}{b_k}, \cdots \frac{1}{2}, \frac{c_{2^n-1}}{b_{2^n-1}}, \frac{c_{2^n-2}}{b_{2^n-2}}, \cdots, \frac{c_k}{b_k}, \cdots, \frac{c_3}{b_3}, \frac{c_2}{b_2}, \frac{c_1}{b_1}, \frac{c_0}{b_0} \right\}$$

Then by our assumption

$$c_k = b_{2k} \text{ and } c_{2^n-k} = b_{2k} \text{ for any } k \in \{0, \cdots, 2^{n-1}\}. \tag{5.8}$$

We want to show that

$$c_k + c_{k+1} = b_{2k+1} \text{ and } c_k = b_{2k}.$$

$$c_{2^n-k} + c_{2^n-(k+1)} = b_{2k+1} \text{ and } c_{2^n-k} = b_{2k} \text{ for any } k \in \{0, 2^{n-1}\}.$$

By the equation 5.8 $c_k + c_{k+1} = b_{2k} + b_{2k+2} = b_{2k+1}$ by the rule of the Farey sum, and also $c_k = b_{2k}$. Moreover, $c_{2^n-k} + c_{2^n-(k+1)} = b_{2k} + b_{2k+2} = b_{2k+1}$ by Farey sum, and $c_{2^n-k} = b_{2k}$ for any $k \in \{0, \cdots, 2^{n-1}\}$.
So it gives us

$$V(\mathcal{F}_{n+1}) = \{\frac{0}{1}, \frac{a_1}{b_1+1}, \frac{a_1}{b_1}, \frac{a_1+a_2}{b_1+b_2}, \frac{a_2}{b_2}, \frac{a_3+a_2}{b_3+b_2}, \frac{a_3}{b_3}, \cdots,$$
$$\frac{a_k+a_{k-1}}{b_k+b_{k-1}}, \frac{a_k}{b_k}, \cdots, \frac{1}{2}, \frac{b_1}{b_{2^{n-1}}+2}, \frac{b_2}{b_{2^{n-1}}} \cdots,$$
$$\frac{b_4}{b_2}, \frac{b_3}{b_1+b_2}, \frac{b_2}{b_1}, \frac{b_1}{b_1+1}, \frac{1}{1}\}$$

**Lemma 5.2.3.** Assume

$$V(\mathcal{F}_n) = \{\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots, \frac{a_{2^{n+1}}}{b_{2^{n+1}}}\}$$

Then

$$V(\mathcal{F}_{n+1}) = \{\frac{a_0}{b_0}, \frac{a_1}{b_0+b_1}, \frac{a_2}{b_1}, \cdots, \frac{a_{2^{n+1}}}{b_{2^n}}, \frac{c_{2^{n+1}-1}}{b_{2^n-1}+b_{2^n}}, \cdots, \frac{c_1+c_0}{b_0+b_1}, \frac{c_0}{b_0}\}$$

*Proof.* We will just focus on the nominators of the elements of $V(\mathcal{F}_{n+1})$ which are less than $\frac{1}{2}$. We have $V(\mathcal{F}_0) = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$ and $V(\mathcal{F}_1) = \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$. So the hypothesis is true for this case. Assume that it is true for $V(\mathcal{F}_n)$. We will show it for $V(\mathcal{F}_{n+1})$. We must show

$$a_k + a_{k+1} = a_{2k+1} \text{ and } a_k = a_{2k}.$$

By our assumption $a_k = a_{2k}$ satisfies. By the rule of Farey sum
$a_{2k+1} = a_{2k} + a_{2k+2} = a_k + a_{k+1}$.

We see that in the Farey tree every end can be describbed also by a rational number, and also every vertices of a tree labeled with a rational number and these give us an end. Let $\mathcal{F}_n$ be the subtree of the Farey tree and
$V(\mathcal{F}_n) \setminus V(\mathcal{F}_{n-1}) = \{\frac{a_1}{b_1}, \frac{a_3}{b_3}, \frac{a_5}{b_5}, \ldots, \frac{a_k}{b_k}, \cdots, \frac{a_{2^{n+1}-1}}{b_{2^{n+1}-1}}\}$. Denote $\mathbb{X}_n$ be the set of the all ends which are labeled with the rational numbers in $V(\mathcal{F}_n) \setminus V(\mathcal{F}_{n-1})$. Assume $\frac{a_k}{b_k} \in V(\mathcal{F}_n) \setminus V(\mathcal{F}_{n-1})$. Then it represents two ends in the tree : $\frac{a_k}{b_k}^+$ and $\frac{a_k}{b_k}^-$. Denote that $\mathbb{T}_{\mathbb{X}_n} = \mathbb{T}_{\frac{a_1}{b_1}^+ \vee \frac{a_1}{b_1}^- \vee \cdots \vee \frac{a_{2^{n+1}-1}}{b_{2^{n+1}-1}}^+ \vee \frac{a_{2^{n+1}-1}}{b_{2^{n+1}-1}}^-}$. For example $\mathbb{T}_{\frac{1}{2}^+ \vee \frac{1}{2}^-} = \mathbb{T}_{\mathbb{X}_0}$

The following theorem was raised and proved by us.

**Theorem 5.2.4.** *The density of an invariant measure under the transition map $\mathbb{T}_{\mathbb{X}_n}$ is $f_\mu(y) = \frac{1}{y}$ for any $n \in \mathbb{N}$.*

*Proof.* First, let us to try the inverse branches of the transition map. Assume

$\mathbb{T}_{\mathbb{X}_n} := \dfrac{ax+b}{cx+d}$ for some $a,b,c,d \in \mathbb{N}$ with $c \neq 0$ or $d \neq O$ and $|ad-bc| = 1$. So the inverse of this map is $\dfrac{dy-b}{a-cy}$. Let $\frac{a_k}{b_k} \in V(\mathcal{F}_n) \setminus V(\mathcal{F}_{n-1})$. If $x \in (\frac{a_{k-1}}{b_{k-1}}, \frac{a_{k+1}}{b_{k+1}})$ then $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(x) = \mathbb{T}_{\mathbb{X}_n}(x)$. Let us to divide this interval and try to find the inverse branches.

$$\left(\frac{a_{k-1}}{b_{k-1}}, \frac{a_{k+1}}{b_{k+1}}\right) = \left(\frac{a_{k-1}}{b_{k-1}}, \frac{a_{k-1}+a_k}{b_{k-1}+b_k}\right] \bigcup \left(\frac{a_{k-1}+a_k}{b_{k-1}+b_k}, \frac{a_k}{b_k}\right] \bigcup$$
$$\left(\frac{a_k}{b_k}, \frac{a_k+a_{k+1}}{b_k+b_{k+1}}\right] \bigcup \left(\frac{a_k+a_{k+1}}{b_k+b_{k+1}}, \frac{a_{k+1}}{b_{k+1}}\right)$$

First, let the path $x \in \left(\dfrac{a_{k-1}}{b_{k-1}}, \dfrac{a_{k-1}+a_k}{b_{k-1}+b_k}\right)$, the map $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(x)$ cuts the first $n$ edges of $x$. The depth of $\dfrac{a_{k-1}}{b_{k-1}}$ as a vertex is $n$, so $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}$ maps the interval $\left(\dfrac{a_{k-1}}{b_{k-1}}, \dfrac{a_{k-1}+a_k}{b_{k-1}+b_k}\right)$ to $(0,1)$. Then we reach the following values of the map $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}$:

- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{a_{k-1}}{b_{k-1}}) = \dfrac{0}{1}$
- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{a_{k-1}+a_k}{b_{k-1}+b_k}) = \dfrac{1}{1}$
- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{2a_{k-1}+a_k}{2b_{k-1}+b_k} = \dfrac{1}{2}$
- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{3a_{k-1}+a_k}{3b_{k-1}+b_k} = \dfrac{1}{3}$

After some calculations we obtain the following equalities: $b = -a\dfrac{a_{k-1}}{b_{k-1}}$, $c = -a\dfrac{b_k}{b_{k-1}}$, $d = -c\dfrac{a_k}{b_k}$, $a = a$. We know $|ad-bc| = 1$, then $a^2 = b_{k-1}^2$. Without loss of generality $a = b_{k-1}$. Thus, $d = a_k$, $b = -a_{k-1}$ and $c = -b_k$. Hence we obtain

$$\mathbb{T}^{-1}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(y) = \frac{a_k y + a_{k-1}}{b_k y + b_{k-1}}.$$

Now assume that $x \in \left(\dfrac{a_{k-1}+a_k}{b_{k-1}+b_k}, \dfrac{a_k}{b_k}\right)$. Then $x$ has at least $n+1$ common edges with $\dfrac{a_k}{b_k}$.

If $x \in \left(\dfrac{a_{k-1}+a_k}{b_{k-1}+b_k}, \dfrac{a_{k-1}+2a_k}{b_{k-1}+2b_k}\right)$ then $x$ has exactly $n+1$ common edges with the end $\dfrac{a_k}{b_k}$, if we apply the transition map, it means we will cut the common edges. So we obtain:

- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{a_{k-1}+a_k}{b_{k-1}+b_k}) = \dfrac{0}{1}$
- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{a_{k-1}+2a_k}{b_{k-1}+2b_k}) = \dfrac{1}{1}$
- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{2a_{k-1}+3a_k}{2b_{k-1}+3b_k} = \dfrac{1}{2}$
- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{3a_{k-1}+4a_k}{3b_{k-1}+4b_k} = \dfrac{1}{3}$

After some calculations we obtain $a = b_{k-1}+b_k$, $b = -(a_{k-1}+a_k)$, $c = -b_k$ and $d = a_k$. So $\dfrac{dy-b}{a-cy} = \dfrac{a_k y + a_{k-1}+a_k}{b_k y + b_{k-1}+b_k}$ which gives us the inverse branch of the map.
Now assume that $x \in \left(\dfrac{a_{k-1}+2a_k}{b_{k-1}+2b_k}, \dfrac{a_{k-1}+3a_k}{b_{k-1}+3b_k}\right)$ then $x$ has exactly $n+2$ common edges with the end $\dfrac{a_k}{b_k}$, if we apply the transition map, it means we will cut the common edges. So we obtain:

- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{a_{k-1}+2a_k}{b_{k-1}+2b_k}) = \dfrac{0}{1}$
- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{a_{k-1}+3a_k}{b_{k-1}+3b_k}) = \dfrac{1}{1}$
- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{2a_{k-1}+5a_k}{2b_{k-1}+5b_k} = \dfrac{1}{2}$

After some calculations we obtain $\dfrac{dy-b}{a-cy} = \dfrac{a_k y + a_{k-1}+2a_k}{b_k y + b_{k-1}+2b_k}$ which gives us the inverse branch of the map.
For generaliser the inverse branches in this case, we assume $x$ has $n+m$ common edges with $\dfrac{a_k}{b_k}$. Then we will cut this edges and in this case
$x \in \left(\dfrac{a_{k-1}+ma_k}{b_{k-1}+mb_k}, \dfrac{a_{k-1}+(m+1)a_k}{b_{k-1}+(m+1)b_k}\right)$.

- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{a_{k-1}+ma_k}{b_{k-1}+mb_k}) = \dfrac{0}{1}$

- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\frac{a_{k-1}+(m+1)a_k}{b_{k-1}+(m+1)b_k}) = \frac{1}{1}$

- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\frac{2a_{k-1}+(2m+1)a_k}{2b_{k-1}+(2m+1)b_k} = \frac{1}{2}$

After similar steps we obtain the inverse branch: $\frac{dy-b}{a-cy} = \frac{a_k y + a_{k-1} + ma_k}{b_k y + b_{k-1} + mb_k}$. In summary we divise the interval $\left(\frac{a_{k-1}+a_k}{b_{k-1}+b_k}, \frac{a_k}{b_k}\right)$ infinitely many intervals such that

$$x \in \left(\frac{a_{k-1}+a_k}{b_{k-1}+b_k}, \frac{a_k}{b_k}\right) = \bigcup_{i=1}^{\infty} \left(\frac{a_{k-1}+ia_k}{b_{k-1}+ib_k}, \frac{a_{k-1}+(i+1)a_k}{b_{k-1}+(i+1)b_k}\right).$$

Since if $x \in \left(\frac{a_{k-1}+ma_k}{b_{k-1}+mb_k}, \frac{a_{k-1}+(m+1)a_k}{b_{k-1}+(m+1)b_k}\right)$ it has exactly $n+m$ common edges with the end $\frac{a_k}{b_k}$.

Now assume that $x \in \left(\frac{a_k}{b_k}, \frac{a_k+a_{k+1}}{b_k+b_{k+1}}\right)$. By the similar reason with the previous interval we will divise this interval as following :

$$\left(\frac{a_k}{b_k}, \frac{a_k+a_{k+1}}{b_k+b_{k+1}}\right) = \bigcup_{i=1}^{\infty} \left(\frac{(i+1)a_k+a_{k+1}}{(i+1)b_k+b_{k+1}}, \frac{ia_k+a_{k+1}}{ib_k+b_{k+1}}\right).$$

If $x$ has exactly $n+m$ common edges with $\frac{a_k}{b_k}$ then $x \in \left(\frac{(m+1)a_k+a_{k+1}}{(m+1)b_k+b_{k+1}}, \frac{ma_k+a_{k+1}}{mb_k+b_{k+1}}\right)$. And in this case we have the following values of the transition map:

- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\frac{a_{k+1}+ma_k}{b_{k+1}+mb_k}) = \frac{0}{1}$

- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\frac{a_{k+1}+(m+1)a_k}{b_{k+1}+(m+1)b_k}) = \frac{1}{1}$

- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\frac{2a_{k+1}+(2m+1)a_k}{2b_{k+1}+(2m+1)b_k} = \frac{1}{2}$

Then we find the inverse branch:

$$\frac{dy-b}{a-cy} = \frac{a_k y + ma_k + a_{k+1}}{b_k y + mb_k + b_{k+1}}.$$

Finally if $x \in \left(\dfrac{a_k+a_{k+1}}{b_k+b_{k+1}}, \dfrac{a_{k+1}}{b_{k+1}}\right)$.

Then $x$ has exactly $n$ common edges with the end $\dfrac{a_k}{b_k}$. Then

- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{a_{k+1}}{b_{k+1}}) = \dfrac{0}{1}$
- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{a_{k+1}+a_k}{b_{k+1}+b_k}) = \dfrac{1}{1}$
- $\mathbb{T}_{\frac{a_k}{b_k}^+ \vee \frac{a_k}{b_k}^-}(\dfrac{2a_{k+1}+a_k}{2b_{k+1}+b_k} = \dfrac{1}{2}$

After similar calculations the inverse branch of the transition map is given by

$$\frac{dy-b}{a-cy} = \frac{a_k y + a_{k+1}}{b_k y + b_{k+1}}.$$

$V(\mathcal{F}_n) = \left\{\frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots, \frac{a_{2^{n+1}}}{b_{2^{n+1}}}\right\}$. $V(\mathcal{F}_n)/V(\mathcal{F}_{n-1}) = \{\frac{a_1}{b_1}, \frac{a_3}{b_3}, \cdots, \frac{a_{2^{n+1}-1}}{b_{2^{n+1}-1}}\}$ Then the equation $f(y) - f(y+1)$ for the map $\mathbb{T}_{\mathbb{X}_n}$ is given as follows:

$$\begin{aligned} f(y) - f(y+1) = \sum_{k=0}^{2^n-1} & \frac{1}{(b_{2k+1}y + b_{2k})^2} f(\frac{a_{2k+1}y + a_{2k}}{b_{2k+1}y + b_{2k}}) \\ & + \frac{1}{(b_{2k+1}y + b_{2k+2})^2} f(\frac{a_{2k+1}y + a_{2k+2}}{b_{2k+1}y + b_{2k+2}}) \end{aligned} \tag{5.9}$$

We showed that $f(y) = \frac{1}{y}$ is the density of an invariant measure of $\mathbb{T}_{\mathbb{X}_0}$. Assume that $f(y) = \frac{1}{y}$ satisfies the equation 5.9 then we obtain

$$\frac{1}{y(y+1)} = \sum_{k=0}^{2^n-1} \frac{1}{(a_{2k+1}y + a_{2k})(b_{2k+1}y + b_{2k})} + \frac{1}{(a_{2k+1}y + a_{2k+2})(b_{2k+1}y + b_{2k+2})}$$

By the previous lemmas 5.2.1, 5.2.2 and 5.2.3 $V(\mathcal{F}_{n+1})$ can be given as follows:

$$\begin{aligned} V(\mathcal{F}_{n+1}) = \{ & \frac{a_0}{b_0}, \frac{a_1}{b_1+b_0}, \frac{a_2}{b_1}, \frac{a_3}{b_1+b_2}, \cdots, \frac{a_{2^{n+1}-1}}{b_{2^n-1}+b_{2^n}}, \frac{a_{2^{n+1}}}{b_{2^n}}, \\ & \frac{b_{2^{n+1}-1}}{b_{2^n-1}+b_{2^n}}, \frac{b_{2^{n+1}-2}}{b_{2^n-1}}, \cdots, \frac{b_3}{b_1+b_2}, \frac{b_2}{b_1}, \frac{b_1}{b_1+b_0}, \frac{b_0}{b_0}\} \end{aligned}.$$

$V(\mathcal{F}_{n+1})/V(\mathcal{F}_n) = \{\frac{a_1}{b_1+b_0}, \frac{a_3}{b_2+b_1}, \frac{a_5}{b_3+b_2}, \cdots, \frac{a_{2^{n+1}-1}}{b_{2^n-1}+b_{2^n}}, \frac{b_{2^{n+1}-1}}{b_{2^n-1}+b_{2^n}}, \cdots, \frac{b_3}{b_1+b_2}, \frac{b_1}{b_1+b_0}\}$.
Then the equation $f(y) - f(y+1)$ for the map $\mathbb{T}_{X_{n+1}}$ is given as follows:

$$f(y)-f(y+1)=\sum_{k=0}^{2^n-1}\frac{1}{((b_k+b_{k+1})y+b_k)^2}f(\frac{a_{2k+1}y+a_{2k}}{(b_k+b_{k+1})y+b_k})$$
$$+\frac{1}{((b_k+b_{k+1})y+b_{k+1})^2}f(\frac{a_{2k+1}y+a_{2k+2}}{(b_k+b_{k+1})y+b_{k+1}})$$
$$+\frac{1}{((b_k+b_{k+1})y+b_k)^2}f(\frac{b_{2k+1}y+b_{2k}}{(b_k+b_{k+1})y+b_k})$$
$$+\frac{1}{((b_k+b_{k+1})y+b_{k+1})^2}f(\frac{b_{2k+1}y+b_{2k+2}}{(b_k+b_{k+1})y+b_{k+1}})$$

Let $f(y)=\frac{1}{y}$. The right side of the equality is given as follows:

$$\sum_{k=0}^{2^n-1}\frac{1}{((b_k+b_{k+1})y+b_k)(a_{2k+1}y+a_{2k})}+\frac{1}{((b_k+b_{k+1})y+b_{k+1})(a_{2k+1}y+a_{2k+2})}$$
$$+\frac{1}{((b_k+b_{k+1})y+b_k)(b_{2k+1}y+b_{2k})}+\frac{1}{((b_k+b_{k+1})y+b_{k+1})(b_{2k+1}y+b_{2k+2})}$$
$$=\sum_{k=0}^{2^n-1}\frac{(a_{2k+1}+b_{2k+1})y+a_{2k}+b_{2k}}{((b_k+b_{k+1})y+b_k)(a_{2k+1}y+a_{2k})(b_{2k+1}y+b_{2k})}+$$
$$\frac{(a_{2k+1}+b_{2k+1})y+a_{2k+2}+b_{2k+2}}{((b_k+b_{k+1})y+b_{k+1})(a_{2k+1}y+a_{2k+2})(b_{2k+1}y+b_{2k+2})}$$

By the Lemma 5.2.1 $\frac{b_{2k+1}}{b_k+b_{k+1}}+\frac{a_{2k+1}}{b_k+b_{k+1}}=1$ then $b_{2k+1}+a_{2k+1}=b_k+b_{k+1}$ and $\frac{a_{2k}}{b_k}+\frac{b_{2k}}{b_k}=1$ then $a_{2k}+b_{2k}=b_k$. Then we have the following equality:

$$\sum_{k=0}^{2^n-1}\frac{(a_{2k+1}+b_{2k+1})y+a_{2k}+b_{2k}}{((b_k+b_{k+1})y+b_k)(a_{2k+1}y+a_{2k})(b_{2k+1}y+b_{2k})}+$$
$$\frac{(a_{2k+1}+b_{2k+1})y+a_{2k+2}+b_{2k+2}}{((b_k+b_{k+1})y+b_{k+1})(a_{2k+1}y+a_{2k+2})(b_{2k+1}y+b_{2k+2})}$$
$$=\sum_{k=0}^{2^n-1}\frac{(b_k+b_{k+1})y+b_k}{((b_k+b_{k+1})y+b_k)(a_{2k+1}y+a_{2k})(b_{2k+1}y+b_{2k})}+$$
$$\frac{(b_k+b_{k+1})y+b_{k+1}}{((b_k+b_{k+1})y+b_{k+1})(a_{2k+1}y+a_{2k+2})(b_{2k+1}y+b_{2k+2})}$$
$$=\sum_{k=0}^{2^n-1}\frac{1}{(a_{2k+1}y+a_{2k})(b_{2k+1}y+b_{2k})}+\frac{1}{(a_{2k+1}y+a_{2k+2})(b_{2k+1}y+b_{2k+2})}$$

So by the Equation 5.9 we have

$$\sum_{k=0}^{2^n-1}\frac{1}{(a_{2k+1}y+a_{2k})(b_{2k+1}y+b_{2k})}+\frac{1}{(a_{2k+1}y+a_{2k+2})(b_{2k+1}y+b_{2k+2})}=\frac{1}{y(y+1)}.$$

So $f(y)=\frac{1}{y}$ satisfies all the equalities $f(y)-f(y+1)$ for our general map if $f$ is an analytic funtion. As a result we can say that $f(y) = \frac{1}{y}$ is the density of an invariant measure under these special case of a generalization of continued fraction maps.

# 6 ASSOCIATED POWER SERIES OF THE CONTINUED FRACTIONS

This chapter is not directly related to the other chapters in this thesis. Our motivation is to associate a power series to continued fractions, but we want to this power series has some proprieties such as continuity and convergence.

The modular group $PSL(2,\mathbb{Z})$ is the group of 2X2 matrices which of the coefficients are integers and the operation is matrix multiplication. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an element of $PSL(2,\mathbb{Z})$, it represents a linear fractional transformation such that $z \mapsto \frac{a\cdot z+b}{c\cdot z+d}$. The group which occurs by these linear transformations and composition operation is called modular group. Moreover we can generate this modular group by the linear transformations $T : z \mapsto 1+z$ and $S : z \mapsto -\frac{1}{z}$ such that $S^2 = (TS)^3 = Id$. So apparently we can say that $PSL(2,\mathbb{Z}) = \langle S,T|S^2 = (TS)^3 = Id\rangle$. Let $L := TS$.
There is a natural homomorphism between the group $PSL(2,\mathbb{Z})$ and real numbers such that

$$\gamma : PSL(2,\mathbb{Z}) \longrightarrow \mathbb{R}^+ \cup \mathbb{R}^-$$

$$(LS)^{n_1}(L^2S)^{n_2}(LS)^{n_3}\ldots \longmapsto [n_1,n_2,n_3,\cdots]$$

$$S(LS)^{n_1}(L^2S)^{n_2}(LS)^{n_3}\ldots \longmapsto [-n_1-1,1,n_2-1,n_3,n_4\cdots].$$

such that $LS = 1+z$ and $L^2S = \cfrac{1}{1+\cfrac{1}{z}}$.

It is clear that $(LS)^n(z) = n+z$ for any $n \in \mathbb{Z}$. We know that

$(L^2S)(z) = \dfrac{1}{1+\dfrac{1}{z}}$. Assume that $(L^2S)^{n-1}(z) = \dfrac{1}{n-1+\dfrac{1}{z}}$ then by induction

$$\begin{aligned}(L^2S)^n(z) &= L^2S(L^2S)^{n-1}(z)\\ &= \cfrac{1}{1+\cfrac{1}{\cfrac{1}{n-1+\cfrac{1}{z}}}} = \cfrac{1}{n+\cfrac{1}{z}}.\end{aligned}$$

So we obtain the following equality

$$\begin{aligned}(LS)^{n_1}(L^2S)^{n_2}(LS)^{n_3}(z) &= n_1 + \cfrac{1}{n_2+\cfrac{1}{n_3+z}}\\ &= [n_1,n_2,n_3]\end{aligned} \tag{6.1}$$

We can generalize this result for infinite continued fraction expansions. Moreover if we multiply by $S$ we can obtain negative real numbers such that

$$S(LS)^{n_1}(L^2S)^{n_2}(LS)^{n_3}(z) = -n_1 + \cfrac{1}{n_2+\cfrac{1}{n_3+z}} = [-n_1-1,1,n_2-1,n_3,\ldots]. \tag{6.2}$$

## 6.1 Some Power Series Candidates

Now our aim is to come up with a well defined power series for each continued fraction expansion of reel numbers. First of all finding for $\mathbb{R}\cap(0,1)$ is adequate, after we will enhance and generalize for all real numbers.

Former, basicly for $x = [0,n_1,n_2,\ldots]$ we can suggest the power series $h_x(t) = t^{n_1} + t^{n_1+n_2} + t^{n_1+n_2+n_3} + \cdots$ for any $t$. But it is obvious that this series is not well defined, we will explain the reason by a simple example. Let $x = \frac{1}{2}$ then it can be represented by two continued fractions but actually they are the same, $[0,1,1,\infty] = [0,2,\infty]$. Our expactation is $h_x(t)$ should be the same power series for two expansion but it is not: for $[0,1,1,\infty]$ $h_x(t) = t^1 + t^2$ whereas for $[0,2,\infty]$, $h_x(t) = t^2$.

Latter for $x = [0,n_1,n_2,\ldots]$ we put forward the power series $g_x(t)$ which is ordinary generating function of the sequence $\underbrace{0,0,\ldots,0}_{n_1\text{-times}},\underbrace{1,1,\ldots,1}_{n_2\text{-times}},\underbrace{0,0,\ldots,0}_{n_3\text{-times}}\ldots$ In this case

$$g_x(t)=\sum_{k=0}^{\infty}a_kt^k=t^{n_1}+t^{n_1+1}+t^{n_1+2}++t^{n_1+n_2-1}+t^{n_1+n_2+n_3}+t^{n_1+n_2+n_3+1}+$$
$$\cdots+t^{n_1+n_2+n_3+n_4-1}+t^{n_1+\cdots n_5}+\cdots+t^{n_1+\cdots n_6-1}+\cdots$$

Or if we write compactly $g_x(t)=\sum_{k=1}^{\infty}t^{n_1+n_2+\cdots n_{2k-1}}\dfrac{1-t^{n_2k}}{1-t}$. So we can say that $g_x(t)<\sum_{k=0}^{\infty}t^k$ for any $t$. And we know that this series is geometric and it is convergent for $|t|<1$. Then by comparison test $g_x(t)$ is also convergent for $|t|<1$. But this power series is not well defined like previous power series. We can control this result by calculating the power series of the same rational number $\frac{1}{2}$. Hence $g_x(t)=t$ for the premier continued fraction expansion $[0,1,1,\infty]$ whereas $g_x(t)=\frac{t^2}{1-t}$ for $x=[0,2,\infty]$ and they are not equal.

## 6.2 A Well Defined Power Series

**Proposition 6.2.1.** The series $f_x(t)=1+t+t^2+\cdots t^{n_1-1}+t^{n_1}(1+s+s^2+\cdots s^{n_2-1})+t^{n_1}s^{n_2}(1+t+t^2+\cdots t^{n_3-1})+t^{n_1}s^{n_2}t^{n_3}(1+s+\cdots s^{n_4-1})+\cdots$ is a well defined power series for $x=[0,n_1,n_2,n_3,n_4\cdots]$ with $n_i\in\mathbb{Z}\setminus\{0\}$ and $s=t-1$.

*Proof.* Let $q^+=[0,n_1,n_2,\ldots,n_k-1,1,\infty]$ and $q^-=[0,n_1,n_2,\ldots,n_k,\infty]$. We must show that $f_{q^+}(t)=f_{q^-}(t)$ for any $t$.
$f_{q^+}(t)=1+t+\cdots t^{n_1-1}+t^{n_1}(1+s+\cdots+s^{n_2-1})+\cdots+t^{n_1+\cdots n_{k-2}}s^{n_2+\cdots+n_{k-1}}(1+t+t^2+\cdots+t^{n_k-2})+t^{n_1+n_3+\cdots n_{k-2}+n_k-1}s^{n_2+n_4+\cdots+n_{k-1}}$.
So if we calculate $f_{q^-}(t)$ we will observe that $f_{q^+}(t)=f_{q^-}(t)+t^{n_1+\cdots n_{k-2}}s^{n_2+\cdots+n_{k-1}}(1+t+\cdots+t^{n_k-2})+t^{n_1+n_3+\cdots+n_k-1}s^{n_2+n_4+\cdots+n_{k-1}}-t^{n_1+n_3+\cdots n_{k-2}}s^{n_2+n_4+\cdots+n_{k-1}}(1+t+\cdots+t^{n_k-1})$. Then it is obvious that $f_{q^+}(t)=f_{q^-}(t)$ for any $t$.

**Example 6.2.1.**

$$f_0(t)=\sum_{k=0}^{\infty}t^k=\frac{1}{1-t}\text{ for }|t|<1.$$

$$f_1(t)=1+t\sum_{k=0}^{\infty}s^k=1+t\frac{1}{1-s}=2\text{ for }|s|<1$$

.

$$f_{\frac{1}{2}}(t)=1+t+t^2(\sum_{k=0}^{\infty}s^k)=1+t+t^2(\frac{1}{1-s})=1+2t\text{ for }|s|<1$$

.

*Let* $x = [0, 1, 1, \cdots]$

$$\begin{aligned} f_x(t) &= 1 + t + ts + t^2 s + t^2 s^2 + t^3 s^2 + t^3 s^3 + \cdots \\ &= \sum_{k=0}^{\infty} (ts)^k + t \sum_{i=0}^{\infty} (ts)^i \\ &= \frac{1}{1-ts} + \frac{t}{1-ts} \frac{1+t}{1-t+t^2} \end{aligned}$$

**Lemma 6.2.1.** *For* $x = [0, \overline{n}]$, $f_x(t) = \dfrac{1}{1-(ts)^n}\left(\dfrac{1-t^n}{1-t} + t^n \dfrac{1-s^n}{1-s}\right)$ *with* $|t| < 1$.

*Proof.*

$$\begin{aligned} f_x(t) &= 1 + t + \cdots + t^{n-1} + t^n (1 + s + \cdots s^{n-1}) + t^n s^n (1 + t + \cdots + t^{n-1}) + \cdots \\ &= (1 + t + \cdots + t^{n-1})(1 + (ts)^n + (ts)^{2n} + (ts)^{3n} + \cdots) + \\ &\quad t^n (1 + s + \cdots s^{n-1})(1 + (ts)^n + (ts)^{2n} + (ts)^{3n} + \cdots) \\ &= \frac{1}{1-(ts)^n}\left(\frac{1-t^n}{1-t} + t^n \frac{1-s^n}{1-s}\right) \text{ with } |t| < 1. \end{aligned}$$

**Proposition 6.2.2.** Let $x = [0, \overline{n_1, n_2, \cdots, n_k}]$ with k is an even number then

$$\begin{aligned} f_x(t) = \frac{1}{1 - t^{n_1} s^{n_2} t^{n_3} \cdots s^{n_k}} \Bigg( & \frac{1-t^{n_1}}{1-t} + t^{n_1} \frac{1-s^{n_2}}{1-s} + t^{n_1} s^{n_2} \frac{1-t^{n_3}}{1-t} \\ & + \cdots + t^{n_1} s^{n_2} \cdots t^{n_{k-1}} \frac{1-s^{n_k}}{1-s} \Bigg) \end{aligned}$$

.

*Proof.*

$$
\begin{aligned}
f_x(t) &= 1+t+\cdots t^{n_1-1}+t^{n_1}(1+s+\cdots+s^{n_2-1})+t^{n_1}s^{n_2}(1+t+\cdots+t^{n_3-1})+\\
&t^{n_1+n_3}s^{n_2}(1+s+\cdots+s^{n_4-1})+\cdots+t^{n_1+n_3+\cdots+n_{k-1}}s^{n_2+\cdots+n_k}(1+t+\cdots+t^{n_1-1})+\\
&t^{n_1+n_3+\cdots+n_{k-1}}s^{n_2+\cdots+n_k}t^{n_1}(1+s+\cdots+s^{n_2-1})+\cdots\\
&=(1+t+\cdots+t^{n_1-1})(1+t^{n_1+\cdots+n_{k-1}}s^{n_2+\cdots+n_k}+(t^{n_1+\cdots+n_{k-1}}s^{n_2+\cdots+n_k})^2+\\
&(t^{n_1+\cdots+n_{k-1}}s^{n_2+\cdots+n_k})^3+\cdots)+(1+s+\cdots+s^{n_2-1})t^{n_1}(1+t^{n_1+\cdots+n_{k-1}}s^{n_2+\cdots+n_k}+\\
&(t^{n_1+\cdots+n_{k-1}}s^{n_2+\cdots+n_k})^2+(t^{n_1+\cdots+n_{k-1}}s^{n_2+3\cdots+n_k})^3+\cdots)+\\
&(1+t+\cdots+t^{n_3-1})t^{n_1}s^{n_2}(1+t^{n_1+\cdots+n_{k-1}}s^{n_2+\cdots+n_k}+(t^{n_1+\cdots+n_{k-1}}s^{n_2+\cdots+n_k})^2+\cdots)\\
&=\frac{1}{1-t^{n_1}s^{n_2}+\cdots s^{n_k}}\left(\frac{1-t^{n_1}}{1-t}+t^{n_1}\frac{1-s^{n_2}}{1-s}+t^{n_1}s^{n_2}\frac{1-t^{n_3}}{1-t}+\cdots+\right.\\
&\left.t^{n_1}s^{n_2}\cdots t^{n_{k-1}}\frac{1-s^{n_k}}{1-s}\right).
\end{aligned}
$$

Now we fix $x=[0,n_1,n_2,\cdots]$.

**Proposition 6.2.3.** We have the following functional equation for any $n$.

$$f_{\frac{1}{n+x}}(t)=\frac{1-t^n}{1-t}+t^n f_x(s).$$

*Proof.* $\dfrac{1}{n+x}=\cfrac{1}{n+\cfrac{1}{n_1+\cfrac{1}{n_2+\cfrac{1}{\vdots}}}}=[0,n,n_1,n_2,\cdots].$

So
$f_{\frac{1}{n+x}}(t)=1+t+t^2+\cdots t^{n-1}+t^n(1+s+s^2+\cdots+s^{n_1-1})+t^ns^{n_1}(1+t+t^2+\cdots+t^{n_2-1})+\cdots)=\dfrac{1-t^n}{1-t}t^n(1+s+s^2+\cdots+s^{n_1-1}+s^{n_1}(1+t+t^2+\cdots t^{n_2-1})+\cdots=\dfrac{1-t^n}{1-t}t^n\cdot f_x(s).$

*Corollary* 6.2.1. The following equalities are provided

$$f_{\frac{1}{1+x}}(t)=1+tf_x(s)$$

$$f_{\frac{1}{2+x}}(t)=1+t+t^2f_x(s)$$

*Corollary* 6.2.2. For $n = 0$ we find $f(\frac{1}{x}, t) = f(x, 1-t) = f(x, s)$ then we can say that the series $f(x,t)$ is defined for $x \in (0, \infty) \cap \mathbb{R}$.

**Proposition 6.2.4.** The following equation satisfies for any $n \in \mathbb{N} \cup \{0\}$

$$f\left(\frac{x}{nx+1}, t\right) = \frac{1-t^n}{1-t} + t^n \cdot f(x,t)$$

*Proof.* $$\frac{x}{nx+1} = \frac{1}{\frac{nx+1}{x}} = \frac{1}{n + \frac{1}{x}} = \frac{1}{n + n_1 + \frac{1}{n_2 + \frac{1}{\vdots}}} = [0, n+n_1, n_2, \cdots]$$

So

$$\begin{aligned} f(\frac{x}{nx+1}, t) &= 1 + t + \cdots t^{n+n_1-1} + t^{n+n_1}(1 + s + \cdots s^{n_2}) + t^{n+n_1}s^{n_2}(1 + t + \cdots t^{n_3-1}) + \cdots \\ &= t^n f(x,t) + 1 + t + t^2 + \cdots + t^{n-1} \\ &= \frac{1-t^n}{1-t} + t^n \cdot f(x,t) \end{aligned}$$

**Proposition 6.2.5.** For any $x \in (0,1)$, $f(x, \frac{1}{2}) = 2$.

*Proof.* Let $x = [0, n_1, n_2, \cdots]$ with $n_i \in \mathbb{Z}$. We know $s = 1 - t = \frac{1}{2}$. Then

$$\begin{aligned} f(x, \frac{1}{2}) &= 1 + \frac{1}{2} + (\frac{1}{2})^2 + \cdots + (\frac{1}{2})^{n_1-1} + \frac{1}{2}^{n_1}(1 + \frac{1}{2} + \cdots \frac{1}{2}^{n_2-1}) + \cdots \\ &= \sum_{i=0}^{\infty} (\frac{1}{2})^i \\ &= 2 \end{aligned}$$

**Proposition 6.2.6.** $f(x,t)$ is a convergent power series where $t \in (0,1) \cap \mathbb{R}$.

*Proof.* Assume that $t \in (0,1)$ since $s = 1-t$, $s \in (0,1)$. We will try to write the series in a more appropriate form to see that it is convergent:

$$
\begin{aligned}
f_x(t) &= \frac{1-t^{n_1}}{1-t} + t^{n_1}\frac{1-s^{n_2}}{1-s} + t^{n_1}s^{n_2}\frac{1-t^{n_3}}{1-t} + t^{n_1}s^{n_2}t^{n_3}\frac{1-s^{n_4}}{1-s} + t^{n_1}s^{n_2}t^{n_3}s^{n_4}\frac{1-t^{n_5}}{1-t} \\
&+ t^{n_1}s^{n_2}t^{n_3}s^{n_4}t^{n_5}\frac{1-s^{n_6}}{1-s} + t^{n_1}s^{n_2}t^{n_3}s^{n_4}t^{n_5}s^{n_6}\frac{1-t^{n_7}}{1-t}\cdots \\
&= \frac{1-t^{n_1}}{1-t} + t^{n_1-1}(1-s^{n_2}) + t^{n_1}s^{n_2-1}(1-t^{n_3}) + t^{n_1}s^{n_2}t^{n_3-1}(1-s^{n_4}) \\
&+ t^{n_1}s^{n_2}t^{n_3}s^{n_4-1}(1-t^{n_5}) + t^{n_1}s^{n_2}t^{n_3}s^{n_4}t^{n_5-1}(1-s^{n_6}) + t^{n_1}s^{n_2}\cdots s^{n_6-1}(1-t^{n_7})\cdots \\
&= \frac{1-t^{n_1}}{1-t} + t^{n_1-1} + t^{n_1-1}s^{n_2-1}(t-s) + t^{n_1}s^{n_2-1}t^{n_3-1}(s-t) \\
&+ t^{n_1}s^{n_2}t^{n_3-1}s^{n_4-1}(t-s) + t^{n_1}s^{n_2}t^{n_3}s^{n_4-1}t^{n_5-1}(s-t) \\
&+ t^{n_1}s^{n_2}t^{n_3}s^{n_4}t^{n_5-1}s^{n_6-1}(t-s) + t^{n_1}s^{n_2}t^{n_3}s^{n_4}t^{n_5}s^{n_6-1}t^{n_7-1}(s-t) + \cdots \\
&= \frac{1-t^{n_1}}{1-t} + t^{n_1} + (s-t)(-t^{n_1-1}s^{n_2-1} + t^{n_1}s^{n_2-1}t^{n_3-1} - t^{n_1}s^{n_2}t^{n_3-1}s^{n_4-1} \\
&+ t^{n_1}s^{n_2}t^{n_3}s^{n_4-1}t^{n_5-1} - t^{n_1}s^{n_2}t^{n_3}s^{n_4}t^{n_5-1}s^{n_6-1} + t^{n_1}s^{n_2}t^{n_3}s^{n_4}t^{n_5}s^{n_6-1}t^{n_7-1}) - \cdots \\
&< \frac{1-t^{n_1}}{1-t} + t^{n_1} + (s-t)\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} t^j\right)s^i \\
&= 1+t+t^2+\cdots+2t^{n_1-1} + (s-t)\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} t^j\right)s^i.
\end{aligned}
$$

When $t \in (0,1)$ we obtain $\sum_{j=0}^{\infty} t^j = \dfrac{1}{1-t}$.
So we have

$$
\begin{aligned}
1+t+t^2+\cdots+2t^{n_1-1} + (s-t)\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} t^j\right)s^i \\
= 1+t+t^2+\cdots+2t^{n_1-1} + \frac{s-t}{1-t}\sum_{i=0}^{\infty} s^i
\end{aligned}
$$

If $t \in (0,1)$ then $s \in (0,1)$. So $\sum_{i=0}^{\infty} s^i = \dfrac{1}{1-s}$.
Then we reach

$$
\begin{aligned}
1+t+t^2+\cdots+2t^{n_1-1} + \frac{s-t}{1-t}\sum_{i=0}^{\infty} s^i \\
= 1+t+t^2+\cdots+2t^{n_1-1} + \frac{s-t}{st}
\end{aligned}
$$

It is a finite sum when $n_1 < \infty$. So $\dfrac{1-t^{n_1}}{1-t} + t^{n_1} + (s-t)\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} t^j\right)s^i$ converges.

Then $f_x(t)$ is convergent by comparison test. If $n_1 = \infty$ since $|t| < 1$ the sum $1+t+t^2+\cdots+2t^{n_1-1}$ converges. By the same reason $f_x(t)$ is convergent.

**Proposition 6.2.7.** $f(-x,t) = \frac{t}{s^2}\Big(f(x,s)-1\Big)$ for $x \in (0,1) \cap \mathbb{R}$.

*Proof.* If $x = [0,n_1,n_2,n_3,\ldots]$ then $-x = [-1,1,n_1-1,n_2,\ldots]$. So $\frac{-1}{x} = [0,-1,1,n_1-1,n_2,\ldots]$ by the corolloary 6.2.2.

$$\begin{aligned}
f\left(\frac{-1}{x},t\right) &= t^{-1}s(1+t+t^2+t^{n_1-2}) + t^{-1}st^{-n_1-1}(1+s+\cdots+s^{n_2-1})+ \\
& t^{-1}st^{n_1-1}s^{n_2}(1+t+\cdots+t^{n_3-1})+\cdots \\
&= \frac{s}{t}(1+t+t^2+\cdots+t^{n_1-2}+t^{n_1-1}(1+s+s^2+\cdots+s^{n_2-1}) \\
&+t^{n_1-1}s^{n_2}(1+t+\cdots+t^{n_3-1}))+\cdots \\
&= \frac{s}{t^2}(t+t^2+\cdots+t^{n_1-1}+t^{n_1}(1+s+s^2+\cdots+s^{n_2-1}) \\
&+t^{n_1}s^{n_2}(1+t+\cdots+t^{n_3-1}))+\cdots \\
&= \frac{s}{t^2}(f(x,t)-1).
\end{aligned} \tag{6.3}$$

So $f(\frac{-1}{x},s) = \frac{t}{s^2}(f(x,s)-1) = f(-x,t)$ by the corollary 6.2.2.

### 6.2.1 Conclusion for The Chapter 6

We said that at the equality 6.1 every continued fraction expansion can be identified by a multiplication of the elements of the modular group $\mathrm{PSL}_2\mathbb{Z}$. And also we associate every continued fraction expansion by the power series $f(x,t)$. Now we can associate the power series $f(x,t)$ with $\mathrm{PSL}_2\mathbb{Z}$ by the following way:

- $X := LS = 1+z \longleftrightarrow X_t : p(t) \mapsto 1+sp(t)$.
- $Y := L^2S = \dfrac{1}{1+\frac{1}{z}} \longleftrightarrow Y_t : p(t) \mapsto 1+tp(t)$
- $U := \frac{1}{z} \longleftrightarrow U_t : p(t) \mapsto p(s)$.

where the maps $X_t, Y_t, U_t$ defines on $\mathbb{C}[t]$ which is the ring of the formal power series with complex coefficients.
So for $x = [0,n_1,n_2,\cdots]$

$$\begin{aligned}
f(x,t) &= 1+t+t^2+\cdots t^{n_1-1}+t^{n_1}(1+s+\cdots+s^{n_2-1})+t^{n_1}s^{n_2}(1+t+\cdots+t^{n_3-1})+\cdots \\
&= {Y_t}^{n_1}{X_t}^{n_2}{Y_t}^{n_3}\cdots
\end{aligned}.$$

But for the power series which associate to negative numbers can not be associate to these maps.

In this study we couldn't find any other power series such that series expansion of negative numbers associate to these maps. It can be tried in the other works.

# 7 CONCLUSION

In this thesis we studied on the trees especially the Farey Tree. We defined the topology on the boundary of a tree. We showed that if a tree is perfect then its topology is homeomorphic to the Cantor Set. Furthermore, we gave an equivalence relation on the boundary of a perfect tree and we showed that the quotient topology is homeomorphic to the unit circle. Secondly, we studied the Borel measures on the tree, we give some of them. Then we studied the automorphism groups of Farey Tree whose edges can be described by the generators of modular group $\mathrm{PSL}_2(\mathbb{Z})$ in a nice way. Finally, we generalized the Gauss map and the Fibonacci map which is called the continued fraction maps and we examined the dynamics of this map such that we tried to find an invariant measure under this map. At final, we gave a more generalization of this map and for a special case we found the density of an invariant measure under the continued fraction maps, and we proved it.

In this study, we focused on the Farey tree. For other special trees or all trees, automorphism groups can be observed or the measure on the boundary of these trees can be studied. We know every vertex is the degree 3 in Farey tree (except the root vertex) so we define the probability $\pi$ function by two siblings. If there were more than 2 we could modify this function and we could try to give another special measures on the boundary of such trees.

Furthermore, we found an invariant measure under the more general continued fraction map but we choose the paths conveniently. We could try to find an invariant measure for more general cases.

# REFERENCES

[1] Bates, B. P., Bunder, M. W. and Tognetti, K.P. (2010). ”Linking the Calkin–Wilf and Stern–Brocot trees.” European Journal of Combinatorics 31.7 : 1637-1661.

[2] Bates, B. P., Bunder, M. W. and Tognetti, K.P. ”Linkages between the Gauss map and the Stern-Brocot tree.” (2006). Acta Mathematica Academiae Paedagogicae Nyíregyháziensis22.3: 217-235.

[3] Bestvina, M. (2002) ”R-trees in topology, geometry, and group theory.” Handbook of geometric topology : 55-91.

[4] Brouwer, L. E. J. (1910) On the structure of perfect sets of points.,Proc. Koninklijke Akademie van Wetenschappen, 12:785–794

[5] Brunner, A. M., and Sidki, S. (1997). ”On the automorphism group of the one-rooted binary tree.” Journal of Algebra 195.2 : 465-486.

[6] Burger, M. and Mozes, S. (2000). ”Groups acting on trees: from local to global structure.” Publications Mathématiques de l'IHÉS 92 113-150.

[7] Diestel, R. Graph Theory, Springer-Verlag Heidelberg New York 1997, 2000, 2005, pg:1-16

[8] Einsiedler, M. and Ward, T. Ergodic Theory: With a View Towards Number Theory, Springer, 13-21.

[9] Alkauskas, G. The Minkowski question mark function: explicit series for the dyadic period function and moments arXiv:0805.1717 [math.NT], 86 pages

[10] Graham, R.E. and Knuth, D.E. and Patashnik, O. (1994). Concrete Mathematics, A Foundation for Computer Science, second ed., Addison-Wesley, Reading, MA, 1994

[11] Grigorchuk, R. I., Nekrashevich, V. V. and Sushchanskii, V. I. (2000). ”Automata, dynamical systems, and groups.” Trudy Matematicheskogo Instituta Imeni VA Steklova 231 (2000): 134-214.

[12] Hocking, J. G. and Young, G. S. (1961). Topology, Addison-Wesley Publishing Company Inc., pg: 55, theorem: 2.28

[13] Iosifescu, M. and Kraaikamp, C. (2013). Metrical theory of continued fractions. Vol. 547. Springer Science and Business Media, 2013.

[14] Khinchin, A. Ya. (1997). Continued Fractions,. Dover Publications,(reproduction of 1964 english translation of the original 1935 russian edition) edition, 1997

[15] Pemantle, R. (1992). ”Automorphism invariant measures on trees.” The Annals of Probability : 1549-1566.

[16] Serre, J. P. and Bass, H. (1977). Arbres, amalgames, SL2. Paris: Société mathématique de France, 1977.

[17] Uludağ, M. and Zeytin, A. and Durmuş, M. (2012). Binary quadratic forms as dessins, arXiv:1508.01677 [math.NT].

[18] Uludağ, M. and Ayral, H. (2016). A subtle symmetry of Lebesgue's measure, arXiv :1605.07330 [math.NT].

[19] Uludağ, M. and Ayral, H. (2017). Dynamics of a family of continued fraction maps, arXiv:1704.06912v1 [math.DS].

[20] Quansheng, L. and Rouault, A.,(1997). On Two Measures Defined on the Boundary of a Branching Tree,Springer

[21] Vepstas, L.(2008). The gauss-kuzmin-wirsing operator. http://linas.org/math/gkw.pdf, Oct 2008.

[22] Vepstas, L. (2008). On the Minkowski Measure. arXiv:0810.1265, Oct 2008.

[23] Zeytin, A. and Uludağ, M. (2017). Boundaries of Planar Trees, unpublished manuscript

# Appendix

More informations about continued fractions:

**Proposition 7.0.1.** Every finite continued fraction has a unique canonical representation. ie. every continued fraction can be presented by simple fraction.

*Proof.* Let $[a_0] = a_0 = \frac{a_0}{1}$. We will prove it by induction. Assume that it satisfies for the $(n-1)^{th}$ order continued fraction. Let $a = [a_0, a_1, \cdots, a_n]$ be finite continued fraction. The **order** of $a$ is $n$. Let $r_1 = [a_1, a_2, \cdots, a_n]$ then we have $a = a_0 + \dfrac{1}{r_1}$. Now the order of $r_1$ is $n-1$. So $r_1 = \frac{p'}{q'}$ for some $p, q \in \mathbb{N}$. Then $a = \frac{a_0 p' + q'}{p'}$. So $a$ can be represented by simple fraction. If $a = [a_0, a_1, \cdots, a_n] = \frac{p}{q}$. We reach the following equation:

$$p = a_0 p' + q' \text{ and } q = p' \tag{7.1}$$

So the canonical representation of $a$ is unique.

Let $\alpha = [a_0, a_1, \cdots]$ be a finite or infinite continued fraction we can give the canonical representation of the segment $S_k = [a_0, a_1, \cdots, a_k]$ by $\frac{p_k}{q_k}$. $S_k$ is called the $k^{th}$ order convergent of the continued fraction. So if $a$ is finite it has finite number of convergents.

**Theorem 7.0.1.** *We have the rule of the convergents for $k \geq 2$.*

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2} \\ q_k &= a_k q_{k-1} + q_{k-2} \end{aligned} \tag{7.2}$$

*Proof.* This equation is satisfied for $k = 2$ since $\dfrac{p_2}{q_2} = [a_0, a_1, a_2] = \dfrac{a_2(a_1 a_0 + 1) + a_0}{a_2 a_1 + 1}$ then $p_2 = a_2(a_1 a_0 + 1) + a_0 = a_2 p_1 + p_0$ and $q_2 = a_2 a_1 + 1 = a_2 q_1 + q_0$. Assume that this recursion satisfy for all $k < n$. Let $S_r = [a_1, a_2, \cdots, a_r]$ be the $r^{th}$ order convergent of $[a_1, a_2, \cdots a_n]$. By the equation 7.1 we have

$$\begin{aligned} p_n &= a_0 p'_{n-1} + q'_{n-1} \\ q_n &= p'_{n-1} \end{aligned} \tag{7.3}$$

Moreover by our assumption we have:

$$\begin{aligned} p'_{n-1} &= a_n p'_{n-2} + p'_{n-3} \\ q'_{n-1} &= a_n q'_{n-2} + q'_{n-3} \end{aligned} \tag{7.4}$$

We don't write $a_{n-1}$ since we start with $a_1$ not $a_0$.
Then by the equations 7.3 and 7.4 we obtain

$$\begin{aligned} p_n &= a_0\left(a_n p'_{n-2} + p'_{n-3}\right) + \left(a_n q'_{n-2} + q_{n-3}\right) \\ &= a_n\left(a_0 p'_{n-2} + q'_{n-2}\right) + \left(a_0 p'_{n-3} + q_{n-3}\right) \\ &= a_n p_{n-1} + p_{n-2} \end{aligned} \tag{7.5}$$

And also:

$$\begin{aligned} q_n &= a_n p'_{n-2} + p'_{n3} \\ &= a_n q_{n-1} + q_{n-2} \end{aligned} \tag{7.6}$$

By convention we set $p_{-1} = 1$ and $q_{-1} = 0$

**Theorem 7.0.2.** *For all $k \geq 0$*

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k$$

*Proof.* By the theorem 7.0.1 we have

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2} \\ q_k &= a_k q_{k-1} + q_{k-2} \end{aligned} \tag{7.7}$$

Let us to multiply the first equarion with $q_{k-1}$ and the second with $p_{k-1}$ and then substracting the first from the second we have :

$$q_k p_{k-1} - p_k q_{k-1} = -\left(q_{k-1} p_{k-2} - p_{k-1} q_{k-2}\right)$$

By the convention $q_0 p_{-1} - p_0 q_{-1} = q_0 = 1$. So it proves our theorem.

*Corollary* 7.0.1. For all $k \geq 1$ we have

$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}}.$$

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