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MODERN CONTROL PROBLEMS IN LINEAR TIME-INVARIANT MULTIVARIABLE SYSTEMS

by

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tüm öğrencilik yaşamım boyunca bana destek olan ANNEM'e ve BABAM'a...

TABLE OF CONTENTS

INTRODUCTION

Progresses in any field of scientific research always bring simple and powerful approaches to the solutions of sophisticated problems.In recent years,the developping technology has greatly increased the interest in System Theory.Especially,the problems encountered in the design of complicated control systems have emphasized the necessity of simple and unified approaches to the structural analysis of linear time-invariant multivariable systems.In the sequel to this work,the results obtained in two new frameworks,namely the geometric approach and the frequency domain approach to the treatment of modern control problems,are considered and used.

This work consists of two parts.The first part is a detailed litterature survey on the newly introduced geometric approach and its frequency domain translation.In Chapter I , " the mathematical preliminairies ",some basic concepts of Linear Algebra are reviewed and new geometric concepts (such as (A,B)-invariant subspaces,stabilizability subspaces) are given. A frequency domain characterization of each geometric property is also presented.Chapter II takes into consideration some of modern control problems (such as Disturbance Decoupling,Output stabilization with respect to Disturbance).Each problem is defined first,then the geometric and the related frequency domain formulations of its solvability are given.Special comments,remarks and alternate proofs are also made,whenever it is possible.

In the second part a special structure of linear multivariable systems is considered.This system is a coupling of two basic multiports,the first composed of " lossless " components and the second characterizing the " algebraic " components.I Chapter III some properties of this structure are introduced, based on [10]. Then using strict system equivalence [13], the existence of different " lossless " multiports for which the transfer matrix of the system is invariant when the " algebraic " multiport is kept constant, is investigated. The results are given as theorems and illustrated with an example. In Chapter IV using the results of Chapter III, the solvability of D D P for two different disturbance structures is formulated. In one of these cases a larger degree of freedom, to nullify the effect of disturbances, is obtained. This case is also illustrated with an example.

One general remark is that most of the decoupling problems considered are generically unsolvable; that is the solution space of these problems consist of isolated points in the neighborhood of which the problem is unsolvable.

It should be emphasized here that, the problem stated and formulated in Chapter III is totally new and answer obtained is that this problem is generically solvable. The new properties that are introduced may find many areas of application in System Theory, especially in electrical circuit design. A letter from M L J Hautus also states that the problems are very interesting and relevant.

Finally, I'm personally gratefull to my thesis supervisor C.Göknar, for his orientation and helps in the preparation of this thesis.

M.Salim Arslanalp

I MATHEMATICAL PRELIMINARIES

In order to give the frequency domain characterisations of some geometric concepts widely used in the geometric formulations of modern control problems we will first consider the $(\underline{\xi},\underline{w})$-representation newly introduced in [7 , 9] .

① - $(\underline{\xi},\underline{w})$ - Representation :

We consider the linear time-invariant system Σ given in its state space form by

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \qquad (1)$$

$$\underline{y} = C\underline{x} \qquad (2)$$

where $\underline{x}(t) \in \mathcal{X} \triangleq \mathbb{R}^n$, $\underline{u}(t) \in \mathcal{U} \triangleq \mathbb{R}^m$, $\underline{y}(t) \in \mathcal{Y} \triangleq \mathbb{R}^r$

and $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{X}$, $C : \mathcal{X} \to \mathcal{Y}$ are linear maps.Now let $\underline{x}(t)$ be a time-domain solution of (1) subject to an input function $\underline{u}(t)$ and to an initial state $\underline{x}(t_o) \triangleq x_o$ .Then

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$

and the same equations (1),(2) in Laplace domain can be written as

$$s\underline{\xi}(s) - \underline{x}_o = A\underline{\xi}(s) + B\underline{w}(s) \qquad (3)$$

where $\underline{\xi}(s)$, $\underline{w}(s)$ are the Laplace transforms of $\underline{x}(t)$, $\underline{u}(t)$ respectively. The initial state $\underline{x}(t_o)=\underline{x}_o$ is obviously an element of the state space $\mathcal{X}$, hence $\underline{x}_o \in \mathcal{X}$ and $\underline{\xi}(s)$ and $\underline{w}(s)$ are strictly proper rational functions since $\underline{x}(t)$ and $\underline{u}(t)$ are real functions.

DEFINITION 1.1 [7]

Let $\underline{x}_o \in \mathcal{X}$; the formula

$$\underline{x}_o = (sI - A)\underline{\xi}(s) - B\underline{w}(s) \qquad (4)$$

is called a $(\underline{\xi},\underline{w})$-representation of $\underline{x}_o$,if $\underline{\xi}(s)$ and $\underline{w}(s)$ are strictly proper rational functions. ■

2 - A-invariance :

A-invariance is an important property of subspaces, widely used in linear Algebra. It will be discussed briefly, since it helps to understand the related but more complicated concept of (A,B)-invariance.

Consider the linear space $\mathcal{Z}$, the linear map $A:\mathcal{Z}\rightarrow\mathcal{Z}$ and a subspace $\mathcal{W}\subseteq\mathcal{Z}$. If $\mathcal{T}\triangleq A\mathcal{W}$, then for $\underline{w}_i\in\mathcal{W}, i\in\{1,\dots,g\}$ a basis for $\mathcal{W}$, the set of vectors $\underline{t}_i\in\mathcal{T}$ such that $\underline{t}_i=A\underline{w}_i$, , $i\in\{1,2,\dots,g\}$ spans the subspace $\mathcal{T}$.

DEFINITION 1.2

$\mathcal{W}$ is said to be A-invariant if and only if $A\mathcal{W}\subseteq\mathcal{W}$. ∎

With this definition we will consider the system Σ and always talk about the A-invariance of a subspace $\mathcal{V}$ of the state space $\mathcal{X}$. It's also possible to obtain a matrix characterisation of the above property as follows. Let V be a matrix whose columns are basis vectors for the subspace $\mathcal{V}\subseteq\mathcal{X}$, that is $V\triangleq[\underline{v}_1,\underline{v}_2,\dots \underline{v}_k]$ where $\underline{v}_i\in\mathcal{V}$ for $i\in\{1,\dots,k\}$ are linearly independant and span $\{\underline{v}_i\}=\mathcal{V}$. V will be called a basis matrix for $\mathcal{V}$.

THEOREM 1.3

Given Σ and $\mathcal{V}\subseteq\mathcal{X}$ with a basis matrix V, $\mathcal{V}$ is A-invariant if and only if a solution P of the matrix equation

$$AV=VP \qquad \text{exists.}$$

Proof: "$\Longrightarrow$" Let $A\mathcal{V}\subseteq\mathcal{V}$ and $V=[\underline{v}_1\ \underline{v}_2\ ..\ \underline{v}_k]$ be a basis matrix for $\mathcal{V}$. Then $AV=[A\underline{v}_1\ \ A\underline{v}_2\ ..\ A\underline{v}_k]=[\underline{y}_1\ \underline{y}_2\ ..\ \underline{y}_k]\in\mathcal{V}$

Hence $\underline{y}_i \in \mathcal{V}$, for $i \in \{1,..,k\}, \Rightarrow \underline{y}_i = V\underline{p}_i$ for $i \in \{1,..,k\}$ for some $\underline{p}_i \in \mathbb{R}^k$. Then $AV = V[\underline{p}_1\ \underline{p}_2 .. \underline{p}_k]$ and $P \triangleq [\underline{p}_1 \cdots \underline{p}_k]$ is a solution of the matrix equation.

" $\Leftarrow$ ". Let $AV = VP$ with V a basis matrix for $\mathcal{V}$ and let $\underline{x} \in \mathcal{V}$, then $\underline{x}$ can be written as a linear combination of basis vectors of $\mathcal{V}$ such as

$$\underline{x} = V\underline{r} \ , \text{ for some } \ r \in \mathbb{R}^k .$$

Then as $A\underline{x} = AV\underline{r} = VP\underline{r}$ we see that $A\underline{x}$ is a linear combination of columns of V. Hence $A\underline{x} \in \mathcal{V}$, which implies $A\mathcal{V} \subseteq \mathcal{V}$ since $\underline{x} \in \mathcal{V}$ was arbitrary. ■

In modern control problems, an extension of the idea of invariance has found an area of application. The (A,B)-invariant subspaces of a state space $\mathcal{X}$ as introduced by Wonham and Morse will be the essential mathematical tool in handling the problems to be stated in section II.

3 - <u>(A,B)- invariant subspaces</u>

We consider again the system Σ with its state space $\mathcal{X}$ its input space $\mathcal{U}$ and its output space $\mathcal{Y}$. Let $\mathcal{V}$ be a subspace of $\mathcal{X}$. If $\mathcal{W} \subseteq \mathcal{X}$ and $\mathcal{E} \triangleq \mathcal{W} + \mathcal{V}$; then $\forall \underline{t} \in \mathcal{E}$, $\underline{t} = w + v$ for some $w \in \mathcal{W}$ and $v \in \mathcal{V}$.

<u>DEFINITION 1.4 [5]</u>

$\mathcal{V}$ is an (A,B)-invariant subspace of $\mathcal{X}$ if it's A-invariant mod $(B\mathcal{U})$ i.e

$$A\mathcal{V} \subseteq \mathcal{V} + B\mathcal{U} \qquad (5) \qquad ■$$

The importance of (A,B)-invariant subspaces is that these subspaces can be made (A + BF)-invariant for a suitable choice of the feedback matrix F. This property is very useful since it helps to change a feedback problem to an existence of a subspace which is (A,B)-invariant.

LEMMA 1.5 [1]

Let $\mathcal{V} \subseteq \mathcal{X}$ .There exists $F : \mathcal{X} \to \mathcal{U}$ such that $(A+BF)\mathcal{V} \subseteq \mathcal{V}$ if and only if $\mathcal{V}$ is (A,B)-invariant.

PROOF: [5]

"$\Rightarrow$" Let $(A+BF)\mathcal{V} \subseteq \mathcal{V}$ and $\underline{x} \in \mathcal{V}$.
Then $(A+BF)\underline{x} = \underline{v}$ and $\underline{v} \in \mathcal{V}$;then $A\underline{x} = \underline{v} - BF\underline{x} \in \mathcal{V} + B\mathcal{U}$ since $F\underline{x} \in \mathcal{U}$

"$\Leftarrow$" Let $\mathcal{V}$ be (A,B)-invariant and $\{\underline{v}_1, \dots, \underline{v}_\mu\}$ be a basis for $\mathcal{V}$ .As $A\mathcal{V} \subseteq \mathcal{V} + B\mathcal{U}$,there exist $\underline{w}_i \in \mathcal{V}$ and $\underline{u}_i \in \mathcal{U}$ for $i \in \{1, \dots, \mu\}$ such that

$$A\underline{v}_i = \underline{w}_i - BF\underline{u}_i \quad , \quad i \in \{1, \dots, \mu\}$$

Defining $F_o : \mathcal{V} \to \mathcal{U}$ by $F_o \underline{v}_i = \underline{u}_i$, $i \in \{1, \dots, \mu\}$ and letting F be any extension of the map F_o to $\mathcal{X}$ we obtain

$$A\underline{v}_i = \underline{w}_i - BF\underline{v}_i \quad , \quad i \in \{1, \dots, \mu\} .$$

∎

By the above lemma it's seen that there always exists a feedback F by which an (A,B)-invariant subspace $\mathcal{V}$ in the openloop characterisation can be made $\bar{A}$ -invariant in closed loop characterisation;where $\bar{A} \triangleq A+BF$ .This fact will be later used in finding a solution of disturbance decoupling problem by state feedback.

For frequency domain applications,the frequency domain characterisation of (A,B)-invariant subspaces is needed.This characterisation is given in terms of (ζ , ω)-representation as in [7].

(However we have to mention that a polynomial characterisation of (A,B)-invariant subspaces making use of the Rosenbrock system matrix is discussed in detail in [6]).

THEOREM 1.6 [7,9]

A subspace $\mathcal{V} \subseteq \mathcal{X}$ is an (A,B)-invariant subspace if and only if every $\underline{x}_o \in \mathcal{V}$ has a $(\underline{\xi}, \underline{\omega})$-representation satisfying $\underline{\xi}(s) \in \mathcal{V}$ for all s .

P R O O F : [9]

"$\Leftarrow$"

Let $\underline{x}_o \in \mathcal{V} \subseteq \mathcal{X}$ and $\underline{x}_o = (sI-A)\underline{\xi}(s) - B\,\underline{w}(s)$ with strictly proper $\underline{w}(s) \in \mathcal{U}$ and strictly proper $\underline{\xi}(s) \in \mathcal{V}$ for all s . Since $s\underline{\xi}(s) \in \mathcal{V}$

$$A\underline{\xi}(s) = s\underline{\xi}(s) - \underline{x}_o - B\underline{w}(s) \in \mathcal{V} + B\mathcal{U} \quad \text{for all } s.$$

The functions $\underline{\xi}(s)$ and $\underline{w}(s)$ being strictly proper, it is possible to go back to time domain by inverse Laplace transform and for $\mathcal{L}^{-1}\{\underline{\xi}(s)\} = \underline{x}(t)$, $\mathcal{L}^{-1}\{\underline{w}(s)\} = \underline{u}(t)$ for $t \geq 0$ we have $\underline{x}(t) \in \mathcal{V}$ for $t \geq 0$. Then $\underline{\dot{x}}(0^+) = \lim_{t \to 0} t^{-1}(\underline{x}(t) - \underline{x}_o) \in \mathcal{V}$. Hence for $t = 0^+$ $A\underline{x}_o = \underline{\dot{x}}(0^+) - B\underline{u}(0^+) \in \mathcal{V} + B\mathcal{U}$.

"$\Rightarrow$" Let $\mathcal{V} \subseteq \mathcal{X}$ and $A\mathcal{V} \subseteq \mathcal{V} + B\mathcal{U}$,then by T H M 1.5 there exist an F such that $(A+BF)\mathcal{V} \subseteq \mathcal{V}$ .For $\underline{x}_o \in \mathcal{V}$ choosing $\underline{\xi}(s) \triangleq (sI-A-BF)^{-1}\underline{x}_o$ and $\underline{w}(s) \triangleq F\underline{\xi}(s)$ we have $\underline{x}_o = (sI-A-BF)\underline{\xi}(s) = (sI-A)\underline{\xi}(s) - B\underline{w}(s)$ with $\underline{\xi}(s)$ and $\underline{w}(s)$ strictly proper because of the property of $(sI-A-BF)^{-1}$. ■

When we are interested in (A,B)-invariant subspaces contained in a subspace of $\mathcal{X}$,the property of maximality becomes very important in expressing the solvability criteria of various problems.

(4) - Maximal (A,B)-invariant subspaces in $Ker\, C$

DEFINITION 1.7

Let $\mathcal{V} \subseteq \mathcal{X}$; $\mathcal{V}$ is an (A,B)-invariant subspace contained in $Ker\, C$ if and only if the following two conditions hold:

$$(i)\quad A\mathcal{V} \subseteq \mathcal{V} + B\mathcal{U}$$
$$(ii)\quad \mathcal{V} \subseteq Ker\, C \qquad \blacksquare$$

In general one can talk about an (A,B)-invariant subspace contained in any subspace of $\mathcal{X}$ .However for application purposes (D D P ,etc) we are interested with the inclusion in $Ker\, C$.

It can be shown that it's not necessary that $Ker\, C$ contains a unique (A,B)-invariant subspace.This leads us to talk about the maximal (largest) (A,B)-invariant subspace contained in $Ker\, C$,in order to judge correctly if a problem has a solution or not.In [2] a geometric construction of the maximal (A,B)-invariant subspace in $Ker\, C$ is given as part of theorem.We will consider only the iterative construction formula and will not discuss the proof which is in [1,2].

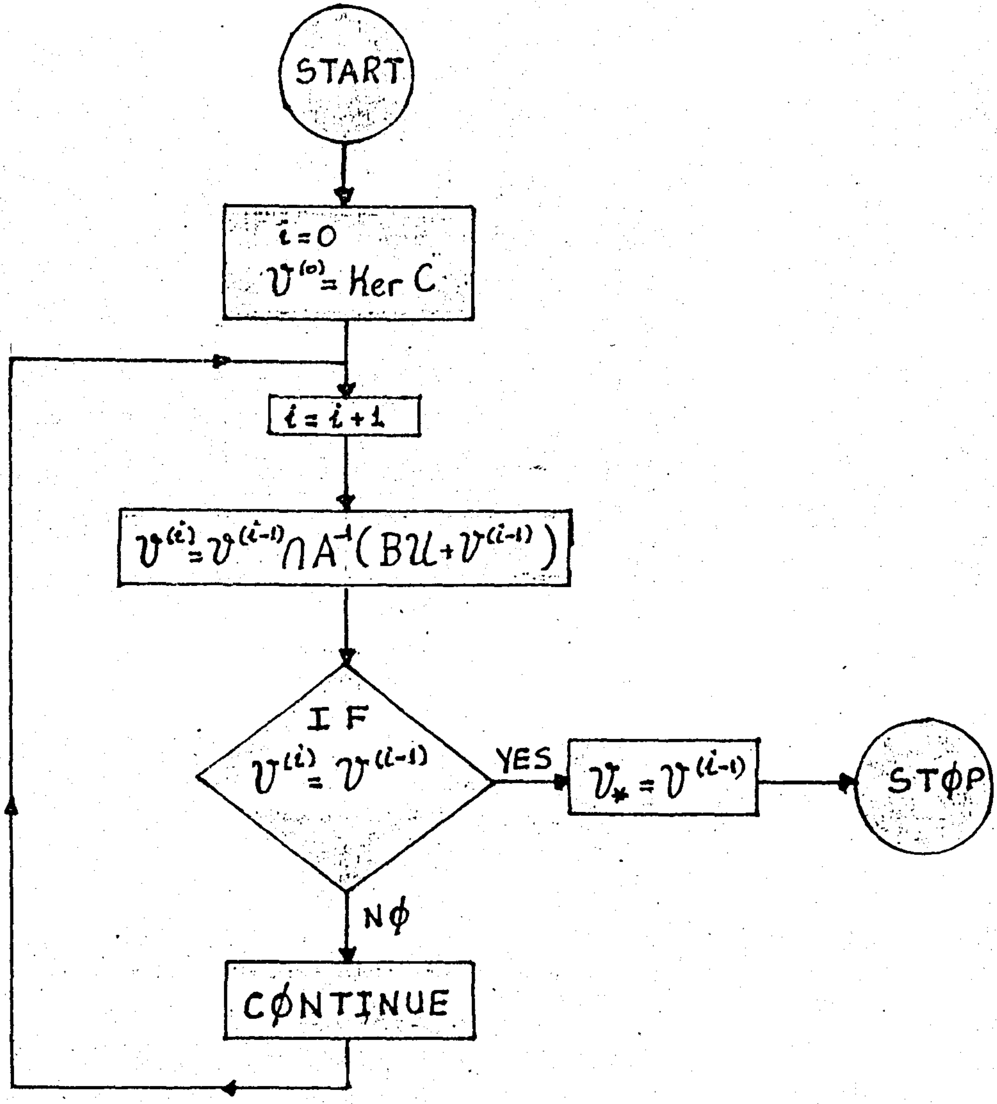
THEOREM 1.8 [2]

Let $\mathcal{V}^{(0)} \triangleq Ker\, C$ and define

$$\mathcal{V}^{(i)} = \mathcal{V}^{(i-1)} \cap A^{-1}(B\mathcal{U} + \mathcal{V}^{(i-1)}) \qquad i = 1, 2, \ldots, \mu \qquad (6)$$

where $\mu = \dim Ker\, C$ .Then $\mathcal{V}_* = \mathcal{V}^{(\mu)}$ is the maximal (A,B)-invariant subspace contained in $Ker\, C$. $\blacksquare$

In the recursive relation (6) A^{-1} denotes the functional inverse of A matrix whenever A is singular.The geometric construction of the maximal (A,B)-invariant subspace contained in $Ker\, C$ is a practical method when working on specific problems.To illustrate T H M 1.8 the following flowchart will be considered together with an example.



FLOWCHART 1.8 : GEOMETRIC CONSTRUCTION ØF $\mathcal{V}_*$

Example 1.8 :

Given

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

we will construct the maximal (A,B)-invariant subspace contained in $Ker\, C$ .For this we begin by computing $Ker\, C$:

$$C\underline{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow Ker\, C = sp\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Following the steps of the flowchart 1.8 ,

$i=0 \qquad \mathcal{V}^{(0)} = \mathrm{Ker}\, C$

$i=1$

$$B\mathcal{U} + \mathcal{V}^{(0)} = \mathrm{sp.}\left\{\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$$

$$A^{-1}(B\mathcal{U} + \mathcal{V}^{(0)}) = \mathrm{sp.}\left\{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$$

$$\mathcal{V}^{(1)} = \mathcal{V}^{(0)} \cap A^{-1}(B\mathcal{U} + \mathcal{V}^{(0)}) = \mathrm{sp.}\left\{\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$$

$\boxed{\text{check}}$ $\quad \mathcal{V}^{(1)} \subset \mathcal{V}^{(0)} \quad$ <u>but</u> $\mathcal{V}^{(1)} \neq \mathcal{V}^{(0)}$

$i=2$

$$B\mathcal{U} + \mathcal{V}^{(1)} = \mathrm{sp.}\left\{\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$$

$$A^{-1}(B\mathcal{U} + \mathcal{V}^{(1)}) = \mathrm{sp.}\left\{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$$

$$\mathcal{V}^{(2)} = \mathcal{V}^{(1)} \cap A^{-1}(B\mathcal{U} + \mathcal{V}^{(1)}) = \mathrm{sp}\left\{\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$$

$\boxed{\text{check}}$ $\mathcal{V}^{(2)} \subset \mathcal{V}^{(1)}$ <u>but</u> $\mathcal{V}^{(2)} \neq \mathcal{V}^{(1)}$

$i = 3$

$$B\mathcal{U} + \mathcal{V}^{(2)} = \mathrm{sp}\left\{\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$$

$$A^{-1}(B\mathcal{U} + \mathcal{V}^{(2)}) = \mathrm{sp}\left\{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$$

$$\mathcal{V}^{(3)} = \mathcal{V}^{(2)} \cap A^{-1}(B\mathcal{U} + \mathcal{V}^{(2)}) = \mathrm{sp}\left\{\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$$

$\boxed{\text{check}}$

$$\mathcal{V}^{(3)} = \mathcal{V}^{(2)} = \mathrm{sp}\left\{\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\} = \mathcal{V}_*$$

As seen from the above example, to obtain the maximal (A,B)-invariant subspace contained in $\mathrm{Ker}\,C$ will be a more difficult problem when systems of larger dimension are considered.

To give a frequency domain characterisation of the maximal (A,B)-invariant subspace contained in $\mathrm{Ker}\,C$, the (z,ω)-representation will play the basic role. In [9], this formulation is given by a definition and a theorem which are combined here.

THEOREM 1.9 [9]

$\mathcal{V}_*$ is the largest (A,B)-invariant subspace contained in $Ker\,C$ if and only if $\mathcal{V}_*$ denotes the space of all points for which there exists a $(\underline{\xi},\underline{w})$-representation satisfying $\underline{\xi}(s) \in Ker\,C$ for all s. (i.e. $C\underline{\xi}(s) = \underline{0}$)

Proof :

"$\Longrightarrow$"

For $\underline{x}_0 \in \mathcal{V}_*$,we have by T H M 1.6

$$\underline{x}_0 = (sI-A)\underline{\xi}(s) - B\underline{w}(s) \quad with \quad \underline{\xi}(s), \underline{w}(s)$$

strictly proper and $\underline{\xi}(s) \in \mathcal{V}_* \quad \forall s$ .Since $\mathcal{V}_* \subseteq Ker\,C$ by definition $\underline{\xi}(s) \in Ker\,C$.

"$\Longleftarrow$"

Let $\underline{x}_0 \in \mathcal{V}_*$ and show first that $\underline{x}_0 \in Ker\,C$.
For this expand $\underline{\xi}(s)$ and $\underline{w}(s)$ as power series in s^{-1},

$$\underline{\xi}(s) = \sum_{k=1}^{\infty} \underline{\xi}_k \, s^{-k}$$

$$\underline{w}(s) = \sum_{k=1}^{\infty} \underline{w}_k \, s^{-k}$$

and note that $\underline{\xi}(s) \in Ker\,C \ \forall s \Rightarrow \underline{\xi}_k \in Ker\,C \quad \forall k \in \{1,2,\dots\}$

Then $\underline{x}_0$ having a $(\underline{\xi},\underline{w})$-representation with $\underline{\xi}(s) \in Ker\,C$,

$$\underline{x}_0 = (sI-A)\underline{\xi}(s) - B\underline{w}(s)$$

or

$$\underline{x}_0 = (sI-A)\sum_{k=1}^{\infty} \underline{\xi}_k s^{-k} - B\sum_{k=1}^{\infty} \underline{w}_k s^{-k}$$

and equating the constant terms

$$\underline{x}_0 = \underline{\xi}_1 \in Ker\,C.$$

are obtained.

To complete the proof there remains only to show that $\mathcal{V}_*$ is (A,B)-invariant (or equivalently, to show that $\underline{\xi}(s) \in \mathcal{V}_*$ for all s because of T H M 1.6). Regarding $\underline{\xi}(s)$ as power series in s^{-1}, it's sufficient again to proove that $\underline{\xi}_k \in \mathcal{V}_* \quad \forall k \in \{1,2,\dots\}$
For this we take the proper parts of both sides of the equation

$$s^k \underline{x}_0 = s^k (sI-A)\,\underline{\xi}(s) - s^k B \underline{\omega}(s) \qquad (7)$$

that is

$$[s^k \underline{x}_0]_{proper} = [(sI-A)(s^{k-1}\underline{\xi}_1 + s^{k-2}\underline{\xi}_2 + \dots + \underline{\xi}_k + s^{-1}\underline{\xi}_{k+1} + \dots\ )]_{proper}$$
$$- [B(s^{k-1}\underline{\omega}_1 + \dots + \underline{\omega}_k + s^{-1}\underline{\omega}_{k+1} + \dots)]_{proper} \quad (8)$$

which is equivalent to

$$0 = -A\underline{\xi}_k + (sI-A)(s^{-1}\underline{\xi}_{k+1} + \dots) - B\underline{w}_k - B(s^{-1}\underline{\omega}_{k+1} + \dots) \quad (9)$$

Then if $(\underline{V}(s))_-$ denotes the strictly proper part of the rational vector $\underline{V}(s)$, by equation (9)

$$(sI-A)(s^{k-1}\underline{\xi}(s))_- - B(s^{k-1}\underline{\omega}(s))_- \qquad (10)$$
$$= \underline{\xi}_k + s^{-1}[-A\underline{\xi}_k + \cancel{(sI-A)(s^{-1}\underline{\xi}_{k+1} + \dots)} - B(\underline{w}_k + s^{-1}\underline{w}_{k+1} + \dots)] = \underline{\xi}_k$$

defining $(s^{k-1}\underline{\xi}(s))_- \triangleq \underline{\eta}(s)$ and $(s^{k-1}\underline{\omega}(s))_- \triangleq \underline{\varphi}(s)$
equation (10) implies

$$\underline{\xi}_k = (sI-A)\,\underline{\eta}(s) - B\,\underline{\varphi}(s) \qquad (11)$$

Now $\underline{\eta}(s)$, $\underline{\varphi}(s)$ are strictly proper functions ; furthermore $\underline{\eta}(s) \in Ker C$ since for $\underline{\xi}(s) \in Ker C$, $\underline{\xi}_k$ is also an element of $Ker C$ (for $k \in \{1,2,\dots\}$). Then $\underline{\xi}_k$ has a ($\underline{\xi}$, $\underline{\omega}$)-representation with $\underline{\eta}(s) \in Ker C$, and by definition $\underline{\xi}_k \in \mathcal{V}_*$.
The maximality of $\mathcal{V}_*$ is immediate since it denotes all such points. ■

Using T H M 1.9 the following result which is more convenient to use in most cases can easily be obtained.

<u>COROLLARY 1.10</u> [9]

$\underline{x}_0 \in \mathcal{V}_*$ if and only if there exists strictly proper rational functions $\underline{\xi}(s)$ and $\underline{w}(s)$ such that

$$\begin{bmatrix} sI-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} \underline{\xi}(s) \\ \underline{w}(s) \end{bmatrix} = \begin{bmatrix} \underline{x}_0 \\ 0 \end{bmatrix} \qquad (12)$$

■

Also the following corollary can be obtained by eliminating $\underline{\xi}(s)$ from equations (12).

COROLLARY 1.11 [9]

$\underline{x}_0 \in \mathcal{V}_*$ if and only if there exists $\underline{w}(s)$ strictly proper such that

$$C(sI-A)^{-1}\underline{x}_0 = -R(s)\,\underline{w}(s)$$

where $R(s) \triangleq C(sI-A)^{-1}B$ is the transfer function matrix of the system Σ described by (1), (2). ■

In fact COROLLARY 1.11 is very explanatory about $\mathcal{V}_*$ and tells that for this $\underline{w}(s)$ the output corresponding to the initial state $\underline{x}_0$ is zero. In other words, whatever be the initial state chosen in $\mathcal{V}_*$, the trajectories followed by the state of the system remain at any instant in $\mathcal{V}_*$, and thus the corresponding output at any instant is zero.

5 - Stabilizability subspaces :

In most of the design problems stability considerations play an important role. A new type of subspace, stabilizability subspaces introduced in [7,8,9] are very useful to treat such problems. There is a close relation between the stabilizability and the controllability subspaces. To emphasize this point later on, it is useful to explain with two brief definitions what is a contollability subspace.

Consider again the system Σ described by (1),(2) and its state space $\mathcal{X}$; we have the following definitions

DEFINITION 1.12 [1]

Given the system Σ the controllable subspace $\mathcal{R}_0 \subset \mathcal{X}$ of the pair (A,B) is

$$\mathcal{R}_0 = \mathcal{B} + A\mathcal{B} + \cdots + A^{n-1}\mathcal{B} \triangleq \langle A/\mathcal{B}\rangle \qquad (13)$$

where $\mathcal{B} \triangleq Im\, B \triangleq B\mathcal{U}$. ■

In otherwords $\mathcal{R}_0$ is the set of states which are reachable from $x_0 = 0$ and is a linear subspace of $\mathcal{X}$. [1] .

Now given the pair (A,B), consider all pairs (A+ BF, BG) which can be obtained by using a state feedback F and connecting again matrix G at the system input. (fig. 1.13). The controllable subspace of the new system pair (A +BF, BG) is called a controllability subspace of the original pair (A,B). The following definition w ill make the concept clearer.

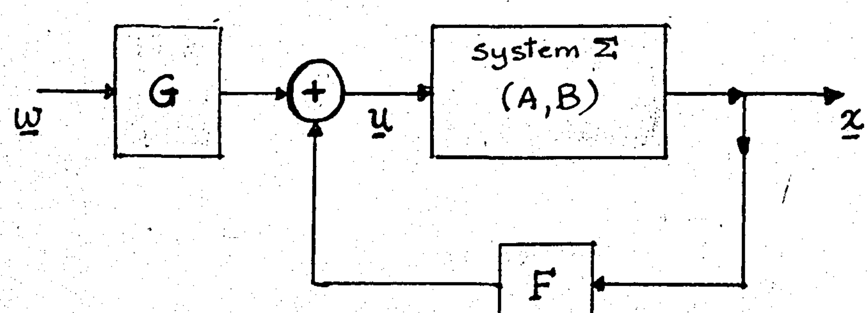


FIGURE 1.13: System (A+BF, BG) obtained from Σ

DEFINITION 1.13 [1]

Let $A: \mathcal{X} \to \mathcal{X}$ and $B: \mathcal{U} \to \mathcal{X}$ be as described in system Σ. A subspace $\mathcal{R} \subset \mathcal{X}$ is a controllability subspace of the pair (A,B) if there exist maps $F: \mathcal{X} \to \mathcal{U}$ and $G: \mathcal{U} \to \mathcal{U}$ such that

$$\mathcal{R} = \langle A + BF / Im\, BG\rangle \qquad (14)$$ ■

Note that

$$\mathcal{R} = Im\, BG + (A+BF)\, Im BG + \cdots + (A+BF)^{n-1} Im BG$$

and $(A+BF)\mathcal{R} = (A+BF)\, Im\, BG + \cdots + (A+BF)^{n} Im\, BG$;

hence $(A+BF)\mathcal{R} \subset \mathcal{R}$ by the Cayley-Hamilton theorem. By LEMMA 1.5

the family of controllability subspaces of a fixed pair (A,B) is a subfamily of the family of (A,B)-invariant subspaces.

We now our attention to stability and stabilizability and we consider stability from a general point of view.We denote by $\mathbb{C}^-$ any subset of $\mathbb{C}$ satisfying the condition $\mathbb{C}^- \cap \mathbb{R} \neq \phi$.This condition is brought in, recalling the property that no stable system having only complex conjugate poles in $\mathbb{C}^-$ exists.As understood $\mathbb{C}^-$ denotes our " stability region" in the general sense and we say that A is a <u>stability map (matrix)</u> if $\sigma(A) \subseteq \mathbb{C}^-$,where $\sigma(\cdot)$ means " spectrum of A ".Again we will say that a rational function is stable if it has no poles outside of $\mathbb{C}^-$. $\mathbb{C}^+ \subset \mathbb{C}$ is the set with the properties $\mathbb{C}^- \cap \mathbb{C}^+ = \phi$ and $\mathbb{C}^- \cup \mathbb{C}^+ = \mathbb{C}$.

<u>DEFINITION - 1.14</u>. [1]

Given the system Σ ,we say that the pair (A,B) is stabilizable if there exists a map $F: \mathcal{X} \rightarrow \mathcal{U}$ such that

$$\sigma(A+BF) \subset \mathbb{C}^-$$

Hence " stabilizing " the pair (A,B) is equivalent to change the unstable map A to a stable map $\bar{A}$ by simply using a state feedback F. In [1] ,the close relationship between the existence of F and controllability is given.Let the minimal polynomial of A be $\alpha(s)$ and factor it as $\alpha(s) = \alpha^-(s)\alpha^+(s)$ where zeroes of $\alpha^-(s) \in \mathbb{C}^-$ and zeroes of $\alpha^+(s) \in \mathbb{C}^+$. The subspace $Ker\, \alpha^+(A)$ of $\mathcal{X}$ is called the subspace of " unstable modes " of A .As shown in [1] ,the pair (A,B) is stabilizable if and only if the " unstable modes " of A are controllable.The proof of this conclusion will not be discussed but the following theorem will be stated for further reference.

THEOREM 1.15 [1]

Given the system Σ, the pair (A,B) is stabilizable if and only if $\mathcal{X}_b \triangleq Ker\,\alpha^+(A) \subset \langle A|\mathcal{B}\rangle$. ■

Consequently there exist a feedback F such that $\sigma(A+BF) \subseteq \mathbb{C}^-$ if and only if the subspace of " unstable modes " of A is included in the contollable subspace of the pair (A,B), [1] .Recalling a property of controllability for linear systems,we can state the following theorem.

THEOREM 1.16 [7, 9]

(A,B) is stabilizable if and only if for every complex $s \in \mathbb{C}^-$ we have

rank $[sI-A \vdots B] = n$,where n is the system dimension. ■

We notice that rank $[sI-A \vdots B]$ is always equal to " n " except at the eigenvalue of A .Then when we restrict the controllability to the unstable modes the eigenvalues are also restricted to those which lie in $\mathbb{C}^+$.Another useful conclusion is that a completely controllable system pair is always stabilizable.

We are now ready to give the definition of stabilizability subspace.

DEFINITION 1.17 [7]

$\mathcal{V} \subseteq \mathcal{X}$ is called a stabilizability subspace if there exists $F: \mathcal{X} \to \mathcal{U}$ such that $(A+BF)\mathcal{V} \subseteq \mathcal{V}$ and $\sigma((A+BF)|_{\mathcal{V}}) \subseteq \mathbb{C}^-$. ■

In the above definition $\sigma((A+BF)|_{\mathcal{V}})$ means the " spectrum of the map (A + BF) restricted to the subspace $\mathcal{V}$ .To be clearer,consider the map $\mathcal{A}: \mathcal{X} \to \mathcal{X}$ .With a basis $\{x_1, x_2, \ldots x_n\}$ for $\mathcal{X}$,let $A_{n\times n}$

be an $n \times n$ matrix representing $\mathcal{A}$ with respect to this basis. Then let $\mathcal{V} \subseteq \mathcal{X}$ be an $\mathcal{A}$-invariant subspace of $\mathcal{X}$ such that $\mathcal{A}\mathcal{V} \subseteq \mathcal{V}$. If $\{v_1, \dots v_r\}$ is a basis for $\mathcal{V}$, completing this basis to a basis for $\mathcal{X}$ we obtain $\{v_1, \dots, v_r, x_{r+1}, \dots x_n\}$. In this new basis the map $\mathcal{A}$ is characterised by the matrix

$$\bar{A} = \begin{bmatrix} \bar{A}_1^{r\times r} & \bar{A}_3 \\ 0 & \bar{A}_2^{(n-r)\times(n-r)} \end{bmatrix}$$

and $\sigma(\mathcal{A}/\mathcal{V}) \triangleq \sigma(\bar{A}_1)$, [1] .

To follow the idea introduiced by DEF 1.17, consider a stabilizability subspace $\mathcal{V} \subseteq \mathcal{X}$; $\forall \underline{x}_0 \in \mathcal{V}$, using a suitable feedback F the response

$$Ce^{t(A+BF)}\underline{x}_0 \rightarrow 0 \quad \text{for} \quad t \rightarrow \infty .$$

In other words we don't require that

$$Ce^{t(A+BF)} \rightarrow 0 \quad \text{for} \quad t \rightarrow \infty ,$$

but obtain a decay to zero only for some of the state vectors that we are interested in. This is equivalent to say that we are interested in stabilizing a part of the state space $\mathcal{X}$, if it's not possible to stabilize the whole state space. This property will be later used to stabilize the output corresponding to a set of disturbance inputs by using state feedback.

By definition a stabilizability subspace is (A,B)-invariant according to LEMMA 1.5. If $\mathcal{V}$ is any (A,B)-invariant subspace with $F: \mathcal{X} \rightarrow \mathcal{U}$ satisfying $(A+BF)\mathcal{V} \subseteq \mathcal{V}$ we can construct a system $\Sigma_{F,G}$ such that

$$\dot{\underline{x}} = (A+BF)\underline{x} + BG\underline{w} \qquad (14)$$

where $G: \mathcal{W} \rightarrow \mathcal{U}$ with $\mathcal{W} = R^{\ell}$ for some ℓ . This is so since there exists $G: \mathcal{W} \rightarrow \mathcal{U}$ such that the time domain response to the input $\underline{u} \in \mathcal{U}$, $\xi_u(t,x_0) \in \mathcal{V}$ for all $t \geq 0$ if and only if $x_0 \in \mathcal{V}$ and $\underline{u}$ is of the form $\underline{u} = F\underline{x} + G\underline{w}$ for some $\underline{w} : [0,\infty) \rightarrow \mathcal{W}$, [7].
So taking any G such that $BG\mathcal{W} = (B\mathcal{U}) \cap \mathcal{V}$ we can obtain a

restricted system $\Sigma'_{F,G}$ with an input value space $\mathcal{W}$ and a state space $\mathcal{V}$. Obviously $\Sigma'_{F,G}$ is not necessarily unique; then let $\Phi_{\Sigma}(\mathcal{V})$ denote the set of such pairs (F,G) given rise to systems $\Sigma_{F,G}$ with a state space $\mathcal{V}$.

LEMMA 1.18 [7]

Let $\mathcal{V} \subseteq \mathcal{X}$ be (A,B)-invariant and let $(F,G) \in \Phi_{\Sigma}(\mathcal{V})$ then $\mathcal{V}$ is a stabilizability subspace if and only if $(A+BF, BG)/_{\mathcal{V}}$ is stabilizabl

The proof is in [7] ∎

The following lemma shows the existence conditions of such (F,G) pairs.

LEMMA 1.18 A [7]

There exists $(F,G) \in \Phi_{\Sigma}(\mathcal{V})$ such that $\sigma(A+BF) \subseteq \mathbb{C}^-$ if and only if the system Σ' is stabilizable and $\mathcal{V}$ is a stabilizability subspace. ∎

Hence the system $\Sigma'_{F,G}$ must be stabilizable, in other words the unstable modes of $\Sigma'_{F,G}$ must be included in the controllable subspace of $\Sigma_{F,G}$ $\Sigma_{F,G}$. In conclusion, we use a linear state feedback F and a gain matrix G to obtain a new system which restricted to a subspace $\mathcal{V}$ has controllable unstable modes. Since $\langle A+BF/_{\mathcal{V}} | Im(BG) \rangle = \langle A+BF | Im(BG) \rangle = \mathcal{R}$ for $\mathcal{V}$ to be a stabilizability subspace there must exist a controllability subspace $\mathcal{R}$ of (A,B) such that it contains the unstable modes of $(A+BF)$ where F matrix is not necessarily unique and induced by the condition. $(A+BF)\mathcal{V} \subseteq \mathcal{V}$.

To give the frequency domain characterisation of a stabilizability subspace we again use the $(\underline{\zeta},\underline{\omega})$ -representation.

THEOREM 1.19 [9]

$\mathcal{V} \subseteq \mathcal{X}$ is a stabilizability subspace if and only if every $\underline{x} \in \mathcal{V}$ has a $(\underline{\xi},\underline{\omega})$-representation such that $\underline{\xi}(s) \in \mathcal{V}$ and $\underline{\xi}(s)$, $\underline{\omega}(s)$ are stable. ∎

As is the case of (A,B)-invariant subspaces, the maximal stabilizability subspace contained in $Ker C$ is important to characterize the solutions of various control problems.

DEFINITION 1.20 A [7, 9]

Given the system Σ and the subset $\mathbb{C}^-$ of $\mathbb{C}$, $\mathcal{V}_*^-$ denotes the set of points for which there exists a stable $(\underline{\xi},\underline{\omega})$-representation satisfying $\underline{\xi}(s) \in Ker C$. ∎

THEOREM 1.20 [9]

$\mathcal{V}_*^-$ is the maximal stabilizability subspace contained in $Ker C$. ∎

REMARK 1.20 B

The DEF 1.20 A and THM 1.20 can be stated for the maximal stabilizability subspace contained in a space $\mathcal{K} \subseteq \mathcal{X}$, in general, by simply changing $Ker C$ to $\mathcal{K}$.

COROLLARY 1.21

The system Σ is stabilizable if and only if $\mathcal{V}_*^-(\mathcal{X}) = \mathcal{X}$, where $\mathcal{V}_*^-(\mathcal{X})$ denotes the maximal stabilizability subspace contained in $\mathcal{X}$. ∎

We believe that this simple introduction to stabilizability subspaces will be sufficient to follow the analysis of the decoupling problems which takes stability criteria in consideration, and this completes the first chapter on mathematical preliminairies. In the next chapter we will discuss in detail control problems in which the concepts stated here will play a basic role.

II DISTURBANCE DECOUPLING & STABILIZATION

The aim of this chapter is to give a detailed presentation of various formulations concerning the disturbance decoupling and stabilization problems.We have to point out that the problems we are going to analyze have recently been introduced with the help of a new approach,the geometric approach to the structural synthesis of linear time-invariant multivariable systems.For each problem being analyzed the goal is to formulate the solvability criteria.In each case,the formulation will be basicly geometric,then its frequency domain translation will be given using the (ζ, ω)-representation and the work done in [6,7,8,9] .Most of the time we will also try to give a matrix polynomial formulation for some comparaison purposes.

① Disturbance Decoupling Problem (DDP):

In system simulation and mathematical modelling,the unwanted effects imposed on the system are known as disturbance parameters.In very simple terms,disturbance decoupling is "to decouple the effect of disturbances acting at the system's input parts " from the system output,using state feedback.Examination of this problem involves essentially the fundamental geometric concept of (A,B)-invariant subspaces.

Consider the system Σ

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) + E\underline{q}(t) \qquad (15)$$

$$\underline{y}(t) = C\underline{x}(t) \qquad (16)$$

for $t \geq 0$;where again $\underline{x}(t)$ is the state vector, $\underline{u}(t)$ is the input vector , $\underline{q}(t)$ represents the disturbance vector and $\underline{y}(t)$ the output vector.

PROBLEM (DDP) :

Find a state feedback $F:\mathcal{X}\to\mathcal{U}$ such that $\underline{q}(\cdot)$ has no influence on the output $\underline{y}(\cdot)$. Note that we denote again the state space by $\mathcal{X}$, the input space by $\mathcal{U}$, the output space by $\mathcal{Y}$ and the space where $\underline{q}(\cdot)$ takes its values by $\mathcal{Q}$.

A definition that takes place in [1] states that " the system Σ is said to be disturbance decoupled if and only if the forced response

$$\underline{y}(t) = C\int_0^t e^{(t-\tau)A} E\, q(\tau)\, d\tau \qquad (17)$$

due to the disturbance is zero for all $\underline{q}(\cdot)\in\underset{\sim}{Q}$ and for all $t\geq 0$, where $\underset{\sim}{Q}$ is a function class. Geometricly, the system Σ is disturbance decoupled if and only if

$$\langle A|\mathcal{E}\rangle \subset \operatorname{Ker}C\ , \text{ where } \mathcal{E}\triangleq \operatorname{Im}E\ , [1,2].$$

Now, consider the case where the system Σ is not originally disturbance decoupled and where the linear state feedback law $F:\mathcal{X}\to\mathcal{U}$ is being used to change the map $A:\mathcal{X}\to\mathcal{X}$ to $\bar{A}\triangleq A+BF:\mathcal{X}\to\mathcal{X}$. The so obtained system is disturbance decoupled if and only if

$$\langle A+BF|\mathcal{E}\rangle \subset \operatorname{Ker}C.$$

THEOREM 2.1 [1, 2]

Given system Σ described by (15), (16), DDP is solvable if and only if $\mathcal{E}\subseteq\mathcal{V}_*$, where $\mathcal{V}_*$ is the maximal (A,B)-invariant subspace contained in $\operatorname{Ker}C$.

PROOF : " $\Longrightarrow$ "

Given the system Σ, DDP is solvable implies that $\mathcal{V}\triangleq\langle A+BF|\mathcal{E}\rangle\subset\operatorname{Ker}C$. Since $\mathcal{E}\subset\mathcal{V}$, $\mathcal{V}$ is (A,B)-invariant and $\mathcal{V}\subset\operatorname{Ker}C$; $\mathcal{V}\subseteq\mathcal{V}_*$ hence $\mathcal{E}\subseteq\mathcal{V}_*$.

"$\Longleftarrow$" Given $\mathcal{E} \subseteq \mathcal{V}_*$, since $\mathcal{V}_*$ is (A,B)-invariant, by LEMMA 1.5 there exists a state feedback $\mathbf{F}$ such that $\mathcal{V}_*$ is (A+BF)-invariant, i.e. $(A+BF)\mathcal{V}_* \subseteq \mathcal{V}_*$. Then since $\mathcal{E} \subseteq \mathcal{V}_*$,

$$\langle A+BF/\mathcal{E}\rangle \subset \langle A+BF/\mathcal{V}_*\rangle \subseteq \mathcal{V}_* \subseteq Ker C ,$$

hence $\langle A+BF/\mathcal{E}\rangle \subseteq Ker C$. ■

So far the analysis of DDP has been abstract. However using the algorithm of THM 1.8 we can always compute $\mathcal{V}_*$ and check if DDP is solvable or not, remembering that $\mathbf{E}$ is a given matrix. The following block diagram gives a clear picture of the disturbance decoupling.

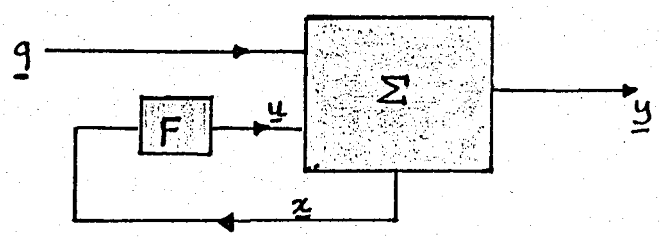


FIGURE 2.1 : DDP

Once it is known how to characterize $\mathcal{V}_*$ in frequency domain it's easy to obtain the frequency domain formulation of DDP. As assumed previously, the class of function $\underset{\sim}{Q}$, where $\underline{q}(\cdot) \in \underset{\sim}{Q}$, is large enough not to give them a special configuration, and the only restriction is assumed to be on the structure of $\mathbf{E}$ matrix. Thus it's wanted in general the disturbance to output transfer function matrix be m.-nullified by using state feedback. The geometric condition being $\mathcal{E} \triangleq E\underset{\sim}{Q} \triangleq Im E \subseteq \mathcal{V}_*$ we use LEMMA 1.10 to obtain to frequency domain formulation of DDP.

THEOREM 2.2 [9]

DDP is solvable if and only if there exists strictly proper matrices $X(s)$ and $U(s)$ such that

$$\begin{bmatrix} sI-A & -B \\ C & 0 \end{bmatrix}\begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix} \qquad (18)$$

PROOF .

The proof is straight forward. By applying LEMMA 1.10 to each column of E ,we obtain that (18) implies each column of $E\underline{q}_o$ for $\forall \underline{q}_o \in Q$ be an element of $\mathcal{V}_*$, and we use THM 2.1 to complete the proof. ∎

At this point it is useful to consider another problem known as the exact model matching problem [3] .Our purpose is to show that the existence of a solution for DDP is equivalent to the existence of a solution for the corresponding EMMP.

Exact model matching problem is motivated in general by the notion of " model following control ",To be more explicite we may think of a model system as a system having all desirable qualities. The large scale realisation of it may introduce unwanted side-effects. The compensation scheme is then used to modify the realised system such that it behaves just like its model.

PROBLEM (EMMP) :

Given a system with the $(p\times m)$ strictly proper,rational transfer matrix $G_1(s)$ and a model system with the $(p\times q)$ strictly proper,rational transfer matrix $G_2(s)$,does there exist a compensation scheme which employs linear state variable feedback in combination with input dynamics such that the transfer matrix of the given system is equal to $G_2(s)$ and when ?

THEOREM 2.3 [3]

EMMP is solvable if and only if there exists a strictly proper rational $(m\times q)$ matrix $Q(s)$ such that

$$G_1(s)\, Q(s) = G_2(s) .$$ ∎

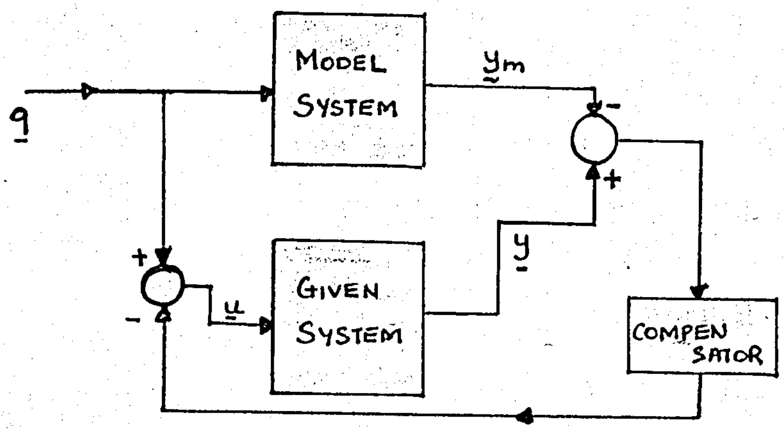


FIGURE 2.3: MODEL FOLLOWING CONTROL

Now to show first the one to one correspondance between EMMP and DDP we make the following definitions. Given the observable system $(A,[B,E],C)$:

$$G_1(s) \triangleq C(sI-A)^{-1}B \qquad (19)$$

$$G_2(s) \triangleq C(sI-A)^{-1}E \qquad (20).$$

In fact (19) characterize the transfer function of the "given system" and (20) the transfer function of the model system, and we want to exactly match the input/output transfer matrix $G_1(s)$ to the disturbance/ output transfer matrix $G_2(s)$. Conversely when data for EMMP is given as $G_1(s)$ and $G_2(s)$, the corresponding data for DDP is constructed as observable realisations (A,B,C) and (A,E,C) combined as $[A,[B,E),C)$. By these definitions we have to

$$\boxed{\left\{\begin{array}{c}\text{SOLVE DDP} \\ \text{FOR } (A,[B,E],C)\end{array}\right\} \Longleftrightarrow \left\{\text{FOR } \begin{array}{c}\text{SOLVE EMMP} \\ G_1(s) \triangleq C(sI-A)^{-1}B \\ G_2(s) \triangleq C(sI-A)^{-1}E\end{array}\right\}}$$

THEOREM 2.4 [6]

EMMP is solvable if and only if the corresponding DDP is solvable

PROOF

For system Σ defined by (15),(16) let $G_1(s) \triangleq C(sI-A)^{-1}B$ and $G_2(s) \triangleq C(sI-A)^{-1}E$ .Then DDP issolvable if and only if THM.2.2 holds.By eliminating $X(s)$ in the equations of (18) we have

$$C(sI-A)^{-1}E = -C(sI-A)^{-1}BU(s)$$

which implies $G_2(s) = -G_1(s)U(s)$,where $U(s)$ is srticly proper by THM.2.2. ∎

The relation between DDP and EMMP is attractive because it gives a second framework to treat the EMMP.In order to solve EMMP we have always the possibility to construct the data $(A,[B,E],C)$ for the corresponding DDP.Then by solving DDP we will have the realisation of the compensation transfer matrix $Q(s)$.

② Modified Disturbance Decoupling Problem (MDDP)

We consider again the system Σ ,but this time we assume that the disturbance $\underline{q}(t)$ is also directly available for measurement such that a feedback from the disturbance input is possible.

PROBLEM (MDDP):

Given the system Σ determine constant matrices F and D such that when the linear state feedback law $\underline{u}(t) = F\underline{x}(t) + D\underline{q}(t)$ is used the output doesnot depend on $\underline{q}(t)$, i.e. $C(sI-A-BF)^{-1}(BD+E) \equiv 0$.

The solvability of MDDP is formulated in a similar way to the one of DDP.

THEOREM 2.5

MDDP is solvable if and only if

$$\mathcal{E} \subseteq \mathcal{V}_* + \mathcal{B} \qquad (21)$$

where $\mathcal{V}_*$ is the maximal (A,B)-invariant subspace contained in $Ker C$.

PROOF : "$\Leftarrow$"

Given $\mathcal{E} \subseteq \mathcal{V}_* + \mathcal{B}$ since $\mathcal{E} \triangleq \mathrm{Im}E$ and $\mathcal{B} \triangleq \mathrm{Im}B$,for any basis matrix V of $\mathcal{V}_*$ there exists maps D_1 and G such that

$$E = VG + BD_1$$

which implies that

$$E - BD_1 = VG \quad \text{, hence } \mathrm{Im}\{E - BD_1\} \subseteq \mathcal{V}_* \subseteq \mathrm{Ker}C \, .$$

Then by LEMMA 1.5 there exists $F: \mathcal{X} \to \mathcal{U}$ such that

$(A+BF)\,\mathrm{Im}\{E - BD_1\} \subseteq \mathcal{V}_*$. Hence sufficiency holds by $C(sI-A-BF)^{-1}(E - BD_1) \equiv 0$ and by taking $D = -D_1$.

"$\Rightarrow$"

Let MDDP be solvable, then there exist F and D such that by $\underline{u} = F\underline{x} + D\underline{q}$ the system is disturbance decoupled. Then by the definition of disturbance decoupling

$$\langle A+BF \mid \mathrm{Im}\,(E+BD) \rangle \subseteq \mathrm{Ker}C$$

and $\quad \mathrm{Im}\,(E+BD) \subseteq \mathcal{V}_*$. $\qquad (22)$

Now since $\quad \mathrm{Im}\,(E+BD) \triangleq (E+BD)\,\mathcal{Q} = E\mathcal{Q} + BD\mathcal{Q}$

and since $\quad -BD\mathcal{Q} \subseteq \mathrm{Im}\,B$

(22) implies that $\quad E\mathcal{Q} + BD\mathcal{Q} - BD\mathcal{Q} \subseteq \mathcal{V}_* + \mathrm{Im}B$ ∎

completing the proof.

Similar to the DDP case, we can talk about the relation between MDDP and the <u>modified exact model matching problem</u> (MEMMP). The MEMMP is defined as follows :

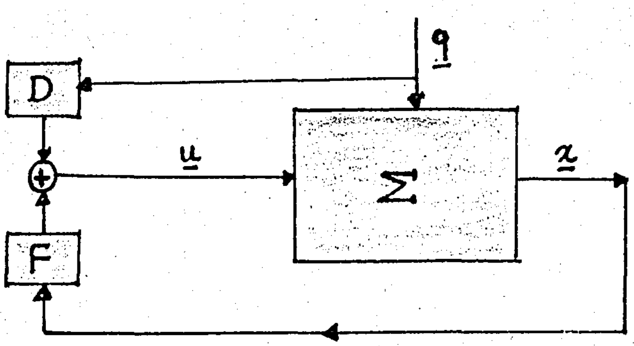


FIGURE 2.5: MDDP

PROBLEM (MEMMP) :

Given the strictly proper transfer matrices $G_1(s)$ and $G_2(s)$ find a compensation scheme that employs a transfer matrix $Q(s)$ proper rational.

REMARK :

In fact in [3] Wolovich defines the EMMP, in the most general case by putting the condition of the existence of a proper rational $Q(s)$. However in [6] E.Emre and MLJ.Hautus have splitted this general formulation into two parts: i) a strictly proper $Q(s)$ (EMMP) and ii) a proper $Q(s)$ (MEMMP).

THEOREM 2.6 [6]

MDDP is solvable if and only if the corresponding MEMMP has a solution. ■

The one to one corresponding and the proof is completely analogous to the one of THM 2.4 and avaible in [6] .

The problems of disturbance decoupling and modified disturbance decoupling intend only to reduce the effect of disturbance at the output, but they don't consider how the system dynamics are changed by the used feedback. Stability is the most important property which should be considered when investigating the effect of feedback on the system dynamics. Next topics will take this into considerations.

3- Stable Disturbance Decoupling Problem

As implied by the title of the section the problem that we are going to analyze is the stable version of DDP. For this, we again consider the system Σ described by the equations (15),(16).

PROBLEM (SDDP):

Find (if possible) a state feedback law $\underline{u}=F\underline{x}$ such that the effect of the disturbance at the output is anihilated and the closed loop system w ith $(A+BF)$ is stable, i.e find $F:\mathcal{X}\rightarrow\mathcal{U}$ such that $\langle A+BF \mid \mathcal{E}\rangle \subset \operatorname{Ker}C$ and $\sigma(A+BF)\subset\mathbb{C}^-$.

In [1] we see the definition of a family of (A,B)-invariant subspaces as follows:

$$\underline{\mathcal{V}} \triangleq \left\{\mathcal{V} : \exists F:\mathcal{X}\rightarrow\mathcal{U} \ni (A+BF)\mathcal{V}\subseteq\mathcal{V},\ \mathcal{V}\subseteq \operatorname{Ker}C \text{ and } \sigma((A+BF)|_{\mathcal{V}})\subset\mathbb{C}^-\right\} \tag{23}$$

One can immediatly notice that $\mathcal{V}\in\underline{\mathcal{V}}$ is a " stabilizability subspace " contained in $\operatorname{Ker}C$;so the maximal element of $\underline{\mathcal{V}}$ is $\mathcal{V}_*$,which was defined in DEF 1.17 combined with DEF 1.20 A ,as the maximal stabilizability subspace contained in $\operatorname{Ker}C$.

The formulation of the solvability of SDDP is given by the following theorem.

THEOREM 2.7 [1]

Given Σ ,suppose (A,B) is controllable.Then SDDP is solvable if and only if $\mathcal{E}\subseteq\mathcal{V}_*^-$. ■

Now,notice that the formulation of the THM.2.7 brings the hypothesis of (A,B) to be controllable,which implies that (A,B) is always stabilizable according to THM 1.16.However,as pointed out in the sequel of the samework in [1] ,this hypothesis is too strong, since by LEMMA 1.18A it is only necessary that (A,B) is stabilizable. Then the following formulation bring less restriction.

THEOREM 2.8 [7]

Given Σ ,SDDP is solvable if and only if (A,B) is stabilizable and $\mathcal{E}\subseteq\mathcal{V}_*^-$. ■

The difference between the formulations of THM 2.7 and THM 2.8 is that in the first one it is wanted that Σ be completely controllable; but in the second, only the unstable modes of Σ are required to be controllable. (Since (A,B) is stabilizable iff $\mathcal{X}_b(A) \subset \langle A/\mathcal{B}\rangle$).

We will turn our attention now, to the frequency domain characterisation of SDDP. Again this will be a direct translation from the geometric formulation. First consider the following LEMMA characterizing the stabilizability of the pair (A,B) in frequency domain.

LEMMA 2.9 [7]

Given Σ, (A,B) is stabilizable if and only if there exists strictly proper stable matrices $\tilde{X}(s)$, $\tilde{U}(s)$ such that

$$(sI-A)\tilde{X}(s) - B\tilde{U}(s) = I . \qquad (24)$$

Proof :

By COROLLARY 1.21 (A,B) is stabilizable if and only if $\mathcal{V}_*^- = \mathcal{X}$. Then applying THM 1.19 to $\mathcal{X}$, with a basis $\{\underline{e}_1,\dots,\underline{e}_n\}$ for $\mathcal{X}$ where $\underline{e}_i$ $i \in \{1,\dots,n\}$ are columns of I_n matrix we obtain (24) with $\tilde{X}(s)$ and $\tilde{U}(s)$ strictly proper stable matrices. ∎

THEOREM 2.10 [7, 9]

Given Σ, SDDP is solvable if and only if (i) there exists $\tilde{X}(s)$ and $\tilde{U}(s)$ strictly proper stable such that

$$(sI-A)\tilde{X}(s) - B\tilde{U}(s) = I$$

and (ii) there exists $X(s)$ and $U(s)$ strictly proper stable such that

$$\begin{bmatrix} sI-A & -B \\ C & 0 \end{bmatrix}\begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}.$$ ∎

Until now we have analyzed three problems DDP, MDDP and SDDP. We have to mention that this three problems are generically unsolvable since the localisation of the disturbances in to the wanted subspace $\mathcal{V}_*$

of $\mathcal{X}$ is very difficult. It is also rather difficult to have a sufficiently large (A,B)-invariant subspace in $Ker\,C$. We also have to notice that the constraints brought in the stable version of DDP are stronger. The modified problem brings the difficulty of measuring the disturbance which is normally not possible in most of the physical systems. However we think again that all of these three problems are very useful when a clever modelling is of consideration, and they lead to more efficient design techniques such as partial decoupling of disturbances.

In the next section we will consider more realistic problems which are generically solv able.

4 - Output Stabilisation with Respect to Disturbance (OSDP)

Our purpose now is to obtain a stable response depending on the disturbance. More clearly the problem is:

PROBLEM (OSDP):

Given Σ determine the linear state feedback law $\underline{u} = F\underline{x}$ such that the disturbance/output transfer matrix

$$C(sI-A-BF)^{-1}E \qquad \text{is stable.}$$

We face two problems together, the problem of finding a feedback (in other words a suitable (A,B)-invariant subspace) and the problem of making the output to disturbances stable using this feedback. Recalling that in the geometric approach we think of $\mathcal{X}$ as the set of all possible initial states; the response corresponding to initial state $\underline{x}_0$ is stable if $\underline{x}_0$ has a $(\underline{\xi},\underline{\omega})$-representation with $C\underline{\xi}(s)$ stable. Hence $\underline{x}_0$ gives a stable response when $\underline{\xi}(s)$ is stable or when the unstable poles of $\underline{\xi}(s)$ are in $Ker\,C$ and decoupled at the output.

DEFINITION 2.11 [9]

Given Σ, $\mathcal{S}^-$ denotes the subspace of points $\underline{x}_0 \in \mathcal{X}$ for which there exists a $(\underline{\zeta}, \underline{\omega})$-representation with $C_{\underline{\zeta}}(s)$ stable. ∎

The above explications make it clear that in extreme cases such points in $\mathcal{X}$ may be chosen from a stabilizability subspace of $\mathcal{X}$ or from $\mathcal{V}_*$. In general $\underline{x}_0 \in \mathcal{S}^-$ can always be written as a sum

$$\underline{x}_0 = \underline{x}_{01} + \underline{x}_{02}$$

where $\underline{x}_{01} \in \mathcal{V}_*$ and $\underline{x}_{02} \in \mathcal{V}_*^-(\mathcal{X})$; $\mathcal{V}_*^-(\mathcal{X})$ being the maximal stabilizability subspace contained in $\mathcal{X}$ and $\mathcal{V}_*$ being the maximal (A,B)-invariant subspace contained in $Ker\, C$ as stated previously. The following theorem characterizes $\mathcal{S}^-$ in terms of $\mathcal{V}_*^-(\mathcal{X})$ and $\mathcal{V}_*$.

THEOREM 2.12 [7]

$$\mathcal{S}^- = \mathcal{V}_* + \mathcal{V}_*^-(\mathcal{X}). \qquad ∎$$

The complete proof is in 7 . In the sequel we also need: -

THEOREM 2.13 [7]

There exists a feedback $F: \mathcal{X} \to \mathcal{U}$ such that $(A+BF)\mathcal{V}_* \subseteq \mathcal{V}_*$ and $C(sI-A-BF)^{-1}\underline{x}_0$ is stable for all $\underline{x}_0 \in \mathcal{S}^-$. ∎

Combining the results of THM 2.12 , THM 2.13 and the DEF 2.11 we have state the solvability of OSDP as follows:

THEOREM 2.14 [8,9]

Given Σ, OSDP is solvable if and only if $\mathcal{S}^- \supseteq \mathcal{E}$. ∎

The above theorem gives the geometric formulation. When the DEF 2.11 is applied to each column of $\mathbf{E}$ we obtain the frequency domain formulation.

THEOREM 2.15 [9]

Given Σ OSDP is solvable if and only if there exists strictly proper matrices $X(s)$ and $U(s)$ such that

$$(sI-A)X(s) - BU(s) = E$$

and $CX(s)$ is stable. ■

Compared with the decoupling problems previously analyzed , OSDP differs by being generically solvable. This is since controllability is a generic property. Then according to COROLLARY 1.21, $\mathcal{S}^- = \mathcal{X}$ is generically satisfied, which immediately satisfies the condition $\mathcal{E} \subseteq \mathcal{S}^-$ trivially.

(5). Output Stabilization Problem (OSP):

Given the system Σ ,we pose the problem as follows:

PROBLEM (OSP):

Stabilize the output y by means of a state feedback F ; more precisely find the conditions for the existence of a state feedback matrix F which can be calculated in terms of the system parameters (A,B,C) and such that the response $Ce^{t(A+BF)}$ tends to zero, as $t \to \infty$

An alternate interpretation of the problem statement is to find such a feedback matrix F ,that the characteristic exponents appearing in the response are in the stable subset $\mathbb{C}^-$ of $\mathbb{C}$. The response function, which is written for arbitrary initial state, make us understand that OSP is a generalisation of OSDP, in which initial states are bounded by the subspace $\mathcal{E} \subseteq \mathcal{X}$. In [1] the geometric formulation is given as follows:

THEOREM 2.16 [1]

Given Σ ,OSP is solvable if and only if
$\mathcal{X}_b(A) \subseteq \langle A|\mathcal{B}\rangle + \mathcal{V}_*$, where $\mathcal{X}_b(A) \triangleq Ker\alpha^+(A)$. ∎

Hence it's wanted that the unstable modes of A be either controllable or unobservable.The proof given in [1] is long and tedious as the formulation is based again on the controllability subspaces. However,using the stabilizability subspace the formulation becomes easier.In [9] the following procedure is used:

The largest stabilizability subspace $\mathcal{X}$ can be written as
$\mathcal{V}_*^-(\mathcal{X}) = \langle A|\mathcal{B}\rangle + \mathcal{X}_g(A)$,where $\mathcal{X}_g(A) \triangleq Ker\alpha^-(A)$.
So $\mathcal{V}_*^-(\mathcal{X})$ is the space of all initial conditions for which the response can be made stable by means of a suitable feedback matrix F or is already stable.The if the condition of THM 2.16 is satisfied, we have by

$$\mathcal{V}_*^-(\mathcal{X}) = \langle A|\mathcal{B}\rangle + \mathcal{X}_g(A) \text{ and } \mathcal{X}_b(A) \subseteq \langle A|\mathcal{B}\rangle + \mathcal{V}_*$$
$$\mathcal{S}^- = \mathcal{V}_* + \mathcal{V}_*^-(\mathcal{X}) = \langle A|\mathcal{B}\rangle + \mathcal{V}_* + \mathcal{X}_g(A) = \mathcal{X}_b(A) + \mathcal{X}_g(A)$$

and since $\mathcal{X}_b(A) + \mathcal{X}_g(A) = \mathcal{X}$, we have
$\mathcal{S} = \mathcal{X}$ appearing as the condition.We obtained the formulation by direct translation,however using the information brought by OSDP we notice that if the initial states are not bounded by $\mathcal{E} \subseteq \mathcal{X}$,but assumed totally arbitrary,we obtain the condition for the solvability of OSP as $\mathcal{S} = \mathcal{X}$;since OSDP is transformed to OSP.

THEOREM 2.17 [9]

OSP is solvable if and only if $\mathcal{S}^- = \mathcal{X}$. ∎

Now,since the columns of I_n matrix span the state space $\mathcal{X}$,the following gives the frequency domain formulation.

THEOREM 2.18 [9]

OSP is solvable if and only if there exists stricly proper matrices $P(s)$ and $Q(s)$ such that

$$(sI-A)P(s) - BQ(s) = I$$

and $CP(s)$ is stable. ∎

The stabilization of output as posed by OSP guarantees only the output $y(\cdot)$ is well-behaved, but it brings no restriction on the behaviour of system map on the unobservable subspace of $\mathcal{X}$; and it's a possibility this map be unstable.

6- Output Null Control

Ever it sounds as a completely new problem, the output null controllability has been derived from applications widely used in control theory, especially on discrete-time systems.

PROBLEM (ONC) :

Given $x_0 \in \mathcal{X}$ find a conditinn for the existence of a control function $u(\cdot)$ such that for solutivn of Σ with x_0 as the initial state there exists $T>0$ such that $y(t)=0 \ (\forall t>T)$.

We first notice that the conditions is weaker than the one required by output controllability, in which any arbitrary state is reachable from $x_0 = 0$. However in ONC we have more freedom on choosing $u(\cdot)$ such that it only brings the response of a given state x_0 to zero at the end of some $T>0$.

THEOREM 2.19 [9]

$\underline{x}_0 \in \mathcal{X}$ is output null controllable if and only if

$$\underline{x}_0 \in \Psi \triangleq \langle A|\mathcal{B}\rangle + \mathcal{V}_* .$$ ■

We think that the following exlanation will be sufficient. Since we understand by ONC to bring the initial state response to zero after some $t > T$ where $T > 0$, we have two choices: either we choose the initial state in the controllable subspace of the system Σ or in the largest (A,B)-invariant subspace contained in $\operatorname{Ker} C$, to decouple of completely at the output even if $\underline{x}_0$ is not controllable.

LEMMA 2.20 [9]

Ψ is an A-invariant subspace of $\mathcal{X}$.

PROOF :

Since by definition $\Psi \triangleq \mathcal{V}_* + \langle A|\mathcal{B}\rangle$, we see that $\langle A|\mathcal{B}\rangle$ is A-invariant and $\langle A|\mathcal{B}\rangle \supset \mathcal{B}$ implies

$$A\Psi = A\mathcal{V}_* + A\langle A|\mathcal{B}\rangle \subseteq \mathcal{V}_* + \mathcal{B} + \langle A|\mathcal{B}\rangle = \Psi$$

as $$A\mathcal{V}_* \subseteq \mathcal{V}_* + \mathcal{B} .$$ ■

The following lemma is used in the characterisation of the frequency domain formulation.

LEMMA 2.21 [9]

$\underline{x}_0 \in \langle A|\mathcal{B}\rangle$ if and only if there exists polynomial vectors $\underline{\xi}(s)$ and $\underline{w}(s)$ such that

$$\underline{x}_0 = (sI-A)\underline{\xi}(s) - B\underline{w}(s).$$

PROOF :

" $\Leftarrow$ "

Let $\underline{\xi}(s), \underline{w}(s)$ be polynomial vectors such that

$$\underline{x}_0 = (sI-A)\underline{\xi}(s) - B\underline{w}(s),$$

which can be written explicitely as

$$\underline{x}_0=(sI-A)[\underline{\xi}_n s^n+\cdots+\underline{\xi}_0] - B[\underline{\omega}_m s^m+\cdots+\underline{\omega}_0].$$

Now $\underline{x}_0$ being a constant vector, obviously $m=n+1$ and we have by equating the coefficients

$$\begin{aligned}\underline{\xi}_n &= B\underline{\omega}_{n+1}\\ \underline{\xi}_{n-1}-A\underline{\xi}_n &= B\underline{\omega}_n\\ &\vdots\\ \underline{\xi}_0-A\underline{\xi}_1 &= B\underline{\omega}_1\\ -A\underline{\xi}_0 &= B\underline{\omega}_0+\underline{x}_0\end{aligned}$$

Solving for $\underline{x}_0$ by successive subtitutions

$$\underline{x}_0=-B\underline{\omega}_0-A\underline{\xi}_0=-B\underline{\omega}_0-A^2\underline{\xi}_1-AB\underline{\omega}_1=\cdots$$

finally we obtain

$$\underline{x}_0=-B\underline{\omega}_0-AB\underline{\omega}_1-A^2B\underline{\omega}_2-\cdots-A^{p-1}B(\underline{\omega}_{p-1}+\cdots+\underline{\omega}_{n+1})$$

where $p=\dim(A)$. Since

$$\langle A|\mathcal{B}\rangle \triangleq \mathcal{B}+A\mathcal{B}+\cdots+A^{p-1}\mathcal{B}$$

$\underline{x}_0\in\langle A|\mathcal{B}\rangle$ as $B\underline{\omega}_0\in\mathcal{B}$, $AB\underline{\omega}_1\in A\mathcal{B}$, $\ldots$ $A^{p-1}B(\underline{\omega}_{p-1}+\cdots\underline{\omega}_{n+1})\in A^{p-1}\mathcal{B}$

"$\Longrightarrow$" Let $\underline{x}_0\in\langle A|\mathcal{B}\rangle$, then we can write

$$\underline{x}_0=B\underline{c}_1+AB\underline{c}_2+\cdots+A^{p-1}B\underline{c}_p \qquad (25)$$

Define $-\underline{\omega}_0\triangleq\underline{c}_1$ and in general $-\underline{\omega}_{i-1}\triangleq\underline{c}_i$ for $i\in\{1,\ldots,p-1\}$

then also defining

$$\begin{aligned}\underline{\xi}_{p-2} &\triangleq B\underline{\omega}_{p-1}\\ \underline{\xi}_{p-3} &\triangleq A\underline{\xi}_{p-2}+B\underline{\omega}_{p-2}\\ &\vdots\\ \underline{\xi}_0 &\triangleq A\underline{\xi}_1+B\underline{\omega}_1\end{aligned}$$

(25) can be written as

$$\underline{x}_0=(sI-A)[\underline{\xi}_0+\cdots\underline{\xi}_{p-2}s^{p-2}]-B[\underline{\omega}_0+\cdots+\underline{\omega}_{p-1}s^{p-1}],$$

completing the proof. ∎

Combining this result with THM 2.19 we obtain:

THEOREM 2.22 [9]

$\underline{x}_0\in\Psi$ if and only if there exists rational vectors $\underline{\xi}(s)$ and $\underline{\omega}(s)$ such that $\underline{x}_0=(sI-A)\underline{\xi}(s)-B\underline{\omega}(s)$ and $C\underline{\xi}(s)$ is a polynomial. ∎

PROOF :

"$\Rightarrow$"

Let $\underline{x}_0 \in \Psi \triangleq \langle A|\mathcal{B}\rangle + \mathcal{V}_*$, then $\underline{x}_0$ can be writeh as $\underline{x}_0 = \underline{x}_{01} + \underline{x}_{02}$

,where $\underline{x}_{01} \in \mathcal{V}_*$, $\underline{x}_{02} \in \langle A|\mathcal{B}\rangle$

which implies that

$\underline{x}_{01} = (sI-A)\underline{\xi}_1(s) - B\underline{\omega}_1(s)$ with $\underline{\xi}_1, \underline{\omega}_1$ strictly proper and $C\underline{\xi}_1(s) = 0$

$\underline{x}_{02} = (sI-A)\underline{\xi}_2(s) - B\underline{\omega}_2(s)$ with $\underline{\xi}_2, \underline{\omega}_2$ polynomial.

Hence $\underline{x}_0 = (sI-A)[\underline{\xi}_1(s) + \underline{\xi}_2(s)] - B[\underline{\omega}_1(s) + \underline{\omega}_2(s)]$

then $\underline{\xi}(s) \triangleq \underline{\xi}_1(s) + \underline{\xi}_2(s)$ is rational and

$C\underline{\xi}(s) = \cancel{C\underline{\xi}_1(s)}^{0} + C\underline{\xi}_2(s) = C\underline{\xi}_2(s)$ is polynomial

and $\underline{\omega}(s) = \underline{\omega}_1(s) + \underline{\omega}_2(s)$ is rational, and

$\underline{x}_0 = (sI-A)\underline{\xi}(s) - B\underline{\omega}(s)$ with $\underline{\xi}(s), \underline{\omega}(s)$ rational and $C\underline{\xi}(s)$ polynomial.

"$\Leftarrow$"

Let $\underline{x}_0 = (sI-A)\underline{\xi}(s) - B\underline{\omega}(s)$ with $\underline{\xi}(s), \underline{\omega}(s)$ rational and $C\underline{\xi}(s)$ polynomial. Defining

$\underline{\xi}_1(s) \triangleq (\underline{\xi}(s))_-$, we have $\underline{\xi}(s) = \underline{\xi}_1(s) + \underline{\xi}_2(s)$

where $\underline{\xi}_2(s)$ polynomial. And since $C\underline{\xi}(s)$ is polynomial and C is a constant matrix, obviously $C\underline{\xi}_1(s) = 0$. Also defining $\underline{\omega}_1(s) \triangleq (\underline{\omega}(s))_-$ we have $\underline{\omega}(s) = \underline{\omega}_1(s) + \underline{\omega}_2(s)$ with $\underline{\omega}_2(s)$ polynomial.

Then $\underline{x}_0 = (sI-A)\underline{\xi}_1(s) - B\underline{\omega}_1(s) + (sI-A)\underline{\xi}_2(s) - B\underline{\omega}_2(s) = \underline{x}_{01} + \underline{x}_{02}$

where $\underline{x}_{01} \triangleq (sI-A)\underline{\xi}_1(s) - B\underline{\omega}_1(s) \in \mathcal{V}_*$.

$\underline{x}_{02} \triangleq (sI-A)\underline{\xi}_2(s) - B\underline{\omega}_2(s) \in \langle A|\mathcal{B}\rangle$. ■

7 - Concluding Remarks

We want to end this chapter with the following table summarising the conditions corresponding to the solvability of the problems that have been considered.

Given $\Sigma: \quad \underline{\dot{x}} = A\underline{x} + B\underline{u} + E\underline{q}$
$\qquad\qquad \underline{y} = C\underline{x}$

Problem $\Longleftrightarrow$	Geometric condition $\Longleftrightarrow$	Frequency domain condition
DDP IS SOLVABLE	$\mathcal{E} \triangleq E\mathcal{Q} \subseteq \mathcal{V}_*$	$\exists\, X(s), U(s)$ STRICTLY PROPER such that $\begin{bmatrix} sI-A & -B \\ C & 0 \end{bmatrix}\begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}$
MDDP IS SOLVABLE	$\mathcal{E} \triangleq E\mathcal{Q} \subseteq \mathcal{V}_* + \mathcal{B}$	$\exists\, Q(s)$ PROPER such that $R_1(s)\,Q(s) = R_2(s)$ where $R_1(s) \triangleq C(sI-A)^{-1}B$, $R_2(s) \triangleq C(sI-A)^{-1}E$
DDP IS SOLVABLE	$\mathcal{E} \triangleq E\mathcal{Q} \subseteq \mathcal{V}_*^-$	(i) $\exists\, \tilde{X}(s), \tilde{U}(s)$ STRICTLY PROPER STABLE such that $(sI-A)\tilde{X}(s) - B\tilde{U}(s) = I$ (ii) $\exists\, X(s), U(s)$ STRICTLY PROPER STABLE such that $\begin{bmatrix} sI-A & -B \\ C & 0 \end{bmatrix}\begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}$
SDP IS SOLVABLE	$\mathcal{E} \triangleq E\mathcal{Q} \subseteq \mathcal{S}^-$	$\exists\, X(s), U(s)$ STRICTLY PROPER such that $(sI-A)X(s) - BU(s) = E$ and $CX(s)$ is STABLE.
OSP IS SOLVABLE	$\mathcal{S}^- = \mathcal{X}$	$\exists\, X(s), U(s)$ STRICTLY PROPER such that $(sI-A)X(s) - BU(s) = I$ and $CX(s)$ is STABLE
NC IS SOLVABLE	$\underline{x}_0 \in \Psi$	$\exists\, \underline{\xi}(s)$ and $\underline{w}(s)$ rational such that $\underline{x}_0 = (sI-A)\underline{\xi}(s) - B\underline{w}(s)$ and $C\underline{\xi}(s)$ is polynomial.

We have mentioned before that disturbance decoupling problems are generically unsolvable. In many system descriptions a large enough $\mathcal{V}_*$ does not exists. However if our purpose is to obtain a partial decoupling we can achieve this very easily. Let us define by $\mathcal{E}_p$ the subspace

$$\mathcal{E}_p \triangleq \mathcal{E} \cap \mathcal{V}_* .$$

Now choosing a suitable feedback F_p it's possible to decouple the noise components $\underline{q}$ such that $E\underline{q} \in \mathcal{E}_p$. Also we notice that if $\mathcal{E} \subseteq \mathcal{S}^-$ is satisfied (generally it is) we have always the possibility to choose the feedback F_p such that $\mathcal{E}_p$ is decoupled and the remaining noise components is stabilized at the output.

III - PROPERTIES OF A LINEAR m-PORT SYSTEM COMPOSED OF SEPARATED "LOSSLESS" AND "ALGEBRAIC" PARTS

In many applications, a linear m-port system determined by a state space description consists of a "lossless component N_D" and a "algebraic component N_A", as in Fig.3.1. As an example consider an electrical network, the lossless components are inductors, capacitors and the algebraic components are resistors, dependent sources, etc. In [10] a detailed work on obtaining the state space description of such an m-port and on the observability and controllability conditions has been done. Here our aim is to investigate further properties of this m-port system; the interconnection of N_D and N_A brings a larger degree of freedom in the solution of the problems considered in chapter II. We will concentrate our investigation on the improvements obtained for the solution of DDP in such an m-port. We begin by giving the system description.

1 - First Level System Decomposition

We consider an m-port obtained by interconnecting an algebraic (m + n)-port N_A and a lossless n-port N_D, as shown in Fig.3.1.

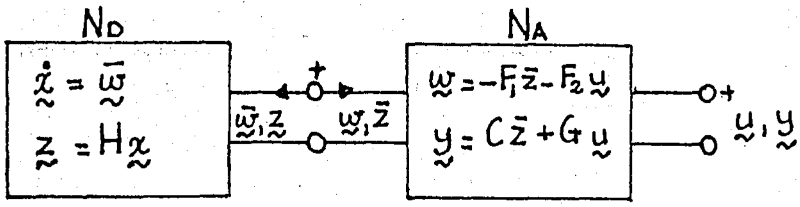


FIGURE 3.1: SYSTEM OBTAINED BY INTERCONNECTING NA & ND

The defining equations of N_D and N_A are

$$\underset{\sim}{w} = -F_1\bar{z} - F_2\underset{\sim}{u} \tag{26}$$

$$\underset{\sim}{y} = C\bar{z} + G\underset{\sim}{u} \tag{27}$$

$$\underset{\sim}{z} = H\underset{\sim}{x} \tag{28}$$

$$\dot{\underset{\sim}{x}} = \underset{\sim}{\bar{w}} \tag{29}$$

The input vector ($\underset{\sim}{\bar{z}}, \underset{\sim}{u}$) and the output vector ($\underset{\sim}{w}, \underset{\sim}{y}$) of N_A are usually hybrid pairs (*i.e* the i^{th} element of ($\underset{\sim}{\bar{z}}, \underset{\sim}{u}$) is a current, then the i^{th} element of ($\underset{\sim}{w}, \underset{\sim}{y}$) is a voltage variable). Same observation can be made for the input vector $\underset{\sim}{\bar{w}}$ and output vector $\underset{\sim}{z}$ of N_D; when equations (26),(27),(28),(29) are combined the state space description of this system is given by

$$\dot{\underset{\sim}{x}} = F_1 H\underset{\sim}{x} + F_2\underset{\sim}{u} \tag{30}$$

$$\underset{\sim}{y} = CH\underset{\sim}{x} + G\underset{\sim}{u} \tag{31}$$

where $\underset{\sim}{x} \in \tilde{\mathcal{X}}$ is the state vector, $\underset{\sim}{u} \in \tilde{\mathcal{U}}$ is the input vector and $\underset{\sim}{y} \in \tilde{\mathcal{Y}}$ is the output vector. These equations give the state equations as a function of charges and flux linkages. A more convenient description can be obtained in terms of capacitor voltages and inductor currents, as

$$\dot{\underset{\sim}{z}} = HF_1\underset{\sim}{z} + HF_2\underset{\sim}{u} \tag{32}$$

$$\underset{\sim}{y} = C\underset{\sim}{z} + G\underset{\sim}{u} \tag{33}$$

An important remark is that H must be invertible. If this condition is not satisfied the above equations are no longer the state equations of the system of Fig.3.1. For simplicity we will neglect the direct coupling of $\underset{\sim}{u}$ to the output $\underset{\sim}{y}$ and also denote by D the inverse of the matrix H, hence $D \triangleq H^{-1}$. We will call the <u>system Σ</u> the following system described by the state equations

$$D\dot{\underline{x}} = F_1\underline{x} + F_2\underline{u} \tag{34}$$

$$\underline{y} = C\underline{x} \tag{35}$$

where $\underline{x}$ is the state vector, $\underline{u}$ is the input vector, $\underline{y}$ is the output vector. When analyzing the disturbance decoupling problem we will introduce the noise component $E\underline{q}$ to the equation (34) to give a description of the disturbance. We also immediately notice that the matrix D is totally a function of the internal properties of N_D. So we can say that in an electrical network the entries of D are determined by the values of inductors and capacitors.

In the sequel we will also need the conditions for which the system Σ is controllable, observable. The following tables, [10], gives us the necessary and sufficient conditions.

rank F_2	rank $[F_1 \vdots F_2]$		Σ is
$\neq 0$	$= n$	$\Leftrightarrow$	$\exists$ D such that Σ is cont.
$= n$	$= n$	$\Leftrightarrow$	$\forall$ D, Σ is cont.
$= 0$	arbitrary	$\Leftrightarrow$	$\nexists$ D such that Σ is cont.
$\neq 0$	$< n$		

TABLE 3.2a : CONTROLLABILITY COND. OF Σ

rank C	rank $\begin{bmatrix} F_1 \\ \cdots \\ C \end{bmatrix}$		Σ is
$\neq 0$	$= n$	$\Leftrightarrow$	$\exists$ D such that Σ is observ.
$= n$	$= n$	$\Leftrightarrow$	$\forall$ D, Σ is observ.
$= 0$	arbitrary	$\Leftrightarrow$	$\nexists$ D such that Σ is observ.
$\neq 0$	$< n$		

TABLE 3.2.b : OBSERVABILITY COND. OF Σ

As can be seen from tables 3.2.a, 3.2.b the controllability and observability conditions are derived independently. However in general we expect a system to be controllable and observable at the same time. The necessary and sufficient conditions for this case is given by the following theorem.

THEOREM 3.1 [10]

Given F_1, F_2 and C there exist matrix D (not necassarily unique) such the system Σ is observable and controllable if and only if both of the following conditions are satisfied:

$$(\text{i}) \quad F_1 \neq \underset{\sim}{O}, \quad \text{rank}\,[F_1 \vdots F_2] = n$$
$$(\text{ii}) \quad C \neq \underset{\sim}{O}, \quad \text{rank}\begin{bmatrix} F_1 \\ \cdots \\ C \end{bmatrix} = n .$$

In the same work, the proof of this theorem also gives an algorithm that shows how such D matrices can be selected when F_1, F_2, C are given. In the sequel we will assume that we know how to choose D matrices and concentrate an other properties of Σ .

(2) - Polynomial System Matrices [13]

Consider the state space equations of a linear system, as

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \qquad (36)$$
$$\underline{y} = C\underline{x} + G\underline{u} \qquad (37)$$

it's obvious that these equations are linear time -invariant differential equations. In control theory, the advantage of using the frequency domain approach leads us to consider the Laplace transform of the state variables for many application purposes. Then assuming zero initial state the equations (36),(37) turns out to be

$$s\bar{x} = A\bar{x} + B\bar{u} \qquad (38)$$
$$\bar{y} = C\bar{x} + G\bar{u} \qquad (39)$$

where $\bar{x} = \mathcal{L}(\underset{\sim}{x}(t))$, $\bar{u} = \mathcal{L}(\underset{\sim}{u}(t))$ and $\bar{y} = \mathcal{L}(\underline{y}(t))$.

Rearranging (38) and (39) we obtain

$$(sI-A)\bar{x} - B\bar{u} = 0 \qquad (40)$$
$$-C\bar{x} - D(s)\bar{u} = -\bar{y} \qquad (41)$$

or

$$\underbrace{\begin{bmatrix} sI-A & B \\ -C & D(s) \end{bmatrix}}_{P(s)} \begin{bmatrix} \bar{x} \\ -\bar{u} \end{bmatrix} = \begin{bmatrix} 0 \\ -\bar{y} \end{bmatrix} \qquad (42)$$

In [13] , the coefficient matrix P(s) of the vector $[\bar{x}\ \bar{u}]^T$ is called a polynomial system matrix. Although the results seem to be trivial, the represetation of a linear time invariant system as a single polynomial matrix has many advantages. First, in the case of general linear constant differential systems it is most of the time difficult to obtain a suitable state space description. Secondly, using the system matrix $P(s)$ all transformations of the system equations can be expressed as operation on $P(s)$. (Third they appear naturally, by simply taking the Laplace transform of the describing differential equations.) Therefore the properties of the operations on $P(s)$ can be more systematically studied. In general a linear constant differential equations system gives the following system matrix

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}$$

We restrict ourselves to the case of system Σ and the corresponding system matrix $P_\Sigma(s)$ is

$$P_\Sigma(s) = \begin{bmatrix} Ds - F_1 & F_2 \\ -C & O \end{bmatrix} .$$

3 - Problem Statement

In a first level system decomposition one can notice that given the triple (F_1, F_2, C) a D matrix givingrise to a controllable and observable system Σ is not necessarily unique. Hence, by changing D matrix in such a way that Σ remains controllable and observable we obtain different state descriptions leading probably to different transfer function matrices. Then the following questions can be asked for realisation purposes:

a) Given (F_1, F_2, C) fixed, are the matrices D giving rise to controllable and observable systems having <u>same transfer function matrices</u> unique ?

b) Assuming that such matrices D are not unique, is it possible to decouple a noise component $E\underline{q}$ using a special D matrix from the above equivalence class w hich leaves the transfer function matrix invariant.

The answer to the first question obviously prepares the investigation of the second one. In fact, the idea is to investigate if a first level system decomposition brings any extra degree of freedom to treat several control problems, and more specifically the disturbance decoupling problem. This degree of freedom can also be useful for realisations of a given transfer function, if many D matrices are available to describe the "lossless" n-port N_D one can choose a proper one for a more suitable physical realisation.

4- Equivalence Transformations of System Matrices

We shall be particularly interested in a transformation which leaves unchanged the transfer function matrix and the system order:

Strict System Equivalence (SSE) [13]

Let an $(r+m)\times(r+\ell)$ polynomial system matrix P(s) be given. Let M(s), N(s) be $(r\times r)$ unimodular matrices; that is their determinants ane nonzero and independent of s. Also let X(s), Y(s) be polynomial matrices respectively $(m\times r)$ and $(r\times \ell)$. If two system matrices

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \quad \text{and} \quad P_1(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}.$$

are related by the transformation

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I_m \end{bmatrix}\begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}\begin{bmatrix} N(s) & Y(s) \\ 0 & I_\ell \end{bmatrix} = \begin{bmatrix} T_1(s) & U_1(s) \\ -V_1(s) & W(s) \end{bmatrix} \qquad (43)$$

then P (s) and P_1 (s) are said to be strictly system equivalent. ∎

A very important property of SSE transformation is given as follows.

THEOREM 3.2 [13]

Two system matrices which are strictly system equivalent give rise to the same transfer function matrix and have same order. ∎

Hence, because of the property stated by THM 3.2, it's very probable that for the system Σ described by the equations (34),(35), $P_{\Sigma}(s)$ matrices giving rise to the same transfer function matrix (i.e. D matrices, since they are the only changing parameter of $P_{\Sigma}(s)$) are related by SSE transformations. More precisely the question is:

QUESTION 3.3:

Given the system Σ_{α} described by the state equations

$$D_{\alpha}\dot{\underline{x}} = F_1\underline{x} + F_2\underline{u} \quad (44)$$

$$\underline{y} = C\underline{x} \quad (45)$$

and the corresponding $P_{\Sigma_{\alpha}}(s)$, are all system matrices giving rise to the same transfer function matrix related by SSE transformations ?

To give the answer of this question we need the following results which are instrumental.

5 - Decoupling Zeroes and Relatively Prime Polynomials

Let's consider first the Smith Form of a polynomial matrix A(s).

THEOREM 3.4 [13,14]

By a combination of elementary row and column operations, an $(m\times n)$ polynomial matrix A(s) can be reduced to its Smith form

$$S(s) = M(s)\ A(s)\ N(s)$$

where M,N are unimodular and represent the elementary row and column operations, and

$$S(s) = [\,Q(s)\ \ O_{m,n-m}\,] \quad \text{for} \quad n>m$$

$$S(s) = Q(s) \quad \text{for} \quad n=m$$

$$S(s) = \begin{bmatrix} Q(s) \\ O_{m-n,n} \end{bmatrix} \quad \text{for} \quad n<m$$

Here Q(s) is a diagonal matrix having as entries on its principal diagonal the invariant polynomials $\mathcal{E}_i(s)$. Each non-zero invariant polynomial has the coefficient of its higest power of s equal to 1. In Smith form if the rank of $A(s)$ is r, there are r non-zero invariant polynomials occupying the leading positions and the remaining invariant polynomials are zero.

THM 3.4 gives a detailed definition of the Smith Form. Now let's consider a system matrix $P(s)$ in which the matrix $[T(s)\ U(s)]$ has a Smith form $S(s) = [Q(s)\ 0]$. Then the determinant $|Q(s)| \triangleq D_r(s)$ is called the greatest monic common divisor of minors of order r in $[T(s)\ U(s)]$. The roots $\{\beta_i\}$, $i \in \{1,..,b\}$ of the equation $D_r(s)=0$ will be called zeroes of $S(s)$. The complete set of these roots is called <u>input-decoupling zeroes</u> [13]. The removal of these roots from $[T(s)\ \ U(s)]$ by dividing one by one by factors of $(s-\beta_i)$, the Smith form turns out to be $S(s)=\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$, $p<r$. In [13,14] we equivalently find the definition for <u>output-decoupling zeroes</u> where the $[T^T(s)\ \ -V^T(s)]$ submatrix of $P(s)$ is taken into consideration. Similarly by removing the output-decoupling zeroes from $[T^T(s)\ \ -V^T(s)]$ the Smith form becomes $S_1'(s)=\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}$, $q<r$. Hence by these definitions the following lemma is immediate.

<u>LEMMA 3.5</u>

A system described by the polynomial system matrix $P(s)$ has no decoupling zeroes if and only if the following conditions hold:

(i) the Smith form of $[T(s)\ \ U(s)]$ is $S(s) = [I_r\ \ 0]$

(ii) the Smith form of $[T^T(s)\ \ -V^T(s)]$ is $S_1(s) = [I_r^T\ \ 0^T]$,

where $r \triangleq$ {dimension of the spuare matrix $T(s)$}. ■

The following theorem is basic to find an answer to the Question 3.3.

THEOREM 3.6 [13]

Let $P(s)$ and $P_1(s)$ be two $(r+m)\times(r+\ell)$ polynomial system matrices having no decoupling zeroes. Then $P(s)$ and $P_1(s)$ are strictly system equivalent if and only if they give rise to the same transfer function matrix. ∎

We also need the following definition of relatively prime polynomial.

DEFINITION 3.6 [13]

Polynomial matrices $T(s)$ and $V(s)$ are called relatively left (right) prime if and only if their greatest common left divisor $G_L(s)$ (g.c.r.d. $G_R(s)$) is unimodular. ∎

A property of the relatively prime polynomials is given by the following theorem.

THEOREM 3.7 [14]

The polynomial matrices $T(s), V(s)$ respectively $(r\times r)$ and $(r\times \ell)$ are relatively left prime if and only if the Smith form of $[T(s) \quad V(s)]$ is $[I_r \quad O]$. A similar result can be stated for right primeness. ∎

Thus if $T(s), V(s)$ are the submatrices of a given polynomial system matrix $P(s)$; they are relatively left prime if and only if this system has no input decoupling zeroes.

THEOREM 3.8 [3]

Given a system described by the polynomial matrix

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}, \text{ it is}$$

(a) completely controllable if and only if any g.c.l.d $G_L(s)$ of $\{T(s),\ V(s)\}$ is unimodular

(b) completely observable if and only if any g.c.r.d $G_R(s)$ of $\{-V(s),\ T(s)\}$ is unimodular

(c) completely controllable and completely observable if and only if both (a) and (b) holds. ∎

The properties of least order systems that we have mentioned one by one lead to a very important result that gives an answer to question 3.3. The next part of the investigation is mainly based on the following result.

THEOREM 3.9

Consider two completely controllable and completely observable systems described by the polynomial matrices $P_1(s)$ and $P_2(s)$. Then $P_1(s)$ and $P_2(s)$ give rise to the same transfer function matrix if and only if they are strictly system equivalent. ∎

PROOF :

After THM 3.6, all we need to show is that a system is completely controllable and completely observable if and only if it has no decoupling zeroes. For this: P(s) is completely controllable & completely observable $\overset{\text{THM 3.8}}{\Longleftrightarrow}$ g.c.l.d of $\{T(s),V(s)\}$ is unimodular and g.c.r.d of $\{-V(s),T(s)\}$ is unimodular $\overset{\text{DEF 3.6}}{\Longleftrightarrow}$ $T(s),V(s)$ are relatively left prime and $-V(s),T(s)$ are relatively right prime $\overset{\text{THM 3.7}}{\Longleftrightarrow}$ Smith form of $[T(s)\ \ V(s)] = [I_r\ \ 0]$ and Smith form of $[T^T(s)\ \ -V^T(s)] = [I_r^T\ \ 0^T]$ $\overset{\text{LEMMA 3.5}}{\Longleftrightarrow}$ P(s) has no decoupling zeroes. ∎

Now consider the systems Σ_α described by the equations (44),(45).

COROLLARY 3.10

Let system Σ_α be described by

$$D_\alpha \dot{\underline{x}} = F_1\underline{x} + F_2\underline{u}$$
$$\underline{y} = C\underline{x}$$

where D and F_1 are $(n\times n)$, F_2 is $(n\times q)$ and C is $(q\times n)$. Let the corresponding system matrices be

$$P_\alpha(s) = \begin{bmatrix} D_\alpha s - F_1 & F_2 \\ -C & 0 \end{bmatrix}.$$

Let also rank $[F_1 \vdots F_2] = n$ and rank $[F_1^T \vdots C^T] = n$.
Consider all systems Σ_α which are controllable and observable; then these systems have the same transfer function matrix if and only if they are strictly system equivalent. ∎

⑥ - Equivalence Class of D_α Matrices Which Leaves The Transfer Function Invariant

The answer to the question 3.3 is given by COR.3.10. Assuming that we have a controllable and observable first level decomposition with an initial D_{α_0}, we can obtain all other D_α giving the same transfer function by using SSE transformations. For this we will use the following procedure.

Let $P_{\alpha_0}(s) = \begin{bmatrix} D_{\alpha_0}s - F_1 & F_2 \\ -C & 0 \end{bmatrix}$ be the initial controllable and observable decomposition. A $P_{\alpha_i}(s)$ which is strictly system equivalent to $P_{\alpha_0}(s)$ will also be completely controllable and observable and related to $P_{\alpha_0}(s)$ by equation (43), such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I_q \end{bmatrix} \begin{bmatrix} D_{\alpha_0}s - F_1 & F_2 \\ -C & 0 \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} D_{\alpha_i}s - F_1 & F_2 \\ -C & 0 \end{bmatrix} \qquad (46)$$

where $M(s)$, $N(s)$ are unimodular and $X(s)$, $Y(s)$ are polynomial.

THEOREM 3.10.a [13]

Two system matrices $P_1(s)$ and $P_2(s)$ which are in state space form are strictly system equivalent if and only if they are system similar. ■

Then using THM 3.10.a and the nonsingularity of D_{α_o} and D_{α_i} we can restrict M(s), N(s) to be constant nonsingular matrices and $X(s)=Y(s)=0$.

THEOREM 3.10.b

Given $P_{\alpha_o}(s)$ and $P_{\alpha_i}(s)$, they are strictly system equivalent if and only if there exists nonsingular constant matrices M and N such that

$$\begin{bmatrix} M & 0 \\ 0 & I_q \end{bmatrix}\begin{bmatrix} D_{\alpha_o}s-F_1 & F_2 \\ -C & 0 \end{bmatrix}\begin{bmatrix} N & 0 \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} D_{\alpha_i}s-F_1 & F_2 \\ -C & 0 \end{bmatrix} \tag{47}$$

Proof :

The equation (46) is equivalent to

$$\begin{bmatrix} D_{\alpha_i}^{-1} & 0 \\ 0 & I_q \end{bmatrix}\begin{bmatrix} M(s) & 0 \\ X(s) & I_q \end{bmatrix}\begin{bmatrix} D_{\alpha_o} & 0 \\ 0 & I_q \end{bmatrix}\begin{bmatrix} sI-D_{\alpha_o}^{-1}F_1 & D_{\alpha_o}^{-1}F_2 \\ -C & 0 \end{bmatrix}\begin{bmatrix} N(s) & Y(s) \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} sI-D_{\alpha_i}^{-1}F_1 & D_{\alpha_i}^{-1}F_2 \\ -C & 0 \end{bmatrix}$$

since $D_{\alpha_o}, D_{\alpha_i}$ are nonsingular matrices. Now letting $\hat{M}(s) = D_{\alpha_i}^{-1}M(s)D_{\alpha_o}$

$$\begin{bmatrix} \hat{M}(s) & 0 \\ X(s) & I_q \end{bmatrix}\begin{bmatrix} sI-D_{\alpha_o}^{-1}F_1 & D_{\alpha_o}^{-1}F_2 \\ -C & 0 \end{bmatrix}\begin{bmatrix} N(s) & Y(s) \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} sI-D_{\alpha_i}^{-1}F_1 & D_{\alpha_i}^{-1}F_2 \\ -C & 0 \end{bmatrix} \tag{48}$$

In the matrix equation (48), the SSE transformation is preserved; further more the polynomial system matrices are in state space form. Then by THM 3.10.a they are SSE if and only if they are system similar. Therefore there exist a constant nonsingular matrix H such that

$$\begin{bmatrix} H^{-1} & 0 \\ 0 & I_q \end{bmatrix}\begin{bmatrix} sI-D_{\alpha_o}^{-1}F_1 & D_{\alpha_o}^{-1}F_2 \\ -C & 0 \end{bmatrix}\begin{bmatrix} H & 0 \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} sI-D_{\alpha_i}^{-1}F_1 & D_{\alpha_i}^{-1}F_2 \\ -C & 0 \end{bmatrix} \tag{49}$$

or equivalently

$$\begin{bmatrix} D_{\alpha_i}H^{-1}D_{\alpha_o}^{-1} & 0 \\ 0 & I_q \end{bmatrix}\begin{bmatrix} D_{\alpha_o}s-F_1 & F_2 \\ -C & 0 \end{bmatrix}\begin{bmatrix} H & 0 \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} D_{\alpha_i}s-F_1 & F_2 \\ -C & 0 \end{bmatrix} \tag{50}$$

and call $N \triangleq H$ and $M \triangleq D_{\alpha_i} H^{-1} D_{\alpha_0}^{-1}$. ■

Thus without loss of generality we can restrict the unimodular matrices M(s) and N(s) to being constant and nonsingular, also choose $X(s)=0$ and $Y(s)=0$.

COROLLARY 3.10.c

Given $P_{\alpha_0}(s)$ and $P_{\alpha_1}(s)$, they are strictly system equivalent and give rise to the same transfer function matrix if and only if there exist M and N constant and nonsingular such that the following equations are satisfied:

(i) $M \; D_{\alpha_0} N = D_{\alpha_1}$ (51)

(ii) $M \; F_1 \; N = F_1$ (52)

(iii) $M F_2 = F_2$ (53)

(iv) $C \cdot N = C$ (54) ■

Obviously, since $(F_1, F_2, C, D_{\alpha_0})$ are given we can compute M and N matrices from equation (52),(53),(54) then using equation (51) we can compute D_{α_1}. The degree of freedom obtained on (M,N) couple will also determine the degree of freedom in choosing the matrices D_{α_i}. An important remark is that (52),(53),(54) always have a trivial sulution, that is the (M,N) couple where $M = I$ and $N = I$. In this case all four equations hold but giving $D_{\alpha_0} = D_{\alpha_1}$. The following theorem gives the necessary and sufficient conditions for which a nontrivial solution of equations (52),(53) (54) exists.

THEOREM 3.11

Given the matrices F_1, F_2, C respectively $(n \times n)$, $(n \times q)$, $(q \times n)$ and such that rank $[F_1 \vdots F_2] = n$ and rank $[F_1^T \vdots C^T] = n$, there exist nontrivial solutions of the matrix equations

$$\begin{bmatrix} M & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} F_1 & F_2 \\ C & 0 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ C & 0 \end{bmatrix} \qquad (55)$$

with nonsingular M,N matrices if and only if rank $F_2 < n$ and rank $C < n$. ∎

Due to THM 3.11 the existence of nontrivial solutions of (52),(53),(54) is a generic property since the theorem excludes only the special cases where rank $F_2 = n$ and/ or rank $C = n$.
The following lemma is needed in the proof of THM 3.11.

LEMMA 3.12

Given the matrix $X\,(q\times n), q \geq n$; there exists a nonsingular and nontrivial (i.e. $\neq I$) solution of the matrix equation

$$X = XY \tag{56}$$

where Y is ($n\times n$), if and only if $\operatorname{Ker} X \neq \{0\}$.

PROOF: "$\Leftarrow$"

Given $\operatorname{Ker} X \neq \{0\}$; rank $[X] < n$ since X has less than n linearly independent columns. By elementary row and column operations represented by $T_1\,(q\times q)$ and $T_2\,(n\times n)$, we bring X to its Smith form

$$T_1 X T_2 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad r = \operatorname{rank}[X] < n.$$

Hence (56) becomes

$$T_1^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} T_2^{-1} Y = T_1^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} T_2^{-1}$$

where for $\hat{Y} = T_2^{-1} Y T_2$, $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\hat{Y} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ (57)

Then partitionning we obtain

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{Y}_{11} & \hat{Y}_{12} \\ \hat{Y}_{21} & \hat{Y}_{22} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

which gives $\hat{Y}_{11} = I_r$, $\hat{Y}_{12} = 0$ with $\hat{Y}_{21}, \hat{Y}_{22}$ totally arbitrary; therfore taking $\hat{Y}_{11}$ to be identity matrix $\hat{Y}$ can be made nonsingular by carefully choosing $\hat{Y}_{21}$ and $\hat{Y}_{22}$.

Finally is obtained as $Y = T_2 Y T_2^{-1}$.

"$\Longrightarrow$"

Let Y be a nontrivial solution of (56), then $X(Y-I)=0$. Since $Y \neq I$ we have $Y-I \neq 0$ then $\operatorname{Ker} X \neq \{0\}$. ∎

PROOF OF THEOREM 3.11

"$\Longrightarrow$"

We will consider all possible cases one by one.

(i) Let rank $F_2 = n$ and rank $C = n$, then $\operatorname{Ker} F_2^T = \{0\}$ and $(M-I)F_2 = 0 \Rightarrow M = I$. Also $\operatorname{Ker} C = \{0\}$ and and $C(N-I)=0 \Rightarrow N = I$.

(ii) Let rank $F_2 < n$ and rank $C = n$, then as in (i)

$N = I$ hence the matrix equations (52),(53) give

$$\left.\begin{array}{l} M F_1 = F_1 \\ M F_2 = F_2 \end{array}\right\} \Rightarrow M[F_1 \,\vdots\, F_2] = [F_1 \,\vdots\, F_2]$$

Since rank $[F_1 \,\vdots\, F_2] = n$ for $(M-I)[F_1 \,\vdots\, F_2] = 0 \Rightarrow M = I$.

(iii) Let rank $F_2 = n$ and rank $C < n$. Similarly, as in (i)

$M = I$ and (52),(54) give

$$\left.\begin{array}{l} F_1 N = F_1 \\ C N = C \end{array}\right\} \Rightarrow \begin{bmatrix} F_1 \\ \hdashline C \end{bmatrix}(N-I) = 0$$

Then since rank $\begin{bmatrix} F_1 \\ \hdashline C \end{bmatrix} = n$, we have $(N-I)=0$ or $N = I$.

"$\Longleftarrow$"

Let rank $F_2 < n$ and rank $C < n$. Proceed as follows:

* Put F_2, C to Smith form keeping the structure of the matrix equations (55).

$$\left[\begin{array}{c:c} F_1 & F_2 \\ \hdashline C & O \end{array}\right] = \left[\begin{array}{c:c} T_1 & O \\ \hdashline O & T_A \end{array}\right] \left[\begin{array}{c:c} \tilde{F}_1 & \begin{smallmatrix} I_a & 0 \\ 0 & 0 \end{smallmatrix} \\ \hdashline \begin{smallmatrix} I_b & 0 \\ 0 & 0 \end{smallmatrix} & O \end{array}\right] \left[\begin{array}{c:c} P_1 & O \\ \hdashline O & P_A \end{array}\right] \qquad (58)$$

Save matrices T_1, P_1

* Partition as

$$
\begin{array}{c}
a \\ n-a \\ b \\ q-b \\ \\
\end{array}
\begin{array}{c}
\left[\begin{array}{c:c:c:c}
\tilde{F}_{11} & \tilde{F}_{12} & I_a & 0 \\ \hdashline
\tilde{F}_{13} & \tilde{F}_{14} & 0 & 0 \\ \hdashline
I_b & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0
\end{array}\right] \\
\begin{array}{cccc} b & n-b & a & q-a \end{array}
\end{array}
\qquad (59)
$$

* Put $\tilde{F}_{14}$ to Smith form.

$$
\left[\begin{array}{c:c:c:c}
\tilde{F}_{11} & \tilde{F}_{12} & I_a & 0 \\ \hdashline
\tilde{F}_{13} & \tilde{F}_{14} & 0 & 0 \\ \hdashline
I_b & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0
\end{array}\right]
=
\left[\begin{array}{c:c}
\overbrace{\left[\begin{array}{c:c} I_a & 0 \\ \hdashline 0 & T_0 \end{array}\right]}^{T_2} & 0 \\ \hdashline
0 & I_q
\end{array}\right]
\left[\begin{array}{c:c:c:c}
\hat{F}_{11} & \hat{F}_{12} & I_a & 0 \\ \hdashline
\hat{F}_{13} & \begin{array}{c:c} I_m & 0 \\ \hdashline 0 & 0 \end{array} & 0 & 0 \\ \hdashline
I_b & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c:c}
\overbrace{\left[\begin{array}{c:c} I_b & 0 \\ \hdashline 0 & P_0 \end{array}\right]}^{P_2} & 0 \\ \hdashline
0 & I_q
\end{array}\right]
\qquad (60)
$$

Save T_2, P_2 matrices.

* Partition as

$$\begin{array}{r c}
 & \\
a & \\
m & \\
n-m-a & \\
b & \\
q-b &
\end{array}
\left[\begin{array}{c:c:c:c:c}
\hat{F}_{11} & \hat{F}_{121} & \hat{F}_{122} & I_a & 0 \\ \hdashline
\hat{F}_{131} & I_m & 0 & 0 & 0 \\ \hdashline
\hat{F}_{132} & 0 & 0 & 0 & 0 \\ \hdashline
I_b & 0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0 & 0
\end{array}\right] \tag{61}$$
$$\begin{array}{ccccc} b & m & n-b-m & a & q-a \end{array}$$

Reduce $\hat{F}_{131}$ and $\hat{F}_{121}$ by elementary row or column operations.

$$\left[\begin{array}{c:c:c:c:c}
\hat{F}_{11} & \hat{F}_{121} & F_{122} & I_a & 0 \\ \hdashline
\hat{F}_{131} & I_m & 0 & 0 & 0 \\ \hdashline
\hat{F}_{132} & 0 & 0 & 0 & 0 \\ \hdashline
I_b & 0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0 & 0
\end{array}\right]
=
\left[\begin{array}{c:c}
\overbrace{\begin{array}{c:c:c} I_a & T & 0 \\ \hdashline 0 & I_m & 0 \\ \hdashline 0 & 0 & I_{n-a-m} \end{array}}^{T_3} & 0 \\ \hdashline
0 & I_q
\end{array}\right]
\left[\begin{array}{c:c:c:c:c}
\bar{F}_{11} & 0 & \bar{F}_{122} & I_a & 0 \\ \hdashline
0 & I_m & 0 & 0 & 0 \\ \hdashline
\bar{F}_{132} & 0 & 0 & 0 & 0 \\ \hdashline
I_b & 0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c:c}
\overbrace{\begin{array}{c:c:c} I_b & 0 & 0 \\ \hdashline P & I_m & 0 \\ \hdashline 0 & 0 & I_{n-m-b} \end{array}}^{P_3} & 0 \\ \hdashline
0 & I_q
\end{array}\right] . \tag{62}$$

Save T_3 , P_3 matrices.

* Finally we obtain equation (55) as

$$
\begin{bmatrix} \hat{M} & 0 \\ 0 & I_q \end{bmatrix}
\begin{array}{c|c}
 & \begin{array}{ccccc} b & m & n-b-m & a & q-a \end{array} \\
\begin{array}{c} a \\ m \\ n-m-a \\ b \\ q-b \end{array} &
\left[\begin{array}{ccc:cc}
\bar{F}_{11} & 0 & \bar{F}_{122} & I_a & 0 \\
0 & I_m & 0 & 0 & 0 \\
\bar{F}_{132} & 0 & 0 & 0 & 0 \\ \hdashline
I_b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{bmatrix} \hat{N} & 0 \\ 0 & I_q \end{bmatrix}
=
\left[\begin{array}{ccc:cc}
\bar{F}_{11} & 0 & \bar{F}_{122} & I_a & 0 \\
0 & I_m & 0 & 0 & 0 \\
\bar{F}_{132} & 0 & 0 & 0 & 0 \\ \hdashline
I_b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right] \qquad (63)
$$

where

$$\hat{M} = T_3^{-1} T_2^{-1} T_1^{-1} M T_1 T_2 T_3$$

$$\hat{N} = P_3 P_2 P_1 N P_1^{-1} P_2^{-1} P_3^{-1} .$$

* Block multiplications in (63) give first:

$$
\begin{array}{cc}
\begin{array}{c} a \\ n-a \end{array} &
\begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} \\
 & \begin{array}{cc} a & n-a \end{array}
\end{array}
\begin{array}{cc}
\begin{array}{c} a \\ n-a \end{array} &
\begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix} \\
 & \begin{array}{cc} a & q-a \end{array}
\end{array}
=
\begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix} \qquad (64)
$$

Thus $\hat{M}_{11} = I_a$ and $\hat{M}_{21} = 0$,and by partitioning

$$
\hat{M} = \begin{bmatrix} I_a & \hat{M}_{12} \\ 0 & \hat{M}_{22} \end{bmatrix}
=
\begin{array}{cc}
\begin{array}{c} a \\ m \\ n-m-a \end{array} &
\begin{bmatrix} I_a & \hat{M}_{121} & \hat{M}_{122} \\ 0 & \hat{M}_{221} & \hat{M}_{222} \\ 0 & \hat{M}_{223} & \hat{M}_{224} \end{bmatrix} \\
 & \begin{array}{ccc} a & m & n-m-a \end{array}
\end{array} \qquad (65)
$$

The second equation in $\hat{N}$ is obtained as :

$$\begin{array}{c} b \\ q-b \\ \\ \end{array}\!\!\begin{array}{c} \left[\begin{array}{c:c} I_b & 0 \\ \hdashline 0 & 0 \end{array}\right] \\ \;\; b \quad n-b \end{array} \begin{array}{c} b \\ n-b \\ \\ \end{array}\!\!\begin{array}{c} \left[\begin{array}{c:c} \hat{N}_{11} & \hat{N}_{12} \\ \hdashline \hat{N}_{21} & \hat{N}_{22} \end{array}\right] \\ \;\; b \quad n-b \end{array} = \left[\begin{array}{c:c} I_b & 0 \\ \hdashline 0 & 0 \end{array}\right] \qquad (66)$$

Thus $\hat{N}_{11} = I_b$, $\hat{N}_{12} = 0$ and by partitioning

$$\hat{N} = \left[\begin{array}{c:c} I_b & 0 \\ \hdashline \hat{N}_{21} & \hat{N}_{22} \end{array}\right] = \begin{array}{c} b \\ m \\ n-b-m \\ \\ \end{array}\!\!\begin{array}{c} \left[\begin{array}{c:c:c} I_b & 0 & 0 \\ \hdashline \hat{N}_{211} & \hat{N}_{221} & \hat{N}_{222} \\ \hdashline \hat{N}_{212} & \hat{N}_{223} & \hat{N}_{224} \end{array}\right] \\ b \quad\; m \quad\; n-b-m \end{array} \qquad (67)$$

Replacing $\hat{M}$ and $\hat{N}$ obtained in (65) and (67) in the third equation $\hat{M}\bar{F}_1\hat{N} = \bar{F}_1$,we obtain

$$\left[\begin{array}{c:c:c} I_a & \hat{M}_{121} & \hat{M}_{122} \\ \hdashline 0 & \hat{M}_{221} & \hat{M}_{222} \\ \hdashline 0 & \hat{M}_{223} & \hat{M}_{224} \end{array}\right] \left[\begin{array}{c:c:c} \bar{F}_{11} & 0 & \bar{F}_{122} \\ \hdashline 0 & I_m & 0 \\ \hdashline \bar{F}_{132} & 0 & 0 \end{array}\right] \left[\begin{array}{c:c:c} I_b & 0 & 0 \\ \hdashline \hat{N}_{211} & \hat{N}_{221} & \hat{N}_{222} \\ \hdashline \hat{N}_{212} & \hat{N}_{223} & \hat{N}_{224} \end{array}\right] = \left[\begin{array}{c:c:c} \bar{F}_{11} & 0 & \bar{F}_{122} \\ \hdashline 0 & I_m & 0 \\ \hdashline \bar{F}_{132} & 0 & 0 \end{array}\right] \qquad (68)$$

* Block multiplications in equation (68) give:

$$\bar{F}_{11} + \hat{M}_{122}\bar{F}_{132} + \hat{M}_{121}\hat{N}_{211} + \bar{F}_{122}\hat{N}_{212} = \bar{F}_{11} \qquad (69)$$

$$\hat{M}_{121}\hat{N}_{221} + \bar{F}_{122}\hat{N}_{223} = 0 \qquad (70)$$

$$\hat{M}_{121}\hat{N}_{222} + \bar{F}_{122}\hat{N}_{224} = \bar{F}_{122} \qquad (71)$$

$$\hat{M}_{222}\bar{F}_{132} + \hat{M}_{221}\hat{N}_{211} = 0 \qquad (72)$$

$$\hat{M}_{221}\hat{N}_{221} = I_m \qquad (73)$$

$$\hat{M}_{221}\hat{N}_{222} = 0 \qquad (74)$$

$$\hat{M}_{224}\bar{F}_{132} + \hat{M}_{223}\hat{N}_{211} = \bar{F}_{132} \qquad (75)$$

$$\hat{M}_{223}\hat{N}_{221} = 0 \qquad (76)$$

$$\hat{M}_{223}\hat{N}_{222} = 0 \qquad (77)$$

Then the following 3 cases covers all possible structures that $\overline{F}_1$ matrix can take.

CASE A : $m = n - a = n - b$

We have immediately $\overline{F}_{122}=0$; $\overline{F}_{132}=0$ in $\overline{F}_1$. Then $\hat{M}$, $\hat{N}$ to be nonsingular a choice

$$\hat{M}_{22} = \hat{N}_{22}^{-1} \neq I \qquad \text{with } \hat{M}_{12}=0 \qquad \text{and} \qquad \hat{M}_{21}=0$$

always exists. Then computing M and N matrices by inverse transformations (T, P, etc.) we end up with a nontrivial (M,N) couples, since ($\hat{M}$, $\hat{N}$) couple is nontrivial.

CASE B : $a \neq b$, $m \neq 0$

In this we have the following implications:

$$\text{Eq.}(73) \Rightarrow \hat{N}_{221} = \hat{M}_{221}^{-1}$$
$$\text{Eq.}(74) \Rightarrow \hat{M}_{223} = 0$$
$$\text{Eq.}(76) \Rightarrow \hat{N}_{222} = 0$$

Then $\hat{M}$, $\hat{N}$ are nonsingular if and only if $\hat{M}_{224}$ and $\hat{N}_{224}$ are nonsingular. Now

- (73),(74),(76) determine $\hat{N}_{221}$, $\hat{M}_{223}$, $\hat{M}_{221}$, $\hat{N}_{222}$, and (77) holds immediately since (74),(76).

- Choosing $\hat{N}_{224}=I$, $\hat{M}_{224}=I$ we see that (71),(75) are trivially satisfied.

- Choosing $\hat{M}_{122}=0$, $\hat{M}_{121}=0$, $\hat{N}_{212}=0$ (69) is trivially satisfied.

- Choosing $\hat{M}_{222}=0$, $\hat{N}_{211}=0$ (72) is satisfied trivially

- Choosing $\hat{N}_{223}=0$ (70) is satisfied trivially.

However even after these choices we have

$$\hat{N} = \begin{bmatrix} I_b & O & O \\ O & \hat{N}_{221} & O \\ O & O & I \end{bmatrix} \quad \text{and} \quad \hat{M} = \begin{bmatrix} I_a & O & O \\ O & \hat{N}_{221}^{-1} & O \\ O & O & I \end{bmatrix}$$

which shows that $(\hat{M}, \hat{N})$ is nontrivial. Hence for $a \neq b$ and $m \neq \theta$ there always exists a nontrivial solutions for $(\hat{M}, \hat{N})$ couple.

CASE C : $m = \theta$

If $m=0$, then $\bar{F}_{14} = 0$. Then the resulting equation is

$$\begin{bmatrix} I_a & \hat{M}_{12} \\ 0 & \hat{M}_{22} \end{bmatrix} \begin{array}{c} a \\ n-a \end{array} \begin{array}{c} \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{13} & 0 \end{bmatrix} \\ \begin{array}{cc} b & n-b \end{array} \end{array} \begin{bmatrix} I_b & 0 \\ \hat{N}_{21} & \hat{N}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{13} & 0 \end{bmatrix} \tag{78}$$

and block multiplication gives

$$\tilde{F}_{11} + \hat{M}_{12}\tilde{F}_{13} + \tilde{F}_{12}\hat{N}_{21} = \tilde{F}_{11} \tag{79}$$

$$\tilde{F}_{12}\hat{N}_{22} = \tilde{F}_{12} \tag{80}$$

$$\hat{M}_{22}\tilde{F}_{13} = \tilde{F}_{13} \tag{81}$$

The block triangular form of $\hat{M}$ and $\hat{N}$ imposes $\hat{M}_{22}$ and $\hat{N}_{22}$ to be nonsingular. Now

- Let $Ker\,\tilde{F}_{12} \neq \{0\}$ and $Ker\,\tilde{F}_{13}^T \neq \{0\}$, then by LEMMA 3.12 the solution of the equation (80) and (81) are nontrivial and nonsingular. Choosing $\hat{M}_{12} = 0$ and $\hat{N}_{21} = 0$ (79) is trivially satisfied. Then since $\hat{M}_{22}$ and $\hat{N}_{22}$ are nontrivial there exists always a nontrivial solution of $(\hat{M}, \hat{N})$ couple.

- Let $Ker\,\tilde{F}_{12} = \{0\}$ and $Ker\,\tilde{F}_{13}^T = \{0\}$, then we have rank $\tilde{F}_{12} = n-b$ and rank $\tilde{F}_{13} = n-a$. By LEMMA 3.12 $\hat{N}_{22} = I$ and $\hat{M}_{22} = I$. Hence the solution of $(\hat{M}, \hat{N})$ couple is nontrivial if and only if (79) is nontrivially satisfied, that is

$$\hat{M}_{12}\tilde{F}_{13} + \tilde{F}_{12}\hat{N}_{21} = 0$$

for $\hat{M}_{12} \neq 0$ and $\hat{N}_{21} \neq 0$. For this by nonsingular transformations we bring $\tilde{F}_{13}$ and $\tilde{F}_{12}$ to their Smith forms

$$\hat{M}_{12} T_4 \left[I_{n-a} \vdots 0 \right] T_5 = - P_4 \begin{bmatrix} I_{n-b} \\ \hdashline 0 \end{bmatrix} P_5 \hat{N}_{21} \tag{82}$$

or

$$P_4^{-1} \hat{M}_{12} T_4 \left[I_{n-a} \vdots 0 \right] = - \begin{bmatrix} I_{n-b} \\ \hline 0 \end{bmatrix} P_5 \hat{N}_{21} T_5^{-1} \tag{83}$$

Therefore

$$\tilde{M}_{12} \left[I_{n-a} \vdots 0 \right] = - \begin{bmatrix} I_{n-b} \\ \cdots \\ 0 \end{bmatrix} \tilde{N}_{21} \tag{84}$$

where $\tilde{M}_{12} = P_4^{-1} \hat{M}_{12} T_4$ and $\tilde{N}_{21} = P_5 \hat{N}_{21} T_5^{-1}$.

Equation (84) implies

$$\left[\tilde{M}_{12} \vdots 0 \right] = - \begin{bmatrix} \tilde{N}_{21} \\ \cdots \\ 0 \end{bmatrix} \tag{85}$$

By partitioning (85) gives

$$\begin{array}{c} n-b \\ a+b-n \end{array} \begin{bmatrix} \tilde{M}_{121} & \vdots & 0 \\ \cdots & & \cdots \\ \tilde{M}_{122} & \vdots & 0 \end{bmatrix} = - \begin{bmatrix} \tilde{N}_{211} & \vdots & \tilde{N}_{212} \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \begin{array}{c} n-b \\ a+b-n \end{array} \tag{86}$$
$$\qquad\quad n-a \quad a+b-n \qquad\qquad\qquad n-a \quad a+b-n$$

The equation (86) implies that $\tilde{M}_{121} = -\tilde{N}_{211}$ and $\tilde{M}_{122} = 0$, $\tilde{N}_{212} = 0$. Since $a < n$, $b < n$; $\tilde{M}_{121}$ and $\hat{N}_{211}$ are never zero. So by inverse transformations nonzero $\hat{M}_{12}, \hat{N}_{21}$ can be calculated.

Therfore when $Ker\, \tilde{F}_{12} = \{0\}$ and $Ker\, \tilde{F}_{13}^{T} = \{0\}$ nontrivial solutions always exist. To complete, when $m=0$ a nontrivial solution for (M , N) couple can always be found.

Finally we have considered all possible cases in which a nontrivial solution of the matrix equation (55) always exists if and only if rank $F_2 < n$ and rank $C < n$. This completes the proof. ■

We will close this chapter by the following result which is immediate after the previous theorems.

THEOREM 3.13

D_α giving rise to controllable and observable systems Σ_α having same transferfunctionmatrix are not unique if and only if rank $F_2 < n$ and rank $C < n$. ■

EXAMPLE 3.13

Consider the system Σ_{α_0}

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\underline{x}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underline{x}$$

and note that rank $F_2 = 2 < 3$ and rank $C = 2 < 3$ .The system is completely controllable and completely observable.The transfer function matrix can be calculated easily and is equal to

$$T(s) = C\,(D_{\alpha_0} s - F_1)^{-1} F_2 = \begin{bmatrix} 1/s & 1/s \\ \frac{(s-1)^2}{s(s^2-3s+1)} & \frac{1-s}{s(s^2-3s+1)} \end{bmatrix}$$

Now we want to find all matrices D_{α_i} such that the new systems are controllable,observable and have the same transfer function matrix T(s).First we have to determine the (M,N)-couples satisfying equations of COROLLARY 3.10.c ,that is

$$M F_1 N = F_1 \tag{87}$$

$$M F_2 = F_2 \tag{88}$$

$$C N = C \quad (\text{or } CN^{-1} = C) \tag{89}$$

Then the matrices D_{α_i} will be obtained from

$$D_{\alpha_i} = M D_{\alpha_0} N. \tag{90}$$

The equation (89) gives:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \\ \hat{n}_4 & \hat{n}_5 & \hat{n}_6 \\ \hat{n}_7 & \hat{n}_8 & \hat{n}_9 \end{bmatrix} = \begin{bmatrix} \hat{n}_7 & \hat{n}_8 & \hat{n}_9 \\ \hat{n}_4 & \hat{n}_5 & \hat{n}_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Therfore

$$N^{-1} = \begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly equation (88) gives

$$\begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} m_2+m_3 & m_3 \\ m_5+m_6 & m_6 \\ m_8+m_9 & m_9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Therefore

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ m_4 & 1 & 0 \\ m_7 & 0 & 1 \end{bmatrix}$$

Substituting M and N in (87)

$$\begin{bmatrix} m_1 & 0 & 0 \\ m_4 & 1 & 0 \\ m_7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

equivalently.

$$\begin{bmatrix} m_1 & 0 & m_1 \\ m_4+1 & 1 & m_4 \\ m_7 & 0 & m_7 \end{bmatrix} = \begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \hat{n}_3+1 \\ \hat{n}_1 & \hat{n}_2+1 & \hat{n}_3 \\ 0 & 0 & 0 \end{bmatrix}$$

then we have

$$m_1 = \hat{n}_1$$
$$m_4 = \hat{n}_1 - 1$$
$$\hat{n}_3 = \hat{n}_1 - 1$$
$$\hat{n}_2 = 0$$
$$m_7 = 0$$

which gives for $\hat{n}_1 = 1/\delta \quad (\delta \neq 0)$

$$M = \begin{bmatrix} 1/\delta & 0 & 0 \\ \frac{1-\delta}{\delta} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad N^{-1} = \begin{bmatrix} 1/\delta & 0 & \frac{1-\delta}{\delta} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} \delta & 0 & \delta-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore

$$D_\alpha(\delta) = \begin{bmatrix} 1/\delta & 0 & 0 \\ \frac{1-\delta}{\delta} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta & 0 & \delta-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D_\alpha(\delta)\quad\begin{bmatrix} 1 & -1/\delta & 1 \\ 1-\delta & \frac{2\delta-1}{\delta} & 1-\delta \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{with } \delta\neq 0$$

We notice that $D_\alpha(1)=D_{\alpha_0}$ and the upper triangular form of the matrix D_{α_0} is not preserved for $\delta\neq 1$. Let us check if for $D_\alpha(\delta)$ in general, the transfer function matrix equals the given T(s). For this we first form

$$\left(D_\alpha(\delta)\,s - F_1\right) = \begin{bmatrix} s-1 & -\frac{1}{\delta}s & s-1 \\ (1-\delta)s-1 & \frac{2\delta-1}{\delta}s-1 & (1-\delta)s \\ 0 & 0 & s \end{bmatrix}$$

and

$$\left(D_\alpha(\delta)\,s - F_1\right)^{-1} = \frac{1}{s(s^2-3s+1)}\begin{bmatrix} \frac{2\delta-1}{\delta}s^2-s & s^2/\delta & -s^2+\frac{3\delta-1}{\delta}-1 \\ -(1-\delta)s^2+s & s(s-1) & 1-s \\ 0 & 0 & s^2-3s+1 \end{bmatrix}$$

finally

$$T'(s) = C\left(D_\alpha(\delta)s-F_1\right)^{-1}F_2 = \begin{bmatrix} 1/s & 1/s \\ \frac{(s-1)^2}{s(s^2-3s+1)} & \frac{(1-s)}{s(s^2-3s+1)} \end{bmatrix} = T(s) \qquad \forall\delta\neq 0\ .$$

IV SOLVABILITY OF DDP IN AN m-PORT SYSTEM WITH SEPERATED ALGEBRAIC AND LOSSLESS PARTS

Before considering the DDP for two different disturbance structures we need to characterize the largest $(D_\alpha^{-1}F_1, D_\alpha^{-1}F_2)$ invariant subspace contained in $\mathrm{Ker}\,C$.

LEMMA 4.1

Given the system Σ_α

$$D_\alpha \dot{\underline{x}} = F_1\underline{x} + F_2\underline{u}$$
$$\underline{y} = C\underline{x}$$

$x_0 \in \mathcal{V}_*(D_\alpha)$ if and only if there exist $\underline{\xi}(s)$ and $\underline{\omega}(s)$ strictly proper rational such that

$$\begin{bmatrix} D_\alpha s - F_1 & F_2 \\ -C & 0 \end{bmatrix} \begin{bmatrix} \underline{\xi}(s) \\ \underline{\omega}(s) \end{bmatrix} = \begin{bmatrix} D_\alpha \underline{x}_0 \\ 0 \end{bmatrix} \qquad (91)$$

where $\mathcal{V}_*(D_\alpha)$ denotes the largest $(D_\alpha^{-1}F_1, D_\alpha^{-1}F_2)$-invariant subspace contained in $\mathrm{Ker}\,C$. ∎

The proof can be achieved by inverting D_α and applying LEMMA 1.10.

Now let Σ_{α_0} be a controllable ano observable system as described before and let rank $F_2 < n$ and rank $C < n$. Then by COROLLARY 3.10 and THM 3.13 it is known that D_{α_i} matrices leaving the transfer matrix invariant under SSE transformations are not unique.

Now let a decomposition $(D_{\alpha_0}, F_1, F_2, C)$ be given and satisfy the above conditions. Then applying SSE

$$\begin{bmatrix} M_\alpha & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_{\alpha_0}s - F_1 & F_2 \\ C & 0 \end{bmatrix} \begin{bmatrix} N_\alpha & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} D_\alpha s - F_1 & F_2 \\ C & 0 \end{bmatrix}$$

(M_α , N_α)-couples and by $M_{\alpha i} D_{\alpha_0} N_{\alpha i} = D_{\alpha i}$ all $D_{\alpha i}$ matrices can be computed giving the following lemma.

LEMMA 4.2

Let $\mathcal{V}_*(D_{\alpha_0})$ be the largest ($D_{\alpha_0}^{-1}F_1, D_{\alpha_0}^{-1}F_2$)-invariant subspace contained in $\operatorname{Ker} C$ of Σ_{α_0}, then

$$N_{\alpha i}^{-1}\mathcal{V}_*(D_{\alpha_0}) = \mathcal{V}_*(D_{\alpha i}) \tag{92}$$

where $N_{\alpha i}$ is such that $D_{\alpha i} = M_{\alpha i} D_{\alpha_0} N_{\alpha i}$.

Proof : Let $\underline{x}_0 \in \mathcal{V}_*(D_{\alpha i})$,by LEMMA 4.1 we have

$$\begin{bmatrix} D_{\alpha i}s-F_1 & F_2 \\ C & 0 \end{bmatrix}\begin{bmatrix} \underline{\xi}(s) \\ -\underline{w}(s) \end{bmatrix} = \begin{bmatrix} D_{\alpha i}\underline{x}_0 \\ 0 \end{bmatrix}$$

where $\underline{\xi}(s), \underline{w}(s)$ are strictly proper. Using SSE transformation we obtain

$$\begin{bmatrix} M_{\alpha i} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} D_{\alpha_0}s-F_1 & F_2 \\ C & 0 \end{bmatrix}\begin{bmatrix} N_{\alpha i} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \underline{\xi}(s) \\ -\underline{w}(s) \end{bmatrix} = \begin{bmatrix} M_{\alpha i}D_{\alpha_0}N_{\alpha i}\underline{x}_0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} D_{\alpha_0}s-F_1 & F_2 \\ C & 0 \end{bmatrix}\begin{bmatrix} N_{\alpha i}\underline{\xi}(s) \\ -\underline{w}(s) \end{bmatrix} = \begin{bmatrix} D_{\alpha_0}N_{\alpha i}\underline{x}_0 \\ 0 \end{bmatrix} \tag{93}$$

Since $N_{\alpha i}\underline{\xi}(s)$ is strictly proper we have by LEMMA 4.1

$$N_{\alpha i}\underline{x}_0 \in \mathcal{V}_*(D_{\alpha_0}) \Rightarrow \underline{x}_0 \in N_{\alpha i}^{-1}\mathcal{V}_*(D_{\alpha_0})$$

equivalently

$$\mathcal{V}_*(D_{\alpha i}) \subset N_{\alpha i}^{-1}\mathcal{V}_*(D_{\alpha_0}) \tag{94}$$

Similarly let $\underline{x}_0 \in N_{\alpha i}^{-1}\mathcal{V}_*(D_{\alpha_0})$,we have by nonsingularity of $N_{\alpha i}$, $N_{\alpha i}\underline{x}_0 \in \mathcal{V}_*(D_{\alpha_0})$ .Using the above procedure we obtain

$$\underline{x}_0 \in \mathcal{V}_*(D_{\alpha i}) \Rightarrow N_{\alpha i}^{-1}\mathcal{V}_*(D_{\alpha_0}) \subseteq \mathcal{V}_*(D_{\alpha i}) \tag{95}$$

Hence (94) and (95) imply:

$$\mathcal{V}_*(D_{\alpha_i}) = N_{\alpha_i}^{-1}\,\mathcal{V}_*(D_{\alpha_0}).$$

Now we will consider two different disturbance component structures for which we will formulate the solvability of DDP.

CASE I :

Consider the system

$$D_{\alpha_0}\dot{\underline{x}} = F_1\underline{x} + F_2\underline{u} + E_{\alpha_0}\underline{q} \qquad (96)$$

$$\underline{y} = C\underline{x} \qquad (97)$$

where $E_{\alpha_0}\underline{q}$ is the disturbance component. We assume that the application of SSE transformations also changes the matrix E_{α_0}. Hence while the input/output transfer function remains the same, disturbance/output transfer function may change.

More precisely

$$\boxed{\begin{array}{l} M_{\alpha_i}D_{\alpha_0}N_{\alpha_i}\dot{\underline{x}} = M_{\alpha_i}F_1N_{\alpha_i}\underline{x} + M_{\alpha_i}F_2\underline{u} + M_{\alpha_i}E_{\alpha_0}\underline{q} \\ \underline{y} = CN_{\alpha_i}\underline{x} \end{array}} \Longrightarrow \boxed{\begin{array}{l} D_{\alpha_i}\dot{\underline{x}} = F_1\underline{x} + F_2\underline{u} + E_{\alpha_i}\underline{q} \\ \underline{y} \doteq C\underline{x} \end{array}}$$

where $E_{\alpha_i} = M_{\alpha_i}E_{\alpha_0}$. The system Σ_α is shown in Figure 4.1.

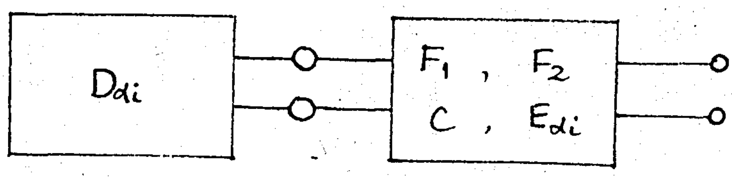


FIGURE 4.1: STRUCTURE OF CASE I

Now in general, by THM 2.1 we have the following lemma for the solvability of DDP in CASE I.

LEMMA 4.3

DDP is solvable if and only if

$$D_{\alpha_i}^{-1}\,\mathcal{E}_{\alpha_i} \subseteq \mathcal{V}_*(D_{\alpha_i}).$$

This trivial result can be completed by

LEMMA 4.4

$$D_{\alpha_0}^{-1}\mathcal{E}_{\alpha_0} \subseteq \mathcal{V}_*(D_{\alpha_0}) \text{ if and only if } D_{\alpha_i}^{-1}\mathcal{E}_{\alpha_i} \subseteq \mathcal{V}_*(D_{\alpha_i}).$$

Proof :

Let $D_{\alpha_i}^{-1}\,\mathcal{E}_{\alpha_i} \subseteq \mathcal{V}_*(D_{\alpha_i}) = N_{\alpha_i}^{-1}\mathcal{V}_*(D_{\alpha_0})$ by LEMMA 4.2.

Since $E_{\alpha_i} = M_{\alpha_i}E_{\alpha_0}$

$$D_{\alpha_i}^{-1}\,\mathcal{E}_{\alpha_i} \subseteq N_{\alpha_i}^{-1}\,\mathcal{V}_*(D_{\alpha_0}) \Leftrightarrow D_{\alpha_i}^{-1}M_{\alpha_i}\mathcal{E}_{\alpha_0} \subseteq N_{\alpha_i}^{-1}\,\mathcal{V}_*(D_{\alpha_0}).$$

Since $D_{\alpha_i}, M_{\alpha_i}$ are nonsingular

$$D_{\alpha_i}^{-1}M_{\alpha_i}\,\mathcal{E}_{\alpha_0} \subseteq N_{\alpha_i}^{-1}\,\mathcal{V}_*(D_{\alpha_0}) \Leftrightarrow N_{\alpha_i}^{-1}D_{\alpha_0}^{-1}M_{\alpha_i}^{-1}M_{\alpha_i}\,\mathcal{E}_{\alpha_0} \subseteq N_{\alpha_i}^{-1}\mathcal{V}_*(D_{\alpha_0})$$

$$\Leftrightarrow D_{\alpha_0}^{-1}\,\mathcal{E}_{\alpha_0} \subseteq \mathcal{V}_*(D_{\alpha_0}).$$

Hence by LEMMA 4.4 we show, assuming the structure of CASE I, that we bring no improvement in the solvability range of DDP. In this case DDP is either solvable in all systems Σ_α obtained by SSE or not solvable at all.

CASE II :

Now let the system be described by

$$D_{\alpha_i}\dot{\underline{x}} = F_1\underline{x} + F_2\underline{u} + E\underline{q} \tag{98}$$

$$\underline{y} = C\underline{x} \tag{99}$$

We assume that we want to decouple a noise component $E\underline{q}$ without changing input/output transfer matrix and without violating the observability and the controllability, by properly selecting D_{α_i}. In other words the general description for the SSE systems is shown in Fig.4.2.

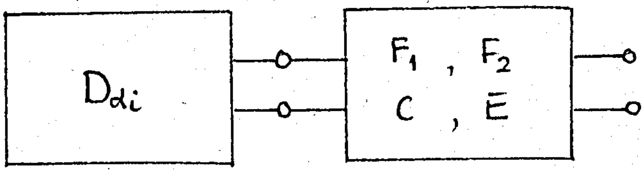


FIGURE 4.2 : STRUCTURE OF CASE II

The following lemmas clearify the range of the solutions for DDP
n the CASE II.

EMMA 4.5

Given the systems Σ_α of CASE II, DDP is solvable for both Σ_{α_o} and Σ_{α_i} if and only if $\mathcal{E} \subseteq \mathcal{V}_{M_{\alpha_i}}$, where $\mathcal{V}_{M_{\alpha_i}}$ is the M_{α_i}-invariant subspace of $D_{\alpha_o} \mathcal{V}_*(D_{\alpha_o})$.

roof : "$\Rightarrow$"

DDP is solvable for D_{α_i} if and only if $D_{\alpha_i}^{-1} \mathcal{E} \subseteq \mathcal{V}_*(D_{\alpha_i})$
r $\mathcal{E} \subseteq D_{\alpha_i} \mathcal{V}_*(D_{\alpha_i})$. By LEMMA 4.2

$$\mathcal{V}_*(D_{\alpha_i}) = N_{\alpha_i}^{-1} \mathcal{V}_*(D_{\alpha_o})$$

nd

$$\mathcal{E} \subseteq M_{\alpha_i} D_{\alpha_o} N_{\alpha_i} N_{\alpha_i}^{-1} \mathcal{V}_*(D_{\alpha_o}) = M_{\alpha_i} D_{\alpha_o} \mathcal{V}_*(D_{\alpha_o})$$

ince DDP is solvable for D_{α_o}

$$\mathcal{E} \subseteq D_{\alpha_o} \mathcal{V}_*(D_{\alpha_o}).$$

ence DDP is solvable simultaneously for D_{α_o} and D_{α_i} implies

$$\mathcal{E} \subseteq M_{\alpha_i} D_{\alpha_o} \mathcal{V}_*(D_{\alpha_o}) \cap D_{\alpha_o} \mathcal{V}_*(D_{\alpha_o}) \triangleq \mathcal{V}_{M_{\alpha_i}} ,$$

here $\mathcal{V}_{M_{\alpha_i}}$ is necessarily the M_{α_i}-invariant subspace of $D_{\alpha_o} \mathcal{V}_*(D_{\alpha_o})$.

"$\Leftarrow$"

$$\mathcal{E} \subseteq \mathcal{V}_{M_{\alpha_i}} \Rightarrow \begin{cases} \mathcal{E} \subseteq M_{\alpha_i} D_{\alpha_o} \mathcal{V}_*(D_{\alpha_o}) \Rightarrow & \mathcal{E} \subseteq D_{\alpha_i} \mathcal{V}_*(D_{\alpha_i}) \\ \mathcal{E} \subseteq D_{\alpha_o} \mathcal{V}_*(D_{\alpha_o}) \Rightarrow & \mathcal{E} \subseteq D_{\alpha_o} \mathcal{V}_*(D_{\alpha_o}). \end{cases} \blacksquare$$

Thus by LEMMA 4.5 we understand that, given $(D_{\alpha_0}, F_1, F_2, E, C)$ for which DDP is solvable, DDP will also be solvable for all $(D_{\alpha_i}, F_1, F_2, E, C)$ obtained using $(M_{\alpha_i}, N_{\alpha_i})$ couples for which $\mathcal{E}$ is M_{α_i}-invariant.

LEMMA 4.6

Given Σ_{α_0}, let DDP be unsolvable for Σ_{α_0}. Then DDP is not solvable for Σ_{α_i} obtained using $(M_{\alpha_i}, N_{\alpha_i})$-couples for which the subspace $D_{\alpha_0}\mathcal{V}_*(D_{\alpha_0})$ is M_{α_i}-invariant. ∎

The proof of LEMMA 4.6 is straight forward since

$$\mathcal{E} \not\subset D_{\alpha_0}\mathcal{V}_*(D_{\alpha_0}) = M_{\alpha_i} D_{\alpha_0}\mathcal{V}_*(D_{\alpha_0}) = D_{\alpha_i}\mathcal{V}_*(D_{\alpha_i}).$$

Hence by LEMMA 4.6 we see that we may have an improvement for the solvability of DDP. If for the initial decomposition $(D_{\alpha_0}, F_1, F_2, E, C)$ DDP is not solvable, we will find out all $(M_{\alpha_i}, N_{\alpha_i})$ couples for which $D_{\alpha_0}\mathcal{V}_*(D_{\alpha_0})$ is M_{α_i}-invariant and exclude them. The condition for the solvability of DDP can be satisfied in systems Σ_{α_i} obtained using the remaining $(M_{\alpha_i}, N_{\alpha_i})$-couples.

In fact it will be wise to determine first $(M_{\alpha_i}, N_{\alpha_i})$-couples parametrically and check if the condition $\mathcal{E} \subseteq M_{\alpha_i} D_{\alpha_0}\mathcal{V}_*(D_{\alpha_0})$ is satisfied for any choice of free parameters. We will close this chapter by the following example illustrating the above results.

EXAMPLE 4.1 :

Consider the controllable and observable system Σ_{α_0} described by the equations:

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dot{\underline{x}} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{u} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \underline{q}$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underline{x}$$

where $\underline{x}$ is the state vector, $\underline{u}$ is the input vector, $\underline{q}$ is the disturbance vector and $\underline{y}$ is the output vector. We will first try to find out the constraints in the structure of E matrix for DDP to be solvable in Σ_{α_0}. According to LEMMA 4.1 DDP is solvable if and only if there exists strictly proper $\underline{X}(s)$ and $\underline{U}(s)$ such that

$$\left[\begin{array}{cccc:cc} s-1 & -s & s & -2s-1 & 0 & 0 \\ 0 & s-1 & -s & s & 1 & 0 \\ 0 & 0 & s & -s & 0 & 1 \\ 0 & 0 & -1 & s & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1(s) \\ x_2(s) \\ x_3(s) \\ x_4(s) \\ \hdashline u_1(s) \\ u_2(s) \end{array}\right] = \left[\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ \hdashline 0 \\ 0 \end{array}\right]$$

explicitly we obtain tha following equations

$$(s-1)x_1(s) - sx_2(s) + sx_3(s) - (2s+1)x_4(s) = e_1 \qquad (100)$$

$$(s-1)x_2(s) - sx_3(s) + sx_4(s) + u_1(s) = e_2 \qquad (101)$$

$$sx_3(s) - sx_4(s) + u_2(s) = e_3 \qquad (102)$$

$$-x_3(s) + sx_4(s) = e_4 \qquad (103)$$

$$x_1(s) = 0 \qquad (104)$$

$$x_4(s) = 0 \qquad (105)$$

Then (104),(105) implies that

$$\text{Eq.}(100) \Leftrightarrow -sx_2(s) + sx_3(s) = e_1 \qquad (106)$$

$$\text{Eq.}(101) \Leftrightarrow (s-1)x_2(s) - sx_3(s) + u_1(s) = e_2 \qquad (107)$$

$$\text{Eq.}(102) \Leftrightarrow sx_3(s) + u_2(s) = e_3 \qquad (108)$$

$$\text{Eq.}(103) \Leftrightarrow -x_3(s) = e_4 \qquad (109)$$

Now for DDP to be solvable

$Eq.(109) \Rightarrow -x_3(s) = e_4 = 0$ since $x_3(s)$ is required to be strictly proper,

$Eq.(108) \Rightarrow u_2(s) = e_3 = 0$ since $u_2(s)$ must be strictly proper

and then

$Eq.(106) \Rightarrow x_2(s) = -\frac{e_1}{s}$ which is strictly proper,

finally

$Eq.(107) \Rightarrow u_1(s) = \frac{se_2 + se_1 - e_1}{s}$ which is strictly proper only for $e_2 = -e_1$.

Hence DDP is solvable if and only if $E = \begin{bmatrix} e_1 \\ -e_1 \\ 0 \\ 0 \end{bmatrix}$, $e_1 \in \mathbb{R}$.

In fact one can immediately check that

$\mathrm{Im}\, E = \mathrm{span}[1\ -1\ \ 0\ \ 0]^T = D_{\alpha_0} \mathcal{V}_*(D_{\alpha_0})$ .To be more explicit , we can construct the subspace $\mathcal{V}_*(D_{\alpha_0})$ by using the Flowchart 1.8.We define first

$$A \triangleq D_{\alpha_0}^{-1} F_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \qquad B \triangleq D_{\alpha_0}^{-1} F_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then we obtain

$$\mathrm{Ker}\, C = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Setting

$$\mathcal{V}^0 \triangleq \mathrm{Ker}\, C$$

from $$\mathcal{V}^i = \mathcal{V}^{i-1} \cap A^{-1}(\mathcal{V}^{i-1} + \mathcal{B})$$ we obtain iteration

$$\mathcal{V}^1 = \mathcal{V}^0 \cap A^{-1}(\mathcal{V}^0 + \mathcal{B})$$ where

$$= \text{span}\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\} \quad \text{and} \quad \mathcal{V}^0 + \mathcal{B} = \text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}$$

nce A^{-1} denotes the "functional inverse of map A", by forming

$$A\underline{x} = \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1+x_2+x_3+x_4\\x_2\\x_3\\x_3\end{bmatrix}$$

$$A\underline{x} \in \mathcal{V}^0 + \mathcal{B} \quad \text{if and only if} \quad x_3 = 0$$

en

$$(\mathcal{V}^0 + \mathcal{B}) = \text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} \quad \text{and} \quad \mathcal{V}^0 \cap A^{-1}(\mathcal{V}^0 + \mathcal{B}) = \text{span}\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\} = \mathcal{V}^1$$

iteration

$$\mathcal{V}^2 = \mathcal{V}^1 \cap A^{-1}(\mathcal{V}^1 + \mathcal{B}) = \text{span}\{[0 \ 1 \ 0 \ 0]^T\} = \mathcal{V}^1 = \mathcal{V}_*(D_{\alpha_0}).$$

nce $\mathcal{V}_*(D_{\alpha_0}) = \text{span}\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\}$ and according to the

M 2.1, DDP is solvable for Σ_{α_0} if and only if $\mathcal{E} \subseteq D_{\alpha_0}\mathcal{V}_*(D_{\alpha_0})$, i.e.

$$\mathcal{E} \subseteq \text{span}\{[-1 \ 1 \ 0 \ 0]^T\}.$$

Now we will try to out if we have any important improvement to enlarge the subspace $D_{\alpha_0}\mathcal{V}_*(D_{\alpha_0})$,using the procedure defined in CASE II.

Let

$$M_{\alpha_i} \triangleq \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ m_5 & m_6 & m_7 & m_8 \\ m_9 & m_{10} & m_{11} & m_{12} \\ m_{13} & m_{14} & m_{15} & m_{16} \end{bmatrix} \qquad N_{\alpha_i} \triangleq \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \\ n_5 & n_6 & n_7 & n_8 \\ n_9 & n_{10} & n_{11} & n_{12} \\ n_{13} & n_{14} & n_{15} & n_{16} \end{bmatrix}$$

Then according to COROLLARY 3.10.c,we determine the free parameters of M_{α_i} and N_{α_i} using the equations

$$M_{\alpha_i} F_2 = F_2 \tag{110}$$

$$C\, N_{\alpha_i}^{-1} = C \tag{111}$$

$$M_{\alpha_i} F_1 = F_1 N_{\alpha_i}^{-1} \tag{112}$$

where

$$\text{Eq}(110) \Rightarrow M_{\alpha_i} = \begin{bmatrix} m_1 & 0 & 0 & m_4 \\ m_5 & 1 & 0 & m_8 \\ m_9 & 0 & 1 & m_{12} \\ m_{13} & 0 & 0 & m_{16} \end{bmatrix} \text{ and } \text{Eq}(111) \Rightarrow N_{\alpha_i}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \hat{n}_5 & \hat{n}_6 & \hat{n}_7 & \hat{n}_8 \\ \hat{n}_9 & \hat{n}_{10} & \hat{n}_{11} & \hat{n}_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Substituting them in (112) we obtain

$$M_{\alpha i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{n_7 n_{12} - n_5 n_{11}}{n_{11}} & 1 & 0 & -\frac{n_7}{n_{11}} \\ 0 & 0 & 1 & 0 \\ -\frac{n_{12}}{n_{11}} & 0 & 0 & \frac{1}{n_{11}} \end{bmatrix} \qquad N_{\alpha i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ n_5 & 1 & n_7 & n_5 \\ n_{12} & 0 & n_{11} & n_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $n_5, n_7, n_{12} \in \mathbb{R}$ and $n_{11} \in \mathbb{R} - \{0\}$

And in general with

$D_{\alpha i} = M_{\alpha i} D_{\alpha_0} N_{\alpha i}$,keeping F_1, F_2, C fixed we obtain the same transfer function matrix. To judge whether there is an improvement to solve DDP we check

$$D_{\alpha i} \mathcal{V}_*(D_{\alpha i}) = M_{\alpha i} D_{\alpha_0} \mathcal{V}_*(D_{\alpha_0})$$

$$= \operatorname{span}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{n_7 . n_{12} - n_5 . n_{11}}{n_{11}} & 1 & 0 & -\frac{n_7}{n_{11}} \\ 0 & 0 & 1 & 0 \\ -\frac{n_{12}}{n_{11}} & 0 & 0 & \frac{1}{n_{11}} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} -1 \\ \frac{n_7 n_{12} - n_5 n_{11}}{n_{11}} + 1 \\ 0 \\ \frac{n_{12}}{n_{11}} \end{bmatrix}\right\}$$

Now $D_{\alpha i} \mathcal{V}_*(D_{\alpha i}) \subseteq D_{\alpha_0} \mathcal{V}_*(D_{\alpha_0})$ if and only if $n_{12} = 0$ and $n_5 = 0$ .Other wise we obtain

$$D_{\alpha i} \mathcal{V}_*(D_{\alpha i}) = \operatorname{span}\left\{\begin{bmatrix} -1 \\ \alpha \\ 0 \\ \beta \end{bmatrix}\right\}$$

with $(\alpha,\beta) \in \mathbb{R} - \{0\}$. Equivalently

$$\mathcal{E} \subseteq \text{span}\left\{\begin{bmatrix}-1\\ \alpha\\ 0\\ \beta\end{bmatrix}\right\}$$

and the structure of E matrix for DDP to be solvable is

$$E = \begin{bmatrix}-q\\ \alpha q\\ 0\\ \beta q\end{bmatrix} = \begin{bmatrix}e_1\\ e_2\\ 0\\ e_4\end{bmatrix} \qquad \text{with } e_1, e_2, e_4 \in \mathbb{R}\text{, arbitrary.}$$

Hence choosing suitable $(M_{\alpha_i}, N_{\alpha_i})$-couples we can solve DDP for $E = [e_1\ e_2\ 0\ e_4]^T$ by simply changing D_{α_i} matrices and keeping the the transfer function matrix unchanged.

Thus the subspace that $\mathcal{E}$ must be included is determined by the product of $D_{\alpha_0} \mathcal{V}_*(D_{\alpha_0})$ and the degree of freedom obtained in M_{α_i} in general. ∎

V. CONCLUSION

In this work, it is presented a report which consists of two seperated parts: a survey and an investigation. While completing, it will be usefull to give some extra notes to emphasize the importance of various concepts encountered.

The basic reference for the "Geometric Approach" is [1]. However it is full of sophisticated mathematics and concepts which make the study for a beginner, difficult. So, the aim of the survey is to give the essential idea introduced by this new approach. For this, basic concepts (such as A-invariance, (A,B)-invariance, Stabilizability) and basic problems (such as DDP, SDDP, OSDP, ONC) are studied as simply as possible. Thinking of the fact that most of the readers are familiar with the frequency domain approach, frequency domain treatment of these basic concepts and problems is also studied as introduced in [6,7,8,9]. So, a survey of the modern control problems chosen is obtained, complete with their solutions the <u>geometric framework</u> and in the <u>frequency domain</u>.

The "geometric approach" will tend to be the "exponent" of the "Modern Control Theory", in the future. In this new framework, we believe that it will be possible to easily investigate the properties of the solvability criteria of new control problems and the other relations existing in the diverging methods of the Control Theory.

The second part which is an investigation is believed to be new. The problem is essentially based on the work introduced in [10]. The considered linear m-part has a special configuration which is obtained using a first level decomposition,and the obtained state equations are not in state-space form.Hence the essential idea is to investigate if the "extra parameter:matrix D " brings any degree of freedom.It has to be mentioned here that in this study it is only considered completely contollable and observable systems of the above structure,in which state-space form of the state equations can be obtained by simply inverting D matrix.In Chapter III,the mathematical tools used are presented in full detail,and an important property of such systems is presented by THM 3.11.This result is totally new.Based on this property of "the existence of D matrices leaving transfer function invariant while F_1 ,F_2 ,C kept unchanged" another question is asked and answered in Chapter IV : To use D matrices for decoupling the disturbance at the output.The result obtained is that in the disturbance structure of CASE II the dimension of $\mathcal{V}_*$,the maximal (A,B)-invariant subspace in Ker C,does not increased,however a larger degree of freedom may be obtained for some D . As seen in the example 4.1 a $\mathcal{V}_*$ which is initially equal to span$\{[-1\ 1\ 0\ 0]^T\}$ can be mapped to span $\{[1\ \alpha\ 0\ \beta]^T\}$ with $\alpha,\beta \in \mathbb{R}-\{0\}$arbitrary for a suitable choice of D ;furthermore we also know that this choice D does not change the transfer function matrix of the system.

As we stated in the context,it can be expected that it finds an efficient area of application in the electrical an electronic circuitry designes.The analysis related to the solvability of DDP may be extended,to more generic decoupling problems.For further investigations the determination of the boundaries of the equivalence class of (M,N) matrix couples using the information brought by the stucture of F_1 ,F_2 ,C matrices,can be suggested.

In general, the study includes an introductive survey, concerning both the geometric and frequency domain treatment of some new concepts; and the second part may be considered as a partial application of the methods that make part of the survey.

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