TOPOLOGY OF ALGEBRAIC FUNCTIONS AS TORUS KNOTS

(CEBİRSEL FONKSİYONLARIN TORUS DÜĞÜMLERİ
TOPOLOJİSİ)

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Ümran Gülbin AYDIN, B.S.

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prepared by **Ümran Gülbin AYDIN** in partial fulfillment of the requirements for the degree of **Master of Science in Mathematics** at the **Galatasaray University** is approved by the

Examining Committee :

Prof. Dr. Susumu TANABÉ (Supervisor)
Department of Mathematics
Galatasaray University

Prof. Muhammed Uludağ
Department of Mathematics
Galatasaray University

Assoc. Prof. Çağrı Karakurt
Department of Mathematics
Boğaziçi University

Date: 15.05.2019

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LIST OF SYMBOLS

π_1	:Fundamental Group
K_i	:The iterated knot of order i
T_1	:Monodromy transformation
$f(x, y)$	: Function of two complex variable x and y
(p_1, q_1)	: Puiseux Pairs
(m_1, n_1)	: Newton Pairs
g_i	: Loop around y_i
L_i	: Longitiduional turn of K_{i+1} in the neighbourhood of K_i
M_i	: Meridianal turn of K_{i+1} in the neighbourhood of K_i

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ABSTRACT

The aim is to understand the topology of an algebraic function around it's singular point which leads to study the topology of the knot. While studying a knot K, one can consider the topology of the complement of it,which appears as an invariant since all information about the homotopy of it is contained in its fundamental group.
First the Newton pairs are computed using Newton Polygon Method. Then, the fundamental group of the branch is stated by computing its generators and relations between these generators. It has been shown that the characteristic exponents carry out the information about the topology of the branch. More explicitly, if two branches have the same characteristic exponent, then the knots associated to the branches are isotopic.

Keywords: Fundamental Group, Newton Polygon, Puiseux Pairs, Newton Pairs, Iterated Torus Knots

ÖZET

Bu çalışmada cebirsel bir fonksiyonun singüler noktası etrafındaki topolojisinin anlaşılması hedeflenmiştir. Düğümler, cebirsel fonksiyonların tekilliğinin topolojik imajı olarak görülür. Bir düğüm çalışılırken, düğümün kendisi yerine, tümleyeni çalışılabilir. Temel grubunda homotopisi ile ilgili bütün bilgiler olduğundan, bu tümleyen aslında bir düğüm invaryantıdır.
Öncelikle Newton Poligonu yöntemi ile Newton çiftleri bulunur. Daha sonra dalların temel grupları yazılır, bu temel grubun üreteçleri arasındaki ilişkiler Newton çiftleri ile gösterilebildiğinden karakteristik kuvvetlerin dalın topolojisi hakkında bütün bilgiyi taşıdığı sonucuna varılır. Daha farklı bir ifade ile, eğer karakteristik kuvvetleri aynı olan iki dal varsa, bu dallarla ilişkilendirilen düğümler aynıdır.

Anahtar Sözcükler: Temel Grup, Newton Poligonu, Puiseux çifleri, Newton Çiftleri, Yinelemeli Torus Düğümleri

1. INTRODUCTION

We begin with an algebraic function $f(x,y)$ of two complex variable x,y of the form

$$f(x,y) = y^n + a_1(x)y^{n-1} + ... + a_{n-1}y^1 + a_n(x) = 0$$

where $a_i(x) \in C[x]$. We utilized Newton's algorithm to compute Puiseux and Newton pairs of this algebraic function.

In chapter 2, we compute the fundamental groups of branches and show that we have $k+1$ generators and k relations for k^{th} fundamental group of the branch. We use Zariski (1932)'s construction for that. We show that the characteristic pairs determine the relation between generators of fundamental group.

In Chapter 3, after giving a brief summary of knots, we consider the set of zeros of the polynomial $f(x,y)$. A point (x_0, y_0) in the zero set for which y_0 is a multiple root of the polynomial $f(x_0, y)$ is called a branch point and we prove that the stereographic projection of the intersection of the singularity consisting of one branch with the boundary of a small neighbourhood is a torus knot via Brauner (1928)'s construction. In the second section, we introduce iterated torus knots and compute the linking number of these knots, both in terms of Puiseux pairs and also of Newton pairs using homological tools.

2. PRELIMINARIES

2.1 Homotopy

Definition 2.1.1. *Two directed curves on a surface F are called homotopic if they can be continously deformed into each other on F while their beginning points and end points remain fixed.*

Remark that beginning and end points of one curve must coincide with the beginning and end points of the other curve and F is a connected surface in space or Riemann region over the z-sphere.
If A and B are two homotopic curves, we say that $A \sim B$.

Definition 2.1.2. *Let I be the closed interval $[0, 1]$ of R and let X, Y be two topological spaces and $f_0 : X \to Y$ and $f_1 : X \to Y$ be two continuous maps. A continuous map $F : X \, x \, I \to Y$ is called homotopy between continuous maps from f_0 to f_1 if it satisfies the following condition for all $x \in X$;*

$$F(x, 0) = f_0(x)$$
$$F(x, 1) = f_1(x)$$

If there exists a homotopy from f_0 to f_1, then we say that they are homotopic.

Definition 2.1.3. *A closed curve issuing from η is homotopic to zero if and only if with n no longer fixed, it is possible to contract the curve continuously on the surface to any point.*

Theorem 2.1.1. *Homotopy is an equivalence relation.*

Proof:

- $A \sim B$ implies that $B \sim A$.
- $A \sim B$ and $B \sim C$ implies that $A \sim C$.
- $A \sim A$ for all A.

Since we showed that homotopy is an equivalence relation, we can talk about homotopy classes. If the beginning point of a curve A_2 is same as the end point of another curve A_1, then A_1A_2 is the joint curve where we first traverse A_1 and then A_2.
If $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1A_2 \sim B_1B_2$.
Remark that multiplication is well defined and associative, i.e. $(A_1A_2)A_3 = A_1(A_2A_3)$.
A^{-1} is the curve obtained from A by reversing its direction. $(A^{-1})^{-1} = A$,
$(AB)^{-1} = B^{-1}A^{-1}$ and $A \sim B$ implies that $A^{-1} \sim B^{-1}$.

Consider curves A, B on the surface F with common beginning and end points η, that is closed curves issuing from η. First introduce the degenerate curve E consisting of the single point η. The curve E has the property $AE = EA = A$ If $A \sim E$, then A is homotopic to zero. Furthermore $C.C^{-1} \sim E$ and $C \sim D$ implies that $DC^{-1} \sim E$.
We seperate the closed curves issuing from η into homotopy classes, α, β, etc. where A_1 and A_2 belong to the same class if, and only if $A_1 \sim A_2$.
We define the product of two homotopy classes, $\alpha\beta$ as the class containing the product AB of a curve A in the class α and B in the class β. Remark that A and B are chosen arbitrarily, hence $\alpha\beta$ is independent of the particular choice of A and B. Now, we have proved the following theorem;

Theorem 2.1.2. *Homotopy class form a group under multiplication with unit element, E, consisting of the curves homotopic to zero.*

Definition 2.1.4. *A surface is simply connected if and only if its fundamental group consists of identity alone.*

The group described above is called the Fundamental Group of the given surface. Let η^* be another point of the surface and let H be a fixed curve on F, joining η to η^*. If A^* is a closed curve issuing from η^* the $HA^*H^{-1} = A$ is a closed curve issuing from η. If $A^* \sim B^*$ and $B = HB^*H^{-1}$, then $A \sim B$. The relation $HA^*H^{-1} = A$ associates with every homotopy class α^* relative to η^* a unique homotopy class α relative to η.
Conversely, if A is a closed curve issuing from η, then $A^* = H^{-1}AH$ is a closed curve issuing from η and we have $HA^*H^{-1} = HH^{-1}AHH^{-1} \sim EAE = A$ Hence our correspondence is one-to-one. Since, in addition,
$(A_1A_2)^* = H^{-1}A_1A_2H \sim H^{-1}A_1HH^{-1}A_2H = A_1^*A_2^*$. So, we conclude that the correspondence from A to A^* defines an isomorphism of the two fundamental groups. This implies that the homotopy group of a surface determined up to isomorphism by the surface alone (Siegel, 1969).

3. NEWTON POLYGONS AND PUISEUX EXPANSIONS

3.1 Introduction

Let $f(x, y)$ be a complex polynomial in 2 complex variables, x and y. In the neighbourhood of a smooth point of the curve $f(x, y) = 0$, the equation can be solved for one of the variables in terms of the other, but around a singular point one needs further calculations. In this case the power series expansion of the function $f(x, y)$ at the singular point is very useful to make the local investigation of plane curves. The straightforward method for doing this is *The Newton-Polygon Method.* The method can be explained briefly as solving the polynomial $f(x, y)$ for y in terms of x by means of a power series whose powers are rational numbers near the singular point, namely near the point $x = y = 0$, i.e. view $f \in K[x, y]$ as a polynomial of y with coefficients from $K[x, y]$, where $K[x, y]$ is the field of formal Puisseux series (Brieskorn and Knörrer, 1986).

3.2 The Algorithm

The general description of the algorithm is by following the below steps (Willis et al., 2008);

1. Let the equation below be given.

$$f(x, y) = \sum_{\substack{i \\ i \text{ finite}}} k_i x^{a_i} y^{b_i} = 0$$

   Draw the Newton polygon of $f(x, y)$ by plotting (a_i, b_i) for each term of f, where k_i is any complex coefficient.

2. Take a segment of the Newton polygon from the set of segments, such that no point is plotted below or to the left side of segments.

3. The first exponent, γ_1, is the negative of the slope of this segment.

4. Find $f(x, x^{\gamma_1}(c_1 + y_1))$

5. Take the lowest terms in x alone. Since $f(x, y)$=0, these must cancel and therefore solving for c_1 is possible.

6. Taking the values of γ_1 and c_1 and $\beta = x$-intercept on the Newton polygon of the segment that has been chosen, $f_1(x, y_1)$ has been found;

$$f_1(x, y_1) = x^{-\beta} f(x, x^{\gamma_1}(c_1 + y_1))$$

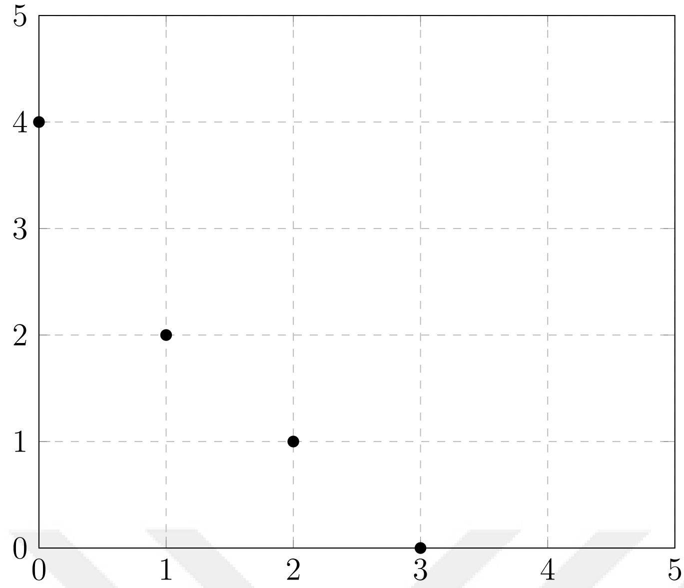


Figure 3.1: Plotting the points

7. Repeat the process for $f_1(x, y_1)$ to find γ_2 and c_2.
8. Continue until one of two happens;
   - $f_n(x, y_n)$ has a factor of y_n.
   - Newton polygon consists of a single segment with only two vertices one on each axis.

Example 3.2.1. $f(x, y)$= $2x^4 + x^2y + 4xy^2 + 4y^3$=0

Solution: This polynomial has a solution of the form y = $c_1x^{\gamma_1} + c_2x^{\gamma_1+\gamma_2}$ + ..., where γ_i's are negatives of the slopes of the lower segments of Newton polygon.
Step 1: First of all, we plot the the points (0,4), (1,2), (2,1) and (3,0) as shown in Figure 3.1
Now we can draw the convex hull using these points Figure 3.2;
Step 2: We take the leftmost segment such that no point is plotted below or to the left of it. This is the blue segment in Figure Figure 3.3.
Step 3: The slope of the blue segment is -2 and the first exponent, γ_1, is the negative of the slope of this segment. So γ_1= $-(-2)$=2.
Step 4: We now factor x^{γ_1} out of $y = c_1x^{\gamma_1} + c_2x^{\gamma_1+\gamma_2} + ...$
We get y = $x^2(c_1 + y_1)$ and substitute that into $f(x, y)$;

$$\begin{aligned} f(x, y) &= 2x^4 + x^2.x^2(c_1 + y_1) + 4x.x^4(c_1 + y_1)^2 + 4x^6(c_1 + y_1)^3 \\ &= x^4(2 + (c_1 + y_1) + 4x(c_1 + y_1)^2 + 4x^2(c_1 + y_1)^3) \end{aligned}$$

Step 5: The terms with lowest degree in x is $2 + c_1$. Then $2 + c_1 = 0$ i.e. $c_1 = -2$.
Step 6: Subsitute c_1 and divide by x^4. So we get;

$$\begin{aligned} f_1(x, y) &= y_1 + 4x(y_1 - 2)^2 + 4x^2(y_1 - 2)^3 \\ &= y_1 + 4xy_1^2 - 16xy_1 + 16x + 4x^2(y_1^3 - 6y_1^2 + 12y_1 - 8) \end{aligned}$$

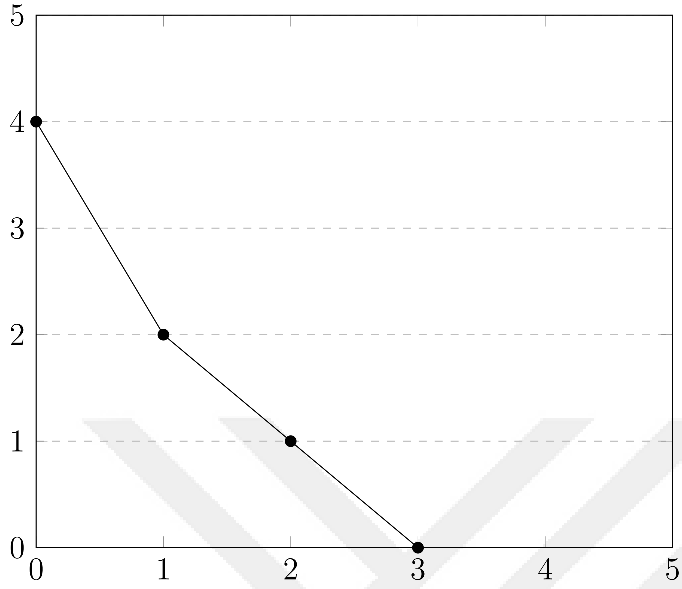


Figure 3.2: Newton Polygon for $f(x, y)$

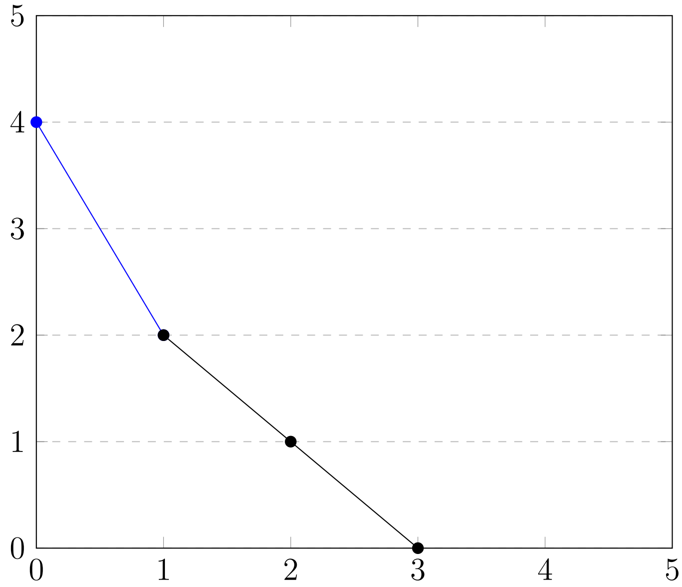
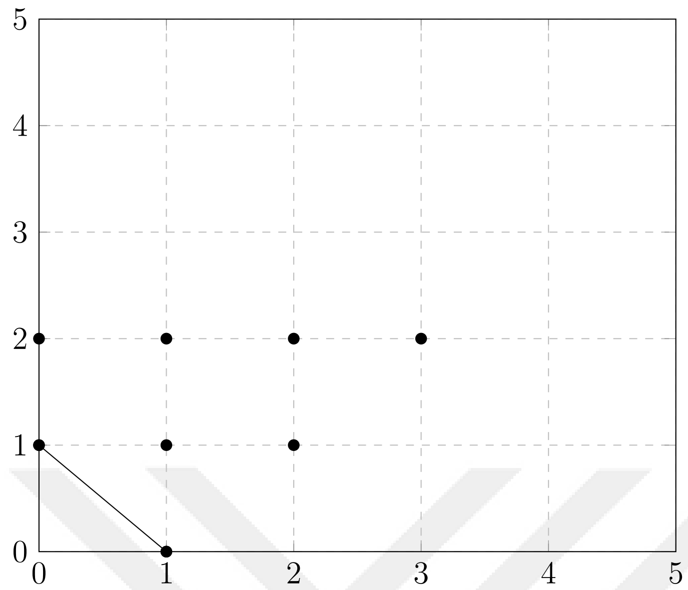


Figure 3.3: Newton Polygon for $f(x, y)$

Figure 3.4: Plotting the points



Now, we repeat the same process for y_2 and γ_2 to find f_2. The slope of the segment in Figure: 3.4 is -1, which implies that $\gamma_2 = 1$ and x-intercept of the line is 1.

$$y_1 = x(c_2 + y_2) \qquad \beta_2 = 1 \qquad y_1 = c_2 x + c_3 x^2 + ...$$

Substitute y_1 into $f_1(x, y)$.

$$\begin{aligned} f_1(x, y) &= y_1 + 4x(y_1 - 2)^2 + 4x^2(y_1 - 2)^3 \\ &= y_1 + 4xy_1^2 - 16xy_1 + 16x + 4x^2(y_1^3 - 6y_1^2 + 12y_1 - 8) \\ &= x(c_2 + y_2) + 4x^3(c_2 + y_2)^2 - 16x^2(c_2 + y_2) + 16x + 4x^5(c_2 + y_2)^3 - \\ &24x^4(c_2 + y_2)^2 + 48x^3(c_2 + y_2) - 32x^2 \end{aligned}$$

We let the terms of lowest degree in x alone equal zero. $c_2 + 16$=0 implies that c_2=-16. Substitute c_2 and divide by x.

$$\begin{aligned} f_2(x, y) = y_2 + 4x^2(y_2 - 16)^2 - 16x(y_2 - 16) + \\ 4x^4(y_2 - 16)^3 - 24x^3(y_2 - 16)^2 + 48x^2(y_2 - 16) - 32x \end{aligned}$$

$$\gamma_3 = 1 \qquad \beta_3 = 1 \qquad y_2 = x(c_3 + y_3)$$

$c_3 - 16c_2 - 32$=0 implies that c_3=224
This process can be continued similarly, so far we reached the below solution;

y = -2x$^2 - 16x^3 - 224x^4 + ...$

We have examined the case γ_1 = 2 and now we must return to the original polynomial $f(x,y)$ and examine the case $\gamma_1 = 1$.

$$\begin{aligned} f(x,y) &= 2x^4 + x^2y + 4xy^2 + 4y^3 = 0 \\ y &= x(c_1 + y_1) \\ f(x,y) &= 2x^4 + x^3(c_1+y_1) + 4x^3(c_1+y_1)^2 + 4x^3(c_1+y_1)^3 = 0 \end{aligned}$$

So let the coefficients of x^3 terms equal to zero;

$$\begin{aligned} c_1x^3 + 4c_1^2x^3 + 4c_1^3x^3 = 0 \\ \Rightarrow c_1 = 0, -1/2 \end{aligned}$$

Since we are considering only nonzero cases, assume $c_1 = -1/2$.
By substituting $c_1 = -1/2$ into $f(x,y)$ we get;

$$\begin{aligned} x^3(2x + (y_1 - 1/2) + 4(y_1 - 1/2)^2 + 4(y_1 - 1/2)^3) = 0 \\ f_1(x,y) = 4y_1^3 - 2y_1^2 + 2x \end{aligned}$$

Here we have the following points to plot; $(0,1), (2,0), (3,0)$. Clearly the slope of the graph is $-1/2$, so $\gamma_2 = 1/2$ and $y_1 = x^{1/2}(c_2 + y_2)$. Now, subsitute these into $f_1(x,y)$;

$$f_1(x,y) = 4x^{3/2}(c_2+y_2)^3 - 2x(c_2+y_2)^2 + 2x$$

Let the coefficients of x terms equal to zero;

$$\begin{aligned} -2c_2^2 + 2 &= 0 \\ c_2 &= 1 \end{aligned}$$

Substitute c_2 into the equation of $f_1(x,y)$ and divide by x;

$$4x^{1/2}(1+y_2)^3 - 2(1+y_2)^2 + 2 = 4x^{1/2} + 12x^{1/2}y_2 + 12x^{1/2}y_2^2 + 4x^{1/2}y_2^3 - 2 - 4y_2 - 2y_2^2 + 2$$

$\gamma_3 = 1/2$ and $y_2 = x^{1/2}(c_3 + y_3)$, subsitute these into $f_2(x,y)$ and let the coefficients of $x_{1/2}$ terms equal to zero;

$$\begin{aligned} -4c_3 + 4 &= 0 \\ c_3 &= 1 \end{aligned}$$

Continue similarly and get another solution of y for $f(x,y)$ as in the former case;

$$y = -1/2x + x^{3/2} + x^2 + ..$$

Definition 3.2.1. *The series $y(x) = \Sigma a_i x^{i/n}$ is a Puiseux expansion for the curve with equation $f(x,y) = 0$.*

3.3 Puiseux Pairs and Newton Pairs

Definition 3.3.1. *The pairs satisfying the following conditions* $(p_1, q_1), ..., (p_k, q_k)$ *are called the* ***Puiseux pairs*** *of f.*

- $q_1 < p_1$
- $p_{j-1}q_j < p_j$ *for* $j \geq 2$
- *gcd* $(p_j, q_j) = 1$ *for* $j = 1, ..., k$

Example 3.3.1. *Let* $y = x^{3/2} + x^{7/4}$*, rewrite this equation keeping* q_i*'s in the denominators of the latter powers;*

$$y = x^{3/2} + x^{7/2.2}$$

The Puiseux pairs are; (2,3) and (2,7)

Example 3.3.2. *Let* $y = x^{3/2} + x^{5/3} + x^{37/2}$ *Rewrite this equation;*

$$y = x^{3/2} + x^{5.2/2.3} + x^{37.3/2.3}$$

$$(p_1, q_1) = (2, 3), \qquad (p_2, q_2) = (3, 10)$$

That is to say that we obtain a sequence of approximations of the form;

$$y_1 = b_1 x^{m_1/n_1} \qquad y_2 = x^{m_1/n_1}(b_1 + b_2 x^{m_2/n_1 n_2})) \qquad etc.$$

where $gcd(m_i, n_i) = 1$ and both m_i and n_i are greater than zero. By this method, all solutions of $f(x, y) = 0$ near origin can be found. Writing these solutions in the form of fractional power series, i.e. in the form of Puiseux series, we get;

$$y = c_1 x^{q_1/p_1} + c_2 x^{q_2/p_1 p_2} + ... \qquad gcd(q_i, p_i) = 1, c_i \neq 0, \qquad q_1/p_1 < q_2/p_1 p_2 < ...$$

If we write the solutions in the multiplicative form we get;

$$y = x^{m_1/n_1}(a_1 + x^{m_2/n_1 n_2}(a_2 + ...(a_{s-1} + x^{m_s/n_1 ... n_2}(a_s + ...)...)))$$

with $m_i, n_i > 0$ and $gcd(m_i, n_i) = 1$ for all i.

Definition 3.3.2. *The pairs* (m_i, n_i) *are called the Newton pairs of the expansion.*

If we rearrange the expression in the above example and write it as described above, we get;

$$y = x^{3/2}(1 + x^{1/2.2})$$

So, we have the following Newton pairs;

$$(n_1, m_1) = (2, 3)$$
$$(n_2, m_2) = (2, 1)$$

Newton pairs can easily be computed from Puiseux pairs. Obviously, $n_i = p_i$ for all i and $m_1 = q_1$. $q_2 = m_1 n_2 + m_2$, substitute p_2 for n_2 and q_1 for m_1 and then we get the following equation for m_2

$$m_2 = q_2 - q_1 p_2$$

We continue in the same pattern for q_3,

$$\begin{aligned} q_3 = m_1 n_2 n_3 + m_2 n_3 + m_3 = q_1 p_2 p_3 + (q_2 - q_1 p_2) p_3 + m_3 \\ \Rightarrow m_3 = q_3 - p_3 q_2 \end{aligned}$$

We have then an algorithm between these pairs by doing similar computations for the rest of them.

Corollary 3.3.0.1. *Puiseux pairs (p_i, q_i) can be determined by the Newton pairs (m_i, n_i) by the following formulas;*

$$p_i = n_i, \qquad q_1 = m_1, \qquad q_i = m_i + m_{i-1} n_i$$

for $i > 1$.

From this point on, we can associate singular points of plane curves with Puiseux pairs. The number of the characteristic pairs is the genus of the branch.

3.4 About the Association of Puiseux Pairs, Braids and Coverings

The substitution of $x^{1/n}$ with z enables to get rid of the fractional exponent and it converts the Puiseux expansion into power series. Furthermore, this parametrisation is actually just the resolution of singularities. Let X be a space such that

$$X = \{(x, y) \in \mathbb{C} | y^n = x^m\}$$

So, we have a solution of y in terms of x, namely $y = x^{(m/n)}$. Let's do the parametrisation;

$$x = z^n.$$

obviously we have that;

$$y = z^m$$

To sum up, for any closed curve x, we have m different continuous functions y. Now consider the curve $C = \{(x, y) \in \mathbb{C}_2 | f(x, y) = 0\}$ and let $f(x, y) = y^m - x^n$. This curve has m branches. Let x makes one turn around the unit circle, i.e.

$$x(t) = e^{2\pi i t}$$

We have m different solutions for y.

$$\begin{array}{cccc} y_1(t) = e^{2\pi i \frac{n}{m} t} & y_2(t) = w e^{2\pi i \frac{n}{m} t} & y_3(t) = w^2 e^{2\pi i \frac{n}{m} t} & \dots \\ y = x^{\frac{n}{m}} & y_i = w^i x^{\frac{n}{m}} & w = e^{\frac{2\pi i}{m}} & \end{array}$$

For example, consider $y^2 - x^3 = 0$. Since there are two solutions for y, we have;

$$y_1(t) = e^{2\pi i \frac{3}{2} t} \qquad y_2(t) = -e^{2\pi i \frac{3}{2} t}$$

The associated Puiseux expansion is

$$y = x^{\frac{3}{2}} \qquad y_i = w^i x^{\frac{3}{2}} \qquad w = e^{\frac{2\pi i}{2}}$$

Let $C = \{(x, y) \in \mathbb{C}^2 | f(x, y) = 0\}$. Consider the maps;

$$\begin{array}{ccc} \pi : C \subset \mathbb{C}^2 \to D & \phi : \tilde{C} \to D & \tilde{\pi} : \tilde{C} \to C \\ (x, y) \mapsto x & x \mapsto x^m & x \mapsto (x^m, y(x)) \end{array}$$

For all x in $D - \{0\}$, we have m different solutions of the equation $f(x, y) = 0$

$$y_i(x) = y(e^{2\pi i/m} x^{1/m})$$

The graphs of these lie on a cylinder and if we consider the projection of this graph, we remark that it is a braid of m strands. Hence we have the result that a singularity can be associated with a braid, where braids are the results of the association of each point in the interval $[0, 1]$ with m different complex numbers in a continuous way. In other words,each closed curve $x(t) \in D - \{0\}$ may be lifted to m distinct paths; $y_1(t), ..., y_m(t)$, where y_i's are complex valued continuous functions from $[0, 1]$ interval to $\mathbb{C}$ such that $y_i \neq y_j$ for $i \neq j$.

$$\{y_1(0), ..., y_m(0)\} = \{y_1(1), ..., y_m(1)\}$$

We consider the set Y_m as the space of all complex polynomials of degree m with distinct roots. Now we can give the definition of braid;

Definition 3.4.1. *A braid, being the homotopy class of a closed path with initial and final point, is an element of the group $\pi_1(Y_m, \bar{y})$, where is the initial point.*

Puiseux pairs are also called **characteristic pairs**, since they report all the necessary topological information of the given algebraic equation. By comparing two examples, it becomes clearer;

Example 3.4.1. *Let the function f have the Puiseux expansion*

$$y = x^{3/2}$$

As discussed earlier, the Puiseux pair is (2,3) and this gives a two strand braid.

Example 3.4.2. *Let the function f_2 have the Puiseux expansion*

$$y = x^{3/2} + x^{37/2}$$

Although there are two terms in the expansion, there is only one characteristic pair and that is (2,3). This gives also a two strand braid.

We obtain Puiseux pairs from the Puiseux expansion. If we try to obtain Puiseux expansion from Puiseux pairs, that is also possible but one loses some information in this case such that the higher order terms, which cause oscillations in braid and the coefficients of the terms in the expansion. Now we are ready to prove the following theorem which states that Puiseux pairs suffice to describe the braid.

Theorem 3.4.1. *Puiseux expansions with the same Puiseux pair yield equivalent braids.*

Proof. Let M be a curve around the point $x = y = 0$ such that the caharacteristic pair (m, n) for this point is relatively prime. Around this point we do the following convergent parametrization;

$$x = t^m(a_m + ta_{m+1} + ...)y \qquad\qquad = t^n$$

where $gcd(m, n) = 1$ and $a_m \neq 0$. Now we allocate another curve M_0 with same pair such that it has the following parametrization;

$$x = t^m a_m$$
$$y = t^n$$

We choose (x_1, x_2)-plane where y is constant and intersects both of the curves. We denote the intersection points with M as $x_1, x_2, ..., x_n$ and the intersection points with M_0 as $x_{01}, x_{02}, ..., x_{0n}$.
t_0 is a parameter value that gives one of $x_1, x_2, ..., x_n$. Without loss of generality, we call it x_1 and so the same t_0 provides also a point from $x_{01}, x_{02}, ..., x_{0n}$, without loss of generality, we call it x_{01}. All the other intersection points in both sequences is resulted then by $\epsilon_i t_0$ for $i = 1, 2, ..., n-1$ where ϵ_i is the n^{th} unit root.
Now we consider the parametrization of M and M_0 and for two points x_i and x_{0i}, we have;

$$\mid x_i - x_{0i} \mid = \mid t^{m+1} \mid\mid a_{m+1} + ta_{m+2} + ... \mid$$

$$\mid x_j - x_{0i} \mid = \mid (\epsilon_j^m - \epsilon_i^m)a_m t^m + (\epsilon_j t^{)m+1} a_{m+1} + ... \mid$$

Since we are in the convergence neighbourhood, for $\mid t \mid < k$ where $k > 0$ we have that;

$$\mid a_{m+1} + ta_{m+2} + ... \mid \leq A$$

This implies for $t < \frac{m(\epsilon_j^m - \epsilon_i^m)}{2A}$

$$\mid x_i - x_{0i} \leq \mid t^{m+1} \mid A$$

$$\mid x_j - x_{0i} \mid \geq \mid (\epsilon_j^m - \epsilon_i^m)a_m \mid\mid t^m \mid + \mid t^{m+1} \mid A$$

Then we obtain this inequality for $i, j, r = 1, 2..., n$ and $i \neq j$:

$$\mid x_j - x_{0i} \mid > \mid x_r - x_{0r} \mid$$

Now we combine the points x_i and x_{0i} by line in each plane near the origin. So the line parts give a bijective transformation of M into M_0 şf we consider the parts of our manifolds that lie near the origin. So we conclude that the topological behaviour of M has been replaced by of M_0 in the small neighbourhood of origin. Hence the topological behaviour is determnined completely by characteristic pair. □

The number of the characteristic pairs is the genus of the branch.

4. FUNDAMENTAL GROUP OF BRANCH

Let the order of the branch be n and let us have non-intersecting loops $g_0, g_1, ..., g_{n-1}$ each surrounding the points $y_0, y_1, ..., y_{n-1}$ and P_1 be the circuit surrounding all the points y_i. Furthermore, P_1 is the product of loops g_i and hence g_i's are the generators of the fundamental group of the singularity. Let T_1 be the transformation when x makes one turn around origin, $x = 0$. The loops g_i's are affected by the turn of x and each g_i is deformed into a new loops $g_i^{'}$. The new set of loops are still non-intersecting. We will obtain general relations of fundamental group by expressing the new loops in terms of old loops using Zariski-Van-Kampen Theorem (Zariski, 1932) by putting

$$g_i^{'} = g_i \qquad\qquad i = 0, 1, ..., n-1.$$

Now, we will find the fundamental group of the branch step by step beginning with an expansion with only one term.

4.1 The Fundamental Group of the Branch with genus one

We compute the fundamental group G_1 of the branch $y = x^{m_1/n_1}$.
If x makes one turn around the origin, g_i will transform into a suitable g_j for $i, j = 0, ..., n_1$. Let $m_1 = h_1 n_1 + l_1$ for $n_1, h_1, l_1 \in \mathbb{Z}$ and $n_1 > l_1 \geq 0$. Then; we have the following relation for the transformation of the loops for $P_1 = g_0 g_1 ... g_{n_1-1}$;

$$g_{i+l_1} = P_1^{h_1} g_i P_1^{-h_1} \qquad\qquad i = 0, 1, ..., n_1 - l_1 - 1 \tag{4.1}$$

$$g_i = P_1^{h_1+1} g_{n_1-l_1+i} P_1^{-h_1-1} \qquad\qquad i = 0, 1, ..., l_1 - 1 \tag{4.2}$$

These are the generating relations for the fundamental group.
We see for the general case;

$$P_1^{m_1} g_0 = g_0 P_1^{m_1} \tag{4.3}$$

Let x_1, y_1 be two positive integers such that

$$x_1 m_1 = y_1 n_1 + 1$$

Then we have the following generating relations obtained by 4.8 and 4.9;

$$g_{i+1} = P_1^{y_1} g_i P_1^{-y_1} \qquad\qquad i = 0, 1, ..., n_1 - 2 \tag{4.4}$$

$$g_i = P_1^{y_1+1} g_{n_1-1+i} P_1^{-y_1-1} \qquad\qquad i = 0 \tag{4.5}$$

$$g_1 = P_1^{y_1} g_0 P_1^{-y_1} \Rightarrow g_0 = P_1^{-y_1} g_1 P_1^{y_1}$$

Using the equation 4.10, we continue in this way;

$$P_1^{m_1} g_0 = P_1^{m_1} P_1^{-y_1} g_1 P_1^{y_1} = P_1^{-y_1} g_1 P_1^{y_1} P_1^{m_1} = g_0 P_1^{m_1}$$

$P_1^{m_1}$ and $P_1^{y_1}$ commute, so;

$$P_1^{-y_1} P_1^{m_1} g_1 P_1^{y_1} = P_1^{-y_1} g_1 P_1^{m_1} P_1^{y_1}$$

Evidently, we have the following result as the first relation between g_0 and P_1;

$$g_1 P_1^{m_1} = P_1^{m_1} g_1 \tag{4.6}$$

Computing g_i's from 4.11, we have;

$$\begin{aligned} g_1 &= P_1^{y_1} g_0 P_1^{-y_1} \\ g_2 &= P_1^{y_1} g_1 P_1^{-y_1} = P_1^{2y_1} g_0 P_1^{-2y_1} \\ g_3 &= P_1^{y_1} g_2 P_1^{-y_1} = P_1^{3y_1} g_0 P_1^{-3y_1} \\ &. \\ &. \\ &. \\ g_{n_1-1} &= P_1^{(n_1-1)y_1} g_0 P_1^{-(n_1-1)y_1} \end{aligned}$$

We have the following equation obtained by the generalization of this algorithm;

$$g_{i+1} = P_1^{(i+1)y_1} g_0 P_1^{-(i+1)y_1} \qquad i = 0, 1, ..., n_1 - 2 \tag{4.7}$$

Recalling that P_1 is the product of loops g_i's, we compute this multiplication gradually;

$$g_1 g_2 = P_1^{y_1} g_0 P_1^{-y_1} P_1^{2y_1} g_0 P_1^{-2y_1} = P_1^{y_1} g_0 P_1^{y_1} g_0 P_1^{-2y_1}$$

$$g_1 g_2 g_3 = P_1^{y_1} g_0 P_1^{y_1} g_0 P_1^{-2y_1} P_1^{3y_1} g_0 P_1^{-3y_1} = P_1^{y_1} g_0 P_1^{y_1} g_0 P_1^{y_1} g_0 P_1^{-3y_1}$$

$$.$$
$$.$$
$$.$$

$$g_0 g_1 g_2 ... g_{n_1-1} = g_0 \underbrace{(P_1^{y_1} g_0)...(P_1^{y_1} g_0)}_{(n_1-1)\text{times}} P_1^{-(n_1-1)y_1} = P_1$$

$$g_0 (P_1^{y_1} g_0)^{n_1-1} P_1^{-(n_1-1)y_1} = P_1$$

$$g_0 (P_1^{y_1} g_0)^{n_1} (P_1^{y_1} g_0)^{-1} = P_1^{(n_1-1)y_1+1}$$

$$(P_1^{y_1}g_0)^{n_1} = g_0^{-1}P_1^{(n_1-1)y_1+1+y_1}g_0$$

Since $x_1m_1 = y_1n_1 + 1$;

$$(P_1^{y_1}g_0)^{n_1} = g_0^{-1}P_1^{x_1m_1}g_0$$

Since $P_1^{m_1}$ and g_0 commute;

$$(P_1^{y_1}g_0)^{n_1} = P_1^{x_1m_1} \tag{4.8}$$

This is the second relation between g_0 and P_1.
We are not just trying to find relations for the fundamental group, we aim to find their most compact expression. Therefore we continue by introducing a new generator Q_1 in addition to the other generator P_1 such that

$$Q_1 = P_1^{-m_1y_1}(P_1^{y_1}g_0)^{m_1} \tag{4.9}$$

due to (4.3)
Compute $Q_1^{x_1}$ due to (4.3);

$$Q_1^{x_1} = (P_1^{y_1}g_0)^{m_1x_1}P_1^{-m_1y_1x_1}$$

$$= (P_1^{y_1}g_0)(P_1^{y_1}g_0)^{m_1x_1-1}P_1^{-m_1y_1x_1}$$

The power of second term in paranthesis, $(P_1^{y_1}g_0)$ is $m_1x_1 - 1$ and that is n_1y_1,

$$Q_1^{x_1} = (P_1^{y_1}g_0)(P_1^{y_1}g_0)^{n_1y_1}P_1^{-m_1y_1x_1}$$

Since $(P_1^{y_1}g_0)^{n_1} = P_1^{x_1m_1}$ by (4.8);

$$Q_1^{x_1} = (P_1^{y_1}g_0)P_1^{m_1x_1y_1}P_1^{-m_1y_1x_1}$$

$$Q_1^{x_1} = (P_1^{y_1}g_0) \tag{4.10}$$

When we compute $Q_1^{n_1}$, we get the following relation due to (4.3);

$$Q_1^{n_1} = (P_1^{y_1}g_0)^{m_1n_1}P_1^{-m_1y_1n_1}$$

By substitution using the equalities $(P_1^{y_1}g_0)^{n_1} = P_1^{x_1m_1}$ and $y_1n_1 = x_1m_1 - 1$ we get;

$$Q_1^{n_1} = (P_1^{m_1x_1})^{m_1}P_1^{-m_1y_1n_1}$$

$$= P_1^{m_1^2x_1-m_1^2x_1+m_1}$$

$$Q_1^{n_1} = P_1^{m_1} \tag{4.11}$$

Hence, we have obtained two generators, P_1 and Q_1 of G_1 and the relation $Q_1^{n_1} = P_1^{m_1}$ for the first fundamental group G_1. In other words G_1 is generated by P_1 and Q_1 and they satisfy the relation (4.11).
Let us examine an example and study it via Zariski's construction (Zariski, 1932).

Example 4.1.1.

$$y^3 = x^7$$

Let x vary on the unit circle $|x| = 1$. We have 3 branches and in general

$$y_k = e^{2\pi ikm/n} \tag{4.12}$$

where $k = 0, ..., n-1$. Hence for our case; we have 3 solutions of y.

$$y_0 = e^{2\pi i0(7/3)} = 1$$
$$y_1 = e^{2\pi i(7/3)}$$
$$y_2 = e^{2\pi i2(7/3)}$$

If m_1 were 1, the equation would look like $y = x^{1/3}$. If x makes one turn around the origin g_0' will transform into g_1 clearly and g_1 will transform into g_2. In that case g_2's transformed version can be expressed as $P_1^{-1}g_0P_1$ where $P_1 = g_0g_1g_2$. We conclude that $T_1(P_1) = P_1$, where T_1 is the induced transformation of the loops in fiber space.

(a) Lops g_0, g_1, g_2, P_1 (b) Transformed loops $T(g_0), T(g_1), T(g_2)$

Figure 4.1: Loops initially and after one complete turn of x around origin

In (4.1.1) m_1 is 7 and $m_1 = h_1n_1 + l_1$ for $n_1, h_1, l_1 \in \mathbb{Z}$ and $n_1 > l_1 \geq 0$. This formulation leads in our example as;

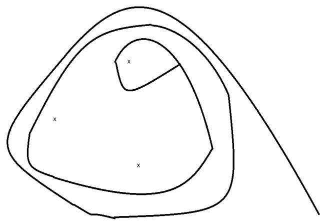
$$7 = 2.3 + 1$$

Now we can write the transformed loops for that case, namely;

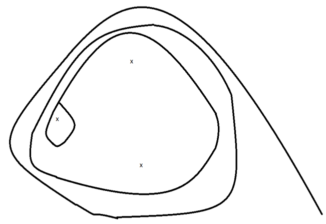
$$g_0 = g_0' = P_1^{-2}g_1P_1^2 \tag{4.13}$$
$$g_1 = g_1' = P_1^{-2}g_2P_1^2 \tag{4.14}$$
$$g_2 = g_2' = P_1^{-3}g_0P_1^3 \tag{4.15}$$

The first equality is due to Zariski Van Kampen Theorem.
The equalities between g_i's and $P_1^{-2}g_jP_1^2$ lead to more useful ones in view of (3.2),(3.3) and (3.4) for $P_1 = g_0g_1...g_{n_1-1}$;

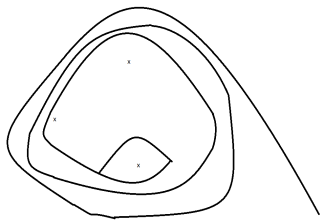
$$g_1 = P_1^2g_0P_1^{-2} \tag{4.16}$$
$$g_2 = P_1^2g_1P_1^{-2} \tag{4.17}$$
$$g_0 = P_1^3g_2P_1^{-3} \tag{4.18}$$



(a) $T_1(g_0)$



(b) $T_1(g_1)$



(c) $T_1(g_2)$

Figure 4.2: Transformed loops

If we substitute $P_1^2 g_1 P_1^{-2}$ in (4.18) and then $P_1^2 g_0 P_1^{-2}$ in that new equation, we get that

$$g_0 = P_1^7 g_0 P_1^{-7} \Rightarrow g_0 P_1^7 = P_1^7 g_0$$

Hence we can conclude that P_1^7 and g_0 commute. Obviously, the power of P_1 is not a coincide in that case.
To make it clear, we state that we have the following values for the given parameters; $m_1 = 7, n_1 = 3, h_1 = 2, l_1 = 1, x_1 = 1, y_1 = 2$ When we apply (3.17) to the example we get;

$$(P_1^2 g_0)^3 = P_1^7$$

We can substitute it into the definition of Q_1

$$Q_1 = (P_1^2 g_0)^7 P_1^{-14} = (P_1^2 g_0)^7 (P_1^2 g_0)^{-6} = P_1^2 g_0$$

$$g_1 = P_1^2 g_0 P_1^{-2}$$

$$g_2 = P_1^4 g_0 P_1^{-4}$$

$$P_1 = g_0 g_1 g_2 = g_0 P_1^2 g_0 P_1^2 g_0 P_1^{-4}$$

$$P_1^5 = g_0 P_1^2 g_0 P_1^2 g_0$$

$$P_1^7 = P_1^2 g_0 P_1^2 g_0 P_1^2 g_0$$

$$= (P_1^2 g_0)^3$$

$$P_1^7 = Q_1^3 \tag{4.19}$$

4.2 The Fundamental Group of the Branch with genus two

We find the fundamental group G_2 of the branch $y = x^{m_1/n_1} + x^{m_1/n_1+m_2/n_1n_2}$. We have n_1n_2 values of y for $x = 1$. For a fixed i, n_2-many points lie on the circle with center y_i. Each of these points is denoted by y_{ij}, where $j = 0, ..., n_2 - 1$. The non-intersecting loops $g_{i,0}, g_{i,1}, ..., g_{i,n_2-1}$ surround the points y_{ij} such that g_i is the circuit surrounding all the points y_{ij}, hence g_i surrounds also the point y_i.

$$g_{i,0}g_{i,1} \cdots g_{i,n_2-1} = g_i$$

In this context we can say that g_i is nothing but P_2.
Let T_2 be the transformation of the loops $g_{i,j}$ affected by a complete turn of the point x around the origin such that it induces a transformation of the loops g_i that coincides with T_1. That means that the first set of generating relations of G_2 are the generating relations of G_1.
Now, we will consider $T_2^{n_1}$ obtained by letting the variable x make n_1 complete turns around the origin. Let us do the following substitution;

$$x = z^{n_1}$$

$$y = x^{m_1/n_1} + x^{m_1/n_1+m_2/n_1n_2} = z^{m_1} + z^{m_1+m_2/n_2}$$

In this case, we consider that z makes one turn around $z = 0$ and we have;

$$z \to e^{2\pi i}z$$
$$T_2^{n_1}(x) = e^{2\pi i n_1}x$$

Let $m_2 = h_2n_2 + l_2$ for $n_2, h_2, l_2 \in \mathbb{Z}$

$$g'_{0,j} = g_0^{-h_2}P_1^{-m_1}g_{0,j+l_2}P_1^{m_1}g_0^{h_2} \qquad j = 0, ..., n_2 - l_2 - 1 \tag{4.20}$$

$$g'_{0,n_2-l_2+j} = g_0^{-h_2-1}P_1^{-m_1}g_{0,j}P_1^{m_1}g_0^{h_2+1} \qquad j = 0, ..., l_2 - 1 \tag{4.21}$$

Similar to the previous case, we do the same substitutions and get the relations below in view of (4.20) and (4.21);

$$g_{0,j+l_2} = P_1^{m_1}g_0^{h_2}g_{0,j}g_0^{-h_2}P_1^{-m_1} \qquad j = 0, ..., n_2 - l_2 - 1 \tag{4.22}$$

$$g_{0,j} = P_1^{m_1}g_0^{h_2+1}g_{0,n_2-l_2+j}g_0^{-h_2-1}P_1^{-m_1} \qquad j = 0, ..., l_2 - 1 \tag{4.23}$$

We substitute 0 in both of the equations above and get;

$$g_{0,0} = P_1^{m_1}g_0^{h_2+1}g_{0,n_2-l_2}g_0^{-h_2-1}P_1^{-m_1}$$

$$g_{0,l_2} = P_1^{m_1}g_0^{h_2}g_{0,0}g_0^{-h_2}P_1^{-m_1}$$

If we repeat the latter operation n_2-times we get;

$$g_{0,l_2n_2} = P_1^{m_1n_2} g_0^{h_2n_2} g_{0,0} g_0^{-h_2n_2} P_1^{-m_1n_2}$$

Furthermore we have

$$g_{0,l_2n_2} = g_0^{-l_2} g_{0,0} g_0^{l_2} \tag{4.24}$$

This is because we have;

$$g_{0,j+n_2} = g_0^{-1} g_{0,j} g_0$$

$$g_{0,0} = P_1^{m_1n_2} g_0^{h_2n_2+l_2} g_{0,0} g_0^{-h_2n_2-l_2} P_1^{-m_1n_2}$$

We rearrange the equation below as follows;

$$g_{0,0} = P_1^{m_1n_2} g_0^{m_2} g_{0,0} g_0^{-m_2} P_1^{-m_1n_2} \tag{4.25}$$

This is the generating relation of G_2 together with (4.11) . So, the only generators of G_2 are $g_{0,0}, P_1, Q_1$. If we rearrange (4.25), we get that

$$P_1^{m_1n_2} g_0^{m_2} g_{0,0} = g_{0,0} P_1^{m_1n_2} g_0^{m_2}$$

Since $P_1^{m_1}$ and g_0 commute by (4.3), we can write (4.25) as below;

$$P_1^{m_1n_2} g_0^{m_2} g_{0,0} = g_{0,0} g_0^{m_2} P_1^{m_1n_2}$$

So we have shown that $P_1^{m_1n_2} g_0$ and $g_{0,0}$ commute by the following equation.

$$P_1^{m_1n_2} g_0^{m_2} g_{0,0} = g_{0,0} P_1^{m_1n_2} g_0^{m_2} \tag{4.26}$$

As (m_2, n_2) are coprime, we can define two positive integers x_2, y_2 such that after Euclid algorithm;

$$x_2m_2 = y_2n_2 + 1 \tag{4.27}$$

A quick proof of existence of such x_2 and y_2 is given by the following procedure. If the equation (4.27) holds, then we have

$$x_2m_2 = x_2(h_2n_2 + l_2) = x_2h_2n_2 + x_2l_2 = y_2n_2 + 1$$

This implies that

$$x_2l_2 \equiv 1(mod n_2)$$

Hence there exists an integer k_2 such that

$$x_2l_2 = k_2n_2 + 1 \tag{4.28}$$

So,

$$x_2m_2 = x_2h_2n_2 + x_2l_2 = x_2h_2n_2 + k_2n_2 + 1$$
$$= (x_2h_2 + k_2)n_2 + 1$$

So, we have proved the existence of x_2 and y_2 for the following case;

$$y_2 = x_2h_2 + k_2 \tag{4.29}$$

Now consider (4.20) and (4.21);

$$g_{0,j} = g_0^{-h_2x_2}P_1^{-m_1x_2}g_{0,j+x_2l_2}P_1^{m_1x_2}g_0^{h_2x_2}$$

by (4.28)

$$= (g_0^{h_2}P_1^{m_1})^{-x_2}g_{0,j+k_2n_2+1}(P_1^{m_1}g_0^{h_2})^{x_2}$$

by (4.24)

$$= (g_0^{h_2}P_1^{m_1})^{-x_2}g_0^{-k_2}g_{0,j+1}g_0^{k_2}(P_1^{m_1}g_0^{h_2})^{x_2}$$

$$= P_1^{-m_1x_2}g_0^{-h_2x_2-k_2}g_{0,j+1}g_0^{h_2x_2+k_2}P_1^{m_1x_2} \tag{4.30}$$

Let us define a new operator B such that;

$$B = P_1^{m_1x_2}g_0^{y_2} \tag{4.31}$$

Then (4.30) can be stated as below;

$$g_{0,j} = B^{-1}g_{0,j+1}B$$

As a result we have;

$$g_{0,j+1} = Bg_{0,j}B^{-1} \qquad j = 0, ..., n_2 - 2 \tag{4.32}$$

If we write down g_{ij}'s according to (4.32), we get;

$$g_{0,1} = Bg_{0,0}B^{-1}$$

$$g_{0,2} = Bg_{0,1}B^{-1} = B^2g_{0,0}B^{-2}$$

$$\vdots$$

$$g_{0,n_2-1} = Bg_{0,n_2-2}B^{-1} = B^{n_2-1}g_{0,0}B^{-(n_2-1)}$$

$$Bg_{0,n_2-1}B^{-1} = B^{n_2}g_{0,0}B^{-n_2} \tag{4.33}$$

By (4.32),

$$g_{0,0} = Bg_{0,-1}B^{-1} \tag{4.34}$$

and by (4.24),

$$g_{0,n_2-1} = g_0^{-1}g_{0,-1}g_0 \tag{4.35}$$

If we rearrange (4.35) we get;

$$g_{0,-1} = g_0g_{0,n_2-1}g_0^{-1} \tag{4.36}$$

Now we substitute (4.36) in (4.34);

$$g_{0,0} = Bg_0g_{0,n_2-1}g_0^{-1}B^{-1} \tag{4.37}$$

Now we substitute (4.37) into (4.33);

$$Bg_{0,n_2-1}B^{-1} = B^{n_2}Bg_0g_{0,n_2-1}g_0^{-1}B^{-1}B^{-n_2}$$

$$g_{0,n_2-1} = B^{n_2}g_0g_{0,n_2-1}g_0^{-1}B^{-n_2} \tag{4.38}$$

By (4.38) and (4.27) we have the following equations;

$$g_{0,j+l_2} = B^{l_2}g_{0,j}B^{-l_2} \qquad j = 0, ..., n_2 - l_2 - 1 \tag{4.39}$$

$$g_{0,j} = B^{l_2}g_0g_{0,n_2-l_2+j}g_0^{-1}B^{-l_2} \qquad j = 0, ..., l_2 - 1 \tag{4.40}$$

We have now two important equations (4.32) and (4.37).
Now we want to prove that $(y_2l_2 - h_2)n_2 = (x_2l_2 - 1)m_2$. We already know by (4.28) that

$$x_2l_2 - 1 = k_2n_2 \tag{4.41}$$

So we need to show that

$$y_2l_2 - h_2 = k_2m_2 = k_2(h_2n_2 + l_2) \tag{4.42}$$

By the last step of the previous proof that leads us to the existence of x_2 and y_2, we can denote y_2 as $x_2h_2 + k_2$. We rewrite (4.41) and get (4.42)

$$(x_2h_2 + k_2)l_2 - h_2 = x_2h_2l_2 + k_2l_2 - h_2 = \underbrace{(x_2l_2 - 1)}_{k_2n_2}h_2 + k_2l_2$$

We will use these algebraic relations by computing the generating relations of G_2 as follows;

$$B^{l_2} =^{(} P_1^{m_1x_2}g_0^{y_2})^{l_2}$$

$$= P_1^{m_1x_2l_2}g_0^{y_2l_2}$$

By (4.28) and (4.29)

$$= P_1^{(k_2n_2+1)m_1} g_0^{k_2m_2+h_2}$$

Since g_0 and $P_1^{m_1}$ commute by (4.3);

$$= (g_0^{m_2} P_1^{m_1n_2})^{k_2} P_1^{m_1} g_0^{h_2}$$

Now it is time for simplifying the generating relations of G_2. So we are going to derive several equations using previous results.
We have by (4.22)

$$g_{0,0} = (P_1^{m_1x_2} g_0^{y_2})^{-1} g_{0,1} P_1^{m_1x_2} g_0^{y_2}$$

We replace $g_{0,0}$ in (4.26) by the above expression and get;

$$g_0^{m_2} P_1^{m_1n_2} (P_1^{m_1x_2} g_0^{y_2})^{-1} g_{0,1} P_1^{m_1x_2} g_0^{y_2} = (P_1^{m_1x_2} g_0^{y_2})^{-1} g_{0,1} P_1^{m_1x_2} g_0^{y_2} g_0^{m_2} P_1^{m_1n_2}$$

If we use the result that $P_1^{m_1}$ and g_0 commute, we can rearrange this equation above and write it as;

$$\underbrace{(P_1^{m_1x_2} g_0^{y_2})^{-1}}_{B^{-1}} g_0^{m_2} P_1^{m_1n_2} g_{0,1} \underbrace{P_1^{m_1x_2} g_0^{y_2}}_{B} = \underbrace{(P_1^{m_1x_2} g_0^{y_2})^{-1}}_{B^{-1}} g_{0,1} g_0^{m_2} P_1^{m_1n_2} \underbrace{P_1^{m_1x_2} g_0^{y_2}}_{B}$$

This means that $g_{0,1}$ and $g_0^{m_2} P_1^{m_1n_2}$ commute.
Using (4.32) we can write equations for the loops $g_{0,j}$ where $j = 0, ..., n_2 - 1$.

$$g_{0,1} = B g_{0,0} B^{-1}$$

$$g_{0,2} = B g_{0,1} B^{-1}$$

$$= B^2 g_{0,0} B^{-2}$$

$$\vdots$$

$$g_{0,n_2-1} = B^{n_2-1} g_{0,0} B^{-(n_2-1)}$$

$$g_{0,1} g_{0,2} = B g_{0,0} B g_{0,0} B^{-2}$$

$$g_{0,1} g_{0,2} g_{0,3} = B g_{0,0} B g_{0,0} B g_{0,0} B^{-3}$$

$$\vdots$$

$$g_{0,0}g_{0,1}...g_{0,n_2-1} = g_{0,0}\underbrace{(Bg_{0,0})(Bg_{0,0})...(Bg_{0,0})}_{(n_2-1)times}B^{-(n_2-1)}$$

Remember that g_0 is the circuit surrounding all points y_{ij} and it is the product of loops $g_{0,j}$'s for $j = 0, ..., n_2 - 1$.

$$g_0 = g_{0,0}(Bg_{0,0})^{n_2-1}B^{-(n_2-1)}$$

We substitute the expression of B into the above equation and rearrange it;

$$g_0(P_1^{m_1x_2}g_0^{y_2})^{n_2-1} = g_{0,0}(P_1^{m_1x_2}g_0^{y_2}g_{0,0})^{n_2-1}$$

$$g_0(g_0^{y_2}P_1^{m_1x_2})^{n_2} = (g_{0,0}g_0^{y_2}P_1^{m_1x_2})^{n_2}$$

$$g_0^{y_2n_2+1}P_1^{m_1x_2n_2} = (g_{0,0}g_0^{y_2}P_1^{m_1x_2})^{n_2}$$

By (4.27),

$$P_1^{m_1x_2n_2}g_0^{x_2m_2} = (g_{0,0}g_0^{y_2}P_1^{m_1x_2})^{n_2} \tag{4.43}$$

Let us define Q_2 in a way similar to the definition of Q_1 in (4.9)

$$\begin{aligned} Q_2 &= (g_{0,0}g_0^{y_2}P_1^{m_1x_2})^{m_2}(g_0^{m_2}P_1^{m_1n_2})^{-y_2} \\ &= (Bg_{0,1})^{m_2}(g_0^{m_2}P_1^{m_1n_2})^{-y_2} \end{aligned}$$

In this case,

$$Q_2^{x_2} = (g_{0,0}g_0^{y_2}P_1^{m_1x_2})^{x_2m_2}(g_0^{m_2}P_1^{m_1n_2})^{-x_2y_2}$$

By (4.43)

$$= g_{0,0}^{x_2m_2}(g_0^{y_2x_2m_2}P_1^{m_1x_2m_2x_2})(g_{0,0}g_0^{y_2}P_1^{m_1x_2})^{-n_2y_2}$$

$$= g_{0,0}g_0^{y_2}P_1^{m_1x_2} \tag{4.44}$$

$$= g_{0,0}B \tag{4.45}$$

The last step of the equation shows us that Q_2 can be used as one of the generators of G_2, instead of $g_{0,0}$.

$$Q_2^{n_2} = (g_{0,0}g_0^{y_2}P_1^{m_1x_2})^{m_2n_2}(g_0^{m_2}P_1^{m_1n_2})^{-y_2n_2}$$

By (4.43)

$$= (g_0^{m_2x_2}P_1^{m_1n_2x_2})^{m_2}(g_0^{m_2}P_1^{m_1n_2})^{-y_2n_2}$$

$$= g_0^{m_2}P_1^{m_1n_2}$$

By (4.11)

$$= g_0^{m_2}Q_1^{n_1n_2} \tag{4.46}$$

Hence we have obtained three elements, P_1, Q_1 and Q_2 as generators of G_2 satisfying the following conditions;

$$Q_1^{n_1} = P_1^{m_1}$$

$$Q_2^{n_2} = P_2^{m_2}Q_1^{n_1n_2}$$

where P_2 surrounds n_2 points $y_{0,j}$ and P_1 surround n_1n_2 points $y_{i,j}$ for $0 \leq i \leq n_1 - 1$, and $0 \leq j \leq n_2 - 1$

Example 4.2.1.

$$y = x^{3/2} + x^{7/4} = x^{3/2} + x^{3/2+1/2.2}$$

In this case we have the following values for our parameters;

$$m_1 = 3 \qquad n_1 = 2$$
$$m_2 = 1 \qquad n_2 = 2$$

$$m_2 = h_2n_2 + l_2 \qquad h_2 = 0 \qquad l_2 = 1$$

First of all we write the relations for this example based on the results we have obtained in previous section; By (4.3),

$$g_0P_1^3 = P_1^3g_0$$

By (4.11),

$$Q_1^2 = P_1^3$$

By definition of Q_1,

$$Q_1 = (P_1g_0)^3P_1^{-3}$$

Now we write the relations of loops $g_{0,0}$ and $g_{0,1}$ with help of (4.20) and (4.21);

$$g'_{0,0} = P_1^{-3}g_{0,1}P_1^3 \tag{4.47}$$

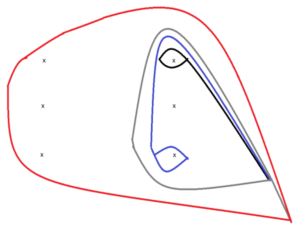


Figure 4.3: Loops in genus two case

$$g'_{0,1} = g_0^{-1} P_1^{-3} g_{0,0} P_1^3 g_0 \tag{4.48}$$

By Zariski-Van-Kampen,

$$g'_{0,0} = g_{0,0} \qquad\qquad g'_{0,1} = g_{0,1}$$

In the figure below, the loops for this example has been drawn. The red one is P_1 surrounding y_0 and y_1, the gray one is g_0 surrounding y_0 and black and blue loops are g_{00} and g_{01} surrounding $y_0 0$ and y_{01}.
We rearrange (4.47) and (4.53) and get the following equations respectively;

$$g_{0,1} = P_1^3 g_{0,0} P_1^{-3} \tag{4.49}$$

$$g_{0,0} = P_1^3 g_0 g_{0,1} g_0^{-1} P_1^{-3} \tag{4.50}$$

If we replace $g_{0,1}$ in (4.50) with (4.49) we get that;

$$g_{0,0} = P_1^3 g_0 P_1^3 g_{0,0} P_1^{-3} g_0^{-1} P_1^{-3}$$

Since g_0 and P_1^3 commute; we get the generating relation of G_2 as follows;

$$g_{0,0} = P_1^6 g_0 g_{0,0} g_0^{-1} P_1^{-6}$$

This is (4.25), indeed.
This enables us to write the following equation;

$$g_{0,0} g_0 P_1^6 = P_1^6 g_0 g_{0,0}$$

This is obviously (4.26).
The positive integers x_2 and y_2 are 3 and 1,respectively and k_2 is 1. B is $P_1^9 g_0$.
By (4.49), we have that

$$g_{0,0} = P_1^{-3} g_{0,1} P_1^3$$

and if we repeat this 3 times since $x_2 = 3$;

$$g_{0,0} = P_1^{-9} g_{0,3} P_1^9$$

We replace $g_{0,3}$ in this equation by another expression of it using (4.24);

$$g_{0,0} = \underbrace{P_1^{-9} g_0^{-1}}_{B^{-1}} g_{01} \underbrace{g_0 P_1^9}_{B}$$

$$g_{0,1} = Bg_{0,0}B^{-1} \tag{4.51}$$

This is the result we obtained in (4.32).
Since (4.51) and the fact that $g_{0,0}$ and $P_1^6 g_0$ commute, we can write the following equation;

$$P_1^6 g_0 (P_1^9 g_0)^{-1} g_{0,1} P_1^9 g_0 = (P_1^9 g_0)^{-1} g_{0,1} P_1^9 g_0$$

Since P_1^3 and g_0 commute we can rearrange the above equation and obtain the latter one;

$$g_{0,1} = (P_1^6 g_0)^{-1} g_{0,1} P_1^6 g_0$$

Hence $g_{0,1}$ and $P_1^6 g_0$ commute, also.
Now we can say that

$$g_{0,0} = Bg_{0,-1}B^{-1}$$

Furthermore, we have the following equation by (4.24);

$$g_{0,-1} = g_0 g_{0,1} g_0^{-1}$$

Now, we substitute the latter equation in the former one and get;

$$g_{0,0} = Bg_0 g_{0,1} g_0^{-1} B^{-1} \tag{4.52}$$

This is the result we obtained in (4.37).
We can have the following equation by (4.51), obviously;

$$Bg_{0,1}B^{-1} = B^2 g_{0,0} B^{-2}$$

If we replace $g_{0,0}$ here with its expression in (4.52) , we get that;

$$Bg_{0,1}B^{-1} = B^2 B g_0 g_{0,1} g_0^{-1} B^{-1} B^{-2}$$

This implies immediately;

$$g_{0,1} = B^2 g_0 g_{0,1} g_0^{-1} B^{-2}$$

This is the result we obtained in (4.38).
Now we can use the fact that g_0 is obtained by multiplication of the loops $g_{0,0}$ and $g_{0,1}$.

$$g_0 = g_{0,0} g_{0,1} = g_{0,0} B g_{0,0} B^{-1}$$

This implies that

$$g_0 B = g_{0,0} B g_{0,0}$$

If we replace B by its expansion;

$$g_0(g_0 P_1^9) = g_{0,0}(g_0 P_1^9 g_{0,0})$$

Since $g_{0,0}$ and $g_0 P_1^6$ commute we may write;

$$g_0(g_0 P_1^9)^2 = g_{0,0}^2 (g_0 P_1^9)^2$$

$$g_0^3 P_1^{18} = (g_{0,0} g_0 P_1^9)^2 \tag{4.53}$$

This is the result we obtained in (4.43).
Now we introduce the new parameter Q_2 defined as;

$$Q_2 = (g_{0,0} g_0 P_1^9)(g_0 P_1^6)^{-1}$$

Then Q_2^3 yields to;

$$Q_2^3 = (g_{0,0} g_0 P_1^9)^3 (g_0 P_1^6)^{-3}$$

If we write this equation clearly;

$$= g_{0,0}^3 g_0^3 P_1^{27} (g_0^{-3} P_1^{-18})$$

By (4.53),

$$= g_{0,0}^3 g_0^3 P_1^{27} (g_{0,0} g_0 P_1^9)^{-2}$$

Hence we get that;

$$Q_2^3 = g_{0,0} g_0 P_1^9$$

Similarly;

$$Q_2^2 = (g_{0,0} g_0 P_1^9)^2 (g_0 P_1^6)^{-2}$$

By (4.53),

$$= (g_0^3 P_1^{18})(g_0 P_1^6)^{-2}$$

Hence

$$Q_2^2 = g_0 P_1^6$$

By the relations we obtained for Q_1 and P_1

$$Q_2^2 = g_0 Q_1^4$$

According to our notation; g_0 is P_2, hence

$$Q_2^2 = P_2 Q_1^4 \tag{4.54}$$

4.3 The Fundamental Group of the Branch with genus k

We find the fundamental group G_p of the branch
$y = x^{m_1/n_1} + x^{m_1/n_1 + m_2/n_1 n_2} + \ldots + x^{m_1/n_1 + m_2/n_1 n_2 + \ldots m_k/n_1 n_2 \ldots n_k}$. Our aim in this chapter is to prove the following theorem.

Theorem 4.3.1. *The group G_p is generated by $p+1$ elements, $P_1, Q_1, Q_2, ...Q_p$. The generating relations of G_p are the following;*

$$\begin{aligned} Q_i^{n_i} &= P_i^{m_i} Q_{i-1}^{n_{i-1}n_i} && i = 1,2,...,p && Q_0 = 1 \\ P_{i+1} P_i^{y_i} Q_{i-1}^{n_{i-1}x_i} &= Q_i^{x_i} && i = 1,2,...,p-1 \end{aligned}$$

Here x_i and y_i are positive integers such that

$$x_i m_i = y_i n_i + 1$$

Moreover, if an element P_{p+1} is defined as;

$$P_{p+1} P_p^{y_p} Q_{p-1}^{n_{p-1}x_p} = Q_p^{x_p}$$

the elements P_i's are the loops surrounding $n_i n_{i+1}...n_p$ of the values of the function y for $x = 1$.

We are proving this theorem by induction. Section 4.1 is for the case where $p = 1$ and Section 4.2 is for the case $p = 2$. We have got the following results; For the case $p = 1$;

$$Q_1^{n_1} = P_1^{m_1} \tag{4.11}$$

$$P_1^{y_1} \underbrace{g_0}_{P_2} = Q_1^{x_1} \tag{4.10}$$

For the case $p = 2$;

$$\underbrace{g_{0,0}}_{P_3} \underbrace{g_0^{y_2}}_{P_2^{y_2}} \underbrace{P_1^{m_1 x_2}}_{Q_1^{n_1 x_2}} = Q_1^{x_1} \tag{4.44}$$

$$Q_2^{n_2} = P_2^{m_2} Q_1^{n_1 n_2} \tag{4.46}$$

Now in this third section we will study the inductive step. Assume the statement is true for $p-1$ and namely for the branch Γ_{p-1}. Then we have for the group G_{p-1} the following relations;

$$Q_i^{n_i} = P_i^{m_i} Q_{i-1}^{n_{i-1}n_i} \qquad i = 1,2,...,p-1 \tag{4.55}$$

$$P_{i+1} P_i^{y_i} Q_{i-1}^{n_{i-1}x_i} = Q_i^{x_i} \qquad i = 1,2,...,p-1 \tag{4.56}$$

Here P_i's are the loops of y for $x = 1$ surrounding $n_i ... n_{p-1}$ points. They can also be considered as generators/elements of G_p, where each P_i surrounds $n_i n_{i+1} ... n_p$ points of y_{ij}. Here, we introduce an simpler notation, namely, instead of writing $y_{\underbrace{0,...,0}_{p-1-many},j}$ we write $y_{0,j}$ and for the loops surrounding them we write $g_{0,j}$ instead of $g_{\underbrace{0,0,...,0}_{p-1-many},j}$. They surround points $y_{0,0}, y_{0,1}, ..., y_{0,p-1}$ respectively such that

$$g_{0,0}g_{0,1}...g_{0,n_p-1} = P_p$$

The elements $g_{0,j}$ together with $P_1, Q_1, ..., Q_{p-1}$ are the generators of G_{p-1}.
Now, for the inductive step we may think that x makes $n_1n_2...n_{p-1}$ turns around origin. In other words, we think of the transformation T_p of the loops g_0, j which will give the new generating relations of G_p that coincide with generating relations of G_{p-1}. Obviously we can keep the algebraic tools that we use;

$$m_p = h_pn_p + l_p \qquad\qquad 0 \leq l_p < n_p \tag{4.57}$$

$$g_{0,j+l_p} = Ag_{0,j}A^{-1} \qquad\qquad j = 0, 1, ..., n_p - l_p - 1 \tag{4.58}$$

$$g_j = AP_pg_{n_p-l_p+j}P_p^{-1}A^{-1} \qquad\qquad j = 0, 1, ..., l_p - 1 \tag{4.59}$$

where

$$A = P_1^{m_1n_2n_3...n_{p-1}}P_2^{m_2n_3...n_{p-1}}...P_{p-1}^{m_{p-1}}P_p^{h_p}$$

We try to find a simpler expression for A;

$$\begin{aligned} P_1^{m_1n_2n_3...n_{p-1}}P_2^{m_2n_3...n_{p-1}} &= (P_1^{m_1n_2}P_2^{m_2})^{n_3...n_{p-1}} \\ &= Q_2^{n_2n_3...n_{p-1}} \end{aligned}$$

$$\begin{aligned} Q_2^{n_2n_3...n_{p-1}}P_3^{m_3n_4...n_{p-1}} &= (Q_2^{n_2n_3}P_3^{m_3})^{n_4...n_{p-1}} \\ &= Q_3^{n_3n_4...n_{p-1}} \end{aligned}$$

This pattern inductively yields;

$$A = Q_{p-1}^{n_{p-1}}P_p^{h_p} \tag{4.60}$$

We prove that P_i commutes with $Q_{i-1}^{n_{i-1}}$.
For $i = 2$, P_2 is g_0 which commutes with $P_1^{m_1}$ by (4.3) and which is equal to $Q_1^{n_1}$ by (4.11). We assume this is true for a given i and prove it for $i + 1$. Then by hypothesis, we have that P_i commutes with $Q_{i-1}^{n_{i-1}}$. That means;

$$P_iQ_{i-1}^{n_{i-1}} = Q_{i-1}^{n_{i-1}}P_i \tag{4.61}$$

Lemma 4.3.2. *P_i commmutes with $Q_i^{n_i}$;*

Proof. In the first step, we just use the expression for $Q_i^{n_i}$ given in (4.55)

$$Q_i^{n_i} P_i = P_i^{m_i} Q_{i-1}^{n_{i-1} n_i} P_i$$

By (4.61);

$$= P_i Q_{i-1}^{n_{i-1} n_i} P_i^{m_i}$$

$$= P_i Q_i^{n_i}$$

□

Lemma 4.3.3. *$Q_i^{n_i}$ commutes with $Q_{i-1}^{n_{i-1}}$*

Proof. Similar to the above lemma's proof, we use the expression for $Q_i^{n_i}$ given in (4.55);

$$Q_i^{n_i} P_i = P_i^{m_i} Q_{i-1}^{n_{i-1} n_i} P_i$$

By (4.61),

$$= P_i Q_{i-1}^{n_{i-1} n_i} P_i^{m_i}$$

= P$_i Q_i^{n_i}$

□

Lemma 4.3.4. *P_{i+1} commutes with $Q_i^{n_i}$.*

Proof. First; we write an expression for P_{i+1} using (4.56) first;

$$P_{i+1} = Q_i^{x_i} Q_{i-1}^{-n_{i-1} x_i} P_i^{-y_i}$$

Secondly, we compute $P_{i+1} Q_i^{n_i}$;

$$P_{i+1} Q_i^{n_i} = Q_i^{x_i} Q_{i-1}^{-n_{i-1} x_i} P_i^{-y_i} Q_i^{n_i}$$

By (4.3.2),

$$= Q_i^{x_i} Q_{i-1}^{-n_{i-1} x_i} Q_i^{n_i} P_i^{-y_i}$$

By (4.3.3),

$$= Q_i^{x_i} Q_i^{n_i} Q_{i-1}^{-n_{i-1} x_i} P_i^{-y_i}$$

$$= Q_i^{n_i} P_{i+1}$$

□

Now we try to find new relations between generators $P_1, Q_1, Q_2, ...Q_{p-1}$ following similar paths that we have done in previous sections. We consider (4.58),

$$g_{l_p} = A g_0 A^{-1}$$

$$g_{2l_p} = A^2 g_0 A^{-2}$$

$$\vdots$$

$$g_{n_p l_p} = A^{n_p} g_0 A^{-n_p}$$

By (4.24) and (4.60),

$$P_p^{-l_p} g_0 P_p^{l_p} = (Q_{p-1}^{n_{p-1}} P_p^{h_p})^{n_p} g_0 (Q_{p-1}^{n_{p-1}} P_p^{h_p})^{-n_p}$$

We rewrite this equation for g_0;

$$g_0 = P_p^{l_p} Q_{p-1}^{n_{p-1} n_p} P_p^{h_p n_p} g_0 Q_{p-1}^{-n_{p-1} n_p} P_p^{-h_p n_p} P_p^{-l_p}$$

Since P_i commutes with $Q_{i-1}^{n_{i-1}}$;

$$g_0 = Q_{p-1}^{n_{p-1} n_p} P_p^{m_p} g_0 (Q_{p-1}^{n_{p-1} n_p} P_p^{m_p})^{-1} \tag{4.62}$$

Let x_p and y_p are two non-negative integers such that;

$$x_p m_p = y_p n_p + 1$$

Then define B as;

$$B = P_1^{x_p m_1 n_2 n_3 \ldots n_{p-1}} P_2^{x_p m_2 n_3 \ldots n_{p-1}} \ldots P_{p-1}^{x_p m_{p-1}} P_p^{y_p}$$

After an arrangement similar to that used in (4.60)

$$B = Q_{p-1}^{x_p n_{p-1}} P_p^{y_p}$$

In this case we have, by (4.58) and (4.59) the following relations, respectively,

$$g_{0,j+1} = B g_{0,j} B^{-1} \qquad j = 0, ..., n_p - 2 \tag{4.63}$$

$$g_{0,0} = B P_p g_{0,n_p - 1} P_p^{-1} A^{-1} \tag{4.64}$$

(4.63) and (4.64) are actually the relations for the transformation $T_p^{x_p}$. Now we can compute g_i's according to the relation given in (4.63) for $i = 0, ..., n_p - 1$;

$$g_{0,1} = B g_{0,0} B^{-1}$$

$$\begin{aligned} g_{0,2} &= B g_{0,1} B^{-1} \\ &= B^2 g_{0,0} B^{-2} \end{aligned}$$

$$\vdots$$

$$g_{0,j} = B^j g_{0,0} B^{-j} \qquad\qquad j = 0, ..., n_p - 1$$

We know that $P_p = g_0 g_1 ... g_{n_p - 1}$, so we compute multiplication of these loops;

$$g_{0,1} g_{0,2} = B g_0 B g_0 B^{-2}$$

$$g_{0,1} g_{0,2} g_{0,3} = B g_{0,0} B g_{0,0} B g_{0,0} B^{-3}$$

$$P_p = g_{0,0} g_{0,1} g_{0,2} ... g_{0,n_p-1} = g_0 \underbrace{(Bg_{0,0})(Bg_{0,0})...(Bg_{0,0})}_{(n_p-1)-times} B^{-(n_p-1)}$$

$$P_p = g_0 (Bg_0)^{n_p-1} B^{-(n_p-1)}$$

$$P_p (Q_{p-1}^{x_p n_{p-1}} P_p^{y_p})^{n_p-1} = g_0 (Q_{p-1}^{x_p n_{p-1}} P_p^{y_p})^{n_p-1} g_0^{n_p-1}$$

$$(P_p^{m_p} Q_{p-1}^{n_p n_{p-1}})^{x_p} = (g_0 P_p^{y_p} Q_{p-1}^{x_p n_{p-1}})^{n_p} \tag{4.65}$$

Taking this relation into account we define Q_p as follows;

$$Q_p = (g_0 P_p^{y_p} Q_{p-1}^{x_p n_{p-1}})^{m_p} (P_p^{m_p} Q_{p-1}^{n_p n_{p-1}})^{-y_p}$$

Then we compute $Q_p^{x_p}$ by the above relation;

$$Q_p^{x_p} = (g_0 P_p^{y_p} Q_{p-1}^{x_p n_{p-1}})^{m_p x_p} (P_p^{m_p} Q_{p-1}^{n_p n_{p-1}})^{-y_p x_p}$$

By (4.65);

$$= (g_0 P_p^{y_p} Q_{p-1}^{x_p n_{p-1}})^{m_p x_p} (g_0 P_p^{y_p} Q_{p-1}^{x_p n_{p-1}})^{-n_p y_p}$$

By (4.57);

$$= g_0 P_p^{y_p} Q_{p-1}^{x_p n_{p-1}} \tag{4.66}$$

Q_p is a generator of G_p and it satisfies the following relation

$$Q_p^{n_p} = (g_0 P_p^{y_p} Q_{p-1}^{x_p n_{p-1}})^{m_p n_p} (P_p^{m_p} Q_{p-1}^{n_p n_{p-1}})^{-y_p n_p}$$

Similar to the above case, by (4.65),

$$= (P_p^{m_p} Q_{p-1}^{n_p n_{p-1}})^{m_p x_p} (P_p^{m_p} Q_{p-1}^{n_p n_{p-1}})^{-y_p n_p}$$

$$= P_p^{m_p} Q_{p-1}^{n_p n_{p-1}}$$

Hence we proved the theorem (Zariski, 1932).

5. KNOTS

A link L is a closed smooth submanifold in the three dimensional sphere S^3 such that each connected component is homeomorphic to S^1. A connected link is called a knot (Dimca, 1992).
A knot K is a topological embedding of S^1 into $\mathbb{R}^3$ or S^3. In other words it is a subset of S^3 homeomorphic to S^1. Two knots are equivalent, when there is an orientation preserving homeomorphism;

$$\begin{aligned} \phi : S^3 &\rightarrow S^3 \\ K &\mapsto K' \end{aligned}$$

Here S^3 is regarded in two ways;

$$\begin{aligned} &S^3 = \{(x, y) \in \mathbb{C}^2 || x|^2 + |y|^2 = r\} \\ &S^3 = \mathbb{R}^3 \cup \{\infty\} \end{aligned}$$

Furthermore it can be written as a union of two toris

$$\begin{aligned} &T_1 = \{(x, y) \in \mathbb{C}^2 || x|^2 + |y|^2 = 1, |x| \leq |y|\} \\ &T_2 = \{(x, y) \in \mathbb{C}^2 || x|^2 + |y|^2 = 1, |x| \geq |y|\} \end{aligned}$$

By orientation preserving we mean that the map respects over and undercrossings. The image of the embedding of S^1 into S^3 is also regarded as knot i.e. the knot $K = C \cap S$ is the image of a circle S^1, where

$$S^1 = \{t \in \mathbb{C} || t| = c\}$$

- The simplest knot, S^1x $\{0\} \subset S^1$ x $D^2 \subset S^3$, is the trivial knot which can be defined by the complex equation $x = 0$. This is the torus knot of O^{th} order.

- The simplest nontrivial knot is the so-called trefoil knot. It is $(2, 3)$ torus knot of first order.

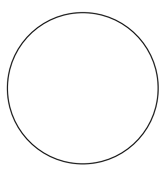


Figure 5.1: Trivial Knot

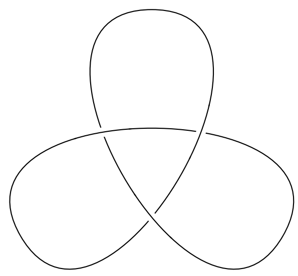


Figure 5.2: Trefoil Knot

We may consider the map j such that

$$j : S^1 x S^1 \to S^3$$
$$j(x, y) = (x/\sqrt{2}, y/\sqrt{2})$$

and another map e such that

$$e : S^1 \to S^1 x S^1$$
$$e(z) = (z^q, z^p)$$

where p, q are two positive integers such that $gcd(p, q) = 1$. Then (q, p)- torus knot, denoted as $K_{q,p}$ is resulted by composition of the two maps above; $j \circ e$. $K_{q,p}$ can also be expressed as below;

$$K_{q,p} = \{(x, y) \in S^3 | x^p + y^q = 0\}$$

- In general, K_i is the torus knot of i^{th} order with the Puiseux pairs $(q_1, p_1), ..., (q_i, p_i)$. These torus knots of higher order are iterated torus knots i.e. cable knots.

A cable knot is obtained in the following manner, firstly we start with an ordinary torus knot, K_1, placed on an unknotted torus and with a tubular neighbourhood around K_1. We replace K_1 by a knot K_2 placed on the boundary of neighbourhood of K_1. So, we construct a cable knot K_2 around K_1 and so on, we construct inductively a cable knot K_i around K_{i-1}. In this context; K_i is called an iterated torus knot.
Let us state formally what we have described;

Definition 5.0.1. *The knot with the given below Puiseux expansion*

$$y = x^{\frac{q_1}{p_1}} + x^{\frac{q_2}{p_1 p_2}} + ... + x^{\frac{q_k}{p_1 \cdots p_k}}$$

with Puiseux pairs

$$(q_1, p_1), ..., (q_k, p_k)$$

corresponds to the iterated torus knot of order k.

This definition makes more sense when one recalls that knots are resulted by braids when the initial and final points are identified.

5.1 Torus knots

Brauner (1928) regards torus knots as topological images of singularities of algebraic functions of two complex variables. He shows that the stereographic projection of the intersection of the singularity consisting of one branch with the boundary of a small neighbourhood is a knot. Let us investigate the case for an irreducible curve.
We first take an analytic function of complex variables x and y such that

$$f(x, y) = 0$$

as the equation of 2-parameter real curve in 4-dimensional space $R_4(x_1, x_2, y_1, y_2)$
Let this aforementioned irreducible curve equation be the following;

$$ax^n + by^m = 0$$

where $gcd(n, m) = 1$ and x, y are complex variables such that;

$$x = x_1 + ix_2$$
$$y = y_1 + iy_2$$

These variables should satisfy the following equation;

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2 \tag{5.1}$$

Around the point $x = y = 0$ we may have the following parametrization with respect to the complex parameter t;

$$x = \alpha t^m \qquad\qquad y = \beta t^n$$

where α and β are constants that depend only to a and b. Therefore we can go on by the following parametrizations;

$$\alpha = ae^{ic} \qquad\qquad \beta = be^{id} \qquad\qquad t = \rho e^{i\phi}$$

Then two-parametric real curve M in R_4 has the following parametrization;

$$x = ae^{ic}(\rho e^{i\phi})^m \qquad\qquad y = be^{id}(\rho e^{i\phi})^n$$

It gives rise to the following parametrizations;

$$x_1 = a\rho^m cos(m\phi + c) \qquad\qquad x_2 = a\rho^m sin(m\phi + c)$$
$$y_1 = b\rho^n cos(n\phi + d) \qquad\qquad y_2 = b\rho^n sin(n\phi + d)$$

If the curve M intersects with the projection's sphere z, we will get a condition for the parameter. If we substitute the parameters in (5.1) for x_1, x_2, x_3 and x_4, the parameter will have the following condition;

$$b^2\rho^{2n} + a^2\rho^{2m} = r^2$$

where ϕ is free. Here one should remark that the equation above gives a monotone increasing function of ρ. For positive and real values of r, ρ has two roots.

Let σ be a positive root of ρ. So we find for our intersection curve Γ of our curve with the sphere z,

$$x_1 = a\sigma^m cos(m\phi + c) \qquad x_2 = a\sigma^m sin(m\phi + c)$$
$$y_1 = b\sigma^n cos(n\phi + d) \qquad y_2 = b\sigma^n sin(n\phi + d)$$

We consider the following substitution;

$$\tau = m\phi + c \qquad \psi = n\phi + d$$

We project the points x_1, x_2, y_1 and y_2 on a point such that

$$x_1 = x_2 = y_1 = 0$$
$$y_2 = r$$

As orthogonal projection's coordinate axes we choose ξ, η, ζ such that they are parallel to x_1, x_2, y_1 axes of R_4, respectively. Hence;

$$\xi = \frac{2r}{r - y_2}x_1 \qquad \eta = \frac{2r}{r - y_2}x_2 \qquad \zeta = \frac{2r}{r - y_2}y_1 \tag{5.2}$$

Now we can substitute the parameters into (5.2) and so we get the following equations;

$$\xi = \frac{2ra\sigma^m cos\phi}{r - b\sigma^n sin\psi} \qquad \eta = \frac{2ra\sigma^m sin\phi}{r - b\sigma^n sin\psi} \qquad \zeta = \frac{2rb\sigma^n cos\psi}{r - b\sigma^n sin\psi}$$

where ϕ and ψ are independent parameters. So we have a torus equation of the form;

$$\zeta^2 + (\mu - \frac{2r^2}{a\sigma^m})^2 = (\frac{2rb\sigma^{n-m}}{a})^2$$

where μ is

$$\mu = \sqrt{\xi^2 + \eta^2}$$

Therefore, we conclude that the surface is a torus.

5.2 Iterated Torus Knots

Let $f(x, y)$ be a complex polynomial vanishing at the origin and let $C = \{(x, y)|f(x, y) = 0\}$ be the equation of the plane algebraic curve. For all sufficiently small ϵ, we have a sphere

$$S_\epsilon = \{(x, y)|\sqrt{x^2 + y^2} = \epsilon\}$$

So far, we have found out that if the curve C intersects this sphere, we get the knot K. To describe it, we may solve $f(x, y) = 0$ for y in terms of x and obtain the aforementioned Puiseux series. Each fractional power series solution gives rise to a branch of the curve.

Let us denote the trivial knot by K_0 and take the first approximation. This is indeed the (q_1, p_1) torus knot namely;

$$y = a_1 x^{q_1/p_1}$$

Let $x = \epsilon t^{\theta}$ such that $Arg(t) \in [0, 2\pi]$ i.e. it turns once around the complex unit circle S^1. Then we have for y;

$$y = a_1(\epsilon)^{q_1/p_1} t^{q_1}$$

Clearly, y is a constant times t^{q_1}. So, while t makes one turn around S^1, (x,y) makes p_1 turns in longitudional direction in R, which is x-axis and q_1 turns in meridianal drection of R, which is y-axis. This is the knot K_1 lying in the neighbourhood of K_0.
For the general case, we can write the following expression as;

$$K_1 \sim n_1 L_0 + m_1 M_0 \tag{5.3}$$

As a second approximation to the knot K, we consider the knot K_2, which lies in the ϵ-neighbourhood of K_1 and it will be a cable on K_1 such that it makes n_2 turns in longitudional direction. Generally speaking; since longitudional turn of K_i around K_{i-1} needs to be n_i, we have the following for $i = 1, ..., k$ where k is the number of Newton pairs;

$$L_i \sim n_1 L_{i-1} \tag{5.4}$$

Since $M_0 - n_1 M_1$ is zero-homologous in the neighbourhood of K_1, we have the following for all i;

$$M_{i-1} \sim n_i M_i \tag{5.5}$$

K_2 is given by the following equation;

$$y = a_1 x^{m_1/n_1} + a_2 x^{m_1/n_1 + m_2/n_1 n_2}$$

In this case, we update the parametrization of x to

$$x = \epsilon t^{n_1 n_2}$$

So, we have for y;

$$y = a_1 \epsilon^{m_1/n_1} t^{m_1 n_2} + a_2 \epsilon^{m_1/n_1 + m_2/n_1 n_2} t^{m_1 n_2 + m_2}$$

Similar to K_1 in the neighbourhood of K_0, K_2 turns $m_2 + n_1 n_2 m_1$ in meridianal direction in neighbourhood of K_1

Definition 5.2.1. *The number of meridianal turns of K_{i+1} in the neighbourhood of K_i is a_i.*

So K_2 is infact (n_2, a_2) cable on K_1 for $a_2 = m_2 + n_1 n_2 m_1$. Our aim is to find a general formula for a_i for all i.

Theorem 5.2.1. *The a_i above are given by Newton pairs such that*

- $a_1 = m_1$
- $a_{i+1} = m_{i+1} + n_i n_{i+1} a_i$ *for i≥1*

For K_2 we can say that

$$K_2 \sim n_2 K_1 + m_2 M_1$$

Substitute (5.3) in this expression;

$$K_2 \sim n_2(n_1 L_0 + m_1 M_0) + m_2 M_1$$

By (5.4) and (5.5);

$$K_2 \sim n_2(L_1 + m_1 n_1 M_1) + m_2 M_1$$

Finally we get;

$$K_2 \sim n_2 L_1 + (m_1 n_1 + m_2) M_1$$

Hence K_2 makes n_2 turns in longitudional direction and $m_1 n_1 + m_2$ turns in meridianal direction.
Let L be the knot obtained by shifting each point of K_i for a small distance directly away from K_{i-1}, hence it is perturbation of K_i and therefore they are homologous. L can be expressed parametrically as following for sufficiently small δ;

$$y = x^{m_1/n_1}(a_1 + (a_2 x^{m_2/n_2} + ...(a_{i-1} + x^{m_i/n_1 n_2 \dots n_i}(a_i + \delta)))...))$$

$$L \sim K_i \sim n_{i+1} L_i + a_{i+1} M_i$$

K_2 is homologous to $p_2 L_1 + (q_2 + p_1 p_2 q_1) M_1$ where L_1 and M_1 are longitude and meridian of K_2 in the neighbourhood of K_1 respectively.
Continuing in this manner, we can say that K_i is a cable knot (n_i, a_i) in the neighbourhood of (n_{i-1}, a_{i-1})-knot where a_i's are suitable integers in this context.
So, we have proved the theorem (5.2.1) (Eisenbud and Neumann, 1985).
The topological meaning of a_i is that it is the linking number between K_i and K_{i+1}, where the linking number can be explained briefly as the number of times each pair of components turn about each other (Rolfsen, 2003). The linking number is symmetric by its definition and changes sign if the orientation of one of the knots K_i or K_{i-1} is reversed.
When we consider that the result we proved in (5.2.1) is for Newton pairs, we can find a similar algorithm for Puiseux pairs as well. In other words, we seek for another algorithm to compute w_i's, which are same as a_i's for Puiseux pairs. Let v_i be the number of double points in the projection of torus knots. In this case, v_0 will be zero, since the double point for a circle is zero. For K_1 a torus knot of type (n_1, m_1) it is $(n_1 - 1)m_1$. If we consider the trefoil knot, the number of double points is just 3. This algorithm for w_i is as follows (Burau, 1933) for $i = 1, ..., k$;

$$w_i = p_i v_{i-1} + q_i \tag{5.6}$$

Theorem 5.2.2. *We can find a recurrence relation for* v_i *and work out it's value recursively. Namely, it yields to be;*

$$v_0 = 0 \tag{5.7}$$
$$v_i = v_{i-1}p_i^2 + (p_i - 1)q_i \tag{5.8}$$

Proof. By (5.2.1) we have for a_{i+1};

$$a_{i+1} = m_{i+1} + n_i n_{i+1} a_i$$

Since a_{i+1} is equal to w_{i+1} and by (5.6) we have that

$$m_{i+1} + n_i n_{i+1} a_i = p_{i+1} v_i + q_{i+1}$$

We replace the expression of a_i in the above relation with its definition given in (5.6);

$$p_{i+1} v_i + q_{i+1} = m_{i+1} + n_i n_{i+1}(p_i v_{i-1} + q_i)$$

By (3.3.0.1), for all i, we have $n_i = p_i$; and $m_i = q_i - p_i q_{i-1}$. So we do the necessary substituions to the above relation and get;

$$p_{i+1} v_i + q_{i+1} = q_{i+1} - p_{i+1} q_i + p_i^2 p_{i+1} v_{i-1} + p_i p_{i+1} q_i$$

Clearly we get that;

$$v_i = v_{i-1} p_i^2 + p_i q_i - q_i \tag{5.9}$$

□

Example 5.2.1.

$$y = x^{3/2} + x^{7/4}$$

In order to have a consistent notation, we rearrange the expression and get;

$$y = x^{3/2}(1 + x^{1/2.2})$$

So, we have the following Newton pairs recalling definition (3.3.2);

$$n_1 = 2 \qquad\qquad m_1 = 3$$
$$n_2 = 2 \qquad\qquad m_2 = 1$$

K_0 is the trivial knot, K_1 is $(2,3)$-torus knot. That enables us to consider K_1 as it makes 2 turns in longitudional direction and 3 turns in meridianal direction. In other words;

$$K_1 \sim 2L_0 + 3M_0$$

Similar to that, we have the following;

$$K_2 \sim 2L_1 + (1 + 2.2.3)M_1$$

$$y = x^{3/2} + x^{3/2+1/22}$$

We have

$$L \sim L_1 + n_1 m_1 M_1 = L_1 + 6M_1$$

$$K_2 \sim n_2 L + m_2 M_1 = 2L + M_1 = 2L_1 + 13M_1$$

The topological meaning of these expressions are; K_2 i.e. $(4, 7)$-knot makes twice longitudional turns around L_1 and 13 meridianal turn around M_1. Let $x = t^{n_1 n_2} = t^4$, then $y = t^6 + t^7$ where $t = e^{2\pi i(7/4)t}$.
Now we can compute the linking number for our example:

$$a_1 = m_1 = q_1 = w_1 = 3$$

$$\begin{aligned} a_2 &= 1 + 2.2.3 = 13 \\ &= 2.3 + 7 = w_2 \end{aligned}$$

6. CONCLUSION

Our efforts to understand the topology of an algebraic function around it's singular point, lead us to study the topology of the knot. While studying a knot K, we can consider the topology of the complement of it, denoted as $S^3 \setminus K$, which appears as an invariant since all information about the homotopy type of this space is contained in its fundamental group $\pi = \pi_1(S^3)$. For example we showed that for the $(q, p)-$ torus knot K we have

$$\pi = \pi_1(S^3 \setminus K) = \{a, b; a^q = b^p\}$$

That means; π is the group with two generators and just one relation. We showed this in Chapter 2 and computed the generators of the fundamental groups of branches. The relation between these generators are determined by the characteristic exponents, that carry out the information about the topology of the branch. More explicitly, if two branches have the same characteristic exponent, then the knots associated to the branches are isotopic.

References

Brauner, K. (1928). Zur Geometrie der Funktionen zweier komplexer Veränderlicher. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 6(1):1–55.

Brieskorn, E. and Knörrer, H. (1986). *Plane Algebraic Curves*. International series of monographs on physics. Birhkäuser Verlag, Basel.

Burau, W. (1933). Kennzeichnung der schlauchknoten. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 9(1):125–133.

Dimca, A. (1992). *Knots and Links*. Springer Verlag, New York.

Eisenbud, D. and Neumann, W. (1985). *Three Dimenssional Link Theory and Invariants of Plane Curve Singularities*. Princeton University Press.

Rolfsen, D. (2003). *Knots and Links*. American Mathematical Society.

Siegel, C. L. (1969). *Topics in Complex Function Theory*. John Wiley Sons Inc.

Willis, N. J., Didier, A. K., and Sonnanburg, K. M. (2008). How to compute a puiseux expansion. **URL:** *https://arxiv.org/pdf/0807.4674.pdf*.

Zariski, O. (1932). On the topology of algebroid singularities. *American Journal of Mathematics*, 54(3):453–465.

BIOGRAPHICAL SKETCH

Umran Gulbin AYDIN

Fatih, Turkey, 1991

Education

2010-2016	Bachelor of Science in MATHEMATICS **Bogazici University**, Istanbul, Turkey
2005-2010	High School Diploma in SCIENCE DEPARTMENT and Abitur **Istanbul (Erkek) High School**, Istanbul, Turkey