ON MANIFOLDS WITH BOUNDARY AND CORNERS

(SINIRLI VE KÖŞELİ MANIFOLDLAR)

by

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ABSTRACT

In this thesis, smooth manifold, manifold with boundary and manifolds with corners are examined from differential calculus point of view. As it is known, manifolds with boundary and corners are not smooth manifold. We have examined these spaces as an example of smooth spaces developed in recent years in terms of diffeology. First, we have shown that these are diffeological spaces, then they are diffeological manifolds. We examined the smooth functions and differential forms by utilizing the tools provided by diffeology on these spaces. We characterized the differential forms on manifolds with corners.

Keywords : Differantial Forms, Manifolds with Corners

ÖZET

Bu tezde, düzgün manifold, sınırlı manifold ve köşeli manifoldlar diferansiyel geometri açısından incelenmiştir. Bilindiği üzere sınırlı ve köşeli manifoldlar düzgün manifoldlar değildir. Bu uzayları son dönemde geliştirilen düzgün uzayların bir örneği olarak difeolojik açıdan inceledik. İlk olarak bunların difeolojik uzay olduğunu, daha sonra difeolojik manifold olduklarını gösterdik. Bu uzaylar üzerinde difeolojinin bizlere sağladığı araçlardan faydalanarak düzgün fonksiyonları ve diferansiyel formları inceledik. Köşeli manifoldlar üzerinde diferansiyel formları karakterize etmeye çalıştık ve bu uzaylar üzerinde karakterize ettiğimiz diferansiyel formlarla ilgili hesaplamalar yaptık.

Anahtar Kelimeler : Diferansiyel Formlar, Köşeli Manifoldlar

1 INTRODUCTION

In this chapter, we will introduce the topological manifold and the smooth manifold. We will give a few examples of smooth manifolds. Finally, we will introduce the manifolds with boundary and corners.

1.1 Topological Manifolds

First, we will define the topological spaces, since topological manifolds are defined on topological spaces.

Definition 1.1. Let M be a set and let τ be a set of subsets of M. The set M is called a **topological space**, if τ satisfy the following three axioms :

1. $\emptyset, M \in \tau$.
2. If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$.
3. If $U_i \in \tau$ for $i \in I$, then $\bigcup_{i\in I} U_i \in \tau$.

Definition 1.2. The topological space M is said to be a **topological manifold** of dimension n or n-manifold, for $n \in \mathbb{N}$ if it satisfies the following three conditions :

1. M is a Hausdorff space, i.e. for $x, y \in M$ with $x \neq y$ and there exists U, V open subsets of M such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.
2. M is second countable.

   If τ has a countable topological basis, i.e. this topological space has a countable open base, then M is called a second countable.
3. M is locally Euclidean of dimension n.

   Every point in M has an open neighbourhood homeomorphic to an open subset of Euclidean space $\mathbb{R}^n$, i.e. there exists a homeomorphism $f : U \to V$ where U is an open subset of M and V is an open subset of $\mathbb{R}^n$.

   Remember that the definition of homeomorphism, f is a homeomorphism between two topological spaces if and only if f is a bijection and both f and f^{-1} are continuous.

Example 1.1.1. The Euclidean space $\mathbb{R}^n$ is topological manifold for which every point $x \in \mathbb{R}^n$ has a neighbourhood homeomorphic to Euclidean space by the identity map $\mathbb{I}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$.

Example 1.1.2. Let (X, τ) be a topological space with a topology τ. If Y is a subset of X, the collection $\tau_Y = \{Y \cap U \mid U \in \tau\}$ is a topology on Y. This topology is called the **subspace topology**. Y is called a subspace of X with this topology.

1.1.1 Charts and Atlas

Definition 1.3. Let M be a topological manifold. Let $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ be two open subsets. A homeomorphism $\varphi : U \to V$, $\varphi(u) = (x_1(u), \cdots, x_n(u))$ is called a coordinate map on U with the coordinate functions $x_1, \cdots, x_n$. The pair (U, φ) is called a **chart** on M. The inverse map φ^{-1} is called a parametrization of U.

The collection of charts (U_n, φ_n) whose domains cover M on a topological manifold M is called an **atlas** for the manifold M.

Definition 1.4. The homeomorphisms $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ with the coordinate maps $\varphi_\beta, \varphi_\alpha$ are called the **transition maps**.

Example 1.1.3. All subspaces of $\mathbb{R}^n$ are Hausdorff and second countable by the subspace topology. Every open subset U of $\mathbb{R}^n$ is also a topological manifold with chart $(U, \mathbb{I}_U)$ where $\mathbb{I}_U$ is a identity map from U to $\mathbb{R}^n$.

Example 1.1.4. We want to define an atlas on the circle $S^1 = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$.

By the subspace topology, $S^1 \subset \mathbb{R}^n$ for $n \geq 2$ is Hausdorff and second countable. Let $N = (0, 1)$ and $S = (0, -1)$ and we will define stereographic maps (U_1, φ_1), (U_2, φ_2) as follows. Consider the line d_1 passes through the point $(x, y) \in S^1 \setminus \{N\}$ and N on the circle and this line intersects the axis x at the point $(u, 0)$ which is a stereographic coordinate.

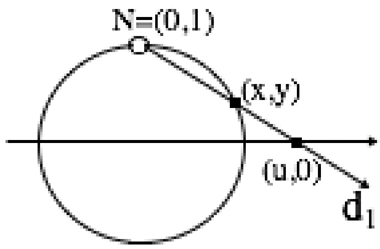

Figure 1.1: The line d_1 which passes through the north pole $N = (0, 1)$ and the point (x, y) on the circle and intersects the axis x at the point $(u, 0)$

We know that for all $(x, y) \in S^1$ we have $x^2 + y^2 = 1$.

The equation of the line d_1 is $\frac{x}{u} + y = 1$.

We have $y = 1 - \frac{x}{u}$ and we substitute this value in the other equation as follows :

$$x^2 + y^2 = x^2 + \left(1 - \frac{x}{u}\right)^2 = 1$$

This equation has two solutions. One of them is the north pole $N = (0, 1)$ and the second solution is $x = \frac{2u}{1+u^2}$, this implies that $y = 1 - \frac{x}{u} = \frac{1-u^2}{1+u^2}$ and $u = \frac{x}{1-y}$.

Now, consider the line d_2 passes through the point $(x, y) \in S^1 \setminus \{S\}$ and S on the circle and this line intersects the axis x at the point $(u', 0)$ is a stereographic coordinate.

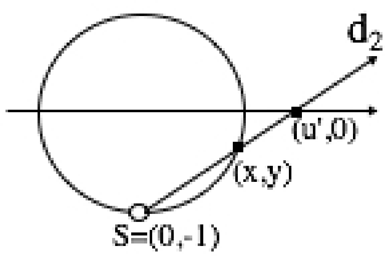

Figure 1.2: The line d_2 which passes through the south pole $S = (0, -1)$ and the point (x, y) on the circle and intersects the axis x at the point $(u', 0)$

The equation of the line d_2 is $\frac{x}{u'} + \frac{y}{-1} = 1$.

So we have $\frac{x}{u'} - 1$ and we substitute this value in the first equation. First solution is the south pole $S = (0, -1)$ for these equaitons, second solution is $(x, y) = \left(\frac{2u'}{1+u'^2}, \frac{u'^2-1}{1+u'^2}\right)$ and we obtain that $u' = \frac{x}{1+y}$.

Since $\frac{x}{1-y}\frac{x}{1+y} = \frac{x^2}{1-y^2} = \frac{x^2}{x^2} = 1$, the relation between coordinates u and u' is $uu' = 1$.

Therefore, we define the coordinates charts $(U_1, \varphi_1), (U_2, \varphi_2)$ on S^1 such that

$$U_1 = S^1 \setminus \{N\}, \varphi_1 : (x, y) \mapsto u = \tfrac{x}{1-y} \text{ for } (x, y) \in U_1$$

$$U_2 = S^1 \setminus \{S\}, \varphi_2 : (x, y) \mapsto u' = \tfrac{x}{1+y} \text{ for } (x, y) \in U_2$$

Now we have to show that these functions are homeomorphisms. To show that they are homeomorphisms, we have to show φ_1, φ_2 are bijections and $\varphi_1, \varphi_1^{-1}, \varphi_2, \varphi_2^{-1}$ are continuous. Since critical points are removed from the domain $U_1 \subset S^1 \setminus \{N\}$ and $U_2 \subset S^1 \setminus \{S\}$, these maps are continuous. We will prove that $\varphi_1 \circ \varphi_1^{-1}(u)$ and $\varphi_2 \circ \varphi_2^{-1}(u')$ are identity functions.

Since $u = \frac{x}{1-y}$ and $\varphi_1^{-1}(\frac{x}{1-y}) = u' = \frac{x}{1+y}$, the composition $\varphi_1 \circ \varphi_1^{-1}$ is defined by $\varphi_1 \circ \varphi_1^{-1}(u) : \mathbb{R} \to S^1 \to \mathbb{R}, u \mapsto (x, y) \mapsto \frac{x}{1+y}$

$$\begin{aligned}
\varphi_1 \circ \varphi_1^{-1}(u) &= \varphi_1(\varphi_1^{-1}(u)) \\
&= \varphi_1(x, y) \\
&= \varphi_1\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right) \\
&= \frac{\frac{2u}{1+u^2}}{1 + \frac{1-u^2}{1+u^2}} \\
&= u
\end{aligned}$$

We proved that this function is an identity function.

Since $u' = \frac{x}{1+y}$ and $\varphi_2^{-1}(\frac{x}{1+y}) = u = \frac{x}{1-y}$,

$$\begin{aligned}
\varphi_2 \circ \varphi_2^{-1}(u') : \mathbb{R} &\to S^1 \to \mathbb{R} \\
u' &\mapsto (x, y) \mapsto \frac{x}{1-y} \\
\varphi_2 \circ \varphi_2^{-1}(u') &= \varphi_2(\varphi_2^{-1}(u')) \\
&= \varphi_2\left(\frac{x}{1-y}\right) \\
&= \frac{\frac{2u}{1+u^2}}{1 + \frac{1-u^2}{1+u^2}} \\
&= u
\end{aligned}$$

This shows that φ_1, φ_2 are homeomorphisms. So, $\mathcal{A} = \{(U_1, \varphi_2), (U_2, \varphi_2)\}$ is an atlas on S^1.

Example 1.1.5. Let us consider the subset $X = \{(x, 0) \mid -1 < x < 1\} \cup \{(0, y) \mid -1 < x < 1\} \subset \mathbb{R}^2$.

In this example, we will show that the subset X is not a topological manifold, since there is no homeomorphism from the cross on X to $\mathbb{R}^n$.

Suppose that X is homeomorphic to $\mathbb{R}^n$ for some $n > 0$. Let f be a homeomorphism between X and $\mathbb{R}^n$. Now, consider the map $f' : X \setminus \{(0, 0)\} \to \mathbb{R}^n \setminus \{f(0, 0)\}$, where f' is the restriction of f to the domain $X \setminus \{(0, 0)\}$. Then f' is continuous, bijective and has continuous inverse. So, f' is a homeomorphism between $X \setminus \{(0, 0)\}$ and $\mathbb{R}^n \setminus \{f(0, 0)\}$.

Observe that $X \setminus \{(0, 0)\}$ has 4 connected components, whereas $\mathbb{R}^n \setminus \{f(0, 0)\}$ has 2 connected components when $n = 1$ and 1 connected component when $n \geq 2$. Since homeomorphisms preserver connected components, X is not a homeomorphic to $\mathbb{R}^n$ for any $n > 0$. So, it is not a topological manifold.

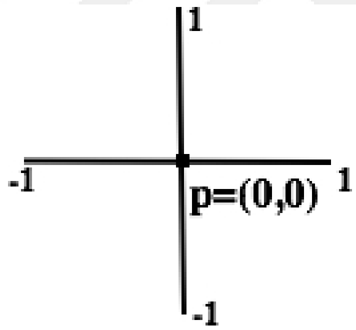

Figure 1.3: A cross on $\mathbb{R}^2$

1.2 Smooth Manifolds

A second countable, Hausdorff topological space M is a n-dimensional topological manifold if it admits an atlas $\{U_\alpha, \varphi_\alpha\}$ with the coordinate map $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ for $n \in \mathbb{N}$. We will see that if all transition maps of M are diffeomorphisms, i.e. all partial derivatives exist and continuous, then it is called a **smooth manifold**.

A map f between two spaces U, U' such that $U \subset \mathbb{R}^n$ and $U' \subset \mathbb{R}^n$ is called a **smooth** if f has continuous partial derivatives at the each component functions for all orders.

$$D(f)(x)(u) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon u) - f(x)}{\varepsilon}$$

$D(f)(x)$ is the partial derivative of F at $x \in U$.

Definition 1.5. Let M be any topological n-manifold. Let U be an open subset on M and U' be an open subset in $\mathbb{R}^n$. We define a map f on M such that $f : M \to \mathbb{R}$. The map f is **smooth** map if and only if $h = f \circ g^{-1} : U' \to \mathbb{R}$ is smooth with the coordinate map $g : U \to U'$.

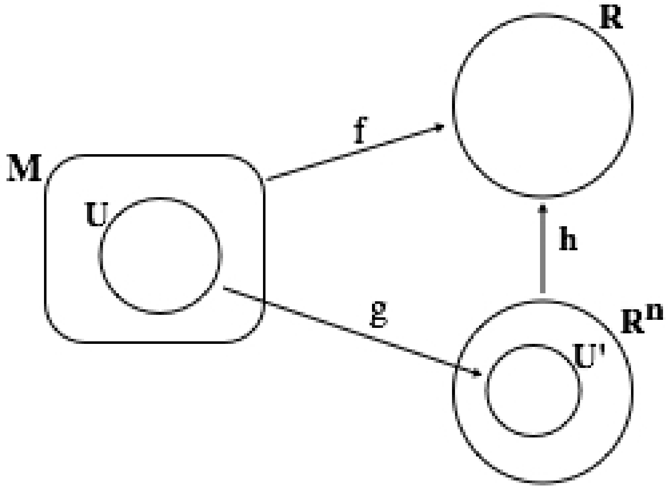

Figure 1.4: A smooth map on the topological manifold M

Definition 1.6. Let ψ, ϕ be two charts of a topological manifold M. Two charts ψ, ϕ are compatible if and only if

1. $\psi^{-1}(\phi(U'))$ and $\phi^{-1}(\psi(U))$ are open maps.
2. The transition maps $\phi^{-1} \circ \psi$ and $\psi^{-1} \circ \phi$ are smooth.

$$\phi^{-1} \circ \psi : \psi^{-1}(\phi(U')) \to \phi^{-1}(\psi(U))$$

$$\psi^{-1} \circ \phi : \phi^{-1}(\psi(U)) \to \psi^{-1}(\phi(U'))$$

Definition 1.7. If the transition map $\psi \circ \phi^{-1}$ is smooth, then (U, ψ) and (U', ϕ) are called *smoothly compatible.*

Definition 1.8. The set $\mathcal{A}$ of smooth charts whose domains cover the manifold M, is called an **atlas** for M. If any two charts are smoothly compatible, then $\mathcal{A}$ is a **smooth atlas**. Two smooth atlases are equivalent if their union is a smooth atlas. We can also say that if the transition map is smooth for any two coordinate maps of a manifold, it implies that $\mathcal{A}$ is a smooth atlas.

If there is no another atlas $\mathcal{A}'$ such that $\mathcal{A} \subset \mathcal{A}'$, then $\mathcal{A}$ is **maximal atlas**. A smooth structure on the manifold M may be defined as a maximal smooth atlas.

Definition 1.9. Let M be a topological n-manifold and let $\mathcal{A}$ be a smooth atlas of M, then a pair $(M, \mathcal{A})$ is called a **smooth manifold**.

Let U, U' be two open subsets in M such that $U \cap U' \neq \emptyset$ and let M be a topological n-manifold. Let us define two charts (U, ψ) and (U', ϕ) and the composition map $\psi \circ \phi^{-1} : \phi(U \cap U') \to \psi(U \cap U')$ is called a *transition map*. Since ψ, ϕ are homeomorphisms, $\psi \circ \phi^{-1}$ is also homeomorphism.

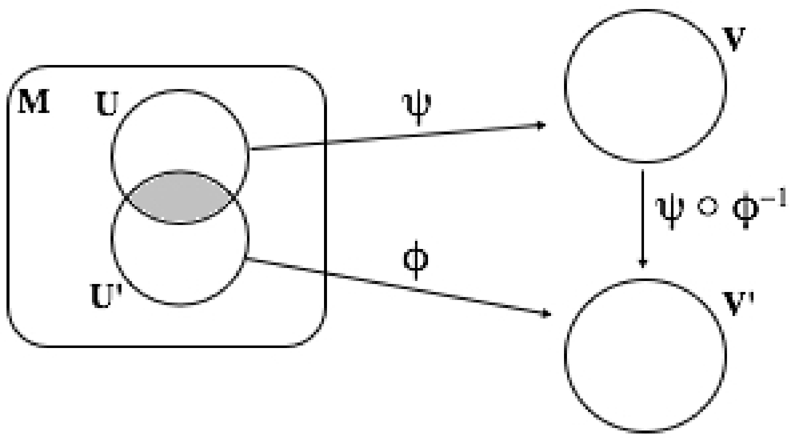

Figure 1.5: A transition map of a smooth manifold

Example 1.2.1. Let M be a topological 0-dimensional manifold. Then M is a countable discrete space. A neighborhood of each point $p \in M$ that is homeomorphic to an open subset of $\mathbb{R}^0$ is $\{p\}$ itself. There exists exactly one chart $\varphi : \{p\} \to \mathbb{R}^0$. Then the set of all charts on M satisfies the smooth compatibility condition. Thus, 0-dimensional manifold M is a smooth manifold.

Example 1.2.2. The n-sphere is defined as the subspace of unit vectors in $\mathbb{R}^{n+1}$

$$S^n = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$$

Let $N = (1, 0, \cdots, 0)$ be the north pole and let $S = (-1, 0, \cdots, 0)$ be the south pole in S^n. Then we may write $S^n = U_N \cup U_S$, where $U_N = S^n \setminus \{S\}$ and $U_S = S^n \setminus \{N\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^n$, given by the stereographic projections from S, N respectively

$$\varphi_N : (x_0, \vec{x}) \mapsto (1 + x_0)^{-1}\vec{x}$$

$$\varphi_S : (x_0, \vec{x}) \mapsto (1 - x_0)^{-1}\vec{x}$$

The charts for the n-sphere given above form a smooth atlas, since

$$\varphi_N \circ \varphi_S^{-1} : \vec{z} \mapsto \frac{1 - x_0}{1 + x_0}\vec{z} = \frac{(1 - x_0)^2}{|\vec{x}|^2}\vec{z} = |\vec{z}|^{-2}\,\vec{z}$$

which is smooth on $\mathbb{R}^n \setminus \{0\}$. Then the n-sphere is a smooth manifold.

1.3 Manifold with boundary

In differential geometry, manifolds with boundary are important as well. For example, they are used in the Stokes theorem.

Definition 1.10. A topological n-manifold M with boundary is defined to be a Hausdorff space and second countable such that every point in M has an open neighborhood which is homeomorphic to an open subset in the upper half space $\mathbb{H}^n$. The upper half space $\mathbb{H}^n$ is closed in $\mathbb{R}^n$ and this space is defined by

$$\mathbb{H}^n = \left\{ \left(x^1, \cdots, x^n\right) \in \mathbb{R}^n \mid x^n \geq 0 \right\}$$

The subspace topology is defined on $\mathbb{H}^n$. Let's assume that $\sum \subset \mathbb{H}$. Any $\sum$ in $\mathbb{H}$ is bounded by a circle. Since $\sum$ which is an embedded with boundary a convex curve is included in the Euclidean space $\mathbb{R}$, then the half space $\mathbb{H}$ is embedded in $\mathbb{R}$. A chart of a topological n-manifold M with boundary is defined by a homeomorphism $\varphi : U \to V$ where U is an open subset of M and V is an open subset of $\mathbb{H}^n$.

Definition 1.11. The preimages of points $(x_1, \cdots, x_{n-1}, 0) \in \mathbb{H}^n$ are the *boundary* ∂M of M and $M - \partial M$ is the *interior* of M.

Lemma 1.1. If a topological n-manifold M with boundary,then ∂M is a topological $(n-1)$-manifold without boundary.

Proof. If x is in ∂M and an open neighborhood U which is homeomorphic to an open subset of $\mathbb{H}^n$, then $\partial M \cap U$ is homeomorphic to an open subset of $\mathbb{R}^{n-1}$. $\square$

Lemma 1.2. A manifold M with boundary is a manifold if and only if ∂M is empty.

Proof. Firstly, if ∂M is empty, then M is the manifold without boundary.

By the previous lemma, we have that if M is a topological manifold with boundary, then ∂M is a topological $(n-1)$-manifold without boundary. Since M is a topological manifold, then we say that there is no boundary of M. So, $\partial M = \{\emptyset\}$. □

Definition 1.12. A function $f : M \to N$ is a map of topological manifolds if f is continuous. It is a **smooth** map of smooth manifolds M, N if the function $\phi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \phi(V)$ is a diffeomorphism for any smooth charts (U, φ) of M and (V, ϕ) of N on the open subset $U \cap f^{-1}(V)$.

If a topological n-manifold M with boundary is smooth, then M is called a smooth n-manifold with boundary.

Example 1.3.1. Let M be a topological n-manifold. The manifold M is called a n-manifold with boundary if $int(M) = M$ and $\partial M = \emptyset$.

Example 1.3.2. Let $M = [0, 1]$ be 1-manifold. We have that the set of all regular points of M, $int(M) = (0, 1)$ and the set of all boundary points of M, $\partial M = \{0, 1\}$, then M is a 1-manifold with boundary.

Example 1.3.3. The manifold $D^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$ as a manifold has no boundary. The closed n-ball $\overline{D}^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is a n-manifold with boundary $\partial \overline{D}^n = S^{n-1}$.

1.4 Manifold with corners

Definition 1.13. A topological n-manifold M with corners is identical to a topological n-manifold with boundary. A topological n-manifold M with corners is a second countable Hausdorff space in which each point has a neighborhood homeomorphic to an open subset of the corner $\mathbb{K}^n$.

The corner $\mathbb{K}^n$ is a subset of $\mathbb{H}^n$ and it is defined by

$$\mathbb{K}^n = \{(x_i)_{i=1}^n \in R^n \mid x_i \geq 0, i = 1, \cdots, n\}$$

A chart of a topological n-manifold M with corners is defined by a homeomorphism $\varphi : U \to V$ where U is an open subset of M and V is an open subset of the corners $\mathbb{K}^n$.

Definition 1.14. Let (U, ψ) and (U', ϕ) be two charts with corners on the manifold M with corners. If the transition map $\psi \circ \phi^{-1} : \phi(U \cap U') \to \psi(U \cap U')$ is smooth, then (U, ψ) and (U', ϕ) are smoothly compatible. The maximal collection $\mathcal{A}$ of smoothly compatible charts with corners whose domains cover the manifold M with corners, is called an **atlas** on M with corners. A pair $(M, \mathcal{A})$ is a **smooth manifold with corners**.

Definition 1.15. A point $p \in M$ is called a **corner point** if $\varphi(p)$ is a corner point in $\in \mathbb{K}^n$ with the smooth chart (U, φ) with corners.

Lemma 1.3. Every smooth manifold is a smooth manifold with corners. A smooth manifold with corners is a smooth manifold with boundary if and only if it has no corners points.

Example 1.4.1. Let any closed rectangle be in $\mathbb{R}^n$. This is a smooth n-manifold. A closed retangle has more than one point which at least one coordinates vanishes. These points are said to be corners of a closed rectangle. Then, it is a smooth n-manifold with corners.

Example 1.4.2. The set $M = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y^2 \leq x^2 - x^4\}$ is a 2-manifold with corners.

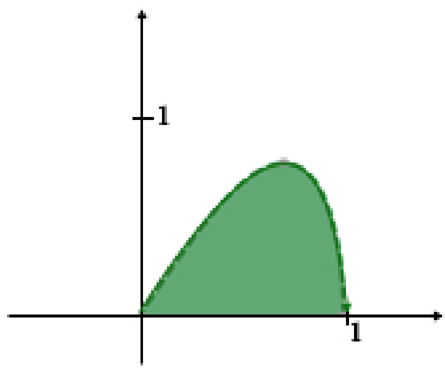

Figure 1.6: A 2-manifold with corners

Example 1.4.3. Consider the exterior surface of a cube. Since the boundary of cube has edges and corners, such a cube looks like a manifold with corners. The exterior surface of a cube is defined by $[0,1] \times [0,1] \times [0,1] \subset \mathbb{R}^3$.

The homeomorphism condition tells us that, let f be a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^m$ and if $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$, then $n = m$. So, a map from the corner point p of the exterior surface of a cube must be homeomorphic to $\mathbb{K}^3$. But, any map from an open neighborhood of p is in $\mathbb{K}^2$. There is no homeomorphic neighborhood of a corner point p to an open subset of $\mathbb{K}^3$. Then, the exterior surface of a cube is not a manifold with corners.

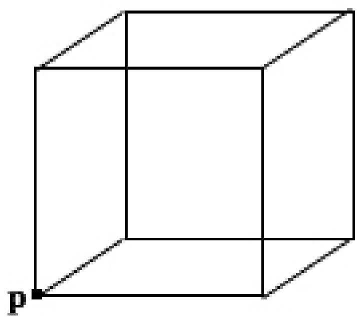

Figure 1.7: The exterior surface of a cube

2 DIFFEOLOGY

First, we will give the basic vocabulary needed : domains and parametrization.

Definition 2.1. An **open subset** in $\mathbb{R}^n$ is any union of open balls $B_a(r)$ which is defined by $B_a(r) = \{x \in \mathbb{R}^n \mid \|x - a\| \leq r\}$ for each $a \in \mathbb{R}^n$ and $r > 0$ with $n \in \mathbb{N}$.

Definition 2.2. An open subset U of Euclidean spaces $\mathbb{R}^n$ is called a **real domain**.

Definition 2.3. Any map $P : U \to X$ is called a **parametrization** of X where U is a real domain and X is a non-empty set. We denote $Param(X)$ is the set of all parametrizations P of X.

2.1 Diffeology

Given a subset $\mathfrak{D}$ of $Param(X)$ is called a **diffeology** if $\mathfrak{D}$ satisfies the following three axioms ; *covering, locality* and *smooth compatibility.*

1. *Covering.* Every constant parametrization is in $\mathfrak{D}$. There exists a constant parametrization $\mathbf{x} : r \mapsto x$ defined on $\mathbb{R}^n$ for all $x \in X$.
2. *Locality.* Let $P : U \to X$ be a parametrization. For every point r of U, if the restriction $P \upharpoonright V_r$ of P on a neighborhood V_r of r in U is in $\mathfrak{D}$, then P belongs to $\mathfrak{D}$.
3. *Smooth compatibility.* For every element $P : U \to X$ of $\mathfrak{D}$, $P \circ F$ belongs to $\mathfrak{D}$ for every $F \in \mathcal{C}^\infty(U, V)$ and every real domain V.

   Let's define a smooth parametrization F between two real domains U, U' and a parametrization $P \in \mathfrak{D}$ such that the composite $h = P \circ f \in \mathfrak{D}$.

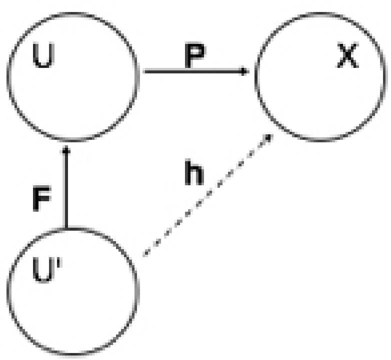

Figure 2.1: Smooth compatibility of parametrizations of X

2.1.1 Smoothness of Plots

Definition 2.4. An element of the diffeology is called a **plot**. The set of the plots of the diffeological space X defined on the domain U is denoted by $\mathfrak{D}(U, X)$.

Let P be a plot of the diffeology $\mathfrak{D}$ of X. If the composite $P \circ f$ is in $\mathfrak{D}$ for every smooth parametrization $f : X \to X'$, then P is called a smooth.

We can describe a smooth map $f \in \mathfrak{D}(X, X')$ with the figure below.

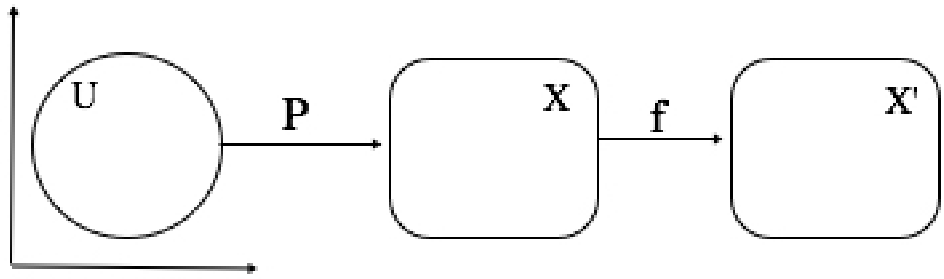

Figure 2.2: Plots are smooth

2.2 Diffeological Spaces

A pair $(X, \mathfrak{D})$ is a **diffeological space**. $(X, \mathfrak{D})$ is a nonempty set X equipped with a diffeology $\mathfrak{D}$.

Definition 2.5. Let X, X' be two diffeological spaces and let $\mathfrak{D}, \mathfrak{D}'$ be two diffeologies of X, X' respectively. $\mathfrak{D}(X, X') = \{F \in Maps(X, X') \mid F \circ \mathfrak{D} \subset \mathfrak{D}'\}$ is defined by the set of smooth parametrizations F between two diffeological spaces. So, if $F \circ P$ is an element of the diffeology $\mathfrak{D}'$ of X' for each parametrization P of X, it implies that F is smooth parametrization between these two diffeologial spaces.

Lemma 2.1. Let's consider three diffeological spaces X, X', X'' with their diffeo logies $\mathfrak{D}, \mathfrak{D}', \mathfrak{D}''$ respectively. Let f, g be two elements of smooth parametrizations such that

$$f : X \to X',\ g : X' \to X''$$

By the definition of diffeology and smooth, we have that $f \circ \mathfrak{D} \subset \mathfrak{D}'$ and $g \circ \mathfrak{D}' \subset \mathfrak{D}''$. Thus the compositon $g \circ f$ is also smooth

$$(g \circ f) \circ \mathfrak{D} = g \circ f \circ \mathfrak{D} \quad \subset \quad g \circ \mathfrak{D}' \quad \subset \quad \mathfrak{D}''$$

Example 2.2.1. Diffeology for the circle. The circle $S^1 = \{z \in \mathbb{C} \mid \bar{z}z = 1\}$ and suppose that the parametrizations $P : U \to S^1$ satisfy the condition : $P \upharpoonright V_r : r \to exp(2i\pi\varphi(r))$ with an open neighborhood V_r for all $r \in U$ where $\varphi : V \to \mathbb{R}$ is a smooth parametrization. Let $\mathfrak{D}$ be a set of these parametrizations.

Now, we are talking about whether or not the set of these parametrization is diffeology. Firstly, let's check being covering. We define a constant parametrization $\mathbf{z} : r \to z$ for all $z \in S^1$. We can find a real number $\theta = \varphi(r)$ such that $exp(2i\pi\theta) = z$ for every r, accordingly the exponential function is surjective from $\mathbb{R}$ to S^1. We can say that this result for every r, so if we choose $V = \mathbb{R}^n$, every constant parametrization satisfying this condition is in $\mathfrak{D}$.

By the hypothesis, $P \upharpoonright V_r$ is in $\mathfrak{D}$ and P is also. For the third condition, we can show with the following figure

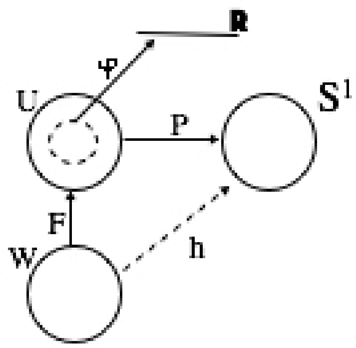

Figure 2.3: Diffeology for the circle S^1

We want to show that the parametrization h is in $\mathfrak{D}$. Let $r \in U$ and V be an open neighborhood of r. We define a smooth parametrization F from a real domain W to U. By the hypothesis,we have the parametrization $\varphi : V \to \mathbb{R}$. Suppose that $P : U \to S^1$ is a plot, we will show that the composition $h : P \circ F$ is in $\mathfrak{D}$. Since F is a smooth and φ is a parametrization, we can say that $\varphi \circ F : W \to \mathbb{R}$ is a smooth parametrization and F is continuous. So, we have that a real domain $V' = F^{-1}(V)$ and we define a smooth parametrization $\varphi^{'} = \varphi \circ F : V' \to \mathbb{R}$ as follows :

$\forall s_0 \in W, \exists V', \exists \varphi' = \varphi \circ F$ such that

$$(P \circ F) \upharpoonright V' : s_0 \to exp(2i\pi\varphi\,(r))$$

Then, $h \in \mathfrak{D}$. In this case, $\mathfrak{D}$ is a diffeology of S^1.

Example 2.2.2. An example of diffeology on $\mathbb{R}$.

Suppose that $X = \mathbb{R}$ and $\mathfrak{D}$ is a set of all smooth parametrizations from a real domain U to X.

Let $\mathbf{x} : U \to X$ be a constant map for any point $x \in U$. Since $\mathbf{x}$ is a constant map we have that $\mathbf{x}(r) = x$ for every $r \in U$. This implies that there are two parametrizations φ, φ' such that $\varphi' \circ \mathbf{x} \circ \varphi$ is smooth. So, $\mathfrak{D}$ is covering.

Let $P : U \to \mathbb{R} \in \mathfrak{D}$. If for any $r \in U$ there exists an open neighborhood V of r and the restriction of P on V is smooth, i.e. $P \upharpoonright V$ is locally smooth, then P is smooth by the definition of smoothness. Thus $\mathfrak{D}$ is locality.

Let F be a parametrization from any real domain W to U. Since F and P are smooth, the composition $P \circ F$ is also smooth. So that $\mathfrak{D}$ is smooth compatibility. The set $\mathfrak{D}$ specifies a difeology for X.

Example 2.2.3. An example of diffeology on topological space

Axiom 1 : Suppose that X is a topological space and $\mathfrak{D}$ is a set of all continuous parametrizations from a real domain U to X.

Axiom 2 : Let f be a constant map defined by $f(r) = x$ for all $r \in U$ and for any point $x \in X$. The inverse image f^{-1} of any open subset in X is either $\emptyset$ or U, which are open. Then f is always continuous and $f \in \mathfrak{D}$. By definition, every constant parametrization is locally constant.

Axiom 3 : Let F be a continuous map between two real domains W and U in $\mathbb{R}^n$, the composition of a parametrization from U to X and F is continuous since the composition of two continuous functions is also continuous.

Then $\mathfrak{D}$ is smooth compatibility and $\mathfrak{D}$ is diffeology on the nonempty set X.

As we can understand in the examples, we can describe more than one diffeology because diffeology may vary according to the characteristics of the parametrizations.

2.3 Comparison of Diffeologies

Let X be a diffeological space equipped with two diffeologies $\mathfrak{D}$ and $\mathfrak{D}'$. If the elements of $\mathfrak{D}$ are in $\mathfrak{D}'$, $\mathfrak{D} \subset \mathfrak{D}'$, then $\mathfrak{D}$ is finer than $\mathfrak{D}'$. The locally constant parametrizations of X are the **finest diffeology**. Conversly, if the elements of $\mathfrak{D}'$ are in $\mathfrak{D}$, $\mathfrak{D} \supset \mathfrak{D}'$, $\mathfrak{D}$ is coarser then $\mathfrak{D}'$. The set $Param(X)$ of all parametrizations of X is called a **coarsest diffeology** of X.

2.3.1 Subset Diffeology

Let A be any subset of a diffeological space X, the *subset diffeology* of A, induced by X, is made of all the plots of X whose take their values in A.

2.3.2 Pushforwards of Diffeologies

Let f be a map from a diffeological space X to a set X'. Our aim is to carry the properties of the diffeological space into the set X' with the map f.

Definition 2.6. There exists a finest diffeology on X' such that f is smooth. This diffeology is called the **pusforward** diffeology by f of the diffeology $\mathfrak{D}$ of X, it is denoted by $f_*(\mathfrak{D})$.

$$f_*(\mathfrak{D}) = \{P : U \to X \mid f \circ P \ \in \mathfrak{D}\}$$

2.3.3 Pullbacks of Diffeologies

Let X be a set. Let X' be a diffeological space and $\mathfrak{D}'$ its diffeology. Let $f : X \to X'$ be some map. There exists a coarsest diffeology of X such that the map f is smooth. This diffeology is called the pullback of the diffeology $\mathfrak{D}'$ by f and it is denoted by $f^*(\mathfrak{D}')$. A parametrization P in X belongs to $f^*(\mathfrak{D}')$ if and only if $f \circ P$ belongs to $\mathfrak{D}'$.

$$f^*(\mathfrak{D}') = \{P \in Param(X) \mid f \circ P \ \in \mathfrak{D}'\}$$

Lemma 2.2. Let $f : X \to Y$ and $g : Y \to Z$ where X, Y are two nonempty sets and Z is a diffeological space. Suppose that $\mathfrak{D}$ is a diffeology on the diffeological space Z. So that $f^*(g^*(\mathfrak{D})) = (g \circ f)^* (\mathfrak{D})$.

2.3.4 Category of Diffeologies

Let $\mathfrak{D}$ be a diffeology of a nonepmty set X. We need the smooth theory when defining the diffeology category $\mathfrak{D}$ of diffeological spaces. On the $\mathfrak{D}$ category, the diffeology $\mathcal{D}$ consisting of smooth maps which inducing isomorphisms on smooth theory.

Definition 2.7. The largest element in the diffeology $\mathfrak{D}$ is said to be the ***indiscrete diffeology*** on X, the smallest element in $\mathfrak{D}$ is the ***discrete diffeology*** on X.

Definition 2.8. The diffeology which has the smallest element of all smooth parametrizations on X is the ***final diffeology***.

Definition 2.9. The final diffeology defined by the quotient map is the ***quotient diffeology*** with an equivalence relation for a diffeological space X.

Definition 2.10. A largest diffeology of all smooth parametrization on a nonempty set X is the ***initial diffeology*** on X.

Definition 2.11. The initial diffeology which is defined by the inclusion map is the ***sub-diffeology***.

2.3.5 Product of Diffeological Spaces

Given diffeological spaces M, M' of dimensions m, m' with atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U'_\beta, \varphi'_\beta)\}$, the cartesian product $M \times M'$ is a diffeological space of dimension $m+m'$. An atlas is given by the product charts $U_\alpha \times U'_\beta$ with the product map $\varphi_\alpha \times \varphi'_\beta : (x, x') \mapsto (\varphi_\alpha(x), \varphi'_\beta(x'))$.

Let $X_1, \cdots, X_n$ be n diffeological spaces. The product of these spaces is defined from $X_1 \times \cdots \times X_n \to$ into individual all spaces $X_1, \cdots, X_n$.

2.4 Generating Families

Let $\mathcal{F} \subset Param(X)$. The finest diffeology including $\mathcal{F}$ is called the diffeology generated by $\mathcal{F}$. Inversely, let $\mathcal{D}$ be a diffeology of a diffeological space X. A family of all plots of X that generates $\mathcal{D}$ is called a **generating family** of $\mathcal{D}$. A set of all the generating family for the space X is denoted as $Gen(X)$. Let $\mathcal{F}$ be a generating family, then

$$Gen(X) = \{\mathcal{F} \subset \mathcal{D} \mid \langle \mathcal{F} \rangle = \mathcal{D}\}$$

The plots of the diffeology generated by $\mathcal{F}$ are characterized by the following property :

— A parametrization $P : U \to X$ is a plot for the diffeology generated by $\mathcal{F}$ if and only if there exists an open neighborhood V for $r \in U$ such that $P \upharpoonright V$ is a constant parametrization or there exists $F : W \to X$ belonging to $\mathcal{F}$ and a smooth parametrization $\theta : V \to W$ such that $P \upharpoonright V = F \circ \theta$.

If $X = \bigcup_{F \in \mathcal{F}} val(F)$, this property is reduced to the following :

— A parametrization $P : U \to X$ is a plot for the diffeology generated by a parametrized cover $\mathcal{F}$ if and only if there exists an open neighborhood V for $r \in U$, a parametrization $F : W \to X$ belonging to $\mathcal{F}$ and a smooth parametrization $\theta : V \to W$ such that $P \upharpoonright V = F \circ \theta$.

A generating family $\mathcal{F}$ which is parametrized cover will be called a *covering generating family.*

Let $f : X \to X'$ be a map between two sets X, X'. Let $\mathcal{F}$ be a family of parametrizations of X and let $\mathcal{F}'$ be a family of parametrizations of X'.

The family $f_*(\mathcal{F})$ of parametrizations of X' is a **pushforward of the family** $\mathcal{F}$ by f.

$$f_*(\mathcal{F}) = \{f \circ F \mid F \in \mathcal{F}\}$$

The family $f^*(\mathcal{F})$ of parametrization F of X' is a **pullback of the family** $\mathcal{F}'$ by f such that $F : U \to X$ that provides there exists a smooth parametrization $\psi : U \to U'$ with an element F' of $\mathcal{F}'$, $F' : U' \to X'$, such that $F' \circ \psi = f \circ F$.

2.5 Diffeological manifolds

We will introduce diffeological manifolds which are diffeological spaces modeled on diffeological vector space.

Definition 2.12. Let E be a diffeological vector space and X be a diffeological space. If X is locally diffeomorphic to E at every point, then X is called a **diffeological manifold** modeled on E. For every $x \in X$ there exists a local diffeomorphism

$F : E \supset U \to X$ such that $x \in F(U)$. Such a local diffeomorphism is called a chart of X.

Now, we will talk about the condition of the manifold as a diffeological space.

Definition 2.13. Let M be a diffeological space. M is a n-manifold if and only if M is locally diffeomorphic at each point to $\mathbb{R}^n$, i.e. for each point $m \in M$, there exist a local diffeomorphism $F : U \subset \mathbb{R}^n \to M$ and a point $r \in U$ such that $F(r) = m$. Such local diffeomorphism are called charts of M. The set of all the charts of M is called the atlas of M.

Definition 2.14. Let M, M' be two classical manifolds and let f be a smooth map between these two classical maifolds. We know that the map f is smooth if and only if $F'^{-1} \circ f \circ F : U \to V$ for each chart of M and M' with the open subsets U, V on any manifold with the figure below as follows :

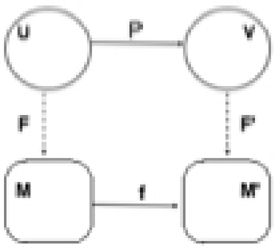

Figure 2.4: A smooth map between manifolds

We have that the following results for every pair of parametrizations $F : U \to M$, $F' : U' \to M'$,

1. $(f \circ F)^{-1}(F'(U'))$ is open
2. $F'^{-1} \circ f \circ F : (f \circ F)^{-1}(F'(U')) \to U'$ is smoth

A family $\mathbf{A}$ is a set of parametrizations $F : U \to M$ such that $\bigcup_{F \in \mathbf{A}} val(F) = M$ and let P is plot of diffeology generated by $\mathbf{A}$. This is to be covering of diffeology. There exists an open neighborhood V such that $\rho \restriction V$ is constant or there exists a parametrization $F \in \mathbf{A}$ such that $P \restriction V = F$. Thus the family $\mathbf{A}$ shows the condition of being locality. We can consider another smooth parametrization ρ' from an open subset W to M such that $P \restriction V = F \circ P'$. Diffeology on a manifold is built in this manner. This family $\mathbf{A}$ is a generating family by this result and $\mathbf{A}$ refers to a diffeology on the manifold.

Let M be a diffeological space.

1. A family $\mathcal{A}$ of local diffeomorphisms from $\mathbb{R}^n$ to M such that $\bigcup_{F \in \mathcal{A}} val(F) = M$ is a generating family of the diffeology of M.

2. The diffeology space M is a n-manifold if and only if there exists a generating family $\mathcal{A}$ of M, made of local diffeomorphisms from $\mathbb{R}^n$ to M.

2.6 Diffeological manifolds with boundary

We show that smooth manifolds with boundary are diffeological spaces modeled on half spaces $\mathbb{H}^n$, equipped with the subset diffeology.

A smooth n-manifold with boundary is a topological space M, together with a family of local homeomorphisms F_i defined on some open sets U_i of the half-space $\mathbb{H}^n$ to M such that the values of F_i cover M for every two elements F_i and F_j of the family, the transition map $F_i^{-1} \circ F_j : F_i^{-1}(F_i(U_i) \cap F_j(U_j)) \to F_j^{-1}(F_i(U_i) \cap F_j(U_j))$, is the restriction of some smooth map defined on an open set of $F_i^{-1}(F_i(U_i) \cap F_j(U_j))$.

The boundary ∂M is the union of $F_i(U_i \cap \partial \mathbb{H}^n)$. Such a family $\mathfrak{F}$ of homeomorphisms is called an atlas of M and its elements are called charts. There exists a maximal atlas $\mathcal{A}$ containing $\mathcal{F}$.

2.7 Diffeological manifolds with corners

The manifolds with corners can also be viewed as diffeological spaces. The D-topology on the corners $\mathbb{K}^n$ coincides with the subset topology induced by $\mathbb{R}^n$. From the fact that the plot $(x_1, \cdots, x_n) \to (x_1^2, \cdots, x_n^2)$ restricts to a homeomorphism from $\mathbb{K}^n$ to itself with the subset topology. The subset diffeology and the subset differential structure on $\mathbb{K}^n$ determine each other. It follows that a map between relatively open subsets of $\mathbb{K}^n$ is a diffeological diffeomorphism if and only if it is a functional diffeomorphism, which is equivalent to being a diffeomorphism in the classical sense. A manifold with corners can be equivalently defined as a diffeological space that is locally diffeomorphic to open subsets of the corners $\mathbb{K}^n$.

Let M be a smooth n manifold with corners. Let us recall that a parametrization P of M is smooth if there exists an open neighborhood V of $r \in U$ such that $P \upharpoonright V = F \circ \psi$ for a chart $F : \Omega \to M$ and a smooth parametrization $\psi : V \to \Omega$. Let $\mathcal{D}$ be a set of all smooth parametrizations of M. Then $\mathcal{D}$ is a diffeology of M. On the other hand, a diffeological space defined on H_n is a manifold with corners. Likewise able to define a manifold on corners $\mathbb{K}^n = \{(x_i)_{i=1}^n \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \cdots, n\}$.

A map f from $\mathbb{K}^n$ to $\mathbb{R}$ is smooth for the subset diffeology of the corners if and only if there exists an open neigborhood $W \supset \mathbb{K}^n$ such that $F \restriction \mathbb{K}^n = f$ for a smooth map $F : W \to \mathbb{R}$.

The smooth maps f on the manifold with corners represent a diffeological space modeled on $\mathbb{K}^n$.

3 MULTILINEAR ALGEBRA

This chapter provides the logic definitions used for differential forms in diffeological spaces.

3.1 Linear Maps

Definition 3.1. Let U be arbitrary nonempty set of vectors defined by vector addition and scalar multiplication such that suppose u and v are vectors in U, so $u+v \in U$ and further that $k \in \mathbb{R}$, then $ku \in U$. If the following 10 axioms hold true for vectors $u, v, w \in U$ and for all scalars $k, m \in \mathbb{R}$, then U is called a **real vector space**.

Axiom 1 : (closure property for addition) If u and v are in U, then $u + v \in U$. Axiom 2 : (associative property for addition) $u + (v + w) = (u + v) + w$. Axiom 3 : (commutative property for addition) $u + v = v + u$. Axiom 4 : (the additive identity property) There exists an element O, called a zero vector such that $O+u = u+O = u$ for all $u \in U$. Axiom 5 : (additive inverse property) For each $u \in U$, there exists $-u$, such that $u + (-u) = (-u) + u = O$. $-u$ is called the additive inverse. Axiom 6 : (closure property for scalar multiplication) If $u \in U$ and $k \in \mathbb{R}$, then $ku \in U$. Axiom 7 : (distributive property for vectors) $k(u + v) = ku + kv$. Axiom 8 : (distributive property for scalars) $(k+m)u = ku+mu$. Axiom 9 : (associative property for scalars) $k(mu) = (km)u$. Axiom 10 : (the multiplicative identity property) $1u = u$.

Definition 3.2. A map is called a **linear map**, if it is defined by an element of a real vector space $E * F$,

$$E * F = \{A : E \to F \mid A(ax + by) = aA(x) + bA(y), \forall a, b \in \mathbb{R}, \forall x, y \in E\}$$

with the following conditions for all $a \in \mathbb{R}$ and $A, B \in E * F$

1. $(aA)(x) = aA(x)$
2. $(A + B)(x) = A(x) + B(x)$

Example 3.1.1. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = ax$ for each $a \in \mathbb{R}$. For all $x, y \in \mathbb{R}$ and any scalar $c \in \mathbb{R}$,

$$f(x + y) = a(x + y) = ax + ay = f(x) + f(y)$$
$$f(cx) = a(cx) = acx = cax = c(ax) = cf(x)$$

Then f is a linear map.

Example 3.1.2. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. For all $x, y \in \mathbb{R}$,

$$f(x+y) = (x+y)^2 = x^2 + 2xy + y^2 = f(x) + 2xy + f(y)$$

$f(x+y) \neq f(x) + f(y)$, then f is not a linear map.

We define a linear map from any real vector space E to $\mathbb{R}$, i.e. $L(E, \mathbb{R})$, this space is called the **dual vector space** of E, denoted by $E^* = E * \mathbb{R}$. An element of E^* is a *covector*.

A vector is expressed by a basis and this basis must be independent and generate this vector. Let $\mathfrak{B}$ be a basis of independent vectors $b_1, b_2 \cdots b_n$ of E,

$$\sum_{i=1}^{n} c_i b_i = 0 \Leftrightarrow c_i = 0$$

for all $s_i \in \mathbb{R}$ with $i = \{1, 2, \cdots, n\}$ and

$$v = \sum_{i=1}^{n} b_i v_i$$

where v_i is coordinate of v in the basis $\mathfrak{B}$ for $v \in E$ with $i = \{1, 2, \cdots, n\}$.

Let w be an element of E^* of finite n-dimensional. For every vector $x \in E$,

$$\begin{aligned} w(x) &= w\left(\sum_{i=1}^{n} x^i e_i\right) \\ &= \sum_{i=1}^{n} x^i w(e_i) \\ &= \sum_{i=1}^{n} x^i w_i \\ &= \sum_{i=1}^{n} w_i e^j(x) \end{aligned}$$

Let us take $\sum_{i=1}^{n} w_i e^j(x) = \mathfrak{B}^*$, we check $\mathfrak{B}^*$ is a basis of E^*.

Thus we say if E is a finite n-dimensional vector space, $dim(E^*) = n$. Furthermore, E^* and E are isomorphic.

3.1.1 Bilinear Maps

Definition 3.3. A linear map A from a real vector space E to the space of linear maps between two real vector space F, G, denoted by $E * (F * G)$, i.e. $A : [x \mapsto [y \mapsto z]]$ for $x \in X, y \in F$ and $Z \in G$, is called a **bilinear map**. We check the linearity of A and $A(x)$ such that

$$\left.\begin{aligned} A(x+x')(y) = A(x)(y) + A(x')(y) \\ A(sx)(y) = sA(x)(y) \end{aligned}\right\} \textit{Linearity of } A$$

$$\left.\begin{aligned} A(x)(y+y') = A(x)(y) + A(x)(y') \\ A(x)(sy) = sA(x)(y) \end{aligned}\right\} \textit{Linearity of } A(x)$$

Let $A \in E * (E * F)$ and $B \in E * (E * F)$ be two linear maps defined by $A(x)(y) = B(x)(y)$. For the case $B(y)(x) = A(x)(y)$, if $B = A$ then A is the ***symmetric operator,*** if $B = -A$ then A is the ***antisymmetric operator***.

We can write a bilinear map $A(x)(y)$ as the sum of the symmetric operator and the antisymmetric operator as follows :

$$A(x)(y) = \frac{1}{2}\left[A(x)(y) + A(y)(x)\right] + \frac{1}{2}\left[A(x)(y) - A(y)(x)\right]$$

If $A(x)(x) = 0$, from the equation above we say this operator A is the antisymmetric.

Example 3.1.3. Let $A(x)(y) = xy$ be a map from a real vector space E to the vector space F.

$$A(x+x')(y) = (x+x')y = xy + x'y = A(x)(y) + A(x')(y)$$

$$A(sx)(y) = sxy = sA(x)(y)$$

$$A(x)(y+y') = x(y+y') = xy + xy' = A(x)(y) + A(x)(y')$$

$$A(x)(sy) = xsy = sxy = sA(x)(y)$$

for all $x, x', y, y' \in E$. So, $A(x)(y)$ is a bilinear map.

Since $A(x)(y) = xy = yx = A(y)(x)$, we say A is a symmetric bilinear map.

3.1.2 Multilinear Maps

We define a multilinear map A with finite N vectors $x_1, x_2, \cdots, x_N \in E_1 \times E_2 \times \cdots \times E_N$,

$A \in E_1 * (E_2 * (E_3 * (\cdots * E_{N+1})))$. We check the linearity of this operator with the conditions

$$[A+B]\,(x_1)(x_2)\cdots(x_N) = A(x_1)(x_2)\cdots(x_N) + B(x_1)(x_2)\cdots(x_N)$$

$$[cA]\,(x_1)(x_2)\cdots(x_N) = c\,[A(x_1)(x_2)\cdots(x_N)]$$

for $s \in \mathbb{R}$ and $A, B \in E_1 * (E_2 * (E_3 * (\cdots * E_{N+1})))$.

If the multilinear operator A doesn't change when exchanges any two vector, A is ***totally symmetric operator***. If its sign changes for any two vectors, then A is a ***totally antisymmetric operator***.

$$A(x_{\sigma(1)})(x_{\sigma(2)})\cdots(x_{\sigma(N)}) = \begin{cases} A(x_1)(x_2)\cdots(x_N) & ,if\ A\ is\ symmetric \\ sgn(\sigma)A(x_1)(x_2)\cdots(x_N) & ,if\ A\ is\ antisymmetric \end{cases}$$

where σ is an element of the group of permutations and $sgn(\sigma)$ is signature of the permutation σ.

A is an antisymmetric operator if and only if $A\cdots(x)\cdots(x)\cdots = 0$.

3.2 Tensors

Definition 3.4. A p-multilinear operator A defined p times on the real vector space E with real values of $\mathbb{R}$ is said to be a **covariant p-tensor** of E. An operator A is defined in the vector space $E * (E * (\cdots * E) * \mathbb{R}$ with p times E. The operator $A(x_1)(x_2)\cdots(x_p)$ is in $\mathbb{R}$ for vectors $x_1, x_2, \cdots, x_p \in E$. The dual space E^* of E.

Definition 3.5. If a multilinear operator B of E is defined p times on the dual space E^* with real values in $\mathbb{R}$, $E^* * (E^* * (\cdots * E^*) * \mathbb{R}$, is called a ***contravariant p-tensor*** of E. $B(w_1)(w_2)\cdots(w_p) \in \mathbb{R}$ for vectors $w_1, w_2, \cdots, w_p \in E^*$.

Definition 3.6. If a tensor of E has contravariant tensor and covariant tensor, it implies that this tensor is a ***mixed tensor*** of E.

For example, let C be a mixed tensor of E which have p-covariant tensor and q-contravariant tensor. We describe the mixed tensor

$$C(x_1)(x_2)\cdots(x_p)(w_1)(w_2)\cdots(w_p) \in \mathbb{R}$$

for $x_1, x_2, \cdots, x_p \in E$ and $w_1, w_2, \cdots, w_p \in E^*$.

Let us take $p = 1$ for a covariant p-tensor A of E, so A is 1-tensor of E and is defined in $E * \mathbb{R}$. It means that a covariant 1-tensor of E is an element of the dual space of E.

0-tensor is a map defined from 0 to $\mathbb{R}$, i.e. 0-tensor form a number and the space of 0-tensor is $\mathbb{R}$.

A linear map defined from the dual space E^* to $\mathbb{R}$ is called a bidual of E, and denoted by $(E^*)^*$. Let $x \in E$. By the hypothesis, w is a linear map. Suppose that $\tilde{x} = 0 \in (E^*)^*$, so that $\tilde{x}(w) = w(x)$ and $\tilde{x}(w) = 0$ for all $w \in E^*$. If $dim(E) < \infty$, then $x = 0$. This is the condition of a one-to-one map. Since $dim(E) = dim(E^*) = dim((E^*)^*)$, the map $x \mapsto \tilde{x}$ is surjective. Thus, $x \mapsto \tilde{x}$ is bijective and E and $(E^*)^*$ are isomorphic. An element of E can be defined by an element of the bidual space $(E^*)^*$.

3.2.1 Tensor Product

The product of any two tensor is called tensor product, and denoted by $\otimes$. A tensor product is defined by tensors and their orders. Let A be a covariant p-tensor of the real vector space E and B be a covariant q-tensor of E. The product of $A \otimes B$: $\overbrace{E * (E * (\cdots * E)}^{p-times} * R \times \overbrace{E * (E * (\cdots * E)}^{q-times} * R \to \overbrace{E * (E * (\cdots * E)}^{p+q-times} * R,$

$$(A \otimes B)(x_1) \cdots (x_p)(y_1) \cdots (y_q) = A(x_1) \cdots (x_p) \times B(y_2) \cdots (y_q) \in \mathbb{R} \times \mathbb{R}$$

Thus, $A \otimes B$ is $p + q$-tensor of E.

Let A be of order p, B be of order q and C be of order l. The product of $(A \otimes B) \otimes C$,

$$\begin{aligned}
((A \otimes B) \otimes C)(x_1) \cdots (x_p)(y_1) \cdots (y_q)(z_1) \cdots (z_l) \\
&= (A(x_1) \cdots (x_p) \times B(y_1) \cdots (y_q)) \times C(z_1) \cdots (z_l) \\
&= A(x_1) \cdots (x_p) \times B(y_1) \cdots (y_q) \times C(z_1) \cdots (z_l) \\
&= A(x_1) \cdots (x_p) \times (B(y_1) \cdots (y_q) \times C(z_1) \cdots (z_l)) \\
&= (A \otimes (B \otimes C))(x_1) \cdots (x_p)(y_1) \cdots (y_q)(z_1) \cdots (z_l)
\end{aligned}$$

The tensor product is associative. One of two tensors can be taken as 0-tensor. Let A be 0-tensor and B be any tensor, then we have $s \otimes B = s \times B = B \times s = B \otimes s$.

3.2.2 Symmetrization and Antisymmetrization of Tensors

Let E be a real vector space. We define two operations Sym and Alt of a covariant p-tensor T of E as follows :

$$Sym(T)(x_1)\cdots(x_p) = \frac{1}{p!}\sum_{\sigma\in\sigma_p} T(x_{\sigma(1)})\cdots(x_{\sigma(p)})$$

$$Alt(T)(x_1)\cdots(x_p) = \frac{1}{p!}\sum_{\sigma\in\sigma_p} sgn(\sigma)\times T(x_{\sigma(1)})\cdots(x_{\sigma(p)})$$

where sgn is the signature of a permutation σ. The permutation can be written as a product of r transpositions such that $sgn(\sigma) = (-1)^r$. If the permutations are *even*, then $sgn(\sigma) = 1$. Otherwise, $sgn(\sigma) = (-1)$.

Corollary 3.1. $Sym(T) = T$ if and only if T is ***symmetric*** p***-tensor*** of E. $Alt(T) = T$ if and only if T is ***antisymmetric*** p***-tensor*** of E.

Proof. We need to check $Sym(T)$ is a symmetric operator. By the definition,

$$Sym(T)(x_1)\cdots(x_p) = \frac{1}{p!}\sum_{\sigma\in\sigma_p} T(x_{\sigma(1)})\cdots(x_{\sigma(p)})$$

Let $\sigma = \sigma' \circ \vartheta$ for $\vartheta \in \sigma_p$.

$$= \frac{1}{p!}\sum_{\sigma\in\sigma_p} T(x_{\sigma'\circ\vartheta(1)})\cdots(x_{\sigma'\circ\vartheta(p)})$$

$$= Sym(T)(x_{\vartheta(1)})\cdots(x_{\vartheta(p)})$$

$Sym(T)$ is a symmetric operator. We check that $Alt(T)$ is an antisymmetric operator. By the definition, we have

$$Alt(T)(x_1)\cdots(x_p) = \frac{1}{p!}\sum_{\sigma\in\sigma_p} sgn(\sigma)\times T(x_{\sigma(1)})\cdots(x_{\sigma(p)})$$

Let $\sigma = \sigma' \circ \vartheta$.

$$= \frac{1}{p!}\sum_{\sigma'\in\sigma_p} sgn(\sigma'\circ\vartheta)\times T(x_{\sigma'\circ\vartheta(1)})\cdots(x_{\sigma'\circ\vartheta(p)})$$

$$= Alt(T)(x_{\vartheta(1)})\cdots(x_{\vartheta(p)})$$

$Alt(T)$ is an antisymetric operator. □

3.3 Linear p-forms

An antisymmetric covariant p-tensor of the real vector space E is called a ***linear p-form*** of E. The vector space of all linear p-forms of E is denoted by $\Lambda^p(E)$.

Let p be 0. Since these antisymmetric covariant 0-tensor is any linear map from 0 to $\mathbb{R}$, this tensor form a number. So,

$$\Lambda^0(E) = \mathbb{R}$$

Let us take $p = 1$, the vector of all linear p-forms is defined by $E * \mathbb{R}$. Then we have

$$\Lambda^1(E) = E * \mathbb{R} = E^*$$

$\Lambda^p(E)$ is defined on subspaces of $E * (E * (\cdots * E) * \mathbb{R}$ with p times E, containing the antisymmetric covariant p-tensors.

Definition 3.7. A form α of k-form is defined by $\sum_I f_I dx_I$.

3.3.1 Inner Product

Let A be a covariant p-tensor of E. This tensor can be written as $p-1$-tensor with a fixed element x of E as follow :

$$[A(x)](x_1)\cdots(x_{p-1}) = A(x)(x_1)\cdots(x_{p-1})$$

$A(x)$ is the ***inner product*** of A with x.

3.3.2 Exterior Product of Forms

Definition 3.8. The exterior product $\wedge$ of $A \in \Lambda^p(E)$ and $B \in \Lambda^q(E)$ is in $\Lambda^{p+q}(E)$ and denoted by $A \wedge B$. The product $A \wedge B$ is defined as

$$A \wedge B = \frac{(p+q)!}{p!.q!} Alt(A \otimes B)$$

Proof. Given a basis $e_1, \cdots, e_p$ for the space of the p-tensors E, we define a basis for the space of the antisymmetric covariant p-tensors at the point x for each $i_1, \cdots, i_p$ with $A \in \Lambda^p(E)$ such that

$$dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} = p!A(dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_p})$$

Then we define $Alt(A\otimes B)$ by the previous equivalence as follows :

$$\begin{aligned} Alt(A\otimes B)(x_1) \quad \cdots \quad (x_{p+q}) & \\ &= \frac{1}{(p+q)!}\sum_{\sigma\in\sigma_p} sgn(\sigma)A(x_{\sigma(1)})\cdots(x_{\sigma(p)})\wedge B(x_{\sigma(p+1)})\cdots(x_{\sigma(p+q)}) \\ &= \frac{1}{(p+q)!}(p!A(x_1)\cdots(x_p)\times q!B(x_{p+1})\cdots(x_{p+q})) \\ &= \frac{p!\cdot q!}{(p+q)!}(A\wedge B)(x_1)\cdots(x_{p+q}) \end{aligned}$$

□

Example 3.3.1. Let A, B be two 1-forms, then we have $p = 1$ and $q = 1$. So, the exterior product $A\wedge B$ defined by

$$A\wedge B = \frac{(1+1)!}{1!.1!}Alt(A\otimes B) = 2Alt(A\otimes B)$$

Let A, B, C be three forms of $\Lambda^p(E), \Lambda^q(E)$ and $\Lambda^l(E)$ respectively.

$$\begin{aligned} A\wedge(B\wedge C) &= \frac{(p+(q+l))!}{p!.(q!.l!)}Alt(A\otimes(B\otimes C)) \\ &= \frac{(p+q+l)!}{p!.q!.l!}Alt(A\otimes B\otimes C) \\ &= \frac{((p+q)+l)!}{(p!.q!).l!}Alt((A\otimes B)\otimes C) \\ &= (A\wedge B)\wedge C \end{aligned}$$

The tensor product is associative.

Suppose that one of these forms is 0-form. Since a 0-form is a number c of $\mathbb{R}$, then we have the exterior product given by

$$c\wedge A = A\wedge c = c\times A$$

The commutation of A, B is defined as a **graded commutativity**. Every k-form α and l-form β give us the result

$$\alpha\wedge\beta = (-1)^{kl}\,\beta\wedge\alpha$$

for $k, l \in \mathbb{Z}^+$

Proof. Let $I = (m_1, m_2, \cdots, m_k)$ and $J = (n_1, n_2, \cdots, n_l)$ be two coordinates for α and β respectively, i.e. α is a k-form and β is a l-form. Since $dx_I = dx_{m_1} \wedge dx_{m_2} \wedge \cdots \wedge dx_{m_k}$ and $dx_J = dx_{n_1} \wedge dx_{n_2} \wedge \cdots \wedge dx_{n_l}$, we product the coordinate of $\alpha \cdot \beta$ as follows :

$$dx_I \wedge dx_J = dx_{m_1} \wedge dx_{m_2} \wedge \cdots \wedge dx_{m_k} \wedge dx_{n_1} \wedge dx_{n_2} \wedge \cdots \wedge dx_{n_l}$$

We change the order of dx_{n_1} for k-times

$$dx_I \wedge dx_J = (-1)^k \, dx_{n_1} \wedge dx_{m_1} \wedge dx_{m_2} \wedge \cdots \wedge dx_{m_k} \wedge dx_{n_2} \wedge \cdots \wedge dx_{n_l}$$

We do the same for dx_{n_2}

$$dx_I \wedge dx_J = (-1)^k (-1)^k \, dx_{n_1} \wedge dx_{n_2} \wedge dx_{m_1} \wedge dx_{m_2} \wedge \cdots \wedge dx_{m_k} \wedge dx_{n_3} \wedge \cdots \wedge dx_{n_l}$$

We continue to do the same for l terms and we are getting the following result

$$\begin{aligned} dx_I \wedge dx_J &= (-1)^{kl} \, dx_{n_1} \wedge dx_{n_2} \wedge \cdots \wedge dx_{n_l} \wedge dx_{m_1} \wedge dx_{m_2} \wedge \cdots \wedge dx_{m_k} \\ &= (-1)^{kl} \, dx_J \wedge dx_I \end{aligned}$$

The definiton of the form α is $\sum_I f_I dx_I$ with the smooth function f_I, similarly we define the form $\beta = \sum_J g_J dx_J$ with the smooth function g_J. So,

$$\begin{aligned} \alpha \cdot \beta &= \sum_{I,J} f_I g_J dx_I dx_J = (-1)^{kl} \sum_{I,J} g_J f_I dx_J dx_I \\ &= (-1)^{kl} \, \beta \cdot \alpha \end{aligned}$$

□

Exterior product of an element e of the dual space E^* and $A \in \Lambda^p(E)$ is given by

$$\begin{aligned} (e \wedge A)(x)(x_1) \cdots (x_p) &= e(x) \times A(x_1) \cdots (x_p) \\ &- e(x_1) \times A(x)(x_2) \cdots (x_p) \\ &- \cdots \\ &- e(x_p) \times A(x_1)(x_2) \cdots (x_{p-1})(x) \end{aligned}$$

Let $dim(E) = n$. We can calculate the dimension of the vector space of linear p-forms of E, $\Lambda^p(E)$ with the combination for every element of $\Lambda^p(E)$ such that

$$dim(\Lambda^p(E)) = \frac{n!}{(n-p)!.p!} = \mathrm{C}_n^p$$

Suppose that $p = n+1$, then the linear p-form is written as $a_0 \wedge (a_1 \wedge \cdots \wedge a_n)$. In this case, two of these forms become elements of the same vector space. Since a linear p-form is a antisymmetric covariant p-tensor, we have zero forms. Thus, $dim(\Lambda^p(E)) = 0$ with $dim(E) = n$. It is the same result for all $p > n$.

3.3.3 Pullbacks of Forms

Definition 3.9. A linear map M is defined from the real vector space E to the real vector space F, $M \in E * F = L(E, F)$ with a covariant p-tensor A of F. We define a linear operator such that

$$M^*(A)(y_1)\cdots(y_p) = A(M(y_1)\cdots M(y_p))$$

for $y_1, \cdots, y_p \in E$. $M^*(A)$ is a **pullback operator** of A by M.

Let A, B be two covariant p-tensors of F. M^* satisfy the two conditions for $\lambda \in \mathbb{R}$ as follows :

1. $M^*(A + B) = M^*(A) + M^*(B)$
2. $M^*(\lambda \times A) = \lambda \times M^*(A)$

Furthermore, let $N : F \to G$ be a linear map with a contravariant p-tensor A on G. The pullback operation $(N \circ M)^*$ satisfy

$$(N \circ M)^* = M^* \circ N^*$$

Then, $(N \circ M)^*(A) = M^*(N^*(A))$ is the pullback operation of a form A. If A is p form, it implies that the pullback of this form by M is also p form. Since the pullback is a morphism, we have $M^*(A \wedge B) = M^*(A) \wedge M^*(B)$.

4 SMOOTH FORMS ON REAL DOMAINS

We construct the diffeology consisting of smooth parametrizations on a real domain U and we can define the smoothness on $\mathbb{R}^n$ with the smooth structure of finite dimensional vector space defined on $\Lambda^p(\mathbb{R}^n)$.

Definition 4.1. A smooth map $w : U \to \Lambda^p(\mathbb{R}^n)$ is called a ***smooth*** p***-form*** of U. $\mathcal{C}^\infty(U, \Lambda^p(\mathbb{R}^n))$ is the set of all smooth p forms on U and let w and w' be two elements of $\mathcal{C}^\infty(U, \Lambda^p(\mathbb{R}^n))$ with any element $s \in \mathbb{R}$. The sum of these two elements and scalar multiplication are as follows :

$$(w + w')(x) = w(x) + w'(x)$$

$$(s \times w)(x) = s \times w(x)$$

Definition 4.2. The pullback of any p form $w \in \mathcal{C}^\infty(V, \Lambda^p(\mathbb{R}^n))$ is defined by a smooth parametrization f such that $f^*(w)(u)(x_1)\cdots(x_p)$ for $u \in U$ and $x_1, \cdots, x_p \in \mathbb{R}^n$

$$f^*(w)(u)(x_1)\cdots(x_p) = wf(u)\,(D(f)(u)(x_1))\cdots(D(f)(u)(x_p))$$

where the tangent map $D(f)(u)$ of f at the point u is the vector on V with the direction vectors x_i, $i \in 1, \cdots, n$.

4.1 Exterior Product of Smooth Forms

Let U be a real domain and let α, β be two smooth forms such that $\alpha \in \mathcal{C}^\infty(U, \Lambda^p(\mathbb{R}^n))$ and $\beta \in \mathcal{C}^\infty(U, \Lambda^q(\mathbb{R}^n))$. The exterior product $\wedge$ of α and β,

$$\wedge : \mathcal{C}^\infty(U, \Lambda^p(\mathbb{R}^n)) \times \mathcal{C}^\infty(U, \Lambda^q(\mathbb{R}^n)) \to \mathcal{C}^\infty(U, \Lambda^{p+q}(\mathbb{R}^n))$$

denoted by $\alpha \wedge \beta$, is a $p+q$ form of U and it is defined as $(\alpha \wedge \beta)(x) = \alpha(x) \wedge \beta(x)$ for all $x \in U$.

4.2 Exterior Derivative of Smooth Forms

The exterior differentiation operation d is defined from $\mathcal{C}^\infty(U, \Lambda^p(\mathbb{R}^n))$ to $\mathcal{C}^\infty(U, \Lambda^{p+1}(\mathbb{R}^n))$ for all real domain U with any integers n.

Let α be a form on U such that $\alpha : x \mapsto ae^i \wedge \cdots \wedge e^k$ for any coordinates $i, \cdots, k$ and a smooth map a with the smooth parametrization $x \mapsto a \in \mathcal{C}^\infty(U, \mathbb{R})$. The exterior derivative of α with x,

$$
\begin{aligned}
(d\alpha)(x) &= \sum_{l=1}^{n} \frac{\partial a}{\partial x^l} e^l \wedge e^i \wedge \cdots \wedge e^k \\
&= \sum_{l=1}^{n} \frac{\partial a}{\partial x^l} dx^l \wedge dx^i \wedge \cdots \wedge dx^k
\end{aligned}
$$

Utilizing the above definition, we have

$$
\begin{aligned}
d(\alpha + \alpha')(x) &= \sum_{l=1}^{n} \left(\frac{\partial \alpha}{\partial x^l} + \frac{\partial \alpha'}{\partial x^l} \right) e^l \wedge e^i \wedge \cdots \wedge e^k \\
&= \sum_{l=1}^{n} \frac{\partial \alpha}{\partial x^l} e^l \wedge e^i \wedge \cdots \wedge e^k + \sum_{l=1}^{n} \frac{\partial \alpha'}{\partial x^l} e^l \wedge e^i \wedge \cdots \wedge e^k \\
&= d(\alpha)(x) + d(\alpha')(x) \\
d(s\alpha)(x) &= \sum_{l=1}^{n} \frac{\partial (s\alpha)}{\partial x^l} e^l \wedge e^i \wedge \cdots \wedge e^k \\
&= \sum_{l=1}^{n} s \frac{\partial \alpha}{\partial x^l} e^l \wedge e^i \wedge \cdots \wedge e^k \\
&= s \sum_{l=1}^{n} \frac{\partial \alpha}{\partial x^l} e^l \wedge e^i \wedge \cdots \wedge e^k \\
&= sd(\alpha)(x)
\end{aligned}
$$

for any smooth p forms α, α'. So, d is a linear map and $d\alpha$ is called the exterior derivative of any form α on a real domain U.

The pullback operation f^* between $\mathcal{C}^\infty(U, \Lambda^p(\mathbb{R}^n)), \mathcal{C}^\infty(U', \Lambda^p(\mathbb{R}^n))$ commutes the pullback operation f^* between $\mathcal{C}^\infty(U, \Lambda^{p+1}(\mathbb{R}^n)), \mathcal{C}^\infty(U', \Lambda^{p+1}(\mathbb{R}^n))$ with the exterior derivative d such that $f^* \circ d = d \circ f^*$.

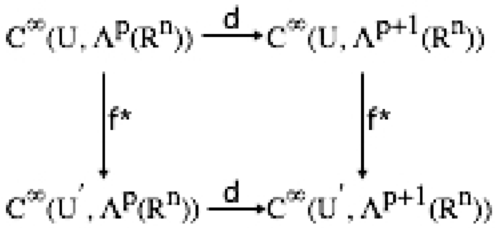

Figure 4.1: The commutation between f^* and d

4.3 Differential Forms on Real Domains

Definition 4.3. A **differential form** of degree k on $\mathbb{R}^n$ is an expression which produced by tensor product and smooth functions for $k \in \mathbb{Z}^+$. Let α be a k-form, then we define the expression of α as follows with the smooth functions f_I on $\mathbb{R}^n$

$$\alpha = \sum_I f_I dx_I$$

where I is in the multi-index $\{i_1, i_2, \cdots, i_k\}$.

Example 4.3.1. A form $\alpha = 3x_1dx_1dx_3 - 7x_3dx_1dx_4 + 2dx_2dx_3$ is an example of a 2-form on $\mathbb{R}^4$ and $\beta = x_2dx_1dx_3dx_5$ is an example of a 3-form on $\mathbb{R}^5$.

Let w be a k-form on $\mathbb{R}^n$. Then the maximum number of terms we can generate is calculated as follows ;

$$\frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

for $n, k \in \mathbb{Z}^+$.

4.3.1 Closed and Exact Differantial Forms

Definition 4.4. A differantial form such that its exterior derivative is zero, is called ***a closed form***.

Example 4.3.2. Let $\alpha = 2dx \wedge dy$ be a differential form, then the exterior derivative of α as follows :

$$d\alpha = d[2dx \wedge dy] = 2dx \wedge dx \wedge dy = 0$$

We have that $dx \wedge dx = 0$, since a form is an antisymmetric covarint tensor.

Definition 4.5. A differential k-form w such that $dw = \alpha$ with a $(k-1)$-form α, then α is said to be a ***exact differential form***.

For example, let $\alpha = ydx + xdy$, α can be written as $d(xy)$. So, α is a exact form.

Proposition 4.1. The exterior derivative of a exterior derivative of a form is zero, i.e. $d(d(\alpha)) = 0$ for any form α.

Proof. Let α be a k-form. The exterior derivative of this form is given by

$$d(\alpha) = \sum_{l=1}^{n} \frac{\partial \alpha}{\partial x^l} dx^l \wedge dx^i \wedge \cdots \wedge dx^k$$

For the exterior derivate of $d(\alpha)$, we have

$$\begin{aligned} d(d(\alpha)) &= d\left(\sum_{l=1}^{n} \frac{\partial \alpha}{\partial x^l} dx^l \wedge dx^i \wedge \cdots \wedge dx^k\right) \\ &= \sum_{l=1}^{n} d\left(\frac{\partial \alpha}{\partial x^l} dx^l \wedge dx^i \wedge \cdots \wedge dx^k\right) \\ &= \sum_{l=1}^{n} \frac{\partial^2 \alpha}{\partial (x^l)^2} (dx^l \wedge dx^i \wedge \cdots \wedge dx^k) \wedge (dx^l \wedge dx^i \wedge \cdots \wedge dx^k) \\ &= \sum_{l=1}^{n} 0 = 0 \end{aligned}$$

□

Proposition 4.2. Every exact form is a closed form.

Proof. Let α be a exact form. Then we can say that there exist a form β such that $\alpha = d(\beta)$. Since $d(d(\beta)) = 0$, α is a exact form. □

Proposition 4.3. Let α, β be any closed form. Then the product of α and β is always closed.

Proof. Let α, β be two closed form. We calculate the derivative exterior of $\alpha \wedge \beta$ by definition of derivative exterior such that

$$d(\alpha \wedge \beta) = \alpha d(\beta) - \beta d(\alpha)$$

Since α and β are closed forms, we obtain $d(\beta)$ and $d(\alpha)$ are zero. So,

$$d(\alpha \wedge \beta) = \alpha \cdot 0 - \beta \cdot 0 = 0$$

Then, $\alpha \wedge \beta$ is also closed form. □

Proposition 4.4. If α is a closed form and β is exact form, then $\alpha \wedge \beta$ is always exact form.

Proof. Let α be a closed form and let β be a exact form in $\Omega^{n-1}(X)$. Since β is exact form, we can also define the form β as $\beta = d(g)$ with g is a n-form. Then,

$$\alpha \wedge \beta = \alpha \wedge d(g)$$

We know that the exterior derivative of α is zero when α is a closed form, i.e. $d(\alpha) = 0$. We can also say that $g \wedge d(\alpha) = 0$. So, we can add the term $g \wedge d(\alpha)$ to $\alpha \wedge d(g)$ by the definiton of Leibniz as follows

$$\begin{aligned} \alpha \wedge \beta &= \alpha \wedge d(g) - g \wedge d(\alpha) \\ &= d(\alpha \wedge g) \end{aligned}$$

By the definition of the exact form, $\alpha \wedge \beta$ is exact form. □

Example 4.3.3. Let α, β and γ be three forms such that $\alpha = xdx - ydy$, $\beta = ydx \wedge dy + xdy \wedge dz$ and $\gamma = zdz$.

We calculate $\alpha \wedge \beta$

$$\begin{aligned} \alpha \wedge \beta &= xydx \wedge dx \wedge dy + x^2 dx \wedge dy \wedge dz - y^2 dy \wedge dx \wedge dy + yxdy \wedge dy \wedge dz \\ &= 0 + x^2 dx \wedge dy \wedge dz + 0 + 0 \\ &= x^2 dx \wedge dy \wedge dz \end{aligned}$$

We use the above result to calculate $(\alpha \wedge \beta) \wedge \gamma$

$$\begin{aligned} \alpha \wedge \beta \wedge \gamma &= (x^2 dx \wedge dy \wedge dz)(zdz) \\ &= x^2 zdx \wedge dy \wedge dz \wedge dz \\ &= 0 \end{aligned}$$

Example 4.3.4. Let $\alpha = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n = \sum_{i=1}^{n} dx_i \wedge dy_i$

We compute $w^n = w \cdot w \cdots w$.

Let's start with w^2 to compute and then we will continue to multiply by w.

$$
\begin{aligned}
w^2 &= (dx_1 \wedge dy_1 + dx_2 \wedge dy_2)(dx_1 \wedge dy_1 + dx_2 \wedge dy_2) \\
&= dx_1 \wedge dy_1 \wedge dx_1 \wedge dy_1 + dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 + dx_2 \wedge dy_2 \wedge dx_1 \wedge dy_1 \\
&\quad + dx_2 \wedge dy_2 \wedge dx_2 \wedge dy_2 \\
&= -dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2 - dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2 \\
&= -2dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2 \\
w^3 &= w^2 \cdot w \\
&= (-2dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2)(dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3 \\
&= 0 + 0 - 2dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_1 \wedge dy_2 \wedge dy_3 \\
&= -2dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_1 \wedge dy_2 \wedge dy_3 \\
w^n &= -2dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n
\end{aligned}
$$

5 DIFFERENTIAL FORMS ON DIFFEOLOGICAL SPACES

Let X be a diffeological space and let U, V be two real domains. Let $P : U \to X$ be a plot of X and a smooth parametrization $F : V \to U$. If a p form α on X satisfy the following two conditions, then α is called a ***diffeological*** p ***form***.

i) $\alpha : \mathfrak{D} \to \mathcal{C}^\infty(U, \Lambda^p(\mathbb{R}^n))$

ii) The pullback of F, $F^* : \mathcal{C}^\infty(U, \Lambda^p(\mathbb{R}^n)) \to \mathcal{C}^\infty(V, \Lambda^p(\mathbb{R}^n))$, satisfies

$$F^*(\alpha(P)) = \alpha(P \circ F)$$

The form $\alpha(P)$ express a differential form α on the domain of a plot P of X.

$$\begin{array}{l} \alpha : \mathcal{D} \to \mathcal{C}^\infty(U, \Lambda^p(\mathbb{R}^n)) \\ \qquad\quad\;\; \downarrow F^* \\ \qquad\quad\;\; \mathcal{C}^\infty(V, \Lambda^p(\mathbb{R}^n)) \end{array}$$

Figure 5.1: A diffeological p form α of X in the plot P of $\mathcal{D}$

The set of all differential p forms α of the diffeological space X is denoted by $\Omega^k(X)$.

Let X be a diffeological space and let α, α' be two diffeological p forms in $\Omega^k(X)$. For all plots P with $s \in \mathbb{R}$, we have

1. $(\alpha + \alpha')(P) = \alpha(P) + \alpha'(P)$
2. $(s \times \alpha)(P) = s \times \alpha(P)$

5.1 Pullback of Differential Forms

Let f be a smooth map between two diffeological spaces X, X', $f : X \to X'$.

A diffeological p form on X shown as $f^*(\alpha')$ with $\alpha' \in \Omega^p(X')$ such that $(f^*(\alpha'))(P) = \alpha'(f \circ P)$ for all plots P of X, is called the pullback of α' by f.

In addition, let $g : X' \to X''$ be a smooth map with the diffeological space X''. Let $\alpha'' \in \Omega^p(X'')$, then we have

$$(g \circ f)^*(\alpha'') = f^*(g^*(\alpha''))$$

So, $f^* : \Omega^p(X') \to \Omega^p(X')$ is a smooth linear map.

5.2 Exterior Product of Differential Forms

Let $\alpha \in \Omega^p(X)$ and $\beta \in \Omega^q(X)$ where X is a diffelogical space. A exterior product operation $\wedge$ is smooth and bilinear map, it is defined as follow :

$$\wedge : \Omega^p(X) \times \Omega^q(X) \to \Omega^{p+q}(X)$$

The exterior product of α and β defined on X, $\alpha \wedge \beta$ is

$$(\alpha \wedge \beta)(P) = \alpha(P) \wedge \beta(P)$$

for all plots P of X and this exterior product is $p+q$ form on X.

5.3 Exterior Derivative of Forms

The exterior derivative operator d is a smooth linear operator and

$$d : \Omega^p(X) \to \Omega^{p+1}(X)$$

where X be a diffeological space and α is a p form of X such that

$$(d\alpha)(P) = d(\alpha(P))$$

for all plots P of X. $d\alpha$ is the exterior derivative of α. Let $f : X \to X'$ be a smooth map with the diffeological space X', we have that

$$\begin{aligned}
(d\alpha)(P \circ f) &= d(\alpha(P \circ f)) \\
&= d(f^*(\alpha(P))) \\
&= f^*(d(\alpha(P))) \\
&= f^*((d\alpha)(P))
\end{aligned}$$

So, for all differential forms α of X'

$$d(f^*(\alpha)) = f^*(d(\alpha))$$

5.4 Differential Forms on Manifolds

Let M be a smooth manifold in $\mathbb{R}^N$, $n \leq N$.

Let us define a smooth function f from M to $\mathbb{R}$. Then we have to show that the partial derivative of f, $\frac{\partial f}{\partial x_j} = \lim_{t \mapsto 0} \frac{f(x+te_j)-f(x)}{t}$, is in $\mathbb{R}$ for any $x \in M$ and the basis e_j. But we don't know te_j is in M or not. Let $(U_i, \psi_i), (U_j, \psi_j)$ be two charts on M for each indices i, j such that $\psi_i : U_i \to \mathbb{R}^N$ and $M = \bigcup_i \psi_i(U_i)$. We can define also a map $f_i : \mathbb{R}^n \to \mathbb{R}$ such that $f_i(t) = f(\psi_i(t))$. Now, we have to show that this result exists for a overlap of two charts $(U_i, \psi_i), (U_j, \psi_j)$. Suppose that $x \in M$ is in the overlap of these two charts such that $x = \psi_i(t) = \psi_j(u)$ for two vectors $t \in U_i$ and $u \in U_j$. Since the maps on the manifold are injective maps, we have that $t = \psi_i^{-1}(\psi_j(u))$. Then

$$f(x) = f(\psi_i(t)) = f(\psi_j(u))$$

Since $f_i(t) = f_j(u)$, $f(\psi_i(t)) = f_i(t)$ and $f(\psi_j(u)) = f_j(u)$, we have that

$$f_i(\psi_i^{-1} \circ \psi_j(u)) = f_j(u)$$

$$(f_j \circ f_i)(\psi_i^{-1} \circ \psi_j(u)) = u$$

We use the pullback operation to determine $f_j(u)$ on $\psi_j^{-1}(\psi_i(U_i))$ given by

$$f_j(u) = (\psi_i^{-1} \circ \psi_j(u))^* f_i$$

for all u such that $\psi_j(u) \in \psi_i(U_i)$.

For all $x \in \psi_i(U_i)$

$$f(x) = f(\psi_i(\psi_j^{-1}(x))) = f_i(\psi_i^{-1}(x)$$

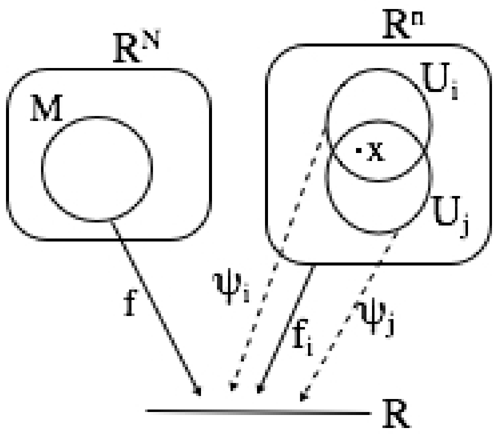

Figure 5.2: A differential k form on a manifold M

Definition 5.1. A given k form f on a manifold M such that f is a collection of f_i for indices i, that provides the following transition law, is called a **differential** k **form** on M

$$f_j = (\psi_i^{-1} \circ \psi_j)^* f_i$$

The vector space $\Omega^p(M)$ is the collection of all differential p forms on M.

Transitions maps are defined as

$$\psi_i^{-1} \circ \psi_j : \psi_j^{-1}(U_1 \cap U_2) \to \psi_i^{-1}(U_1 \cap U_2)$$

$$\psi_j^{-1} \circ \psi_i : \psi_i^{-1}(U_1 \cap U_2) \to \psi_j^{-1}(U_1 \cap U_2)$$

Example 5.4.1. Let $\alpha = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ on S^2. Then α is an example of a differential 2 form on the manifold S^2, $\alpha \in \Omega^2(S^2)$.

5.4.1 Exterior Derivative of Differantial Forms on Manifolds

Definition 5.2. Let α be a k-form and let β be l-form are defined by $\sum_I f_I dx_I$ and $\sum_J g_J dx_J$ respectively. Then the exterior derivative of α is defined by $\sum_I df_I dx_I$ and $d(\beta) = \sum_J dg_J dx_J$. We define this derivation also in a chart as follows :

$$d(\alpha) = \frac{1}{k!} \partial_i \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

We have $d(f \wedge g) = f \wedge dg + g \wedge df$ by the Leibniz rule. The exterior derivative of

$\alpha \wedge \beta$ is defined by

$$\begin{aligned} d(\alpha \wedge \beta) &= \sum_{I,J} d(f_I \wedge g_J) dx_I dx_J \\ &= \sum_{I,J} (d(f_I) g_J + f_I d(g_J)) dx_I dx_J \\ &= \sum_{I,J} (d(f_I) dx_I g_J dx_J + (-1)^k f_I dx_I d(g_J) dx_J) \\ &= d(\alpha) \wedge \beta + (-1)^k \alpha d(\beta) \end{aligned}$$

Corollary 5.1. Let $\alpha = \beta$. By the previous definition, we obtain $\alpha \wedge \alpha = (-1)^k \alpha \wedge \alpha$. So, $\alpha^2 = 0$.

5.5 Differential Forms on Manifold with Boundary and Corners

Definition 5.3. $\mathbb{K}^n$ which is the diffeological n-power of the half-line $\mathbb{K} = [0, \infty[$ with the subset diffeology, is called the **corners**. $\mathbb{K}^n$ is defines as

$$\mathbb{K}^n = \{(x_i)_{i=1}^n \in R^n \mid x_i \geq 0, i = 1, \cdots, n\}$$

Let θ be an open neighborhood on $\mathbb{K}^n$ and θ' be an open neighborhood on $\mathbb{R}^n$ such that $\theta' \cap \mathbb{K}^n = \theta$. A plot P in $\mathbb{K}^n$ is a smooth parametrization to $\mathbb{R}^n$ taking values in $\mathbb{K}^n$. For every $P \in \mathcal{C}^\infty(U, \mathbb{K}^n)$, $P^{-1}(\mathbb{K}^n)$ is open in $\mathbb{R}^n$, then the subset $\mathbb{K}^n$ of $\mathbb{R}^n$ is D-open. So, $\mathbb{K}^n$ is embedded and closed in R^n by the induced topology. $X_0 = \{0\} \subset X_1 \subset \cdots \subset X_n = \mathbb{K}^n$ is the natural filtration of $\mathbb{K}^n$ and X_j is defined by

$$X_j = \{(x_i)_{i=1}^n \in \mathbb{K}^n \mid \textit{there exists } i_1 < \cdots < i_{n-j} \text{ such that } X_{i_l} = 0\}$$

For example, let us take $n = 2$. We have $\mathbb{K}^2 = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\} \subset R^2$. Let us define X_0, X_1, X_2 in K^2

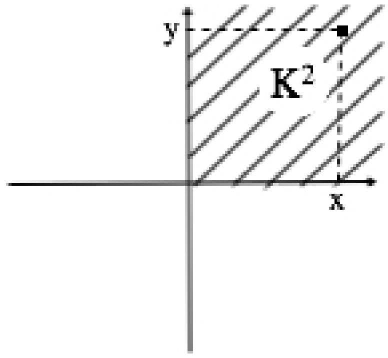

Figure 5.3: The Corners $\mathbb{K}^2$

$$\begin{aligned} X_0 &= \{(x_i)_{i=1}^2 \in K^2 \mid i_1 < i_2 \; such \; that \; X_{i_l} = 0\} \\ &= \{(0,0)\} \\ X_1 &= \{(x_i)_{i=1}^2 \in K^2 \mid i_1 \; such \; that \; X_{i_l} = 0\} \\ &= \partial K^2 - \{(0,0)\} \\ X_2 &= \{(x_i)_{i=1}^2 \in K^2 \mid i_1 < i_0 \; such \; that \; X_{i_l} = 0\} \\ &= K^2 \end{aligned}$$

A plot in $\mathbb{K}^n$ can be defined in three different ways. It can be defined as an open neigborhood in $\mathbb{K}^n$, defined on a half plane or defined in a corner.

Consider again $\mathbb{K}^2$. Let P be a plot in $\mathbb{K}^2$. A smooth parametrization $P : U \to \mathbb{K}^2$ is shown in three ways as follows

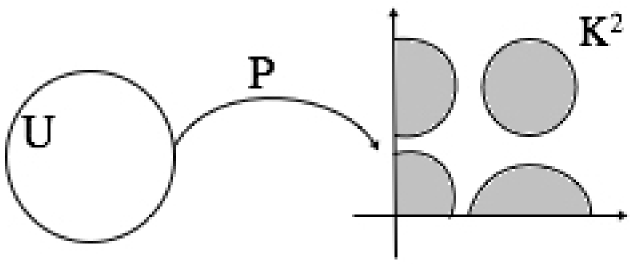

Figure 5.4: Smooth parametrizations of the corner $\mathbb{K}^2$

We define the subset of points in $\mathbb{R}^n$ such that $X_i - X_{i-1}$, it is denoted by S_j and

S_j is called a **strata**. $S_j = X_i - X_{i-1}$ have only j coordinates strictly positive and the rest are 0. Then, S_j is defined by

$$S_j = \{ \quad (x_i)_{i=1}^n \in \mathbb{R}^n \mid \text{there exists } i_1 < \cdots < i_j \text{ such that } X_{i_l} > 0$$

$$\text{and } X_m = 0 \text{ for all } m \notin \{i_1, \cdots, i_j\}\}$$

Let us take again $n = 2$ and let us define S_2 as folllows :

$$\begin{aligned} S_2 &= \{(x_i)_{i=1}^2 \in \mathbb{R}^2 \mid \text{there exists } i_1 < i_2 \text{ such that } X_{i_l} > 0 \\ &\quad \text{and } X_m = 0 \text{ for all } m \notin \{i_1, i_2\}\} \\ &= \{(x_1, x_1) \in \mathbb{R}^2, x_1 > 0 \text{ and } x_2 > 0\} \end{aligned}$$

For $j = 1$,

$$\begin{aligned} S_1 &= \{(x_i)_{i=1}^2 \in \mathbb{R}^2 \mid \text{there exists } i_1 \text{ such that } X_{i_l} > 0 \\ &\quad \text{and } X_m = 0 \text{ for all } m \notin \{i_1\}\} \\ &= \{(0, a) \cup (b, 0) \mid a, b \in \mathbb{R} - \{0\}\} \end{aligned}$$

For $j \geq 1$, X_j is the union of the strata and the strata S_j is D-open in X_j.

Definition 5.4. A parametrization $f : \mathbb{K}^n \to \mathbb{R}^k$ is a smooth by the subset diffeology if and only if there exists $F : \mathbb{R}^n \to \mathbb{R}^k$ is smooth such that $F \restriction \mathbb{K}^n = f$.

Theorem 5.1. *(Differential forms on manifolds with corners, 2017)* Let $f : \mathbb{K}^n \to \mathbb{R}^k$ be a any map. If for all smooth parametrization $P : U \to \mathbb{R}^n$ such that $P(U) \subset \mathbb{K}^n$ with $f \circ P \in \mathcal{C}^\infty(U, \mathbb{R}^k)$, then there exists an open neighborhood θ of $\mathbb{K}^n$ and $F \in \mathcal{C}^\infty(\theta, \mathbb{R}^k)$ such that $f = F \restriction \mathbb{K}^n$

Proof. Let us consider the smooth parametrization P defined on $\mathbb{R}^2$ taking values on $\mathbb{K}^n$ such that

$$P : (t_1, \cdots, t_n) \mapsto (t_1^2, \cdots, t_n^2)$$

Let $f : \mathbb{K}^n \to \mathbb{R}^k$ be a any map. Suppose that the composition $f \circ P$ is smooth and we have $f \circ P$ is an even function, since P is an even function in each t_i. We can define a function $g(x^2)$ for all even function $f(x) = f(-x)$ for each x. The Whitney Theorem shows us if f is of class $\mathcal{C}^\infty$, then g is also of class $\mathcal{C}^\infty$.

Firstly, we want to show this result for f which is of class $\mathcal{C}^{2s}$

The Taylor's formula of the even function f

$$f(x) = a_0 + a_1x^2 + a_2x^4 + \cdots + a_{s-1}x^{2s-2} + \phi(x)x^{2s}$$

By the above Taylor's formula, we have $\lim_{x\mapsto 0}\phi(x)x^{2s} = 0$

For $g(x^2) = f(x)$, we pick $x^2 = u$ and the Taylor's formula of g is given by

$$g(x) = a_0 + a_1u + a_2u^2 + \cdots + a_{s-1}u^{s-1} + \psi(u)u^s$$

We need to show that $\lim_{u\mapsto 0}\psi(u)u^s$ exists for showing g is of class $\mathcal{C}^s$. We have that $\psi(x^2) = \phi(x)$ and we can calculate $\psi(x^2)$ for some constants a_{ki} and $x > 0$

$$\phi^k(x) = \sum_{1\leq i\leq \frac{k}{2}} a_{ki}x^{k-2i}\psi^{k-i}(x) + 2^kx^k\psi^k(x^2)$$

$$\phi^k(x) - \sum_{1\leq i\leq \frac{k}{2}} a_{ki}x^{k-2i}\psi^{k-i}(x) = 2^kx^k\psi^k(x^2)$$

for some constants β_{ki}

$$\phi^k(x) - \sum_{1\leq i\leq k-1} \beta_{ki}x^{-i}\phi^{k-i}(x) = 2^kx^k\psi^k(x^2)$$

for some constants β'_{ki}

$$\sum_{1\leq i\leq k-1} \beta'_{ki}x^{k-i}\phi^{k-i}(x) = x^{2k}\psi^k(x^2)$$

Since $\lim_{x\mapsto 0}\phi(x)x^{2s} = 0$, $\lim_{x\mapsto 0}x^{k-i}\phi^{k-i}(x) = 0$ and by the equality we say that $\lim_{x\mapsto 0}x^{2k}\psi^k(x^2) = 0$ for $x > 0$. So, $\lim_{u\mapsto 0}\psi(u)u^s = 0$ exists and g is of class $\mathcal{C}^s$.

We showed this result for an even function f which is of class $\mathcal{C}^{2s}$, then we can find a function g for any even function f which is of class $\mathcal{C}^\infty$.

Therefore, there exists a smooth parametrization F defined from $\mathbb{R}^n$ to $\mathbb{R}$ such that $f(t_1^2, \cdots, t_n^2) = F(t_1^2, \cdots, t_n^2)$ and $f = F \upharpoonright K^n$. □

Example 5.5.1. Let f be the Cauchy's function such that $f(x) = exp\left(-\frac{1}{x^2}\right)$ for $x \neq 0$. f is of class $\mathcal{C}^\infty$. Let us take $u = x^2$ and $g(u) = exp\left(-\frac{1}{u}\right)$ is of $\mathcal{C}^\infty$ for $u \neq 0$.
Theorem 5.2. *(Differential forms on manifolds with corners, 2017)*

Let f be a local diffeomorphism of $\mathbb{K}^n$, i.e. if $x \in S_j$, then $f(x) \in S_j$.

Proof. Let $P : U \to \mathbb{R}^n$ be a parametrization. The dimension of the image of the tangent linear map $D(P)(x)$ is denoted by $rk(P)_x$ at the point $x \in U$.

Let $P : U \to K^n$ be a plot. If $P(r') \in S_j$, we describe $P(r')$ as $(P_1(r'), P_2(r'), \cdots, P_n(r'))$ such that $P_{i_k}(r') = 0$. For all $r \in U$, we know that $P_{i_k}(r) \geq 0$ and $rk(P)_{r'}$ can be up to j. Then we have that $D(P_{i_k})(r') = 0$, i.e. $rk(P)_r \leq 0$.

Let $x \in S_j$ and $f(x) = x' \in S_k$ with $k \neq j$.

Suppose that $k > j$. Since f is a local diffeomorphism, there exists a smooth parametrization F on an open neigborhood $\theta \supset \mathbb{K}^n$ such that $F \restriction \mathbb{K}^n = f$ and there exists a smooth parametrization G on an open neigborhood $\theta' \supset \mathbb{K}^n$ such that $G \restriction \mathbb{K}^n = f^{-1}$.

$G \restriction S_k : x' \mapsto x \in S_j$ is a plot of $\mathbb{K}^n$ and $rk(G \restriction S_k)_{x'} \leq j$. But $G \restriction S_k = G \circ j_k$ where j_k is identified with a plot such that $j_k : S_k \to \mathbb{K}^n$. j_k takes its values in the border of $\mathbb{K}^n$ and

$$\begin{aligned}(F \circ G \restriction S_k)(t) &= F \circ G \circ j_k(t) \\ &= F \circ G(j_k(t))\end{aligned}$$

Since $f \in Diff_{loc}(\mathbb{K}^n)$, f carries a border to the border for the D-topology. Then G and f^{-1} coincide and F and f coincide on $\partial\mathbb{K}^n$. So,

$$\begin{aligned}F \circ G(j_k(t)) &= f \circ f^{-1}(j_k(t)) \\ &= j_k(t)\end{aligned}$$

We have that $rk(F \circ G \restriction S_k)_{x'} = rk(j_k)_{x'} = k$. But $rk(F \circ G \restriction S_k)_{x'} \leq rk(G \restriction S_k)_{x'} \leq j$. It is contradiction, thus $k = j$. □

Theorem 5.3. *(Differential forms on manifolds with corners, 2017)* Any differential k-form on the corner $\mathbb{K}^n$ equipped with the subset diffeology of $\mathbb{R}^n$, is the restriction of a smooth differential k-form defined on some neighborhood of the corner.

Proof. Let w be a differential k-form on a manifold with corner. Then the restriction of a differential k-form w on some open neighborhood on the corner is defined by

$$w \restriction \mathbb{K}^n = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k}(x_1, \cdots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

with $i_j \in \{1, \cdots, n\}$ and $a_{i_1 \cdots i_k} \in \mathcal{C}^{\infty}(\overset{o}{K^n}, \mathbb{R})$.

Now we will use the square function lemma. By the lemma, we have that a smooth parametrization $sq : \mathbb{R}^n \to \mathbb{K}^n$ such that $sq(x_1, \cdots, x_n) = ((x_1)^2, \cdots, (x_1)^n)$. The the pullback of sq is injective, i.e. $sq^* : \Omega^k(\mathbb{K}^n) \to \Omega^k(\mathbb{R}^n)$ and if $sq^*(\alpha) = 0$, then $\alpha = 0$ for all $\alpha \in \Omega^k(\mathbb{K}^n)$. So, we obtain that the pullback of sq^* as follows :

$$sq^*(w) = \sum_{i_1 < \cdots < i_k} A_{i_1 \cdots i_k}(x_1, \cdots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where $A_{i_1 \cdots i_k} \in \mathcal{C}^{\infty}(R^n, \mathbb{R})$.

Suppose a parametrization $\epsilon_j = (x_1, \cdots, x_j, \cdots, x_n) \mapsto (x_1, \cdots, -x_j, \cdots, x_n)$

$$\begin{aligned} \epsilon_j^*(sq^*(w)) &= \sum_{i_1 < \cdots < i_k} A_{i_1 \cdots i_k}(x_1, \cdots, -x_j, \cdots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &- \sum_{i_1 < \cdots \leq j \leq \cdots < i_k} A_{i_1 \cdots j \cdots i_k}(x_1, \cdots, -x_j, \cdots, x_n)\, dx_{i_1} \wedge \cdots dx_j \cdots \wedge dx_{i_k} \end{aligned}$$

Since $sq \circ \epsilon_j = sq$, we say that $sq^*(w) = \epsilon_j^*(sq^*(w))$. Then,

$$A_{i_1 \cdots i_k}(x_1, \cdots, -x_j, \cdots, x_n) = A_{i_1 \cdots i_k}(x_1, \cdots, x_j, \cdots, x_n)$$

$$A_{i_1 \cdots j \cdots i_k}(x_1, \cdots, -x_j, \cdots, x_n) = -A_{i_1 \cdots j \cdots i_k}(x_1, \cdots, x_j, \cdots, x_n)$$

Hence, $A_{i_1 \cdots j \cdots i_k}(x_1, \cdots, x_j = 0, \cdots, x_n) = 0$. This implies that

$$A_{i_1 \cdots j \cdots i_k}(x_1, \cdots, x_j, \cdots, x_n) = 2x_j B_{i_1 \cdots j \cdots i_k}(x_1, \cdots, x_j, \cdots, x_n)$$

with $B_{i_1 \cdots j \cdots i_k} \in \mathcal{C}^{\infty}(R^n, \mathbb{R})$. Then $A_{i_1 \cdots j \cdots i_k}$ is defined by the real smooth function $\widehat{A}_{i_1 \cdots j \cdots i_k}$ defined on $\mathbb{R}$ such that

$$A_{i_1 \cdots i_k}(x_1, \cdots, x_n) = 2^k x_{i_1}, \cdots, x_{i_k} \widehat{A}_{i_1 \cdots j \cdots i_k}(x_1, \cdots, x_n)$$

Then the pullback of sq of the restriction of a differential k-form w on some open neighborhood on the corner is

$$sq^*(w \restriction \overset{o}{K^n}) = sq^*(w) \restriction \{x_i \neq 0\}$$

$$\sum_{i_1 < \cdots < i_k} 2^k a_{i_1 \cdots i_k} x_{i_1 \cdots i_k} \left(x_1^2, \cdots, x_n^2\right) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$= \sum_{i_1 < \cdots < i_k} 2^k x_{i_1 \cdots i_k} \widehat{A}_{i_1 \cdots j \cdots i_k}(x_1, \cdots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and

$$\widehat{A}_{i_1\cdots j\cdots i_k}(x_1,\cdots,x_n) = a_{i_1\cdots i_k}\left(x_1^2,\cdots,x_n^2\right)$$

for $x_i \neq 0$, $i \in \{1,\cdots,n\}$. By the Whitney theorem, $\widehat{A}_{i_1\cdots j\cdots i_k}(x_1,\cdots,x_n) = \underline{a}_{i_1\cdots i_k}(x_1^2,\cdots,x_n^2)$ with $\underline{a}_{i_1\cdots i_k} \in \mathcal{C}^\infty(R^n,\mathbb{R})$.

For all $(x_1,\cdots,x_n) \in \overset{o}{K^n}$, we have that $\underline{a}_{i_1\cdots i_k}(x_1,\cdots,x_n) = a_{i_1\cdots i_k}(x_1,\cdots,x_n)$. Since the k-form $\underline{w} = \sum_{i_1<\cdots<i_k} \underline{a}_{i_1\cdots i_k}(x_1,\cdots,x_n)dx_{i_1}\wedge\cdots\wedge dx_{i_k}$,then $\underline{w} \upharpoonright \overset{o}{K^n} = w \upharpoonright \overset{o}{K^n}$.

For all plot $P : U \to \mathbb{R}^n$, $p^*(w) = w(p)$.

$$\begin{aligned} sq^*(w) &= \sum_{i_1<\cdots<i_k} A_{i_1\cdots i_k}(x_1,\cdots,x_n)dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\ &= \sum_{i_1<\cdots<i_k} 2^k x_{i_1\cdots i_k}\widehat{A}_{i_1\cdots i_k}(x_1,\cdots,x_n)dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\ &= \sum_{i_1<\cdots<i_k} 2^k x_{i_1\cdots i_k}\underline{a}_{i_1\cdots i_k}(x_1^2,\cdots,x_n^2)dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\ &= sq^*(\underline{w} \upharpoonright \mathbb{K}^n) \end{aligned}$$

So, we show that $\underline{w} \upharpoonright K^n = w$. Thus, $sq^*(w - \underline{w} \upharpoonright K^n) = 0$. Furthermore, $w - \underline{w} \upharpoonright K^n = 0$ and w is the restriction on corner of the smooth k-form $\underline{w}$ on $\mathbb{R}$. $\square$

6 CONCLUSION

In this thesis, the conditions of a topological manifold are given and we describe the smooth structure on the topological manifold. We give the definition of a smooth manifold. We show that the smooth manifold is a diffeological space. Manifold with boundary and corners are examined and we see that they are not a smooth manifold.

We want to examine these spaces in terms of differential and we examine them in terms of diffeology. In the light of the results obtained concerning the diffeology of a manifold with boundary and corners, we show that these are diffeological spaces. Furthermore, these are a diffeological manifold.

We give the definition of the differential form on this space and we characterize these forms defined on the manifold with boundary and corners.

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BIOGRAPHICAL SKETCH

Gülşah Bakı, 25 years old. Graduated from İstanbul Yaşar Dedeman High School, received university diploma from Galatasaray University, İstanbul, Mathematics and Lille 1 University, France, Economy.