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DIFFERENTIAL FORMS ON STRATIFIED SPACES

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DIFFERENTIAL FORMS
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DİFERANSİYEL FORMLAR)

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List of Symbols

$\mathfrak{D}$	: Diffeology
$(X, \mathfrak{D})$	: Diffeological Space
P	: Plot
$\mathcal{D}(U, X)$	: Set of smooth maps from U to X
M	: Manifold
X_{reg}	: Regular part of X
$\mathcal{S}$	: Stratification on a diffeological space
S	: Strata
TS	: Tube over S
$\Omega_0^k[\mathrm{X}]$	: The space of stratified differential k-forms
$\Omega^k(\mathrm{X})$	: The space of differential k-form on diffeological spaces.

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Abstract

In this thesis, we studied the differential forms on stratified spaces by diffeological methods. A stratified space is a topological space that can be decomposed into pieces called strata which are required to fit together in a certain way. These spaces are not a manifold, and since there are no geometric definitions of concepts such as tangent bundles in singular spaces so differantial forms on these spaces are meaningless. However, in cases where the strata are manifold, it is possible to define the differential forms of these spaces on the regular parts by putting tube systems and appropriate retraction on these strata. In their study, Goresky-MacPherson described differential forms on stratified spaces by this method.

Some of the tools we use in classical differential geometry can be meaningless and difficult in some cases due to the reasons mentioned above. For this reason, we will revise stratified spaces as diffeological spaces in order to use the tools that are meaningful on diffeological spaces. As a result, we will compare the differential forms on previously defined stratified spaces and the differential forms on diffeological spaces.

Keywords: Manifold, Diffeological spaces, Stratified spaces, Differential forms.

Özet

Bu tez çalışmasında stratifiye uzaylar üzerindeki diferensiyal formları difeolojik yöntemlerle tanımlamaya çalıştık. Stratifiye uzaylar, stratalarının birbirine bağlanması açısından güzel yapıya sahip özel filtre edilmiş uzaylardır. Bu uzaylar bir manifold değillerdir ve tekil uzaylarda tanjant demetleri gibi kavramların geometrik tanımları olmadığı için bu uzaylar üzerinde differensiyal formlar anlamsız kalmaktadır. Fakat strataların manifold olduğu durumlarda bu strataların üzerine tüp sistemleri ve uygun büzülmeler koyarak diferensiyal formları bu uzayların düzgün bölümlerinde tanımlamamız mümkün. Goresky-MacPherson çalışmalarında bu bahsettiğimiz yöntemle diferensiyal formları tanımlamışlardır.

Klasik diferensiyal geometride kullandığımız bazı araçlar yukarıda bahsettiğimiz sebeplerden dolayı bazı durumlarda anlamsız ve zor kalabilmektedir. Bu sebeple biz difeolojik uzaylar üzerinde anlamlı olan araçları kullanabilmek adına stratifiye uzayları difeolojik uzaylar olarak görüp revize edeceğiz. Sonuç olarak daha önce tanımlanmış stratifiye uzaylar üzerindeki diferensiyal formlar ile difeolojik uzaylar üzerindeki diferensiyal formları karşılaştıracağız.

Anahtar Sözcükler : Manifold, Difeolojik uzaylar, Stratifiye uzaylar, Diferensiyal formlar.

1 INTRODUCTION

In this thesis we will examine differential forms on locally fibered stratified spaces. Because of that we introduce with differential forms on Euclidean spaces and manifolds.

1.1 Smooth Manifolds

First, we will define briefly the smooth manifolds. These are topological spaces that locally look like some euclidean space $\mathbb{R}^n$ on which we can do calculus. Following the definition, we give some examples.

Definition 1.1.1. *Let M be a topological space. We say that M is a* ***topological manifold*** *if M satisfies the following properties;*

- *M is a Hausdorff space.*
  That is for all $p, q \in M$, there exists disjoint open subsets U, V of M such that $p \in U$ and $q \in V$.
- *There exists a countable basis for the topology of M. (Second countable)*
- *M is locally Euclidean of dimension n. In other words, every point of M has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.*

By the definition of a topological manifold, we see that some homeomorphisms naturally arise. Let us define them more specifically:

Definition 1.1.2. *Let M be a topological manifold. A* ***chart*** *on M is a pair (U, ϕ) where U is an open subset of M and ϕ is a homeomorphism from U to an open subset of $\mathbb{R}^n$.*

Definition 1.1.3. *The collection of charts $A = \{(U_\alpha, \phi_\alpha)\}$, α indexed by some set B, on M such that $\bigcup_{\alpha \in B} U_\alpha = M$, is called an* ***atlas*** *for the topological space M.*

To be able to compare two charts of an atlas, we define the transition map. We consider the composition of one chart with the inverse of the other one. More precisely;

Definition 1.1.4. *Suppose that $(U_\alpha, , \phi)$ and (U_β, γ) are two charts for a manifold M such that $U_\alpha \cap U_\beta$ is non-empty. The **transition map**, denoted by $\tau_{\alpha,\beta}$, is a map from $\phi(U_\alpha \cap U_\beta)$ to $\gamma(U_\alpha \cap U_\beta)$ defined by $\tau_{\alpha,\beta} = \gamma \circ \phi^{-1}$, $\tau_{\alpha,\beta}$ is also a homeomorphism.*

As M is locally Euclidean of dimension n, we obtain that ϕ and γ are homeomorphisms. Now, to construct a smooth structure on an atlas M, we need to define the smoothness on functions and so on charts.

Definition 1.1.5. *Let U and V be two open subsets of euclidean space $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A function F from U to V is said to be **smooth** if each of its components has continuous partial derivative of all orders.*

In particular, if F is bijective and has a smooth inverse map then F is called a **diffeomorphism**.

Definition 1.1.6. *Two charts $(U_\alpha, , \phi)$ and (U_β, γ) are said to be **smoothly compatible** if either $U_\alpha \cap U_\beta = \varnothing$ or $\tau_{\alpha,\beta}$ is a diffeomorphism.*

Definition 1.1.7. *An atlas A is called a **smooth atlas** if any two charts in A are smoothly compatible with each other.*

By the above definition, to show that any atlas is smooth, we need to verify that each transition map according to this smooth atlas are smooth. The transition map is a diffeomorphism because its inverse map is also a transition map and it is smooth. There will be many possible atlases which give the same smooth structure. They determine the same collection of smooth functions on M. We give an example about this situation.

Example 1.1.1. *Let $A_1 = \{(\mathbb{R}^n, Id_{\mathbb{R}^n})\}$ and $A_2 = \{\{(B(x), Id_{B(x)}\} \,|x \in \mathbb{R}^n\}$ be two atlases where $B(x)$ is unit ball in $\mathbb{R}^n$ centered x, determine the same collection of smooth functions on M.*

Thus; we must define smooth structure as an appropriate equivalence classes. Two smooth atlases are said to be compatible if their union is still a smooth atlas. This relation is an equivalence relation. By this equivalence relation, a smooth atlas A on M is maximal if it is not properly contained in any larger atlas. It means that any chart which is smoothly compatible with every chart in A, is already in A.
If M is a topological manifold, a smooth structure on M is a maximal smooth atlas so; a smooth manifold is a pair (M, A), where M is a topological manifold and A is a maximal smooth atlas on M.

Example 1.1.2. *Consider the pair $(\mathbb{R}^n, Id)$: Note that $\mathbb{R}^n$ is a manifold and the identity function is a chart. Since this map is surjective, we conclude that it is an atlas.*

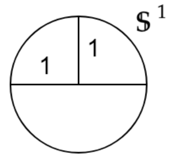

Figure 1.1: Unit sphere

Example 1.1.3. *We will show that the unit circle given by*

$$\mathbb{S}^1 = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| = 1\}$$

is locally Euclidean. First, note that $\mathbb{S}^1$ is Hausdorff and second countable since it is a subspace of $\mathbb{R}^2$. Let U_i^+, respectively U_i^- denote the subset of $\mathbb{S}^1$ where i-th coordinate is positive, respectively negative for $i \in \{1, 2\}$. Basically, U_i^+ and U_i^- are the following sets:

$$\begin{aligned} U_i^+ &= \{x = (x_1, x_2) \in \mathbb{R}^2 : x_i > 0\} \\ U_i^- &= \{x = (x_1, x_2) \in \mathbb{R}^2 : x_i < 0\} \end{aligned}$$

Define

$$\begin{aligned} \phi_i^\pm : U_i^\pm &\to \mathbb{R} \\ (x_1, x_2) &\mapsto x_j \text{ where } j \in \{1, 2\} \text{ and } j \neq i \end{aligned}$$

Observe that $\phi_i^\pm$ is continuous and $(\phi_i^\pm)^{-1}(x_j) = \left(x_j, \sqrt{1 + |x_j|^2}\right)$ is also continuous. So $\phi_i^\pm$ is a homeomorphism. We conclude that if $(x_1, x_2) \in \mathbb{S}^1$, then (x_1, x_2) is in the domain of one of the two charts. Hence $\mathbb{S}^1$ is locally Euclidean of dimension 1 and so it is a topological manifold.

Example 1.1.4. *We will show that the unit circle 1.1 which is given by above example is manifold with another atlas.*
This time, we will use stereographic projection in order to define the charts by projection from north point $(0, 1)$ and south point $(0, -1)$.
More precisely ; (U_1, ϕ_1) and (U_2, ϕ_2) are such charts where

$$U_1 = \mathbb{S}^1 \backslash \{(0, 1)\}$$

$$U_2 = \mathbb{S}^1 \backslash \{(0, -1)\}$$

$$\begin{aligned} \phi_1 : U_1 &\rightarrow \mathbb{R} \\ (x,y) &\mapsto \frac{x}{1-y} \text{ is continuous} \end{aligned}$$

$$\begin{aligned} \phi_2 : U_2 &\rightarrow \mathbb{R} \\ (x,y) &\mapsto \frac{x}{1+y} \text{ is continuous} \end{aligned}$$

This functions are homeomorphisms with this inverses,

$$\begin{aligned} \phi_1^{-1} : \mathbb{R} &\rightarrow U_1 \\ u &\mapsto \left(\frac{2u}{1+u^2}, \frac{u^2-1}{1+u^2}\right) \text{ is continuous} \end{aligned}$$

$$\begin{aligned} \phi_2^{-1} : \mathbb{R} &\rightarrow U_2 \\ u &\mapsto \left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right) \text{ is continuous} \end{aligned}$$

so; $\mathcal{A} = \bigcup_{i \in \{1,2\}} (U_i, \phi_i)$ *. For all* $(x,y) \in \mathbb{S}^1$*,* (x,y) *is in the one of the charts of* U_1 *and* U_2*. Thus* $\mathbb{S}^1$ *is locally Euclidean.* $\mathbb{S}^1$ *is a manifold.*

Example 1.1.5. *Double cone is not a smooth manifold.*

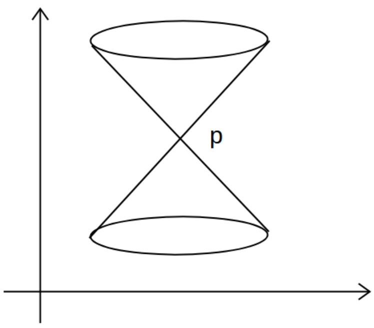

Figure 1.2: Double Cone

Consider a double cone given by the equation $\frac{z^2}{c^2} = \frac{x^2+y^2}{a^2}$*. We will now show that this double cone is not a smooth manifold. Assume not and say it is a smooth manifold so there exists an open subset* U *such that* ϕ *is smooth map from* U *to* V*, where* V *is open neighborhood of vertex* p*. we can write;* ϕ *from* $U - \phi^{-1}(p)$ *to* $V - \{p\}$ *is a* *homeomorphism. But $U-\phi^{-1}(p)$ is connected and $V-\{p\}$ is disconnected. Thus, ϕ cannot be a homeomorphism, so cannot be a smooth map. This leads to a contradiction.*

1.2 Differential Forms on Euclidean Spaces

A differential k-form α on $\mathbb{R}^n$ is an expression

$$\alpha = \sum_I f_I dx_I$$

where f_I 's are smooth functions on $\mathbb{R}^n$ and I is multi-index $(i_1, i_2, \ldots, i_k)$ of degree k.

Example 1.2.1. *A 4-form on $\mathbb{R}^6$;*

$$\alpha = x_1 x_3 x_5 dx_1 dx_6 dx_2 dx_3$$

A 2-form on $\mathbb{R}^3$;

$$\alpha = x_2 dx_1 dx_3 + x_3^2 x_5 dx_2 dx_3$$

But as we can see, domain of differential form can not be understood by expression and smooth functions may not be defined on whole space. We should specify the domain of the differential form.

Example 1.2.2. $\ln(x+y)zdz$ *is not a 1-form on $\mathbb{R}^3$ because ln is not defined for* $x+y=0$.

Now, we think dx_i as an oriented volume of k-dimensional rectangle block with sides $dx_{i_1}, dx_{i_2}, \ldots, dx_{i_k}$, therefore if we change the order of two variables, then the sign of side changes. This property is called alternating. That is

$$dx_{i_1} dx_{i_2} \cdots dx_{i_q} \cdots dx_{i_p} \cdots dx_{i_k} = -dx_{i_1} dx_{i_2} \cdots dx_{i_p} \cdots dx_{i_q} \cdots dx_{i_k}.$$

By this property, we can change the order of same variable. It means

$$dx_i dx_i = -dx_i dx_i$$

it implies that

$$dx_i dx_i = 0 \tag{1.1}$$

for all i. After that we can define the product of two diffenretial forms which is called exterior product.

Definition 1.2.1. *An exterior product of a k-form $\alpha = \sum_I f_I dx_I$ and a l-form $\beta = \sum_J g_J dx_J$ is $k+l$-form such that*

$$\alpha\beta = \sum_{I,J} f_I g_J dx_I dx_J$$

Remark. This product is graded commutative. That is

$$\beta\alpha = (-1)^{kl}\alpha\beta$$

It's clear that this result is about alternating property. l-many variables interchanges k-many so the sign changes $(-1)^{kl}$.

Now , we will give another important notion of differential forms. The exterior derivative of a k-form $\alpha = \sum_I f_I dx_I$ is a $k+1$-form such that

$$d\alpha = \sum_I df_I dx_I$$

Remark. This operator is linear which satisfy the Leibniz rule. It means

$$d(k\alpha + j\beta) = kd\alpha + jd\beta$$

for all k-forms α, β and all scalars k, j.

The following is the important property of derivative of differential forms.

Proposition 1.2.1. *For any differential form α ,*

$$d(d\alpha) = 0.$$

Proof. [6, Proposition 2.6]

Finally, we give the definition of exact and closed form.

Definition 1.2.2. *A form α is closed if $d\alpha = 0$.*
A form α is exact if $\alpha = d\beta$ for some form β.

1.3 Differential Forms On Manifolds

Let M be a n-manifold in $\mathbb{R}^N$ where $N \in \mathbb{N}$ and $f : M \to \mathbb{R}$. Since M is a manifold there exists open sets U_i in $\mathbb{R}^n$ and embeddings $\psi_i : U_i \to \mathbb{R}^N$ such that $M = \bigcup_i \psi_i(U_i)$. Now

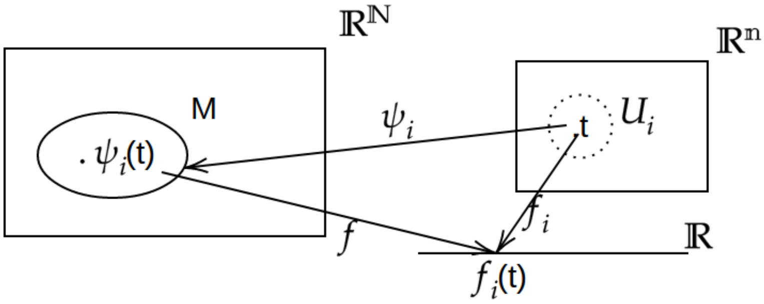

Figure 1.3: Differential forms on Manifold

for all i we define functions $f_i : U_i \to \mathbb{R}$ by

$$f_i(\mathrm{t}) = f\left(\psi_i(\mathrm{t})\right) \tag{1.2}$$

That is pullback of f by ψ_i. Assume that $x \in \psi_i\left(U_i\right) \cap \psi_j\left(U_j\right)$ such that

$$\mathbf{x} = \psi_i(\mathbf{t}) = \psi_j(\mathbf{u}). \tag{1.3}$$

where $\mathbf{t} \in U_i$ and $\mathbf{u} \in U_j$. Then $f(x) = f\left(\psi_i(t)\right) = f\left(\psi_j(u)\right)$. By 1.2 we have,

$$f(x) = f_i(\mathbf{t}) = f_j(\mathbf{u}) \tag{1.4}$$

Also we have 1.3 and we know that ψ_i's are bijective, so

$$\mathbf{t} = \psi_i^{-1} \circ \psi_j(\mathbf{u}) \tag{1.5}$$

Since 1.4 and 1.5,

$$f_j(\mathbf{u}) = f_i\left(\psi_i^{-1} \circ \psi_j(\mathbf{u})\right) \tag{1.6}$$

Naturally this hold for all u in $\psi_j^{-1}\left(\psi_i\left(U_i\right)\right)$ and on this set we can write the following;

$$f_j = \left(\psi_i^{-1} \circ \psi_j\right)^*\left(f_i\right) \tag{1.7}$$

In consideration of this construction we can give the definition of differential forms on manifolds.

Definition 1.3.1. *A differential k-form α on a manifold M is a collection of k-forms α_i on U_i satisfying 1.7. That is,*

$$\alpha_j = \left(\psi_i^{-1} \circ \psi_j\right)^* (\alpha_i)$$

2 LITERATURE REVIEW

The word ”stratification” has been introduced by R. Thom in his book ”La stabilité topologique des applications polynomiales” in 1962. He suggested conditions for how to unite these strata regularly for any stratification and isotopy lemmas. After that Mark Goresky and Robert MacPherson began their intersection theory studies in 1974 and published their first paper in intersection theory in the 80s. In this paper, they introduced intersection homology in order to extend Poincare Duality to some singular spaces. They also introduced intersection cohomology from the point of view intersection differential forms. Later, MacPherson introduced a more general notion perversity about intersection differential forms.
Diffeology is a relatively new area of mathematics which was started to be constructed at the late 20^{th} century which combines differential geometry and theoretical physics and leads to find some answers more ”easier”.
This concept was first introduced by Jean-Marie Souriau in 80’s and developed by his students Paul Donato and Patrick-Iglesias Zemmour.
After that Patrick Iglesias Zemmour and Serap Gürer give the condition about stratified differential forms which satisfy perversity-0 condition from diffeological point of view in the papers [2] and [3].

3 DIFFEOLOGY AND DIFFEOLOGICAL SPACES

This chapter can be seen as a brief introduction to diffeology. We will mainly talk about two concepts: diffeological spaces and smooth maps between them.

3.1 Diffeology

A diffeology on a non-empty set X is declaration of smooth maps from U to X where U is an open subset of $\mathbb{R}^n$ for all $n \in \mathbb{N}$. Before introducing the axioms of diffeology we will give some preliminary definitions.

Definition 3.1.1. *We say that an open subset U of n-dimensional euclidean space is n-**domain.***

Definition 3.1.2. *Let X be any non-empty set. Every map P from a n-domain to X, is called a n-**parametrization** in X. We denote the set of all parametrizations in X by $Param(X)$.*

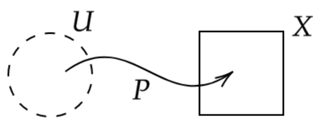

Figure 3.1: A parametrization

Now, we will define smooth parametrizations.

Definition 3.1.3. *Let U and V be a n-domain and m-domain respectively, f be a continuous map from U to V. f is said to be **differentiable** if there exists a map $D(f) : U \to L(\mathbb{R}^n, \mathbb{R}^m)$ such that $D(f)(x)(u) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon u) - f(x)}{\epsilon}$, for all $x \in U$ and all $u \in \mathbb{R}^n$.*

$D(f)$ is called **the derivative of** f.

Definition 3.1.4. *Let f be any map. f is said to be **class of** C^k, if it satisfies the following conditions:*

- *f is continuous when $k = 0$.*
- *if $0 < k < \infty$, then f is continuous, differentiable and its derivative is of class C^{k-1}.*
- *if $k = \infty$, then f is class of C^n, for all $n \in \mathbb{N}$.*

We will denote the set of C^k mappings from U to V by $C^k(U,V)$. We say that f is **smooth**, for $k = \infty$. The map is smooth if and only if its all partial derivatives are smooth.

Definition 3.1.5. *Let V be a domain. Every smooth map f from U to V is called* ***smooth parametrization****, where U is some domain.*

Now, we are ready to give the definition of diffeology on an arbitrary set X.

Definition 3.1.6. *Let X be an arbitrary non-empty set. A* ***diffeology of*** *X is any subset $\mathfrak{D}$ of $Param(X)$ which satisfies the following three axioms.*

(D1) *The set $\mathfrak{D}$ contains all the constant parametrizations.*

(D2) *Let P be a parametrization from U to X. If for all point r in U, there exists an open neighborhood V of r such that restriction of P to V in $\mathfrak{D}$, then P in $\mathfrak{D}$.*

(D3) *for all P from U to X of $\mathfrak{D}$, for every real domain V, for every $F \in C^\infty(V,U)$, $P \circ F \in \mathfrak{D}$.*

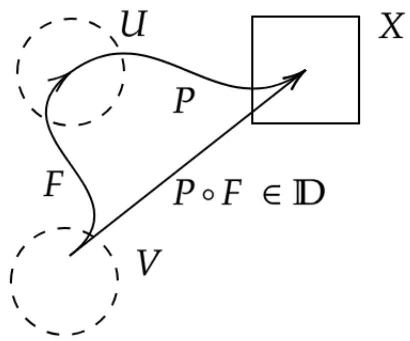

Figure 3.2: D3- Smooth Compabillity

Any non-empty set X equipped with a diffeology $\mathfrak{D}$ is called **a diffeological space**, so diffeological space is a pair $(X,\mathfrak{D})$ which will be denoted by X. The elements of diffeology $\mathfrak{D}$ are called **plots**. The set of plots is denoted by $Plots(X)$. If we want to restrict $Plots(X)$ on some domain U, then we formulate it by $Plots(U,X)$.

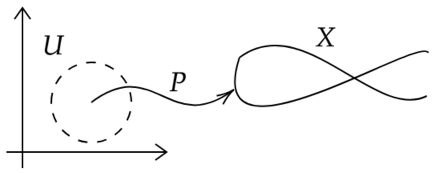

Figure 3.3: A Plot

In some situations we want to regard diffeology as a set, so it can be written of the following form:

$$Diffeology(X) = \{\mathfrak{D} \subset Param(X) | \ \mathfrak{D} \ satisfies \ D1, D2, D3\}$$

Example 3.1.1. *The set of all smooth parametrizations in a domain U is a diffeology on U. ($C^\infty(U)$: every smooth map f from V to U, where U is some domain.) It is also called* ***standart diffeology*** *of the domain U.*

Example 3.1.2. *Let equip $\mathbb{R}\times\mathbb{R}$ with standart diffeology. It means the set of all smooth parametrizations in $\mathbb{R}\times\mathbb{R}$. We define a square by $\{0,1\}\times[0,1]\cup[0,1]\times\{0,1\}\subset\mathbb{R}\times\mathbb{R}$. We want to regard the parametrizations of square as the parametrizations of $\mathbb{R}\times\mathbb{R}$. That is, the parametrizations of $\mathbb{R}\times\mathbb{R}$ whose values in square are the parametrizations of square. We claim that the set of all parametrizations in square which are smooth is diffeology. We will show that the set of such parametrizations is diffeology.*

(D1) *Every constant parametrization of square is constant parametrization of $\mathbb{R}\times\mathbb{R}$ which are smooth so diffeology of square contains constant parametrizations.*

(D2) *Take an locally smooth parametrization P of square at each point of its domain. That is locally smooth parametrization of $\mathbb{R}\times\mathbb{R}$ at each point of its domain. Since $\mathbb{R}\times\mathbb{R}$ is a diffeology, P is smooth.*

(D3) *Let P' be a plot of square and F be any smooth parametrization from any domain to domain of P'. Since P' is a plot, it is a smooth parametrization so $P\circ f$ is smooth.*
Thus, the square is a diffeological space equipped with this diffeology.

Now, we will give another example.

Example 3.1.3. *Let α be some irrational number. Let T_α be quotient space of $\mathbb{R}$ by the following equivalence relation;*

$$x \sim x' \ \ if \ and \ only \ if \ there \ exists \ n,m \in \mathbb{Z} \ \ such \ that \ x' = x + n + \alpha m.$$

We say that T_α is an irrational torus.
Let $\pi_\alpha : \mathbb{R} \longrightarrow T_\alpha$ be the canonical projection. The set of parametrizations, $P : U \longrightarrow T_\alpha$ such that;
for all $r \in U$, there exists a neighborhood V of r and a smooth parametrizations $Q : V \rightarrow R$ such that $\pi_\alpha \circ Q = P\lceil V$, is a diffeology of irrational torus.

(D1) *Let $t : r \mapsto t$ be a constant parametrization from a domain U to T_α. Since π_α is surjective there exists $x \in \mathbb{R}$, an open neighborhood V of r and a constant parametrizations $x : r \longmapsto x$ in $\mathbb{R}$ such that $t\lceil V = \pi_\alpha \circ x$.*

(D2) *Clearly the axiom of locality is satisfied.*

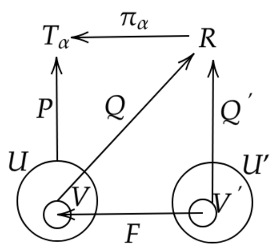

Figure 3.4: D3

(D3) *Let $P : U \rightarrow T_\alpha$ be a plot of T_α and $F : U' \rightarrow U$ be a smooth parametrization. Let $r' \in U'$ such that $r = F(r')$ and V be an open neighborhood of r. Since P is a plot there exists a smooth parametrization Q in $\mathbb{R}$ such that $\pi_\alpha \circ Q = P\lceil V$. Let $V' = F^{-1}(V)$ and $Q' = Q \circ F$ defined on V'. Thus, $\pi_\alpha \circ Q' = (P \circ F)|V'$. Hence, irrational torus is a diffeological space equipped with this diffeology.*

3.2 Smooth Maps Between Diffeological Spaces

We defined diffeological spaces in the previous section and now we examine the maps between these spaces.

Definition 3.2.1. *Let $(X, \mathfrak{D})$ and $(X', \mathfrak{D}')$ be two diffeological spaces. A map f from X to X' is said to be* ***smooth*** *if for each $P \in \mathfrak{D}$, then $f \circ P \in \mathfrak{D}'$.*

The set of smooth maps from X to X' is denoted by;

$$\mathcal{D}(X, X') = \{f \in Maps(X, X') | f \circ \mathfrak{D} \subset \mathfrak{D}'\}$$

Proposition 3.2.1. *The composition of two smooth maps is smooth.*

Proof. Let X, X',X'' be three diffeological spaces whose diffeologies are $\mathfrak{D}$, $\mathfrak{D}'$, $\mathfrak{D}''$ respectively. Let f, g be smooth maps from X to X' and from X' to X'' respectively. Let P be a plot of $\mathfrak{D}$. By the definition of smoothness of f, $f \circ P$ is a plot of $\mathfrak{D}'$. In the same way $g \circ (f \circ P)$ is plot of $\mathfrak{D}''$. Since the composition is associative, $(g \circ f) \circ P$ is plot of $\mathfrak{D}''$, so $g \circ f$ is smooth.

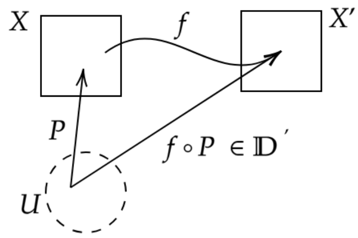

Figure 3.5: Smooth Maps

Proposition 3.2.2. *Let X be a diffeological space and U be a non-empty domain. The smooth maps from U to X are the plots of X defined on U. That is,*

$$\mathcal{D}(U, X) = \mathcal{D}(X)$$

Proof. First we take an element of $\mathcal{D}(U, X)$, it means a smooth map from U to X and show that this map is a plot of X. Let f be a smooth parametrization from U to X. We have identity map 1_u of U is a plot. By the axiom $D3$, $f \circ 1_u = f$ is a plot.
Now, take a plot and show that it is a smooth map. Let P be a plot from U to X. By the axiom $D3$ again, for every smooth parametrizations F in U, $P \circ F$ is plot of X. Then by definition P is smooth.

Definition 3.2.2. *Let X and X' be two diffeological spaces. A map f from X to X' is called* ***a diffeomorphism*** *if f is bijective and both f and f^{-1} are smooth.*

We denote the set of diffeomorphism from X to X' by $Diff(X, X')$.
Now, we talk about locality. In this case we define local smooth maps and local diffeomorphisms.

Definition 3.2.3. *Let X, X' be two diffeological spaces. Let $A \subset X$ be some subset of X. A map f from A to X is called* ***local smooth*** *map if; for every plot P of X, the parametrization $f \circ P$ defined on $P^{-1}(A)$ is a plot of X'.*

Definition 3.2.4. *Let X, X' be two diffeological spaces. A map f from X to X' is called **locally smooth** at the point $x \in X$ if there exists a subset $B \subset X$ containing x such that restriction of f to B is a local smooth map.*

Definition 3.2.5. *Let X, X' be two diffeological spaces. Let $A \subset X$ be some subset. A map f from A to X' is called **local diffeomorphism** if and only if f is injective, itself and its inverse are local smooth.*

Definition 3.2.6. *Let $x \in X$ be a point of X. We say that a map f from X to X' is a **local diffeomorphism at x** if there exists a subset $B \subset X$ containing x such that restriction of f to B is a local diffeomorphism.*

3.3 Comparing Diffeologies

We know that diffeology can be regarded as a set of plots. So we can compare diffeologies which are defined on the same set. This situation naturally bring the concept of partial ordering on diffeological spaces. This relation is called fineness. Now we give the definition of fineness.

Definition 3.3.1. *Let X be a non-empty set and $\mathfrak{D}$, $\mathfrak{D}'$ be two diffeologies of X. We say that $\mathfrak{D}$ is finer than $\mathfrak{D}'$ if $\mathfrak{D} \subset \mathfrak{D}'$.*

In other words, fineness means that $\mathfrak{D}$ has fewer elements than $\mathfrak{D}'$. Alternatively, we can say that $\mathfrak{D}'$ is coarser than $\mathfrak{D}$. After this definition the first question that comes to our mind is the existence of finest or coarsest diffeologies. If we think about coarsest diffeology, we need a diffeology which contains all other diffeologies which means that this diffeology have to contain all parametrizations on the set. Clearly, the set of all parametrizations is a diffeology. Thus that is the coarsest diffeology. Finest diffeology is more complicated than coarsest diffeology because this diffeology must be included in all other possible diffeologies. For this situation we will define discrete diffeology.

Definition 3.3.2. *Let X be a non-empty set. The locally constant parametrizations of X are defined as follows:*
A paramatrization P from a domain U to X is said to be locally constant if for all $r \in U$ there exists an open neighborhood V of r such that restriction of P to V is constant.

Plots of discrete diffeology are the locally constant parametrizations.

3.4 Transfer of Diffeology

In this section we talk about transfer of diffeology by some maps. We want to carry a diffeology from one set to another by using maps. There are common ways. These are pulling back, pushing forward, subset and quotient diffeologies.

3.4.1 Pulling Back Diffeologies

Let X be a non-empty set and $(X',\mathfrak{D}')$ be a diffeological space. Let f be a map from X to X'. There exists a coarsest diffeology of X, which is called pullback of $\mathfrak{D}'$ by f, such that f is smooth. It is denoted by $f^*(\mathfrak{D}')$. Thus this diffeology can be regarded as following:

$$f^*(\mathfrak{D}') = \{P \in Param(X) \,|\, f \circ P \in \mathfrak{D}'\}$$

Proposition 3.4.1. *A map f from X to X' is smooth if and only if $\mathfrak{D} \subset f^*(\mathfrak{D}')$*

Proof. Let $P \in \mathfrak{D}$, that is P is a plot, and consider a smooth map f. By definition of smoothness $f \circ P \in \mathfrak{D}'$, so such a P is also in $f^*(\mathfrak{D}')$ by definition of pulling back diffeology, Thus $\mathfrak{D} \subset f^*(\mathfrak{D}')$.
Conversely, if $\mathfrak{D} \subset f^*(\mathfrak{D}')$ for all P in $\mathfrak{D}$, then $f \circ P \in \mathfrak{D}'$. This implies that f is smooth.

3.4.2 Push Forward Diffeologies

Let X' be a non-empty set and $(X,\mathfrak{D})$ be a diffeological space. Let f be a map from X to X'. There exists finest diffeology of X which is called pushforward of $\mathfrak{D}$ by f, such that f is smooth. We denote it by $f_*(\mathfrak{D})$. A parametrization P from U to X' is a plot of $f_*(\mathfrak{D})$ if and only if for all r in U there exists an open neighborhood V of r which satisfy at least one property of the following;

1. The restriction of P to V is constant parametrization.

2. There exists a plot Q from V to X such that $P|V = f \circ Q$.

3.4.3 Subset Diffeology

As we know diffeology can be regarded as a set, naturally there exists subsets of this set. We will examine transfer of diffeology to its subsets.

Definition 3.4.1. *Let X be a diffeological space and $\mathfrak{D}$ be its diffeology. Pick any subset A of X and let j_A be the inclusion map from A to X. The pullback diffeology of the diffeology $\mathfrak{D}$ by inclusion j_A is called subset diffeology which is denoted by $j_A^{\star}(\mathfrak{D})$.*

Basically, the plots of subset diffeology are just the plots of X with values in A.
In other words,

$$j_A^{\star}(\mathfrak{D}) = \{P \in \mathfrak{D} | val(P) \subset A\}$$

As a result, **diffeological subspace** is any subset equipped with the subset diffeology.

Example 3.4.1. *Let $\mathbb{K}^2 = \left\{(x, y) \in \mathbb{R}^2 |\ x, y \geq 0\right\}$. Let $\mathfrak{D}$ be the standart diffeology of $\mathbb{R}^2$. We know that $\mathbb{K}^2 \subset \mathbb{R}^2$. By the definition of subset diffeology,*

$$j^*_{\mathbb{K}^2}\left(\mathbb{R}^2\right) = \left\{P \in \mathfrak{D}|\ val(P) \subset \mathbb{K}^2\right\}$$

is subset diffeology. So $\mathbb{K}^2$ is diffeological subspace equipped with this diffeology.

3.4.4 Quotient Diffeology

Definition 3.4.2. *Let X be a diffeological space, and let $\mathfrak{D}$ be its diffeology. Pick an equivalence relation $\mathfrak{R}$ on X. The quotient set $X/\mathfrak{R}$ carries a natural diffeology which is pushforward of diffeology $\mathfrak{D}$ by the natural projection from X to $X/\mathfrak{R}$. This diffeology is called the **quotient diffeology**.*

Example 3.4.2. *Double cone is a diffeological space.*
Let I be an open interval $]0, 1[$ and $\mathbb{S}^1$ be a circle. $\mathbb{S}^1 \times]0, 1[$ is a cylinder. We define an equivalance relation $\mathfrak{R}$ on $\mathbb{S}^1 \times]0, 1[$ by all points $(x, 1/2) \in \mathbb{S}^1 \times]0, 1[$ are equivalent the center of this circle. It define the quotient diffeology of the double cone equipped with smooth parametrizations class from $\mathbb{S}^1 \times]0, 1[$ to $\mathbb{S}^1 \times]0, 1[/\mathfrak{R}$.

3.5 Topology on Diffeological Space

Several topologies may be defined on diffeological spaces but one of them is important for the diffeological point of view. On every diffeological space X, there exists a topology such that plots are continous. This topology is called D-topology of X. The open sets for D-topology is called D-open which are identified by the following:
A subset $A \subset X$ is D-open if and only if for all plot P of X, P^{-1}(A) is open.

Definition 3.5.1. *Let X and Y be two diffeological spaces. We say that a map f from X to Y is D-continous if for all D-open V in Y, pre-image of V under f is D-open in X.*

Definition 3.5.2. *Let X and Y be two diffeological spaces. We say that a map f from X to Y is D-homeomorphism if f is bijective, D-continous and its inverse f^{-1} is D-continous.*

Proposition 3.5.1. *Let X, X' be two diffeological spaces. Every smooth map f from X to X' is D-continous.*

Proof. Consider a D-open $A' \subset X'$ and let $f^{-1}(A') = A$. For all plot P in $\mathfrak{D}$, we have $P^{-1}(A) = P^{-1}\left(f^{-1}(A')\right) = (f \circ P)^{-1}(A')$. Since f is smooth, $f \circ P$ belongs to $\mathfrak{D}'$ and A' is D-open by assumption, then $(f \circ P)^{-1}(A')$ is open. It means $P^{-1}(A)$ is open. By definiton, A is D-open. Hence f is D-continous.

Proposition 3.5.2. *Let X, X' be two diffeological spaces. Every diffeomorphism f from X to X' is D-homeomorphism.*

Proof. By the definition of the diffeomorphism f is bijective, smooth, and its inverse is smooth. By previous proposition, it is D-continous and its inverse also D-continous map which implies that it is D-homeomorphism.

Now, we talk about surjecive maps between diffeological spaces which are called subductions. At the same time, the local examination of these maps will provide us useful properties on the fibrations which are local subductions.

Definition 3.5.3. *Let X and $X^{'}$ be two diffeological spaces. Let f be a map from X to $X^{'}$. We say that f is a* ***subduction*** *if it satisfies the following;*

- *f is surjective.*
- *The diffeology of $X^{'}$ is the pushforward of the diffeology of X.*

Definition 3.5.4. *Let X and $X^{'}$ be two diffeological spaces. Let f be a smooth surjection from X to $X^{'}$. We say that f is* ***local subduction*** *at the point $x \in X$ if for every plot P from a domain U to $X^{'}$ pointed at $P'(0) = x' = f(x)$, there exist an open neighborhood V of 0 and a plot Q from V to X such that* $\mathrm{Q}(0) = x$ *and* $\mathrm{f} \circ \mathrm{Q} = \mathrm{P}|\mathrm{V}$.

If f is a local subduction at every point of X, then f is a subduction. The inverse is not true. One can find the proof and counter example in [8, 2.16] and [8, Exercice 61].

3.6 Diffeological Fibration

Definition 3.6.1. *Let T and B be two diffeological spaces. Let f be a smooth surjection from T to B. We say that f is* ***trivial with fiber*** *F if f is equivalent to the first projection $pr_1 : B \times F \to B$.*

Definition 3.6.2. *Let T and B be two diffeological spaces. Let f be a smooth surjection from T to B. We say that f is* ***locally trivial with fiber*** *F if, there exists a cover by D-opens $\{U_i\}_{i \in j}$ of B such that the restriction of f to each D-open U_i is trivial with fiber F.*

Definition 3.6.3. *Let T and B be two diffeological spaces. A smooth map f from a diffeological space T to another B is a* ***fibration*** *if and only if there exists a diffeological space F such that pullback of f by any plot P of B is locally trivial with fiber F.*

Lemma 3.6.1. *A diffeological fibration is a local subduction.*

Proof. Let T and B be two diffeological spaces and π be a fibration from T to B. For all plot P from a domain U to B, for all $r \in U$, for all $t = \pi^{-1}(\mathrm{b})$ with $b = P(r)$, there exists a plot Q of diffeology of T defined on some domain V of r such that $\mathrm{P} \lceil \mathrm{V} = \pi \circ \mathrm{Q}$ where $\mathrm{Q(r)} = \mathrm{t}$. This is the definition of local subduction.

4 STRATIFIED SPACE

We will introduce the notion of stratified spaces. Roughly speaking, a stratified space is a topological space that is not necessarily a manifold but which is filtered into strata which are manifolds. They are introduced by René Thom in [7] . The polyhedra, orbit spaces, algebraic varieties are some examples of stratified spaces.

4.1 Filtered Spaces

Definition 4.1.1. *Let X be a topological space. A* **filtration** *is the sequence of closed subspaces*

$$\varnothing = X^{-1} \subset X^0 \subset X^1 \subset \ldots \subset X^{n-1} \subset X^n = X$$

for some integer $n \geq -1$.

Definition 4.1.2. *A Hausdorff topological space X together with the filtration is called* ***filtered*** *space.*

The closed subspace X^i in the filtration is called an **i-skeleton**. The index i denote the formal dimension of skeleton. We say formal dimension by reason of keeping away from the confusion with topological dimension. The subset $X^n - X^{n-1}$ of the filtration is called the **regular part** of X which is denoted by X_{reg} and so X^{n-1} is called **singular part** which also denoted by X_{sing}.

Example 4.1.1. *A simplicial complex, according to the definition 5 in appendix, of triangle (with its interior) with the following is a filtered space.*

$$\varnothing \subset vertices \subset edges \subset triangle$$

Remark. The subspace $X^i - X^{i-1}$ is denoted by X_i.

Definition 4.1.3. *The connected components of X_i are called the* ***strata*** *of X of formal dimension i.*

Remark. We utilize **stratum** for a single connected component.

For defining stratified space we need a condition which give us a nice structure on filtered space without some possible pathologies about the concatenation of the strata.

Definition 4.1.4. *The filtered space X satisfies the* **Frontier Condition** *if the following condition is satisfied.*
For any two strata S, T of X such that $S \cap \bar{T} \neq \varnothing$ then $S \subset \bar{T}$ where $\bar{T}$ is closure of T.

Definition 4.1.5. *If a filtered space X satisfies the Frontier Condition, we call it* ***stratified space****.*

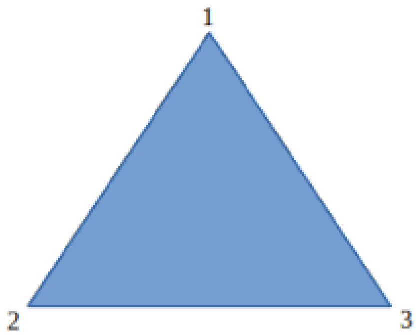

Figure 4.1: Triangle

Example 4.1.2. *We continue with the previous example 4.1.1. First we specialise the triangle by giving name to its vertex from left to right 2, 1, 3 respectively. Thus the filtration,*

$$\varnothing \subset 1,2,3 \subset [1,2],[2,3],[1,3] \subset [1,2,3]$$

Now we compose strata X_0, X_1 and X_2.
$X_0 = X^0 - X^{-1} = \{1,2,3\}$
$X_1 = X^1 - X^0 = \{(1,2),(2,3),(1,3)\}$
$X_2 = X^2 - X^1 = \{(1,2,3)\}$
If one checks the frontier condition for each pair of strata, one will notice that this condition is indeed satisfied.

Example 4.1.3. *We will give an example of a space which does not satisfy the Frontier Condition. Let Y be the y -axis and X be the union of the open upper half plane $\mathbb{H}^2$ and Y where $\mathbb{H}^2 = \left\{(x,y) \in \mathbb{R}^2 | \ \ y > 0\right\}$. We equip X with the following filtration.*

$$\varnothing \subset Y \subset X$$

The strata are Y and $X - Y$. Here Y intersects with the closure of $X - Y$ but it's not contained in the closure $X - Y$.

Definition 4.1.6. *A manifold stratified space is a stratified space if all its i-formal dimensional strata are i-dimensional manifolds.*

By our definition 'manifold' not possess manifold with boundary.

Example 4.1.4. *The example 4.1.2 is a manifold stratified space. It suffices to show that strata are manifold. The strata X_0 are points, X_1 are edges without boundary points and X_2 is triangle without border. Thus they are homeomorphic to 0-dimensional manifolds, $\mathbb{R}$ and $\mathbb{R}^2$ respectively.*

4.2 Locally Conelike Spaces

One may want to create a filtration by order the manifolds, with dimensions from less to more. But in this case pathological condition may cause complications between strata.So to refuse this situation we impose the conditions such as local flatness. Thus this is the main reason to define locally conelike space.

Definition 4.2.1. *Let Z be a compact space. The open cone cZ is the quotient of cZ $[0,1)\times Z$ by the relation $(0,w)\sim(0,z)$ where $w,z\in Z$.*

Definition 4.2.2. *A filtered space X of formal dimension n is* ***locally cone like*** *if ; for all i, $0\leq i\leq n$ and for each $x\in X_i$, there exist*

- *open neigbourhood I of x in X_i*
- *a neighbourhood D of x in X*
- *a compact filtered space L*
- *a homeomorphism $h: I\times cL\to D$ such that;*

$$h(I\times c(L^k)) = X^{i+k+1}\cap D$$

L and D are called link and distinguished neighbourhood of x respectively. Remark that L is not necessarily uniquely determined.

Definition 4.2.3. *Locally cone like spaces whose i-formal dimensional strata are i-dimensional manifolds are called CS sets.*

Remark. By the definition of locally cone like space, it is possible to say previous definition is equivalent to every point in i-formal dimensional stratum having a neighbourhood D homeomorphic to $\mathbb{R}^i\times cL$.

Example 4.2.1. *For showing that example 4.1.2 is locally conelike space, it suffice to find link L for all x for all cases.*

- *if point p is in the interior of triangle. So neighbourhoods U_1 and D_1 have same form.Thus L_1 must be empty.*
- *if point p is in the face of any edge. An open U_2 in strata of its edge is open in $\mathbb{R}$ and open D_2 in strata of X is half open ball in $\mathbb{R}^2$. For satisfying the homeomorphism L_2 must be $p \times [0,1]/(p,0) \sim (point,0)$*
- *if point p is in the special point of triangle(any of 1,2,3). An open U_3 in strata of special point is only p and an open D_3 in X is open ball in $\mathbb{R}^2$. It's like pieces of a cake. Thus the link L_3 is just previous $L_2 \times [0,1]$.*

Now, we will extend the stratified space to the diffeology.

4.3 Diffeological Stratified Space

Definition 4.3.1. *A **stratification** on a diffeological space X is a partition $\mathcal{S}$ of X into strata;*

$$X = \bigcup_{S \in \mathcal{S}} S \quad with \quad S \neq S' \Rightarrow S \cap S' = \varnothing$$

which satisfy the frontier condition.

Remark. Since X is a diffeological space, the topological condition of frontier condition must be changed for the D-topology.

Remark. We will assume that the diffeological space X is connected, Hausdorff and metrizable for the D-topology and all strata S are manifolds.

For specify this stratification we should talk about geometric and formal stratification. A geometric stratification is a natural partition on diffeological space X. We split this space in to connected components of orbits of pseudo-group of local diffeomorphisms. That is,

$$X = \bigcup_{x \in X} O_x \text{ with } \quad O_x = \{g(x) | g \in \mathrm{Diff}(X)\}$$

where O_x is such orbits.

Definition 4.3.2. *Let X be diffeological space equipped with stratification $\mathcal{S}$. We say that the stratification is **locally fibered** if there exists a tube system $\{\pi_s : TS \to S\}_{S \in \mathcal{S}}$ such that;*

- *TS is an open neighborhood of S, called tube over S*
- *The map $\pi_s : TS \to S$ is a smooth retraction.*
- *For all $x \in TS \cap TS' \cap \pi_{S'}^{-1}(TS)$, we have $\pi_S(\pi_{S'}(x)) = \pi_S(x)$*

5 DIFFERENTIAL FORMS ON STRATIFIED SPACES

In this chapter, we will examine the relation between diffeological differential forms and stratified forms on the stratified spaces.

5.1 Differential Forms

This section will introduce the definitions of stratified differential forms and diffeological differential forms.

Definition 5.1.1. *A **stratified** k-form on a locally fibered stratified space X is a differential k-form α defined on the regular part X_{reg} such that;*
for every stratum $S \in \mathcal{S}$, for all points $x \in TS \cap X_{reg}$, for all vectors $w \in ker D(\pi_S)_x$,

$$\alpha_x(w) = 0 \quad and \quad d\alpha_x(w) = 0$$

where $d\alpha_x(w)$ denotes the contraction of $d\alpha_x$ with w. This condition is also called perversity-0.

We will denote the space of stratified differential k-form by $\Omega_0^k[X]$.

Definition 5.1.2. *A differential k-form α on a diffeological space X is a mapping that associates with every plot P from U to X, a smooth k-form $\alpha(P)$ on U such that;*
for any smooth parametrization F in U,

$$\alpha(P \circ F) = F^*(\alpha(P)) \tag{5.1}$$

We will denote the space of differential k-form on diffeological space X by $\Omega^k(X)$ so by definition a stratified differential k-form α a priori belongs to $\Omega^k(X_{reg})$.
We can also mention the pullback and pushforward of the differential forms on any diffeological space by any smooth map. These notions are very practical to comparing differential forms on diffeological space and stratified differential forms. Because pullback of any differential form on diffeological space by smooth map is also differential form. This provide us to passing stratified differential forms.

Proposition 5.1.1. *Let X and $X^{'}$ be two diffeological spaces. Let $\alpha^{'} \in \Omega^k(X^{'})$ and f be a smooth map from X to $X^{'}$. There exists a differential k-form $f^*(\alpha^{'})$ on X such that for all plot P of diffeology of X,*

$$f^*(\alpha^{'})(P) = \alpha^{'}(f \circ P)$$

Proof. Let P be a plot from U to X and F be a smooth parametrization from a domain V to U. By the definition of pullback;

$$f^*(\alpha^{'})(P \circ F) = \alpha^{'}(f \circ P \circ F) = F^*(\alpha^{'}(f \circ P)) = F^*(f^*(\alpha^{'})(P))$$

Then $f^*(\alpha^{'})$ has the same form with the equation 5.1.

Proposition 5.1.2. *Let X and $X^{'}$ be two diffeological spaces. Let $\alpha \in \Omega^k(X)$ and $\pi : X \to X'$ be a subduction. The k-form α is the pullback of a k-form β defined on $X^{'}$ if and only if for any plots P, Q of diffeology of X such that $\pi \circ \mathrm{P} = \pi \circ \mathrm{Q}$, $\alpha(\mathrm{P}) = \alpha(\mathrm{Q})$.*

Proof. [8, 6.38]

Definition 5.1.3. *We say that any k-form α on X which satisfying the above property is **basic** form with respect to π.*

Proposition 5.1.3. *Let X and $X^{'}$ be two diffeological spaces. Let f be a subduction from X to $X^{'}$. For any two k-forms α and β in $X^{'}$, if $\mathrm{f}^*(\alpha) = \mathrm{f}^*(\beta)$, then $\alpha = \beta$.*
In other words the pullback of any subduction is injective.

Proof. [8, 6.39]

5.2 Strata Forms

Now, we will begin a construction that is used in our theorem which shows under some condition that a stratified differential form which defined on the regular part of some locally fibered stratified space X is a restriction of a differential form on X in the sense of diffeology.

Let X be a diffeological space equipped with stratification $\mathcal{S}$ and a tube system $\{\pi_s : TS \to S\}_{S \in \mathcal{S}}$. Let α be a stratified differential k-form on X and $S \in \mathcal{S}$, pr from $\tilde{S}$ to S be its universal covering. Consider the restriction of π_S to $TS \cap X_{reg}$ is a fiber bundle over S, denote its fiber by F. We want to associate a set $\mathcal{A}_S$ of differential k-forms on $\tilde{S}$ with α indexed by the connected components of F. We can formulate it by ;

$$\mathcal{A}_S = \{\bar{\alpha}_a\}_{a \in \pi_0(F)} \;\; with \;\; \bar{\alpha}_a \in \Omega^k(\tilde{S})$$

For the first part;
Let $pr : \tilde{S} \to S$ be the universal covering of S. By assumption strata are always assumed to be connected so $\tilde{S}$ is simply connected. Let $pr^*(TS \cap X_{reg})$ be the pullback of the above restriction by pr.

$$\begin{tikzcd}
pr^*(TS \cap X_{reg}) \arrow[r, "pr_2"'] \arrow[d, "pr_1"'] & TS \cap X_{reg} \arrow[d, "\pi_S"] \arrow[dl, dashed, dash] \\
\tilde{S} \arrow[r, "pr"'] & S
\end{tikzcd}$$

Figure 5.1: Commutative Diagram

That is;

$$pr^*(TS \cap X_{reg}) = \left\{(\tilde{x}, y) \in \tilde{S} \times TS \cap X_{reg} | pr(\tilde{x}) = \pi_S(y)\right\}$$

Let pr_1 and pr_2 be first and second projections respectively. Since pr_1 is a fibration that gives us the exact homotopy sequence which we show by following;

$$\ldots \pi_1(\tilde{S}) \to \pi_0(F) \to \pi_0(pr^*(TS \cap X_{reg})) \to \pi_0(\tilde{S})$$

Since $\tilde{S}$ is simply connected, $\pi_1(\tilde{S}) = \{0\}$ and again by same reason $\pi_0(\tilde{S}) = \{\tilde{S}\}$. By exactness of sequence we have;

$$\pi_0(F) \simeq \pi_0(pr^*(TS \cap X_{reg}))$$

It means each connected component of fiber F defines a connected component of $pr^*(TS \cap X_{reg})$ over $\tilde{S}$. We will denote it by;

$$\mathrm{pr}^*\left(\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}}\right) = \coprod_{a \in \pi_0(\mathrm{F})} \left\{\mathrm{pr}^*\left(\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}}\right)\right\}_a$$

Since restriction of a fibration is a fibration, the restriction of pr_1 to $\left\{pr^*(TS \cap X_{reg})\right\}_a$ is a fiber bundle with fiber $a \in \pi_0(F)$.

Now we continue with second part;
Consider the k-forms $\tilde{\alpha} = pr_2^*(\alpha) \in \Omega^k(pr^*(TS \cap X_{reg}))$ and the form $\tilde{\alpha_a}$ is the restriction of the form $\tilde{\alpha}$ to the connected components of $\left\{(pr^*(TS \cap X_{reg})\right\}_a$ with $a \in \pi_0(F)$.
For any element in $ker(D(pr_1))$, pr maps it to 0 in S. By commutative of diagram 5.1,

the elements are in $ker(D(\pi_S))$. It means $D(pr_2)$ maps $ker(D(pr_1))$ to $ker(D(\pi_S))$. It implies that $\tilde{\alpha_a}$ satisfy the perversity-0 condition. But $\left\{(pr^*(TS \cap X_{reg})\right\}_a$ is connected, in this case perversity-0 condition means $\tilde{\alpha_a}$ is basic, that is,
there exists a k-form $\bar{\alpha}_a$ on the covering $\tilde{S}$ such that;

$$\tilde{\alpha}_a = pr_1^*(\bar{\alpha}_a) \;\; and \;\; then, \;\; \mathcal{A}_S = \{\bar{\alpha}_a\}_{a \in \pi_0(F)}.$$

That is to say the restriction of $pr_2^*(\alpha)$ to $\left\{(pr^*(TS \cap X_{reg})\right\}_a$ is equal to $pr_1^*(\bar{\alpha}_a)$. In the final part we will talk about the action of $\pi_1(S)$. First of all, because of $\mathrm{pr}(\underline{k}(\tilde{x})) = \mathrm{pr}(\tilde{x})$, $\pi_1(S)$ acts on $\mathrm{pr}^*\left(\mathrm{TS} \cap \mathrm{X_{reg}}\right)$ by for all $k \in \pi_1(S)$, $\underline{k}(\tilde{x}, y) = (\underline{k}(\tilde{x}), y)$ where $\underline{k}$ symbolizes indifferently the two actions of k. We note following to give a lemma which has an important role for our proof:

- The action of $\pi_1(S)$ exchanges the connected components on $\mathrm{pr}^*\left(\mathrm{TS} \cap \mathrm{X_{reg}}\right)$.
- for all $k \in \pi_1(S)$, $\mathrm{pr}_1 \circ \underline{k} = \underline{k} \circ \mathrm{pr}_1$.
- The projection pr_2 is invariant by $\pi_1(S)$. It means $\mathrm{pr}_2 \circ \underline{k} = \mathrm{pr}_2$

Lemma 5.2.1. *If $\underline{k}$ sends the component relative to $a \in \pi_0(F)$ onto the component relative to b, then $\overline{\alpha}_a = \underline{k}^*\left(\overline{\alpha}_b\right)$.*

Proof. We have $\tilde{\alpha} = \mathrm{pr}_2^*(\alpha) \in \Omega^k\left(\mathrm{pr}^*\left(\mathrm{TS} \cap \mathrm{X_{reg}}\right)\right)$ from the second part of our construction. Now look at this pullback, $\underline{k}^*\left(\mathrm{pr}_2^*(\alpha)\right)$. By definition of pullback and thirth fact of action;

$$\underline{k}^*\left(\mathrm{pr}_2^*(\alpha)\right) = \alpha(pr_2(\underline{k})) = \alpha(pr_2) = pr_2^*(\alpha)$$

It implies that $\underline{k}^*(\tilde{\alpha}) = \tilde{\alpha}$.
Let $k \in \pi_1(S)$ and k permutes the components such that;

$$\underline{k} : \left\{\mathrm{pr}^*\left(\mathrm{TS} \cap \mathrm{X_{reg}}\right)\right\}_a \to \left\{\mathrm{pr}^*\left(\mathrm{TS} \cap \mathrm{X_{reg}}\right)\right\}_b$$

Let for all $i \in \pi_0(F)$, $j_i : \left\{\mathrm{pr}^*\left(\mathrm{TS} \cap \mathrm{X_{reg}}\right)\right\}_i \to \mathrm{pr}^*\left(\mathrm{TS} \cap \mathrm{X_{reg}}\right)$ be the inclusion of the components. So that is;

$$\underline{k} \circ j_a = j_b \circ \underline{k}$$

so pullback of $\tilde{\alpha}$ by $k \circ j_a$ and $j_b \circ k$ are equal. That is, $\left(\underline{k} \circ j_a\right)^*(\tilde{\alpha}) = \left(j_b \circ \underline{k}\right)^*(\tilde{\alpha})$. By the definition of pullback,

$$j_a^*\left(\underline{k}^*(\tilde{\alpha})\right) = \underline{k}^*\left(j_b^*(\tilde{\alpha})\right)$$

But we have $\underline{k}^*(\tilde{\alpha}) = \tilde{\alpha}$ so $j_a^*(\tilde{\alpha}) = \underline{k}^*\left(j_b^*(\tilde{\alpha})\right)$. It means $\tilde{\alpha}_a = \underline{k}^*(\tilde{\alpha}_b)$ where $\tilde{\alpha}_i = \tilde{\alpha}|\left\{\mathrm{pr}^*\left(\mathrm{TS}\cap X_{\mathrm{reg}}\right)\right\}_i$. Accordingly $\tilde{\alpha}_i = \mathrm{pr}_1^*(\overline{\alpha}_i)$, $\mathrm{pr}_1^*(\overline{\alpha}_a) = \underline{k}^*\left(\mathrm{pr}_1^*(\overline{\alpha}_b)\right)$. By the second fact of action we obtain $\mathrm{pr}_1^*(\overline{\alpha}_a) = \mathrm{pr}_1^*\left(\underline{k}^*(\overline{\alpha}_b)\right)$. Since pr_1 is a fibration, $\overline{\alpha}_a = \underline{k}^*(\overline{\alpha}_b)$.

Now we can give the theorem.

Theorem 5.2.2. *There exists a differential k-form α_S on the stratum S such that $\alpha\lceil\mathrm{TS} = \pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}})$ if and only if, for all $a, b \in \pi_0(\mathrm{F}), \alpha_a = \alpha_b$. In this case, for all $\boldsymbol{a} \in \pi_0(\mathrm{F})$, $\overline{\alpha}_a = \mathrm{pr}^*(\alpha_{\mathrm{S}})$.*

On this point, we introduce the **index** of the form $\alpha \in \Omega_0^*[\mathrm{X}]$ at the stratum S by ;

$$\nu_{\mathrm{S}}(\alpha) = \mathrm{card}\left(\mathcal{A}_{\mathrm{S}}\right)$$

In brief, by definition the set $\mathcal{A}_S$ is indexed by connected components of $\pi_0(F)$, so cardinality of this set give us the number of connected components. Therefore we can give this theorem as follow;
There exists a differential k-form α_S on the stratum S such that $\alpha\lceil\mathrm{TS} = \pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}})$ if and only if $\nu_{\mathrm{S}}(\alpha) = 1$.

Proof. Let assume that there exists α_S such that $\alpha\lceil\mathrm{TS} = \pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}})$. Let $\mathrm{pr}_{1,a} = \mathrm{pr}_1\lceil\left\{\mathrm{pr}^*\left(\mathrm{TS}\cap\mathrm{X}_{\mathrm{reg}}\right)\right\}_a$ and $\mathrm{pr}_{2,a} = pr_2\lceil\left\{pr^*\left(\mathrm{TS}\cap\mathrm{X}_{\mathrm{reg}}\right)\right\}_a$ be first and second projections. By commutative diagram $\pi_{\mathrm{S}}\circ pr_{2,a} = pr\circ pr_{1,a}$ and by assumption $\alpha = \pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}})$ on $\left\{\mathrm{pr}^*\left(\mathrm{TS}\cap\mathrm{X}_{\mathrm{reg}}\right)\right\}_a$ so we have,

$$\mathrm{pr}_{2,a}^*(\alpha) = \mathrm{pr}_{2,a}^*\left(\pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}})\right) = \mathrm{pr}_{1,a}^*\left(\mathrm{pr}^*(\alpha_{\mathrm{S}})\right)$$

But we know that by our construction above, the restriction of $pr_2^*(\alpha)$ to $\left\{(pr^*(TS\cap X_{reg})\right\}_a$ is equal to $pr_1^*(\bar{\alpha}_a)$. In this case, $\mathrm{pr}_{2,a}^*(\alpha) = \mathrm{pr}_{1,a}^*(\overline{\alpha}_a)$. Thus we have,

$$\mathrm{pr}_{1,a}^*\left(\mathrm{pr}^*(\alpha_{\mathrm{S}})\right) = \mathrm{pr}_{1,a}^*(\overline{\alpha}_a)$$

Since $\mathrm{pr}_{1,a}$ is a fibration, $\mathrm{pr}^*(\alpha_S) = \overline{\alpha}_a$. This process is the same for another connected components b because it's independent the choice of components, so; $\overline{\alpha}_a = \overline{\alpha}_b$ for all $a, b \in \pi_0(\mathrm{F})$. It means there is only one connected component and cardinality is 1 so $\nu_{\mathrm{S}}(\alpha) = 1$.
Conversely, assume that $\nu_{\mathrm{S}}(\alpha) = 1$. It implies that $\overline{\alpha}_a = \overline{\alpha}_b$ for all $a, b \in \pi_0(\mathrm{F})$. Let us denote this differential form on $\tilde{S}$ by $\overline{\alpha}$. By the lemma 5.2.1, for all $k \in \pi_1(\mathrm{S})$ $\underline{k}^*(\overline{\alpha}) = \overline{\alpha}$. This means that $\overline{\alpha}$ is invariant by $\pi_1(S)$. Hence $\overline{\alpha}$ is basic differential form, then there

exists $\alpha_{\mathrm{S}} \in \Omega^k(\mathrm{S})$ such that $\overline{\alpha} = \mathrm{pr}^*(\alpha_S)$. The commutative diagram give us $\pi_{\mathrm{S}} \circ \mathrm{pr}_2 = \mathrm{pr} \circ \mathrm{pr}_1$, so we have its pullback equation

$$\mathrm{pr}_2^*\left(\pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}})\right) = \mathrm{pr}_1^*(\mathrm{pr}^*(\alpha_{\mathrm{S}})) = \mathrm{pr}_1^*(\overline{\alpha}) \tag{5.2}$$

By our construction the restriction of $pr_2^*(\alpha)$ to $\left\{(pr^*(TS \cap X_{reg})\right\}_a$ is equal to $pr_1^*(\bar{\alpha}_a)$. In this case we can denote it by $\mathrm{pr}_{2,a}^*(\alpha) = \mathrm{pr}_1^*(\overline{\alpha}_a)$, but we have assumed that $\overline{\alpha}_a = \overline{\alpha}_b = \overline{\alpha}$, so $\mathrm{pr}_2^*(\alpha) = \mathrm{pr}_1^*(\overline{\alpha})$. Put $\mathrm{pr}_2^*(\alpha)$ instead of $\mathrm{pr}_1^*(\overline{\alpha})$ in above pullback equation 5.2. We obtain $\mathrm{pr}_2^*\left(\pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}})\right) = \mathrm{pr}_2^*(\alpha)$. Since pr_2 is a fibration, its pullback is injective so $\alpha|\mathrm{TS} = \pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}})$.

6 CONCLUSION

In this thesis, we study stratified differential forms and differential forms on diffeological spaces. For this purpose, we basically introduce diffeological spaces and their smooth maps between us. After that we give the definition subduction and diffeological fibration which have the important properties. These properties are the key points of our proofs. We know that Goresky-MacPherson have already defined stratified differential forms by using perversity-0 condition. In order to use this definition, we adapt stratified spaces to the diffeological point of view. On this point, we talk about what is the stratification of diffeological spaces and locally fibered stratified spaces. Then, we introduce the index of the stratified differential form so we prove a theorem with this new notion. The theorem tell us the number of the connected components of $\pi_0(F)$ is equal to 1 if and only if there exists a differential k-form on the strata S such that the restriction of stratified differential k-form on X to tube over S is equal to the pullback of this form. Starting from this point of view, we reach the following theorems as a conclusion. Briefly, this theorem show us stratified differential form which defined on the regular part of some locally fibered stratified space X is a restriction of a differential form on X in the sense of diffeology.

Theorem 6.0.1. *Let* $\alpha \in \Omega^k_0[\mathrm{X}]$. *If* $\nu_S(\alpha) = 1$ *for all* $\mathrm{S} \in \mathcal{S}$, *then there exists a (unique) differential form* $\underline{\alpha} \in \Omega^k(\mathrm{X})$ *such that* $\alpha = \underline{\alpha}|\mathrm{X}_{\mathrm{reg}}$.

Proof. Let $\alpha \in \Omega^k_0[\mathrm{X}]$ such that $\nu_S(\alpha) = 1$ for all $S \in \mathcal{S}$. Let S and $S^{'}$ be two strata such that $TS \cap TS' \neq \varnothing$. By previous theorem 5.2.2 for all $\mathrm{S} \in \mathcal{S}$, there exists a differential k-form α_S on the stratum S such that $\alpha \lceil \mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}} = \pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}})$ so on $\mathrm{TS} \cap \mathrm{TS}' \cap \mathrm{X}_{\mathrm{reg}}$ which is open,

$$\pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}}) = \pi^*_{\mathrm{S}'}(\alpha_{\mathrm{S}'}) = \alpha. \tag{6.1}$$

Then, consider $x \in \mathrm{TS} \cap \mathrm{TS}'$ such that $x \notin \mathrm{X}_{\mathrm{reg}}$ so $x \in \mathrm{TS} \cap \mathrm{TS}' \cap \mathrm{X}_{\mathrm{sing}}$, write it $x \in \mathrm{S}'' \cap \mathrm{TS} \cap \mathrm{TS}'$ with $S'' \subset X_{\mathrm{sing}}$. Since $\mathrm{S}'' \cap \mathrm{TS} \cap \mathrm{TS}'$ is non-empty, $S'' \cap TS \neq \varnothing$ it implies that $\bar{S''} \cap S \neq \varnothing$ by frontier condition $\mathrm{S} \subset \overline{\mathrm{S}}''$ and by definition of locally fibered stratified space,

$$\pi_{\mathrm{S}} \circ \pi_{\mathrm{S}''} = \pi_{\mathrm{S}}$$

on $\mathrm{TS} \cap \mathrm{TS}'' \cap \pi_{\mathrm{S}''}^{-1}(\mathrm{TS}) \cap \mathrm{X}_{\mathrm{reg}}$. We can write it;

$$\pi_{\mathrm{S}''}^{*}\left(\pi_{\mathrm{S}}^{*}(\alpha_{\mathrm{S}}) - \alpha_{\mathrm{S}''}\right) |\mathrm{TS} \cap \mathrm{TS}'' \cap \pi_{\mathrm{S}''}^{-1}(\mathrm{TS}) \cap \mathrm{X}_{\mathrm{reg}} = 0$$

But $\pi_{\mathrm{S}''}|\mathrm{TS} \cap \mathrm{TS}'' \cap \pi_{\mathrm{S}''}^{-1}(\mathrm{TS}) \cap \mathrm{X}_{\mathrm{reg}}$ is submersion on $S'' \cap TS$. The pullback of any submersion is injective and this map is linear so we have

$$\alpha_{\mathrm{S}''}|\mathrm{S}'' \cap \mathrm{TS} = \pi_{\mathrm{S}}^{*}(\alpha_{\mathrm{S}}).$$

and at the same time, on $\mathrm{TS}'' \cap \mathrm{TS}$:

$$\pi_{\mathrm{S}''}^{*}(\alpha_{\mathrm{S}''}) = \pi_{\mathrm{S}''}^{*}\left(\pi_{\mathrm{S}}^{*}(\alpha_{\mathrm{S}})\right) = (\pi_{\mathrm{S}} \circ \pi_{\mathrm{S}''})^{*}(\alpha_{\mathrm{S}}) = \pi_{\mathrm{S}}^{*}(\alpha_{\mathrm{S}})$$

On the other hand, we have 6.1 on $\mathrm{TS}'' \cap \mathrm{TS}'$ so

$$\pi_{\mathrm{S}}^{*}(\alpha_{\mathrm{S}}) = \pi_{\mathrm{S}'}^{*}(\alpha_{\mathrm{S}'})$$

should be defined on an open neighborhood of x so that for this situation $\mathrm{TS} \cap \mathrm{TS}' \cap \mathrm{X}_{\mathrm{sing}}$. Thus, $\pi_{\mathrm{S}}^{*}(\alpha_{\mathrm{S}}) = \pi_{\mathrm{S}'}^{*}(\alpha_{\mathrm{S}'})$ on $\mathrm{TS} \cap \mathrm{TS}'$. Since differential forms on diffeological spaces are local, there exists a differential k-form $\underline{\alpha}$ defined on X such that for all $\mathrm{S} \in \mathcal{S}$, $\underline{\alpha}|\mathrm{TS} = \pi_{\mathrm{S}}^{*}(\alpha_{\mathrm{S}})$ and then $\alpha = \underline{\alpha}\lceil \mathrm{X}_{\mathrm{reg}}$.

Theorem 6.0.2. *Let* $\underline{\alpha} \in \Omega^{k}(\mathrm{X})$ *such that* $\alpha = \underline{\alpha}|\mathrm{X}_{\mathrm{reg}} \in \Omega_{\mathrm{o}}^{k}[\mathrm{X}]$. *If for any two points in* X_{reg} *there is a smooth path connecting them that cuts* X_{sing} *into a finite number of points, then* $\nu_{\mathrm{S}}(\alpha) = 1$ *for all* $\mathrm{S} \in \mathcal{S}$.

Proof. Let $\alpha = \underline{\alpha}|\mathrm{X}_{\mathrm{reg}}$ such that $\underline{\alpha} \in \Omega^{k}(\mathrm{X})$. By the commutative diagram 5.1 the pullback $\mathrm{pr}_1 : \mathrm{pr}^{*}(\mathrm{TS}) \to \tilde{\mathrm{S}}$ is a locally trivial fiber bundle. Let's call its fiber $\underline{F}$ and in this case $F = \underline{F}_{reg}$ be the regular part. The differential form $\mathrm{pr}_2^{*}(\underline{\alpha})$ is defined on the pullback $\mathrm{pr}^{*}(\mathrm{TS})$ and so we have

$$\mathrm{pr}_2^{*}(\alpha) = \mathrm{pr}_2^{*}(\underline{\alpha})\lceil \mathrm{pr}^{*}\left(\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}}\right) \tag{6.2}$$

As we obtained in construction before giving first theorem, the restriction of $pr_2^{*}(\alpha)$ to $\left\{(pr^{*}(TS \cap X_{reg})\right\}_a$ is equal to $pr_1^{*}(\bar{\alpha}_a)$. Then according to this situation, for each component $a \in \pi_0(\mathrm{F})$, we have $\mathrm{pr}_2^{*}(\alpha)\lceil\left\{\mathrm{TS} \cap X_{\mathrm{reg}}\right\}_a = \mathrm{pr}_1^{*}(\overline{\alpha}_a)$. Since equality 6.2;

$$\mathrm{pr}_2^{*}(\underline{\alpha})\lceil\left\{\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}}\right\}_a = \mathrm{pr}_1^{*}(\overline{\alpha}_a) \tag{6.3}$$

Now, let F_a and F_b express different connected components of F. Let $y \in F_a$ and $y' \in \mathrm{F}_b$.

Assume that there exists a smooth map $t \longmapsto y_t$ in $\underline{F}$ such that connect y to $y^{'}$and cut singular part of F in a finite number of points. Then the interval $]0,1[$ is divided into finite set of open intervals denoted by I_a where y_t belongs to the component $a \in \pi_0(F)$, seperated by points belonging to $\underline{F}_{\text{sing}}$. Since pr_1 is a locally trivial fiber bundle, we have local trivialization of pr_1. Let it be $(r,y) \longmapsto (\tilde{x}_r, y)$, so $(r,t) \longmapsto (\tilde{x}_r, y_t)$ is a plot of $\text{pr}^*(\text{TS})$. On the open subset of (r,t) such that $t \in \text{I}_a$ we have;

$$\text{pr}_2^*(\underline{\alpha})((r,t) \mapsto (\tilde{x}_r, y_t)) \begin{pmatrix} u_i \\ \varepsilon_i \end{pmatrix}_{i=1}^{k} = \text{pr}_1^*(\overline{\alpha}_a)((r,t) \mapsto (\tilde{x}_r, y_t)) \begin{pmatrix} u_i \\ \varepsilon_i \end{pmatrix}_{i=1}^{k} \tag{6.4}$$

And then;

$$\overline{\alpha}_a(r \mapsto \tilde{x}_r)_r(u_1)\ldots(u_k) \tag{6.5}$$

where $\underline{\alpha}$ is a k-fom and the (u_i, ε_i) are tangent vectors at (t,r).
But for each r and $u_1 \ldots u_k$

$$t \mapsto \text{pr}_2^*(\underline{\alpha})((r,t) \mapsto (\tilde{x}_r, y_t))_{(t)} \begin{pmatrix} u_i \\ \varepsilon_i \end{pmatrix}_{i=1}^{k} \tag{6.6}$$

is smooth but it is constant on each I_a. Since $\underline{\text{F}}$ is connected and it is the closure of F,

$$\overline{\alpha}_a(r \mapsto \tilde{x}_r)_r(u_1)\ldots(u_k) = \overline{\alpha}_b(r \mapsto \tilde{x}_r)_r(u_1)\ldots(u_k) \tag{6.7}$$

Hence, $\nu_{\text{S}}(\alpha) = 1$ for all $\text{S} \in \mathcal{S}$.

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Appendix

Some definitions and theorems which we used as follows:

Definition 1. *A **fiber bundle** structure on a topological space E , with fiber F , consists of a projection map $p : E \longrightarrow B$, which are called total and base space respectively, such that each point of B has a neighborhood U such that there exists a homeomorphism $h : p^{-1}(U) \longrightarrow U \times F$ making the following diagram at the right commute:*

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\text{first projection}"] \\ & U & \end{tikzcd}$$

Figure 6.1: Diagram at right commute

Homeomorphism h is called **local trivialization** of the bundle . h carries each fiber $p^{-1}(b)$ where $b \in B$, homeomorphically to $\{b\} \times F$.

Fact 1. Fiber bundles over manifolds are fibrations

Definition 2. *A covering space of a topological space X is a space $\tilde{X}$ together with a map $p : \tilde{X} \longrightarrow X$ satisfying;*
for all $x \in X$ there exists a neighborhood of x such that $p^{-1}(U)$ is union of disjoint open sets.

Definition 3. *A simply-connected covering space of a path-connected space X is a covering space of every other path-connected covering space of X . A simply-connected covering space of X is called a **universal cover**.*

Lemma 1. A fiber bundle with a fiber which is discrete space is a covering space.

Definition 4. *A subset $\{v_0, \cdots, v_r\} \subset \mathbb{R}^n$ is said to be affinely independent if the set of vectors $\{v_1 - v_0, v_2 - v_0, \cdots, v_r - v_0\}$ is linearly independent. Given any affinely independent set $\{v_0, \cdots, v_r\}$. The r-**simplex** is a expression;*

$$< v_0, \cdots, v_r >= \left\{ \sum_{i=0}^{r} t_i v_i | \sum_{i=0}^{r} t_i = 1,\ t_i \geq 0 \right\}.$$

Definition 5. *A* ***simplicial complex*** *is a non-empty finite set K of simplices (plural of simplex) in $\mathbb{R}^n$ such that*

- *if $\sigma \in K$ and T is a subcomplex of σ then $T \in K$.*
- *if σ_1 , $\sigma_2 \in K$, then $\sigma_1 \cap \sigma_2 = \varnothing$ or $\sigma_1 \cap \sigma_2$ is a subcomplex of σ_1 or σ_2.*

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