FUZZY EXTENSIONS AND APPLICATION ON DECISION MAKING

(KARAR VERME YÖNTEMLERİNDE BULANIK SİSTEM GENİŞLEMELERİ VE UYGULAMASI)

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LIST OF SYMBOLS

DM	**:** Decision Maker
MCDM	**:** MCDM
TFN	**:** Triangular Fuzzy Number
AHP	**:** Analytic Hierarchy Process
ANP	**:** Analytic Network Process
UBC	**:** Upper Bound Condition
MAD	**:** Mean Absolute Deviation
ACC	**:** Accuracy
MACC	**:** Mean Accuracy
APD	**:** Absolute Percentage Deviation

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ABSTRACT

Cognitively speaking, some processes or steps included in decision- making can be a serious burden for DMs. Pairwise comparisons are one of those steps to be handled carefully in order to obtain useful results. During pairwise comparisons of elements such as decision criteria, let alone providing exact quantitative judgments, a complete linguistic set of judgments may sometimes be hard to retrieve from DMs. From that perspective, completing an incomplete pairwise comparison matrix is an interesting problem that many attempted to cover. For that purpose, when DMs provided incomplete interval-valued fuzzy preference relations, experimental approach was used to obtain the completed matrix. Also, the fuzzy synthetic extent was used to calculate the fuzzy weights of criteria.

The purpose of this exploratory study is to analyse what happens when triangular fuzzy numbers (TFNs) are used for judgments and how powerful is the method in order to represent the will of the DMs. In the study, a numerical application is provided using incomplete pairwise comparison matrices of dimension four and five collected from 40 decision makers where their preferences are given using a linguistic set. The Accuracy of the approach is then tested for different levels of confidence in order to provide insightful concluding remarks.

ÖZET

Kavramsal açıdan bakarsak; karar vermeye dahil olan bazı süreçler veya adımlar karar vericiler için ciddi bir yük olabilir. İkili karşılaştırmalar, yararlı sonuçlar elde etmek için dikkatle kullanılması gereken adımlardan biridir. Karar kriterlerinin ikili olarak karşılaştırılması sırasında, kesin nicel kararlar verilmesine rağmen, tam bir dilbilimsel kararlar kümesinin karar verici tarafından doldurulması bazen zor olabilir. Bu açıdan bakıldığında, tamamlanmamış bir ikili karşılaştırma matrisi tamamlamak, birçok akademisyen tarafından ele alınmış bir problemdir. Bu amaçla karar vericiler ikili karşılaştırma matrislerini bulanık tercih ilişkileri ile eksik doldurduklarında, tamamlanmış matrisi elde etmek için deneysel yaklaşım kullanıldı. Ayrıca, kriterlerin bulanık ağırlıklarını hesaplamak için bulanık suni boyut metodu kullanıldı.

Bu keşif çalışmasının amacı, üçgen bulanık sayıların (ÜBS'ler) değerlendirmeler için kullanıldığı zaman ne sonuçlar verdiği ve karar vericilerin iradesini temsil etmek için yöntemin ne kadar güçlü olduğunu analiz etmektir.

Çalışmada; kendi içlerinde ikili kıyaslanması gereken, araba seçimi için hazırlanan dört kriter ve telefon seçimi için hazırlanan beş kriter kırk karar verici ile paylaşılmıştır. Karar vericiler kriterleri ikili olarak karşılaştırırken dilbilimsel bir küme kullanmışlardır. Karar vericilerin çeşitli sebeplerden ötürü matrisleri tamamlama fırsatlarının olamayacağı ön görülerek çalışmada; karar vericilerden matrislerdeki tek satırı tam olarak doldurmaları istenmiştir. Dilbilimsel olarak oluşturulan karşılaştırma matrisi üçgen bulanık sayılara çevrilerek önerilen yaklaşım uygulanmış ve eksik satırlar excel üzerinden hazırlanan makro kodlama ile tamamlanmıştır.

Tamamlanan matrislerin gerçeği yansıtma düzeylerinin incelenebilmesi için karar vericilere matrisleri tekrardan verilmiş ve tamamlamaları istenmiştir.

Analiz esnasında farklı güven düzeyleri kullanılarak karar vericiler tarafından ve çalışmada kullanılan metot tarafından tamamlanan matrisler karşılaştırılmıştır. Matrislerin karşılaştırılması ağırlıklarının hesaplanarak orijinali yansıtmasının yüzdesel karşılığı üzerinden gerçekleştirilmiştir.

Bu çalışma altı boyutlu matris analizini de içeren ve örneklem sayısının genişletildiği bir çalışma ile ayrıntılı bir hale getirilebilir.

1. INTRODUCTION

Decision making is one of the most important life skills. Decisions made on-site appropriately lead to positive changes in the life of the individual, while incorrectly issued decisions can affect life adversely. In the increasingly complex social relations, individuals face constant problems and options, trying to make the most appropriate decision for themselves. In the event of a need to decide, we can define the options that are best suited to the situation in order to meet the needs. We can define the decision-making as in case of a need to select the one that is most suitable to the situation in order to meet the needs.

In Cambridge Dictionary the verb of decision described as a choice that you make about something after thinking about several possibilities. When we make a research deeply about the verb of ‘’decide’’ and behind the scenes of its meaning, we see that the verb “decide” is based on the Latin origin “cide”. Latin meaning of “cide” is “killer,” “act of killing/eliminating” and we can also encounter several uses in English verbs such as homicide and suicide. We can see that the act means; killing, destroying or eliminating. Decision making is to achieve the right by eliminating the wrong or weak. Therefore, we can think that we “kill” rest of our choices when we decide.

The decision - making process consists of certain steps according to the article of Lunenburg (2010), which can be listed as:

- Determination of the purpose,
- Determination of the controllable variables,
- Determination of the uncontrollable variables,
- Determining the relation of controllable variables with uncontrollable variables,
- Determining the effect of each possible decision based on the purpose,

- Decision-making,
- Interpretation of the results,
- The renewal of the decision process for the next time.

Most of the time, people tend to choose between the options that have the highest positive (maximum likelihood) and the lowest negative (least undesirable) value, and the strongest option.

In individual decisions, the individual feels the existence of a problem alone, considers the options that appear for the solution path, and in doing so he/she also refers to his/her own memory, knowledge, values and the knowledge of other resources as needed. But he/she makes the choice himself/herself. Although such decisions may appear to be individual decisions, they are not the result of an individual's independent personality. Social factors also influence the decision of the individual. The decision is the intersection of the individual's own values, the trilogy of society and personality.

Time factor is one of the most significant factors that can affect the decisions. Enough time is required to make effective decisions. Both in the decision-making process, and before and after the individual should use his limited resources in the best way and avoid confining the limiting obstacles. In this way, decision makers (DMs) can prevent them from achieving their goals; manage constraints, prejudice and personal tendencies.

Elderly individuals can compare the potential benefits of the two decision choices with respect to one another in active decisions compared to other young people. Thus, from a group of complex options, the possibility of selecting the best option may be greater for individuals older than the age. However, when the decisions are simple, or the subjects are equally close to the two age groups, the older individuals are not more likely to make the best choice. Nowadays, the average amount of remotely conscious decisions an adult makes each day equals about 35,000 according to the several Internet types of research.

Personality characteristics are also an important factor. Personality traits are the most important features that distort the discovery process or distract the DM from the most important goals.

During the history of humanity, people always need to make several decisions such as personal, social and sometimes as corporate decisions. Every single decision makes results and our main aim is to make the right decisions in life. Yet in our daily basis life, we always need to make a choice to pursue our life, the choices that could have both positive and negative consequences.

In today's easily changeable conditions, people are forced to make the best decisions to pursue their life how they want it to be. A good decision is important for people in their personal life and either in their business life to be successful. In other words, it is a necessity to be in play in this environment, to gain sustainable competitive advantage, and a healthy decision-making. According to Herbert A. Simon, who examined the management based on the decision-making activity, stated that the decision-making activity was the heart of the management (Simon, 1956).

In contrast to complete rationality in decision-making, bounded rationality implies the following steps (Simon, 1982; Simon, 1997; Simon, 2009):

- Decisions will always be based on an incomplete and, to some degree, inadequate comprehension of the true nature of the problem being faced.
- DMs will never succeed in generating all possible alternative solutions for consideration.
- Alternatives are always evaluated incompletely because it is impossible to predict accurately all consequences associated with each alternative.
- The ultimate decision regarding which alternative to choose must be based on some criterion other than maximization or optimization because it is impossible to ever determine which alternative is optimal.

According to Henry Mintzberg (1968), managers including interpersonal relations, information gathering and distribution, and decision-making, they play roles that can be collected into the main title. Regardless of what the managers are doing, it is always a matter of deciding on the works that are of a prominent nature. Henry Mintzberg claims that management is not a set various disconnected parts of the job as perceived by managers, nor a whole universal job of listed individual tasks manager does as understand by academic scholars (Mintzberg, 1968).

If there is more than one criterion in the decision-making process, it is called as a 'Multi Criteria Decision Making (MCDM)' problem. Many methods have been developed since the early seventies in order to solve the MCDM problems. In order to achieve the best solution in a MCDM problem, it can be used different methods. Different methods may suggest different solutions. Finding which method provides the best solution for the problem is a new problem.

In this research, the focus is on pairwise comparison matrices which are mainly used in Analytic Hierarchy Process (AHP) and Analytic Network Process (ANP) methods. Moreover, cases where DMs cannot or will not provide complete pairwise comparison matrices will be the concern. The proposed method was studied to complete those pairwise incomplete comparison matrices and with the completed matrices to calculate fuzzy weights of criteria. The method is based on Khalid and Beg (2016)'s study in which they completed matrices where only one complete row is given using interval valued fuzzy numbers.

The originality of this study is two folds:

- In their study, Khalid and Beg suggested the further research which involves TFN extension of their method. This study provides that research.
- On the other hand, not only the study provides that extension, but presents an exploratory research of how well this approach represents a DM's will by testing the efficiency by giving a comprehensive numerical application including an

efficiency analysis of the results for various certainty level of DMs about their judgments.

The rest of the thesis is arranged as follows: Section 2 provides literature survey and third section includes methodology with definitions and proposed methods. Numerical application and results are presented in section 4. Concluding remarks and future research directions are mentioned in section 5.

2. LITERATURE REVIEW

Under today's competitive business conditions, effective process management and the right decision-making process become more important to sustainable growth and success for all companies.

In daily life, people can make their own decision by itself, but when it comes to business life decisions, it can be made by several DMs to make it more effective.

Steps to be followed when solving a decision-making problem (Mckanna, 1980):

- To identify the problem,
- To create alternatives and selection criteria,
- To evaluate the alternatives,
- To select the top alternative, (Alternatives are ordered from the most preferred to the least preferred.)

According to Vassilev et al. (2005) multicriteria analysis problems can be examined in three types:

- Problems of multicriteria choice
- Problems of multicriteria ranking
- Problems of multicriteria sorting

In MCDM approaches, a significant number and characteristics of candidates, plans, policies, strategies, movements are compared by comparing alternatives and the best of them is tried to be chosen. MCDM methods, using relative importance of criteria, provide solutions to complex problems with conflicting qualities.

First, the definition of alternatives and qualifications is made. Then, according to each criterion, the measurements of each alternative (separately) are obtained and their weights are assigned according to the criteria. The criterion weights assigned and the single-criterion value measurements of the alternatives - with an integration model - are combined to determine the overall values of the alternatives. Finally, sensitivity analyses are carried out and results suggestions and evaluations are presented.

According to Zadeh (1965), fuzzy sets basically defined, as a class of objects with a continuum of grades of membership. Such a set is characterized by a membership function which assigns to each object a degree of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. A separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint (Zadeh, 1965).

The basic of his introduction is all about generalizing the notion of ordinary fuzzy sets. This definition gives us a general overview on Fuzzy Sets to understand it in a clearer way. The fuzzy set theory is a very attractive method used in years, in which subjective evaluation of any event to be dealt with by probabilistic or mathematical models is necessary or if the event is not clear.

Nevertheless, the fact that subjective considerations are handled in the form of conditional probabilities, though complex, or decision-making is a restrictive and more intuitive approach, may be attractive as in fuzzy set theory.

In whatever circumstances and dimensions DMs would decide, they must fulfil these functions in an environment of uncertainty. The accuracy of the decisions made shall be provided to the extent that such uncertainty can be converted to risk.

However, if the DMs are using the classical scientific approach and the methods involved in the decision-making process, then the resulting decisions will be good - bad, beautiful - ugly, right - wrong, yes - no, black - white or 0 - 1. Yet, real life is not based on absolute distinction. In other words, the presence of thousands of grey tones in absolute black and absolute white should not be forgotten.

According to Roberts (1979) you can get information in three ways from DM(s). First one is pairwise comparisons of alternatives with respect to the same experimental conditions. The second way is to consider the preferences of a group of DMs on the same set of alternatives and aggregate the opinions of the DMs to obtain a fuzzy relation. The third one, if the alternatives have multi-attributes, preferences along the attributes can be aggregated to get a degree of preference for the alternative itself which gives rise to a fuzzy relation (Fodor & Roubens, 1994).

The way how DMs perceive events explains how they develop situations and their tasks. In short, it expresses the hidden aspects of people. It could be the reason why DMs would prefer to hide their true opinions and give missing information while they're deciding. According to Park and Kim (1997) the reasons that DMs provide only incomplete information are: (1) decisions are made under time pressure and lack of data; (2) several attributes are intangible or non-monetary, because they reflect social and environmental impacts; (3) DM has limited attention and information processing capabilities and the like.

In business management, multidisciplinary DMs may have to make decisions in a common problem under time pressure or inadequate information. Under these conditions, DMs can provide incomplete information. This issue can be discussed in two directions. Initial situation; criteria are shared with a manager for a complete comparison, but he/she can give incomplete information or refuse the method because of time pressure. The second situation; applying the method is important to make an effective decision, so to perform the partial comparison (therefore settling with incomplete information) can be suggested to the manager because of time pressure.

The consistency of fuzzy preference relations was investigated in several studies (Xu & Da, 2003; Xu, 2004; Herrera-Viedma et al., 2004; Ma et. al., 2006). Peneva and Popchev (2007) worked on conditions of consistent choice or rank. Also, Herrera-Viedma et al.

(2004), Ma et al. (2006) and Basile (1990) studied on consistency with the concept of transitivity.

The interval-valued preference was studied by Bilgic (1998) and Xu (2004) for eliminating uncertainty. Also, Bilgic (1998) worked on consistency in interval-valued fuzzy preference relations and to express vague human preferences in interval-valued languages.

Khalid and Beg (2016) worked on incomplete interval-valued fuzzy preference relations with an upper bound condition. Based on the future research proposals of Khalid and Beg (2016); in this paper, multiplicative TFNs' have been used through their approach with the addition of the analysis of the method's represent ability of the DMs' will. Also, the revised version of Chang's (1996) fuzzy synthetic extent by Wang et al. (2008) was used to calculate the fuzzy weights of criteria. The study provides mostly exploratory research and analysis of the results in terms of strength of the method used.

3. METHODOLOGY

3.1 Preliminaries

Definition 1. A fuzzy set *A* on *X* is characterized by a membership function $\mu_A: X \rightarrow [0, 1]$ where $\mu_A(x)$ is defined as the degree of membership of element x in fuzzy set A for each $x \in X$ (Zadeh, 1965).

Definition 2. Let $\mu : A \times A \rightarrow [0, 1]$ be a membership function of a fuzzy relation and $a, b, c \in A$. Some definitions for transitivity are:

• Max–min transitivity (Dubois & Prade, 1980; Zimmermann, 1993):

$$\mu(a, c) \geq min(\mu(a, b),\mu(b, c)); \quad (1)$$

• Max–max transitivity (Tanino, 1988):

$$\mu(a, c) \geq max(\mu(a, b),\mu(b, c)); \quad (2)$$

• Restricted max–min transitivity (Tanino, 1988):

$$\mu(a, b) \geq 0.5,\ \mu(b, c) \geq 0.5 \Rightarrow \mu(a, c) \geq min(\mu(a, b),\mu(b, c)); \quad (3)$$

• Restricted max–max transitivity (Tanino, 1988):

$$\mu(a, b) \geq 0.5,\ \mu(b, c) \geq 0.5 \Rightarrow \mu(a, c) \geq max(\mu(a, b),\mu(b, c)); \quad (4)$$

• Additive transitivity (Tanino, 1984; Tanino, 1988):

$$\mu(a, c) = \mu(a, b) + \mu(b, c) - 0.5 \tag{5}$$

Definition 3. An additive fuzzy preference relation P on X is characterized by a function $\mu_P: X \times X \rightarrow [0, 1]$ where $\mu_P\ (x_i , x_j) = p_{ij}$ indicates the preference intensity with which alternative x_i is preferred over x_j (Bazdek et al., 1978; Tanino, 1984; Tanino, 1988; Park & Kim, 1997).

Definition 4. A multiplicative fuzzy preference relation A on X is characterized by a function $\mu_A: X \times X \rightarrow [1/9, 9]$ where $\mu_A\ (x_i\ ,\ x_j) = a_{ij}$ indicates the preference intensity with which alternative x_i is preferred over x_j (Tanino, 1984; Saaty, 1994; Saaty, 2000).

Definition 5. Let L ([0, 1]) denote the set of all closed subintervals of [0, 1]. Let A^* denote an interval valued fuzzy set on X. Then $A^*: X \rightarrow L$ ([0, 1]) and the membership of each $x \in X$ is given by $A^*(x) = [\underline{A^*}(x)\ ,\ \overline{A^*}(x)]$ where $\underline{A^*}(x)\ ,\ \overline{A^*}(x) \in [0\ ,\ 1]$ and $\underline{A^*}(x\ ) \leq \overline{A^*}(x)$ (Zadeh, 1965; Turksen & Bilgic, 1996; Bilgic, 1998).

Definition 6. A fuzzy relation is said to be interval valued fuzzy preference relation when $R \subset X \times X$ is characterized by a membership function $\tilde{\mu}_R: X \times X \rightarrow L$ ([0 , 1]) with $\tilde{\mu}_R$ $(x_i\ ,\ x_j) = \overline{r_{ij}} = [r_{ij}^l\ ,\ r_{ij}^r]$ where r_{ij}^l and r_{ij}^r are the left and right limits of $\overline{r_{ij}}$ respectively (Xu, 2004; Wang et al., 2005; Alonso et al., 2008).

Definition 7. A fuzzy relation is said to be multiplicative when X is a mapping $A{:}X \rightarrow L$ ([1/9, 9]) where membership of each $x \in X$ is given by $A(x\ ) = [\underline{A}(x)\ ,\bar{A}(x)]$ such that $\underline{A}(x),\bar{A}(x) \in [1\ /\ 9\ ,\ 9]$ and $\underline{A}(x) \leq \bar{A}(x)$ (Khalid & Beg, 2016).

Definition 8. One can transform additive interval valued fuzzy sets to multiplicative (or vice versa) using the following transformation functions (Khalid & Beg, 2016):

$$\overline{P_{ij}} = F\ (\overline{a_{ij}}) = [\tfrac{1}{2}(1+log_9\ a_{ij}^{l}),\tfrac{1}{2}(1+log_9\ a_{ij}^{r})] \quad (6)$$

$$\overline{a_{ij}} = G\ (\overline{P_{ij}}) = [9^{2p_{ij}^{l}-1}, 9^{2p_{ij}^{r}-1}] \quad (7)$$

Definition 9. A TFN $\tilde{A} = (l, m, u)$ is a fuzzy set with the following membership function.

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < l; \\ \frac{x-l}{m-l}, & l \le x \le m; \\ \frac{u-x}{u-m}, & m \le x \le u; \\ 0, & x > u; \end{cases} \quad (8)$$

Definition 10. $R= [P_{ij}]_{nxn}$, where P_{ij} is a TFN, is said to be an incomplete fuzzy relation matrix when DMs only fill one specific row - of their choice - of the matrix (Khalid & Beg, 2016).

3.2 Fuzzy Synthetic Extent

Chang (1996) worked on an extent analysis method for calculating a crisp priority weights from a triangular fuzzy comparison matrix. True weights cannot be estimated with this method due to normalization formula and this method was later revised by Wang et al. (2008).

In this study, revised method by Wang et al. (2008) is used to calculate and compare fuzzy weights of revised and original matrices.

The normalization formula (Wang et al., 2008) for a set of triangular fuzzy weights:

$$\widetilde{S_i} = \frac{RS_i}{\sum_{j=1}^{n} RS_j} =$$

$$\left(\frac{\sum_{j=1}^{n} l_{ij}}{\sum_{j=1}^{n} l_{ij} + \sum_{j=1,k\neq i}^{n}\ \sum_{j=1}^{n} u_{kj}}, \frac{\sum_{j=1}^{n} m_{ij}}{\sum_{j=1}^{n} l_{ij} + \sum_{j=1}^{n}\ \sum_{j=1}^{n} m_{kj}}, \frac{\sum_{j=1}^{n} u_{ij}}{\sum_{j=1}^{n} l_{ij} + \sum_{j=1,k\neq i}^{n}\ \sum_{j=1}^{n} l_{kj}}\right),\ i=1,2\ldots n \quad (9)$$

The normalization formula for a set of interval fuzzy weights:

$$(\widetilde{S_i^*}) = \frac{RS_i}{\sum_{j=1}^{n} RS_j} = \left(\frac{\sum_{j=1}^{n} l_{ij}}{\sum_{j=1}^{n} l_{ij} + \sum_{j=1,k\neq i}^{n} \sum_{j=1}^{n} u_{kj}}, \frac{\sum_{j=1}^{n} u_{ij}}{\sum_{j=1}^{n} l_{ij} + \sum_{j=1,k\neq i}^{n} \sum_{j=1}^{n} l_{kj}}\right), i=1,2,\ldots n \qquad (10)$$

3.3 Proposed Revision and Completion Method

Khalid and Awais (2014) worked on consistency in fuzzy preference relations. Their proposed method completes an incomplete preference relation to avoid inconsistency with upper bound condition.

The proposed method of Khalid and Beg (2016) based on upper bound condition and this method is used in this paper to revise and complete in complete preferences.

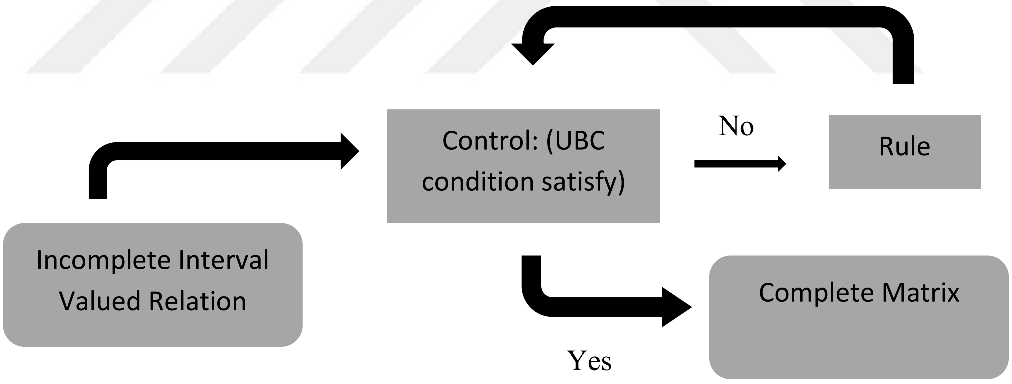

Figure 3.1: Method of Khalid and Beg (2016)

3.3.1 Revision Rule

DMs provide n $\overline{r_{i'j}}$ for fixed i' and $j \epsilon$ {1,2,…,n} with $\overline{r_{ii}}$ = [0.50,0.50]. Each interval has left, and right end limit denoted as r_{ij}^{l}, r_{ij}^{r} respectively. So *(n – 1)* preferences are reviewed and revised if following sets are created in order to explain the revision rules.

ß= $\{r^l_{i'1}, r^r_{i'1}, ... r^l_{i'k}, r^r_{i'k}, ... r^l_{i'n}, r^r_{i'n}\}$ $k \neq i'$

$\Im = \{r^r_{i'1}, r^r_{i'2}, ..., r^r_{i'n}\}$

$\Im^l = \{r^l_{i'1}, r^l_{i'2}, ..., r^l_{i'n}\}$

$\gamma = \{r^r_{i'k}: r^r_{i'k}$ *satisfy property (UBC)*$\} \subset \Im$

$\gamma^l = \{r^l_{i'k}: r^l_{i'k}$ *satisfy property (UBC)*$\} \subset \Im^l$

$\overline{r_{kJ}}$ is provided in the preferences from DMs with fixed k and $\forall j=1,2,.....,n$ in which δ is presented the greatest right end limit and $\epsilon < 0.5$ is presented the smallest left end limit. $\delta \leq 0.5 + \epsilon$ inequation should be ensured to satisfy property UBC.

Case A. If $|\gamma| \geq \left\lceil \frac{(n-1)}{2} \right\rceil$

δ_i : i^{th} right end limit which do not satisfy UBC. So δ_i should be revised and replaced with δ_i' where

$$\delta_i' = inf\,(ß) + 0.5 \tag{11}$$

If left $(\delta_i) \leq \delta_i'$ do not revise. If not left δ_i should be revised and replaced with left δ_i'

$$left\,(\delta_i') = \delta_i' + min_j \left\{ \left| r^l_{i'j} - r^r_{i'j} \right| \right\} \tag{12}$$

This process should be performed for each j.

Case B. If $|\gamma^l| < \left\lceil \frac{(n-1)}{2} \right\rceil$

ϵ_i: i^{th} left end limit which do not satisfy UBC. So ϵ_i should be revised and replaced with ϵ_i' where

$$\epsilon_i' = sup\,(\Im) - 0.5 \tag{13}$$

If left $(\epsilon_i) \geq \epsilon_i'$ do not revise. Otherwise left ϵ_i should be revised and replaced with left ϵ_i' as follows, and this process should be performed for each j.

$$left\ (\epsilon_i') = \epsilon_i' - min_j \left\{ \left| r_{i'j}^{l} - r_{i'j}^{r} \right| \right\} \tag{14}$$

3.3.2 Estimated Missing Preferences

After revision of the given row, calculate its corresponding column with given formula;

$$\overline{r_{ij}} + \overline{r_{ji}} = 1\ ;\ r_{ij}^{l} + r_{ji}^{r} = 1\ ,\ r_{ji}^{l} + r_{ij}^{r} = 1 \quad \forall\ i,j,k=1,2,.....,n \tag{15}$$

$$\overline{r_{ij}} = \overline{r_{ik}} + \overline{r_{kj}} - 0.5\ ;\ r_{ij}^{l} = r_{ik}^{l} + r_{kj}^{l} - 0.5\ ,\ r_{ij}^{r} = r_{ik}^{r} + r_{kj}^{r} - 0.5 \tag{16}$$

3.4 Application Steps

Proposed method works following the step by step procedure given below:

Step 1. Collect incomplete fuzzy preference relation matrix from DMs.

Step 2. Convert linguistic values to TFNs using Table 3.1.

Table 3.1: Linguistic Value - TFN Conversion Scale

	Linguistic	TFN
EX	Extremely More Important	(8,9,9)
VSE	Very Strongly to Extremely More Important	(7,8,9)
VS	Very Strongly More Important	(6,7,8)
SVS	Strongly to Very Strongly More Important	(5,6,7)
S	Strongly More Important	(4,5,6)
MS	Moderately to Strongly More Important	(3,4,5)
M	Moderately More Important	(2,3,4)
EM	Equally to Moderately More Important	(1,2,3)
E	Equally Important	(1,1,1)

Step 3. Use α-cuts to obtain multiplicative interval values.

Step 4. Conversion of multiplicative preferences to additive fuzzy interval values using transformation function given in (6).

Step 5. Complete the matrix using Khalid and Beg's (2016) method: First, given row is checked for the UBC. If UBC is satisfied, then the remainder of the relation matrix is expressible. Otherwise, a revision of judgments is performed in order to meet the UBC. Then the matrix is filled using additive transitivity property (we kindly suggest to the reader to analyze the reference study for more details).

Step 6. Calculate fuzzy weights of criteria with the method Wang et al. (2008).

Step 7. Collect completed matrices from decision makers and calculate fuzzy synthetic extent in order to retrieve fuzzy weights for the criteria.

Step 8. Calculate accuracies of the values found in step 6 from different α-cuts of fuzzy synthetic extents found in step 7.

$$ACC = \begin{cases} 1; & (l_{ij}^* > l_{ij}, u_{ij}^* < u_{ij}) \\ \frac{u_{ij} - l_{ij}}{u_{ij}^* - l_{ij}^*}; & (l_{ij}^* < l_{ij}, u_{ij}^* > u_{ij}) \\ \frac{u_{ij}^* - l_{ij}}{u_{ij}^* - l_{ij}^*}; & (l_{ij}^* < l_{ij}, u_{ij}^* > l_{ij}, u_{ij}^* < u_{ij}) \\ \frac{u_{ij} - l_{ij}^*}{u_{ij}^* - l_{ij}^*}; & (l_{ij}^* > l_{ij}, l_{ij}^* < u_{ij}, u_{ij}^* > u_{ij}) \\ 0; & otherwise \end{cases} \quad (17)$$

4. NUMERICAL APPLICATION

4.1 Step by Step Application

In this section, a numerical illustration will be presented following the step by step procedure mentioned above. For sake of simplicity in presentation, only one instance for each dimension (i.e. four and five) for a given α-cut will be explained in detail. The remaining calculations can be found in Appendices A-D.

Consider of the following example from DM2 for 4x4 dimension and DM30 for 5x5 dimension who gave incomplete linguistic information about the relative importance of criteria for a car and mobile phone selection problem with following criteria set:
X_{4x4}= *{x1: Comfort, x2: Engine Power, x3: Design, x4: Brand Reputation},*
X_{5x5}= *{x1: Cost, x2: Screen, x3: Weight, x4: Camera, x5: Battery}.*

Step 1 Linguistic incomplete pairwise comparison matrices are collected from DMs.

Table 4.1: Linguistic Incomplete Pairwise Comparison, DM 2

	Comfort	Engine Power	Design	Brand Reputation
Comfort	E	EM	MS	M
Engine Power		E		
Design			E	
Brand Reputation				E

Table 4.2: Linguistic Incomplete Pairwise Comparison, DM 30

	Cost	Screen	Weight	Camera	Battery
Cost	E				
Screen		E			
Weight			E		
Camera				E	
Battery	S^{-1}	M^{-1}	VS	MS^{-1}	E

Step 2. Convert linguistic values to TFNs.

Table 4.3: Incomplete Pairwise Comparison Matrix with TFN, DM 2

	Comfort	Engine Power	Design	Brand Reputation
Comfort	[1.00,1.00,1.00]	[1.00,2.00,3.00]	[3.00,4.00,5.00]	[2.00,3.00,4.00]
Engine Power		[1.00,1.00,1.00]		
Design			[1.00,1.00,1.00]	
Brand Reputation				[1.00,1.00,1.00]

Table 4.4: Incomplete Pairwise Comparison Matrix with TFN, DM 30

	Cost	Screen	Weight	Camera	Battery
Cost	[1.00,1.00,1.00]				
Screen		[1.00,1.00,1.00]			
Weight			[1.00,1.00,1.00]		
Camera				[1.00,1.00,1.00]	
Battery	[0.17,0.20,0.25]	[0.25, 0.33, 0.50]	[6.00,7.00,8.00]	[0.20, 0.25, 0.33]	[1.00,1.00,1.00]

Step 3. Find multiplicative interval valued fuzzy preference matrix using different α-cut levels (in the following example α is set to 0.2 for DM2 and 0.4 for DM30)

Table 4.5: Multiplicative Interval Values with α=0.2, DM 2

	Comfort	Engine Power	Design	Brand Reputation
Comfort	[1.00,1.00]	[1.20,2.80]	[3.20,4.80]	[2.20,3.80]
Engine Power		[1.00,1.00]		
Design			[1.00,1.00]	
Brand Reputation				[1.00,1.00]

Table 4.6: Multiplicative Interval Values with α=0.4, DM 30

	Cost	Screen	Weight	Camera	Battery
Cost	[1.00, 1.00]				
Screen		[1.00, 1.00]			
Weight			[1.00, 1.00]		
Camera				[1.00, 1.00]	
Battery	[0.18, 0.23]	[0.28, 0.42]	[6.40, 7.60]	[0.22, 0.29]	[1.00, 1.00]

Step 4. Using the transformation function (6) the following additive interval valued fuzzy comparison matrix is calculated.

Table 4.7: Additive Interval Value, DM 2

	Comfort	Engine Power	Design	Brand Reputation
Comfort	[0.50,0.50]	[0.76,0.86]	[0.88,0.94]	[0.68,0.80]
Engine Power		[0.50,0.50]		
Design			[0.50,0.50]	
Brand Reputation				[0.50,0.50]

Table 4.8: Additive Interval Value, DM 30

	Cost	Screen	Weight	Camera	Battery
Cost	[0.50,0.50]				
Screen		[0.50,0.50]			
Weight			[0.50,0.50]		
Camera				[0.50,0.50]	
Battery	[0.11, 0.16]	[0.21, 0.30]	[0.92, 0.96]	[0.15, 0.22]	[0.50,0.50]

Step 5. Incomplete interval valued matrices are completed using Khalid and Beg (2016)'s method and can be observed in the following table.

Table 4.9: Revised and Completed Matrix, DM 2

	Comfort	Engine Power	Design	Brand Reputation
Comfort	[0.50,0.50]	[0.76,0.86]	[0.88,0.94]	[0.68,0.80]
Engine Power	[0.27,0.46]	[0.50,0.50]	[0.53,0.82]	[0.45,0.76]
Design	[0.14,0.24]	[0.18,0.47]	[0.50,0.50]	[0.32,0.54]
Brand Reputation	[0.20,0.32]	[0.24,0.55]	[0.46,0.68]	[0.50,0.50]

Table 4.10: Revised and Completed Matrix, DM 30

	Cost	Screen	Weight	Camera	Battery
Cost	[0.50,0.50]	[0.55, 0.69]	[0.91, 0.99]	[0.49, 0.61]	[0.84, 0.89]
Screen	[0.31, 0.45]	[0.50,0.50]	[0.77, 0.90]	[0.35, 0.51]	[0.70, 0.79]
Weight	[0.00, 0.09]	[0.10, 0.23]	[0.50,0.50]	[0.04, 0.15]	[0.39, 0.43]
Camera	[0.39, 0.51]	[0.49, 0.65]	[0.85, 0.89]	[0.50,0.50]	[0.78, 0.85]
Battery	[0.11, 0.16]	[0.21, 0.30]	[0.57, 0.61]	[0.15, 0.22]	[0.50,0.50]

Step 6. At this point, complete original matrices are collected from decision makers. After appropriate α-cut and multiplicative – additive transformation, they are represented in tables 4.11 and 4.12.

Table 4.11: The Original Matrix from DM 2

	Comfort	Engine Power	Design	Brand Reputation
Comfort	[0.50,0.50]	[0.54,0.73]	[0.76,0.86]	[0.68,0.80]
Engine Power	[0.27,0.46]	[0.50,0.50]	[0.68,0.80]	[0.27,0.46]
Design	[0.14,0.24]	[0.20,0.32]	[0.50,0.50]	[0.14,0.24]
Brand Reputation	[0.20,0.32]	[0.54,0.73]	[0.76,0.86]	[0.50,0.50]

Table 4.12: The Original Matrix from DM 30

	Cost	Screen	Weight	Camera	Battery
Cost	[0.50,0.50]	[0.82, 0.91]	[0.94, 1.00]	[0.13, 0.25]	[0.82, 0.91]
Screen	[0.09, 0.18]	[0.50,0.50]	[0.82, 0.91]	[0.06, 0.13]	[0.66, 0.82]
Weight	[0.00, 0.06]	[0.09, 0.18]	[0.50,0.50]	[0.06, 0.13]	[0.03, 0.09]
Camera	[0.75, 0.87]	[0.87, 0.94]	[0.87, 0.94]	[0.50,0.50]	[0.75, 0.87]
Battery	[0.09, 0.18]	[0.18, 0.34]	[0.91, 0.97]	[0.13, 0.25]	[0.50,0.50]

Step 7. Using (17), fuzzy weights retrieved from complete DMs' matrices and completed matrices were compared. For DM2 and DM30 the values are collected in the following tables:

Table 4.13: $\widetilde{S_i}$ and $\widetilde{S_i^*}$ Values and Accuracies, DM 2

	$\widetilde{S_i}$ with α=0.2		$\widetilde{S_i^*}$		
	l_{ij}	u_{ij}	l_{ij}^*	u_{ij}^*	%
1	0.301	0.524	0.305	0.619	69%
2	0.145	0.309	0.137	0.430	56%
3	0.058	0.111	0.068	0.202	32%
4	0.214	0.409	0.089	0.277	33%

Table 4.14: $\widetilde{S_i}$ and $\widetilde{S_i^*}$ Values and Accuracies, DM 30

	$\widetilde{S_i}$ with α=0.4		$\widetilde{S_i^*}$		
	l_{ij}	u_{ij}	l_{ij}^*	u_{ij}^*	%
1	0.279	0.368	0.295	0.452	46%
2	0.128	0.190	0.162	0.296	20%
3	0.024	0.032	0.040	0.065	0%
4	0.303	0.400	0.220	0.349	35%
5	0.126	0.175	0.059	0.095	0%

4.2 Analysis of Results

This study was performed with three different α-cuts for 40 DMs' matrices of dimension four and five.

4.2.1 Analysis for n=4

- As the α-cut increased, the accuracy of weights decreased from 39.76% to 28.41%. As the α-cut increased from 0 to 0.40.
- 33% of the completed 360 data are not accurate at all. But this is quite acceptable because the DM provided only 50% of the complete preference data in the case of matrices of dimension four.
- 55% of the all data have representability success below 40%. But 27% of the completed 360 data have representability success over 60% and 0.09% of the total have representability success over 80%.
- Accuracy of fuzzy weights' success representability with respect to α-cuts are collected in the following table.

Table 4.15: Accuracy of Fuzzy Weights, n=4

	Accuracy		
	α=0.0	α=0.2	α=0.4
$S^*_{(i)} - S_{(i)}$	39.76%	34.06%	28.41%

- It was checked whether centroid from original matrix is between $S^*_{(i)}$ intervals. Average success for each α-cuts are in the following table.

Table 4.16: Average Success Number n=4

Average Success Number n=4		
α=0.0	α=0.2	α=0.4
2.3	2.13	1.85

4.2.2 Analysis for n=5

- As the α-cut increased, the accuracy of weights decreased from 47.51% to 36.67%. As the α-cut increased from 0 to 0.40.

- 23% of the completed 720 data are not accurate at all. But this is quite acceptable because the DM provided only 40% of the complete preference data in the case of matrices of dimension five.
- 40% of the all data have representability success below 40%. But 40% of the completed 720 data have representability success over 60% and 15% of the total have representability success over 80%.
- Accuracy of fuzzy weights' success representability with respect to α-cuts are collected in the following table.

Table 4.17: Accuracy of Fuzzy Weights, n=5

	Accuracy		
	α=0.0	α=0.2	α=0.4
$S^*_{(i)} - S_{(i)}$	47.51%	43.50%	36.67%

- It was checked whether centroid from original matrix is between $S^*_{(i)}$ intervals. Average success for each α-cuts are in the following table.

Table 4.18: Average Success Number n=5

Average Success Number n=5		
α=0.0	α=0.2	α=0.4
3.3	3.1	2.7

5. CONCLUSION

Decision making process can be difficult for DMs if they should use [0,1] limit in order to compare a given set of criteria. Although the [1/9,9] scale is cognitively easier than [0,1] limit, it is not preferable for some DMs. Therefore, it would be better for them to use linguistic scale to compare the criteria. Also, DMs are willing and feel surer with filling partial comparison.

Our analysis showed that the method was able to represent the DMs preferences will with an average accuracy rate of 34.07% for dimension four and 42.56% for dimension five.

Therefore, reaching an accuracy level as mentioned in section 4.2 Analysis of Results with such missing information can be considered as successful and promising when one knows that in case of requiring complete information, the DMs can be unwilling to use the method.

For future research, representability of this method can be analysed for 6x6 and 7x7 matrices using a bigger sample of DMs and a software to complete the incomplete matrices and calculate the weights of criteria can be developed.

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APPENDICES

Appendix A.

The original incomplete 4x4 car selection matrices from 5 DMs (sample of 40) are shown below:

Table A.1: The Original Incomplete 4x4 Matrix, DM 1

	Comfort	Engine Power	Design	Brand Reputation
Comfort	[0.50,0.50]			
Engine Power	[0.09,0.18]	[0.50,0.50]	[0.82,0.91]	[0.82,0.91]
Design			[0.50,0.50]	
Brand Reputation				[0.50,0.50]

Table A.2: The Original Incomplete 4x4 Matrix, DM 2

	Comfort	Engine Power	Design	Brand Reputation
Comfort	[0.50,0.50]	[0.50,0.75]	[0.75,0.87]	[0.66,0.82]
Engine Power		[0.50,0.50]		
Design			[0.50,0.50]	
Brand Reputation				[0.50,0.50]

Table A.3: The Original Incomplete 4x4 Matrix, DM 3

	Comfort	Engine Power	Design	Brand Reputation
Comfort	[0.50,0.50]	[0.75,0.87]	[0.87,0.94]	[0.66,0.82]
Engine Power		[0.50,0.50]		
Design			[0.50,0.50]	
Brand Reputation				[0.50,0.50]

Table A.4: The Original Incomplete 4x4 Matrix, DM 4

	Comfort	Engine Power	Design	Brand Reputation
Comfort	[0.50,0.50]			
Engine Power	[0.82,0.91]	[0.50,0.50]	[0.82,0.91]	[0.75,0.87]
Design			[0.50,0.50]	
Brand Reputation				[0.50,0.50]

Table A.5: The Original Incomplete 4x4 Matrix, DM 5

	Comfort	Engine Power	Design	Brand Reputation
Comfort	[0.50,0.50]			
Engine Power		[0.50,0.50]		
Design	[0.91,0.97]	[0.13,0.25]	[0.50,0.50]	[0.66,0.82]
Brand Reputation				[0.50,0.50]

The original incomplete 5x5 mobile selection matrices from 5 DMs (sample of 40) are shown below:

Table A.6: The Original Incomplete 5x5 Matrix, DM 1

	Cost	Screen	Weight	Camera	Battery
Cost	[0.50,0.50]	[0.66,0.82]	[0.75,0.87]	[0.50,0.75]	[0.50,0.75]
Screen		[0.50,0.50]			
Weight			[0.50,0.50]		
Camera				[0.50,0.50]	
Battery					[0.50,0.50]

Table A.7: The Original Incomplete 5x5 Matrix, DM 2

	Cost	Screen	Weight	Camera	Battery
Cost	[0.50,0.50]	[0.66,0.82]	[0.25,0.50]	[0.50,0.75]	[0.50,0.75]
Screen		[0.50,0.50]			
Weight			[0.50,0.50]		
Camera				[0.50,0.50]	
Battery					[0.50,0.50]

Table A.8: The Original Incomplete 5x5 Matrix, DM 3

	Cost	Screen	Weight	Camera	Battery
Cost	[0.50,0.50]				
Screen		[0.50,0.50]			
Weight			[0.50,0.50]		
Camera				[0.50,0.50]	
Battery	[0.66,0.82]	[0.75,0.87]	[0.50,0.50]	[0.66,0.82]	[0.50,0.50]

Table A.9: The Original Incomplete 5x5 Matrix, DM 4

	Cost	Screen	Weight	Camera	Battery
Cost	[0.50,0.50]				
Screen		[0.50,0.50]			
Weight			[0.50,0.50]		
Camera	[0.25,0.50]	[0.25,0.50]	[0.50,0.75]	[0.50,0.50]	[0.50,0.75]
Battery					[0.50,0.50]

Table A.10: The Original Incomplete 5x5 Matrix, DM 5

	Cost	Screen	Weight	Camera	Battery
Cost	[0.50,0.50]				
Screen		[0.50,0.50]			
Weight			[0.50,0.50]		
Camera				[0.50,0.50]	
Battery	[0.66,0.82]	[0.50,0.75]	[0.66,0.82]	[0.50,0.75]	[0.50,0.50]

Appendix B.

The revised and completed 4x4 car selection matrices with interval weights are shown below:

Table B.1: The Revised and Completed 4x4 Matrix with Interval Weights, DM 1, α=0.0

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.50,0.59]	[0.82,1.00]	[0.82,1.00]	0.345	0.602
Eng. Pw.	[0.41,0.50]	[0.50,0.50]	[0.82,0.91]	[0.82,0.91]	0.275	0.502
Design	[0.00,0.18]	[0.09,0.18]	[0.50,0.50]	[0.41,0.50]	0.050	0.104
Brand Rep.	[0.00,0.18]	[0.09,0.18]	[0.41,0.50]	[0.50,0.50]	0.050	0.104

Table B.2: The Revised and Completed 4x4 Matrix with Interval Weights, DM 1, α=0.2

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.53,0.60]	[0.85,1.00]	[0.85,1.00]	0.386	0.594
Eng. Pw.	[0.40,0.47]	[0.50,0.50]	[0.83,0.90]	[0.83,0.90]	0.283	0.464
Design	[0.00,0.15]	[0.10,0.17]	[0.50,0.50]	[0.43,0.50]	0.052	0.094
Brand Rep.	[0.00,0.15]	[0.10,0.17]	[0.43,0.50]	[0.50,0.50]	0.052	0.094

Table B.3: The Revised and Completed 4x4 Matrix with Interval Weights, DM 1, α=0.4

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.55,0.61]	[0.89,1.00]	[0.89,1.00]	0.429	0.586
Eng. Pw.	[0.39,0.45]	[0.50,0.50]	[0.84,0.89]	[0.84,0.89]	0.291	0.426
Design	[0.00,0.11]	[0.11,0.16]	[0.50,0.50]	[0.45,0.50]	0.054	0.085
Brand Rep.	[0.00,0.11]	[0.11,0.16]	[0.45,0.50]	[0.50,0.50]	0.054	0.085

Table B.4: The Revised and Completed 4x4 Matrix with Interval Weights, DM 2, α=0.0

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.50,0.75]	[0.75,0.87]	[0.66,0.82]	0.260	0.648
Eng. Pw.	[0.25,0.50]	[0.50,0.50]	[0.50,0.87]	[0.41,0.82]	0.120	0.499
Design	[0.13,0.25]	[0.13,0.50]	[0.50,0.50]	[0.29,0.57]	0.058	0.230
Brand Rep.	[0.18,0.34]	[0.18,0.59]	[0.43,0.68]	[0.50,0.50]	0.075	0.308

Table B.5: The Revised and Completed 4x4 Matrix with Interval Weights, DM 2, α=0.2

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.54,0.73]	[0.76,0.86]	[0.68,0.80]	0.305	0.619
Eng. Pw.	[0.27,0.46]	[0.50,0.50]	[0.53,0.82]	[0.45,0.76]	0.137	0.430
Design	[0.14,0.24]	[0.18,0.47]	[0.50,0.50]	[0.32,0.54]	0.068	0.202
Brand Rep.	[0.20,0.32]	[0.24,0.55]	[0.46,0.68]	[0.50,0.50]	0.089	0.277

Table B.6: The Revised and Completed 4x4 Matrix with Interval Weights, DM 2, α=0.4

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.58,0.72]	[0.78,0.85]	[0.70,0.79]	0.352	0.588
Eng. Pw.	[0.28,0.42]	[0.50,0.50]	[0.56,0.77]	[0.48,0.71]	0.158	0.371
Design	[0.15,0.22]	[0.23,0.44]	[0.50,0.50]	[0.35,0.51]	0.079	0.178
Brand Rep.	[0.21,0.30]	[0.29,0.52]	[0.49,0.65]	[0.50,0.50]	0.104	0.242

Table B.7: The Revised and Completed 4x4 Matrix with Interval Weights, DM 3, α=0.0

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.75,0.87]	[0.87,0.94]	[0.66,0.82]	0.415	0.688
Eng. Pw.	[0.13,0.25]	[0.50,0.50]	[0.50,0.69]	[0.29,0.57]	0.086	0.237
Design	[0.06,0.13]	[0.31,0.50]	[0.50,0.50]	[0.21,0.45]	0.059	0.151
Brand Rep.	[0.18,0.34]	[0.43,0.71]	[0.55,0.79]	[0.50,0.50]	0.115	0.327

Table B.8: The Revised and Completed 4x4 Matrix with Interval Weights, DM 3, α=0.2

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.76,0.86]	[0.88,0.94]	[0.68,0.80]	0.463	0.668
Eng. Pw.	[0.14,0.24]	[0.50,0.50]	[0.52,0.67]	[0.32,0.54]	0.098	0.214
Design	[0.06,0.12]	[0.33,0.48]	[0.50,0.50]	[0.24,0.43]	0.067	0.138
Brand Rep.	[0.20,0.32]	[0.46,0.68]	[0.57,0.70]	[0.50,0.50]	0.127	0.269

Table B.9: The Revised and Completed 4x4 Matrix with Interval Weights, DM 3, α=0.4

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.78,0.85]	[0.88,0.93]	[0.70,0.79]	0.487	0.646
Eng. Pw.	[0.15,0.22]	[0.50,0.50]	[0.54,0.65]	[0.35,0.51]	0.107	0.193
Design	[0.07,0.12]	[0.35,0.46]	[0.50,0.50]	[0.27,0.41]	0.073	0.126
Brand Rep.	[0.21,0.30]	[0.49,0.65]	[0.59,0.71]	[0.50,0.50]	0.140	0.254

Table B.10: The Revised and Completed 4x4 Matrix with Interval Weights, DM 4, α=0.0

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.09,0.18]	[0.41,0.59]	[0.34,0.55]	0.079	0.330
Eng. Pw.	[0.41,0.91]	[0.50,0.50]	[0.41,0.91]	[0.41,0.87]	0.184	0.707
Design	[0.41,0.59]	[0.09,0.18]	[0.50,0.50]	[0.34,0.55]	0.079	0.330
Brand Rep.	[0.45,0.66]	[0.13,0.25]	[0.45,0.66]	[0.50,0.50]	0.097	0.410

Table B.11: The Revised and Completed 4x4 Matrix with Interval Weights, DM 4, α=0.2

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.10,0.17]	[0.43,0.57]	[0.36,0.53]	0.086	0.311
Eng. Pw.	[0.40,0.90]	[0.50,0.50]	[0.40,0.90]	[0.40,0.86]	0.191	0.689
Design	[0.43,0.57]	[0.10,0.17]	[0.50,0.50]	[0.36,0.53]	0.086	0.311
Brand Rep.	[0.47,0.64]	[0.14,0.24]	[0.47,0.64]	[0.50,0.50]	0.106	0.387

Table B.12: The Revised and Completed 4x4 Matrix with Interval Weights, DM 4, α=0.4

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.11,0.16]	[0.45,0.55]	[0.39,0.51]	0.093	0.293
Eng. Pw.	[0.39,0.89]	[0.50,0.50]	[0.39,0.89]	[0.39,0.85]	0.197	0.670
Design	[0.45,0.55]	[0.11,0.16]	[0.50,0.50]	[0.39,0.51]	0.093	0.293
Brand Rep.	[0.49,0.61]	[0.15,0.22]	[0.49,0.61]	[0.50,0.50]	0.116	0.364

Table B.13: The Revised and Completed 4x4 Matrix with Interval Weights, DM 5, α=0.0

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.00,0.13]	[0.03,0.09]	[0.18,0.41]	0.044	0.086
Eng. Pw.	[0.87,0.96]	[0.50,0.50]	[0.46,0.53]	[0.62,0.84]	0.297	0.498
Design	[0.91,0.97]	[0.47,0.54]	[0.50,0.50]	[0.66,0.82]	0.327	0.521
Brand Rep.	[0.59,0.66]	[0.16,0.38]	[0.18,0.34]	[0.50,0.50]	0.089	0.170

Table B.14: The Revised and Completed 4x4 Matrix with Interval Weights, DM 5, α=0.2

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.00,0.10]	[0.03,0.08]	[0.21,0.39]	0.045	0.078
Eng. Pw.	[0.90,0.98]	[0.50,0.50]	[0.48,0.53]	[0.66,0.84]	0.329	0.498
Design	[0.92,0.97]	[0.47,0.52]	[0.50,0.50]	[0.68,0.80]	0.331	0.490
Brand Rep.	[0.61,0.66]	[0.16,0.34]	[0.20,0.32]	[0.50,0.50]	0.093	0.158

Table B.15: The Revised and Completed 4x4 Matrix with Interval Weights, DM 5, α=0.4

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i^*}$	
					l_{ij}^*	u_{ij}^*
Comfort	[0.50,0.50]	[0.00,0.08]	[0.04,0.08]	[0.24,0.37]	0.046	0.071
Eng. Pw.	[0.92,1.00]	[0.50,0.50]	[0.50,0.54]	[0.70,0.83]	0.362	0.499
Design	[0.92,0.96]	[0.46,0.50]	[0.50,0.50]	[0.70,0.79]	0.334	0.458
Brand Rep.	[0.63,0.67]	[0.17,0.30]	[0.21,0.30]	[0.50,0.50]	0.097	0.147

The revised and completed 5x5 mobile selection matrices with interval weights are shown below:

Table B.16: The Revised and Completed 5x5 Matrix with Interval Weights, DM 1, α=0.0

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.66,0.82]	[0.75,0.87]	[0.50,0.75]	[0.50,0.75]	0.180	0.589
Screen	[0.18,0.34]	[0.50,0.50]	[0.43,0.71]	[0.18,0.59]	[0.18,0.59]	0.052	0.288
Weight	[0.13,0.25]	[0.29,0.57]	[0.50,0.50]	[0.13,0.50]	[0.13,0.50]	0.040	0.214
Camera	[0.25,0.50]	[0.41,0.75]	[0.50,0.75]	[0.50,0.50]	[0.25,0.75]	0.074	0.410
Battery	[0.25,0.50]	[0.41,0.75]	[0.50,0.75]	[0.25,0.75]	[0.50,0.50]	0.080	0.469

Table B.17: The Revised and Completed 5x5 Matrix with Interval Weights, DM 1, α=0.2

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.68,0.80]	[0.76,0.86]	[0.54,0.73]	[0.54,0.73]	0.218	0.552
Screen	[0.20,0.32]	[0.50,0.50]	[0.46,0.68]	[0.24,0.55]	[0.24,0.55]	0.063	0.252
Weight	[0.14,0.24]	[0.32,0.54]	[0.50,0.50]	[0.18,0.47]	[0.18,0.47]	0.049	0.183
Camera	[0.27,0.46]	[0.45,0.76]	[0.53,0.77]	[0.50,0.50]	[0.31,0.69]	0.093	0.377
Battery	[0.27,0.46]	[0.45,0.76]	[0.53,0.77]	[0.31,0.69]	[0.50,0.50]	0.093	0.377

Table B.18: The Revised and Completed 5x5 Matrix with Interval Weights, DM 1, α=0.4

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.70,0.79]	[0.78,0.85]	[0.58,0.72]	[0.58,0.72]	0.254	0.513
Screen	[0.21,0.30]	[0.50,0.50]	[0.49,0.65]	[0.29,0.52]	[0.29,0.52]	0.075	0.214
Weight	[0.15,0.22]	[0.35,0.51]	[0.50,0.50]	[0.23,0.44]	[0.23,0.44]	0.057	0.157
Camera	[0.28,0.42]	[0.48,0.71]	[0.56,0.77]	[0.50,0.50]	[0.36,0.64]	0.111	0.330
Battery	[0.28,0.42]	[0.48,0.71]	[0.56,0.77]	[0.36,0.64]	[0.50,0.50]	0.111	0.330

Table B.19: The Revised and Completed 5x5 Matrix with Interval Weights, DM 2, α=0.0

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.66,0.75]	[0.25,0.50]	[0.50,0.75]	[0.50,0.75]	0.106	0.458
Screen	[0.25,0.34]	[0.50,0.50]	[0.00,0.34]	[0.25,0.50]	[0.25,0.50]	0.039	0.198
Weight	[0.50,0.75]	[0.66,1.00]	[0.50,0.50]	[0.50,1.00]	[0.50,1.00]	0.194	0.715
Camera	[0.25,0.50]	[0.41,0.50]	[0.00,0.50]	[0.50,0.50]	[0.25,0.50]	0.046	0.239
Battery	[0.25,0.50]	[0.41,0.50]	[0.00,0.50]	[0.25,0.50]	[0.50,0.50]	0.046	0.239

Table B.20: The Revised and Completed 5x5 Matrix with Interval Weights, DM 2, α=0.2

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.68,0.77]	[0.27,0.46]	[0.54,0.73]	[0.54,0.73]	0.124	0.411
Screen	[0.23,0.32]	[0.50,0.50]	[0.00,0.28]	[0.28,0.50]	[0.28,0.50]	0.043	0.167
Weight	[0.54,0.73]	[0.72,1.00]	[0.50,0.50]	[0.58,0.97]	[0.58,0.97]	0.242	0.678
Camera	[0.27,0.46]	[0.45,0.53]	[0.03,0.42]	[0.50,0.50]	[0.31,0.53]	0.053	0.206
Battery	[0.27,0.46]	[0.45,0.53]	[0.03,0.42]	[0.31,0.53]	[0.50,0.50]	0.053	0.206

Table B.21: The Revised and Completed 5x5 Matrix with Interval Weights, DM 2, α=0.4

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.70,0.78]	[0.28,0.42]	[0.58,0.72]	[0.58,0.72]	0.143	0.367
Screen	[0.22,0.30]	[0.50,0.50]	[0.00,0.22]	[0.29,0.50]	[0.29,0.50]	0.046	0.143
Weight	[0.58,0.72]	[0.78,1.00]	[0.50,0.50]	[0.65,0.93]	[0.65,0.93]	0.289	0.639
Camera	[0.28,0.42]	[0.48,0.57]	[0.07,0.35]	[0.50,0.50]	[0.36,0.57]	0.062	0.185
Battery	[0.28,0.42]	[0.48,0.57]	[0.07,0.35]	[0.36,0.57]	[0.50,0.50]	0.062	0.185

Table B.22: The Revised and Completed 5x5 Matrix with Interval Weights, DM 3, α=0.0

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.43,0.68]	[0.18,0.34]	[0.34,0.66]	[0.18,0.34]	0.064	0.214
Screen	[0.29,0.50]	[0.50,0.50]	[0.13,0.25]	[0.29,0.50]	[0.13,0.25]	0.049	0.135
Weight	[0.66,0.82]	[0.75,0.87]	[0.50,0.50]	[0.66,0.82]	[0.50,0.50]	0.224	0.473
Camera	[0.34,0.66]	[0.43,0.68]	[0.18,0.34]	[0.50,0.50]	[0.18,0.34]	0.064	0.214
Battery	[0.66,0.82]	[0.75,0.87]	[0.50,0.50]	[0.66,0.82]	[0.50,0.50]	0.224	0.473

Table B.23: The Revised and Completed 5x5 Matrix with Interval Weights, DM 3, α=0.2

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.46,0.68]	[0.20,0.32]	[0.38,0.62]	[0.20,0.32]	0.072	0.192
Screen	[0.32,0.50]	[0.50,0.50]	[0.14,0.24]	[0.32,0.50]	[0.14,0.24]	0.055	0.126
Weight	[0.68,0.80]	[0.76,0.86]	[0.50,0.50]	[0.68,0.80]	[0.50,0.50]	0.245	0.447
Camera	[0.38,0.62]	[0.46,0.68]	[0.20,0.32]	[0.50,0.50]	[0.20,0.32]	0.072	0.192
Battery	[0.68,0.80]	[0.76,0.86]	[0.68,0.80]	[0.68,0.80]	[0.50,0.50]	0.245	0.447

Table B.24: The Revised and Completed 5x5 Matrix with Interval Weights, DM 3, α=0.4

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.49,0.65]	[0.21,0.30]	[0.41,0.59]	[0.21,0.30]	0.080	0.168
Screen	[0.35,0.50]	[0.50,0.50]	[0.15,0.22]	[0.35,0.50]	[0.15,0.22]	0.061	0.118
Weight	[0.70,0.79]	[0.78,0.85]	[0.50,0.50]	[0.70,0.79]	[0.50,0.50]	0.268	0.421
Camera	[0.41,0.59]	[0.49,0.65]	[0.21,0.30]	[0.50,0.50]	[0.21,0.30]	0.080	0.168
Battery	[0.70,0.79]	[0.78,0.85]	[0.50,0.50]	[0.70,0.79]	[0.50,0.50]	0.268	0.421

Table B.25: The Revised and Completed 5x5 Matrix with Interval Weights, DM 4, α=0.0

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.50,0.50]	[1.00,1.00]	[0.50,0.50]	[0.50,0.50]	0.314	0.360
Screen	[0.50,0.50]	[0.50,0.50]	[0.50,0.50]	[0.50,0.50]	[0.50,0.50]	0.121	0.138
Weight	[0.00,0.00]	[0.00,0.00]	[0.50,0.50]	[0.00,0.00]	[0.00,0.00]	0.035	0.040
Camera	[0.25,0.50]	[0.25,0.50]	[0.50,0.75]	[0.50,0.50]	[0.50,0.75]	0.102	0.217
Battery	[0.50,0.50]	[0.50,0.50]	[1.00,1.00]	[0.50,0.50]	[0.50,0.50]	0.314	0.360

Table B.26: The Revised and Completed 5x5 Matrix with Interval Weights, DM 4, α=0.2

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.31,0.69]	[0.58,0.81]	[0.54,0.73]	[0.58,0.81]	0.200	0.533
Screen	[0.31,0.50]	[0.50,0.50]	[0.46,0.50]	[0.46,0.50]	[0.46,0.50]	0.113	0.267
Weight	[0.03,0.42]	[0.03,0.42]	[0.50,0.50]	[0.27,0.46]	[0.31,0.53]	0.061	0.219
Camera	[0.27,0.46]	[0.27,0.46]	[0.54,0.73]	[0.50,0.50]	[0.54,0.73]	0.130	0.379
Battery	[0.03,0.42]	[0.03,0.42]	[0.31,0.53]	[0.27,0.46]	[0.50,0.50]	0.061	0.219

Table B.27: The Revised and Completed 5x5 Matrix with Interval Weights, DM 4, α=0.4

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.36,0.64]	[0.65,0.86]	[0.58,0.72]	[0.65,0.86]	0.165	0.542
Screen	[0.36,0.64]	[0.50,0.50]	[0.43,0.93]	[0.43,0.72]	[0.43,0.93]	0.109	0.545
Weight	[0.07,0.35]	[0.07,0.35]	[0.50,0.50]	[0.28,0.42]	[0.36,0.57]	0.046	0.189
Camera	[0.28,0.42]	[0.28,0.42]	[0.58,0.72]	[0.50,0.50]	[0.58,0.72]	0.098	0.336
Battery	[0.07,0.35]	[0.07,0.35]	[0.36,0.57]	[0.28,0.42]	[0.50,0.50]	0.046	0.189

Table B.28: The Revised and Completed 5x5 Matrix with Interval Weights, DM 5, α=0.0

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.13,0.50]	[0.13,0.50]	[0.13,0.50]	[0.13,0.25]	0.036	0.213
Screen	[0.50,0.75]	[0.50,0.50]	[0.25,0.75]	[0.25,0.75]	[0.25,0.50]	0.068	0.426
Weight	[0.50,0.75]	[0.25,0.75]	[0.50,0.50]	[0.25,0.75]	[0.25,0.50]	0.068	0.426
Camera	[0.50,0.75]	[0.25,0.75]	[0.25,0.75]	[0.50,0.50]	[0.25,0.50]	0.068	0.426
Battery	[0.75,0.87]	[0.50,0.75]	[0.50,0.75]	[0.50,0.75]	[0.50,0.50]	0.158	0.581

Table B.29: The Revised and Completed 5x5 Matrix with Interval Weights, DM 5, α=0.2

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.18,0.47]	[0.18,0.47]	[0.18,0.47]	[0.14,0.24]	0.048	0.183
Screen	[0.47,0.50]	[0.50,0.50]	[0.31,0.50]	[0.31,0.50]	[0.27,0.46]	0.076	0.227
Weight	[0.53,0.77]	[0.31,0.69]	[0.50,0.50]	[0.31,0.69]	[0.27,0.46]	0.093	0.375
Camera	[0.53,0.77]	[0.31,0.69]	[0.31,0.69]	[0.50,0.50]	[0.27,0.46]	0.093	0.375
Battery	[0.76,0.86]	[0.54,0.73]	[0.54,0.73]	[0.54,0.73]	[0.50,0.50]	0.217	0.547

Table B.30: The Revised and Completed 5x5 Matrix with Interval Weights, DM 5, α=0.4

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_i^*}$	
						l_{ij}^*	u_{ij}^*
Cost	[0.50,0.50]	[0.23,0.44]	[0.23,0.44]	[0.23,0.44]	[0.15,0.22]	0.051	0.153
Screen	[0.56,0.77]	[0.50,0.50]	[0.36,0.64]	[0.36,0.64]	[0.28,0.42]	0.099	0.323
Weight	[0.56,0.77]	[0.36,0.64]	[0.50,0.50]	[0.36,0.64]	[0.28,0.42]	0.099	0.323
Camera	[0.56,0.77]	[0.36,0.64]	[0.36,0.64]	[0.50,0.50]	[0.28,0.42]	0.099	0.323
Battery	[0.78,0.85]	[0.58,0.72]	[0.58,0.72]	[0.58,0.72]	[0.50,0.50]	0.224	0.499

Appendix C.

The original completed by 5 DMs (sample of 40) 4x4 car selection matrices and interval weights are shown below:

Table C.1: The Original 4x4 Matrix Completed by DM 1 and interval weights

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i}$	
					l_{ij}	u_{ij}
Comfort	[0.50,0.50]	[0.82,0.91]	[0.91,0.97]	[0.82,0.91]	0.411	0.583
Eng. Pw.	[0.09,0.18]	[0.50,0.50]	[0.82,0.91]	[0.82,0.91]	0.239	0.389
Design	[0.03,0.09]	[0.09,0.18]	[0.50,0.50]	[0.75,0.87]	0.106	0.200
Brand Rep.	[0.09,0.18]	[0.09,0.18]	[0.13,0.25]	[0.50,0.50]	0.036	0.061

Table C.2: The Original 4x4 Matrix Completed by DM 2 and interval weights

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i}$	
					l_{ij}	u_{ij}
Comfort	[0.50,0.50]	[0.50,0.75]	[0.75,0.87]	[0.66,0.82]	0.273	0.552
Eng. Pw.	[0.25,0.50]	[0.50,0.50]	[0.66,0.82]	[0.25,0.50]	0.129	0.335
Design	[0.13,0.25]	[0.18,0.34]	[0.50,0.50]	[0.13,0.25]	0.053	0.120
Brand Rep.	[0.18,0.34]	[0.50,0.75]	[0.75,0.87]	[0.50,0.50]	0.191	0.435

Table C.3: The Original 4x4 Matrix Completed by DM 3 and interval weights

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_i}$	
					l_{ij}	u_{ij}
Comfort	[0.50,0.50]	[0.75,0.87]	[0.87,0.94]	[0.66,0.82]	0.363	0.581
Eng. Pw.	[0.13,0.25]	[0.50,0.50]	[0.66,0.82]	[0.82,0.91]	0.224	0.414
Design	[0.06,0.13]	[0.18,0.34]	[0.50,0.50]	[0.66,0.82]	0.100	0.223
Brand Rep.	[0.18,0.34]	[0.09,0.18]	[0.18,0.34]	[0.50,0.50]	0.047	0.094

Table C.4: The Original 4x4 Matrix Completed by DM 4 and interval weights

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_\iota}$	
					l_{ij}	u_{ij}
Comfort	[0.50,0.50]	[0.09,0.18]	[0.25,0.50]	[0.25,0.50]	0.057	0.155
Eng. Pw.	[0.82,0.91]	[0.50,0.50]	[0.82,0.91]	[0.75,0.87]	0.431	0.705
Design	[0.50,0.75]	[0.09,0.18]	[0.50,0.50]	[0.25,0.50]	0.080	0.236
Brand Rep.	[0.50,0.75]	[0.13,0.25]	[0.50,0.75]	[0.50,0.50]	0.108	0.310

Table C.5: The Original 4x4 Matrix Completed by DM 5 and interval weights

	Comfort	Eng. Pw.	Design	Brand Rep.	$\widetilde{S_\iota}$	
					l_{ij}	u_{ij}
Comfort	[0.50,0.50]	[0.09,0.18]	[0.03,0.09]	[0.75,0.87]	0.121	0.243
Eng. Pw.	[0.82,0.91]	[0.50,0.50]	[0.75,0.87]	[0.50,0.75]	0.285	0.495
Design	[0.91,0.97]	[0.13,0.25]	[0.50,0.50]	[0.66,0.82]	0.275	0.469
Brand Rep.	[0.13,0.25]	[0.25,0.50]	[0.18,0.34]	[0.50,0.50]	0.049	0.112

The original completed by 5 DMs (sample of 40) 5x5 mobile selection matrices and interval weights are shown below:

Table C.6: The Original 5x5 Matrix Completed by DM 1 and interval weights

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S_\iota}$	
						l_{ij}	u_{ij}
Cost	[0.50,0.50]	[0.66,0.82]	[0.75,0.87]	[0.50,0.75]	[0.50,0.75]	0.197	0.502
Screen	[0.18,0.34]	[0.50,0.50]	[0.50,0.75]	[0.18,0.34]	[0.25,0.50]	0.062	0.222
Weight	[0.13,0.25]	[0.25,0.50]	[0.50,0.50]	[0.13,0.25]	[0.25,0.50]	0.044	0.144
Camera	[0.25,0.50]	[0.66,0.82]	[0.75,0.87]	[0.50,0.50]	[0.50,0.75]	0.175	0.458
Battery	[0.25,0.50]	[0.50,0.75]	[0.50,0.75]	[0.25,0.50]	[0.50,0.50]	0.085	0.308

Table C.7: The Original 5x5 Matrix Completed by DM 2 and interval weights

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S}_\iota$	
						l_{ij}	u_{ij}
Cost	[0.50,0.50]	[0.66,0.82]	[0.25,0.50]	[0.50,0.75]	[0.50,0.75]	0.132	0.419
Screen	[0.18,0.34]	[0.50,0.50]	[0.18,0.34]	[0.25,0.50]	[0.25,0.50]	0.048	0.168
Weight	[0.50,0.75]	[0.66,0.82]	[0.50,0.50]	[0.66,0.82]	[0.50,0.75]	0.179	0.500
Camera	[0.25,0.50]	[0.50,0.75]	[0.18,0.34]	[0.50,0.50]	[0.18,0.34]	0.065	0.238
Battery	[0.25,0.50]	[0.50,0.75]	[0.25,0.50]	[0.66,0.82]	[0.50,0.50]	0.112	0.366

Table C.8: The Original 4x4 Matrix Completed by DM 3 and interval weights

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S}_\iota$	
						l_{ij}	u_{ij}
Cost	[0.50,0.50]	[0.50,0.75]	[0.50,0.75]	[0.25,0.50]	[0.18,0.34]	0.085	0.280
Screen	[0.25,0.50]	[0.50,0.50]	[0.66,0.82]	[0.18,0.34]	[0.13,0.25]	0.086	0.240
Weight	[0.25,0.50]	[0.18,0.34]	[0.50,0.50]	[0.18,0.34]	[0.50,0.50]	0.062	0.150
Camera	[0.50,0.75]	[0.66,0.82]	[0.66,0.82]	[0.50,0.50]	[0.18,0.34]	0.154	0.394
Battery	[0.66,0.82]	[0.75,0.87]	[0.50,0.50]	[0.66,0.82]	[0.50,0.50]	0.220	0.477

Table C.9: The Original 5x5 Matrix Completed by DM 4 and interval weights

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S}_\iota$	
						l_{ij}	u_{ij}
Cost	[0.50,0.50]	[0.50,0.75]	[0.50,0.75]	[0.50,0.75]	[0.50,0.75]	0.139	0.451
Screen	[0.25,0.50]	[0.50,0.50]	[0.66,0.82]	[0.50,0.75]	[0.66,0.82]	0.170	0.473
Weight	[0.25,0.50]	[0.18,0.34]	[0.50,0.50]	[0.25,0.50]	[0.50,0.50]	0.069	0.201
Camera	[0.25,0.50]	[0.25,0.50]	[0.50,0.75]	[0.50,0.50]	[0.50,0.75]	0.095	0.344
Battery	[0.25,0.50]	[0.18,0.34]	[0.50,0.50]	[0.25,0.50]	[0.50,0.50]	0.069	0.201

Table C.10: The Original 5x5 Matrix Completed by DM 5 and interval weights

	Cost	Screen	Weight	Camera	Battery	$\widetilde{S}_\iota$	
						l_{ij}	u_{ij}
Cost	[0.50,0.50]	[0.50,0.50]	[0.50,0.75]	[0.25,0.50]	[0.13,0.25]	0.085	0.268
Screen	[0.50,0.50]	[0.50,0.50]	[0.50,0.75]	[0.25,0.50]	[0.25,0.50]	0.089	0.289
Weight	[0.25,0.50]	[0.25,0.50]	[0.50,0.50]	[0.25,0.50]	[0.25,0.50]	0.056	0.212
Camera	[0.50,0.75]	[0.50,0.75]	[0.50,0.75]	[0.50,0.50]	[0.25,0.50]	0.115	0.400
Battery	[0.75,0.87]	[0.50,0.75]	[0.50,0.75]	[0.50,0.75]	[0.50,0.50]	0.193	0.520

Appendix D.

The Accuracy values of fuzzy synthetic extents of 5 DMs (sample of 40) for α=0.0, α=0.2 and α=0.4 are shown below:

Table D.1: Accuracy of Fuzzy Synthetic Extents of 4x4 Matrix, DM 1

	Accuracy		
	α=0.0	α=0.2	α=0.4
Comfort	67.09%	66.43%	66.00%
Eng. Power	50.18%	49.69%	49.15%
Design	0.00%	0.00%	0.00%
Brand Reputation	19.71%	12.87%	0.17%

Table D.2: Accuracy of Fuzzy Synthetic Extents of 4x4 Matrix, DM 2

	Accuracy		
	α=0.0	α=0.2	α=0.4
Comfort	71.82%	69.66%	61.27%
Eng. Power	54.30%	56.16%	57.85%
Design	35.87%	32.08%	23.46%
Brand Reputation	50.03%	33.64%	3.92%

Table D.3: Accuracy of Fuzzy Synthetic Extents of 4x4 Matrix, DM 3

	Accuracy		
	α=0.0	α=0.2	α=0.4
Comfort	60.78%	47.19%	31.75%
Eng. Power	8.60%	0.00%	0.00%
Design	55.79%	38.79%	9.14%
Brand Reputation	0.00%	0.00%	0.00%

Table D.4: Accuracy of Fuzzy Synthetic Extents of 4x4 Matrix, DM 4

	Accuracy		
	α=0.0	α=0.2	α=0.4
Comfort	30.47%	24.51%	16.69%
Eng. Power	52.34%	43.95%	34.74%
Design	61.82%	55.15%	46.74%
Brand Reputation	64.57%	57.62%	48.83%

Table D.5: Accuracy of Fuzzy Synthetic Extents of 4x4 Matrix, DM 5

	Accuracy		
	α=0.0	α=0.2	α=0.4
Comfort	0.00%	0.00%	0.00%
Eng. Power	98.86%	86.17%	67.15%
Design	73.43%	74.23%	76.06%
Brand Reputation	28.24%	15.55%	0.00%

Table D.6: Accuracy of Fuzzy Synthetic Extents of 5x5 Matrix, DM 1

	Accuracy		
	α=0.0	α=0.2	α=0.4
Cost	74.56%	73.05%	70.47%
Screen	67.44%	67.73%	68.90%
Weight	57.56%	59.39%	57.89%
Camera	70.15%	62.58%	48.46%
Battery	57.33%	62.87%	61.08%

Table D.7: Accuracy of Fuzzy Synthetic Extents of 5x5 Matrix, DM 2

	Accuracy		
	α=0.0	α=0.2	α=0.4
Cost	81.27%	80.04%	76.47%
Screen	75.49%	76.86%	73.78%
Weight	58.73%	51.33%	40.98%
Camera	89.75%	84.80%	78.47%
Battery	65.78%	48.91%	27.80%

Table D.8: Accuracy of Fuzzy Synthetic Extents of 5x5 Matrix, DM 3

	Accuracy		
	α=0.0	α=0.2	α=0.4
Cost	85.94%	75.28%	57.61%
Screen	56.63%	39.61%	15.86%
Weight	0.00%	0.00%	0.00%
Camera	40.13%	13.38%	0.00%
Battery	100.00%	100.00%	100.00%

Table D.9: Accuracy of Fuzzy Synthetic Extents of 5x5 Matrix, DM 4

	Accuracy		
	α=0.0	α=0.2	α=0.4
Cost	100.00%	65.55%	49.63%
Screen	0.00%	45.76%	41.77%
Weight	0.00%	66.81%	55.42%
Camera	100.00%	73.65%	62.74%
Battery	0.00%	66.81%	55.42%

Table D.10: Accuracy of Fuzzy Synthetic Extents of 5x5 Matrix, DM 5

	Accuracy		
	α=0.0	α=0.2	α=0.4
Cost	72.35%	62.77%	40.23%
Screen	55.71%	81.73%	53.47%
Weight	40.37%	34.09%	29.95%
Camera	79.31%	80.80%	70.43%
Battery	77.21%	79.20%	71.36%

BIOGRAPHICAL SKETCH

The author was born in Iran on July 08, 1988. She has finished her high school education in Cagaloglu Anatolian High School in 2006. She received her B. Sc. degree in Industrial Engineering from Isik University, Istanbul, in 2012. Since 2014, she is in the master's degree program in Logistics and Financial Management in Galatasaray University. Her research interests and focus are in the areas of multiple criteria decision making, fuzzy set theory, decision making with incomplete information.