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THE DECOMPOSITION OF CLASSICAL SEMISIMPLE LIE ALGEBRAS

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THE DECOMPOSITION OF CLASSICAL SEMISIMPLE LIE ALGEBRAS

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LIST OF SYMBOLS

$\mathfrak{g}, \mathfrak{t}$	: Lie algebra
$\mathfrak{gl}_n(\mathbb{C})$	: General linear Lie algebra
$\mathfrak{sl}_n(\mathbb{C})$	: Special linear Lie algebra
$O_n(\mathbb{R})$	: Orthogonal group
V_λ	: Eigenspace for the eigenvalue λ
$[X,Y]$	: Lie bracket of X and Y
Ad_x	: Adjoint representation of a Lie group
ad_X	: Adjoint representation of a Lie algebra
$K(X,Y)$	: Killing form of X and Y
R	: Root system
W	: Weyl group

Abstract

A Lie algebra is a vector space over a field k ($\mathbb{C}$ or $\mathbb{R}$) with a given bilinear operation satisfying the Jacobi identity and its operation is called the Lie bracket. In chapter 1, we observe the structure of the Lie algebras $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{so}_n(\mathbb{C})$, we work on their subalgebras, ideals. After, we explain the nilpotent and solvable Lie algebras using the lower and derived series with examples, non-examples. The end of the chapter 1 we give the definition of the simple and semisimple Lie algebras. In chapter two, we are talking about Sophus Lie, a Norwegian mathematician who lived in the the later of the 19^{th} century.

In chapter three, we give the threorems of Lie and Engel using the notions of the nilpotent and solvable Lie algebras which is given in chapter 1. In the next chapter, we observe the properties of the adjoint representation for a given Lie algebra and we look at its relation with the Lie bracket, i.e. $ad_X(Y) = [X,Y] = XY - YX$. Further we give the definition of the Killing form, which is $K(X,Y) = Tr(ad(X) \circ ad(Y))$. In the end of this chapter, we see a theorem which contains the notions such as semisimple, abelian ideals, nondegenerate and simple ideals.

Later, we give the definitions of a Cartan subalgebra and the root for a given semisimple Lie algebra, we consider the special linear Lie algabras $\mathfrak{sl}(n+1,\mathbb{C})$, a compact real form for it, is $\mathfrak{su}(n+1)$ that is a skew-hermitian matrix. i.e. $(\overline{A})^T = A$. In this way, we construct the root systems for semisimple Lie algebras.

In summary, in this dissertation, we examine the root system decomposition of classical semi-simple Lie algebras with the aim of establishing a relation between the above-mentioned chapters. Furthermore, I calculated explicitly the roots and root vectors of the semisimple Lie algebras $\mathfrak{so}(2n,\ \mathbb{C})$, $\mathfrak{sl}(n+1,\ \mathbb{C})$, $\mathfrak{so}(2n+1,\ \mathbb{C})$, $\mathfrak{sp}(n,\ \mathbb{C})$; added to the study.

Key words : Lie algebra, Lie bracket, nilpotent and solvable Lie algebras, Cartan subalgebra, Killing form, root system.

Özet

Lie cebiri; Lie parantezi olarak adlandırdığımız ikili işlemle birlikte Jacobi eşitliğini sağlayan bir vektör uzayıdır. İlk bölümde $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{gl}_n(\mathbb{C})$ ve $\mathfrak{so}_n(\mathbb{C})$ Lie cebirlerini, onların alt cebirlerini ve ideallerini ele alıyoruz. Sonra tanım ve örneklerle Lie cebirlerinin yapısını anlamaya çalışıyoruz. Devamında nilpotent ve çözülebilir Lie cebirlerinin yapısını örneklerle açıklıyoruz. İlk bölümün sonunda, basit ve yarı-basit Lie cebirlerinin tanımlarını görüyoruz. İkinci bölümde, Lie cebiri kavramının Sophus Lie tarafından ortaya atıldığını ve geliştirildiğini söyleyerek, Sophus Lie hakkında kısa bilgilere yer veriyoruz

Üçüncü bölümde, daha önce ğördüğümüz nilpotent ve çözülebilir Lie cebirlerinin özelliklerini kullanarak Lie ve Engel teoremlerini veriyoruz. Bir sonraki bölümde, verilen bir Lie cebiri için adjoint temsili yazıp, onun Lie parantezi ile ilişkisini inceliyoruz. Örnek verirsek; $X,Y \in \mathfrak{g}$ olacak şekilde, bu adjoint temsili $ad_X(Y) = [X,Y] = XY - YX$ olarak ifade edebiliriz. Bu temsil yardımıyla aşağıda verilen **Killing form** kavramından bahsediyoruz.

$$K(X,Y) = Tr(ad(X) \circ ad(Y)).$$

İlerleyen bölümlerde, verilen bir Lie cebiri için Cartan alt cebiri ve kökler bulup, Lie cebirleri için kök sistemleri oluşturuyoruz. Ayrıca verilen bir yarı-basit Lie cebiri için kompakt reel form bulmamız gerekiyor. Örneğin; Lie cebiri olarak $\mathfrak{sl}(n+1,\mathbb{C})$ alırsak, kullanacağımız kompakt reel form $\mathfrak{su}(n+1)$ olur.

Özetle, bu çalışmada, yukarıda bahsedilen bölümler arasında ilişki kurmayı amaçlayarak, klasik yarı-basit Lie cebirlerinin kök sistem ayrışmasını inceliyoruz. Ayrıca $\mathfrak{so}(2n,\ \mathbb{C})$, $\mathfrak{sl}(n+1,\ \mathbb{C})$, $\mathfrak{so}(2n+1,\ \mathbb{C})$, $\mathfrak{sp}(n,\ \mathbb{C})$ nin kök ve kök vektörlerini açık bir şekilde hesaplayıp, çalışmaya ekledim.

Anahtar Sözcükler : Lie cebiri, Lie'nin parantezi, nilpotent ve çözülebilir Lie cebirleri, Cartan alt cebiri, Killing form, kök sistemi.

1 INTRODUCTION

1.1 Basic Definitions and Examples

Our aim is to study on the Lie algebras and to observe the decompositions of the semisimple Lie algebras. So we will give basic definitions and examples.

Definition 1.1.1. *Let k be a field.* ***An algebra*** *X is a vector space over k together with the following map*

$$\begin{array}{rcl} *: X \times X & \to & X \\ (x,y) & \mapsto & x * y \end{array}$$

which satisfies the following conditions:
i) *$x*(y+z) = x*y + x*z$ and $(x+y)*z = x*z + y*z$ for all $(x,y,z) \in X^3$.*
ii) *$(ax)*(by) = (ab)(x*y)$ for all $(a,b) \in k^2$ and $(x,y) \in X^2$.*

Example 1.1.1. *Let V be a finite dimensional vector space over k and $End_k(V)$ denote the set of linear transformations $V \to V$ over k. Then $End_k(V)$ is a finite dimensional algebra over k with the operation of composition.*

Definition 1.1.2. *A subvector space Y of X which is stable for the multiplication has a natural structure of algebra inherited from the algebra structure of X. Such a subvector space Y is called* ***a subalgebra of X****.*

Remark. Let V be a finite dimensional vector space, let $Hom(V,V)$ denote the ring of all endomorphisms of V. Let $\{e_1, e_2, ..., e_n\}$ be a basis of V. To each $T \in Hom(V,V)$ we associate a matrix

$$M_T = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

where the coefficients are determined by

$$T(e_j) = \sum_{i=1}^{n} a_{ij} e_i \ (1 \leq j \leq n).$$

We call the matrix M_T the matrix representation of T in terms of the basis $\{e_1, e_2, ..., e_n\}$.

Lemma 1.1.1. *The mapping $T \to M_T$ is an isomorphism of $Hom(V,V)$ onto the ring $M_n(k)$ of all matrices in k.*

Definition 1.1.3. ***An upper triangular matrix*** *is a square matrix in which all entries below the main diagonal are zero.* ***A lower triangular matrix*** *is a square matrix in which all entries above the main diagonal are zero. A matrix which is both upper triangular and lower triangular is* ***a diagonal matrix.***

Definition 1.1.4. *A scalar $\lambda \in k$ is called an* ***eigenvalue*** *of the $n \times n$ matrix M if there is a non trivial solution $x \in V$ of $Mx = \lambda x$. Such an* ***x*** *is called an* ***eigenvector*** *corresponding to the eigenvalue λ. The set of the elements $x \in V$ denoted by V_λ and its union with the zero set is called* ***eigenspace*** *of M for the eigenvalue λ. For the identity matrix I, the equation $det(\lambda I - M) = 0$ is called* ***the characteristic equation*** *of M.*

Example 1.1.2. *We have the matrix* $M = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$.

- *Its characteristic equation is $\lambda^2 - 7\lambda + 6 = 0$.*
- *Its eigenvalues are $\lambda = 6$ and $\lambda = 1$.*
- *Its eigenspaces are*

$$V_{\lambda=6} = \{(2t, t) : t \in \mathbb{R}\}$$

$$V_{\lambda=1} = \{(t, -2t) : t \in \mathbb{R}\}.$$

Definition 1.1.5. *An endomorphism $T \in Hom(V,V)$ is called* ***nilpotent*** *if $T^k = 0$ for some integer $k > 0$. Similarly, a matrix N is called* ***nilpotent*** *if $N^m = 0$ for some integer $m > 0$.*

Example 1.1.3. *Consider* $N = \begin{pmatrix} 3 & 4 & -7 \\ 1 & 2 & -3 \\ 2 & 3 & -5 \end{pmatrix} \in M_3(\mathbb{R})$. *Then we find*

$N^2 = \begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ -1 & -1 & 2 \end{pmatrix}$ *and $N^3 = 0$, so N is nilpotent.*

Proposition 1.1.1. *If $T \in Hom(V,V)$ is nilpotent, then T has exactly one eigenvalue, namely* 0.

Proof. We know that any given T we can associate a matrix N and there is an isomorphism between them therefore if T is nilpotent then N is nilpotent. Suppose N is nilpotent by definition, there exists some $k \in \mathbb{N}$ such that $N^k = 0$. Let λ be an eigenvalue of N and let x be the eigenvector corresponding to λ. So, we can say that they satisfy the equality $Nx = \lambda x$. Multiplying this equality by N on the left, we have $N^2x = \lambda Nx = \lambda^2 x$
Now, the claim is that $N^k x = \lambda^k x$.
Proof of the claim: We know $Nx = \lambda x$ and $N^2x = \lambda^2 x$ for some $k \in \mathbb{N}$. We want to prove that $N^k x = \lambda^k x$. Assume that $N^{k-1}x = \lambda^{k-1}x$

$$N^k x = NN^{k-1}x = N\lambda^{k-1}x \ (by\ hypothesis).$$

Since λ^{k-1} is scalar, then we can write

$$\begin{aligned} N\lambda^{k-1}x &= \lambda^{k-1}Nx \\ &= \lambda^{k-1}\lambda x \\ &= \lambda^k x. \end{aligned}$$

We proved that $N^k x = \lambda^k x$.
Now, since $N^k = 0$, we get $N^k x = 0$. By the equality $N^k x = \lambda^k x$, then $\lambda^k x = 0$.
Since x is an eigenvector and hence nonzero by definition. We obtain that $\lambda^k x = 0$ and thus $\lambda = 0$.

Definition 1.1.6. *Let σ be a subset of $Hom(V,V)$. A subspace W of V is called* ***invariant*** *(under σ), if $T(W) \subset W$ for each $T \in \sigma$.*

Example 1.1.4. *Suppose $T \in \sigma \subset Hom(V,V)$.*

- *If $v \in \{0\}$, then $v = 0$ and hence $Tv = 0 \in \{0\}$, since T is homomorphism. Thus, $\{0\}$ is invariant under T.*

- *If $v \in V$, $Tv \in V$. Thus, V is invariant under T.*

- *$KerT = \{v \in V; Tv = 0\}$. If $v \in KerT$, then $Tv = 0$ and $T(Tv) = T(0) = 0$. Hence $Tv \in KerT$. Thus, $KerT$ is invariant under T.*

1.2 Lie Algebra

In this section, first of all, we will see a relation between a Lie group and a Lie algebra giving the definition of a Lie group and its examples.

Definition 1.2.1. *A **Lie group** is a smooth manifold G endowed with a group structure such that the following maps are smooth:*

1. *The group multiplication is for $x, y \in G$*

$$G \times G \to G$$
$$(x,y) \mapsto xy.$$

2. *The inverse map is for $x \in G$*

$$G \to G$$
$$x \mapsto x^{-1}.$$

Example 1.2.1. *We denote the set of $n \times n$ matrices with complex entries by $M(n;\mathbb{C})$ and the general linear group $GL(n;\mathbb{C})$ is the subset of $M(n;\mathbb{C})$, that is*

$$GL(n;\mathbb{C}) = \{X \in M(n;\mathbb{C}) : det(X) \neq 0\}$$

is a Lie group. For a proof, see (Charters, 2008). The other examples for Lie groups

- *The special linear group $SL(n;\mathbb{C})$ defined as follows:*

$$SL(n;\mathbb{C}) = \{X \in GL(n;\mathbb{C}) : det(X) = 1\}.$$

- *The orthogonal group $O(n)$ is:*

$$O(n) = \{X \in GL(n;\mathbb{R}) : X^T = X^{-1}\}.$$

Let G be a Lie group, the tangent space of G at identity is the Lie algebra of G i.e $T_eG \cong \mathfrak{g}$.

Example 1.2.2. *The fact that $GL(n;\mathbb{R})$ is an open subset of $M(n;\mathbb{R}) \cong \mathbb{R}^{n^2}$ also implies that the Lie algbera of $GL(n;\mathbb{R})$, as the tangent space at identity is the set $M(n;\mathbb{R})$ itself, that is,*

$$\mathfrak{gl}(n;\mathbb{R}) = \{X : X \ is \ \ an \ n \times n \ \ real \ \ matrix\}.$$

Another example, consider the special linear group $SL(n;\mathbb{C})$, its Lie algebra is denoted by $\mathfrak{sl}(n;\mathbb{C})$ which is the set of matrices of trace zero.

Notation. We can denote by $\mathfrak{gl}_n(\mathbb{R})$, $\mathfrak{sl}_n(\mathbb{R})$ instead of $\mathfrak{gl}(n;\mathbb{R})$, $\mathfrak{sl}(n;\mathbb{R})$, respectively.

Now, we will see another definition of a Lie algebra which is more algebraically

Definition 1.2.2. *Let k be a field.* ***A Lie algebra*** *is a vector space over k with an operation*

$$\begin{aligned}[.,.]: \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (X,Y) &\mapsto [X,Y]\end{aligned}$$

which satisfies the following axioms:

- $[.,.]$ *is bilinear.*
- $\forall X,Y \in \mathfrak{g}$, $[X,Y] = -[Y,X]$ *(skew-symmetric).*
- *It satisfies the Jacobi identity.*

$$\forall X,Y,Z \in \mathfrak{g},\ [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

We note that the product $[X,Y]$ is called ***a Lie bracket*** *of X and Y in $\mathfrak{g}$.*

Example 1.2.3. *Consider $k = \mathbb{R}$, we write $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$ in $\mathbb{R}^3$. The cross product is defined as*

$$\begin{aligned}\wedge : \mathbb{R}^3 \times \mathbb{R}^3 &\to \mathbb{R}^3 \\ (x,y) &\mapsto x \wedge y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).\end{aligned}$$

$(\mathbb{R}^3, \wedge)$ is a Lie algebra over $\mathbb{R}$.

Example 1.2.4. *Let k be a field, the general linear algebra $\mathfrak{gl}_n(k)$ which is the space $M_n(k)$ of all $n \times n$ matrices with the entries in k, is a Lie algebra. Since*

- $[.,.]$ *is bilinear.*

$$\begin{aligned}\mathfrak{g}l_n(k) \times \mathfrak{g}l_n(k) &\to \mathfrak{g}l_n(k) \\ (A,B) &\mapsto [A,B] = AB - BA\end{aligned}$$

for every $A, B, C \in \mathfrak{g}l_n(k)$, $\lambda \in k$.

$$\begin{aligned}[\lambda A + B, C] &= (\lambda A + B)C - C(\lambda A + B) \\ &= \lambda AC + BC - C\lambda A - CB \\ &= \lambda(AC - CA) + BC - CB \\ &= \lambda[A,C] + [B,C].\end{aligned}$$

Similarly, $[A, \mu B + C] = \mu[A,B] + [A,C]$.
Then, it is bilinear.

- $[A,B] = -[B,A]$ *for every* $A, B \in \mathfrak{gl}_n(k)$. *It is skew-symmetric.*
- *We check that it verifies the Jacobi identity. Since, for all* $a, b, c \in \mathfrak{gl}_n(k)$,

$$\begin{aligned}[A,[B,C]] + [B,[C,A]] + [C,[A,B]] &= [A, BC - CB] + [B, CA - AC] + [c, AB - BA] \\ &= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CA \\ &- CBA - ABC + BAC = 0.\end{aligned}$$

Hence, $\mathfrak{gl}_n(k)$ *is a Lie algebra.*

Example 1.2.5. *The set of upper triangular matrices in* $\mathfrak{gl}_n(k)$ *is*

$$\mathfrak{b}_n(k) = \{A \in \mathfrak{gl}_n(k) | \; x_{ij} = 0, \; if \; i > j \text{ where } x_{ij} \in k, \forall i, j \in \mathbb{N}\}$$

is a Lie algebra under Lie bracket.

Example 1.2.6. *The special linear algebra*

$$\mathfrak{sl}_n(k) = \{A \in \mathfrak{gl}_n(k) : trace(A) = 0\}$$

is a Lie algebra under Lie bracket.

Definition 1.2.3. *Let* $\mathfrak{g}$ *be a Lie algebra. A subset* $\mathfrak{h} \subseteq \mathfrak{g}$ *is a Lie* ***subalgebra*** *if:*

1. $\mathfrak{h}$ *is a vector subspace of* $\mathfrak{g}$.
2. *It preserves the Lie brackets which means that* $[A,B] \in \mathfrak{h}$, $A, B \in \mathfrak{h}$.

Example 1.2.7. $\mathfrak{sl}_n(k)$ *and* $\mathfrak{b}_n(k)$ *are Lie subalgebras of* $\mathfrak{g}l_n(k)$.
For $k = \mathbb{C}$, *we prove that* $\mathfrak{sl}_n(\mathbb{C})$ *is a Lie subalgebra of* $\mathfrak{gl}_n(\mathbb{C})$.

Proof. 1. Let $A,B \in \mathfrak{sl}_n(\mathbb{C})$ then $Tr(A) = Tr(B) = 0$. From the linearity of the trace, we have

$$Tr(A+B) = Tr(A) + Tr(B) = 0.$$

Then, $A+B \in \mathfrak{sl}_n(\mathbb{C})$. Furthermore, $A \in \mathfrak{sl}_n(\mathbb{C})$ for all $\lambda \in \mathbb{C}$, $Tr(\lambda.A) = \lambda.Tr(A)$ thus $\lambda.A \in \mathfrak{sl}_n(\mathbb{C})$. Hence, $\mathfrak{sl}_n(\mathbb{C})$ is a vector subspace $\mathfrak{gl}_n(C)$.

2. Let $A,B \in \mathfrak{sl}_n(\mathbb{C})$ and we know that $Tr(A.B) = Tr(B.A)$. Then,

$$Tr([A,B]) = Tr(A.B) - Tr(B.A) = 0.$$

This gives that $[A,B] \in \mathfrak{sl}_n(\mathbb{C})$. Thus, $\mathfrak{sl}_n(\mathbb{C})$ is a subalgebra of $\mathfrak{gl}_n(\mathbb{C})$.

Example 1.2.8. *Consider the Orthogonal group*

$$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : A^T.A = \mathbb{I}\}.$$

Let $\mathfrak{o}_n(\mathbb{R})$ be the Lie algebra of $O_n(\mathbb{R})$ consisting of skew-symmetric matrices which is, $A^T = -A$, for all $A,B \in \mathfrak{o}_n(\mathbb{R})$

$$\begin{aligned}(AB-BA)^T = B^T.A^T - A^T.B^T &= (-B).(-A) - (-A)(-B)\\ &= -(AB-BA).\end{aligned}$$

Then, $\mathfrak{o}_n(\mathbb{R})$ is closed under the Lie bracket. Hence, $\mathfrak{o}_n(\mathbb{R})$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$.

Definition 1.2.4. *Let $\mathfrak{g}$ be Lie algebra, a subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called* ***an ideal of*** *$\mathfrak{g}$, if,*

$$[A,B] \in \mathfrak{h} \textit{ for all}, \ A \in \mathfrak{h}, \ B \in \mathfrak{g}.$$

Example 1.2.9. *We give some examples of ideal.*

- *$\{0\}$ is an ideal of $\mathfrak{g}$.*
- *The Lie algebra is itself an ideal.*
- *$\mathfrak{sl}_n(k)$ is an ideal of $\mathfrak{gl}_n(k)$.*

Proof. We prove that $\mathfrak{sl}_n(k)$ is an ideal of $\mathfrak{gl}_n(k)$. We saw that $\mathfrak{sl}_n(k)$ is a subalgebra of $\mathfrak{gl}_n(k)$ in 1.2.7. It is enough to show that $[A,B] \in \mathfrak{sl}_n(k)$ for all $A \in \mathfrak{sl}_n(k)$, $B \in \mathfrak{gl}_n(k)$, we have $Tr(A) = 0$ and we get

$$Tr([A,B]) = Tr(AB) - Tr(BA) = 0, \textit{ then } [A,B] \in \mathfrak{sl}_n(k).$$

Then, $\mathfrak{sl}_n(k)$ is an ideal of $\mathfrak{gl}_n(k)$.

Definition 1.2.5. *Let $\mathfrak{g}$ be a Lie algebra, it is **abelian** if $[A,B]=0$ for every A,B in $\mathfrak{g}$.*

Example 1.2.10. *Let $\mathfrak{g}$ be a Lie algebra over a field, every one-dimensional vector subspace of $\mathfrak{g}$ is an abelian Lie subalgebra.*

1.2.1 Derived algebra of a Lie algebra

Definition 1.2.6. *Let $\mathfrak{g}$ be a Lie algebra over a field k and consider in $\mathfrak{g}$ the set $\mathfrak{g}^{'}=[\mathfrak{g},\mathfrak{g}]$. This is the set of elements of the form $[x,y]$; $x,y\in\mathfrak{g}$ and possible linear combinations of such elements. It is called the **derived algebra** of $\mathfrak{g}$.*

Proposition 1.2.1. *Let $\mathfrak{g}^{'}:=[\mathfrak{g},\mathfrak{g}]$ then this is an ideal in $\mathfrak{g}$.*

Proof. The derived algebra is by definition a subspace of $\mathfrak{g}$ since $\mathfrak{g}^{'}=[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{g}$ and we have $[\mathfrak{g}^{'},\mathfrak{g}^{'}]\subset[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}^{'}$. We have then using $\mathfrak{g}^{'}\subset\mathfrak{g}$

$$[\mathfrak{g}^{'},\mathfrak{g}]\subset[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}^{'}$$

Thus $\mathfrak{g}^{'}$ is an ideal in $\mathfrak{g}$.

Definition 1.2.7. *The **lower series** of a Lie algebra $\mathfrak{g}$ is given by:*

$$\mathfrak{g}^1=\mathfrak{g}\supseteq\mathfrak{g}^2=[\mathfrak{g},\mathfrak{g}^1]\supseteq....\supseteq\mathfrak{g}^j=[\mathfrak{g},\mathfrak{g}^{j-1}].... \tag{1.1}$$

*and the **derived series** of a Lie algebra is given by:*

$$\mathfrak{g}^{(0)}=\mathfrak{g}\supseteq\mathfrak{g}^{(1)}=[\mathfrak{g},\mathfrak{g}]\supseteq\mathfrak{g}^{(2)}=[\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}]\supseteq....\supseteq\mathfrak{g}^{(j)}=[\mathfrak{g}^{(j-1)},\mathfrak{g}^{(j-1)})]\supseteq.... \tag{1.2}$$

Proposition 1.2.2. *Let $\mathfrak{g}$ be a Lie algebra, consider 1.1 and 1.2.*
i) *All $\mathfrak{g}^j$ and $\mathfrak{g}^{(j)}$ are ideals of in $\mathfrak{g}$.*
ii) *$\mathfrak{g}^{(j)}\subseteq\mathfrak{g}^j$ for $n\geq 1$.*

Definition 1.2.8. *A Lie algebra is called **nilpotent** if $\mathfrak{g}^j=0$ for some j.*

Example 1.2.11. *Any abelian Lie algebra $\mathfrak{g}$ is nilpotent. Let $a,b\in\mathfrak{g}$, we have $[a,b]=0$ by the definition of the abelian Lie algebra and $\mathfrak{g}^2=[\mathfrak{g},\mathfrak{g}]=0$ since [a,b]=0. Then, $\mathfrak{g}$ is nilpotent.*

Example 1.2.12. *Let k be a field and assume that $\mathfrak{g} = ka + kb$ with $[a,b] = b$, $\mathfrak{g}$ is not nilpotent. Since, $\mathfrak{g}^2 = [\mathfrak{g},\mathfrak{g}] = [ka+kb, ka+kb]$, we have then*

$$\begin{aligned}\mathfrak{g}^2 &= k[a,a] + k[a,b] + k[b,b] + k[b,a] \\ &= k[a,b] + k[b,a] \quad ([a,b] = b \text{ and } [a,a] = [b,b] = 0) \\ &= kb - kb \ \ (k \text{ is a field}) \\ &= kb.\end{aligned}$$

Similarly, we see that $\mathfrak{g}^3 = [\mathfrak{g},\mathfrak{g}^2] = [ka+kb, kb] = k[a,b] + k[b,b] = kb$. We can prove that $\mathfrak{g}^j = kb \neq 0$ by induction. Suppose that $\mathfrak{g}^j = kb$ for $j > 0$, we prove that $\mathfrak{g}^{j+1} = kb$

$$\mathfrak{g}^{j+1} = [\mathfrak{g},\mathfrak{g}^j] = [ka+kb, kb] = kb \neq 0.$$

Hence, $\mathfrak{g} = ka + kb$ is not nilpotent.

Definition 1.2.9. *Suppose that $\mathfrak{g}^{(j)}$ is derived series as 1.2. A Lie algebra is **solvable** if $\mathfrak{g}^{(j)} = 0$ for some j.*

Example 1.2.13. *Any abelian Lie algebra is solvable. Since $a,b \in \mathfrak{g}$, $[a,b] = 0$ and $\mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}]$, we have then $\mathfrak{g}^{(1)} = 0$. Hence, it is solvable Lie algebra.*

Example 1.2.14. *We see that $\mathfrak{g}^{(1)} = \mathfrak{g}^2$ from the definitions of the nilpotent and solvable Lie algebras, and we return the example 1.2.12. We have $\mathfrak{g}^{(1)} = \mathfrak{g}^2 = kb$ and $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}] = [kb,kb] = 0$. Then, it is solvable.*

Definition 1.2.10. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. A **radical** of $\mathfrak{g}$ is a maximal solvable ideal of $\mathfrak{g}$. It is denoted by $R(\mathfrak{g})$.*

1.2.2 Simple and Semisimple Lie algebras

Definition 1.2.11. *A Lie algebra $\mathfrak{g}$ is called **simple** if $\mathfrak{g}$ is non-abelian and has no proper ideals and dim $\mathfrak{g} \geq 2$.*

Corollary 1.2.1. *If $\mathfrak{g}$ is simple then $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ is equal to $\mathfrak{g}$.*

Proof. We have seen that $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$. $\mathfrak{g}$ has to be a trivial ideal since $\mathfrak{g}$ is simple. In this case there are two cases either $\mathfrak{g}' = 0$ or $\mathfrak{g}' = \mathfrak{g}$. But $\mathfrak{g}' \neq 0$ since $\mathfrak{g}$ is non-abelian. Thus $\mathfrak{g}' = \mathfrak{g}$.

Example 1.2.15. *The special linear Lie algebra*

$$\mathfrak{sl}_2(k) = \{A \in \mathfrak{gl}_2(k) : Trace(A) = 0\}$$

is simple. (See e.g. Humphreys, 1972).

Definition 1.2.12. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra.*

1. *$\mathfrak{g}$ is **semisimple** if it has no a nonzero solvable ideals.*
2. *$\mathfrak{g}$ is **semisimple** if $R(\mathfrak{g}) = 0$ where R is a radical of $\mathfrak{g}$ which is a solvable ideal of $\mathfrak{g}$ of maximal possible dimension.*

Example 1.2.16. *Semisimple Lie algebras over $\mathbb{C}$:*

- $\mathfrak{sl}_n(\mathbb{C}) = \{A \in \mathfrak{gl}_n(\mathbb{C}) |\ Tr(A) = 0\} \qquad for\ n \geq 2.$
- $\mathfrak{so}_n(\mathbb{C}) = \{A \in \mathfrak{gl}_n(\mathbb{C}) |\ A + A^t = 0\} \qquad for\ n \geq 3.$

Semisimple Lie algebras over $\mathbb{R}$:

- $\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathfrak{gl}_n(\mathbb{R}) |\ Tr(A) = 0\} \qquad for\ n \geq 2$
- $\mathfrak{so}(p,q) = \{A \in \mathfrak{gl}_{p+q}(\mathbb{R}) |\ A^* I_{p,q} + I_{p,q}\, A = 0\} \qquad for\ p+q \geq 3$

$$where \quad \mathbb{I}_{p,q} = \begin{bmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{bmatrix}.$$

- $\mathfrak{su}(p,q) = \{A \in \mathfrak{sl}_{p+q}(\mathbb{R}) |\ A^* I_{p,q} + I_{p,q}\, A = 0\} \qquad for\ p+q \geq 2$

1.2.3 Lie Algebras Homomorphism

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be any two Lie algebras over $\mathbb{C}$. A homomorphism of Lie algebras between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is a function $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

1. ψ is a linear map, $\psi(\lambda.X + \mu.Y) = \lambda.\psi(X) + \mu.\psi(Y)$ where $X, Y \in \mathfrak{g}_1$ and $\lambda, \mu \in \mathbb{C}$
2. $\psi[X,Y]_{\mathfrak{g}_1} = [\psi(X), \psi(Y)]_{\mathfrak{g}_2}$ where $[.,.]_{\mathfrak{g}_1}$ is the Lie bracket of $\mathfrak{g}_1$ and $[.,.]_{\mathfrak{g}_2}$ is the Lie bracket of $\mathfrak{g}_2$

Example 1.2.17. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be any two Lie algebras, the function $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\psi(X) = 0$ for all $X \in \mathfrak{g}_1$ is a Lie algebra homomorphism.*

Definition 1.2.13. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be any two Lie algebras, the function $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra isomorphism if ψ is homomorphism and bijective.*

Proposition 1.2.3. *Let V be a vector space of dimension n, $\mathfrak{gl}(V)$ is isomorphic to $\mathfrak{gl}_n(\mathbb{C})$ which means that any linear map can be written as a matrix.*

Proof. Take a basis in $\{e_1, e_2, ..., e_n\}$ *in* V. For all $T \in \mathfrak{gl}(V)$. Consider its matrix $A = A_T = [a_{ij}]_{i,j=1}^n$. We can describe

$$Te_j = \sum_{i=1}^{n} a_{ij} e_j, \quad \forall j = 1, 2, ..., n. \tag{1.3}$$

Define

$$\begin{aligned} \psi : \mathfrak{gl}(V) &\to \mathfrak{gl}_n(\mathbb{C}) \\ T &\mapsto \psi(T) = A_T. \end{aligned}$$

We show that ψ is isomorphism.

- We show that ψ is a homomorphism. First ψ is linear, for all $T, S \in \mathfrak{gl}(V)$ and $\lambda, \mu \in \mathbb{C}$, $\psi(\lambda T + \mu S) = \lambda\, \psi(T) + \mu\, \psi(S)$ since T and S are linear maps. Now, we look at $\psi([T,S]_{\mathfrak{gl}(V)})$

$$\begin{aligned} \psi([T,S]_{\mathfrak{gl}(V)}) &= \psi(TS - ST) \\ &= \psi(TS) - \psi(ST) \\ &= \psi(T)\psi(S) - \psi(S)\psi(T) \\ &= [\psi(T), \psi(S)]_{\mathfrak{gl}_n(\mathbb{C})} \end{aligned}$$

  Then, ψ is a homorphism.

- We want to prove that ψ is bijective. First, we prove that ψ is injective. For $T, S \in \mathfrak{gl}(V)$ and consider a basis $\{e_1, e_2, ..., e_n\}$ in V and we have

$$\begin{aligned} \psi(Te_j) &= \psi(Se_j) \\ a_{11}e_1 + a_{12}e_2 + ... + a_{nn}e_n &= b_{11}e_1 + b_{12}e_2 + ... + b_{nn}e_n \\ (a_{11} - b_{11})e_1 + (a_{12} - b_{12})e_2 + ... + (a_{nn} - b_{nn})e_n &= 0e_1 + 0.e_2 + ... + 0e_n. \end{aligned}$$

  Since $\{e_1, e_2, ..., e_n\}$ *is a basis and it is linearly independent.*

  We see that $a_{11} = b_{11}, a_{12} = b_{12}, ..., a_{nn} = b_{nn}$. ψ is injective. Now, we show that ψ is surjective. For $A \in \mathfrak{gl}_n(\mathbb{C})$ which means that $A = (a_{ij})$ and by 1.3 ψ is surjective. Then, ψ is bijective. Hence, $\mathfrak{gl}(V)$ is isomorphic to $\mathfrak{gl}_n(C)$.

2 LITERATURE REVIEW

Lie algebras are named after Marius Sophus Lie, a Norwegian mathematician who lived between 1842 and 1899. His first mathematical work which is Repräsentation der Imaginären der Plangeometrie, published in 1869. He won the medal of Lobaçevski with his mathematical studies in 1897.

He was interested in continuous symmetries of geometric objects called manifolds and the element of Lie algebras using their derivatives. It was not only important on mathematics, contributed twentieth century mathematical physics.

We have a vast algebraic theory studying objects as Lie algebras, Root systems, Weyl groups etc.

Sophus Lie described simple and semisimple Lie algebras but Elie Cartan and Wilhelm Killing completed it with the new notions (i.e. Killing form, root vector, Cartan subalgebra). They gave the structure of some semisimple Lie algebras with its properties and rules using these structures they classified classical semisimple Lie algebras $\big(\mathfrak{sl}(n;\mathbb{C}), \mathfrak{so}(2n;\mathbb{C}); \mathfrak{so}(2n+1;\mathbb{C})\big)$.

Nowadays mathematicians use in their researchs the results of the given theorems and propositions.

3 THE THEOREMS OF LIE AND ENGEL

3.1 The Theorem of Lie

Theorem 3.1.1. *Let k be a field and assume that $\mathfrak{g}$ is a solvable Lie algebra over k. Let $V \neq \{0\}$ be a finite dimensional vector space over $\tilde{k}$, the algebraic closure of k. Consider a homomorphism π of $\mathfrak{g}$ into $\mathfrak{gl}(V)$. Then there exists a vector $v \neq 0$ in V which is an eigenvector of all the $\pi(\mathfrak{g})$.* $\big($*See e.g. (Helgason, 1978)* $\big)$

Proof. We will prove the theorem by induction on dim $\mathfrak{g}$. If dim $\mathfrak{g} = 1$ the theorem is consequence of **proposition 1** in the appendix; we suppose that the theorem holds for all Lie algebra over k of dimension $<$ dim $\mathfrak{g}$. Let $\mathfrak{h}$ be an ideal in $\mathfrak{g}$ of codimension 1. Since $\mathfrak{g}$ is solvable then $\mathfrak{g}^{(i)} = 0$ for some i and if $\mathfrak{h} \subset \mathfrak{g}$, then $\mathfrak{h}^{(i)} \subset \mathfrak{g}^{(i)} = 0$, we have $\mathfrak{h}^{(i)} = 0$ thus $\mathfrak{h}$ is solvable. Also the eigenvalue condition holds for $\mathfrak{h}$ if it holds for $\mathfrak{g}$. By inductive hypothesis we can choose $e \in V$ with $\pi(H)e = \lambda(H)e$ for all $H \in \mathfrak{h}$, where $\lambda(H)$ is scalar-valued function defined for $H \in \mathfrak{h}$.
Fix $X \in \mathfrak{g}$ with $\mathfrak{g} = kX + \mathfrak{h}$ and $X \notin \mathfrak{h}$, we define recursively

$$e_{-1} = 0, \quad e_0 = e, \quad e_p = \pi(X)e_{p-1}$$

and let $E = span\{e_0, e_1, ..., e_p, ...\}$ then $\pi(X)E \subseteq E$. Let v be an eigenvector for $\pi(X)$ in E, we show that v is an eigenvector for each $\pi(H),\ H \in \mathfrak{h}$.
First we show that

$$\pi(H)e_p \equiv \lambda(H)e_p \ \ mod\ span\{e_0, ..., e_{p-1}\} \tag{3.1}$$

We do so by induction on p. Formula 3.1 is valid for $p = 0$ using $e_0 = e$. Suppose that formula 3.1 is valid for p and we need to prove that it is valid for p+1. We have then

$$
\begin{aligned}
\pi(H)e_{p+1} &\equiv \lambda(H)e_{p+1} \qquad (e_{p+1} = \pi(X)e_p)\\
&= \lambda(H)\ \pi(X)e_p \qquad (\lambda(H) \equiv \pi(H))\\
&= \pi(H)\ \pi(X)e_p\\
&= \pi(H)\ \pi(X)e_p - \pi(X)\ \pi(H)e_p + \pi(X)\ \pi(H)e_p\\
&= \pi([H,X])e_p + \pi(X)\ \pi(H)e_p\\
&\equiv \lambda([H,X])e_p + \pi(X)\ \pi(H)e_p \qquad mod\ span\{e_0,...,e_{p-1}\}\\
&\equiv \lambda([H,X])e_p + \lambda(H)\ \pi(X)e_p \qquad mod\ span\{e_0,...,e_{p-1},\pi(X)e_0,...,\pi(X)e_{p-1}\}\\
&\equiv \lambda(H)\pi(X)e_p \qquad mod\ span\{e_0,...,e_p\}\\
&\equiv \lambda(H)e_{p+1} \qquad mod\ span\{e_0,...,e_p\}.
\end{aligned}
$$

This proves 3.1 for p+1 and completes the induction. Now, we show that

$$\lambda([H,X]) = 0 \quad \text{for all} \ \ H \in \mathfrak{h} \tag{3.2}$$

In fact, (3.1) says that $\pi(H)E \subseteq E$ and that, relative to the basis $e_0, e_1, ...$ the linear transformation $\pi(H)$ has matrix

$$\pi(H) = \begin{pmatrix} \lambda(H) & & & * \\ & \lambda(H) & & \\ & & \ddots & \\ 0 & & & \lambda(H) \end{pmatrix}.$$

Thus $Tr(\pi(H)) = \lambda(H)\ dim\ E$, we obtain

$$\lambda([H,X])\ dimE = Tr\ \pi([H,X]) = Tr[\pi(H),\pi(X))] = 0$$

Since the field have characteristic 0, we obtain that $\lambda([H,X]) = 0$, (3.2) follows.
Now we can sharpen (3.1) to

$$\pi(H)e_p = \lambda(H)e_p \qquad for\ all\ H \in \mathfrak{h} \tag{3.3}$$

To prove (3.3), by induction on p. If $p = 0$, the formula is the definition of e_0.

Assume that 3.3 is valid for p. Then

$$\begin{aligned}
\pi(H)e_{p+1} &= \pi(H)\pi(X)e_p \\
&= \pi([H,X])e_p + \pi(X)\pi(H)e_p \\
&= \lambda([H,X])e_p + \pi(X)\lambda(H)e_p && \textit{by induction} \\
&= 0 + \lambda(H)e_{p+1} && \textit{by } 3.2.
\end{aligned}$$

This completes the induction and proves 3.3. Because of 3.3, $\pi(H)x = \lambda(H)x$ for all $x \in E$ and in particular for $x = v$. Hence the eigenvector v of $\pi(X)$ is also an eigenvector of $\pi(\mathfrak{h})$. The theorem follows.

Corollary 3.1.2. *Let $\mathfrak{g}$ be a solvable Lie algebra over a field k and π a representation of $\mathfrak{g}$ on a finite dimensional vector space $V \neq \{0\}$ over $\tilde{k}$, the algebraic closure of k. Then there exists a basis $\{e_1,...,e_n\}$ of V, in terms of which all the endomorphisms $\pi(X)$, $X \in \mathfrak{g}$ are expressed by upper triangular matrices.*

3.2 The Theorem of Engel

Theorem 3.2.1. *Let V be a nonzero finite dimensional vector space k and let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$ consisting of nilpotent elements. Then,*
i) *$\mathfrak{g}$ is nilpotent.*
ii) *There exists a vector $v \neq 0$ in V such that $Zv = 0$ for all $Z \in \mathfrak{g}$.*
iii) *There exists a basis $\{e_1,...,e_n\}$ of V in terms of which all the endomorphisms $X \in \mathfrak{g}$ are expressed by matrices with zeros on and below the diagonal.* (*See (Helgason,1978)*)

Proof. **i)** We show that $\mathfrak{g}$ is nilpotent. For $Z \in \mathfrak{gl}(V)$ consider the endomorphisms L_Z and R_Z given by

$$L_Z X = ZX \ \ \textit{and} \ \ R_Z X = XZ \quad X \in gl(V).$$

Furthermore let $Z_1, Z_2, X \in \mathfrak{gl}(V)$, we have

$$\begin{aligned}
L_{Z_1}R_{Z_2}X &= L_{Z_1}XZ_2 = Z_1XZ_2 \\
R_{Z_2}L_{Z_1}X &= R_{Z_2}Z_1X = Z_1XZ_2.
\end{aligned}$$

Since $L_{Z_1}R_{Z_2}X = R_{Z_2}L_{Z_1}X$ then L_Z and R_Z commute. If *ad* denotes the adjoint representation of $\mathfrak{gl}(V)$, we have $ad\ Z = L_Z - R_Z$. It follows that for $X \in \mathfrak{g}$ and any integer $p \geq 0$

$$(ad\ Z)^p(X) = \sum_{i=0}^{p}(-1)^i \binom{p}{i} Z^{p-i}\, XZ^i. \tag{3.4}$$

Suppose $Z \in \mathfrak{g}$ then Z is nilpotent we have then $Z^k = 0$ and there are $(p+1)$ terms in the relation 3.4. Also the power of Z decrease up to $(p+1)/2)$ and after that it continues to increase. In this case we can find a number p such that $k = (p+1)/2$. Therefore ad Z is nilpotent by relation 3.4 and being Z is nilpotent. Since $ad_{\mathfrak{g}}\, Z$ is the restriction of ad Z to $\mathfrak{g}$, it follows that $ad_{\mathfrak{g}}\, Z$ is nilpotent. Thus $\mathfrak{g}$ is nilpotent.

ii) Let $r = dim\ \mathfrak{g}$, we shall induct on r. If $r = 1$, $Zv = 0$ since $Z \in \mathfrak{g}$ and Z is nilpotent. Assume that **(ii)** holds for algebras of dim < r. Let $\mathfrak{h}$ be a proper subalgebra of $\mathfrak{g}$ of maximum dimension. If $H \in \mathfrak{h}$, then by **(i)**, $ad_{\mathfrak{g}}H$ is a nilpotent endomorphism of $\mathfrak{g}$ and maps $\mathfrak{h}$ into itself, hence $ad_{\mathfrak{g}}H$ induces a nilpotent endomorphism H^* on the vector space $\mathfrak{g}/\mathfrak{h}$. The set $\{H^* : H \in \mathfrak{h}\}$ is a subalgebra of $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ having dimension < r and consisting of nilpotent elements. Using the induction hypothesis we conclude that there exists an element $X \in \mathfrak{g}$, $X \notin \mathfrak{h}$, such that $ad_{\mathfrak{g}}H(X) \in \mathfrak{h}$ for all $H \in \mathfrak{h}$. The subspace $\mathfrak{h} + kX$ of $\mathfrak{g}$ coincide with $\mathfrak{g}$. Thus $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

Now let W be the subspace of V given by

$$W \ = \ \{e \in V : He = 0\ for\ all\ H \in \mathfrak{h}\}.$$

Owing to the induction hypothesis, $W \neq \{0\}$. Moreover, if $e \in W$ we have

$$HXe = [H,X]e \ + \ XHe \ = 0 \tag{3.5}$$

so $X.W \subset W$. By the definion of W and by 3.5, $Xe \in W$. The restriction of X to W is nilpotent, we have then $X^k e = 0$ and $X\ X^{k-1}e = 0$. If we choose $v = X^{k-1}e \neq 0$, we obtain that $Xv = 0$. This vector v has property required in **(ii)**.

(iii) Let e_1 be any vector in V such that $e_1 \neq 0$ and $Ze_1 = 0$ for all $Z \in \mathfrak{g}$. Let E_1 be the subspace of V spanned by e_1. Then each $Z \in \mathfrak{g}$ induces a nilpotent endomorphism Z^* of the vector space V/E_1. If $V/E_1 \neq \{0\}$ we can select $e_2 \in V$, $e_2 \notin E_1$ such that $e_2 + E_1 \in V/E_1$ is annihilated by all $Z^*, (Z \in \mathfrak{g})$. Continuing in this manner we find a basis $e_1, ..., e_n$ of V such that for each $Z \in \mathfrak{g}$

$$Ze_1 = 0, \quad Ze_i = 0 \quad mod(e_1, ..., e_{i-1}), \quad 2 \leq i \leq n. \tag{3.6}$$

The matrix expressing Z in terms of the basis $e_1, ..., e_n$ has zeros on and below the diagonal.

Corollary 3.2.2. *In the notation of Theorem 3.2.1 we have*

$$X_1X_2...X_s = 0$$

if $s \geq dimV$ *and* $X_i \in \mathfrak{g}(1 \leq i \leq s)$.

In fact, this is an immediate consequence of (**3.6**).

4 ADJOINT REPRESENTATION AND THE KILLING FORM

Definition 4.0.1. *Let $\mathfrak{g}$ be a Lie algebra over a field k then a* ***derivation*** $D : \mathfrak{g} \to \mathfrak{g}$ *is a linear map which satisfies the Leibniz rule*

$$D(XY) = D(X)Y + XD(Y) \quad \textit{for all } X, Y \in \mathfrak{g}.$$

Furthermore, Der($\mathfrak{g}$) the vector space of all derivations of $\mathfrak{g}$ is a Lie algebra whose Lie bracket is given by the commutator bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ for all $D_1, D_2 \in Der(\mathfrak{g})$. We define a very important derivation known as the adjoint operator.

Definition 4.0.2. *Let $\mathfrak{g}$ be a Lie algebra over k and $X \in \mathfrak{g}$. The following application* ***ad*** *is called* ***adjoint homomorphism*** *which is defined as*

$$\begin{array}{rcl} ad_X : \mathfrak{g} & \to & \mathfrak{g} \\ Y & \mapsto & ad_X(Y) := [X, Y]. \end{array}$$

Proposition 4.0.1. *For any Lie algebra $\mathfrak{g}$ we have $ad_X \in Der(\mathfrak{g})$ for all $X \in \mathfrak{g}$.*

Proof. Let $X \in \mathfrak{g}$, we show that ad_X is linear. For any $\alpha, \beta \in k$ and $Y, Z \in \mathfrak{g}$ we have

$$ad_X(\alpha Y + \beta Z) = [X, \alpha Y + \beta Z] = \alpha.[X, Y] + \beta.[X, Z] = \alpha.ad_X(Y) + \beta.ad_X(Z)$$

Hence, the map is linear. We now show that this map satisfies the Leibniz rule using the Jacobi identity. We recall the Jacobi identity as follows

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

For all $X, Y, Z \in \mathfrak{g}$ we have

$$\begin{aligned} ad_X([Y,Z]) = [X,[Y,Z]] &= -[Y,[Z,X]] - [Z,[X,Y]] \\ &= [Y,[X,Z]] + [[X,Y],Z] \\ &= [ad_X(Y),Z] + [Y,ad_X(Z)]. \end{aligned}$$

Then $ad_X \in Der(\mathfrak{g})$.

Definition 4.0.3. *Let $\mathfrak{g}$ be a Lie algebra over a field k and let V be a vector space Let $ad : \mathfrak{g} \to \mathfrak{gl}(V)$ be the adjoint representation and the **Killing form** is defined by*

$$\begin{aligned} K : \mathfrak{g} \times \mathfrak{g} &\to k \\ (A,B) &\mapsto K(A,B) = Tr(ad(A) \circ ad(B)) \end{aligned}$$

Remark. It is associative which means that $K([A,B],C) = K(A,[B,C])$.

Proof. To prove remark above, we use the definition of K, for all $A,B,C \in \mathfrak{g}$,

$$K\big([A,B],C\big) - K\big(A,[B,C]\big) = Tr\big(ad[A,B] \circ ad(C) - ad(A) \circ ad[B,C]\big).$$

Using

$$ad([A,B]) = [ad(A),ad(B)] = ad(A) \circ ad(B) - ad(B) \circ ad(A).$$

We get

$$\begin{aligned} K\big([A,B],C\big) - K\big(A,[B,C]\big) &= Tr\big(ad(A) \circ ad(B) \circ ad(C) - ad(B) \circ ad(A) \circ ad(C) \\ &\quad - ad(A) \circ ad(B) \circ ad(C) + ad(A) \circ ad(C) \circ ad(B) \\ &= Tr(ad(A) \circ ad(C) \circ ad(B) - ad(B) \circ ad(A) \circ ad(C)) \end{aligned}$$

Let $X = ad(A) \circ ad(C)$ and $Y = ad(B)$, we know also that $Tr(XY) = Tr(YX)$. Then, $K\big([A,B],C\big) - K\big(A,[B,C]\big) = Tr(XY - YX) = 0$
Hence, $K\big([A,B],C\big) = K\big(A,[B,C]\big)$.

Example 4.0.1. *We consider $\mathfrak{sl}_2(\mathbb{C})$ the special linear algebra. It has three basis vectors A,B,C where $[A,B] = 2C$*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The relations between the brackets are as follows:

$$[A,C] = 2B\ ;\ [B,C] = 2A.$$

We want to find the matrix representation for these basis vectors

- *For* ad_A,

$$\begin{aligned} ad_A(A) &= [A,A] = 0 = 0.A + 0.B + 0.C \\ ad_A(B) &= [A,B] = 2C = 0.A + 0.B + 2.C \\ ad_A(C) &= [A,C] = 2B = 0.A + 2.B + 0.C\ . \end{aligned}$$

  Then, the matrix of ad_A *with respect to the basis* $\{A,B,C\}$ *is ;*

$$M_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}.$$

- *For* ad_B,

$$\begin{aligned} ad_B(A) &= [B,A] = -2H = 0.A + 0.B - 2.C \\ ad_B(B) &= [B,B] = 2C = 0.A + 0.B + 0.C \\ ad_B(C) &= [B,C] = 2X = 2.A + 0.B + 0.C. \end{aligned}$$

  Then, the matrix of ad_B *with respect to the basis* $\{A,B,C\}$ *is*

$$M_B = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

- *For* ad_C,

$$\begin{aligned} ad_C(A) &= [C,A] = -2B = 0.A + -2.B + 0.C \\ ad_C(B) &= [C,B] = -2A = -2.A + 0.B + 0.C \\ ad_C(C) &= [C,C] = 2C = 2.A + 0.B + 0.C \end{aligned}$$

Then, the matrix of ad_C is

$$M_C = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, we calculte the Killing forms for M_A, M_B, M_C.

$$M_A.M_A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$K(A,A) = Tr(ad(A) \circ ad(A)) = 8$.

$$M_B.M_B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$K(B,B) = Tr(ad(B) \circ ad(B)) = -8$.
By the same method, $K(C,C) = 8,\ K(A,B) = 0,\ K(A,C) = 0,\ K(B,C) = 0$.

Definition 4.0.4. *Let $\mathfrak{g}$ be a Lie algebra over k and let $K : \mathfrak{g} \times \mathfrak{g} \to k$ be the Killing form. The kernel of K is defined by*

$$Ker(K) = \{X \in \mathfrak{g} : \forall\, Y \in \mathfrak{g},\ K(X,Y) = 0\}.$$

The form is non-degenerate if its kernel is zero.

Example 4.0.2. *Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}$ we have $Tr(X) = Tr(Y) = 0$ $K(X,Y) = Tr(ad(X) \circ ad(Y)) = Tr(M_X.M_Y)$ where M_X and M_Y are two matrices corresponding to $ad(X)$ and $ad(Y)$, respectively.*

$$M_X = \begin{pmatrix} x_1 & x_2 & 0 \\ x_3 & -x_1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ M_Y = \begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By definition of $Ker(K)$, X in $\mathfrak{g}$ and for every Y in $\mathfrak{g}$ we have $K(X,Y) = 0$ i.e.

$$K(X,Y) = Tr(ad(X) \circ ad(Y)) = Tr(M_X.M_Y) = 0. \tag{4.1}$$

From (4.1), we have the following equation

$$2ax_1 + cx_2 + bx_3 = 0.$$

If we choose $a = 0$ *and* $c = 0$*, we have then* $bx_3 = 0$ *for every* $b \in \mathbb{C}$*. So we see that* $x_3 = 0$*. Similarly, we have* $x_1 = x_2 = 0$*. Then,* $Ker(K)$ *is zero. Thus the Killing form is non-degenerate.*

Proposition 4.0.2. *Let* $\{X_1, X_2, ..., X_n\}$ *be a basis of* $\mathfrak{g}$*. The Killing form K is non-degenerate if and only if the* $n \times n$ *matrix whose* $(i, j)^{th}$ *entry* $K(X_i, X_j)$ *has nonzero determinant.*

Example 4.0.3. *We compute the Killing form of* $\mathfrak{sl}_2(\mathbb{C})$*, (its characteristic* $\neq 2$*) using the standart basis in the example 4.0.1, which we write in the order* (A, B, C)*. The matrices for* ad_A*,* ad_B*,* ad_C *are* M_A*,* M_B*,* M_C*, respectively and as follows:*

$$M_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \quad M_B = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad M_C = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore K has the following matrix, with determinant $-512 \neq 0$

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

So by the proposition 4.0.2, K is non-degenerate.

Lemma 4.0.1. *Let K be a Killing form of* $\mathfrak{g}$ *and let* ψ *be any automorphism of* $\mathfrak{g}$*. Then, K is invariant under* ψ*.*

Proof. We know that any automorphism is a linear transformation $\psi : \mathfrak{g} \to \mathfrak{g}$ that respect the bracket. From this, we obtain $ad(A)(X) = [A, X]$ for all $A, X \in \mathfrak{g}$.
We get

$$\begin{aligned} ad(\psi A)(X) = [\psi A, X] &= [A, \psi^{-1} X] \\ &= \psi\big(ad(A) \circ \psi^{-1}(X)\big) \\ &= \psi \circ ad(A) \circ \psi^{-1}(X). \end{aligned}$$

Similarly, $ad(\psi B)(X) = \psi \circ ad(B) \circ \psi^{-1}(X)$.

$$
\begin{aligned}
ad(\psi A)\circ ad(\psi B) &= \psi\circ ad(A)\circ\psi^{-1}\circ\psi\circ ad(B)\circ\psi^{-1}\\
&= \psi\circ ad(A)\circ ad(B)\circ\psi^{-1}.
\end{aligned}
$$

Also, we know that if two matrices are equivalents, then their traces are equals. So,

$$
\begin{aligned}
Tr\big(ad(\psi A)\circ ad(\psi B)\big) &= Tr\big(\psi\circ ad(A)\circ ad(B)\circ\psi^{-1}\big)\\
&= Tr\big(ad(A)\circ ad(B)\big).
\end{aligned}
$$

Therefore, $K(\psi A\circ\psi B)=K(A,B)$.
Hence, the Killing form K is invariant under a given automorpism ψ.

Proposition 4.0.3. *The kernel of the Killing form K of $\mathfrak{g}$ is an ideal.*

Proof. Assume that $A\in\mathfrak{g}$ and $B\in Ker(K)$. We want to show that $[A,B]\in Ker(K)$. We have then $K\big([A,B],C\big)=K\big(A,[B,C]\big)=0$ for all $C\in\mathfrak{g}$.
Then, $[A,B]\in Ker(K)$. Thus, Ker(K) is an ideal.

Lemma 4.0.2. *Every abelian ideal in $\mathfrak{g}$ is contained in the $Ker(K)$ where K is the Killing form of $\mathfrak{g}$.*

Proof. Assume that $I\subset\mathfrak{g}$ is an abelian ideal. We take $X\in I$, $Y\in\mathfrak{g}$ then the endomorphism $ad(X)\circ ad(Y)$ sends $\mathfrak{g}$ into I and $ad(X)\big(ad(Y)(I)\big)\subseteq ad(X)(I)=0$, then $\big(ad(X)\circ ad(Y)\big)^2=0$ and this endomorphism is nilpotent. Since nilpotent endomorphism have trace zero, $K(X,Y)=Tr(ad(X)\circ ad(Y))=0$, this shows that $I\subseteq Ker(K)$.

Lemma 4.0.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra. If I is an ideal of $\mathfrak{g}$ then there is an ideal $I^{\perp}$ of $\mathfrak{g}$ such that $\mathfrak{g}=I\oplus I^{\perp}$.*

Proof. We define a subspace of $\mathfrak{g}$ as follows

$$I^{\perp}=\{X\in\mathfrak{g}:\ K(X,Y)=0 \text{ for all } Y\in I\}$$

which is also an ideal, since $X\in I^{\perp}$, $Y\in\mathfrak{g}$ and $Z\in I$, we have

$$K([X,Y],Z)=K(X,[Y,Z])=0.$$

Theorem 1 in the appendix (Cartan criterion) shows that the ideal $I\cap I^{\perp}$ of $\mathfrak{g}$ is solvable hence it is 0. Therefore, since $dim\,I+dim\,I^{\perp}=dim\mathfrak{g}$. We must have $\mathfrak{g}=I\oplus I^{\perp}$.

Theorem 4.0.4. *Let $\mathfrak{g}$ be a Lie algebra over a field k. The followings are equivalent for $\mathfrak{g}$.*

1. *$\mathfrak{g}$ is semisimple.*

2. *$\mathfrak{g}$ has no nonzero abelian ideals.*

3. *The Killing form of $\mathfrak{g}$ is nondegenerate.*

4. *$\mathfrak{g}$ is a direct sum of simple ideals.*

Proof. $(1 \Rightarrow 2)$ Assume that $\mathfrak{g}$ is semisimple, by definition then $\mathfrak{g}$ has no nonzero solvable ideal and also every abelian ideals are solvable. Thus $\mathfrak{g}$ has no nonzero abelian ideals.

$(2 \Rightarrow 3)$ Suppose that $\mathfrak{g}$ has no nonzero abelian ideals, i.e. its only abelian ideal is zero. Also by the lemma 4.0.2, every abelian ideal in $\mathfrak{g}$ is the kernel of the Killing form i.e. $Ker(K) = \{0\}$. Thus the Killing form of $\mathfrak{g}$ is nondegenerate.

$(3 \Rightarrow 4)$ It is the consequence of the lemma (4.0.3).

$(4 \Rightarrow 1)$ Assume that $\mathfrak{g}$ is a direct sum of simple ideals, we will prove that $\mathfrak{g}$ is semisimple. Suppose I is a nonzero ideal of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus ..... \oplus \mathfrak{g}_k$ where $\mathfrak{g}_k$'s are simple ideals. Let $0 \neq x = x_1 + x_2 + ... + x_k \in I$ where $x_k \in \mathfrak{g}_k$. We have then

$$\begin{aligned} [x, \mathfrak{g}_k] &= [x_1 + x_2 + ... + x_k, \mathfrak{g}_k] \\ &= [x_1, \mathfrak{g}_k] + ... + [x_k, \mathfrak{g}_k] \neq 0 \end{aligned}$$

which implies that $0 \neq [I, \mathfrak{g}_k]$. Since $\mathfrak{g}_k$ is simple ideal, we get $\mathfrak{g}_k = [I, \mathfrak{g}_k] \subseteq I \neq 0$. We see that I is not an abelian ideal of $\mathfrak{g}$. It means that $\mathfrak{g}$ has no abelian ideals, then $\mathfrak{g}$ is semisimple.

5 ROOT SYSTEMS

5.1 Cartan Subalgebra

Definition 5.1.1. *If $\mathfrak{g}$ is a complex semisimple Lie algebra then a Cartan subalgebra of $\mathfrak{g}$ is a complex subspace $\mathfrak{h}$ of $\mathfrak{g}$ with the following conditions*

- For all H_1 and H_2 in $\mathfrak{h}$, $[H_1, H_2] = 0$
  It means that $\mathfrak{h}$ is commutative subalgebra of $\mathfrak{g}$.
- If, for some $X \in \mathfrak{g}$ we have $[H, X] = 0$ for all $H \in \mathfrak{h}$ then X is in $\mathfrak{h}$
  This says that $\mathfrak{h}$ is a maximal commutative subalgebra.
- ad_H is diagonalizable for all H in $\mathfrak{h}$ by **proposition 1** in the appendix.

Proposition 5.1.1. *Let $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ be a complex semisimple Lie algebra and let $\mathfrak{t}$ be any maximal commutative subalgebra of $\mathfrak{k}$. Define $\mathfrak{h} \subset \mathfrak{g}$ by*

$$\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} + i\mathfrak{t}.$$

Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. See e.g (Hall,2015)

Proof. First, we show that $\mathfrak{h}$ is commutative subalgebra of $\mathfrak{g}$.
Say $X_1, X_2, Y_1, Y_2 \in \mathfrak{k}$. We see that $H_1 = X_1 + iX_2$ and $H_2 = Y_1 + iY_2$ in $\mathfrak{h}$.
We have

$$\begin{aligned}[H_1, H_2] &= [X_1 + iX_2, Y_1 + iY_2] \\ &= [X_1, Y_1] + i[X_1, Y_2] + i[X_2, Y_1] - [X_2, Y_2].\end{aligned}$$

Since $\mathfrak{t}$ is commutative subalgebra of $\mathfrak{g}$, we have then

$$[X_1, Y_1] = [X_1, Y_2] = [X_2, Y_1] = [X_2, Y_2] = 0.$$

We obtain $[H_1, H_2] = 0$. So, $\mathfrak{h}$ is commutative subalgebra of $\mathfrak{g}$.

Now, we must show that $\mathfrak{h}$ is maximal commutative. Assume that $Y \in \mathfrak{g}$ commutes with each element of $\mathfrak{h}$. This says that it commutes with each element of $\mathfrak{t}$. If we write $Y = Y_1 + iY_2$ with Y_1 and Y_2 in $\mathfrak{k}$, for H in $\mathfrak{t}$. We have

$$[H, Y_1 + iY_2] = [H, Y_1] + i[H, Y_2] = 0$$

where $[H, Y_1]$ and $[H, Y_2]$ are in $\mathfrak{k}$.
However, every element of $\mathfrak{g}$ has a unique decomposition as an element of $\mathfrak{k}$ plus an element of $i\mathfrak{k}$ since $\mathfrak{g}$ is semisimple. From this, we say that $[H, Y_1]$ and $[H, Y_2]$ must separately be zero. Since this holds for all H and by being $\mathfrak{t}$ maximal commutative in hypothesis, we must have Y_1 and Y_2 in $\mathfrak{t}$ that is $Y = Y_1 + iY_2$ is in $\mathfrak{h}$. So, $\mathfrak{h}$ is maximal commutative.
Finally, we will show that for all $H \in \mathfrak{h}$, ad_H is diagonalizable.
We consider $\langle \cdot, \cdot \rangle$ an inner product as in **proposition 3** in the appendix then for all Y in $\mathfrak{k}$, the operator $ad_Y : \mathfrak{g} \to \mathfrak{g}$ is skew self-adjoint meaning that

$$\langle ad_Y(X), Z \rangle = -\langle X, ad_Y(Z) \rangle$$

for all $X, Y, Z \in \mathfrak{g}$
For every ad_H , $H \in \mathfrak{t}$, we can say that it is diagonalizable by **the theorem 2** in the appendix.
If $H \in \mathfrak{h}$, so $H = H_1 + iH_2$ with H_1, with H_1 and H_2 in t, we know that H_1 and H_2 commute since $\mathfrak{t}$ is commutative. In this case, ad_{H_1} and ad_{H_2} also commute.
By **proposition 2** in the appendix, they are simultaneously diagonalizable. Then, ad_H is diagonalizable
Thus, $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

Definition 5.1.2. *If $\mathfrak{g}$ is a complex semisimple Lie algebra, the rank (dimension) of $\mathfrak{g}$ is the dimension of any Cartan subalgebra.*

5.2 Roots and Root Spaces

Definition 5.2.1. *Let $\mathfrak{g}$ be a Lie algebra. A **real form** of $\mathfrak{g}$ is a Lie algebra $\mathfrak{g}_{\mathbb{R}}$ over $\mathbb{R}$ such that there exists an isomorphism from $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$. if the Killing form is negative definite then $\mathfrak{g}$ is **compact**.*

Example 5.2.1. *The compact real form of $\mathfrak{sl}_2(\mathbb{C})$ is $\mathfrak{su}(2)$ that is a skew-hermitian matrix with trace zero. $(\overline{A})^T = A$*

Now, we suppose a compact real form $\mathfrak{k}$ of $\mathfrak{g}$ and a maximal commutative subalgebra t of $\mathfrak{k}$, and the Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$. Also, we suppose an inner product on $\mathfrak{g}$ which is real on $\mathfrak{k}$ and invariant under the adjoint action of $\mathfrak{k}$ by **proposition 3** in the appendix. We know that for $H \in \mathfrak{h}$, ad_H commute with the elements of $\mathfrak{g}$ since $H = A + iB$ where $A, B \in t$ and t is commutative subalgebra. Furthermore, each such ad_H is diagonalizable. So, by **proposition 2** in the appendix, says that each ad_H, for $H \in \mathfrak{h}$ are simultaneously diagonalizable. Suppose that Y $\in \mathfrak{g}$ is a simultaneous eigenvector of every ad_H, $H \in \mathfrak{h}$, then the eigenvalue for the eigenvector Y is linearly dependent on $H \in \mathfrak{h}$. Assume that this linear functional is nonzero, it is a root. We explain the notion of root in details giving its definition and proporties.

Definition 5.2.2. *Let $\mathfrak{h}^* = \{\beta : \mathfrak{h} \to \mathbb{C}, \text{such that } \beta \text{ is linear form}\}$. An element $\beta \neq 0$ of $\mathfrak{h}^*$ is a **root** if there exists a vector Y$\neq 0$ in $\mathfrak{g}$ such that*

$$\left[H, Y\right] = \beta(H)\, Y$$

for all H in $\mathfrak{h}$.
Note that the set of all roots is denoted by R.

Remark. We can find an identification between $\mathfrak{h}$ and $\mathfrak{h}^*$ using an inner product. So β's can be seen as element of $\mathfrak{h}$. In this case, we can rewrite the definition of the root as follows:

$$[H, Y] = \langle \beta, H \rangle Y$$

for all H in $\mathfrak{h}$.

Definition 5.2.3. *Assume β is a root, then the **root space** $\mathfrak{g}_\beta$ is the space of each Y in $\mathfrak{g}$ whose $\left[H, Y\right] = \langle \beta, H \rangle\, Y$ for all $H \in \mathfrak{h}$.*

Remark. We consider an element β of $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$. We write $\mathfrak{g}_\beta$ being the space of each Y in $\mathfrak{g}$ for which $\left[H, Y\right] = \langle \beta, H \rangle\, Y$ for every H in $\mathfrak{h}$. If β is not root, we do not say that $\mathfrak{g}_\beta$ is a root space.
We take that β is zero, we obtain that $[H, Y] = 0$ which means that every elements of $\mathfrak{g}$ commute each element of $\mathfrak{h}$. We said to be $\mathfrak{g}_0$ the set of such all elements of $\mathfrak{g}$. Furthermore, we know that $\mathfrak{h}$ is a maximal commutative subalgebra, we obtain that $\mathfrak{g}_0 = \mathfrak{h}$. But, $\beta \neq 0$ is not a root, so we have $\mathfrak{g}_\beta = \{0\}$.

As we have seen, the operators ad_H, $H \in \mathfrak{h}$ are diagonalizable. In conclusion, $\mathfrak{g}$ may be separeted as the sum of $\mathfrak{h}$ and the root spaces $\mathfrak{g}_\beta$. The sum is direct by **Prop A.17** [see e.g. (Hall, 2015)] and we have also constructed the result below.

Corollary 5.2.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra, let R be the set of roots β and let $\mathfrak{g}_\beta$ be the root space, $\mathfrak{g}$ may be decomposed as a direct sum of vectors spaces as seen below:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in R} \mathfrak{g}_\beta.$$

That is to give in the above corollary, all element of $\mathfrak{g}$ may be expressed uniquely as a sum of an element of $\mathfrak{h}$ and one element of $\mathfrak{g}_\beta$.

Lemma 5.2.2. *Consider β and γ the roots in $\mathfrak{h}$. Let $\mathfrak{g}_\beta$,$\mathfrak{g}_\gamma$ be the root spaces for β and γ, respectively. We have*

$$[\mathfrak{g}_\beta, \mathfrak{g}_\gamma] \subset \mathfrak{g}_{\beta+\gamma}.$$

Proof. First, let X be in $\mathfrak{g}_\beta$ and let Y be in $\mathfrak{g}_{-\beta}$, then $[X,Y]$ is in $\mathfrak{h}$. Since, by the definition of the root, we have

$$\begin{aligned} \mathrm{X} \in \mathfrak{g}_\beta &\Rightarrow [H,X] = \langle \beta, H \rangle\; X \text{ for all } H \in \mathfrak{h} \\ \mathrm{Y} \in \mathfrak{g}_{-\beta} &\Rightarrow [H,Y] = \langle -\beta, H \rangle\; Y \text{ for all } H \in \mathfrak{h}. \end{aligned} \tag{5.1}$$

We use the Jacobi identity, we have

$$\big[H,[X,Y]\big] + \big[X,[Y,H]\big] + \big[Y,[H,X]\big] = 0 \textit{ and}$$

$$\begin{aligned} \big[H,[X,Y]\big] &= -\big[X,[Y,H]\big] - \big[Y,[H,X]\big] \\ &= -\big(-\big[X,[H,Y]\big]\big) - \big(-\big[[H,X],Y\big]\big) \\ &= \big[X,[H,Y]\big] + \big[[H,X],Y\big]. \end{aligned} \tag{5.2}$$

Now, we write the equation 5.1 using 5.2

$$\begin{aligned} \big[H,[X,Y]\big] &= \big[X,[H,Y]\big] + \big[[H,X],Y\big] \\ &= \big[X,\ \langle -\beta, H\rangle Y\big] + \big[\langle \beta, H\rangle X,\ Y\big] \\ &= -\langle \beta, H\rangle [X,Y] + \langle \beta, H\rangle [X,Y] \\ &= 0. \end{aligned}$$

Then, $[X,Y] = 0$ and we obtain $[X,Y]$ in $\mathfrak{h}$.

Now, we want to prove that $[\mathfrak{g}_\beta, \mathfrak{g}_\gamma] \subset \mathfrak{g}_{\beta+\gamma}$. We use 5.2 and assume that X in $\mathfrak{g}_\beta$ and Y in

$\mathfrak{g}_\gamma$, we have

$$\begin{aligned}\big[H,[X,Y]\big] &= \big[X,[H,Y]\big]+\big[[H,X],Y\big]\\ &= [X,\langle\gamma,H\rangle Y]+[\langle\beta,H\rangle X,Y]\\ &= \langle\gamma,H\rangle[X,Y]+\langle\beta,H\rangle[X,Y]\\ &= \langle\beta+\gamma,H\rangle[X,Y]\end{aligned}$$

for all $H\in\mathfrak{h}$, proving that $[X,Y]$ is in $\mathfrak{g}_{\beta+\gamma}$.
Hence, $[\mathfrak{g}_\beta,\mathfrak{g}_\gamma]\subset\mathfrak{g}_{\beta+\gamma}$.

Proposition 5.2.1. *1. Let $\mathfrak{h}=\mathfrak{t}+i\mathfrak{t}$ be Cartan subalgebra of $\mathfrak{g}$ and assume that $\beta\in\mathfrak{h}$ is a root, similarly $-\beta$. Particularly, suppose that A is in $\mathfrak{g}_\beta$, then A^* is in $\mathfrak{g}_{-\beta}$ where A^* is defined as follows: let A_1,A_2 be in $\mathfrak{t}$ and A_1+iA_2 in $\mathfrak{h}$*

$$(A_1+iA_2)^* = -A_1+iA_2. \tag{5.3}$$

2. The roots span $\mathfrak{h}$.

Proof. 1. Suppose that $A=A_1+iA_2$ where $A_1,A_2\in\ \mathfrak{t}$, $\overline{A}=A_1-iA_2$. We know that $\mathfrak{k}$ is closed under bracket, assume $H\in t\subset\mathfrak{k}$ and $A\in\mathfrak{g}$, we look
$\overline{[H,A]}=\overline{[H,A_1+iA_2]}=[H,A_1]-i[H,A_2]=[H,\overline{A}]$.
Furthermore, suppose A is a root vector for $\beta\in it$, therefore for all $H\in\mathfrak{h}$, we get

$$[H,\overline{A}]=\overline{[H,A]}=\overline{\langle\beta,H\rangle A}=-\langle\beta,H\rangle\overline{A} \tag{5.4}$$

because of $\langle\beta,H\rangle$ is pure imaginary for $H\in\ \mathfrak{t}$. We find that $[H,\overline{A}]=-\langle\beta,H\rangle\overline{A}$ for all $H\in\mathfrak{h}$. So, $\overline{A}$ is a root vector for the root $-\beta$. Hence, $A^*=-\overline{A}$ be interested in $\mathfrak{g}$.

2. To show this, we will use the contradiction, assume that the roots did not span $\mathfrak{h}$, therefore we have an element $H\neq 0\in\mathfrak{h}$ which werifies $\langle\beta,H\rangle=0$ for every $\beta\in R$. That is to say, $[H,K]=0$, for every $K\in\mathfrak{h}$, furthermore

$$[H,Y]=\langle\beta,H\rangle Y=0$$

for each in $\mathfrak{g}_\beta$. So, using Corollary 5.2.1 and the definition of the center of $\mathfrak{g}$, we see that the element H contradit the definition of the semisimple Lie algbera.

Theorem 5.2.3. *Let β be a root of $\mathfrak{h}$, we may define A_β in $\mathfrak{g}_\beta$, B_β in $\mathfrak{g}_{-\beta}$ and H_β in $\mathfrak{h}$, that is linearly independent which verifies H_β is a multiple of β and also;*

$$\begin{aligned} [H_\beta, A_\beta] &= 2A_\beta \\ [H_\beta, B_\beta] &= -2B_\beta \\ [A_\beta, B_\beta] &= H_\beta \end{aligned} \tag{5.5}$$

Moreover, B_β may be considered as the adjoint operator A_β^*. See e.g. (Hall, 2015) Assume that $A_\beta, B_\beta, H_\beta$ are seen as in (5.5), we have $[H_\beta, A_\beta] = 2A_\beta$. Furthermore, A_β is a root vector and β is a root in $\mathfrak{h}$, we have then $[H_\beta, A_\beta] = \langle \beta, H_\beta \rangle A_\beta$. Using these two equalties, we have then

$$\langle \beta, H_\beta \rangle = 2. \tag{5.6}$$

At the same time, we know that H_β is a multiple of β and there is the unique multiple appropriate with (5.6), that is

$$H_\beta = 2\frac{\beta}{\langle \beta, \beta \rangle} \tag{5.7}$$

Corollary 5.2.4. *Let β be a root, consider $A_\beta, B_\beta, H_\beta$ like in (5.5), with $B_\beta = A_\beta^*$ (i.e. the adjoint of A_β.) We have then the elements*

$$F_1^\beta := \frac{i}{2}H_\beta; \;\; F_2^\beta := \frac{i}{2}(A_\beta + B_\beta); \;\; F_3^\beta := \frac{1}{2}(B_\beta - A_\beta)$$

are lineraly independents elements of $\mathfrak{k}$ and satisfy the following equations

$$[F_1^\beta, F_2^\beta] = F_3^\beta; \;\; [F_2^\beta, F_3^\beta] = F_1^\beta; \;\; [F_3^\beta, F_1^\beta] = F_2^\beta.$$

Proof. First, we show the equivalent of the equations by the calculations.

- $[F_1^\beta, F_2^\beta] = [\frac{i}{2}H_\beta, \frac{i}{2}(A_\beta + B_\beta)] = \frac{i^2 4[H_\beta, A_\beta + B_\beta}{]} = \frac{-1}{4}(2A_\beta - 2B_\beta) = \frac{1}{2}(B_\beta - A_\beta) = F_3^\beta.$

- $[F_2^\beta, F_3^\beta] = [\frac{i}{2}(A_\beta + B_\beta), \frac{1}{2}(B_\beta - A_\beta)] = \frac{i}{4}[(A_\beta + B_\beta), (B_\beta - A_\beta)] = \frac{i}{4}([A_\beta, B_\beta] - [A_\beta, A_\beta] + [B_\beta, B_\beta] - [A_\beta, B_\beta]) = \frac{i}{4}(H\beta - (-H\beta)) = \frac{i}{4}2H - \beta = \frac{i}{2}H_\beta = F_1^\beta.$

- $[F_3^\beta, F_1^\beta] = [\frac{1}{2}(B_\beta - A_\beta), \frac{i}{2}H_\beta] = \frac{i}{4}([B_\beta, H_\beta - [A_\beta, H\beta]) = \frac{i}{4}(2B - \beta + 2A_\beta) = \frac{i}{2}(A_\beta + B_\beta) = F_2^\beta.$

Now, we prove second part i.e. $F_i^{\beta}{'s}$ are linearly independent.
We have seen the root β belongs to it and H_β is also, by 5.7. A real multiple of β , the element $(i/2)H_\beta$ will be in $t \subset \mathfrak{k}$. Moreover, using the property of the adjoint operator which means that $(A+iB)^* = -A+iB$. We can see as follows:

$$(F_2^\beta)^* = \big(\frac{i}{2}(A_\beta + B_\beta)\big)^* = \frac{i}{2}(A_\beta + B_\beta) = F_2^\beta.$$
$$(F_3^\beta)^* = \big(\frac{1}{2}(B_\beta - A_\beta)\big)^* = \frac{1}{2}(A_\beta - B_\beta) = -F_3^\beta.$$

Then, F_2^β and F_3^β are in $\mathfrak{k}$. We obtain $F_1^\beta, F_2^\beta, F_3^\beta$ from the combinations of $A_\beta, B_\beta, H_\beta$ and we know that A_β, B_β and H_β are linearly independent by assumption so, F_1^β, F_2^β and F_3^β are also.

Lemma 5.2.5. *Assume that A is in $\mathfrak{g}_\beta$ that B is in $\mathfrak{g}_{-\beta}$ and that H is in $\mathfrak{h}$. So, $[A,B]$ is in $\mathfrak{h}$ and*

$$\langle [A,B], H\rangle = \langle \beta, H\rangle \langle B, A^*\rangle \tag{5.8}$$

where A^* is adjoint operator in (5.3) . See e.g. (Hall, 2015).

Proof. First, we recall that $[A,B] = ad_A(B)$ and for all $A,B,C \in \mathfrak{g}$, we have then

$$\langle ad_A(B), C\rangle = \langle B, ad_{A^*}(C)\rangle.$$

We have seeen $[A,B]$ is in $\mathfrak{h}$ in the Lemma 5.2.2. Using the above recalls, we have then

$$\langle [A,B], H\rangle = \langle ad_A(B), H\rangle = \langle B, ad_{A^*}(H)\rangle = \langle B, [H, A^*]\rangle. \tag{5.9}$$

By hypothesis, the element A is in $\mathfrak{g}_\beta$, we see that A^* is in $\mathfrak{g}_{-\beta}$ from Proposition 5.2.1. In this case, we have $[H,A^*] = \langle -\beta, H\rangle A^* = -\langle \beta, H\rangle A^*$. If we write it in 5.9, we find what we want to prove, that is;

$$\langle [A,B], H\rangle = \langle \beta, H\rangle \langle B, A^*\rangle.$$

Now, we pass the proof of Theorem 5.2.3, firstly we take $A \neq 0$ in $\mathfrak{g}_\beta$, which is $A^* = -\overline{A}$ is in $\mathfrak{g}_{-\beta}$. We write A^* instead of B in the Lemma 5.2.5, we get

$$\langle [A, A^*], H\rangle = \langle \beta, H\rangle \langle A^*, A^*\rangle. \tag{5.10}$$

We say that $[A,A^*] \in \mathfrak{h}$ is orthogonal to each $H \in \mathfrak{h}$ which is orthogonal to β Also, we consider the element H in $\mathfrak{h}$ and suppose $\langle [A,A^*],H\rangle \neq 0$. We see that $[A,A^*] \neq 0$. Assume that we construct 5.10 using $H = [A,A^*]$. In this case we find

$$\langle [A,A^*],[A,A^*]\rangle = \langle \beta,[A,A^*]\rangle\langle A^*,A^*\rangle.$$

We have $[A,A^*] \neq 0$, we see that $\langle \beta,[A,A^*]\rangle$ is real and stricly positive.
Let $H = [A,A^*]$ and we determine elements of $\mathfrak{g}$ in the following:

$$\begin{aligned} H_\beta &= \frac{2}{\langle \beta,H\rangle}H \\ A_\beta &= \sqrt{\frac{2}{\langle \beta,H\rangle}}A \\ B_\beta &= \sqrt{\frac{2}{\langle \beta,H\rangle}}B \end{aligned}$$

We want to find $[H_\beta,A_\beta] = 2A$ and $[H_\beta,B_\beta] = -2B$ using $\langle \beta,H_\beta\rangle = 2$

$$\begin{aligned} [H_\beta,A_\beta] &= H_\beta A_\beta - A_\beta,H_\beta \\ &= \frac{2}{\langle \beta,H\rangle}H.\sqrt{\frac{2}{\langle \beta,H\rangle}}A - \sqrt{\frac{2}{\langle \beta,H\rangle}}A.\frac{2}{\langle \beta,H\rangle}H \\ &= \frac{2\sqrt{2}}{\langle \beta,H\rangle.\sqrt{\langle \beta,H\rangle}}[H,A] \\ &= [H,A] \\ &= \langle \beta,H\rangle A, (\langle \beta,H\rangle = 2) \\ &= 2A. \end{aligned} \tag{5.11}$$

We see that $[H_\beta,B_\beta] = -2B$ with the method in (5.11). Also,

$$[A_\beta,B_\beta] = \frac{2[A,B]}{\langle \beta,H\rangle} = H_\beta.$$

Notation. Let A_β, B_β and H_β be the elements in the theorem 5.2.3.
We have

$$\mathfrak{s}^\beta := \langle A_\beta,B_\beta,H_\beta\rangle \tag{5.12}$$

acts on $\mathfrak{g}$ by the adjoint representation.

Lemma 5.2.6. *We assume that* β *and* $k.\beta$ *are two roots such that* $\mid k \mid > 1$*, so we have* $k \pm 2$.

Proof. Say $\mathfrak{s}^{\beta}$ in the previous Notation. Suppose that $\gamma = k.\beta$ is a root and that $A \neq 0$ is an element of $\mathfrak{g}_{\gamma}$, using 5.6, we have

$$[H_{\beta}, A] = \langle \gamma, H_{\beta} \rangle A = \bar{k} \langle \beta, H_{\beta} \rangle A = 2\bar{k} A.$$

When $[H_{\beta}, A] = 2\bar{k}A$, we see that $2\bar{k}$ is an eigenvalue of the adjoint action of $\mathfrak{s}^{\beta}$ over $\mathfrak{g}$. Using **point 1 of Theorem 4.34** (see e.g Hall,2015), we see that the eigenvalue which we obtained, must be an integer. In that case, $k = (1/2).\mu$ where μ is an integer. However, we look at the roots in the hypothesis, we have that $1/k$ must be an integer multiple of $1/2$. We observe the case $k = \mu/2$ for a some integer μ. $1/k = 2/\mu$ is also integer multiple of $1/2$, which means that $2/k = 4/\mu$ is an integer. As a result, we find $\mu = \pm 1, \pm 2, or \pm 4$. So, we have $k = \pm 1/2, \pm 1, or \pm 2$. However, we supposed $\mid k \mid > 1$. Thus, $k = \pm 2$.

5.3 The Weyl Group

Let β be a root and let R be the set of roots. This group is a symmetry of the set R. In this section, we examine the Lie algebra approach to the Weyl group.

Definition 5.3.1. *For every* $\beta \in R$*, we define a linear map* $s_{\beta} = \mathfrak{h} \to \mathfrak{h}$ *as follows*

$$s_{\beta}.H = H - 2\frac{\langle \beta, H \rangle}{\langle \beta, \beta \rangle}\beta. \tag{5.13}$$

The ***Weyl group*** *of* R*, we denote* W*, it is the subgroup of* $Gl(\mathfrak{h})$ *spanned by* s_{β}*'s where* $\beta \in R$.

We supposed that every β is in $i\mathfrak{t}$ and the inner product is real on $\mathfrak{t}$ also assume that $H \in i\mathfrak{t}$, when we look at the definition of $s_{\beta}.H$, then we see that $s_{\beta}.H$ is also in $i\mathfrak{t}$. When we consider a map of $i\mathfrak{t}$ to itself, $\mathfrak{s}_{\beta}$ is the reflection about the hyperplane orthogonal to β which means that $\mathfrak{s}_{\beta}.H = H$ when $H \perp \beta$, and we have $\mathfrak{s}_{\beta}.\beta = -\beta$. We know that every reflection is an orthogonal linear transformation, we see that W is a subgroup of the orthogonal group $O(i\mathfrak{t})$.

Proposition 5.3.1. *Let* β *be a root in* R*, let* $\mathfrak{s}_{\beta} : \mathfrak{h} \to \mathfrak{h}$ *and* $H \in \mathfrak{h}, W \subset O(i\mathfrak{t})$. *We have*

1. $\mathfrak{s}_\beta(H) \in O(\mathfrak{it})$.

2. $\mathfrak{s}_\beta = \mathfrak{s}_{-\beta}$.

3. $M \in O(\mathfrak{it})$*, then* $M\mathfrak{s}_\beta M^{-1} = \mathfrak{s}_{M(\beta)}$.

Proof. 1. We want to prove that $\mathfrak{s}_\beta(H) \in O(\mathfrak{it})$ i.e. $\langle \mathfrak{s}_\beta(H), \mathfrak{s}_\beta(H_0)\rangle = \langle H, H_0\rangle$ for all $H, H_0 \in \mathfrak{h}$, using 5.13, we get

$$\begin{aligned}
\langle \mathfrak{s}_\beta(H), \mathfrak{s}_\beta(H_0)\rangle &= \langle H - 2\frac{\langle \beta, H\rangle}{\langle \beta,\beta\rangle}\beta, H_0 - 2\frac{\langle \beta, H_0\rangle}{\langle \beta,\beta\rangle}\beta\rangle \\
&= \langle H, H_0\rangle - 2\frac{\langle \beta, H\rangle\langle \beta, H_0\rangle}{\langle \beta,\beta\rangle} - 2\frac{\langle \beta, H\rangle\langle \beta, H_0\rangle}{\langle \beta,\beta\rangle} + 4\frac{\langle \beta, H\rangle\langle \beta, H_0\rangle\langle \beta,\beta\rangle}{\langle \beta,\beta\rangle^2} \\
&= \langle H, H_0\rangle.
\end{aligned}$$

2. We know that $\mathfrak{s}_\beta(\beta) = \beta$, we have $\mathfrak{s}_{-\beta}(-\beta) = \beta$. Furthermore, assume that H is perpendicular to β, it is also perpendicular to $-\beta$, hence $\mathfrak{s}_{-\beta}(H) = H$. Then, $\mathfrak{s}_\beta$ acts in the same method as $\mathfrak{s}_{-\beta}$ on the all space, so $\mathfrak{s}_\beta = \mathfrak{s}_{-\beta}$. Afterwards , we see that $\mathfrak{s}^2_\beta(\beta) = \mathfrak{s}_\beta(-\beta) = \beta$. Furthermore, for H perpendicular to β, $\mathfrak{s}^2_\beta(\beta) = \mathfrak{s}_\beta(\beta) = \beta$. Then, $\mathfrak{s}^2_\beta$ acts the similar method as 1 does, hence $\mathfrak{s}^2_\beta = 1$.

3. We see that $M\mathfrak{s}_\beta M^{-1}(M(\beta)) = M\mathfrak{s}_\beta(\beta) = M(-\beta) = -M(\beta)$. For any H perpendicular to $M(H)$, $\langle M^{-1}(H), \beta\rangle = \langle H, M(\beta)\rangle = 0$ with $M \in O(\mathfrak{it})$. So, $M\mathfrak{s}_\beta M^{-1}(H) = M\mathfrak{s}_\beta(\beta) = M(M^{-1}(H)) = H$. Hence, $M\mathfrak{s}_\beta M^{-1}$ acts the similar method as $\mathfrak{s}_{M(\beta)}$, so they are equal.

Theorem 5.3.1. *Let* β *be a root, for every* $w \in W$*, so* $w.\beta$ *is also.* (*See e.g. (Hall, 2015)*).

Proof. Let β in R, say the invertible linear operator S_β on $\mathfrak{g}$ as follows

$$S_\beta = e^{ad_{A_\beta}} e^{-ad_{A_\beta}} e^{ad_{A_\beta}}.$$

Suppose that H is in $\mathfrak{h}$, it satisfies $\langle \beta, H\rangle = 0$, when we write $\langle \beta, H\rangle$ in the equation $[H, A_\beta] = \langle \beta, H\rangle A_\beta$, we obtain $[H, A_\beta] = 0$. Thus, H and A_β commute, meaning ad_H and ad_{A_β} also commute, and in the same way, for ad_H and ad_{B_β}. Assume that $\langle \beta, H\rangle = 0$, the operator S_β going to with ad_H, we have then

$$S_\beta ad_H S_\beta^{-1} = ad_H, \quad \langle \beta, H\rangle = 0. \tag{5.14}$$

We consider the point 3 of Theorem 4.34 (see e.g. Hall,2015) to the adjoınt action of $\mathfrak{s}^{\beta} \cong Sl(2;\mathbb{C})$ on $\mathfrak{g}$, we have

$$S_{\beta} ad_{H_{\beta}} S_{\beta}^{-1} = -ad_{H_{\beta}}. \tag{5.15}$$

Using 5.14 and 5.15, for each H, we have

$$S_{\beta} ad_{H_{\beta}} S_{\beta}^{-1} = -ad_{s_{\beta}.H}.$$

We consider an another root γ and the root vector A, we rewrite
$s_{\beta} : \mathfrak{h} \to \mathfrak{h}$ and look at the vector $S_{\beta}^{-1}(A) \in \mathfrak{g}$.

$$\begin{aligned} ad_H(S_{\beta}^{-1}(A)) &= S_{\beta}^{-1}(S_{\beta} ad_H S_{\beta}^{-1})(A) \\ &= S_{\beta}^{-1} ad_{s_{\beta}.H}(A) \\ &= \langle \beta, s_{\beta}.H \rangle S_{\beta}^{-1}(A) \\ &= \langle s_{\beta}^{-1}.\beta, H \rangle S_{\beta}^{-1}(A). \end{aligned} \tag{5.16}$$

So, S_{β}^{-1} is a root vector with $s_{\beta}^{-1}.\gamma = s_{\beta}.\gamma$. tI gives that the set of roots is invariant under every s_{β} and, hence, under W.

5.4 Root Systems

In this section , we give a few properties of the roots, using also the result which we saw until now. For every root β, we have an element H_{β} of $\mathfrak{h}$ included in $[\mathfrak{g}_{\beta}, \mathfrak{g}_{-\beta}]$ as in Theorem 5.2.3. Furthermore, we have seen $\langle \beta, H_{\beta} \rangle = 2$ and $H_{\beta} = 2\beta/\langle \beta, \beta \rangle$. In particular, H_{β} do not depend the choice of A_{β} and B_{β}.

Definition 5.4.1. *Let β be a root, $H_{\beta} = 2\beta/\langle \beta, \beta \rangle$ is called the* ***corroot*** *appropriate to β.*

Proposition 5.4.1. *Consider the roots β and γ, we obtain that*

$$\langle \gamma, H_{\beta} \rangle = 2\frac{\langle \beta, \gamma \rangle}{\langle \beta, \beta \rangle} \tag{5.17}$$

is an integer.

Proof. let $\mathfrak{s}^{\beta} = \langle A_{\beta}, B_{\beta}, H_{\beta} \rangle$ be seen like in Theorem 5.2.3, let γ be a root and let Y be a root vector appropriate to γ, so we have $[H_{\beta}, Y] = \langle \gamma, H_{\beta} \rangle Y$. In that case, we see that

$\langle \gamma, H_\beta \rangle$ is an eigenvalue for the adjoint action on $\mathfrak{s}^\beta \cong \mathfrak{sl}(2,\mathbb{C})$. As we used before, using point 1 of Theorem 4.34 (see e.g. Hall,2015) we say that $\langle \gamma, H_\beta \rangle$ must be an integer.

Remark. As we have seen in the elementary linear algebra, that is, we have an inner product space, and consider β and γ two elements of our inner product space, hence we can talk about the orthogonal projection of γ onto β is as seen like:

$$\frac{\langle \beta, \gamma \rangle}{\langle \beta, \beta \rangle}\beta. \tag{5.18}$$

When we observe (5.17), we see that it is double of the expression in (5.18), which means that, we can talk about twice projection of γ onto β.

Consider R the set of roots and $R \subset E := i\mathfrak{t}$ where E is a real inner product space, we can give the proporties of R in the following theorem.

Theorem 5.4.1. *Let R be the set of roots, assume that it is finite set of nonzero elements of a real inner product space E, and R has the following additional properties.*

1. *The roots span E.*
2. *Suppose that β is in R, so −β is, and the only multiples of β in R are β and −β.*
3. *Assume that $\beta, \gamma \in R$, so $\mathfrak{s}_\beta.\gamma$, where s_β is the reflection defined by (1.14).*
4. *For every β and γ in R, the quantity*

$$2\frac{\langle \beta, \gamma \rangle}{\langle \beta, \beta \rangle}$$

is an integer.

Any such collection of vectors is called a **root system**.

6 CONCLUSION

Up to now we saw the definitions, theorems and examples on the semisimple Lie algebras. In this subsection we work over $\mathbb{C}$ and we consider the classical Lie algebras $\mathfrak{sl}(n;\mathbb{C}), \mathfrak{so}(2n;\mathbb{C}), \mathfrak{so}(2n+1;\mathbb{C})$ and $\mathfrak{sp}(n;\mathbb{C})$. We want to find the structure of its root systems.

6.0.1 The Special Linear Lie Algebras $\mathfrak{sl}(n+1;\mathbb{C})$ $[A_n]$

We recall that $\mathfrak{sl}(n+1;\mathbb{C})$ is the set of all $(n+1)\times(n+1)$ matrices with complex entries having trace zero. The Lie bracket of element of $\mathfrak{sl}(n+1;\mathbb{C})$ is comutator of its matrices, also the dimension of $\mathfrak{sl}(n+1;\mathbb{C})$ is n^2+n. We work with the compact real form $\mathfrak{k} = \mathfrak{su}(n+1)$ and the commutative subalgebra t which is the intersection of the set of diagonal matrices with $\mathfrak{su}(n+1)$; which is,

$$\mathfrak{t} = \left\{ \begin{pmatrix} ia_1 & & & 0 \\ & ia_2 & & \\ & & \ddots & \\ 0 & & & ia_{n+1} \end{pmatrix} \middle| \, a_j \in \mathbb{R}, \quad a_1 + ... + a_{n+1} = 0 \right\}. \tag{6.1}$$

We also consider $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}$, which is described as follows:

$$\mathfrak{h} = \left\{ \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_{n+1} \end{pmatrix} \middle| \, \lambda_j \in \mathbb{C}, \quad \lambda_1 + ... + \lambda_{n+1} = 0 \right\}. \tag{6.2}$$

If a matrix X commutes with each element of t (i.e. $[X,H]=0, \ \forall H \in t$), it will also commute with each element of $\mathfrak{h}$. Also X is an element of the center of $\mathfrak{sl}(n+1;\mathbb{C})$. For any $1 \leq j,k \leq n+1$, let E_{jk} be the matrix with a 1 in the (j,k) spot and zeros elsewhere. Consider the matrix $H_{jk} \in \mathfrak{sl}(n+1;\mathbb{C})$ given by

$$H_{jk} = E_{jj} - E_{kk} \quad j<k.$$

Then we may calculate, since X is in the center

$$0=[H_{jk},X]=2X_{jk}E_{jk}-2X_{kj}E_{kj}.$$

Since E_{jk} and E_{kj} are linearly independant for $j<k$, we conclude that $X_{jk}=X_{kj}=0$. Since this holds for all $j<k$, we see that X must be diagonal. If $X\in\mathfrak{su}(n+1)$, then X would be in t. Thus, t is actually a maximal commutative subalgebra of $\mathfrak{su}(n+1)$. Now E_{jk} denote the matrix that has a 1 in the j th row and k th column and that has zeros elsewhere. If $H\in\mathfrak{h}$ is as in (6.2), then $HE_{jk}=\lambda_j E_{jk}$ and $E_{jk}H=\lambda_k E_{jk}$. Thus,

$$[H,E_{jk}]=(\lambda_j-\lambda_k)E_{jk}. \tag{6.3}$$

We consider $j\neq k$, then E_{jk} is in $\mathfrak{sl}(n+1;\mathbb{C})$ and (6.3) shows that E_{jk} is a simultaneous eigenvector for each ad_H with H in $\mathfrak{h}$, with eigenvalue $\lambda_k-\lambda_k$. Furthermore $X=Y\oplus Z\in\mathfrak{sl}(n+1;\mathbb{C})$ where Y is an element of the Cartan subalgebra and Z is a linear combination of E_{jk}'s with $j\neq k$ by being $\mathfrak{sl}(n+1;\mathbb{C})$ is semisimple.
Let $\mathfrak{h}^*=\{\phi:\mathfrak{h}\to\mathbb{C}\}$, if we look at the roots as elements of $\mathfrak{h}^*$ then according to (6.3) the roots are the linear functionals α_{jk} with H in $\mathfrak{h}$, as in (6.2), the quantity $\lambda_j-\lambda_k$. We identify $\mathfrak{h}$ with the subspace of $\mathbb{C}^{n+1}$ consisting vectors whose components sum to zero. The inner product $\langle X,Y\rangle$ =trace $(X^*Y\rangle$ on $\mathfrak{h}$ is just the restriction to this subspace of the usual inner product on $\mathbb{C}^{n+1}$ If we use this inner product to transfer the roots from $\mathfrak{h}^*$ to $\mathfrak{h}$, we have the vectors

$$\alpha_{jk}=e_j-e_k \quad (j\neq k).$$

The rank of $\mathfrak{sl}(n+1;\mathbb{C})$ is n by the dimension of $\mathfrak{h}$. We say that the length of a root

$$|\alpha_{jk}|=\sqrt{\langle e_j-e_k,e_j-e_k\rangle}=\sqrt{2}$$

and $\langle e_{jk},e_{j'k'}\rangle$ has the value $-2,-1,0,1,2$ depending on whether $\{j,k\}$ and $\{j',k'\}$ have zero, one or two elements in common. We have then

$$2\frac{\langle\alpha_{jk},\alpha_{j'k'}\rangle}{\langle\alpha_{jk},\alpha_{jk}\rangle}\in\{-2,-1,0,1,2\}.$$

If α and β are roots and $\alpha\neq\beta$ and $\alpha\neq-\beta$, then the angle betweeen α and β is either $\pi/3,\pi/2,$ or $2\pi/3$, depending on whether $\langle\alpha,\beta\rangle=-1,0,1$. The root system of this Lie algebra is called A_n. (See e.g. (Hall, 2015)).

6.0.2 The Orthogonal Algebras $\mathfrak{so}(2n;\mathbb{C})$ $[D_n]$

To talk about root system of $\mathfrak{so}(2n;\mathbb{C})$, we give the elements that we will use:

- $\mathfrak{so}(2n;\mathbb{C}) = \{X \in \mathfrak{gl}(2n;\mathbb{C}) |\ X + X^T = 0\}$.
- Its compact real form is $\mathfrak{so}(2n)$
- t is the commutative subalgebra $\mathfrak{so}(2n)$ which consist of 2×2 block-diagonal matrices in which the j^{th} diagonal block is as follows

$$\begin{pmatrix} 0 & a_j \\ -a_j & 0 \end{pmatrix} \tag{6.4}$$

for some $a_j \in \mathbb{R}$. Now we construct a subalgebra in $\mathfrak{so}(2n;\mathbb{C})$ using t, which means that $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$ and its matrices are of the form (6.4) with the entries are in $\mathbb{C}$. The roots vectors are 2×2 block matrices which has a 2×2 matrix C in the (j,k) block $(j < k)$, the matrix $-C^{tr}$ in the (k,j), and zero in all other blocks. They can be as follows:

$$C_1 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

The calculation in the appendix in page 47 shows that the vectors aboves are roots vectors and that the corresponding roots are the linear funcionals on $\mathfrak{h}$ given by $i(a_j + a_k), -i(a_j + a_k), i(a_j - a_k)$, and $-i(a_j - a_k)$, respectively.
Consider the inner product on $\mathfrak{h}$ given by $\langle X, Y \rangle =$ trace $(X^*Y)/2$. If we identify $\mathfrak{h}$ with $\mathbb{C}^n$ i.e.

$$H \mapsto i(a_1, a_2, ..., a_n),$$

and used inner product on $\mathfrak{h}$ will correspond to the usual inner poduct on $\mathbb{C}^n$. In this case we consider the roots as the elements of $\mathbb{C}^n$ and the vectors are then

$$\pm e_j \pm e_k, \quad j < k, \tag{6.5}$$

where $\{e_j\}$ is the standart basis for $\mathbb{R}^n$ and $j = 1, 2, .., n$ Now we prove that t is a maximal commutative subalgebra of $\mathfrak{k}$. Since $\mathfrak{so}(2n;\mathbb{C})$ is semisimple then we see that $X = H + Y \in \mathfrak{so}(2n;\mathbb{C})$ where $H \in \mathfrak{h}$ and Y is a linear combination of the root vectors $\pm e_j \pm e_k$. Furthermore if X commutes with every element of t, then X also commutes with every elment of $\mathfrak{h}$. Because of the linear functionals $i(\alpha_j \pm \alpha_k), j < k$, is nonzero on $\mathfrak{h}$, in this case the coeffients of the root vectors in the expansion of X must be zero; i.e X must be in $\mathfrak{h}$. If X is in $\mathfrak{k}$, then X must be in the intersection of $\mathfrak{h}$ and $\mathfrak{k}$ but this intersection equal to t so $X \in t$. t is maximal commutative subalgebra in $\mathfrak{k}$. By definition of semisimplicity, $\mathfrak{so}(2n;\mathbb{C})$ has no non-zero abelian ideals (i.e it has no non-zero commutative subalgebras), in this case the center of $\mathfrak{so}(2n;\mathbb{C})$ is trivial. If X is in the center of $\mathfrak{so}(2n;\mathbb{C})$ by previous paragraph, X must be in $\mathfrak{h}$. For each root vector X_α corresponding to root α and X in $\mathfrak{h}$, we have

$$0 = [X, X_\alpha] = \langle \alpha, X \rangle \, X_\alpha,$$

we see that $\langle \alpha, X \rangle = 0$. We have identified $\mathfrak{h}$ with $\mathbb{C}^n$, if $n \leq 2$, the roots in (6.5) span $\mathfrak{h} \cong \mathbb{C}^n$ and we conclude that X must be zero. Furthermore if we choose $n = 1$ then there are no roots and $\mathfrak{so}(2;\mathbb{C}) = \mathfrak{h}$ is commutative. The root system of this Lie algebra is denoted by D_n. $\big($ See e.g.(Hall, 2015)$\big)$.

6.0.3 The Orthogonal Algebras $\mathfrak{so}(2n+1;\mathbb{C})$ $[B_n]$

To determine the root systems of $\mathfrak{so}(2n+1;\mathbb{C})$, we give its structure and the compact real form we will use to find the elements of root systems

$$\mathfrak{so}(2n+1;\mathbb{C}) = \{X \in \mathfrak{gl}(2n+1;\mathbb{C}) | \; X + X^T = 0\}$$

and its compact real form is $\mathfrak{so}(2n+1)$. t is commutative subalgebra which consist of block matrices with n block of size 2×2 followed by one block of size 1×1. We use the 2×2 blocks being of the same form as in $\mathfrak{so}(2n)$. Also by definition $\mathfrak{so}(2n+1;\mathbb{C})$ we see that the 1×1 block matrice is zero. Then $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} + i\mathfrak{t}$, and we must prove that $\mathfrak{h}$ is a Cartan subalgebra, it consists of the matrices in $\mathfrak{so}(2n+1;\mathbb{C})$ and similarly in t, also the off-diagonal elements of the 2×2 blocks are complex. The same method in $\mathfrak{so}(2n;\mathbb{C})$ and the similar calculations in the [appendix 10] in the page 47, shows that t is maximal commutative, then $\mathfrak{h}$ is a Cartan subalgebra.
Since 1×1 block in the last is zero, we can consider the Cartan subalgebra in $\mathfrak{so}(2n+1;\mathbb{C})$ as in the $\mathfrak{so}(2n;\mathbb{C})$. Furthermore they have the same rank that of n. Every root for $\mathfrak{so}(2n;\mathbb{C})$ is also a root for $\mathfrak{so}(2n+1;\mathbb{C})$ with this identification of the Cartan subalgebra. But, there are 2n additional roots in $\mathfrak{so}(2n+1;\mathbb{C})$. They have 2×1 block in entries $(2k, 2n+1)$ and $(2k+1, 2n+1)$ as follows:

$$B_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

and to have $-B_1^{tr}$ in entries $(2n+1, 2k)$ and $(2n+1, 2k+1)$, together with the following matrices to have

$$\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

in entries $(2k, 2n+1)$ and $(2k+1, 2n+1)$ and $-B_2^{tr}$ in entries $(2n+1, 2k)$ and $(2n+1, 2k+1)$. We define the inner product $\langle . \, , \, . \rangle : \mathfrak{h}^* \mapsto \mathfrak{h}$ such that the inner product $\langle X, Y \rangle$ =trace $(X^*Y)/2$ and the roots correspond to the above root vectors, seen as elements of $\mathfrak{h}^*$, are given by ia_k and $-ia_k$.
If we use this inner product to identify the roots with elements of $\mathfrak{h}$ and then we can consider $\mathfrak{h} \cong \mathbb{C}^n$ as in $\mathfrak{so}(2n;\mathbb{C})$, the roots are $\pm e_j \pm e_k, \; j < k$, as we saw in $\mathfrak{so}(2n;\mathbb{C})$, with additional roots are

$$\pm e_j, \;\; j = 1, 2, ..., n.$$

We have seen that the length of the roots $\pm e_j \pm e_k$ is 2, these additional roots are shorter by a factor $\sqrt{2}$ than the roots $\pm e_j \pm e_k$ for $\mathfrak{so}(2n;\mathbb{C})$. The root system of $\mathfrak{so}(2n+1;\mathbb{C})$ is denoted B_n. (See e.g. (Hall, 2015)).

6.0.4 The Symplectic Algebras $\mathfrak{sp}(n;\mathbb{C}) \quad [C_n]$

In this section, we will work to understand the structure of the root system of the Lie algebra $\mathfrak{sp}(n;\mathbb{C})$ studying in details on the followings:

- $\mathfrak{sp}(n;\mathbb{C}) = \{X \in \mathfrak{gl}(2n;\mathbb{C}) | \; X^t\Omega \, + \, \Omega X \, = \, 0\}$ for $n \geq 1$ where Ω is the $2n \times 2n$ matrix, that is,

$$\Omega = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \mathit{and} \;\; \mathbb{I} \; \mathit{is\ identity\ matrix\ of\ dimension} \; n \times n$$

  We have $\Omega^2 = -\mathbb{I}$ where $\mathbb{I}$ is identity matrix of dimension $2n \times 2n$ and we see that $X^t\Omega = -\Omega X$ using the equation in $\mathfrak{sp}(n;\mathbb{C})$ also we multiply by Ω the both side, we obtain that $\Omega X^{tr}\Omega = X$ When we consider the all of the above, we have $X \in \mathfrak{sp}(n;\mathbb{C})$ as follows :

$$X = \begin{pmatrix} A & B \\ C & -A^{tr} \end{pmatrix},$$

  where A is an arbitrary $n \times n$ matrix; B and C are arbitrary symmetric matrices of dimension $n \times n$.

- We know that $\mathfrak{u}(2n) = \{ \; A \in \mathfrak{gl}(2n;\mathbb{C}) | \; \overline{A^{tr}} = -A \; \}$ and the compact real form of $X \in \mathfrak{sp}(n;\mathbb{C})$ is $\mathfrak{sp}(n) = \mathfrak{sp}(n;\mathbb{C}) \cap \mathfrak{u}(2n)$

- The commutative subalgebra is to consist the matrix as follows :

$$\begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & -a_1 & & \\ & & & & \ddots & \\ & & & & & -a_n \end{pmatrix} \tag{6.6}$$

  where each a_j is pure imaginary. We construct the subalgebra $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$ of $\mathfrak{sp}(n;\mathbb{C})$, that consists of the similar to the matrix (6.6) but the elements of matrix a_j is arbitrary complex number. As we did in previous subsection, the calculations below that we will do to find roots and roots vectors of $\mathfrak{sp}(n;\mathbb{C})$, show that t is maximal commutative, then $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$ is a Cartan subalgebra.

Let E_{jk} denote the matrix that has a one in the j^{th} row and k^{th} column and has zeros elsewhere. Using the given matrices in the above, when we do the similar calculation in the [Appendix 10], we have the $2n \times 2n$ matrices of the block form

$$\begin{pmatrix} 0 & E_{jk}+E_{kj} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ E_{jk}+E_{kj} & 0 \end{pmatrix}.$$

$(j \neq k)$ are root vector for which the corresponding roots are $(a_j + a_k)$ and $-(a_j + a_k)$, respectively. Other matrices of the block form

$$\begin{pmatrix} E_{jk} & 0 \\ 0 & -E_{kj} \end{pmatrix}$$

$(j \neq k)$ are root vector for which the corresponding roots are $(a_k - a_j)$. Finally, matrices of the block form

$$\begin{pmatrix} 0 & E_{jj} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ E_{jj} & 0 \end{pmatrix}.$$

are root vectors of being roots $2a_j$ and $-2a_j$. As before the subsection, the inner product on $\mathfrak{h}$ is

$$\langle X, Y \rangle = trace(X^*Y)/2.$$

If we identify $\mathfrak{h}$ with $\mathbb{C}^n$ by the map, $H \in \mathfrak{h}$

$$H \mapsto (a_1, ..., a_n),$$

then the inner product on $\mathfrak{h}$ will correspond to the standart inner product on $\mathbb{C}^n$. The roots are the vector of the form

$$\pm e_j \pm e_k, \quad j < k$$

and of the form

$$\pm 2e_j, \quad j = 1, ..., n.$$

As we saw in the above, this rooot system as that for $\mathfrak{so}(2n; \mathbb{C})$, except that instead of $\pm e_j$, we have $\pm 2e_j$. The root system is C_n. See e.g. (Hall, 2015).

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APPENDIX

Some propositions and definitions which we used as folllows:

Proposition 1. Let V be a finite-dimensional vector space over an algebraically closed field k. Let $A \in Hom(V,V)$, and $\lambda_1, ..., \lambda_r \in k$ be the different eigenvalues of A. Put

$$V_i = \{v \in V : (A - \lambda_i \mathbb{I})^k v = 0 \; for \; k \; sufficiently \; large\}.$$

Then

1. $V = \sum_{i=1}^{r} V_i$ (direct sum).
2. Each V_i is invariant under A.
3. The semisimple part of A is given by

$$S\left(\sum_{i=1}^{r} v_i\right) = \sum_{i=1}^{r} \lambda_i v_i \qquad (v_i \in V_i).$$

4. The characteristic polynomial of A is

$$det(\lambda \mathbb{I} - A) = (\lambda - \lambda_1)^{d_1} ... (\lambda - \lambda_r)^{d_r},$$

   where $d_i = dim\, V_i$ $(1 \leq i \leq r)$.

Proposition 2. Let V be a vector space. If A is commuting collection of linear operators on V and each $a \in A$ is diagonalizable, then the elements of A are simultaneously diagonalizable.

Definition 1. A Lie algebra $\mathfrak{g}$ is called reductive if the following equivalence conditions hold:

1. It is the direct sum $\mathfrak{g} \cong \mathfrak{h} \bigoplus a$ of a semisimple Lie algebra $\mathfrak{h}$ and an abelian Lie algebra a
2. Its adjoint representation is completely reducible: every invariant subspace has an invariant complement.

Proposition 3. Let $\mathfrak{g} := \mathfrak{k}_{\mathbb{C}}$ be a reductive Lie algebra. Then there exists an inner product on $\mathfrak{g}$ that is real valued $\mathfrak{k}$ and such that the adjoint of $\mathfrak{k}$ on $\mathfrak{g}$ is unitary meaning that

$$\langle ad_X(Y), Z\rangle = -\langle Y, ad_X(Z)\rangle \tag{6.7}$$

for all $X \in \mathfrak{k}$ and all $Y, Z \in \mathfrak{g}$. If we define an operation $X \mapsto X^*$ on $\mathfrak{g}$ by the formula

$$(X_1 + iX_2)^* = -X_1 + iX_2 \tag{6.8}$$

for $X_1, X_2 \in \mathfrak{k}$, then any inner product satisfying 6.7 also satisfies

$$\langle ad_X(Y), Z\rangle = -\langle Y, ad_{X^*}(Z)$$

for all X,Y and Z in $\mathfrak{g}$.

Theorem 1. (Cartan's criterion) Let $\mathfrak{g}$ be a Lie algebra. $\mathfrak{g}$ is solvable is and only if $K([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) \equiv 0$ where K is Killing form.

Theorem 2. Suppose that $A \in M_n(\mathbb{C})$ has the property that $A^*A = AA^*$ (e.g. if $A^* = A,\ A^* = A^{-1},\ or\ A^* - A$). Then A is diagonalizable.

Definiton 2.

1. Let $\mathfrak{h}$ be a real Lie algebra. The complexification of $\mathfrak{h}$ noted $\mathfrak{h}^{\mathbb{C}}$ is a vector space equipped with the Lie bracket

   $$[a+ib, c+id] = [a,c] - [b,d] + i\big([b,c] + [a,d]\big)$$

   for $a, b, c, d \in \mathfrak{h}$.

2. Let $\mathfrak{g}$ a complex Lie algebra and $\mathfrak{h}$ a real Lie subalgebra, $\mathfrak{h}$ is a **real form** of $\mathfrak{g}$ if there exists a $\mathbb{C}$-linear isomorphism $\psi : \mathfrak{h}^{\mathbb{C}} \to \mathfrak{g}$ such that $\psi_{|\mathfrak{h}} = Id$. Every complex Lie algebra has a compact real form.

Calculation for the roots and root vectors of the Orthogonal Algebras $\mathfrak{so}(2n;\mathbf{C})$

Consider $H \in \mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$ and the matrix represantiton of H is as follows :

$$H = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & 0\cdots & 0 & 0 \\ -a_1 & 0 & 0 & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0\cdots & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3\cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_3 & 0\cdots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0\cdots & 0 & a_n \\ 0 & 0 & 0 & 0 & 0 & 0\cdots & -a_n & 0 \end{pmatrix}, \quad a_i \in \mathbb{C}$$

Consider $X \in \mathfrak{s}o(2n;\mathbb{C})$ and let $m = 2n$, the matrix representation is as follows :

$$X = \begin{pmatrix} 0 & x_{21} & x_{31} & x_{41} & x_{51} & x_{61} & \cdots & x_{m1} \\ -x_{21} & 0 & x_{32} & x_{42} & x_{52} & x_{62} & \cdots & x_{m1} \\ -x_{31} & -x_{32} & 0 & x_{43} & x_{53} & x_{63} & \cdots & x_{m3} \\ -x_{41} & -x_{42} & -x_{43} & 0 & x_{54} & x_{64} & \cdots & x_{m4} \\ -x_{51} & -x_{52} & -x_{53} & -x_{54} & 0 & x_{65} & \cdots & x_{m5} \\ -x_{61} & -x_{62} & -x_{63} & -x_{64} & -x_{65} & 0 & \cdots & x_{m6} \\ \vdots & & & & & & & \\ -x_{m1} & -x_{m2} & -x_{m3} & -x_{m4} & -x_{m5} & -x_{m6} & \cdots & 0 \end{pmatrix}, \quad x_{ij} \in \mathbb{C}$$

Now we calculate $[H,X] = HX - XH$ using above matrices, that is,

$$[H,X] = \begin{pmatrix} 0 & 0 & -a_{31} & -a_{41} & -a_{51} & -a_{61} & \cdots & & -a_{k1} \\ 0 & 0 & -a_{32} & -a_{42} & -a_{52} & -a_{62} & \cdots & & -a_{k2} \\ a_{31} & a_{32} & 0 & 0 & -a_{53} & -a_{63} & \cdots & & -a_{k3} \\ a_{41} & a_{42} & 0 & 0 & -a_{54} & -a_{64} & \cdots & & -a_{k4} \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 & 0 & \cdots & & -a_{k5} \\ a_{61} & a_{62} & a_{63} & a_{64} & 0 & 0 & \cdots & & -a_{k6} \\ \vdots & & & & & & & & \\ & & & & & & \cdots & 0 & 0 \\ a_{k1} & a_{k2} & a_{k3} & a_{k4} & a_{k5} & a_{k6} & \cdots & 0 & 0 \end{pmatrix}$$

where a_{ij} is en function de a_k and x_{ij}. The four elements a_{31}, a_{32}, a_{41} and a_{42} in $[H,X]$ are explicitly as follows:

- $a_{31} = -a_2 x_{41} - a_1 x_{32} \quad a_{32} = -a_2 x_{41} + a_1 x_{31}$
- $a_{41} = a_2 x_{31} - a_1 x_{42} \quad a_{42} = a_2 x_{32} + a_1 x_{41}$

We can determine the other entries of $[H,X]$ as in the above. We have also the equation

$$[H,X] = \langle \alpha, H \rangle X \quad \text{where } \langle \alpha, H \rangle \textit{ is} \text{ scalar and } H \in \mathfrak{h},\ X \in \mathfrak{so}(2n,\mathbb{C}). \tag{6.9}$$

If we write the above matrices H and X in the equation (6.9), we obtain the following equaions:

$$x_{31}^2 + x_{41}^2 = x_{32}^2 + x_{42}^2 = 0 \quad \text{and} \quad x_{32}x_{41} = x_{31}x_{42}. \tag{6.10}$$

When we find the possible solutions of the equations in (6.10), we obtain the root vectors X which has 2×2 block C in the $(j < k)$ block , the matrix $-C^{tr}$ in the (k,j) block, and zero in the other block and C is one of the four matrices as follows :

$$C_1 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

Also the roots α are linear funcitionals on $\mathfrak{h}$ given by :

$$i(a_j + a_k),\ -i(a_j + a_k),\ i(a_j - a_k), and\ -i(a_j - a_k),\ \text{ respectively.}$$

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