MONOTONOUS CELLULAR DECOMPOSITION

(MONOTON HÜCRESEL AYRIŞMA)

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ABSTRACT

This thesis focuses on some results regarding the axioms of o-minimality which give rise to a restriction on definable sets ensuring the tameness of the topology, and also two regularity notions defined by van den Dries and by Gabrielov that we will simply call vdD-regularity which imposes some monotonicity properties on functions involved in the definition of open cells and definable functions, and Gabrielov-regularity which provides strong topological properties on cells. In this work, we first give some nice features of definable maps such as the Monotonicity Theorem and the Cell Decomposition Theorem which follow from the finiteness properties of definable sets in o-minimal structures. Then we present vdD-regularity which combines these two results and yields to state a more powerful theorem, namely Regular Cell Decomposition Theorem. In our work, we prove this theorem which has no proof in the literature. Afterwards, we are interested in a new regularity notion which is defined by Gabrielov to obtain cells that are topologically regular, whereas vdD-regularity does not necessarily imply this property. Indeed, we will examine an example in detail, which is given by Gabrielov, of a cell that is regular in the sense of van den Dries but not in the sense of Gabrielov.

Keywords : O-minimality, Monotonicity Theorem, Cell Decomposition Theorem, Regularity, Regular Cell Decomposition Theorem

ÖZET

Bu tez, tanımlanabilir kümelere bir kısıtlama getiren ve üzerinde çalışılan topolojinin daha kolay olmasını sağlayan o-minimallik aksiyomlarına ilişkin bazı sonuçlara ve ayrıca van den Dries ve Gabrielov tarafından tanımlanan, açık hücrelere ve tanımlanabilir fonksiyonlara monotonluk özelliği taşıyan vdD-düzenliliği ve hücrelere güçlü bir topolojik özellik getiren Gabrielov-düzenliliği olarak adlandıracağımız iki ayrı düzenlilik kavramına odaklanmaktadır. Bu çalışmada, tanımlanabilir fonksiyonların, o-minimal yapılardaki tanımlanabilir kümelerin sonlu olma özellikleri sayesinde açığa çıkan Monotonluk Teoremi ve Hücresel Ayrışma Teoremi gibi bazı önemli özelliklerini veriyoruz. Daha sonra, bu iki teoremi birleştiren ve daha güçlü bir teoreme ulaşmamızı sağlayan- Düzenli Hücresel Ayrışma Teoremi- vdD-düzenliliği ile ilgileniyoruz. Çalışmamızda literatürde kanıtı bulunmayan Düzenli Hücresel Ayrışma Teoremi'ni kanıtlıyoruz. Sonrasında, vdD-düzenli olan hücreler zorunlu olarak topolojik anlamda düzenli olmazken, topolojik olarak düzenli olan hücrelere ulaşmak amacıyla Gabrielov tarafından tanımlanan yeni bir düzenlilik kavramı üzerinde çalışıyoruz. Son olarak, Gabrielov tarafından verilen, vdD-düzenli olan ancak Gabrielov-düzenli olmayan bir örneği detaylıca inceliyoruz.

Anahtar Kelimeler : O-minimallik, Monotonluk Teoremi, Hücresel Ayrışma Teoremi, Düzenlilik, Düzenli Hücresel Ayrışma Teoremi

LIST OF ABBREVIATIONS

MT	Monotonicity Theorem
CDT	Cell Decomposition Theorem
RCDT	Regular Cell Decomposition Theorem

1 INTRODUCTION

Model theory, a branch of mathematical logic, concerning with sentences and sets in structures that are defined by a formula in the sense of the first order logic.
O-minimality is one of the areas in this branch, studying the order structure for which definable sets in one variable are a finite unions of intervals and points. The occurrence of this domain of research dates to the works of Lou van den Dries ; he discovered that many properties of semialgebraic sets and maps can be obtained by the axioms of o-minimal structures. He showed the excellent framework of o-minimal structures in (van den Dries, 1998) which we will follow as the primary source. For example, one of the key theorems on o-minimality is that this restriction on the topology of definable subsets of the line carries up to a restriction to the topology of definable sets in any number of variables ; this is one of the main motivation to work with definable sets in an o-minimal structure : definable sets in an o-minimal structure give convenient setting due to the flexibility of the notion of definability by first order formulas, and at the same time stay amenable to our understanding due to the constraints on their topology. In addition, Wilkie's solution to Tarski's problem showed in particular that the real exponential field structure is o-minimal (Wilkie, 1996).

Even though o-minimality provides a control on the topology of definable sets, computation of topological invariants remains difficult, mainly because the topology of embedded cells - which is a key tool for the control of the topology of definable sets - can be more complicated than wanted. Therefore, Gabrielov, Basu and Vorobjov aim to find a stronger cell decomposition theorem so as to obtain cells endowed with a simpler topology. For this purpose, they introduce a new regularity notion, see (Gabrielov et al., 2010).

Throughout this thesis, we will be interested in the axioms of o-minimality and its consequences. After that, we will consider the notions of two regularities defined by van den Dries and Gabrielov. Finally, we will compare the definitions by considering an example, and observe that the two definitions do not match.

The outline of this thesis is as follows.

In Chapter 2, we give a survey of the literature.

In Chapter 3, we present necessary background material on model theory needed for the next chapters. In particular, we introduce the notion of definable sets and the theory of dense linear orders. We also prove some basic lemmas about definability.

In Chapter 4, we focus on o-minimality and give examples which help understand the geometry brought along o-minimality. Furthermore, we prove a lemma which shows tameness of the topology of definable sets in o-minimal structures. Afterwards, we are interested in the main results of o-minimality on definable maps, namely Monotonicity and Cell Decomposition Theorems. We give their detailed proofs, following (van den Dries, 1998).

Chapter 5 consists of three main parts. In the first part, we deal with the regularity definition due to Lou van den Dries which makes the Regular Cell Decomposition Theorem stronger than the Cell Decomposition Theorem. We prove Regular Cell Decomposition Theorem which is given as an exercise in (van den Dries, 1998, p. 58) and has no proof in literature. Then we pass to the second part of this chapter where we introduce another regularity notion due to Gabrielov. We will give the important definitions regarding this new notion. Before comparing the two regularities, we prove a topological property of k-cells in a field expansion of o-minimal structures. Finally, we show that these two regularity definitions are not equivalent in the last section. For this, we examine an example in detail, which is given briefly in (Gabrielov et al., 2010).

In Appendix, we touch upon some topological preliminaries.

2 LITERATURE REVIEW

In 1951, Tarski published his proof about the decidability of the real field $(R; +, -, ., 0, 1)$ and asked whether it is also possible to prove this for the structure $(R; +, -, ., 0, 1, \exp x)$ (Tarski, 1951). This question inspired Lou van den Dries to propose the development an axiomatic approach (van den Dries, 1984).

There have been other developments such as studies around subanalytic sets by Łojasiezics (Łojasiezics, 1964), Gabrielov (Gabrielov, 1968) and Hironaka (Hironaka, 1973). The CDT was proven for some cases in these new frameworks but it had not been generalized to an axiomatic framework at the time. In 1984, Grothendieck suggested an axiomatic unification in "Esquisse d'un Programme" (Grothendieck, 1984).

During the same period, Pillay and Steinhorn developed an axiomatization of o-minimal theory (Pillay and Steinhorn, 1984), and they proved the CDT in (J.Knight et al., 1986) along with other fundamental consequences of o-minimality, together with J. Knight. In parallel to their works, Wilkie proved o-minimality of the structure $(R; +, -, ., 0, 1, \exp x)$, as a response to the version proposed by van den Dries of the problem of Tarski (Wilkie, 1996).

Lou van den Dries recapitulates what is known about o-minimality in his book (van den Dries, 1998) and gives another proof of the CDT. He also introduces a regularity notion which makes cells have sort of a monotonicity property. Then he leaves RCDT as an exercise (van den Dries, 1998). In 2013, Gabrielov-Basu and Vorobjov published an article (Gabrielov et al., 2010) where they give an example to show that the regularity defined by van den Dries does not necessarily imply the topological regularity (Gabrielov et al., 2010).

3 PRELIMINARIES

This chapter consists of the required model theoretic preliminary.

3.1 Basics on Model Theory

Definition 3.1. *A language $\mathcal{L}$ is a finite or infinite collection of*

- *a set of function symbols $\mathcal{F}$ and a positive integer n_f for each $f \in \mathcal{F}$,*
- *a set of relation symbols $\mathcal{R}$ and a positive integer n_R for each $R \in \mathcal{R}$,*
- *a set of constant symbols $\mathcal{C}$.*

The numbers n_f and n_R denote the variable numbers of f and R, respectively.

Example 3.1.

- *The language of rings is $\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$ where $+$ and $\cdot$ are binary function symbols , $-$ is a function symbol of arity one and 0 and 1 are constant symbols.*
- *The language of ordered rings is $\mathcal{L}_{or} = \mathcal{L}_r \cup \{<\}$.*
- *The language of pure sets is $\mathcal{L} = \emptyset$.*

Now we explain what an $\mathcal{L}$-formula and a definable set are without details.

An $\mathcal{L}$-formula Φ is obtained by symbols in the language $\mathcal{L}$ and variables , by using first order logical symbols which are the quantifier symbols $\forall$ and $\exists$, the logical connectives $\wedge, \vee, \implies, \iff$ and $\neg$, the punctuation symbols such as parentheses , brackets etc., an infinite set of variables and the equality symbol $=$ and following the synthetic rules obviously required (Marker, 2000). Here we give a few examples of first order formulas.

Example 3.2. *Take the language $\mathcal{L}_o = \{<\}$ where $<$ denotes a linear ordering. Then $\forall x \forall y\ (x < y \implies \exists z\ x < z < y)$ is an $\mathcal{L}_o$-formula, and $\forall x\ x + 0 = x$ is an $\mathcal{L}_g = (+, -, 0)$-formula.*

Definition 3.2. *($\mathcal{L}$-sentence) An $\mathcal{L}$-formula is said to be a sentence if it has no free variables ; that is, all variables in the formula are bounded by a quantifier : $\forall$ or $\exists$.*

Example 3.3. *Consider the language $\mathcal{L}_{or}$ and the following two $\mathcal{L}_{or}$-formulas :*

- $\exists v_2 \;\; v_2.v_2 = v_1$
- $\forall v_1 \; (v_1 = 0 \vee \; \exists v_2 \;\; v_2.v_1 = 1)$

In the second formula, we see that v_1 is bounded by a quantifier while it is not in the first formula and see also that v_2 is bounded in both formulas. Thus the second formula is an $\mathcal{L}$-sentence while the first one is not.

Definition 3.3. *($\mathcal{L}$-structure) An $\mathcal{L}$-structure $\mathcal{M}$ is given by the following data :*

- *a nonempty set M, called the universe of $\mathcal{M}$,*
- *a function $f^{\mathcal{M}} : M^{n_f} \longrightarrow M$ for each function symbol $f \in \mathcal{F}$,*
- *a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$,*
- *an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$.*

We denote structures as $\mathcal{M} = (M; f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}})$ where $f \in \mathcal{F}$, $R \in \mathcal{R}$ and $c \in \mathcal{C}$.

Example 3.4. *Consider the language $\mathcal{L}_g = \{+, -, e\}$ where $+$ is a binary function symbol, $-$ is a unary function symbol and e is a constant symbol. $\mathcal{Z} = (\mathbb{Z}; +, -, 0)$ is an $\mathcal{L}_g$-structure.*

Example 3.5. *Consider the language of rings $\mathcal{L}_r = \{+, \cdot, -, 0, 1\}$. Then $\mathcal{R} = (\mathbb{R}; +, -, \cdot\,, 0, 1)$ is an $\mathcal{L}_r$-structure.*

Let $\mathcal{M} = (M; \ldots.)$ be an $\mathcal{L}$-structure, ϕ be an $\mathcal{L}$-formula and $(a_1, \ldots, a_m) \in M^m$. We write $\mathcal{M} \models \phi(a_1, \ldots, a_m)$ if $\phi(a_1, \ldots, a_m)$ is true in $\mathcal{M}$ or equivalently, if $\mathcal{M}$ *satisfies* $\phi(a_1, \ldots, a_m)$.

Definition 3.4. *Let $\mathcal{M} = (M; \ldots.)$ be an $\mathcal{L}$-structure. We say that a subset X of M^n is definable with parameters $\bar{b} = (b_1, \ldots, b_m) \in M^m$ if there is an $\mathcal{L}$-formula $\Phi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ such that $X = \{\bar{a} \in M^n : \mathcal{M} \models \Phi(\bar{a}, \bar{b})\}$. If X is defined without parameters, then we say that X is $\emptyset$-definable.*

Example 3.6. *Take the structure $\mathcal{R} = (\mathbb{R}; +, -, \cdot, 0, 1)$. Then the set $X = \{x \in \mathbb{R} : x.x + a_1.x + a_2 = 0\}$ is definable with parameters (a_1, a_2).*

Definition 3.5. *A function $f : M^n \rightarrow M^m$ is said to be definable in $\mathcal{M} = (M; \ldots.)$ where $n, m \in \mathbb{N}$ and $\mathcal{M}$ is an $\mathcal{L}$-structure if its graph is a definable set in $\mathcal{M}$.*

Definition 3.6. *($\mathcal{L}$-theory) An $\mathcal{L}$-theory T is a set of $\mathcal{L}$-sentences.*

Definition 3.7. *(Model) Let T be an $\mathcal{L}$-theory and $\mathcal{M}$ be an $\mathcal{L}$-structure. We say $\mathcal{M}$ is a model of T and write $\mathcal{M} \models T$ if for any sentences $\Phi \in T$, $\;\mathcal{M} \models \Phi$.*

Example 3.7. *(Dense Linear Orders Without Endpoints) Let $\mathcal{L} = \{<\}$ where $<$ is a binary relation symbol. The axioms of dense linear orders without endpoints are the following $\mathcal{L}$-sentences :*

- $\forall x \, \neg(x < x)$
- $\forall x \forall y \forall z \;\; ((x < y \wedge y < z) \implies x < z)$
- $\forall x \forall y \;\; (x < y \vee x = y \vee y < x)$
- $\forall x \forall y \; (x < y \implies \exists z \;\; x < z < y)$
- $\forall x \exists y \exists z \;\; y < x < z$

Example 3.8. *The structure $\mathcal{Q} = (\mathbb{Q}; <)$ is a model of the theory of dense linear orders where the structure $\mathcal{Z} = (\mathbb{Z}; <)$ is not as $\mathbb{Z}$ is not dense in itself.*

We finish this chapter with two basic lemmas which will be used in the next chapter.

Lemma 3.1. *Let $\mathcal{R} = (R; \dots)$ be any structure. If $A \subseteq R^n$ and $B \subseteq R^m$ are definable sets then $A \times B$ is also definable in R^{n+m}.*

Proof Let $\Phi_1(a_1, \dots, a_n)$ and $\Phi_2(b_1, \dots, b_m)$ be the formulas that define A and B, then the formula $\Psi(a_1, \dots, a_n, b_1, \dots, b_m) : \Phi_1(a_1, \dots, a_n) \;\wedge\; \Phi_2(b_1, \dots, b_m)$ defines the set $A \times B$ which finishes the proof.

□

Lemma 3.2. *If $f : A \longrightarrow R^m$ is a definable function and $A \subseteq R^n$ is a definable set then $f(A) \subseteq R^m$ is also definable.*

Proof Let $\Gamma(f)$ be the graph of f. Then we have :

$$f(A) = \{\bar{y} \in R^m : \exists \, \bar{x} \in A, \; f(\bar{x}) = \bar{y}\}$$

$$= \{\bar{y} \in R^m : \exists \, \bar{x} \in A, \; (\bar{x}, \bar{y}) \in \Gamma(f)\}.$$

Since both A and $\Gamma(f)$ are definable, then so is $f(A)$.

□

4 O-MINIMALITY AND MAIN RESULTS

Throughout this chapter, we give the definition of *o-minimality* and work on the consequences of the axioms of the definition.

4.1 O-minimality

In this section, we will introduce o-minimal structures and give examples.

Consider a structure $\mathcal{M} = (M; <, \dots)$ where $<$ is a dense linear order without endpoints on M.

Definition 4.1. *(O-minimal Structure) A structure* $\mathcal{M} = (M; <, \dots)$ *is said to be* ***o-minimal*** *if every definable subset of* M *is a finite union of intervals and points.*

Example 4.1. *It follows from Tarski-Seidenberg Theorem (Tarski, 1951) that the expansion of ordered field of real numbers* $\mathcal{R} = (\mathbb{R}; +, -, \cdot, 0, 1, <)$ *is o-minimal.*

Example 4.2. *It follows from Wilkie's Theorem (Wilkie, 1996) that the ordered field structure of real numbers with the exponential function* $\mathcal{R} = (\mathbb{R}; +, -, \cdot, 0, 1, <, \exp(x))$ *is o-minimal.*

Example 4.3. *Ordered structure of rational numbers* $\mathcal{Q} = (\mathbb{Q}; <)$ *is also o-minimal.*

Definition 4.2. *A subset of a real closed field (a field which has the same first-order properties with the field of real numbers) is called* ***semialgebraic set*** *if it is defined by finitely many polynomial equations and inequalities, or any finite union of such sets.*

Example 4.4. *The definable sets in the o-minimal structure* $\mathcal{R} = (\mathbb{R}; +, -, \cdot, 0, 1, <)$ *are semialgebraic sets and the Figure 4.1 below, Bonhomme, is an example of such sets. (Tarski-Seidenberg Theorem) (Tarski, 1951). Because it can be defined as :*

$A = \{(x, y) \in \mathbb{R}^2 : [x^2 + y^2 = 1] \vee [x \in (\frac{-1}{2}, \frac{1}{2}) \wedge y = \frac{-1}{2}] \vee [x = \frac{1}{3} \wedge y = \frac{1}{2}] \vee [x = \frac{-1}{3} \wedge y = \frac{1}{2}]\}$.

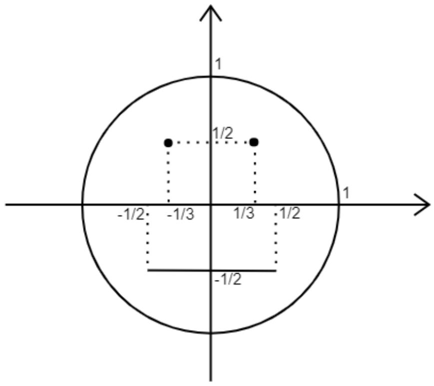

Figure 4.1: Bonhomme

Example 4.5. $\mathcal{R} = (\mathbb{R}; +, -, \cdot, 0, 1, <, f)$ *where* $f : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto \sin x$ *is not an o-minimal structure since the definable set* $\{x \in \mathbb{R} : f(x) = 0\}$ *is an infinite union of points.*

Now to understand better the restriction on definable sets in an o-minimal structure, we give a number of examples of sets which cannot be definable in such structures.

Example 4.6. *Take the subset* $A_0 := \{(m, n) \in \mathbb{N}^2 : m \leq n\}$ *in* $\mathbb{R}^2$. A_0 *is not a definable set in the o-minimal structure* $\mathcal{R} = (\mathbb{R}; <)$ *because the projection of* A_0 *on the* x*-axis is defined by the formula* $\Phi(v) : \exists n \ (v, n) \in A_0 \ \wedge \ (v < n \vee v = n)$ *which is not a finite union of intervals and points.*

Example 4.7. *Take a subset of* $\mathbb{R}^2$

$$A_1 := \{(x, y) \in \mathbb{R}^2 : x \in (1, 3) \cup (4, 6) \text{ and } y \in \mathbb{R} \setminus \mathbb{Q} \text{ if } x \in (1, 3) \text{ and } y \in \mathbb{Q} \text{ if } x \in (4, 6)\}.$$

We will show that A_1 *is not definable by contradiction. Assume that* A_1 *is definable. Consider the set* $B = \{(x, y) \in A_1 : x < y\}$ *which is clearly definable. Now look at the projection of* B *on the* y*-axis; that is,* $\Pi_y(B) = \{y \in \mathbb{R} : (x, y) \in A_1 \text{ and } x < y\}$. *Remark that it can be defined by the formula* $\Phi(v) : \exists x \ (x, v) \in A_1 \ \wedge \ x < v$ *as* A_1 *is definable but on the other hand we have* $\Pi_y(B) = (4, \infty) \cup ((1, 4) \cap \mathbb{R} \setminus \mathbb{Q})$ *which contradicts with being definable in an o-minimal structure. Hence* A_1 *is not definable in* $\mathcal{R} = (\mathbb{R}; <)$.

Example 4.8. *Take a subset* $A_2 := \{(x, y) \in \mathbb{R}^2 : y = x + 2 \sin x\}$ *in* $\mathbb{R}^2$.
Similar to the previous example 4.7, we see that the projection of B *on the* y*-axis is not definable since it is an infinite union of intervals where* $B = \{(x, y) \in A_2 : x < y\}$. *Hence* A_2 *is not definable in* $\mathcal{R} = (\mathbb{R}; <, +, -, \cdot, 0, 1)$.

We finish this section with the following two lemmas which show how tame the definable sets are in o-minimal structures. They will help us prove the CDT in the next chapter.

Lemma 4.1. *Let $\mathcal{R} = (R; <, \ldots)$ be an o-minimal structure and $A \subseteq R$ be definable. Then*

i. $\inf(A)$ *and* $\sup(A)$ *exist in* $R_\infty := R \cup \{\infty, -\infty\}$.

ii. the boundary $\text{bd}(A) := \{x \in R : \text{each interval containing } x \text{ intersects both } A \text{ and } R \setminus A\}$ *is finite, and if* $-\infty = a_0 < a_1 < \cdots < a_k < a_{k+1} = \infty$ *are the points of* $\text{bd}(A)$ *in order, then for each* $i \in \{0, \ldots, k+1\}$, *interval* (a_i, a_{i+1}) *is either part of* A *or disjoint from* A.

Proof Since $A \subseteq R$ is definable then by o-minimality A is a finite union of intervals and points. Let $A = (a_0, a_1) \cup (a_2, a_3) \cup \cdots \cup (a_{n-1}, a_n) \cup \{b_0, \ldots, b_m\}$ for some $n, m \in \mathbb{N}$ with $a_0 < \cdots < a_n$ and $b_0 < \ldots. < b_m$.

Then it's easy to see that $\inf(A) = min\{a_0, b_0\}$ and $\sup(A) = max\{a_n, b_m\}$. So we proved $(i.)$.

Now we prove $(ii.)$. If $a_0, a_n \in R$ then it is easy to see that

$$\text{cl}(A) = [a_0, a_1] \cup \cdots \cup [a_{n-1}, a_n] \cup \{b_0, \ldots, b_m\}, \text{ and}$$

$$\text{int}(A) = (a_0, a_1) \cup \cdots \cup (a_{n-1}, a_n).$$

Thus one can easily obtain that $\text{bd}(A) = \{a_0, \ldots, a_n, b_0, \ldots, b_m\}$.

If $a_0 = -\infty$ or $a_n = \infty$, without loss of generality assume $a_0 = -\infty$. Then

$$\text{cl}(A) = (a_0, a_1] \cup \cdots \cup [a_{n-1}, a_n] \cup \{b_0, \ldots, b_m\}, \text{ and}$$

$$\text{int}(A) = (a_0, a_1) \cup \cdots \cup (a_{n-1}, a_n).$$

It can be easily seen again that $\text{bd}(A) = \{a_1, \ldots, a_n, b_0, \ldots, b_m\}$.

Therefore we proved that $\text{bd}(A)$ is finite.

Without loss of generality, assume that $\text{bd}(A) = \{a_0, \ldots, a_n, b_0, \ldots, b_m\}$. Now we examine the cases :

1^{st} case : If $a_n < b_0$, then we see that $(a_n, b_0), (b_0, b_1), \ldots, (b_{m-1}, b_m)$ are disjoint from A and besides these ones, other intervals will be a part of A. (The case where $b_m < a_0$ goes exactly in the same way.)

2^{nd} case : If there is any b_j and a_i, a_{i+1} such that $a_i < b_j < a_{i+1}$ for some i, j with $(a_i, a_{i+1}) \cap A = \emptyset$, then the intervals (a_i, b_j) and (b_j, a_{i+1}) will be disjoint from A. This establishes the proof of $(ii.)$.

□

Lemma 4.2. *If $A \subseteq R^m$ is definable, then so are the topological closure $\overline{A}$ and the interior A° of A.*

Proof The following two formulas define $\overline{A}$ and A°, respectively :

$$\Phi(v_1, \ldots, v_m) : \forall x_1 \ldots \forall x_m \forall y_1 \ldots \forall y_m \Big[x_1 < v_1 < y_1 \wedge \cdots \wedge x_m < v_m < y_m\Big] \implies$$

$$\exists z_1 \ldots \exists z_m \big(x_1 < z_1 < y_1 \wedge \cdots \wedge x_m < z_m < y_m \wedge (z_1, \ldots, z_m) \in A\big)$$

$$\Psi(v_1, \ldots, v_m) : \exists x_1 \ldots \exists x_m \exists y_1 \ldots \exists y_m \Big[\big(x_1 < v_1 < y_1 \wedge \cdots \wedge x_m < v_m < y_m\big) \wedge$$

$$\forall z_1 \ldots \forall z_m \big(x_1 < z_1 < y_1 \wedge \cdots \wedge x_m < z_m < y_m \implies (z_1, \ldots, z_m) \in A\big)\Big]$$

□

4.2 Main Results of O-minimality

In this section, we state two important consequences of o-minimality ; MT and CDT. By the aid of these theorems, we will be able to see the strength of o-minimality. We will show that the constraints on the definable sets of R impose strong constraints on the definable sets of R^n.

4.2.1 Monotonicity Theorem

We can understand the nature of definable functions with one variable through the MT. Before giving the statement of MT, we establish three lemmas on which the theorem is based.

For this chapter, we fix an arbitrary o-minimal structure $\mathcal{R} = (R; <, \ldots)$. Recall from the Definition 4.1 that $<$ is a dense linear order on the set R and that every definable subset of R

is a finite union of intervals and points. We will consider the Order Topology on R and the Product Topology on R^m.

For the following three lemmas, consider that $f : I \longrightarrow R$ is a definable function on a non-empty interval I.

Lemma 4.3. *There is a subinterval of I on which f is constant or injective.*

Proof If $f^{-1}(y)$ is infinite for some $y \in R$, then as f is definable, by o-minimality, the preimage would contain a subinterval in I on which the function is constant.

Then we can assume that for each y in the range of f, $f^{-1}(y)$ is finite. It follows then $f(I)$ is infinite. As it is definable then it contains an interval $J \subseteq f(I)$. Now we define a function $g : J \longrightarrow I$ by $g(y) := \min\{x \in I : f(x) = y\}$. Note that g is definable and injective. Then $g(J)$ is infinite. Thus by o-minimality we can find a subinterval of $g(J)$ contained in I on which the function f is necessarily injective.

□

Lemma 4.4. *If f is injective, then f is strictly monotone on a subinterval of I.*

Proof Let $I = (a, b)$ where $a, b \in R$. For each $x \in I$, we can write (a, x) as a disjoint union of the following two sets :

$$(a, x) = \{y \in (a, x) : f(y) < f(x)\} \bigsqcup \{y \in (a, x) : f(y) > f(x)\}.$$

At least one of these disjoint sets is infinite, then one of them contains a subinterval (c, x) with $a < c < x$. The interval (x, b) splits up similarly.

Then we obtain that each $x \in I$ satisfies exactly one of the following four formulas :

$$\Phi_{++}(x) := \exists c_1, c_2 \in I\ [c_1 < x < c_2,\ \forall y \in (c_1, x) : f(y) > f(x),$$

$$\forall y \in (x, c_2) : f(y) > f(x)],$$

$$\Phi_{+-}(x) := \exists c_1, c_2 \in I\ [c_1 < x < c_2,\ \forall y \in (c_1, x) : f(y) > f(x),$$

$$\forall y \in (x, c_2) : f(y) < f(x)],$$

$$\Phi_{-+}(x) := \exists c_1, c_2 \in I\, [c_1 < x < c_2,\ \forall y \in (c_1, x) : f(y) < f(x),$$

$$\forall y \in (x, c_2) : f(y) > f(x)],$$

$$\Phi_{--}(x) := \exists c_1, c_2 \in I\, [c_1 < x < c_2,\ \forall y \in (c_1, x) : f(y) < f(x),$$

$$\forall y \in (x, c_2) : f(y) < f(x)].$$

I has infinitely many points that satisfy one of the four formulas above which are definable. Then we can find a subinterval $J \subseteq I$, all of whose points satisfy the same formula. This leads to four cases. Let $J = (c, d)$.

Easy Case : $\Phi_{-+}(x)$ for all $x \in J$.

For each $x \in J$, define $s(x) := \sup\{s \in (x, d) : f(s) > f(x) \text{ on } (x, s]\}$. Then clearly $s(x) = d$, since $s(x) < d$ contradicts $\Phi_{-+}(s(x))$. Thus f is strictly increasing on J.

The case that $\Phi_{+-}(x)$ for all $x \in J$ goes the same way.

Difficult Case : $\Phi_{++}(x)$ for all $x \in J$.

Consider the set $B := \big\{x \in J : \forall y \in J \ \big(y > x \implies f(y) > f(x)\big)\big\}$.

If B is infinite then B contains a subinterval. So f is strictly increasing on this subinterval which finishes the proof of this case.

If B is finite then we can find an interval which is at the right of all points of B. Let us denote this subinterval by J'. We may assume then

$$\forall x \in J'\ \exists y \in J'\ \big(y > x \text{ and } f(y) < f(x)\big). \tag{4.1}$$

Claim : Let $a' \in J'$. For all large enough $y \in J'$, we have $f(y) < f(a')$.

Suppose not and say that for all large enough $y \in (a', d)$, we have $f(y) > f(a')$. Then we can find a minimal element $e \in [a', d)$ such that $\forall y\ \big(e < y < d \implies f(y) > f(a')\big)$.

We must have $f(e) < f(a')$ because e is minimal and $\Phi_{++}(e)$. By assumption (4.1), we find some k with $e < k < d$ and $f(k) < f(e)$ which gives that $f(k) < f(a')$. But since $e < k$ then we must have $f(k) > f(a')$. Contradiction.

Define now $y(a')$ as the least element in $[a', d)$ where the claim holds ; that is, $f(y) < f(a')$ if $y(a') < y < d$. Since $\Phi_{++}(a')$ and $y(a')$ is the minimal element, then we have that $a' < y(a')$ and $f(y(a')) < f(a')$. Thus we see that $y(a')$ satisfies the following formula :

$$\Psi_{+-}(v) := \exists v_1, v_2 \in J' \left[v_1 < v < v_2 \text{ and } \forall z_1, z_2 \left(v_1 < z_1 < v < z_2 < v_2 \implies f(z_1) > f(z_2)\right)\right]$$

Because the minimality of $y(a')$ and $\Phi_{++}(a')$ yields that if $y(a') < y$ then $f(y) < f(a')$ and if $y < y(a')$ then $f(a') < f(y)$.

Since a' was arbitrary, then we have sown that $\forall a' \in J' \; \exists v \in J' \; (v > a'$ and $\Psi_{+-}(v))$.

Therefore, $\Psi_{+-}(v)$ holds for all v in an interval of the form (f, d) where $f \in J'$. So we have found a subinterval on which f is strictly decreasing.

Similar argument shows that we can find a smaller subinterval on which $\Psi_{-+}(v)$ holds. This yields a contradiction because we cannot have both $\Psi_{+-}(v)$ and $\Psi_{-+}(v)$. So we showed that B cannot be finite.

The case that $\Phi_{--}(x)$ holds for all $x \in J$ is shown in a similar way with the previous. This finishes the proof of the lemma.

□

Lemma 4.5. *If f is strictly monotone, then f is continuous on a subinterval of I.*

Proof Without loss of generality, assume that f is strictly increasing. As $f(I)$ is infinite, we can find a subinterval in $f(I)$. Then by taking two points r, s with $r < s$ from this subinterval such that $f(c) = r, f(d) = s$, since f is strictly increasing, we get $c < d$. Remark that the function defines an order preserving bijection between the intervals (c, d) and (r, s). Since R has the order topology, f is continuous on (c, d).

□

Theorem 4.1. *(Monotonicity Theorem) Let $f : (a, b) \longrightarrow R$ be a definable function on the interval (a, b). Then there are points $a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b$ in (a, b) such that the function is either constant or strictly monotone and continuous on each subinterval (a_i, a_{i+1}).*

Proof Let $f : (a, b) \longrightarrow R$ be a definable function on the interval (a, b) and
$X := \{x \in (a, b) :$ on some interval of (a, b) containing x, the function f is either constant or strictly monotone and continuous$\}$.

Claim : The set $(a, b) \setminus X$ is finite.

We prove by contradiction. Suppose that $(a, b)\setminus X$ is infinite. Since $(a, b)\setminus X$ is definable in an o-minimal structure $\mathcal{R} = (R; <, \ldots)$, then it must contain a subinterval I. Then it follows from the Lemmas 4.3, 4.4, and 4.5 that we can find a subinterval J included in I on which the function f is either constant or strictly monotone and continuous. But then by definition, J must be in X. We have a contradiction, thus $(a, b) \setminus X$ is finite.

Since $(a, b) \setminus X$ is finite, we can reduce the proof to the case where $(a, b) = X$. With this reduction, we have replaced (a, b) by finitely many intervals of which the open set X consists. Again since there are finite number of points that does not belong to X, we may assume that f is continuous. By splitting up (a, b) into subintervals, we see that on each subinterval f must satisfy one of the following three cases. We will use (a, b) instead of subintervals to avoid new notations.

1^{st} Case : For all $x \in (a, b)$, f is constant on some neighborhood of x.

Take $x_0 \in (a, b)$ and put

$$s := \sup\{x : x_0 < x < b,\ \text{f is constant on } [x_0, x)\}$$

If $s < b$, then by assumption f must be constant on some neighborhood of s. Hence we get $s = b$, so f is constant on $[x_0, b)$.

Now put the following set

$$t := \inf\{x : a < x < x_0,\ f \ \text{ is constant on } (a, x_0]\}.$$

Similarly, we obtain that $t = a$ and so f is constant on $(a, x_0]$.

Therefore, we conclude that f is constant on (a, b).

2^{nd} Case : For all $x \in (a, b)$, f is strictly increasing on a neighborhood of x.

3^{rd} Case : For all $x \in (a, b)$, f is strictly decreasing on a neighborhood of x.

The proofs of these two cases are similar to the first case. Hence we finished the proof of the Monotonicity Theorem. $\square$

Lemma 4.6. *(Finiteness Lemma) Let $A \subseteq R^2$ be a definable set. Suppose that for each $x \in R$, the fiber $A_x := \{y \in R : (x, y) \in A\}$ is finite. Then there is $N \in \mathbb{N}$ such that for all $x \in R$, $|A_x| \leq N$.*

Proof See (van den Dries, 1998, p. 47) □

Before moving on to the CDT, we use the MT and the Finiteness Lemma to deduce the following, which is a description of the definable sets in R^2 :

Let $A \subseteq R^2$ be a definable set such that each fiber A_x is finite. By Finiteness Lemma 4.6, we know that there is a number $N \in \mathbb{N}$ such that $|A_x| \leq N$. Since A is definable, then its projection to the x-axis is a finite union of intervals and points. Then we can divide this projection into the points $a_1 < \cdots < a_k$ so that for each subinterval (a_i, a_{i+1}), $|A_x| = n(i)$ where $x \in (a_i, a_{i+1})$ and $i \in \{1, \ldots, k\}$.

Now for any subinterval (a_i, a_{i+1}), we define the functions $f_{ij} : (a_i, a_{i+1}) \to R$ where $j \in \{1, \ldots, n(i)\}$.

By MT, we know there is finitely many points $b_1 < \cdots < b_m$ in (a_i, a_{i+1}) so that each function f_{ij} is continuous on (b_l, b_{l+1}) where $l \in \{1, \ldots, m\}$.

Therefore, we can conclude that the intersection of A with each vertical strip $(b_l, b_{l+1}) \times R$ is of the form $\Gamma(f_{i1}) \cup \ldots . \cup \Gamma(f_{in(i)})$ for certain definable continuous functions $f_{ij} : (b_l, b_{l+1}) \longrightarrow R$ with $f_{i1} < \cdots < f_{in(i)}$.

Example 4.9. *Here we can give a similar example to Bonhomme 4.1 again. Figure 4.2 is a good example to understand the previous sequence of arguments.*

We divide the interval $(-1, 1)$ into the subintervals (a_i, a_{i+1}) as for any $x \in a_i, a_{i+1})$, A_x has the same number of points where $i \in \{1, \ldots, 3\}$, $a_1 = -1, a_2 = -3/5, a_3 = -3/5$ and $a_4 = 1$. Consider the subinterval (a_2, a_3). Note that $|A_x| = 3$ for any $x \in (a_2, a_3)$.

*Now we have the functions $f_{21} : (a_2, a_3) \to \mathbb{R} : x \mapsto -\sqrt{1 - x^2}$,
$f_{22} : (a_2, a_3) \to \mathbb{R}$ defined by $f_{22}(x) = -2/5$ if $x \in (-2/5, 2/5)$ and $f_{22}(x) = 2/5$ otherwise, and $f_{23} : (a_2, a_3) \to \mathbb{R} : x \mapsto \sqrt{1 - x^2}$.*

*Remark that f_{22} is not continuous but as in the MT, we find the points $b_1 = a_2$,
$b_2 = -2/5, b_3 = 2/5$ and $b_4 = 3/5$ so that f_{22} is continuous on each subinterval (b_l, b_{l+1})*

where $l \in \{1, 2, 3, 4\}$.

Therefore, the intersection of the Frowning Bonhomme with $(b_l, b_{l+1}) \times \mathbb{R}$ *is* $\Gamma(f_{21}|(b_l, b_{l+1})) \cup \Gamma(f_{22}|(b_l, b_{l+1})) \cup \Gamma(f_{23}|(b_l, b_{l+1}))$.

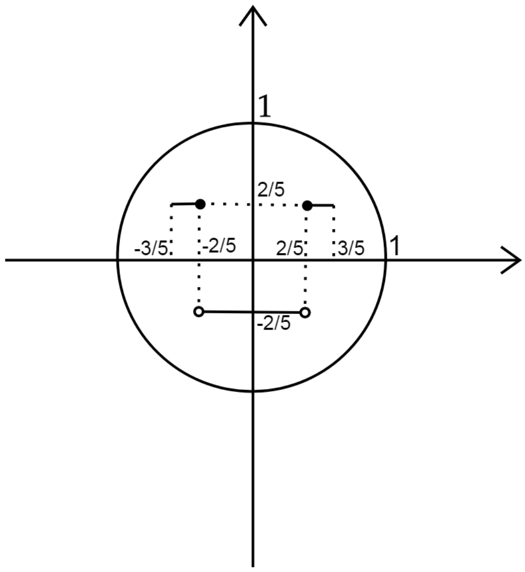

Figure 4.2: Frowning Bonhomme

4.2.2 Cell Decomposition Theorem

In this section, our aim is to give a brief proof of CDT which helps us develop an understanding of definable sets of higher dimensions and definable multi-variable functions in an o-minimal structure.

Before the proof of the theorem, we present the notions of a cell and a decomposition.

For any definable set $X \subseteq R^m$, we put the following sets

$$C(X) := \{f : X \longrightarrow R : \text{f is definable and continuous}\},$$

$$C_\infty(X) := C(X) \cup \{-\infty, +\infty\}$$

where we consider $-\infty$ and $+\infty$ as constant functions on X.

Let $f, g \in C(X)$. We write $f < g$ if for all $x \in X,\ f(x) < g(x)$, and put

$$(f, g)_X := \{(x, r) \in X \times R : f(x) < r < g(x)\}.$$

Note that $(f, g)_X$ is a definable set of R^{m+1} by Lemma 3.1, since both X and the set $\{r \in R : f(x) < r < g(x)\}$ are definable.

If X is clear from the context, we will write (f, g) instead of $(f, g)_X$.

Definition 4.3. *(Cell) Let $(i_1, \ldots, i_m)$ be a sequence of zeros and ones of length m. An $(i_1, \ldots, i_m)$-cell is a definable subset of R^m obtained by induction on m as follows :*

i. a (0)-cell is a set whose element is a point in R, a (1)-cell is an interval $(a, b) \subseteq R$

ii. suppose that $(i_1, \ldots, i_m)$-cells are defined, then an $(i_1, \ldots, i_m, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in C(X)$ where X is an $(i_1, \ldots, i_m)$-cell; further, an $(i_1, \ldots, i_m, 1)$-cell is a set $(f, g)_X$ where X is an $(i_1, \ldots, i_m)$-cell and $f, g \in C_\infty(X)$ with $f < g$.

Here we have an image of a $(f, g)_I$, which is a $(1, 1)$-cell, for some $f, g \in C(I)$ where I is an interval :

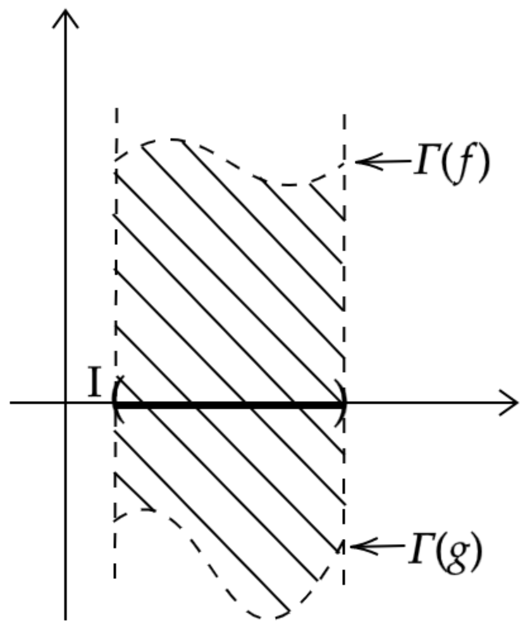

Figure 4.3: $(1, 1)$-cell

Example 4.10. *Now we give an example for a $(1, 0, 1)$-cell :*

We construct it inductively. Let (1)-cell be the interval $(0, 1) \subseteq \mathbb{R}$. Now we need to construct a $(1, 0)$-cell. Take the function $f : (0, 1) \longrightarrow R : x \mapsto x^2$ which is definable and continuous. Then by definition the graph $\Gamma(f) = \{(x, f(x)) \in \mathbb{R}^2 : f(x) = x^2\}$ is a $(1, 0)$-cell.

Now find the $(1, 0, 1)$-cell where $(1, 0)$-cell is the graph $\Gamma(f)$ above. Take two functions $g : \Gamma(f) \longrightarrow R : (x, y) \mapsto (xy)^2$ and $h : \Gamma(f) \longrightarrow R : (x, y) \mapsto (xy)^{1/3}$. See that $g < h$.

Thus the set

$$(g,h)_{\Gamma(f)} = \{(x,y,z) \in \mathbb{R}^3 : g(x,y) < z < h(x,y)\}$$

is by definition a $(1,0,1)$*-cell.*

Remark that a cell in R^m is an $(i_1,\dots,i_m)$-cell for some unique sequence $(i_1,\dots,i_m)$. Since $(i_1,\dots,i_m)$-cells are open in R^m when all $i_j = 1$ where $j \in \{1,2,\dots,m\}$, we call these **open cells**. As we have product topology on R^m, we will also call these open cells **box**.

Proposition 4.1. *Each cell is homeomorphic under a coordinate projection to an open cell.*

Proof See (van den Dries, 1998, p. 51) □

Definition 4.4. *A set* X *in* R^m *is called definably connected if it is definable and is not a disjoint union of any two non-empty definable open subsets of* X*.*

Example 4.11. *The set* $X = \{(x,y) \in \mathbb{R}^2 : xy = 1\}$ *is not definably connected in the structure* $(\mathbb{R}; <, +, \cdot, 0, 1)$ *as we can write* $X = U_1 \cup U_2$ *where*

$$U_1 = \{(x,y) \in \mathbb{R}^2 : xy = 1 \wedge x < 0\}, \text{and}$$

$$U_2 = \{(x,y) \in \mathbb{R}^2 : xy = 1 \wedge x > 0\}$$

which are both definable and open in X*.*

Proposition 4.2. *Each cell is definably connected.*

Proof See (van den Dries, 1998, p. 51) □

Definition 4.5. *(Cell Decomposition) We say* $\mathcal{D}$ *is a decomposition of* R^m *if it is a finite partition of* R^m *whose elements are cells in* R^m*. We define a decomposition by induction on* m*,*

i. a decomposition of R *is a collection of finitely many disjoint intervals and points*

$$\{(-\infty, a_1), \dots, (a_k, +\infty), \{a_1\}, \{a_2\}, \dots, \{a_k\}\}$$

where $a_1 < \dots < a_k$ *are points in* R*.*

ii. a decomposition of R^{m+1} *is a finite partition of* R^{m+1} *into cells A such that the set of the usual projections* $\Pi(A)$ *is a decomposition of* R^m*.*

Remark 4.1. *By the definition of cell decomposition, we can deduce the following : Let* $\mathcal{D} = \{A(1), \dots, A(k)\}$ *be a decomposition of* R^m *where* $A(i) \neq A(j)$ *if* $i \neq j$*, and let for each* $i \in \{1, \dots, k\}$ *functions* $f_{i1} < \cdots < f_{in_i}$ *in* $C(A_i)$ *be given. Then* $\mathcal{D}^* := \mathcal{D}_1 \cup \mathcal{D}_2 \cup \cdots \cup \mathcal{D}_k$ *is a decomposition of* R^{m+1} *where* $\mathcal{D}_i := \{(-\infty, f_{i1}), (f_{i1}, f_{i2}), \dots, (f_{in_i}, \infty), \Gamma(f_{i1}), \dots, \Gamma(f_{in_i})\}$ *is a partition of* $A(i) \times R$*.*

Definition 4.6. *A decomposition* $\mathcal{D}$ *of* R^m *is said to* ***partition*** *a set* $S \subseteq R^m$ *if each cell in* $\mathcal{D}$ *is either in* S *or disjoint from* S*; that is,* S *is a union of cells in* $\mathcal{D}$*.*

Example 4.12. *Let* $\mathcal{D} = \{(-\infty, a_0), (a_0, a_1), (a_1, \infty), \{a_0\}, \{a_1\}\}$ *is a decomposition of* $\mathbb{R}$ *for some* $a_0, a_1 \in \mathbb{R}$*. Observe that* $\mathcal{D}$ *partitions* (a_0, ∞) *whereas it does not partition the set* (a_2, ∞) *where* $a_0 < a_2 < a_1$*. Because the cell* (a_0, a_1) *is neither a subset of* (a_2, ∞) *nor included in it.*

Definition 4.7. *Let* Y *be a definable subset of* R^{m+1}*. We say that* Y *is* ***finite over*** R^m *if for all* $x \in R^m$*, the fiber* $Y_x := \{r \in R : (x, r) \in Y\}$ *is finite.*

Definition 4.8. *Let* Y *be a definable subset of* R^{m+1}*. We say that* Y *is* ***uniformly finite over*** R^m *if there is* $N \in \mathbb{N}$ *such that* $|Y_x| \leq N$ *for all* $x \in R^m$*.*

Example 4.13. *Consider the set* A *as in the Example 4.4 which defines Figure 4.1. It is easy to see that for any* $x \in (-1, 1)$*, each fiber* $A_x = \{y \in \mathbb{R} : (x, y) \in A\}$ *is finite. We also see that* A *is uniformly finite over* $\mathbb{R}$ *because* $|A_x| \leq 4$ *for any* $x \in (-1, 1)$*.*

Lemma 4.7. *(Uniform Finiteness Lemma) Suppose that* $Y \subseteq R^{m+1}$ *is a definable set. If* Y *is finite over* R^m*, then* Y *is uniformly finite over* R^m*.*

Proof See (van den Dries, 1998, p. 56) □

Lemma 4.8. *Let* X *be a topological space,* $(R_1, <), (R_2, <)$ *dense linear orderings without endpoints and* $f : X \times R_1 \longrightarrow R_2$ *a function such that for each* $(x, r) \in X \times R_1$

i. $f(x, .) : R_1 \longrightarrow R_2$ *is continuous and monotone on* R_1

ii. $f(., r) : X \longrightarrow R_2$ *is continuous at* x*.*

Then, f *is continuous.*

Proof See (van den Dries, 1998, p. 56) □

Now we state the CDT and give the main steps of the proof without going into too much detail.

Theorem 4.2. *(Cell Decomposition Theorem)*

(I_m) *Given any definable sets* $A_1, \ldots, A_k \subseteq R^m$*, there is a decomposition of* R^m *partitioning each of* $A_1, \ldots, A_k$*.*

(II_m) *For each definable function* $f : A \longrightarrow R,\ A \subseteq R^m$*, there is a decomposition* $\mathcal{D}$ *of* R^m *partitioning* A *such that the restriction* $f|_B : B \longrightarrow R$ *to each cell* $B \in \mathcal{D}$ *with* $B \subseteq A$ *is continuous.*

Proof We will prove by induction on m :

(I_1) and (II_1) holds by o-minimality and by the MT, respectively.

Now suppose that $(I_1), \ldots, (I_m)$ and $(II_1), \ldots, (II_m)$ hold. First we show (I_{m+1}) also holds :

Let $A_1, \ldots, A_k$ be definable sets in R^{m+1}. We will find a decomposition of R^{m+1} that partitions each A_i where $i \in \{1, \ldots, k\}$.

For a definable set $A \subseteq R^{m+1}$, put

$$\text{bd}_m(A) := \{(\bar{x}, r) \in R^{m+1} : r \in \text{bd}(A_{\bar{x}})\}$$

where bd$(A_{\bar{x}})$ denotes the boundary of $A_{\bar{x}}$ and $A_{\bar{x}} = \{r \in R : (\bar{x}, r) \in A\}$.

By Lemma 4.1, we know that bd$(A_{\bar{x}})$ is definable and observe that
$\text{bd}_m(A)_{\bar{x}} := \{r \in R : (\bar{x}, r) \in \text{bd}_m(A)\}$ is finite. Then we obtain that $\text{bd}_m(A)$ is definable and finite over R^m.

Put $Y := \text{bd}_m(A_1) \cup \cdots \cup \text{bd}_m(A_k)$. Since for each $i \in \{1, \ldots, k\},\ \text{bd}_m(A_i)$ is definable and finite over R^m, then by Uniform Finiteness Lemma 4.7, there is a number $M \in \mathbb{N}$ such that for all $\bar{x} \in R^m$, $|Y_{\bar{x}}| \leq M$.

For each $i \in \{0, \ldots, M\}$, let $B_i := \{\bar{x} \in R^m : |Y_{\bar{x}}| = i\}$ and define functions $f_{i1}, \ldots, f_{ii}$ on B_i by $Y_{\bar{x}} = \{f_{i1}(\bar{x}), \ldots, f_{ii}(\bar{x})\}$ such that $f_{i1}(\bar{x}) < \cdots < f_{ii}(\bar{x})$ with $f_{i0} := -\infty$ and $f_{i(i+1)} := \infty$ which are constant functions on B_i.

Remark that B_i's are defined independently from A_i's. We want to define such sets so that they depend on A_i's and find a decomposition that partitions each set we defined. Therefore, we will obtain a decomposition partitioning each A_i.

Now define following sets for each $\lambda \in \{1, \ldots, k\}$, $i \in \{0, \ldots, M\}$ and $1 \leq j \leq i$;

$$C_{\lambda_{ij}} := \{\bar{x} \in B_i : f_{ij} \in (A_\lambda)_{\bar{x}}\}$$

and for each $\lambda \in \{1, \ldots, k\}$, $i \in \{0, \ldots, M\}$ and $0 \leq j \leq i$;

$$D_{\lambda_{ij}} := \{\bar{x} \in B_i : (f_{ij}, f_{i(j+1)}) \subseteq (A_\lambda)_{\bar{x}}\}.$$

Since each of B_i, $C_{\lambda_{ij}}$ and $D_{\lambda_{ij}}$ is in R^m, then by (I_m), there is a decomposition $\mathcal{D}$ of R^m which partitions each set $C_{\lambda_{ij}}$ and $D_{\lambda_{ij}}$ for all $i \in \{1, \ldots, M\}$ and for all $\lambda \in \{1, \ldots, k\}$. We know also by assumption (II_m) that if a cell $E \in \mathcal{D}$ is contained in B_i, then each restriction of the functions $f_{i1}|_E, \ldots, f_{ii}|_E$ is continuous.

For each cell $E \in \mathcal{D}$, define $\mathcal{D}_E$ as follows :

$$\mathcal{D}_E := \{(f_{i0}|_E, f_{i1}|_E), \ldots, (f_{ii}|_E, f_{i(i+1)}|_E), \Gamma(f_{i1}|_E), \ldots, \Gamma(f_{ii}|_E)\}$$

where $i \in \{0, \ldots, M\}$ is such that $E \subseteq B_i$. Here we see that $\mathcal{D}_E$ is a decomposition partitioning $E \times R$.

Then by Remark 4.1, we get $\mathcal{D}^* := \bigsqcup\{\mathcal{D}_E : E \in \mathcal{D}\}$ is a decomposition of R^{m+1} which partitions each set $A_1, \ldots, A_k$. This finishes the proof of (I_{m+1}).

Now we prove that (II_{m+1}) holds :

Let $f : A \longrightarrow R$ be a definable function on a definable set $A \subseteq R^{m+1}$.

We must show that f is cellwise continuous. By (I_{m+1}), if we take $k = 1$, we obtain that A can be partitioned into finitely many cells. Then we can consider A as a cell to avoid new notations. Now it is enough to prove that f is cellwise continuous on this cell A. Thus we have two cases to examine :

1^{st} Case : If A is a non-open cell in R^{m+1} :

By Proposition 4.1, let $p_A : A \longrightarrow p(A)$ be the definable homeomorphism where $p(A)$ is an open cell in R^n with $n \leq m$. By assumption (II_n), we find a decomposition partitioning $p(A)$ into finitely many definable cells $B_1, \ldots, B_k$ such that $(f \circ p_A^{-1})|_{B_j}$ is continuous for each $j \in \{1, \ldots, k\}$. Since p_A is a homeomorphism then for each B_j, $p_A^{-1}(B_j)$ is definable. Remark that the union of the definable cells $p_A^{-1}(B_j)$ form a partition of A. Hence A is partitioned into $p_A^{-1}(B_1), \ldots, p_A^{-1}(B_k)$, and so the restriction of f to each of these sets is continuous. This concludes the first case.

2^{nd} Case : If A is an open cell in R^{m+1} :

To prove this case, first we define the following notion :

Call f **well-behaved** at a point $(\bar{x}, r) \in A$ if $\bar{x} \in C$ for some box $C \subseteq R^m$ and $a < r < b$ for some $a, b \in R$ such that

i. $C \times (a, b)$ is contained in A

ii. For all $\bar{x} \in C$, the function $f(x, .)$ is continuous and monotone on (a, b)

iii. The function $f(., r)$ is continuous at p.

Let A^* be the set of all points of A at which f is well-behaved. Note that A^* is definable since each i., ii. and iii. can be formulated.

By the following claim and by (I_{m+1}), we will obtain that any open cell contained in A is included in A^*. Then f will be well-behaved on this open cell. Using the Lemma 4.8 we will obtain that f is continuous on this open cell which is what we want to prove.

Claim : A^* is dense in A.

It is enough to show that for any given box B in R^m and $-\infty < a < c < \infty$ such that $B \times (a, c)$ is contained in A, the box $B \times (a, c)$ intersects A^*, then we will be proved the claim.

By MT, for all $x \in B$, there is a largest $\lambda(x) \in (a, c]$ such that the one-variable function $f(x, .)$ is continuous and monotone on $(a, \lambda(x))$. As $\lambda(x) : B \longrightarrow R$ is definable, then by (II_m), there is a box $C \subseteq B$ on which λ is continuous.

If we take C small enough, we may assume that $b \leq \lambda(x)$, for all $x \in C$. So fix such a $b \in (a, c)$ and choose any element $r \in (a, b)$. By (II_m), the function $f(., r) : C \longrightarrow R$ is continuous on some smaller box. If we replace C by this smaller box, then we see that f is well-behaved at each point (p, r) with p in C. Because :

i. $(p, r) \in C' \times (a, b)$ where C' is the smaller box,

ii. for all $x \in C'$, the function $f(x, .)$ is continuous and monotone on $(a, \lambda(x))$, and

iii. $f(., r)$ is continuous on C'.

This establishes the claim.

Now, by (I_{m+1}), take a decomposition $\mathcal{D}$ of R^{m+1} that partitions A and A^*. Since A is open, there is at least one open cell contained in A. Let D be such an open cell.

It remains to show that f is continuous on D. We know that $\mathcal{D}$ partitions A^* and $D \cap A^* \neq \emptyset$ by the previous claim, then we obtain that $D \subseteq A^*$ as $D \subseteq A$.

Since $D \subseteq A$, for each point $(\bar{x}, r) \in D$ the function $f(., r)$ is continuous at $\bar{x}$. Therefore, D is the union of the boxes $C \times (a, b)$ satisfying i., ii. and iii. for each $\bar{x} \in C$ and $r \in (a, b)$. By Lemma 4.8, the function f is continuous on each such box, thus f is continuous on D. This concludes the proof of (II_{m+1}), hence the proof of the theorem.

$\square$

5 COMPARISON OF VAN DEN DRIES AND GABRIELOV REGULARITIES

In this chapter, we will introduce regularity in the sense of Lou van den Dries. Then we will state the Regular Cell Decomposition Theorem which is given in (van den Dries, 1998) as an exercise and prove it. Furthermore, we will give the definition of regularity as Gabrielov defines. In the end of the chapter, we will show these two definitions are not equivalent.

Throughout this chapter, we will call a cell (or function) **vdD-regular** or **Gabrielov-regular** in order to avoid any misunderstanding.

5.1 vdD- Regularity

In this section, we give the definition of vdD-regular cells (also vdD-functions) and then state the Regular Cell Decomposition Theorem which is stronger then CDT.

Definition 5.1. *(vdD-Regular Cell) An open cell $C \subseteq R^m$ is called regular for any two points $x, y \in C$ which differ only in the i^{th} coordinate and for any point $z \in R^m$ that differs from x and y only in the i^{th} coordinate, if we have that $x_i < z_i < y_i$ then $z \in C$.*

Definition 5.2. *(vdD-Regular Function) Consider a function $f : C \mapsto R$ where $C \subseteq R^m$ is a regular cell and $i \in \{1, \ldots, m\}$. We say that f is*

i. strictly increasing in the i^{th} coordinate if for any points $x, y \in C$ that differ only in the i^{th} coordinate with $x_i < y_i$ then $f(x) < f(y)$.

ii. strictly decreasing in the i^{th} coordinate if for any points $x, y \in C$ that differ only in the i^{th} coordinate with $x_i < y_i$ then $f(x) > f(y)$.

iii. independent of the i^{th} coordinate if for any points $x, y \in C$ that differ only in the i^{th} coordinate with $x_i < y_i$ then $f(x) = f(y)$.

The function f is called a regular function if f is continuous and for each $i \in \{1, \ldots, m\}$, f is either strictly increasing, or strictly decreasing in the i^{th} coordinate or independent of the i^{th} coordinate.

Before stating the Regular Cell Decomposition Theorem, we prove a lemma which is given as an exercise in (van den Dries, 1998).

Lemma 5.1. *Let $f, g : C \to R$ be two regular definable functions with $f < g$ where $C \subseteq R^m$ is a regular cell. Then the open cell $(f, g)_C$ is regular in R^{m+1}.*

Proof We have two cases to examine :

***CASE* 1** : If $\bar{a} = (x_1, \ldots, x_m, a_m), \bar{b} = (x_1, \ldots, x_m, b_m) \in (f, g)_C$ and $\bar{c} = (x_1, \ldots, x_m, c_m) \in R^{m+1}$ differ only in the $(m+1)^{th}$ coordinate with $a_{m+1} < c_{m+1} < b_{m+1}$:

Since $f(x_1, \ldots, x_m) < a_m < c_m < b_m < g(x_1, \ldots, x_m)$ then by definition $\bar{c} \in (f, g)_C$. Thus $(f, g)_C$ is regular.

***CASE* 2** : If $\bar{a} = (x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_{m+1}), \bar{b} = (x_1, \ldots, x_{i-1}, b_i, x_{i+1}, \ldots, x_{m+1})$ are in $(f, g)_C$ and $\bar{c} = (x_1, \ldots, x_{i-1}, c_i, x_{i+1}, \ldots, x_{m+1}) \in R^{m+1}$ differ only in the i^{th} coordinate with $a_i < c_i < b_i$ for some $i \in \{1, \ldots, m\}$:

Since C is regular then $(x_1, \ldots, x_{i-1}, c_i, x_{i+1}, \ldots, x_m) \in C$. Now we need to show that

$$f(x_1, \ldots, x_{i-1}, c_i, x_{i+1}, \ldots, x_m) < x_{m+1} < g(x_1, \ldots, x_{i-1}, c_i, x_{i+1}, \ldots, x_m).$$

For each function, there are 3 kind of monotonicity in the i^{th} coordinate which uncover 9 cases to examine. Since each of these cases is shown by the same way, we only give a proof for one of them.

Case : If f and g are both strictly decreasing in the i^{th} coordinate :

We know that $\bar{a}, \bar{b} \in (f, g)_C$, then

$$f(x_1, \ldots, x_{i-1}, c_i, x_{i+1}, \ldots, x_m) < f(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_m) < x_{m+1}$$

$$< g(x_1, \ldots, x_{i-1}, b_i, x_{i+1}, \ldots, x_m) < g(x_1, \ldots, x_{i-1}, c_i, x_{i+1}, \ldots, x_m).$$

Thus $\bar{c} \in (f, g)_C$ which implies that $(f, g)_C$ is regular. $\square$

Now state the Regular Cell Decomposition Theorem. We then give a proof of it. (No proof of this theorem is given in (van den Dries, 1998) or, as far as we could see, in the literature.)

Theorem 5.1. *(Regular Cell Decomposition Theorem)*

(I_m) *Given any definable sets* $A_1, \ldots, A_k \subseteq R^m$*, there is a decomposition of* R^m *partitioning each of* $A_1, \ldots, A_k$*, all of whose open cells are regular.*

(II_m) *For each definable function* $f : A \longrightarrow R,\ A \subseteq R^m$*, there is a decomposition* $\mathcal{D}$ *of* R^m *partitioning* A *all of whose open cells are regular, and such that for each open cell* $C \in \mathcal{D}$ *with* $C \subseteq A$*, the restriction* $f|_C : C \longrightarrow R$ *is regular. (van den Dries, 1998, p. 58)*

Proof We will prove this by induction on m.

First show (I_1) and (II_1) : hold

Let $A_1, \ldots, A_k \subseteq R$ be definable sets. By (I_1) of the CDT, we find a common decomposition partitioning each A_i, whose elements are (0)-cells and (1)-cells where $i \in \{1, \ldots, k\}$. Since the only open cells of this decomposition are (1)-cells, then we should show these are regular.

Take a (1)-cell ; that is, an interval $(a, b) \subseteq R$ with $a < b$. Let $x, y \in (a, b)$ and $z \in R$ such that $x < z < y$, then $z \in (a, b)$ which shows that (a, b) is a regular cell. Thus, all of open cells of the decomposition are regular.

Now we will show (II_1) holds. Let $f : A \longrightarrow R$ be a definable function with $A \subseteq R$. As A is definable in R then it is a finite union of intervals and points. Now take an open cell $C = (a.b) \subseteq A$. Then by MT, we find points $a = a_1 < \cdots < a_n = b$ in $(a.b)$ such that f is continuous and strictly monotone or constant on each subinterval (a_j, a_{j+1}) where $j \in \{1, \ldots, (n-1)\}$. Then the decomposition $\mathcal{D}$ which contains each such subinterval and the points as elements gives the desired decomposition. These establish proofs of (I_1) and (II_1).

For the rest of the proof, first we will suppose (I_m) holds and show (II_m) also holds, then assume (II_m) holds and show (I_{m+1}) holds. Hence this will finish the proof of the theorem.

Now suppose (I_m) holds and show (II_m) holds.

Let $f : A \longrightarrow R$ be a definable function with $A \subseteq R^m$. We should find a decomposition $\mathcal{D}$ of R^m partitioning A, all of whose open cells are regular and such that for each open cell

$C \in \mathcal{D}$ with $C \subseteq A$, the restriction $f|_C$ is regular.

For each $i \in \{1, \ldots, m\}$, we define following sets :

$$A_{i1} := \big\{(x_1, \ldots, x_m) \in A : \exists y_1, y_2,\ y_1 < x_i < y_2,\ \forall y_0 \in (y_1, y_2),$$
$$(x_1, \ldots, x_{i-1}, y_0, x_{i+1}, \ldots, x_m) \in A,$$
$$y_0 < x_i \implies f(x_1, \ldots, x_{i-1}, y_0, x_{i+1}, \ldots, x_m) < f(x_1, \ldots, x_i, \ldots, x_m) \text{ and}$$
$$y_0 > x_i \implies f(x_1, \ldots, x_{i-1}, y_0, x_{i+1}, \ldots, x_m) > f(x_1, \ldots, x_i, \ldots, x_m)\big\},$$

$$A_{i2} := \big\{(x_1, \ldots, x_m) \in A : \exists y_1, y_2,\ y_1 < x_i < y_2,\ \forall y_0 \in (y_1, y_2),$$
$$(x_1, \ldots, x_{i-1}, y_0, x_{i+1}, \ldots, x_m) \in A,$$
$$y_0 < x_i \implies f(x_1, \ldots, x_{i-1}, y_0, x_{i+1}, \ldots, x_m) > f(x_1, \ldots, x_i, \ldots, x_m) \text{ and}$$
$$y_0 > x_i \implies f(x_1, \ldots, x_{i-1}, y_0, x_{i+1}, \ldots, x_m) < f(x_1, \ldots, x_i, \ldots, x_m)\big\},$$

$$A_{i3} := \big\{(x_1, \ldots, x_m) \in A : \exists y_1, y_2,\ y_1 < x_i < y_2,\ \forall y_0 \in (y_1, y_2),$$
$$(x_1, \ldots, x_{i-1}, y_0, x_{i+1}, \ldots, x_m) \in A,$$
$$y_0 < x_i \implies f(x_1, \ldots, x_{i-1}, y_0, x_{i+1}, \ldots, x_m) = f(x_1, \ldots, x_i, \ldots, x_m) \text{ and}$$
$$y_0 > x_i \implies f(x_1, \ldots, x_{i-1}, y_0, x_{i+1}, \ldots, x_m) = f(x_1, \ldots, x_i, \ldots, x_m)\big\}.$$

So the sets A_{i1}, A_{i2}, A_{i3} contain the points in A where the function f is *locally* strictly increasing in the i^{th}, strictly decreasing in the i^{th} and independent from the i^{th} coordinate, respectively.

It's clear that each A_{ij} is definable where $i \in \{1, \ldots, m\}$, $j \in \{1, 2, 3\}$.

Let $\Omega := A_{1j_1} \cap \cdots \cap A_{mj_m}$ where $i \in \{1, \ldots, m\}$ and $j_i \in \{1, 2, 3\}$. Since for each coordinate there are 3 kinds of monotonicity, there are at most 3^m-many such Ω.

Now look at the set $A \backslash \bigcup \Omega$ which contains all points $(x_1, \ldots, x_m)$ in A such that the function f is not strictly increasing, strictly decreasing or independent from at least one component of $(x_1, \ldots, x_m)$.

Note that each set Ω as well as the set $A \setminus \bigcup \Omega$ is definable in R^m and their union gives the set A.

Lemma 5.2. *$A \setminus \bigcup \Omega$ has no interior point.*

Proof We seek for a contradiction. Suppose that $A \backslash \bigcup \Omega$ has at least an interior point, then we can find an open box $U = (a_1, b_1) \times \cdots \times (a_m, b_m)$ around that point such that $U \subseteq A \backslash \bigcup \Omega$. To get a contradiction, we want to show that $U \not\subseteq A \setminus \bigcup \Omega$, so we need to find a point in U

such that the function f is continuous and strictly monotone or constant at each coordinate of that point. Thus we will find a point in U which cannot be in $A \setminus \bigcup \Omega$, by definition. Therefore, we will work through $f|_U$, the restriction of $f(x_1, \ldots, x_m)$ to U.

Now for each $i \in \{1, \ldots, m\}$, we define following set :

$H_i := \big\{(x_1, \ldots, x_m) \in U : \exists y_1 \, \exists y_2$ with $\; y_1 < x_i < y_2, \;\; \forall j_1, j_2 \in \{1, 2, 3\}$ with $j_1 \neq j_2, \;\; \forall z_1$ with $\; y_1 < z_1 < x_i, \;$ and $\forall z_2$ with $x_i < z_2 < y_2$, $(x_1, \ldots, x_{i-1}, z_1, x_{i+1}, \ldots, x_m) \in A_{ij_1}$ and $(x_1, \ldots, x_{i-1}, z_2, x_{i+1}, \ldots, x_m) \in A_{ij_2}\big\}$.

We clearly see that the set H_i contains the points $(x_1, \ldots, x_m) \in U$ at which the function $f(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_m) : (a_i, b_i) \longrightarrow R$ changes monotonicity in the i^{th} coordinate at the point x_i. Note that each H_i is definable.

We will call $(x_1, \ldots, x_m) \in A$ is **good** in the i^{th} coordinate if $(x_1, \ldots, x_i, \ldots, x_m) \notin H_i$, otherwise we will call it **bad** in the i^{th} coordinate. Furthermore, call $(x_1, \ldots, x_m) \in A$ **good point** if it is good in each coordinate. So, we look for a good point in U to obtain a contradiction.

Let π_i and Π_i be the projection maps of A onto R^{m-1} and R, so we have the following maps

$$\pi_i : A \longrightarrow R^{m-1} : (x_1, \ldots, x_i, \ldots, x_m) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$$

$$\Pi_i : A \longrightarrow R : (x_1, \ldots, x_i, \ldots, x_m) \mapsto x_i$$

We define $\forall x_1 \ldots \forall x_{i-1} \forall x_{i+1} \ldots \forall x_m$, the fiber

$$\Pi_i(\pi_i^{-1}(\bar{y})) := \big\{x_i : (x_1, \ldots, x_m) \in H_i\big\}$$

where $i \in \{1, \ldots, m\}, \; \bar{y} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$.

By MT, we know that $\forall x_1 \ldots \forall x_{i-1} \forall x_{i+1} \ldots \forall x_m$, there exists a number N_i such that $|\Pi_i(\pi_i^{-1}(\bar{y}))| < N_i$. In other words, for each $i \in \{1, \ldots, m\}$, the fiber $\Pi_i(\pi_i^{-1}(\bar{y}))$ is finite over R^{m-1}. This finiteness will let us mention of the infimum.

We will prove by induction on $k \in \{1, \ldots, m\}$ that there is a box D_k on which $f|_{D_k}$ is continuous and strictly monotone or constant in each i^{th} coordinate where $i \in \{1, \ldots, k\}$.

We should first show that D_1 exists. Let $B_1 = (a_2, b_2) \times \cdots \times (a_m, b_m)$. Since $\Pi_1(\pi_1^{-1}(\bar{y}))$ is finite, then we can define the following function :

$$c_1 : B_1 \longrightarrow (a_1, b_1) \text{ by}$$

$$c_1(x_2, \ldots, x_m) = \inf\{x_1 : x_1 \in \Pi_1(\pi_1^{-1}(x_2, \ldots, x_m)) \text{ or } x_1 = b_1\}.$$

So c_1 takes a point from B_1 and sends to the least $x_1 \in \Pi_1(\pi_1^{-1}(\bar{y}))$ that changes the monotonicity of the function $f(\bullet, x_2, \ldots, x_m)$ where $x_2, \ldots, x_m$ are fixed,
$(x_1, \ldots, x_m) \in H_1$ and $\bar{y} = (x_2, \ldots, x_m)$.

Note that c_1 is definable since its graph

$$\Gamma(c_1) = \{(\lambda_1, x_2, \ldots, x_m) : \lambda_1 = \inf\{x_1 : \Pi_1(\pi_1^{-1}(\bar{y})) \text{ or } \lambda_1 = b_1\}$$

is definable. Since $B_1 \subseteq R^{m-1}$ and c_1 are definable, then by (II_{m-1}) of CDT, we can find a decomposition $\mathcal{D}_1$ of R^{m-1} partitioning B_1 such that the restriction $c_1|_{C_1} : C_1 \longrightarrow R$ to each cell $C_1 \in \mathcal{D}_1$ with $C_1 \subseteq B_1$ is continuous.

Since B_1 is an open set in R^{m-1}, then there is at least one such $C_1 \subseteq B_1$ which is open. Thus we take a point $(y_2, \ldots, y_m) \in C_1 \subseteq B_1$. Since c_1 is continuous on C_1, then by definition for any σ_1, σ_2 with $c_1(y_2, \ldots, y_m) \in (\sigma_2, \sigma_1)$, we can find p_i, q_i depending on σ_1, σ_2 with
$y_i \in (p_i, q_i)$ such that if $(x_2, \ldots, x_m) \in (p_2, q_2) \times \cdots \times (p_m, q_m)$, then
$c_1(x_2, \ldots, x_m) \in (\sigma_2, \sigma_1)$ where $i \in \{2, \ldots, m\}$.

We can obtain then for any fixed $(x_2, \ldots, x_m) \in (p_2, q_2) \times \cdots \times (p_m, q_m)$, the function $f(\bullet, x_2, \ldots, x_m)$ is continuous and strictly monotone or constant on (σ_3, σ_2) for some σ_3 with $a_1 < \sigma_3 < \sigma_2 < c_1(y_2, \ldots, y_m)$.

Thus we see that each point $(x_1, \ldots, x_m)$ in $(\sigma_3, \sigma_2) \times (p_2, q_2) \times \cdots \times (p_m, q_m) = D_1$
is good in the first coordinate.

From the previous result, we conclude that $f|_{D_1}$, the restriction of $f(x_1, \ldots, x_m)$ to this new box $D_1 \subseteq U$, is continuous and strictly monotone or constant in the first coordinate.

Now suppose that there exist $D_{k-1} \subseteq D_{k-2} \subseteq \cdots \subseteq D_1 \subseteq U$ such that $f|_{D_{k-1}}$ is continuous and strictly monotone or constant in each i^{th}-coordinate where $i \in \{1, \ldots, (k-1)\}$. We will find a box $D_k \subseteq D_{k-1}$ on which f is continuous and strictly monotone or constant in each i^{th}-coordinate where $i \in \{1, \ldots, k\}$.

Let $D_{k-1} = (\alpha_1, \beta_1) \times \cdots \times (\alpha_m, \beta_m)$ be the box on which $f|_{D_{k-1}}$ is continuous and strictly monotone or constant in each coordinate $i \in \{1, \ldots, (k-1)\}$.

Let $B_k = (\alpha_1, \beta_1) \times \cdots \times (\alpha_{k-1}, \beta_{k-1}) \times (\alpha_{k+1}, \beta_{k+1}) \times \cdots \times (\alpha_m, \beta_m)$ be the projection of D_{k-1} on R^{m-1}.

Define the function

$$c_k : B_k \longrightarrow (\alpha_k, \beta_k), \text{by}$$

$$c_k(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m) = \inf\{x_k : x_k \in \Pi_k(\pi_k^{-1}(\bar{y})) \text{ or } \lambda_k = \beta_k\}$$

where $\bar{y} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m)$. So c_k takes a point from B_k and sends to the least $x_k \in (\alpha_k, \beta_k)$ which changes the monotonicity of the function $f(x_1, \ldots, x_{k-1}, \bullet, x_{k-1}, \ldots, x_m)$ where $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m$ are fixed.

Note that c_k is definable since its graph

$$\Gamma(c_k) = \big\{(x_1, \ldots, x_{k-1}, \lambda_k, x_{k+1}, \ldots, x_m) : \lambda_k = \inf\{x_k : x_k \in \Pi_k(\pi_k^{-1}(\bar{y})) \text{ or } \lambda_k = \beta_k\}\big\}$$

is definable. Remark that $B_k \subseteq R^{m-1}$ and c_k are definable, then by (II_{m-1}) of CDT, we find a decomposition $\mathcal{D}_k$ of R^{m-1} partitioning B_k such that the restriction $c_k|_{C_k} : C_k \longrightarrow R$ to each cell $C_k \in \mathcal{D}_k$ with $C_k \subseteq B_k$ is continuous.

Similarly, we find an open cell $C_k \subseteq B_k$. Take a point $(w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_m) \in C_k$. Since c_k is continuous on C_k, then by definition for any σ_4, σ_5 with $c_k(w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_m) \in (\sigma_5, \sigma_4)$, we can find γ_i, θ_i depending on σ_4 and σ_5 with $w_i \in (\gamma_i, \theta_i)$ such that if

$$(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m) \in (\gamma_1, \theta_1) \times \cdots \times (\gamma_{k-1}, \theta_{k-1}) \times (\gamma_{k+1}, \theta_{k+1}) \times \cdots \times (\gamma_m, \theta_m)$$

then $c_k(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m) \in (\sigma_5, \sigma_4)$ where $i \in \{1, \ldots, (k-1), (k+1), \ldots, m\}$.

We can obtain then for any fixed

$$(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m) \in (\gamma_1, \theta_1) \times \cdots \times (\gamma_{k-1}, \theta_{k-1}) \times (\gamma_{k+1}, \theta_{k+1}) \times \cdots \times (\gamma_m, \theta_m),$$

the function $f(x_1, \ldots, x_{k-1}, \bullet, x_{k+1}, \ldots, x_m)$ is continuous and strictly monotone or constant on (σ_6, σ_5) for some $\sigma_6 < \sigma_5 < c_k(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m)$.

Thus we observe that each point $(x_1, \ldots, x_m)$ in D_k is good in the k^{th}-coordinate, where $D_k = (\gamma_1, \theta_1) \times \cdots \times (\gamma_{k-1}, \theta_{k-1}) \times (\sigma_6, \sigma_5) \times (\gamma_{k+1}, \theta_{k+1}) \times \cdots \times (\gamma_m, \theta_m)$.

By assumption we have that $D_k \subseteq D_{k-1}$. So the restriction $f|_{D_k}$ is continuous and strictly monotone or constant in each coordinate $i \in \{1, \ldots, k\}$, since $D_k \subseteq D_{k-1} \subseteq \cdots \subseteq D_1 \subseteq U$.

Therefore, we continue this process and find $D_m \subseteq D_{m-1} \subseteq \cdots \subseteq D_1$ such that $f|_{D_m}$ is continuous and constant or strictly monotone in each coordinate $i \in \{1, \ldots, m\}$.

Thus any point we take from $D_m \subseteq U$ is a good point. This contradicts with the definition of $A \setminus \bigcup \Omega$. Hence $A \setminus \bigcup \Omega$ does not have any interior point.

□

Using Lemma 5.2, we see that if we have an open cell C, then there exists some Ω such that $C \subseteq \Omega$.

From now on, we will say that $f(x_1, \ldots, x_m)$ is **globally** monotone or constant in the i^{th}-coordinate if $f(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_m)$ is continuous and strictly monotone or constant along the i^{th}-fiber of its domain.

Now look at the sets A_{ij}'s. We see their definitions give that $f(x_1, \ldots, x_m)$ is *locally* strictly monotone or constant in the i^{th}-coordinate. By proving the following Lemma, we will show that $f(x_1, \ldots, x_m)$ is *globally* strictly monotone or constant in the i^{th}-coordinate.

Lemma 5.3. *There is a decomposition $\mathcal{D}$ of R^m partitioning each Ω and $A \setminus \bigcup \Omega$ such that for each open cell $C \in \mathcal{D}$, the restriction $f|_C$ is continuous and* globally *strictly monotone or constant in the i^{th}-coordinate.*

Proof By (II_m) of the CDT, we find a decomposition $\mathcal{D}_1$ partitioning each Ω and $A \setminus \bigcup \Omega$ such that for each $C \in \mathcal{D}_1$ with $C \subseteq \Omega$ or $C \subseteq A \setminus \bigcup \Omega$, the restriction $f|_C$ is continuous.

As $\mathcal{D}_1$ is a decomposition then we can consider $\mathcal{D}_1 = \{D_1, \ldots, D_n\}$ for some number n and $D_i \cap D_j = \emptyset$ if $i \neq j$ where $i, j \in \{1, \ldots, n\}$.

We know that each D_i is a definable set in R^m, then by (I_m) of the Regular Cell Decomposition Theorem 5.1, we find a decomposition $\mathcal{D}$ partitioning each D_i, all of whose open cells are regular. It's clear that $\mathcal{D}$ partitions Ω and $A \setminus \bigcup \Omega$. Thus $\mathcal{D}$ gives the desired decomposition; that is, any open cell $C \in \mathcal{D}$ is regular and the restriction $f|_C$ is continuous because $C \subseteq D_i$ for some $i \in \{1, \ldots, n\}$ and $f|_{D_i}$ is continuous.

Consider an intersection $\Omega = A_{1j_1} \cap \cdots \cap A_{mj_m}$ where $i \in \{1, \ldots, m\}$, $j_i \in \{1, 2, 3\}$. Suppose that for some fixed coordinate i, we have $j_i = 1$, so $\Omega = A_{1j_1} \cap \cdots \cap A_{i1} \cap \cdots \cap A_{mj_m}$. (The proof goes the same way when $j_i = 2$ and $j_i = 3$).

Take an open cell $C \in \mathcal{D}$ then $C \subseteq \Omega$ by Lemma 5.2. Consider the restriction of f to this open cell C. Since C is regular, then its i^{th}-fiber $C_i := \{x_i \in R : (x_1, \ldots, x_i, \ldots, x_m) \in C\}$ can be written as an interval. So if $\bar{x} = (x_1, \ldots, x_m) \in C$, then we can assume that
$x_i \in (\alpha_i, \beta_i)$.

By MT, we find points $\alpha_i = a_1 < \cdots < a_n = \beta_i$ such that the function
$f(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_m) : (\alpha_i, \beta_i) \longrightarrow R$ is continuous and strictly monotone or constant on each subinterval (a_k, a_{k+1}) where $k \in \{1, \ldots, (n-1)\}$. So here a_k's are the points which either make the function $f(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_m)$ discontinuous on (α_i, β_i) or change the monotonicity of the function.

But we know $C \subseteq \Omega = A_{1j_1} \cap \cdots \cap A_{i1} \cap \cdots \cap A_{mj_m}$, then on each subinterval (a_k, a_{k+1}), the function $f(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_m)$ is continuous and strictly increasing. Thus the points a_k's do not change the monotonicity but the continuity, so they are the jumping points of the function.

On the other hand, as $C \in \mathcal{D}$, we know that f is continuous on C, then the function $f(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_m)$ cannot have any jumping points. So we obtain that
$f(x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_m)$ is continuous and strictly increasing on $(a_1, a_n) = (\alpha_i, \beta_i)$. Hence, $f|_C$ is continuous and *globally* strictly increasing in the i^{th}-coordinate.

As i was arbitrary, we obtain that $f|_C$ is continuous and *globally* strictly monotone or constant in each coordinate of C.

□

Hence, by Lemma 5.3, we showed that if we have a restriction of f into any open cell $C \subseteq \Omega$, then we know $f|_C$ is continuous and *globally* strictly monotone or constant in each coordinate of C.

Finally, we finish the proof of (II_m) by showing that there is a decomposition $\mathcal{D}$ of R^m

partitioning A, all of whose open cells are regular and such that for any open cell $C \in \mathcal{D}$ with $C \subseteq A$, the restriction $f|_C$ is regular.

As in the proof of Lemma 5.3, by using (II_m) of CDT and (I_m) of Regular Cell Decomposition Theorem 5.1, we find a decomposition $\mathcal{D}$ of R^m partitioning each Ω and $A \setminus \bigcup \Omega$, all of whose open cells are regular and for any $C \in \mathcal{D}$ with $C \subseteq \Omega$ or $C \subseteq A \setminus \bigcup \Omega$, $f|_C$ is continuous.

Note that $\mathcal{D}$ partitions A. Now take an open cell $C \in \mathcal{D}$ with $C \subseteq A$. By Lemma 5.2, $C \subseteq \Omega \subseteq A$ for some $\Omega = A_{1j_1} \cap \cdots \cap A_{mj_m}$ where $i \in \{1, \ldots, m\}$ and $j_i \in \{1, 2, 3\}$.

We want to show that $f|_C$ is regular.

Since we know that $f|_C$ is continuous, then it is enough to show that for each $i \in \{1, \ldots, m\}$, $f|_C$ is either strictly increasing in the i^{th}-coordinate, or strictly decreasing in the i^{th}-coordinate or independent of the i^{th}-coordinate. We know that $C \subseteq \Omega = A_{1j_1} \cap \cdots \cap A_{mj_m}$ and $f|_C$ is continuous.

By Lemma 5.3, we have that $f|_C$ is continuous and globally strictly monotone or constant in each coordinate. It follows from the lemma then for each $i \in \{1, \ldots, m\}$, $f|_C$ is continuous and depending on j_i, either strictly increasing in the i^{th}-coordinate, or strictly decreasing in the i^{th}-coordinate or independent of the i^{th}-coordinate. Hence $f|_C$ is regular. This finishes the proof of (II_m).

Now we suppose that (II_m) holds and show (I_{m+1}) holds, namely, we will show that for any definable sets $A_1, \ldots, A_k$ in R^{m+1}, there is a decomposition of R^{m+1} partitioning each A_i, all of whose open cells are regular.

By (I_{m+1}) of CDT, we find a decomposition $\mathcal{D} = \{C_1, \ldots, C_n\}$ of R^{m+1} partitioning each A_i. Then, by definition of a decomposition, we know that the set $\mathcal{D}' = \{\Pi(C_1), \ldots, \Pi(C_n)\}$ of the projections of each cell in $\mathcal{D}$ is a decomposition of R^m, where $\Pi : R^{m+1} \to R^m$ is the natural projection. By (I_m) of RCDT, we find a decomposition $\mathcal{E}'$ of R^m partitioning each cell in $\mathcal{D}'$ such that each open cell in $\mathcal{E}'$ is regular.

By Remark 4.1, we find a new decomposition $\mathcal{D}|_{\mathcal{E}'}$ of R^{m+1} such that $\mathcal{E}'$ is the set of projections of each cell in this new decomposition. To simplify, we will write $\mathcal{E}$ instead of $\mathcal{D}|_{\mathcal{E}'}$.

Note that $\mathcal{E}$ partitions each A_i and $\mathcal{E}'$ is a regular decomposition of R^m.

For any decomposition $\mathcal{F} = \{C_1, \ldots, C_n\}$ of R^{m+1} such that $\mathcal{F}$ partitions each A_i and $\mathcal{F}^{'} := \{\Pi(C_1), \ldots, \Pi(C_m)\}$ is a regular decomposition of R^m, we let

- $\mathbf{k}_{\mathcal{F}}(A^{'}) = \#\{f : A^{'} \to R : f$ is not regular, where $A := \Gamma(f) \in \mathcal{F}\}$, for $A^{'} \in \mathcal{F}^{'}$ open,
- $\mathbf{N}(\mathcal{F}) = \max\{\mathbf{k}_{\mathcal{F}}(A^{'}) : A^{'} \in \mathcal{F}^{'}, A^{'}$ is open$\}$,
- $\mathbf{M}(\mathcal{F}) = \#\{A^{'} \in \mathcal{F}^{'}.A^{'}$ open $: \mathbf{k}_{\mathcal{F}}(A^{'}) = \mathbf{N}(\mathcal{F})\}$.

It follows from Lemma 5.1 that if $\mathbf{N}(\mathcal{E}) = 0$ then $\mathcal{E}$ is a regular cell decomposition of R^{m+1} partitioning each A_i.

Assume that $\mathcal{E}$ is not a regular decomposition of R^{m+1}, so $\mathbf{N}(\mathcal{E}) \neq 0$. We will see that with the following process, either $\mathbf{N}(\mathcal{E})$ or $\mathbf{M}(\mathcal{E})$ decreases :

Take one of the open regular cells $A^{'} \in \mathcal{E}^{'}$ which makes $\mathbf{k}_{\mathcal{E}}(A^{'})$ maximum and a function $f : A^{'} \to R$ which is not regular such that $\Gamma(f) \in \mathcal{E}$.

By (II_m) of RCDT, we find a regular decomposition $\mathcal{F}^{'}$ of R^m partitioning $A^{'}$ such that the restriction of f to any open cell in $\mathcal{F}^{'}$ is regular.

Let $\mathcal{G}^{'} = \mathcal{E}^{'} \setminus \{A^{'}\} \cup \{C^{'} : C^{'} \in \mathcal{F}^{'}, C^{'} \subseteq A^{'}\}$ and $\mathcal{G} = \mathcal{E}|_{\mathcal{G}^{'}}$. Remark that $\mathcal{G}^{'}$ is a regular decomposition of R^m. Thus $\mathcal{G}$ is a decomposition of R^{m+1} partitioning each of A_i, where $\mathcal{G}^{'}$ is the set of projections of each cell in $\mathcal{G}$.

Note that each regular cell in $\mathcal{G}^{'}$ is included in a regular cell in $\mathcal{E}^{'}$. It is clear that if a function g is regular on a cell $A^{'} \in \mathcal{E}^{'}$ then its restriction $g|_{B^{'}}$ to a cell $B^{'} \subseteq A^{'}$ is also regular, hence $\mathbf{k}_{\mathcal{G}}(B^{'}) < \mathbf{k}_{\mathcal{E}}(A^{'})$.

Therefore, we obtain that

$$\text{if } \mathbf{M}(\mathcal{E}) > 1 \text{ then } \mathbf{N}(\mathcal{E}) = \mathbf{N}(\mathcal{G}) \text{ and } \mathbf{M}(\mathcal{G}) = \mathbf{M}(\mathcal{E}) = 1$$

$$\text{if } \mathbf{M}(\mathcal{E}) = 1 \text{ then } \mathbf{N}(\mathcal{G}) < \mathbf{N}(\mathcal{E}).$$

This gives us then that

$$(\mathbf{N}(\mathcal{G}), \mathbf{M}(\mathcal{G})) <_{lex} (\mathbf{N}(\mathcal{E}), \mathbf{M}(\mathcal{E}))$$

where $\leq_{lex}$ is the lexicographical order on $\mathbb{N}^2$. Since this is a well-ordering on $\mathbb{N}^2$, this process cannot be repeated infinitely. Hence we find a decomposition $\mathcal{H}$ of R^{m+1} where $\mathbf{N}(\mathcal{H}) = 0$ which gives that $\mathcal{H}$ is a regular decomposition of R^{m+1}.

□

Example 5.1. *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x|y - x|$*. We will find a decomposition for* $\mathbb{R}^2$ *as in* (II_m) *of the Regular Cell Decomposition Theorem 5.1.*

Using basic calculus, we obtain :

- *On the open cells* 1*,* 3*,* 4 *and* 6*,* f *is increasing in the first coordinate and decreasing in the second coordinate.*
- *On the open cells* 2.1 *and* 5.1*,* f *is decreasing in the first coordinate and increasing in the second coordinate.*
- *On the open cells* 2.2 *and* 5.2*,* f *is increasing both in the first and the second coordinate.*

So f *is vdD-regular on these open cells. Thus the decomposition for* $\mathbb{R}^2$ *that we obtained can be seen in the following figure 5.1 on which each cell is denoted by either a colour or a number where the lines are* $y = x$ *and* $y = x\tan(67, 5)$ *:*

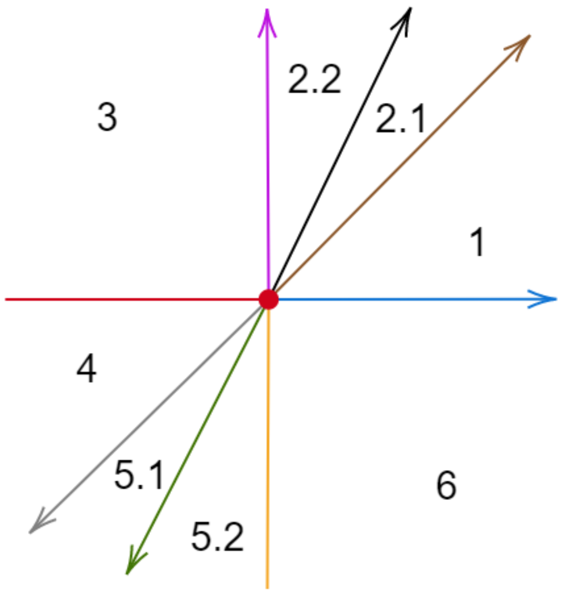

Figure 5.1: Decomposition of $\mathbb{R}^2$

5.2 Gabrielov-Regularity

In this section, we introduce the regularity in the sense of Gabrielov and we give the proposition 5.1.

Definition 5.3. *An* $(i_1, \ldots, i_n)$*-cell in* R^n *is said to be a* k*-cell if* $\sum_{j=1}^{n} i_j = k$*.*

Definition 5.4. *We say that two definable sets $A \subseteq R^n$ and $B \subseteq R^m$ are definably homeomorphic to each other if there is a definable homeomorphism between A and B.*

Definition 5.5. *(Gabrielov-regular cell) A definable set U is called a Gabrielov-regular k-cell if the pair $(\overline{U}, U)$ is definably homeomorphic to the pair $([0,1]^k, (0,1)^k)$ where $[0,1]^k = [0,1] \times \cdots \times [0,1]$, the product of k-many $[0,1]$'s.*

Definition 5.6. *(Gabrielov-regular function) A function $h : U \to R$ is called Gabrielov-regular if h is continuous and the graph of h is a Gabrielov-regular set.*

Before the following proposition, we recall a theorem which will be used in the proof of the proposition and which follows easily from the Intermediate Value Theorem.

Theorem 5.2. *Let f be a real valued function on an open interval I. If f is continuous and increasing (or decreasing) then $f(I)$ is an open interval and*

i. f realizes a bijection between I and $f(I)$, and

ii. $f^{-1} : f(I) \to I$ is continuous and increasing (or decreasing).

We can observe that Gabrielov-regularity makes cells have a strong topological property, by definition. The following proposition shows that open cells have a weaker but still a nice feature.

Proposition 5.1. *Consider the o-minimal structure $(\mathbb{R}; +, -, \cdot, 0, 1, \cdots, <)$. If A is a k-cell in $\mathbb{R}^n$, then A is definably homeomorphic to the product of k-many open intervals.*

Proof We will prove by induction on n. First we show the claim for $n = 1$:

Let A be a cell in $\mathbb{R}$. If A is a (0)-cell, it is trivial. Consider that A is a (1)-cell. Then by definition, it is an open interval in $\mathbb{R}$. Thus it is definably homeomorphic to the open interval $(0, 1)$.

Now we will suppose the assumption holds for the cells in $\mathbb{R}^n$ and show it is also true for the cells in $\mathbb{R}^{n+1}$.

Let $A = (\lambda_1, \ldots, \lambda_n)$ be a k-cell in $\mathbb{R}^n$ and suppose that A is definably homeomorphic to the product of k-many $(0, 1)$'s. We will show that $B = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1})$, which is a cell in $\mathbb{R}^{n+1}$, is also definably homeomorphic to the product of $(k + 1)$-many $(0, 1)$'s if $\lambda_{n+1} = 1$ and it is definably homeomorphic to the product of k-many $(0, 1)$'s if $\lambda_{n+1} = 0$.

First assume that $\lambda_{n+1} = 0$. Then B is a k-cell which is a graph of some function $f \in C(A)$.

By assumption, we know that A is definably homeomorphic to $X_k := (0,1) \times \cdots \times (0,1)$ which is the product of k-many $(0,1)$'s. Let $g : X_k \to A$ be a homeomorphism between A and X_k.

Claim : The function

$$h : X_k \to B$$

$$(t_1, \ldots, t_k) \mapsto \Big(g(t_1, \ldots, t_k), f(g(t_1, \ldots, t_k))\Big)$$

is a homeomorphism between X_k and B.

We show that h is a bijective :

- injectivity : Let $\bar{t} = (t_1, \ldots, t_k), \bar{s} = (s_1, \ldots, s_k) \in X_k$. If $h(\bar{t}) = h(\bar{s})$ then $\Big(g(\bar{t}), f(g(\bar{t}))\Big) = \Big(g(\bar{s}), f(g(\bar{s}))\Big)$.

  Since g is injective, we obtain then $\bar{t} = (t_1, \ldots, t_k) = (s_1, \ldots, s_k) = \bar{s}$.
- surjectivity : Take $(\bar{a}, f(\bar{a})) \in B$ where $\bar{a} = (a_1, \ldots, a_n) \in A$. As g is surjective, we find some $\bar{t} = (t_1, \ldots, t_k) \in X_k$ such that $g(\bar{t}) = \bar{a}$. Thus we found a point $\bar{t} \in X_k$ such that $h(\bar{t}) = \Big(g(\bar{t}), f(g(\bar{t}))\Big) = (\bar{a}, f(\bar{a}))$.

Hence we showed that h is a bijection.

We know that g and f are continuous functions because g is a homeomorphism and $f \in C(A)$. Therefore, h is continuous.

The preimage of h is defined by

$$h^{-1} : B \to X_k$$

$$(\bar{a}, f(\bar{a})) \mapsto h^{-1}(\bar{a}, f(\bar{a})) = g^{-1}(\bar{a}).$$

It's clear that h^{-1} is also continuous which establishes the claim; that is, B is homeomorphic to X_k, the product of k-many $(0,1)$'s, when $\lambda_{n+1} = 0$.

Now we assume that $\lambda_{n+1} = 1$ and show that B is homeomorphic to the product of $k+1$-many $(0,1)$'s. We will denote this product by X_{k+1} .

Remark that B is a $k+1$-cell which is a set $(f,g)_A$ where $f,g \in C_\infty(A)$ with $f < g$. We know that A is definably homeomorphic to X_k, by assumption. Let $p : X_k \to A$ be a homeomorphism between X_k and A. We need to examine four cases :

First Case : If $f,g \in C(A)$:

Then the following function h gives the desired homeomorphism :

$$h : X_{k+1} \to B$$

$$\bar{t} \mapsto \Big(p(t_1,\dots,t_k)\ ,\ t_{k+1}.\big[g\big(p(t_1,\dots,t_k)\big) - f\big(p(t_1,\dots,t_k)\big)\big] + f\big(p(t_1,\dots,t_k)\big)\Big)$$

where $\bar{t} = (t_1,\dots,t_{k+1}) \in X_{k+1}$.

It's clear that h is bijective and continuous.

The inverse of h is defined by $h^{-1}(\bar{a},b) = (p^{-1}(\bar{a}), r(\bar{a}))$ where $(\bar{a},b) \in B$ and $r(\bar{a}).\big(g(\bar{a}) - f(\bar{a})\big) = b - f(\bar{a})$. See that h^{-1} is a continuous function since p^{-1} and r are continuous.

Thus, we showed that B is homeomorphic to X_{k+1} when $f,g \in C(A)$.

Second Case : If $f \in C(A)$ and g is the constant function ∞ :

Then define the following function

$$h : X_{k+1} \to B$$

$$\bar{t} \mapsto \Big(p(t_1,\dots,t_k),\ r\big(p(t_1,\dots,t_k)\big)\Big)$$

is a homeomorphism where $\bar{t} = (t_1,\dots,t_{k+1}) \in X_{k+1}$ and $r\big(p(t_1,\dots,t_k)\big).t_{k+1} = f\big(p(t_1,\dots,t_k)\big)$.

It's clear that h is bijective and continuous.

The inverse of h is defined by $h^{-1}(\bar{a},b) = h^{-1}(\bar{a}, r(\bar{a})) = (p^{-1}(\bar{a}), r(\bar{a}))$ where $b = r(\bar{a})$. Similarly, h^{-1} is also continuous. Thus we showed that B is homeomorphic to X_{k+1} when $f \in C(A)$ and $g = \infty$.

Third Case : If f is the constant function $-\infty$ and $g \in C(A)$:

Then the function

$$h : X_{k+1} \to B$$

$$\bar{t} \mapsto \Big(p(t_1, \ldots, t_k), -r\big(p(t_1, \ldots, t_k)\big)\Big)$$

gives the desired homeomorphism where $\bar{t} = (t_1, \ldots, t_{k+1}) \in X_{k+1}$ and
$r\big(p(t_1, \ldots, t_k)\big).t_{k+1} = f\big(p(t_1, \ldots, t_k)\big)$.

This case is shown by the similar way with the previous one.

Fourth Case : If f is the constant function $-\infty$ and g is the constant function ∞ :

For this case, we define the following function

$$h : X_{k+1} \to B$$

$$(t_1, \ldots, t_{k+1}) \mapsto \big(p(t_1, \ldots, t_k), q(t_{k+1})\big)$$

where $q(t_{k+1}).(1 - (2t_{k+1} - 1)^2) = 2t_{k+1} - 1$.

We show that h is a homeomorphism :

By Theorem 5.2, we obtain that q is bijective as it is continuous and increasing. We know that p is also bijective. Therefore we obtain that h is a bijection.

It's clear that h is continuous and its inverse is defined by $h^{-1}(\bar{a}, b) = \big(p^{-1}(\bar{a}, q^{-1}(b)\big)$ where $(\bar{a}, b) \in B$. We know that p^{-1} is continuous and by Theorem 5.2, q^{-1} is also continuous. Thus we obtain that h^{-1} is continuous. Hence we showed that B is homeomorphic to X_{k+1} when $f = -\infty$ and $g = \infty$.

In each case, we showed that if $\lambda_{k+1} = 1$, then B is homeomorphic to the product of $k+1$-many $(0, 1)$'s. This establishes the proposition.

□

Before passing to the next section, we state a proposition to use in the example, which is one of the main results of continuous functions on topological spaces. Therefore, we will skip the proof.

Proposition 5.2. *Let X and Y be two topological spaces and $f : X \to Y$ be a continuous function. If $U \subseteq X$ is connected then $f(U)$ is also connected.*

Proof See (Wikipedia, 2018) $\square$

5.3 Comparison of The Regularities

In this section, we compare the regularities of van den Dries and Gabrielov. For this purpose, we give a counter-example included in (Gabrielov et al., 2010) and examine it in detail.

Example 5.2. *(Gabrielov et al., 2010, Exercise) Consider the set $X := \{(x, y, z) \in \mathbb{R}^3 : 0 < x, 0 < y, 0 < z < 1, x + y < z\}$, and the continuous function*

$$h : X \to \mathbb{R}$$

$$(x, y, z) \mapsto (x/z)^2 + (y/z)^2.$$

First show that h is vdD-regular. We know h is continuous, so it's enough to show that it is either strictly increasing or decreasing or independent from each coordinate.

- 1^{st} *coordinate : Take $(x_1, y, z), (x_2, y, z) \in X$ such that $x_1 < x_2$. Since $x_1, x_2 \in (0, 1)$, then*
$$h(x_1, y, z) = \frac{x_1^2 + y^2}{z^2} < \frac{x_2^2 + y^2}{z^2} = h(x_2, y, z)$$
So h is strictly increasing at the first coordinate.

- 2^{nd} *coordinate : The proof of that h is strictly in creasing at the first coordinate goes exactly the similar way.*

- 3^{rd} *coordinate : Take $(x, y, z_1), (x, y, z_2) \in X$ such that $z_1 < z_2$.*

  Then we have $h(x, y, z_1) = \dfrac{x^2 + y^2}{z_1^2} > \dfrac{x^2 + y^2}{z_2^2} = h(x, y, z_2)$.

  Thus h is strictly decreasing at the third coordinate.

Hence we showed h is a vdD-regular function.

Now look at the following set :

$$B = \{(x, y, z, t) \in \mathbb{R}^4 : (x, y, z) \in X, 0 < t < h(x, y, z)\}.$$

It follows from Lemma 5.1 that B is vdD-regular. We will prove by contradiction that B is not Gabrielov-regular, so assume that there is a homeomorphism $f : \bar{B} \to [0,1]^4$ such that its restriction to B, $f|_B : B \to (0,1)^4$, is also a homeomorphism.

Fix the point $\bar{a} = (0,0,0,3/4)$. Then $\bar{a} \in \bar{B}$ because any open ball around this point intersects with the set B. Now take an open ball U around this fixed point, so we have $U = \{\bar{b} \in \mathbb{R}^4 : d(\bar{a},\bar{b}) < \epsilon\}$ where $\epsilon > 0$ and d is the standard metric.

We prove first by taking $0 < \epsilon < 1/4$, that U satisfies that if V is open with $\bar{a} \in V$ and $V \subseteq U$ then $V \cap B$ is not connected.

Define the following sets :

$$O_1 := V \cap \{\bar{b} = (x,y,z,t) \in U : x < y\} \cap B \text{ , and}$$

$$O_2 := V \cap \{\bar{b} = (x,y,z,t) \in U : x > y\} \cap B.$$

We claim that $V \cap B = O_1 \cup O_2$. Indeed, since $\epsilon < 1/4$, for all $(x,y,z,t) \in V \cap B$, $t > 1/2$. But then if $(x,y,z,t) \in V \cap B$, $x \neq y$: otherwise, assume $x = y$ for a contradiction. It follows from the definition of B that

$$0 < 2x = x + y < z \textit{ and } tz^2 < x^2 + y^2 = 2x^2,$$

hence $0 < 4tx^2 < tz^2 < 2x^2$, hence $t < 1/2$. Thus we have a contradiction.

Now we know that $V \cap B = O_1 \cup O_2$. It's clear that the sets are disjoint and open in B. Now it is enough to show that O_1 and O_2 are non-empty :

Take $x_0 < \epsilon/3$. Since $\bar{a} \in V$ and V is an open set around $\bar{a}$, then the points $\bar{b}_1 = (\frac{\sqrt{11}x_0}{4}, x_0, \frac{3x_0}{2}, \frac{3}{4})$ and $\bar{b}_2 = (x_0, \frac{\sqrt{11}x_0}{4}, \frac{3x_0}{2}, \frac{3}{4})$ are clearly in O_1 and O_2, respectively.

Note that $f(U \cap \bar{B})$ is open, containing $f(\bar{a})$ because we know that $U \cap \bar{B}$ is open by subspace topology on $\bar{B}$ and f^{-1} is continuous.

Now take an open, connected set W from $f(U \cap \bar{B})$. We know f is continuous, then $f^{-1}(W)$ is open. Let $V = f^{-1}(W)$. Note that $\bar{a} \in V \subseteq U \cap \bar{B}$. By the property that U satisfies, we know that $V \cap B$ is not connected.

We will show now that $f(V \cap B) = W \cap (0,1)^4$:

Take $y \in f(V \cap B)$. Since $f|_B$ is surjective then there is some $x \in V \cap B$ such that $f(x) = y$.

As $x \in V$ *then* $y \in f(V) = W$. *Since* $f|_B$ *is a homeomorphism and* $x \in B$, *then* $y \in (0,1)^4$ *which implies* $y \in W \cap (0,1)^4$.

Take $y \in W \cap (0,1)^4$. *Since* $f|_B$ *is surjective we find some* $x \in V$ *such that* $f(x) = y$. *Also* $y \in (0,1)^4$ *and* $f|_B$ *is a injective then* $x \in B$ *which implies* $y \in f(V \cap B)$. *This finishes the proof of the equality.*

Thus we have $f(V \cap B) = W \cap (0,1)^4$ *but we can find an open ball small enough which is connected in* $W \cap (0,1)^4$ *while* $V \cap B$ *is not connected. But* $f|_B$ *is a homeomorphism then by Proposition 5.2, the image of this open connected ball under the continuous function* f^{-1} *must also be connected in* $V \cap B$. *Therefore we have a contradiction. Hence we proved that* B *is not Gabrielov-regular.*

Therefore, we showed with this example that vdD-regularity and Gabrielov-regularity are not equivalent. If we combine Proposition 5.1 and the example 5.2, we can conclude that any cell is homeomorphic to the corresponding product of open intervals (whether or not the cell is vdD-regular) ; but even if the cell is vdD-regular, its closure for the ambiant topology does not need to be homeomorphic to the corresponding product of closed intervals : the homeomorphism cannot be extended to the boundary.

6 CONCLUSION

This thesis is based on three main motivations. Our first motivation was to understand the tameness of the geometry when we assume the axioms of o-minimal structures. For this purpose, first we tried to comprehend o-minimality by following (van den Dries, 1998) as the primary source which we enriched with examples. We then studied the main results of o-minimality for definable functions such as the Monotonicity and the Cell Decomposition Theorems and gave their detailed proofs.

Secondly, we dealt with the regularity notion defined by van den Dries and stated Regular Cell Decomposition Theorem which becomes stronger than Cell Decomposition Theorem by this new notion. This theorem is left as an exercise in (van den Dries, 1998) and has no proof in literature. We focused on this theorem and proved it since we thought it would be a good exercise for us to understand better vdD-regularity. Then we gave an example of a regular function. Thus, one who has difficulty with this notion could check our study.

Thirdly, we focused on Gabrielov-regularity which is given in the article (Gabrielov et al., 2010). For this part, our aim was to compare the two regularities by working on an example in this article which is not explained in detail. While studying on this new definition, first we realized a topological property of k-cells in o-minimal fields and proved it. Then we finished our work by showing these two regularity definitions are not equivalent, which was our last motivation. To this end, we examined that example and we gave all omitted details in our work.

We must admit that we could not understand how Gabrielov obtained this example. Also, we noticed some other simpler examples and some generalizations of k-cells and vdD-regular cells in lower dimensions when comparing the definitions. Therefore, we think that it may be interesting to work on these and helpful to comprehend how Gabrielov obtained this example.

REFERENCES

Gabrielov, A. (1968). *Projections of semi-analytic sets*, Func. Anal. Appl. 2.

Gabrielov, A., Basu, S. and Vorobjov, N. (2010). Semi-monotone sets, *J. European Mathematicial Society* .

Grothendieck, A. (1984). *Esquisse d'un Programme*, Research Proposal, unpublished.

Hironaka, H. (1973). *Introduction to real-analytic sets and real-analytic maps*, Instituto Mathematico, Pisa.

J.Knight, A.Pillay and C.Steinhorn (1986). *Definable Sets in Ordered Structures II*, American Mathematical Society.

Łojasiezics, S. (1964). *Triangulation of semi-analytic sets*, Ann. Scuola Norm. Sup. , Pisa.

Marker, D. (2000). *Model Theory : An Introduction*, Springer.

Pillay, A. and Steinhorn, C. (1984). *Definable Sets in Ordered Structures I*, American Mathematical Society.

Tarski, A. (1951). *A Decision Method for Elementary Algebra and Geometry*.

van den Dries, L. (1984). *Remarks on Tarski's problem concerning* $(R; +, -, \cdot, 0, 1, \exp x)$.

van den Dries, L. (1998). *Tame Topology and O-minimal Structures*, Cambridge University.

Wikipedia (2018).
URL: *https ://proofwiki.org/wiki/Continuous_Image_of_Connected_Space_is_Connected*

Wilkie, A. J. (1996). Model completeness results for expansions of the ordered field of real numbers by restricted pfaffian functions and the exponential function, *Journal of the American Mathematicial Society* .

APPENDIX A REMINDER OF GENERAL TOPOLOGY

Let $(X, \mathcal{T})$ be a topological space and A be a subset of X.

Definition APPENDIX A.1. *The interior of A is the union of all open sets contained in A, and it is denoted by A°.*

Definition APPENDIX A.2. *The closure of A is the intersection of all closed sets containing A, and it is denoted by $\overline{A}$.*

Definition APPENDIX A.3. *The boundary of A is the set* $\text{bd}(A) = \overline{A} \setminus A^{\circ}$

Let $(R, <)$ be a dense linear ordered set without endpoints.

Definition APPENDIX A.4. *$\mathcal{T}$ is called the order topology on R if it is generated by the subbasis of open rays $(a, \infty) = \{x \in R : a < x\}$ and $(-\infty, b) = \{x \in R : x < b\}$ where $a, b \in R$.*

Definition APPENDIX A.5. *Let X and Y be two topological spaces. The topology generated by the basis*

$$\mathcal{B} = \{U \times V : U \text{ is open in X and } V \text{ is open in Y}\}$$

on $X \times Y$ is called the product topology.

Lemma APPENDIX A.1. *The set $\mathcal{B}$ is a basis for a topology on $X \times Y$.*

Proof It's clear that $X \times Y$ is in the basis $\mathcal{B}$.

Now let $B_1, B_2 \in \mathcal{B}$ and $B_1 \cap B_2 = I$. By definition, $B_1 = U_1 \times V_1$, $B_2 = U_2 \times V_2$ where U_1, U_2 and V_1, V_2 are open in X and Y, respectively.

Take $(x, y) \in I$. Then $x \in U_1 \cap U_2$ and $y \in V_1 \cap V_2$ which implies that $(x, y) \in B_3$ where $B_3 = x \in U_1 \cap U_2 \times V_1 \cap V_2$ and $B_3\mathcal{B}$.

□

Definition APPENDIX A.6. *Let* $(X, \mathcal{T})$ *be a topological space and* $Y \subseteq X$*. The collection*

$$\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$$

is called the subspace topology on Y*.*

Lemma APPENDIX A.2. *The collection* $\mathcal{T}_Y$ *is a topology on* Y*.*

Proof It's clear that $\emptyset$ and Y are in $\mathcal{T}_Y$. Take $U_i \cap Y$ where $i \in I$ with the index set I. Look at the union:

$$\bigcup_{i \in I}(U_i \cap Y) = \bigcup_{i \in I} U_i \cap Y.$$

This gives that $\bigcup_{i \in I}(U_i \cap Y)$ is in $\mathcal{T}_Y$.

Now take $U_i \cap Y$ where $i \in \{1, \ldots, n\}$. Then

$$\bigcap_{i=1}^{n}(U_i \cap Y) = \bigcap_{i=1}^{n} U_i \cap Y.$$

Therefore, $\bigcap_{i=1}^{n}(U_i \cap Y)$ is in $\mathcal{T}_Y$. Hence we showed that $\mathcal{T}_Y$ is a topology on Y.

□

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