POLE ASSIGNMENT IN LINEAR MULTIVARIABLE SYSTEMS

by

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ABSTRACT

In this thesis,pole assignment problem,one of the most commonly used control schemes,is considered and special emphasis is given on the application of avaliable pole assignment algorithms in multi-input systems.

A very convenient method,proven first by Ackermann,for determining the required feedback gains for arbitrary pole assignment in single-input systems, is generalized to include the multivariable systems as well,using a unity rank feedback matrix.The multivariable system is first converted into an "equivalent" single-input system and Ackermann's procedure is subsequently applied.

Furthermore the relationship between various model following control schemes and pole assignment problem is discussed in detail.

ÖZETÇE

Bu tez çalışmasında, en sık kullanılan denetim yöntemlerinden biri olan geri besleme etkisi altında kutup yerleştirilmesi yöntemi incelenmiştir. Çalışmanın büyük bir kısmı kutup yerleştirilmesi yönteminin çok girdili sistemlere uygulanmasına ayrılmıştır.

Şimdiye kadar yalnız tek girdili sistemlerde uygulanabilen Ackermann yöntemi, genelleştirilerek çok girdili sistemlerde de uygulanabilecek duruma getirilmiştir. Denetlenecek sistem önce tek girdili eşdeğer bir sistem haline dönüştürülmüş ve daha sonra Ackermann yöntemi uygulanarak sistemin kutuplarının karmaşık düzlemde istenilen noktalara yerleştirilmesi gerçekleştirilmiştir.

Ayrıca kutup yerleştirme ve model izleme yöntemleri arasındaki benzerlik ve uyumsuzluklar ayrıntıları ile ortaya konmuştur.

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INTRODUCTION

It is a well known fact that the free response of an uncontrolled linear plant is given by a linear combination of the dynamical modes of the system,where the mode shapes are determined by the eigenvectors and the time-domain characteristics by the eigenvalues (poles) of the plant. Therefore most of the commonly used control procedures are based on altering the closed loop pole locations so that a satisfactory system performance is obtained.

In this context the root-locus method has been used and is still being used by numerous control engineers in a vast number of practical and theoretical applications. Basically the root-locus method considers the effect of varying a gain parameter in the feedback loop,on the closed loop pole locations.Unfortunately the use of the root-locus method is somewhat limited since it can only be applied to single-input,single-output systems.However with ever increasing fields of application of the control system theory,the systems dealt with are getting more and more complex to be handled with the tools of classical control theory.As a solution to this problem,techniques for time domain analysis and synthesis of control systems have been developed.In this thesis the pole placement problem of multi-variable systems in state space representation is discussed in detail and basic methods developed in this field are introduced.

In the first chapter the motivating ideas in the pole placement problem are briefly discussed and relations between pole assignability and classical concepts of state space analysis such as controllability and observability are introduced.The second chapter deals with what seems as the fundamental idea of most pole placement algorithms,namely the transformation of system equations with arbitrary structure into controllable companion form.This transformation and its use in pole placement will be discussed both for single-input and also for multi-input systems.The third chapter is basically concerned with Ackermann's procedure for pole placement and its extension to multi-input systems. In the fourth chapter a "model reference control" scheme is applied to the pole placement problem.Also the relations between tracking and pole assignment is discussed in detail.The fifth chapter considers pole placement in stochastic case,the principle of separation and its use in pole placement problem under noisy conditions is also discussed.The last section gives a brief summary of what has been presented in this thesis and suggests topics of further research.

CHAPTER 1

POLE ASSIGNMENT VIA STATE VARIABLE FEEDBACK

Given the state space representation of a multivariable system:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \qquad (1-1)$$

where A and B are matrices of dimension (n x n) and (n x m) respectively;x(t) is an (n x 1) column vector denoting the state of the system,u(t) is the (m x 1) external input vector and y(t) is the (p x 1) output vector.Hence the matrix C is of dimension (p x n). From now on we will assume that all the states of the system (1-1) are available,and therefore the output equation will not be used.The effects of partially inaccesible state variables will be discussed later in this chapter.

The free response of the uncontrolled plant,i.e. when u(t) is equal to zero vector,is given by a linear combination of the dynamical modes of the system,where the mode shapes are determined by the eigenvectors and the time domain characteristics by the pole locations of the system [1].It is possible that for some reason or another the response of the uncontrolled plant is unsatisfactory. The system response may be too slow for a particular purpose, or it may even be unstable due to positive real parts of its poles.

However if control loops are introduced which generate the input vector by linear feedback of the state vector of the plant, then the response characteristics of the resulting closed loop system will no longer be determined by the eigenproperties of the plant matrix A, but those of some new closed loop plant matrix, whose eigenvalues and eigenvectors will depend upon the nature of the feedback loops. In other words we want to modify the external input u(t) ;

$$u(t) = K\, x(t) + r(t) \tag{1-2}$$

where r(t) is an (m x 1) external reference input vector, such that the closed loop system equation becomes

$$\dot{x}(t) = (A + BK)\, x(t) + B\, r(t) \tag{1-3}$$

The main concern of modal control theory is to choose an appropriate feedback gain matrix (A + BK) has a desired set of eigenvalues. In this chapter we want to answer the following questions:

i.) Under which conditions is it possible to find an appropriate K matrix, such that a desired closed loop characteristic polynomial is obtained?

ii.) What are the possible approaches to pole assignment problem if all the state variables are not accessible?

The procedure used to determine the K matrix will be discussed in the next chapter.

When one thinks about the conditions which have to be satisfied, so that the existence of K is guaranteed, one is immediately led to the idea, that the possibility of the existence depends on the controllability of the state x with respect to the external input u. To be precise consider the following. Let

$$\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \tag{1-4}$$

be an arbitrary set of n complex numbers λ_i, such that any λ_i with $Im(\lambda_i) \neq 0$ appears in a conjugate pair. The necessary and sufficient condition for the existence of an (m x n) real matrix K, such that the closed loop system matrix (A + BK) has the set Λ as its eigenvalues is the controllability of the pair (A,B), i.e. the existence of K implies that the (n x mn) controllability matrix of the system (1-1)

$$\mathcal{C}_x = \left[B, AB, \ldots, A^{n-1}B \right] \tag{1-5}$$

is of full rank n. This result has been proved by various authors [2], [3], [4]. Most of these proofs are constructive and some of them, such as the one in [2], are rather involved. The theorem will be stated here without giving a formal proof. But the method shown in Chapter 2, to evaluate K, for single and multi-input systems, is very illustrating for a possible proof.

<u>THEOREM (1-1)</u> For the n-th order dynamical system given in (1-1), let Λ(1-4) be an arbitrary desired set of n complex numbers λ_i, such that any λ_i with $Im(\lambda_i) \neq 0$ appears in Λ as a conjugate pair. The closed loop system (1-3) has Λ for its set of eigenvalues if and only if (A,B) is controllable.

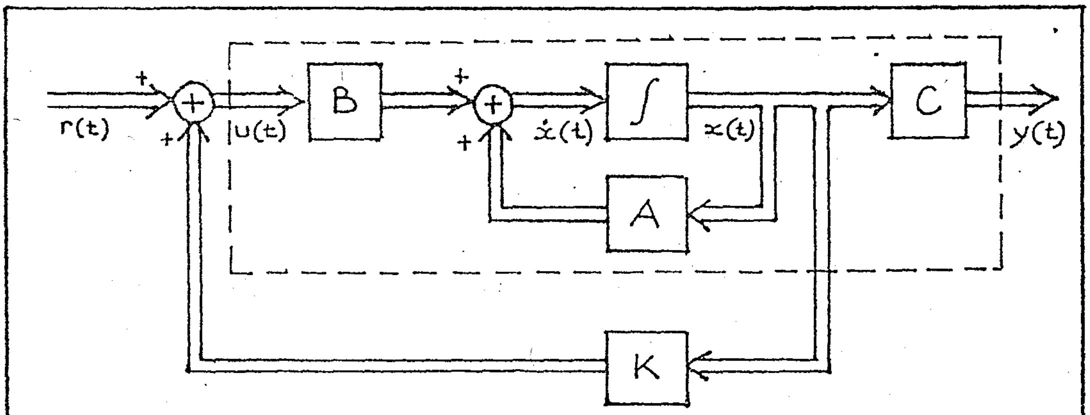


Fig.1-1

The general structure of the linear state variable feedback law given in (1-3) is shown in Fig (1-1).

Linear state variable feedback is an important compensation technique in the synthesis of linear dynamical systems. However one should be aware of one important factor concerning linear state variable feedback, which can in many cases prevent its direct employment for closed loop pole assignment.

In particular, on closer inspection of Fig.(1-1) it its apparent that the feedback path from the state $x(t)$ through the gain matrix K crosses the boundary which encloses the original system. This clearly implies the ability to directly measure the entire internal n-dimensional state vector. In general, however, only the external input $u(t)$ and the output $y(t)$ are directly measurable so that the control scheme given in Fig.(1-1) is not directly realizable. Since all the states of the system are required to implement the control law, we can introduce a state estimator (observer) into the system, such that the states are estimated using only the external input $u(t)$ and output $y(t)$. Hence in the realization of the control law (1-2) the n-dimensional estimated state vector $\hat{x}(t)$ will be used in place of $x(t)$. Obviously this idea of using a state estimator to reconstruct the unavailable states at the output, requires the system to be completely observable. It has been shown in [3] that complete observability of the pair (A,C) is necessary for the realization of an estimator. Certainly the convergence rate of the estimator must be fast compared to the time constant of the system, such that no significant delay is added to the system performance. The block diagram of the system with an estimator causes a slight modification on Fig.(1-1).

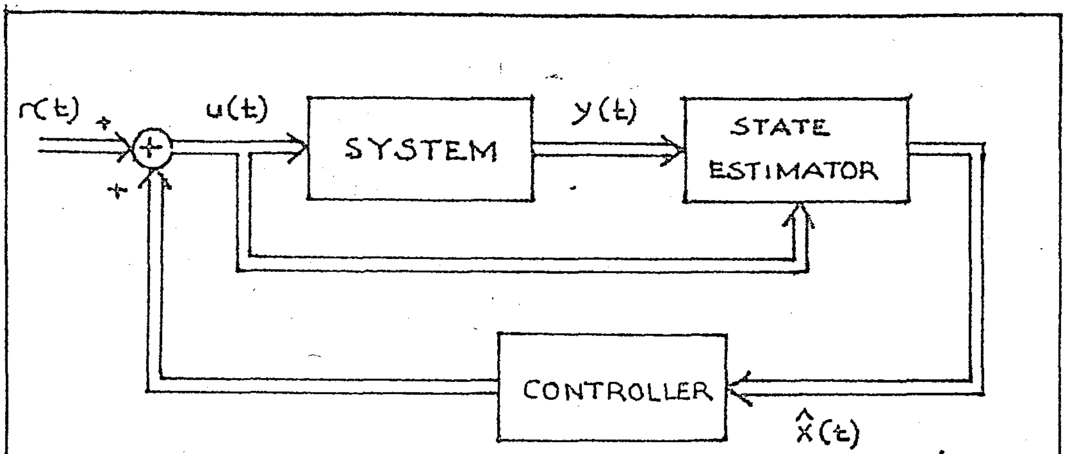


FIG.1-2

Under these considerations we can modify the statement of Theorem (1-1) as follows :

<u>THEOREM (1-2)</u> Consider the n-th order system given in (1-1) and assume that initially not all the states are available. Let Λ(1-4) be an arbitrary desired set of n complex numbers λ_i, such that any λ_i with $Im(\lambda_i) \neq 0$ appears in Λ in cojugate pair. The closed loop system (1-3) has Λ for its set of eigenvalues, i.e. complete and arbitrary pole placement is realizable, if and only if (A,B) is controllable and (A,C) is observable.

However estimating the unavailable states via a state estimator has one major disadvantage; it considerably increases the system order. Let us assume that pole placement is primarily used for plant stabilization. The plant, however, may not need as many feedbacks as there are states for its stabilization, since the response to the normal range of inputs is often determined by a few dominant poles of the system. Therefore one may try to constuct feedback loops only from the available output variables.Pole placement using only output feedback is certainly an alternative approach to using an estimator to establish the necessary state feedback law.For pole placement using only output feedback the external input vector u(t) will be modified,and then it is equal to,

$$u(t) = K_o\, y(t) + r(t)$$
$$u(t) = K_o C\, x(t) + r(t) \qquad (1\text{-}6)$$

The closed loop system equation becomes :

$$\dot{x}(t) = (A + BK_oC)\, x(t) + B\, r(t) \qquad (1\text{-}7)$$

The output feedback matrix K_o must be chosen such that $\det(A + BK_oC)$ will be equal to the desired characteristic polynomial.However determining K_o,such that arbitrary pole placement is achieved,is not that easy.It has been proved in [5],[6]that it is always possible to locate exactly p (p is the rank of the output matrix C) of the closed loop poles to arbitrary locations.If some other additional constraints are also satisfied then all of the n closed loop poles can be arbitrarily placed using only output feedback [7].

CHAPTER 2

COORDINATE TRANSFORMATION and ITS USE IN POLE ASSIGNMENT

2-1 Transformation into controllable companion form

In the previous chapter it has been shown that for a controllable system pair (A,B),all the poles of the closed loop system can be assigned arbitrarily,subject to complex conjugate pairing,by a suitable choice of the feedback gain matrix K.For a single input system the required feedback matrix is unique,whereas for a multi-input system there are practically an infinite number of solutions.Although the existence of such feedback gains is known,determining their values for a particular system is not that easy.Trying to solve for the K matrix usually results in a set of non-linear equations,which becomes almost impossible to solve with an increase in system order.Several algorithms have appeared in the literature on this subject,where the motivating idea of most of these methods is a transformation of the original system equations in a coordinate system where the mathematics is tractable for a particular purpose.

Now in this chapter we are going to introduce a transformation [3],[7] so that the transformed state equations will be in controllable companion form.The use of this form in pole assignment problem will be discussed and illustrated in detail. To start consider the single input system which is expressible by the state space representation of the form :

$$\dot{x}(t) = A\,x(t) + b\,u(t) \qquad (2\text{-}1)$$

It is assumed that the system (2-1), or the pair (A,b) is controllable which implies that the rank of the (n x n) controllability matrix $\mathcal{C}_x$ is n. Now consider the (n x n) matrix T obtained from $\mathcal{C}_x$ by setting t_1^T, the first row of T, equal to the last (n-th) row of $\mathcal{C}_x^{-1}$ and recursively computing the remaining rows of T by successive postmultiplication of each preceding row of T by A. In particular,

$$T = \begin{bmatrix} t_1^T \\ t_1^T A \\ \vdots \\ t_1^T A^{n-1} \end{bmatrix} \qquad (2\text{-}2)$$

where t_1^T is the n-th row of $\mathcal{C}_x^{-1}$. It is thus readily apparent that $t_1^T b = t_1^T A b = \ldots = t_1^T A^{n-2} b = 0$, but that $t_1^T A^{n-1} b = 1$, which immediately implies the relation $T\,b = [0\ 0\ \ldots\ 1]^T$. These observations also imply the nonsingularity of T, since

$$T\mathcal{C}_x = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & x \\ \vdots & & \iddots & & \vdots \\ 1 & x & \ldots & x & x \end{bmatrix}$$

where the x's are possibly non-zero elements. If z is now defined as Tx, it is seen that the first element of z, namely z_1, when differentiated with respect to time, yields the relation (dropping the time arguments for convenience) :

$$\dot{z}_1 = (t_1^T A)\, x + (t_1^T b)\, u$$

which in turn equals to $t_2^T x = z_2$. Furthermore $\dot{z}_2 = (t_1^T A^2)x + (t_1^T A b)u = z_3$ and so forth, or in general $\dot{z}_i = z_{i+1}$ for $i = 1,2,\ldots,(n-1)$. Therefore it follows that the equivalent single-input system representation $(\hat{A},\hat{b})$ or $\dot{z} = \hat{A} z + \hat{b} u$, where $\hat{A} = TAT^{-1}$ and $\hat{b} = Tb$ is in a particular structural form which is termed as the scalar controllable companion form, i.e.

$$\hat{A} = TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & 1 \\ -a_n & -a_{n-1} & \ldots & & -a_1 \end{bmatrix}, \quad \hat{b} = Tb = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad (2\text{-}3)$$

where $\hat{A}$ is the (n x n) companion matrix with the identity matrix in the upper right block and $\hat{b}$ is identically zero except of a non-zero entry 1 in the n-th row. Some immediate benefits are derived from the reduction of (A,b) to controllable companion form. In particular, the characteristic polynomial, $\det(\lambda I - A)$, of the system is apparent from the last row of $\hat{A}$. Expanding the $\det(\lambda I - \hat{A})$ along any but the last row we obtain the characteristic polynomial of the pair (A,b) or $(\hat{A},\hat{b})$,i.e.

$$\Delta(\lambda) = \det(\lambda I - A) = \det(\lambda I - \hat{A}) =$$
$$= \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n \quad (2\text{-}4)$$

Furthermore the input u only effcts the last row of $\hat{b}$,due to its special structure obtained through the transformation $z = Tx$.

The notion of "controllable companion form" is not confined only to scalar systems,and can be extended to more general multivariable cases.In particular, consider any completely state controllable system pair (A,B),with B assumed to be of full rank $m \leq n$.This latter assumption implies that all m available inputs are mutually independent,which is usually the case in practice.We now define $\bar{\mathcal{C}}_x$ as the $(n \times \bar{n})$ matrix obtained by selecting from left to right as many as $\bar{n}$ linearly independent columns of the controllability matrix $\mathcal{C}_x = [B, AB, \dots, A^{n-1}B]$.Since the system was assumed to be controllable,$\bar{\mathcal{C}}_x$ has full rank n.We can construct the nonsingular $(n \times n)$ matrix L by simply reordering the $\bar{n}(=n)$ columns of $\bar{\mathcal{C}}_x$,beginning with a power ordering of the first d_1 columns of $\bar{\mathcal{C}}_x$ which involve b_1,the first column of B,and then employing those d_2 columns of $\bar{\mathcal{C}}_x$ which involve b_2,next and so forth.In particular :

$$L = \left[b_1, Ab_1, \ldots, A^{d_1-1} b_1, b_2, \ldots, A^{d_2-1} b_2, \ldots, A^{d_m-1} b_m \right] \tag{2-5}$$

We now define the m integers d_i as the controllability indices of the system and denote by μ, $\max(d_i)$ for $i = 1, \ldots, m$, which we further define as the controllability of the system. i, e. $\max(d_i) = \mu$. It should now be noted that all m columns of B are present in L since we assumed that B was of full rank m. We now set :

$$\sigma_k = \sum_{i=1}^{k} d_i \quad , \quad k = 1, 2, \ldots, m \tag{2-6}$$

which implies that $\sigma_1 = d_1, \sigma_2 = d_1 + d_2, \sigma_m = d_1 + \ldots + d_m = n$
We can now enlarge the algorithm employed in the single input case to determine an appropriate equivalence transformation matrix T in the multivariable case, i.e. we set t_k^T equal to the σ_k-th row of L^{-1} for $k = 1, \ldots, m$ and consider the following (n x n) matrix T :

$$T = \begin{bmatrix} t_1^T \\ t_1^T A \\ \vdots \\ t_1^T A^{d_1-1} \\ \hline t_2^T \\ \vdots \\ t_2^T A^{d_2-1} \\ \hline \vdots \\ \hline \vdots \\ t_m^T A^{d_m-1} \end{bmatrix} \tag{2-7}$$

The nonsingularity of L implies necessarily T of being full rank n.By now using the same reasoning applied in the development of the single-input case, it follows that T represents an equivalence transformation which reduces the given system to an equivalent representation $\dot{z} = \hat{A}z + \hat{B}u$,where the pair $(\hat{A},\hat{B})$ assumes a particularly useful structured form,namely a multi-input controllable companion form,i.e.

$$\hat{A} = TAT^{-1} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{m1} & \hat{A}_{m2} & \cdots & \hat{A}_{mm} \end{bmatrix} \qquad (2\text{-}8a)$$

$$= \left[\begin{array}{ccccc|ccccc|c|ccccc}
0 & 1 & 0 & \cdots & 0 & 0 & \cdots & & & 0 & & 0 & 0 & \cdots & & 0 \\
0 & 0 & 1 & 0 \cdots & 0 & 0 & \cdots & & & 0 & & \vdots & \vdots & & & \vdots \\
 & & & \ddots & & & & & & & \cdots & & & & & \\
 & & & & 1 & & & & & & & & & & & \\
x & x & \cdots & & x & x & x & \cdots & & x & & x & x & \cdots & & x \\ \hline
0 & 0 & \cdots & & 0 & 0 & 1 & 0 & \cdots & 0 & & 0 & 0 & \cdots & & 0 \\
\vdots & \vdots & & & \vdots & 0 & 0 & 1 & 0 \cdots & 0 & & \vdots & \vdots & & & \vdots \\
 & & & & & \vdots & & & \ddots & & \cdots & & & & & \\
 & & & & & & & & & 1 & & & & & & \\
 & x & \cdots & & x & x & x & \cdots & & x & & x & x & \cdots & & x \\ \hline
 & & \vdots & & & & & \vdots & & & \ddots & & & \vdots & & \\ \hline
0 & 0 \cdots & & & 0 & 0 & 0 \cdots & & & 0 & & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & \vdots & & & \vdots & \cdots & 0 & 0 & 1 & 0 \cdots & 0 \\
 & & & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & & & 1 \\
x & x & \cdots & & x & x & x & \cdots & & x & & x & x & \cdots & & x
\end{array}\right] \qquad (2\text{-}8b)$$

and, $$\hat{B} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & x & \cdots & x \\ \hline 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & x.. & x \\ \hline \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} \qquad (2\text{-}8c)$$

where the m diagonal blocks $\hat{A}_{ii}$ of $\hat{A}$ are each an upper right identity companion matrix of dimension d_i, while the off diagonal blocks, $\hat{A}_{ij}$ for $i \neq j$ are each identically zero except possibly for their respective final rows. We therefore note that all information regarding the equivalent state matrix $\hat{A}$ can be derived from knowledge of the m ordered controllability indices d_i and the m ordered σ_k rows of $\hat{A}$. The same can also be said of $\hat{B}$, since we note that only these same ordered σ_k rows of $\hat{B}$ are nonzero.

<u>EXAMPLE (2-1)</u> Consider the system $\dot{x} = Ax + Bu$, where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 \\ -1 & 1 & 4 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We readily verify that the system is controllable. Furthermore $\bar{\mathcal{C}}_x = [b_1, b_2, Ab_2, A^2b_2]$, the matrix consisting of the first $n=4$ linearly independent columns of $\mathcal{C}_x$. Therefore no reordering of the columns of $\mathcal{C}_x$ is required and hence $\bar{\mathcal{C}}_x = L$. For this example $d_1 = 1$, $d_2 = 3$, $\sigma_1 = 1$ and $\sigma_2 = d_1 + d_2 = n = 4$. The transformation matrix T is computed next by first calculating t_1^T and t_2^T, the first and fourth (corresponding to σ_1 and σ_2) rows of L^{-1}. For this example $t_1^T = [1 \quad 1 \quad 0 \; -2]$, and $t_2^T = [1 \quad 0 \quad 0 \quad 1]$, which implies that :

$$T = \begin{bmatrix} t_1^T \\ t_2^T \\ t_2^T A \\ t_2^T A^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and,}$$

$$T^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Therefore :

$$\hat{A} = TAT^{-1} = \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -3 & 4 \end{array}\right], \hat{B} = TB = \left[\begin{array}{cc} 1 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right]$$

We now note that the pair $(\hat{A}, \hat{B})$ is indeed in controllable companion form. In particular, $\hat{A}_{11}$ is a companion matrix of dimension $d_1 = 1$, and $\hat{A}_{22}$ is a companion matrix of dimension $d_2 = 3$. Consequently, only the first and fourth (corresponding to σ_1 and σ_2) rows of $\hat{A}$ and $\hat{B}$ are nontrivial and thus contain all the pertinent information regarding the equivalent state matrix $\hat{A}$.

2-2 Extension of Controllable Companion Form to Partially State Controllable Systems

We can now consider certain implications and extensions of the preceding results when the multivariable system is only partially state controllable. In particular, we will still assume that rank $(B) = m < n$ but we consider the case when $\text{rank}(\mathcal{C}_x) = \bar{n} < n$. Note that it is still possible to define the $(n \times \bar{n})$ matrix $\bar{\mathcal{C}}_x$ consisting of the first $\bar{n}$ linearly independent columns of $\mathcal{C}_x$, as well as the $(n \times \bar{n})$ matrix L as given by (2-5) but with $\sigma_m = \sum_{i=1}^{m} d_i = \bar{n}$ instead of n. The $\bar{n}$ linearly independent columns of L clearly form a basis of some subspace W of E^n. If we define $W_\perp$ as the orthogonal complement of W, i.e. the subspace of E^n consisting of all vectors in E^n perpendicular to W in the sense of a zero inner product, it follows that any vector v in E^n can be expressed as a linear combination of some vector w in W and some vector $w_\perp$ in $W_\perp$. In particular $v = \alpha w + \beta w_\perp$, for all v in E^n, which implies that E^n can be defined as the direct sum of W and $W_\perp$. It is thus clear that the dimension q of $W_\perp$ is $n-\bar{n}$, since E^n is of dimension n. We let $\beta_1, \beta_2, \ldots, \beta_q$ be any basis of $W_\perp$

and consider the "extended" state representation :

$$\dot{x} = A\,x + B_e\,u_e \qquad (2\text{-}9)$$

where B_e is the $n \times (m+q)$ matrix obtained by appending to B the q basis vectors of $W_\perp$, i.e. $B_e = [B, \beta_1, \ldots, \beta_q]$ while u_e is an $((m+q) \times 1)$ input vector obtained by appending to u,q additional input elements, i.e. $u_e = [u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+q}]^T$. The extended system (2-9), thus defined, is clearly a controllable one, and is therefore possible to employ the algorithm presented earlier, to obtain a n-dimensional equivalence transformation which reduces the extended system to controllable companion form. We denote the appropriate transformation matrix T_e and utilize it to reduce the original system to the equivalent representation $\dot{z} = \hat{A}\,z + \hat{B}\,u$, where $\hat{A} = T_e A T_e^{-1}$ and $\hat{B} = T_e B$. Due to the specific choice of T_e, it follows that the equivalent pair $(\hat{A},\hat{B})$ "partially resembles" the multivariable companion form. In particular

$$\hat{A} = \left[\begin{array}{c|c} \hat{A}_c & \hat{A}_{c\bar{c}} \\ \hline 0 & \hat{A}_{\bar{c}} \end{array}\right] \quad \text{and} \quad \hat{B} = \left[\begin{array}{c} \hat{B}_c \\ \hline 0 \end{array}\right] \qquad (2\text{-}10)$$

where the pair $(\hat{A}_c,\hat{B}_c)$ is in $\bar{n}$-dimensional controllable companion form; i.e. the pair $(\hat{A}_c,\hat{B}_c)$ assumes the structure indicated by (2-8) with $\sum_{i=1}^{m} d_i = \bar{n}$. Furthermore the lower left $(q \times \bar{n})$ block of $\hat{A}$ as well as the final q rows of $\hat{B}$ are identically zero.On closer inspection it becomes apparent that the controllable and the completely uncontrollable "portions" of the system have been separated.More specifically,the $\bar{n}$-dimensional subsystem defined by the first $\bar{n}$ rows of the pair $(\hat{A},\hat{B})$namely $\mathring{z}_c = \hat{A}_c z_c + \hat{A}_{c\bar{c}} z_{\bar{c}} + \hat{B}_c u$ is clearly controllable,since $\hat{A}_{c\bar{c}} z_{\bar{c}}$ can be treated as a known disturbance.Furthermore,the q-dimensional subsystem defined by the remaining rows of $(\hat{A},\hat{B})$,namely $\mathring{z}_{\bar{c}} = \hat{A}_{\bar{c}} z_{\bar{c}}$ is completely uncontrollable.We further note that in view of (2-4) and (2-10) the characteristic polynomial $\det(\lambda I - A)$ of A (and hence of $\hat{A}$) can be written as the product of the characteristic polynomials of the controlable and completely uncontrollable portions of the system;i.e.

$$\det(\lambda I - A) = \det(\lambda I - \hat{A}) = \det(\lambda I - \hat{A}_c)\det(\lambda I - \hat{A}_{\bar{c}}) \tag{2-11}$$

2-3 Pole Assignment via the Controllable Companion Form

We will now consider the general employment of linear state variable feedback for arbitrary assignment of the closed loop poles of the multivariable system

$$\mathring{x} = A x + B u \tag{2-12}$$

In particular if the linear state variable feedback control law,

$$u = K x + r \tag{2-13}$$

is employed to alter the pole configuration of the open loop system,we can readily obtain a state space representation for the dynamical behaviour of the compensation system by simply substituting (2-13) for u into (2-12) :

$$\dot{x} = (A + BK) x + B r \tag{2-14}$$

In general it is not all clear what effect the control law (2-13) has on the system (2-12),since we consider any arbitrary "unstructured" open loop system pair (A,B).However,if the open loop system is in controllable companion form,the effect of the feedback law in (2-13) on the pole locations can be easily clarified.Let us give the main result of this section as a theorem.

<u>THEOREM (2-1)</u> 3 Consider the system (2-12) and the linear state variable feedback law (2-13).All $\bar{n}$ controllable poles of the closed loop system (2-14) can be completely and arbitrarily assigned via linear state variable feedback while the $n-\bar{n}$ uncontrollable poles of the system are unaffected by (2-13).

<u>Proof:</u> Assume that we have already transformed the given system into the controllable companion form (2-10) The pair $(\hat{A}_c,\hat{B}_c)$ is an $\bar{n}$-dimensional controllable com-

panion form, while $\hat{A}_{\bar{c}}$ represents the completely uncontrollable portion of the state matrix. As we have previously noted all (m) σ_k rows of $\hat{A}_c + \hat{B}_c\hat{K}_c$ can be completely and arbitrarily altered via $\hat{K}$. ($\hat{K}$ is the required feedback gain matrix in the transformed coordinate system, and $\hat{K}_c$ is the portion of $\hat{K}$ corresponding to the $\bar{n}$-dimensional controllable system $(\hat{A}_c, \hat{B}_c)$.) We can choose the first $\bar{n}$ columns $\hat{K}_c$, of $\hat{K}$, such that

$$\hat{A}_c + \hat{B}_c\hat{K}_c = \begin{bmatrix} 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ \vdots & & \ddots \quad 1 \\ -\alpha_{\bar{n}} & -\alpha_{\bar{n}-1} & \cdots -\alpha_1 \end{bmatrix} \qquad (2\text{-}15)$$

is an $\bar{n}$-dimensional companion matrix, where the scalars $\alpha_1, \ldots, \alpha_{\bar{n}}$ represent the coefficients of the desired characteristic polynomial, i.e. the coefficients of the polynomial $\det(\lambda I - \hat{A}_c - \hat{B}_c\hat{K}_c)$. Since the remaining $n-\bar{n}$ columns of $\hat{K}$ affect only $\hat{A}_{c\bar{c}}$, the final $n-\bar{n}$ rows of $\hat{A}$ are completely unaffected by $\hat{K}$, which implies that the $n-\bar{n}$ eigenvalues $\hat{A}_{\bar{c}}$, or equivalently the uncontrollable poles of the system, remain unaltered by linear state feedback. This follows formally from the fact that all n poles of the closed loop system are equivalent to the zeros of :

$$\det(\lambda I - A - BK) = \det(\lambda I - \hat{A} - \hat{B}\hat{K}) =$$
$$= \det(\lambda I - \hat{A}_c - \hat{B}_c\hat{K}_c)\ \det(\lambda I - \hat{A}_{\bar{c}}) \qquad (2\text{-}16)$$

In order to explicitly determine a $\hat{K}$ which yields the controllable part of the closed loop system matrix as represented by (2-15), we let $\hat{A}_m^*$ denote the m ordered σ_k rows of $\hat{A}_c + \hat{B}_c\hat{K}_c$ as given by (2-15) and define $\hat{A}_{cm}$ and $\hat{B}_{cm}$ as the same ordered σ_k rows of $\hat{A}_c$ and $\hat{B}_c$, respectively. It therefore follows that

$$\hat{A}_{cm} + \hat{B}_{cm}\hat{K}_c = \hat{A}_m^* \qquad (2\text{-}17)$$

or that the control law (2-13), with the first $\bar{n}$ columns of $\hat{K}$ given by

$$\hat{K}_c = \hat{B}_{cm}^{-1} (\hat{A}_m^* - \hat{A}_{cm}) \qquad (2\text{-}18)$$

yields the desired $\bar{n}$-dimensional closed loop system submatrix (2-15).

The final $n-\bar{n}$ columns of $\hat{K}$ play no part in closed loop pole assignment, since they affect only $\hat{A}_{c\bar{c}}$ which in turn, has no effect on the eigenvalues of the closed loop system matrix. We can therefore set the final $n-\bar{n}$ columns of $\hat{K}$ equal to zero in order to complete our assignment of all (mn) entries of an appropriate. The state feedback gain matrix K, associated with the original system is given by

$$u = \hat{K}\, z + r = \hat{K}T\, x + r = K\, x + r \qquad (2\text{-}19)$$

where,

$$K = \hat{K}T \qquad (2\text{-}20)$$

<u>EXAMPLE (2-2)</u> To illustrate the above constructive procedure for finding a state feedback gain matrix K, which yields any arbitrary set of $\bar{n}$ closed loop poles, consider the following system in the state form (2-12), where

$$A = \begin{bmatrix} -1 & 0 & 0 & -6 & 3 & -1 \\ 1 & -2 & 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 6 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 2 & 1 \\ -2 & 0 & 0 & 2 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & -2 \\ 0 & -1 \\ 0 & 0 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}$$

We first transform this system to an equivalent one via the transformation matrix T, noting that the controllability matrix $\mathcal{C}_x$ for this system has rank $5 < n = 6$, and therefore that the system is not completely controllable. Therefore, in view of the results given in the previous section

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \end{bmatrix}$$

$$\hat{A} = TAT^{-1} = \left[\begin{array}{ccc:cc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & -2 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -4 & -1 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right] = \left[\begin{array}{c|c} \hat{A}_c & \hat{A}_{c\bar{c}} \\ \hline 0 & \hat{A}_{\bar{c}} \end{array}\right]$$

and $$\hat{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ \hdashline 0 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{B}_c \\ \hline 0 \end{bmatrix}$$

Clearly the pair $(\hat{A}_c, \hat{B}_c)$ is in controllable companion form, with $d_1 = 3$ and $d_2 = 2$. Therefore $\sigma_1 = d_1 = 3$ and $\sigma_2 = d_1 + d_2 = 5 = \bar{n}$. We further note that $\hat{A}_{\bar{c}} = [-1]$, or that $n - \bar{n} = 1$ uncontrollable pole at $\lambda = -1$ is an asymptotically stable one. If we now require that the five controllable closed loop poles of the system be given by $\lambda_1 = -0.1, \lambda_2 = -0.2, \lambda_{3,4} = -1 \pm j, \lambda_5 = -2$, it follows that we will require an $\hat{K}_c$ such that

$$\det(\lambda I - \hat{A}_c - \hat{B}_c\hat{K}_c) = (\lambda + 0.1)(\lambda + 0.2)(\lambda^2 + 2 \quad +2)(\lambda + 2)$$
$$= \lambda^5 + (4.3)\lambda^4 + (7.22)\lambda^3 + (5.88)\lambda^2 + (1.32)\lambda + (0.08)$$

$$\hat{A}_c + \hat{B}_c\hat{K}_c = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -0.08 & -1.32 & -5.88 & -7.22 & -4.3 \end{bmatrix}$$

which implies that,

$$\hat{A}_m^* = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ -0.08 & -1.32 & -5.88 & -7.22 & -4.3 \end{bmatrix}$$

since $$\hat{A}_{cm} = \begin{bmatrix} -1 & 2 & 0 & -2 & 0 & 1 \\ 0 & 0 & 3 & -4 & -1 & -1 \end{bmatrix}$$

and $\hat{B}_{cm} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, it follows from (2-18) that :

$$\hat{K}_c = \hat{B}_{cm}^{-1} (\hat{A}_m^* - \hat{A}_{cm}) = \begin{bmatrix} 1.16 & 0.64 & 17.76 & 9.44 & 6.6 \\ -0.08 & -1.32 & -8.88 & -3.22 & -3.3 \end{bmatrix}$$

An appropriate K can now be found by adding a zero sixth column to $\hat{K}_c$ and postmultiplying the resulting matrix $\hat{K}$, by T as indicated in (2-20), which equals to

$$K = \begin{bmatrix} 7.76 & 0.64 & 1.16 & 9.44 & 18.4 & 0. \\ -3.38 & -1.32 & -0.08 & -3.22 & 10.2 & 0. \end{bmatrix}$$

2-4 Application of Linear State Variable Feedback to System Stabilization

Another important point which deserves special attention is the application of state feedback to system stabilization. In the previous section it has been proved that all the controllable poles of any system can be altered via a suitably chosen feedback matrix K. But the $n-\bar{n}$ uncontrollable poles remain unaltered by this feedback, since the uncontrollable portion of the system has no coupling with the external input u. Even if the $\bar{n}$ controllable poles have initially positive real parts, i.e. they cause the system to be unstable, they can be moved in the complex plane via state feedback, so that they all have negative real parts; they are stabilizable.

But if any one of the $n-\bar{n}$ uncontrollable poles is initially unstable, the complete system remains unstable, since the $n-\bar{n}$ poles are completely unaffected by the feedback of (2-13). Here at this point a definition of system stabilizability-which is a weaker condition than controllability-can be given as :

<u>DEFINITION</u> [3] : A system as in (2-12) is called asymptotically stabilizable via state feedback if and only if the $n-\bar{n}$ uncontrollable poles of the system (the eigenvalues of $\hat{A}_{\bar{c}}$) lie in the left half complex plane.

2-5 Final Remarks

It should be noted that the procedure outlined in the previous sections to determine the required feedback gain matrix $\hat{K}_c$ and K as in (2-18) and (2-20) will be extremely simplified for single input systems. For any single input system $d_1 = \bar{n}$, hence $\sigma_m = \bar{n}$. Therefore there exists only a single row of the matrix $\hat{A}_{cm}, \hat{A}^*_m$ and $\hat{B}_{cm}$

$$\hat{A}_{cm} = \left[-a_{\bar{n}}, -a_{\bar{n}-1}, \; \ldots \; , -a_1 \right] \qquad (2\text{-}21)$$

$$\hat{A}^*_m = \left[-\alpha_{\bar{n}}, -\alpha_{\bar{n}-1}, \; \ldots \; , -\alpha_1 \right] \qquad (2\text{-}22)$$

$$\hat{B}_{cm} = \left[1 \right] \qquad (2\text{-}23)$$

Hence the formula for determining $\hat{K}_c$ is reduced to the computation of a row vector

$$\hat{k}^T_c = \hat{B}^{-1}_{cm} (\hat{A}^*_m - \hat{A}_{cm}) =$$

$$= \left[-(\alpha_{\bar{n}} - a_{\bar{n}}), \; \ldots \; , -(\alpha_1 - a_1) \right] \qquad (2\text{-}24)$$

and $k^T = \hat{k}^T T$ (2-25)

where $\hat{k}$ is obtained from $\hat{k}_c$ by placing appropriate number of zero entries to the end of $\hat{k}_c$.

In various books on modern control engineering one can see several different procedures designed to determine the transformation matrix which will convert the single input system of (2-1) into the controllable companion form. The procedure presented in the first section and given by the equation (2-2) is a very convenient one, if one considers equation (2-25) where the transformation back to the original coordinate system has to be realized via T. The matrix T, determined by (2-2) will give us immediately the value of k. In most other procedures however, T^{-1}, the inverse of the transformation matrix, is determined first, from which T must be evaluated. Therefore the procedure presented in this chapter is a very convenient one to use for arbitrary pole assignment in single input systems.

CHAPTER 3

POLE ASSIGNMENT in LINEAR MULTIVARIABLE SYSTEMS USING UNITY RANK FEEDBACK MATRICES

3-1 Introduction

In the previous chapter the basic approach of most pole assignment procedures has been discussed. The main part of the procedure introduced in the second chapter, is determining a non-singular transformation such that the canonical system equations in the transformed coordinate system are convenient from the point of mathematical tractability of computations. But in general determining the required transformation matrix especially for higher order systems is a very time consuming computational process.

In this chapter a new procedure, proved first by Ackermann [9] will be introduced which will eliminate all of the difficulties described above.

The original procedure of Ackermann can only be applied to single input systems, which is a severe restriction on the applicability of this method. But in the third section of this chapter a procedure will be outlined [2],[8], through which multi-input controllable systems can be transformed into an equivalent single-input system. This procedure enables us to apply Ackermann's formula for pole placement in multi-input systems as well.

3-2 Ackermann's Procedure for Pole Assignment in Single Input Systems

THEOREM (3-1) Given the controllable single input system :

$$\dot{x} = A\,x + b\,u \tag{3-1}$$

it is desired to construct a feedback law of the form

$$u = k^T x \tag{3-2}$$

such that

$$\det\left[\lambda I - (A + bk^T)\right] = \Delta(\lambda) \tag{3-3}$$

where the roots of $\Delta(\lambda)$ are the desired poles of the closed loop system, subject to complex pairing. Then the feedback gain vector k^T is given by the following equation :

$$k^T = -e_n^T \mathcal{C}_x^{-1} \Delta(A) \tag{3-4}$$

where e_n is the (n x 1) column unit vector whose all entries are all zero except the last entry which is equal to 1, $\mathcal{C}_x$ is the controllability matrix of the controllable pair (A,b) and is defined by :

$$\mathcal{C}_x = (b, Ab, \ldots, A^{n-1}b) \tag{3-5}$$

and $\Delta(A)$ is the characteristic polynomial evaluated at $\lambda = A$.

PROOF Under the feedback law given by (3-2) the closed loop system equation becomes

$$\dot{x} = (A + bk^T)\, x \qquad (3\text{-}6)$$

$$\dot{x} = F\, x$$

where $F = A + bk^T$ (3-7)

Let $\Delta(\lambda)$ be the desired closed loop characteristic polynomial of the closed loop system matrix F :

$$\Delta(\lambda) = \det(\lambda I - F) =$$
$$= \lambda^n + (\alpha_1)\lambda^{n-1} + \ldots + (\alpha_{n-1})\lambda + \alpha_n \qquad (3\text{-}8)$$

Since the pair (A,b) is a controllable one, the controllability matrix $\mathcal{C}_x$, as given in (3-5), is invertible and one can write from the basic definition of the inverse of a matrix,

$$\mathcal{C}_x^{-1}\mathcal{C}_x = I$$

Let e^T denote the last row of $\mathcal{C}_x^{-1}$. Then

$$e^T(b, Ab, \ldots, A^{n-1}b) = (0\ 0\ \ldots\ 1) \qquad (3\text{-}9)$$

which is equivalent to the following equalities :

$$e^Tb = e^T(Ab) = \ldots = e^T(A^{n-2}b) = 0$$
$$e^T(A^{n-1}b) = 1 \qquad (3\text{-}10)$$

Using (3-10) we obtain the set of equations,

$$\begin{aligned} e^TF &= e^T(A + bk^T) = e^TA \\ e^TF^2 &= (e^TF)F = (e^TA)(A + bk^T) = e^TA^2 \\ &\vdots \\ e^TF^{n-1} &= (e^TF^{n-2})F = (e^TA^{n-2})(A + bk^T) = e^TA^{n-1} \\ e^TF^n &= e^TA^n + k^T \end{aligned} \tag{3-11}$$

Furthermore from the Cayley-Hamilton theorem we know that every matrix satisfies its own characteristic equation,i.e.

$$\Delta(F) = F^n + (\alpha_1)F^{n-1} + \dots + (\alpha_n)I = 0 \tag{3-12}$$

Multiplying (3-12) by e^T and using (3-11) we get;

$$\begin{aligned} e^T\Delta(F) = e^T(A^n) + e^T(\alpha_1 A^{n-1}) + \dots \\ \dots + e^T(\alpha_n I) + k^T = 0 \end{aligned} \tag{3-13}$$

Solving for k^T we obtain,

$$k^T = - e^T \Delta(A) \tag{3-14}$$

We have to note also the fact that e^T,the last row of $\mathcal{C}_x^{-1}$,can be written as,

$$e^T = (0\ 0\ \dots\ 1)\,\mathcal{C}_x^{-1} = e_n^T\,\mathcal{C}_x^{-1}$$

hence

$$k^T = - e_n^T\,\mathcal{C}_x^{-1}\,\Delta(A) \tag{3-15}$$

In the computation of the feedback gain vector k^T,it is only required to calculate the last row of $\mathcal{C}_x^{-1}$, which saves much from computation time. Furthermore even if there are multiple open loop or closed loop poles, the same theorem can be again applied without any modifications which is not the case in most of the other pole assignment algorithms.

Although Ackermann's original procedure can only be applied to single input and completely state controllable systems, the procedure is later modified [10], so that it can also be applied to partially controllable systems. In this latter case, it is only possible to arbitrarily assign only $\bar{n}$ of the closed loop poles, where $\bar{n}$ is the rank of the controllability matrix $\mathcal{C}_x$.

EXAMPLE (3-1) Given a controllable second order system :

$$\dot{x} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

we want to design a feedback law of the form $u = k^T x$, such that the closed loop poles will be at $\lambda_1 = -3$, and $\lambda_2 = -5$, therefore the closed loop characteristic polynomial $\Delta(\lambda)$ should be $\Delta(\lambda) = \lambda^2 + 8\lambda + 15$.

Using the results of the previous theorem we obtain :

$$\Delta(A) = \begin{bmatrix} 22 & -13 \\ 26 & 61 \end{bmatrix}$$

and,

$$e_n^T \mathcal{C}_x^{-1} = (0 \quad 1/4)$$

so that the required feedback gain vector becomes,

$$k^T = - e_n^T \mathcal{C}_x^{-1} \Delta(A) = (-13/2 \quad -61/4)$$

with the above feedback law the closed loop system equation equals to,

$$\dot{x} = (A + bk^T)\ x = \begin{bmatrix} -12 & -63/2 \\ 2 & 4 \end{bmatrix} x$$

A check of the closed loop poles will assure that the desired pole configuration is realized via the above feedback law.

3-3 Extension of Ackermann's Formula to Multivariable Systems

Some of the existing methods developed for assigning specified values to the poles of a linear multivariable system by means of state or output feedback restrict the feedback matrix (explicitly or implicitly) to having unity rank by predefining its structure as a product of a row and a column vector, which is also referred as the dyadic form. This restriction results in considerable simplifications in the calculation of the required feedback matrix by reducing the multivariable system to an "equivalent" single input system. However, the preservation of the controllability under this reduction of the control space is often taken for granted. In this section we want to discuss the relationship between this problem and the concept of cyclic subspaces from linear algebra. For the sake of completeness preliminary results from linear algebra on cyclic subspaces [11] are collected and presented first.

3-3.1 Cyclicity of a Space with respect to a Matrix Operator

Consider the n-dimensional Euclidean vector space E^n and a linear operator in this space represented by a given constant (n x n) matrix A. Take an arbitrary non-zero x in E^n and the sequence of vectors

$$x, A\,x, A^2x, \ldots \qquad (3\text{-}16)$$

since the space is finite dimensional, there exists an integer $r, 0 < r \leq n$, such that the first r vectors $x, Ax, \ldots, A^{r-1}x$ of this sequence are linearly independent. In other words, r is the greatest integer such that the vectors

$$x, Ax, \ldots, A^{r-1}x \qquad (3\text{-}17)$$

are linearly independent. These vectors form the basis of an r-dimensional subspace E^r. This subspace is called "cyclic" with respect to the operator A, in view of the special character of the basis vectors (3-17). The operator A carries the first vector of (3-17) into the second, the second into the third, and so on. The last basis vector is carried by A into a linear combination of the basis vectors. The vector x is said to generate the r-dimensional cyclic subspace E^r by means of the operator A.

The space E^n is cyclic with respect to A if and only if there exists a vector x in E^n such that its cyclic subspace E^r is the entire space, that is $r = n$. In other words, E^n is cyclic with respect to A if there exists a vector x in E^n such that the set of vectors

$$x, Ax, \ldots, A^{n-1}x \qquad (3\text{-}18)$$

span the entire space E^n. The condition of linear independence of the basis vectors (3-18) can be expressed as

$$\operatorname{rank}(x, Ax, \ldots, A^{n-1}x) = n \qquad (3\text{-}19)$$

when E^n is not cyclic with respect to A, then for all x in E^n

$$\operatorname{rank}(x, Ax, \ldots, A^{n-1}x) < n \qquad (3\text{-}20)$$

3-3.2 Application of Cyclicity in State Feedback Design [12]

A controllable linear multivariable system described by

$$\dot{x} = A\,x + B\,u$$

with the state vector x in E^n and the control vector u in E^m can be effectively controlled by a scalar control h in E^1 which is applied to the m inputs of the system through a constant (m x 1) vector q,u = qh This reduction of the control space E^m to E^1,with the preservation of the controllability is possible except for some rather exceptional cases.

The single input control approach is mostly used in state feedback design in pole assignment in multivariable systems and is referred to as "dyadic" or "unity rank" feedback design.This involves determining an (m x n) state feedback matrix K which is constrained to having unity rank by predefining its structure in the dyadic form $K = qp^T$,where q is an (m x 1) vector specified by the designer,while p^T is a (1 x n) vector of unkhown feedback gains,determined so as to assign specified values to the closed loop eigenvalues of the single input system :

$$\dot{x} = A\,x + (Bq)\,h \qquad (3\text{-}21)$$

with the state feedback control law $h = p^T x$. The restriction of the feedback matrix to having unity rank thus reduces the state feedback design of a multivariable system (A,B) to that of much simpler "equivalent" single input system (A,(Bq)), for which p^T can be readily calculated using the results of the previous section or using any other possible method derived only for single input systems. The state feedback matrix for the multivariable system is then given by $K = qp^T$. The desired closed loop poles are obtained for the multivariable system by an appropriate choice of p^T. since the closed loop system matrices of the equivalent single input system and the multivariable system are the same.

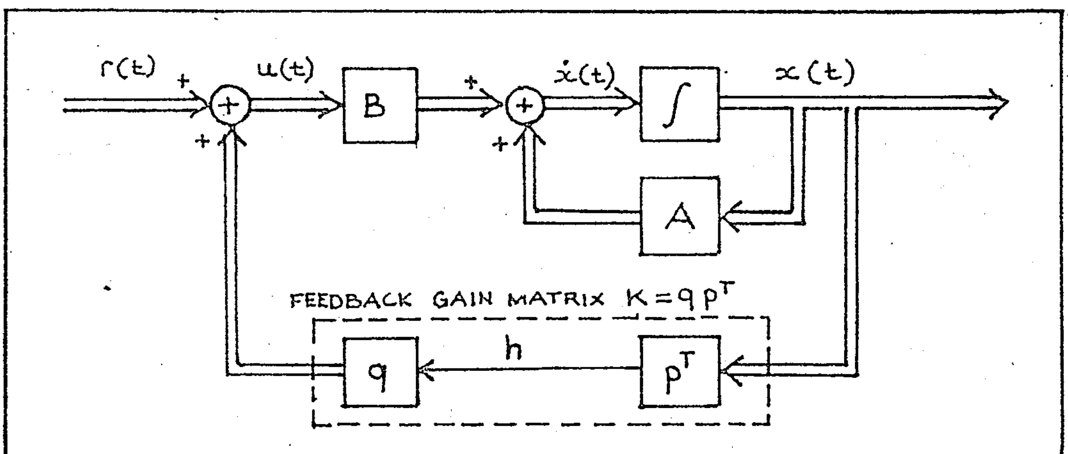


Fig. 3-1 State Feedback Control Structure Using Dyadic Feedback Matrix

The unity rank state feedback matrix K can be determined from the above procedure if and only if the vector q is chosen such that the equivalent single input system (A,Bq) is controllable. This requires the controllability matrix of this system,

$$\mathcal{C}_x^* = \left[(Bq), A(Bq), \ \ldots \ , A^{n-1}(Bq) \right] \qquad (3\text{-}22)$$

to be of full rank n. On comparing expressions (3-19) and (3-20), it is clear that the necessary and sufficient condition for the existence of an appropriate q is that the multivariable system (A,B) is cyclic with respect to the system matrix A. Wonham [2] has shown that if the multivariable system is controllable and A is cyclic, then there exists an (m x 1) vector $b \in \{B\}$ such that (A,b) is controllable, where $\{B\}$ denotes the subspace of E^n spanned by the columns $b_1, \ldots, b_m$ of B. In other words there exists real numbers $q_1, \ldots, q_m$ such that $b = q_1 b_1 + \ldots + q_m b_m = Bq$ forms a controllable pair (A,Bq). In fact Gopinath [13] has pointed out that when (A,B) is controllable and A is cyclic, a randomly chosen q will ensure that (A,Bq) is controllable with probability 1. Once the required transformation from a multivariable system to the equivalent single input system by a suitably chosen q, then the feedback gain vector p^T is determined :

$$p^T = - e_n^T \, (\mathcal{C}_x^*)^{-1} \Delta(A) \qquad (3\text{-}23)$$

and $K = qp^T$, where q is chosen by the designer.

Most multivariable systems are cyclic because the condition of not being cyclic is caused by having two identical subsystems embedded in one system and yet completely decoupled from one another [13]. Hence it is a singular situation. Gopinath has pointed out that only cyclic systems need to be considered in connection with control system design, since, when a system is not cyclic, then a slight amount of feedback will make the system cyclic. More precisely it has been shown [2] that if the system (A,B) is controllable but not cyclic, then for any non-zero vector q, there exists a state feedback matrix K_1, such that the single input system $(A + BK_1, Bq)$ is cyclic and controllable. In fact the controllability of the system (A,B) implies that the closed loop system matrix $(A + BK_1)$ resulting from a randomly chosen feedback matrix K_1 will be cyclic with probability 1.

Consequently, pole assignment in a non-cyclic system is carried out in two stages. In the first stage, the non-cyclicity is removed by applying an arbitrary state feedback matrix K_1. It is noted that K_1 is subject only to the requirement that $(A + BK_1)$ be cyclic, for instance by ensuring it to have distinct eigenvalues (see section 3-3.3). Since controllability is invariant under state feedback, the modified system $(A + BK_1, B)$ is controllable and the feedback gain vector p^T, required for pole assignment, is calculated. The unity rank state feedback for the modified multivariable system is given by $K_2 = qp^T$, where p^T is given by (3-23). The total state feedback matrix for the original system is then,

$$K = K_1 + K_2 = K_1 + qp^T =$$
$$= K_1 - qe_n^T(\mathcal{C}_x^*)^{-1}\Delta(A + BK_1) \qquad (3\text{-}24)$$

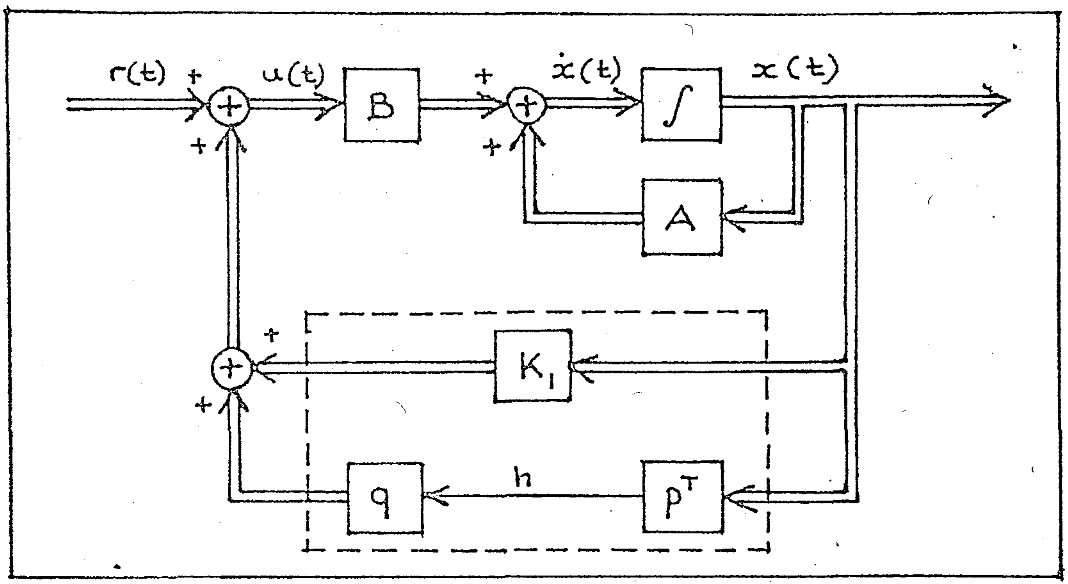


Fig. 3-2 Modified State Feedback Control Scheme for an Initially Non-cyclic System

3-3.3 Criterion for Cyclicity

For an originally non-cyclic system it is important to know how to choose the feedback matrix K_1, such that the closed loop system $(A + BK_1, B)$ will be cyclic with respect to the matrix operator $(A + BK_1)$. Here in this section two efficient criteria to determine cyclicity of the system are presented :

THEOREM (3-2) [11] A space E^n is cyclic with respect to matrix operator A if and only if, its dimension is equal to the degree of its minimal polynomial.

THEOREM (3-3) [12] A sufficient condition for the cyclicity of E^n with respect to A, is that the eigenvalues of A are all distinct.

Though these tests to determine the cyclicity of E^n with respect to A will remove some of the advantages of Ackermann's procedure, it should be noted that most systems are already cyclic and even if they are not, an appropriate feedback matrix K_1 can be easily found which will realize a cyclic system $(A + BK_1, B)$.

EXAMPLE (3-2) Consider the following multivariable system :

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 3 & 2 \\ -1 & -2 \end{bmatrix} u$$

we want to design a feedback control law, such that the closed loop poles are at $\lambda_1 = -2$, $\lambda_2 = -3$.

. Step 1 The original system pair (A,B) is controllable, since

$$\text{rank}\,\mathcal{C}_x = \text{rank}\left[\begin{array}{cc|cc} 3 & 2 & 3 & 2 \\ -1 & -2 & -1 & -2 \end{array}\right] = 2$$

- Step 2 $\varphi(\lambda)$ = minimal polynomial of $A = (\lambda - 1)$

  $\Delta(\lambda)$ = characteristic poly. of $A = (\lambda - 1)^2$

  Hence degree of $\varphi(\lambda) = 1 < 2$ = dimension of E^2.

  $\therefore$ E^2 is not cyclic with respect to A

- Step 3 Let $K_1 = -I$

  then

$$A + BK_1 = \begin{bmatrix} -2 & -2 \\ 1 & 3 \end{bmatrix}$$

  E^2 is now cyclic with respect to $(A + BK_1)$, since the eigenvalues of $(A + BK_1)$ are at $\lambda_1 = (1 - \sqrt{17})/2$ and $\lambda_2 = (1 - \sqrt{17})/2$, i.e. they are distinct.

- Step 4 Let $q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\text{rank}(\mathcal{C}_x^*) = \text{rank}(Bq, ABq) = 2$

  $\therefore (A, Bq)$ is controllable.

- Step 5 $p^T = -e_n^T\,(\mathcal{C}_x^*)^{-1}\;(A + BK_1) =$

  $p^T = (1 \quad 4)$

  The overall closed loop system matrix, becomes

$$A + BK_1 + (Bq)p^T = A + B(K_1 + K_2) = \begin{bmatrix} 0 & 6 \\ -1 & -5 \end{bmatrix}$$

  which has -2 and -3 as its eigenvalues.

CHAPTER 4

MODEL FOLLOWING CONTROL SCHEMES

4-1 Pole Assignment and Explicit Model Following

When we first started this study on pole assignment in multivariable linear systems,our first approach to the problem was simply to build a model reference (MR) system,which basically has the same dynamics as the plant,with the only exception of poles,which are in the desired locations in the complex plane.Using the well-established tools of optimal control theory, with the aim of minimizing the error between the MR system and the plant,we hoped that,once the error between these two systems will be approaching zero,so will the poles of the plant approach the desired pole locations in the complex plane.

In this chapter the significant results of explicit model following control scheme will be summarized (which is nothing but a control scheme based upon the idea as explained in the above paragraph).Then,the effects of the explicit model following type control procedure on the pole locations of the plant will be explained and clarified.The plant and MR system equations can be described by [14],[15]

$$\dot{x}_p = A_p\, x_p + B_p\, u_p \qquad (4\text{-}1)$$
$$\dot{x}_m = A_m\, x_m + B_m\, u_m$$

where the subscript p denotes the plant and m denotes the MR system. x_p and x_m are the n-dimensional column state vectors of the plant and the MR system, respectively. A_p and A_m are square matrices of order n, and B_p and B_m are (n x m) matrices of full rank $m = n$. We also assume that the state variables of the plant and those of the MR system are accessible. Since we are constructing the MR system in such a way, where the MR system and the plant dynamics only differ in their respective system matrices, namely A_p and A_m, it is expected to choose $B_p = B_m$. Therefore we will only use the symbol B to refer to both the plant and MR system input matrices. Another point which must obviously be clarified is the input vectors u_m and u_p. The MR system's input vector u_m is only a reference input to the system, where u_p consists of state feedback terms, feedforward terms and also a reference input, which is the same as the reference input to the model. Hence,

$$\begin{aligned} u_m &= r(t) \\ u_p &= K_p x_p + K_m x_m + r(t) \end{aligned} \qquad (4\text{-}2)$$

where r(t) is a m-dimensional reference input vector. Now we are trying to determine the feedback gain matrix K_p and feedforward gain matrix K_m. Obviously K_p is of more importance for pole assignment, since $K_p x_p$ is the term which can alter the closed loop plant's pole configuration. With the above considerations, a block diagram of the explicit model following control scheme can be drawn as :

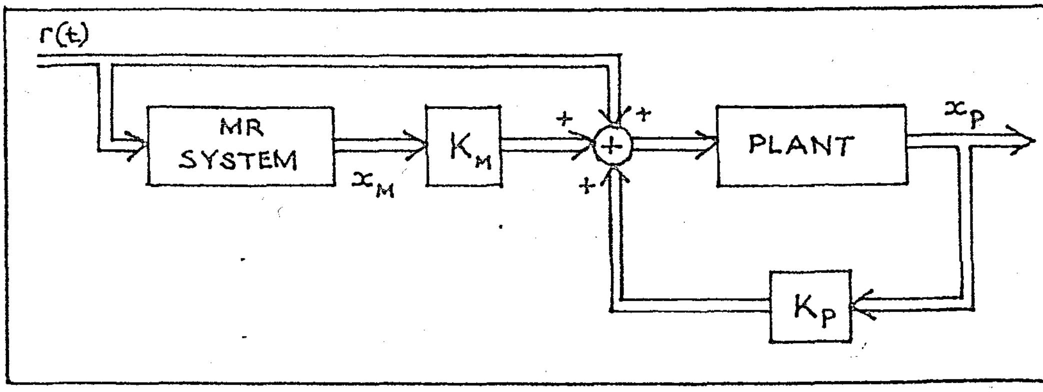


Fig. 4-1

If we define the augmented state vector x^a, which is of dimension 2n, as follows :

$$x^a = \left[\begin{array}{c} x_m \\ \hline x_p \end{array}\right]$$

then the augmented state equations can be written :

$$\dot{x}^a = \left[\begin{array}{c|c} A_m & 0 \\ \hline 0 & A_p \end{array}\right]\left[\begin{array}{c} x_m \\ \hline x_p \end{array}\right] + \left[\begin{array}{c|c} B & 0 \\ \hline 0 & B \end{array}\right]\left[\begin{array}{c} u_m \\ \hline u_p \end{array}\right] \qquad (4\text{-}3)$$

Let us also define the output vector of the augmented system,

$$y = \left[\begin{array}{c|c} I & -I \end{array}\right]\left[\begin{array}{c} x_m \\ \hline x_p \end{array}\right] \qquad (4\text{-}3a)$$

which is obviously nothing but the error vector between the plant and the model. The performance index below minimizes the error between the plant and the model, and also penalizes excess amount of control effort.

$$J = 1/2 \int_0^T (y^T Q y + (u^a)^T R^a (u^a)) \, dt$$

where $u^a = [u_m^T \mid u_p^T]^T$, Q is defined as a symmetric, possibly diagonal, and positive definite matrix of order 2n. R^a is also a symmetric, positive definite 2m-dimensional matrix of the form,

$$R^a = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}$$

For a better understanding of the problem, let us first choose the reference input to both systems, namely r(t), to be equal to zero. Once the results for this case are obtained, they can very easily be extended to include the more general case, when r(t) is an arbitrary vector function of time.

For the case r(t) = 0, the augmented state equations given by (4-3), are reduced to the form :

$$\begin{bmatrix} \dot{x}_m \\ \dot{x}_p \end{bmatrix} = \begin{bmatrix} A_m & 0 \\ 0 & A_p \end{bmatrix} \begin{bmatrix} x_m \\ x_p \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u_p \qquad (4\text{-}3b)$$

$$y = [\, I \mid -I \,] \begin{bmatrix} x_m \\ x_p \end{bmatrix} = x_m - x_p = e$$

The performance index which will minimize the error signal between the MR system and the plant, and also penalize excess amount of control will be in the well known quadratic structure given by :

$$J = 1/2 \int_0^T (y^T Q\, y + u_p^T R\, u_p)\, dt \qquad (4\text{-}4)$$

where Q is a symmetric positive definite matrix and R is an (m x m) symmetric positive definite control weighting matrix.If equation (4-4) is written in terms of the augmented state vector,it becomes

$$J = 1/2 \int_0^T ((x^a)^T Q^a (x^a) + u_p^T R\, u_p) dt \qquad (4\text{-}4a)$$

where

$$Q^a = \left[\begin{array}{c|c} Q & -Q \\ \hline -Q & Q \end{array}\right]$$

The control input u_p which minimizes the performance index (4-4a) under the constraint of the augmented system equations,is given by,

$$u_p^o = - R^{-1}(B^a)^T P(t)\; x^a \qquad (4\text{-}5)$$

where P(t) is the time varying solution of the matrix Ricatti equation,

$$(A^a)^T P + PA^a - PB^a R^{-1}(B^a)^T P + Q = \dot{P} \qquad (4\text{-}6)$$

Since we are only interested in the effect of minimum output error on the pole locations of the plant,only the steady state solution of the above matrix Ricatti equation will be used.To be able to determine the effects of the state feedback on the plant dynamics let us partition P matrix appropriately. In terms of the partioned P,(4-6) becomes :

$$\begin{bmatrix} A_m^T & 0 \\ 0 & A_p^T \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} A_m & 0 \\ 0 & A_p \end{bmatrix} -$$

$$- \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 \\ B \end{bmatrix} R^{-1}\begin{bmatrix} 0 & B^T \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} +$$

$$+ \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix} = 0 \qquad (4\text{-}7)$$

$$A_m^T P_{11} + P_{11}A_m - P_{12}BR^{-1}B^T P_{12} + Q = 0 \qquad (4\text{-}7a)$$

$$A_m^T P_{12} + P_{12}A_p - P_{12}BR^{-1}B^T P_{22} + Q = 0 \qquad (4\text{-}7b)$$

$$A_p^T P_{12} + P_{12}A_m - P_{22}BR^{-1}B^T P_{22} + Q = 0 \qquad (4\text{-}7c)$$

$$A_p^T P_{22} + P_{22}A_p - P_{22}BR^{-1}B^T P_{22} + Q = 0 \qquad (4\text{-}7d)$$

If the above set of matrix equations is solved for P and substituted into (4-3b), the closed loop augmented state equations are obtained as :

$$\dot{x}^a = \begin{bmatrix} \dot{x}_m \\ \dot{x}_p \end{bmatrix} \begin{bmatrix} A_m & 0 \\ -BR^{-1}B^T P_{12} & A_p - BR^{-1}B^T P_{22} \end{bmatrix}\begin{bmatrix} x_m \\ x_p \end{bmatrix} (4\text{-}8)$$

From the above equation it is clear that there is a feedback and also a feedforward term in the plant input vector, which will alter the plant dynamics. Hence the closed loop plant dynamics are :

$$\dot{x}_p = (A_p - BR^{-1}B^T P_{22})\, x_p + (-BR^{-1}B^T P_{12})\, x_m \quad (4\text{-}9)$$

The important point here is that the P_{22} matrix which is expected to alter the plant dynamics is independent of the MR system.This fact can be very easily seen by considering the matrix equation (4-7d).The equation through which P_{22} can be obtained,is repeated here for convenience :

$$A_p^T P_{22} + P_{22} A_p - P_{22} B R^{-1} B^T P_{22} + Q = 0 \quad (4\text{-}10)$$

Let us stop here at this point for a while,and be only concerned with the usual regulator problem,i.e. we want to generate a control law which will drive the output of the plant to the origin of the state-space. Once the solution for the regulator problem will be obtained,its similarity with the previous result,namely equation (4-10),will indicate that the explicit model following control scheme is unsuitable for pole assignment purposes.To obtain the solution for the regulator problem,we must modify the performance index of (4-4a) so that only the plant states are incorporated within the performance index :

$$J = 1/2 \int_0^T (x_p^T Q\, x_p + u_p^T R\, u_p) dt \qquad (4\text{-}11)$$

The optimum control law which will drive the output of the plant to zero is given by the equation :

$$u_p^o = - R^{-1} B^T P_r\, x_p$$

where P_r is the solution (as before only the steady state solutions are used) of the algebraic matrix Ricatti equation,

$$A_p^T P_r + P_r A_p - P_r B R^{-1} B^T P_r + Q = 0 \qquad (4\text{-}12)$$

Comparing the above equation with (4-10),it can be clearly seen that $P_r = P_{22}$.Hence the closed loop plant matrix in the case of regulator problem,$(A_p - BR^{-1}B^T P_r)$, is exactly the same as it would be obtained through the explicit model following scheme.Hence the effect of the feedback term in (4-9) is simply to alter the plant dynamics,such that the plant output will approach to the origin of the state space at the fastest possible rate determined by the relative weights on the state and input terms in the performance index.Therefore it is not expected that the eigenvalues of the closed loop plant matrix will be shifted to the desired pole locations in the complex plane.The same interpretation of the result can also be deduced from the fact that the feedback gain matrix P_{22} is totally independent of the MR system's dynamics.

In the case when both of the systems are driven by a reference input,the results obtained will be of exactly the same form.The feedback terms determined with and without the model are exactly the same,hence the changes in plant dynamics are again independent of the MR system.

4-2 Implicit Model Following

As it has been shown in the previous section explicit model following control scheme is not suitable for pole assignment problem.At this point one is inevitably lead to think of what happens if the state variables of the model are chosen to be the state variables of the original plant.In other words rather than comparing the outputs of the plant and the model, where their dynamics are totally independent from each other,we might as well formulate the model dynamics as follows :

$$\dot{x}_m = A_m x_p \qquad (4\text{-}13)$$

That is we want to see how the plant states x_p will propagate in time if the eigenvalues of the plant would be in the desired locations.A comparison of $\dot{x}_p$,which are obtained through the plant equation (4-1), and $\dot{x}_m$ obtained by (4-13) leads to a new problem formulation,which is known as the implicit model following algorithm [16],or as the model-in-the performance index algorithm [14].

Let us define the error between the model and the plant as

$$e = \dot{x}_m - \dot{x}_p \qquad (4\text{-}14)$$

Since it is desired to have the plant dynamics approach those of the model, the performance index is set up so that one term will consist of the error between the model and the plant derivatives. The performance index is defined as ,

$$J = 1/2 \int_0^T ((\dot{x}_p - \dot{x}_m)^T Q(\dot{x}_p - \dot{x}_m) + u^T R u)dt \tag{4-15}$$

which equals, once (4-13) is substituted into (4-15), to

$$J = 1/2 \int_0^T \{((A_p - A_m)x_p + Bu)^T Q((A_p - A_m)x_p + Bu) + u^T Ru\} dt \tag{4-16}$$

At this point before proceeding any further towards the derivation of the feedback law which will minimize the above performance index, the attention of the reader must be drawn to an important point. From (4-16) it can be clearly seen that the difference between the plant and the model matrices is directly calculated. Since this difference must be a measure of the difference of the eigenvalues of the matrices, A_p and A_m must be of the same form, i.e. if A_p is in companion form, so must be A_m, or if A_p is in diagonal form, so must be A_m. Otherwise a comparison of eigenvalue locations cannot be concluded from the difference $(A_p - A_m)$. This fact, that A_p and A_m are of the same structure, is as fundamental as the idea of adding and subtracting quantities with the same units. To describe the most general case, assume that A_p has initially no special structure. If the system under consideration, i.e. the pair (A_p, B) is a controllable one-which we assume it is, since it is a necessary condition to position all the poles of the plant-then the plant equations can always be transformed into the controllable companion form via a nonsingular transformation denoted by the matrix operator T. In particular if we choose $z = T\,x_p$, the plant equations will be equal to

$$\dot{z} = (TA_pT^{-1})\,z + (TB)\,u = \hat{A}_p\,z + \hat{B}\,u \cdot$$

such that the plant matrices $\hat{A}_p$ and $\hat{B}$ in the transformed coordinate system have the controllable companion form (see 2.1). From this point on we assume that the plant and also the model matrices are already in the controllable companion form.

The derivation of the control for the minimization of the performance index (4-16) is developed in [14]. Once the performance index is defined as in (4-16), the Hamiltonian function $\mathcal{H}$ is set, into which the system differential equations are included as constraints. For the performance index (4-16) and the plant described by (4-1), is :

$$\mathcal{H} = 1/2\left\{((A_p-A_m)x_p+Bu)^T Q\,((A_p-A_m)x_p+Bu) + u^T R\,u + 2\lambda^T(A_p x_p+Bu)\right\} \qquad (4\text{-}17)$$

To find the optimal control, the partial derivative of (4-17) is taken with respect to u and the result is equated to zero. The control u^o which minimizes (4-17) is given by :

$$u^o = -(B^TQB + R)^{-1}(B^T + B^TQ(A_p - A_m)x_p) \quad (4\text{-}18)$$

Pontryagin's maximum principle may be used to evaluate the necessary conditions for optimizing the performance index. These conditions are,

$$\frac{\partial \mathcal{H}}{\partial x_p} = -\dot{\lambda} = \hat{H}^T \hat{Q} \hat{H} x_p + \hat{A}\lambda \quad (4\text{-}19)$$

$$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{x}_p = \hat{A} x_p + \hat{B} u \quad (4\text{-}20)$$

where the new "hat" matrices are defined as

$$\begin{aligned} \hat{R} &= (B^TQB + R) \\ \hat{A} &= A_p - \hat{R}^{-1}B^TQ(A_p - A_m) \\ \hat{B} &= B \\ \hat{Q} &= Q - QB\hat{R}^{-1}B^TQ \\ \hat{H} &= A_p - A_m \end{aligned} \quad (4\text{-}21)$$

Equations (4-19) and (4-20) with the "hat" matrices are in the same form as the canonical equations of the regulator problem. These equations may be solved for the adjoint vector λ and the control of (4-16). The adjoint vector λ depends on the solution of the Matrix Ricatti equation for P(t) and is expressed as

$$\lambda = P(t)\ x_p \tag{4-22}$$

The Ricatti equation for the "hat" matrices has the form,

$$\dot{P} = P\hat{A} + \hat{A}^T P - P\hat{B}\hat{R}^{-1}\hat{B}^T P + \hat{H}^T Q \hat{H} \tag{4-23}$$

The control matrix resulting from the Ricatti equation is then,

$$K_R = \hat{R}^{-1}\hat{B}^T P(t) \tag{4-24}$$

The control from the product of matrices is :

$$K_M = \hat{R}^{-1} B^T Q(A_p - A_m) \tag{4-25}$$

The total control of (4-18) may be expressed as :

$$u^o = -(K_R(t) + K_M)\ x_p = -K(t)\ x_p \tag{4-26}$$

Since it is desirable to have constant feedback gains only the steady state value of the Ricatti equation will be used. This eliminates the need to define the origin of time and program time varying gains. Constant gains are obtained by letting the upper limit of integration in the performance index approach infinity and solving the algebraic rather than the differential Ricatti equation. This algebraic solution is denoted by omitting the indication of time variation on P(t) and K(t) matrices. Besides the simplicity of implementing the control law of (4-26) using only the steady state solution of (4-23), one is forced to let the upper limit of integration in the performance index to approach infinity, since a continous eigenvalue matching between the plant and the model is required. The simplicity of control as well as the way in which this method works may be shown by considering a second order, single input example. A second order system that is to follow the same order model will be considered, The matrices of the plant and the model will have the form,

$$A_p = \begin{bmatrix} 0 & 1 \\ -a_{p1} & -a_{p2} \end{bmatrix}, B = \begin{bmatrix} 0 \\ b \end{bmatrix}, A_m = \begin{bmatrix} 0 & 1 \\ -a_{m1} & -a_{m2} \end{bmatrix} \qquad (4\text{-}27)$$

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}, R = r$$

If these matrices are substituted into the "hat" valued formulas of (4-21)

$$\hat{R} = B^T Q B + R = q_{22} b^2 + r \qquad (4\text{-}28)$$

$$\hat{A} = A_p - B\hat{R}^{-1}B^T Q(A_p - A_m) = \qquad (4\text{-}29)$$

$$= \left[\begin{array}{c|c} 0 & 1 \\ \hline -a_{p1} - \dfrac{b^2(a_{m1}-a_{p1})q_{22}}{q_{22}b^2 + r} & -a_{p2} - \dfrac{b^2(a_{m2}-a_{p2})q_{22}}{q_{22}b^2 + r} \end{array}\right]$$

$$\hat{B} = B = \begin{bmatrix} 0 \\ b \end{bmatrix} \qquad (4\text{-}30)$$

$$\hat{H} = A_p - A_m = \left[\begin{array}{c|c} 0 & 0 \\ \hline a_{m1} - a_{p1} & a_{m2} - a_{p2} \end{array}\right] \qquad (4\text{-}31)$$

and $$\hat{Q} = Q - QB\hat{R}^{-1}B^T Q = \left[\begin{array}{c|c} q_{11} & 0 \\ \hline 0 & q_{22} - \dfrac{b^2 q_{22}}{b^2 q_{22} + r} \end{array}\right] \qquad (4\text{-}32)$$

The control u^o can be found from :

$$u^o = -\left[\hat{R}^{-1}B^TP + \hat{R}^{-1}B^TQ(A_p - A_m) \right] x_p \qquad (4\text{-}33)$$

The first term depends upon the steady state solution of the Ricatti equation, K_R, while the second term, K_M, may be evaluated directly from (4-25) as :

$$K_M = \left[\frac{bq_{22}(a_{m1} - a_{p1})}{b^2q_{22} + r} \;\Big|\; \frac{bq_{22}(a_{m2} - a_{p2})}{b^2q_{22} + r} \right] \qquad (4\text{-}34)$$

For this single input case, the steady state Ricatti equation may be evaluated by letting $\dot{P} = 0$ in (4-23) and solving the individual algebraic equations. If these equations are evaluated for P_{21} and P_{22} the results are,

$$P_{21} = -\frac{\hat{a}_1}{\hat{b}^2}\hat{r} + \sqrt{\left(\frac{\hat{a}_1\hat{r}}{\hat{b}^2}\right)^2 + \frac{\hat{r}}{\hat{b}^2}\hat{q}_{22}(a_{m1} - a_{p1})^2} \qquad (4\text{-}35)$$

$$P_{22} = -\frac{\hat{a}_2}{\hat{b}^2}\hat{r} + \sqrt{\left(\frac{\hat{a}_2\hat{r}}{\hat{b}^2}\right)^2 + \frac{\hat{r}}{\hat{b}^2}\hat{q}_{22}(a_{m2} - a_{p2})^2} \qquad (4\text{-}36)$$

Since P is a symmetric matrix $P_{12} = P_{21}$, therefore only one of these equations must be solved. Also P_{11} need not be found, as it has no effect on control. To observe the effect of the control found in this example assume that the weight put on the error term is much greater than the weight put on the control, i.e. $q_{22} \gg r$. With this assumption we can assume that,

$$\hat{R} = b^2q_{22} + r \cong b^2q_{22} \qquad (4\text{-}37)$$

This effect in $\hat{R}$ affects the other matrices as follows,

$$\hat{A} \cong \begin{bmatrix} 0 & 1 \\ -a_{m1} & -a_{m2} \end{bmatrix} \qquad (4\text{-}38)$$

$$\hat{Q} \cong \begin{bmatrix} q_{11} & 0 \\ 0 & 0 \end{bmatrix} \qquad (4\text{-}39)$$

$$K_M \cong \left[\frac{(a_{m1} - a_{p1})}{b} \;\Big|\; \frac{(a_{m2} - a_{p2})}{b} \right] \qquad (4\text{-}40)$$

Equation (4-38) shows that the equivalent system matrix becomes the model and the q_{22} term which is the only Q element that has any effect, becomes zero. Equations (4-35) and (4-36) show that if $\hat{q}_{22}=0$, P_{21} and P_{22} are also zero. Thus as the weight on the error term becomes greater, the control terms from the Ricatti equation approach zero. For large q_{22}/r, then, the total control is simply the K_M of (4-40). This effect has been found to be true for multivariable systems as well, and may be verified analytically by assuming R of (4-16) to be the null matrix. The result of evaluating the control for this performance index is that the term $K_R(t)$ in (4-26) will be zero. The way in which the model is matched may be seen by evaluating the closed loop system matrix as :

$$A_p - BK_M = \begin{bmatrix} 0 & 1 \\ -a_{p1} & -a_{p2} \end{bmatrix} - \begin{bmatrix} 0 \\ b \end{bmatrix} \left[\frac{bq_{22}(a_{m1} - a_{p1})}{b^2 q_{22} + r} \;\Big|\; \frac{bq_{22}(a_{m2} - a_{p2})}{b^2 q_{22} + r} \right]$$

which for $b^2 q_{22} \gg r$ reduces to :

$$A_p - BK_M = \begin{bmatrix} 0 & 1 \\ -a_{m1} & -a_{m2} \end{bmatrix}$$

As it has been clearly illustrated in this example an exact model matching, which in turn implies eigenvalue matching, is possible if the relative weights assigned on the error terms, i.e. q_{ii}, $i = 1, \ldots, n$, are much greater than the weights placed on the control, r_{ii}, $i = 1, \ldots, m$. If for some reason or another one wishes the poles of the closed loop plant to be exactly equal to the poles of the model, then there is no other choice but setting R in the performance index to be equal to the null matrix. On the other hand if it is acceptable that the poles of the closed loop system are within a reasonable neighbourhood of the model system's poles, then additional freedom is introduced in the controller design, which can be used to limit the feedback gains, by introducing an appropriate non-zero R matrix into the performance index. Hence choosing the correct Q/R ratio depends simply on the particular design and on the required system performance.

4-3 The Relationship Between Implicit Model Following and Eigenvalue Assignment

To show that implicit model following control scheme is equivalent to arbitrary pole assignment we will show that the equations governing the implicit model following will yield a closed loop plant equation which exactly equals to the system equations of the model. However an exact model matching is only possible if the error weighting matrix R in the performance index (4-16) is set equal to zero (this fact has been discussed in the previous section 4-2 further in detail). With the above modification in the performance index, (4-16) is reduced to the form :

$$J = 1/2 \int_0^T \left\{ ((A_p - A_m)x_p + Bu)^T Q((A_p - A_m)x_p + Bu) \right\} dt \qquad (4\text{-}41)$$

The optimum control u^o minimizing the above performance index is found to be,

$$u^o = -(B^TQB)^{-1}B^TQ(A_p - A_m)\, x_p \qquad (4\text{-}42)$$

Hence the control input is only dependent on the difference between the plant's and the model's system matrices. Note that B^TQB is a positive definite matrix, since Q has initially been assumed to be positive definite. and possibly a diagonal matrix.

In general, that is with the system (A_p, B) in an arbitrary unstructured form, it is not possible to prove the equivalency between the implicit model following and eigenvalue assignment. However it has already been explained in the section 4-2, that for the term $(A_p - A_m)$ to be a measure of the difference of the eigenvalue locations, A_p and A_m must be of the same form.

Therefore it is assumed that the plant and the model reference system's equations have been initially transformed into the controllable companion form given in the second chapter. That is, even if the symbols $\hat{A}_p$, $\hat{A}_m$ and $\hat{B}$ refering to the controllable companion form, are not used, it is implicitly meant that the derivation of the control law in section 4-2 and section 4-3, or any mathematical operation on the system equations are carried out in the transformed coordinate system. To be more precise, it is known that A_p and B have the structure given by the equations (2-8b) and (2-8c), respectively, with only the ordered σ_k, k=1, ..., m rows of A_p are important in obtaining information with regard to the pole locations of the open loop plant. The same can also be said of B, since only these same ordered σ_k rows of B are non-zero, where the numbers σ_k, k=1, ..., m are as defined by the equation (2-6).

The main idea in the proof of equivalency between the implicit model following and eigenvalue assignment is to show that the closed loop plant equation,

$$\dot{x}_p = (A_p - B(B^TQB)^{-1}B^TQ(A_p - A_m))\, x_p \qquad (4\text{-}43)$$

can considerably be simplified using the special canonical structure of A_p and B matrices. The matrix B^T has exactly n-m columns with all the entries on these columns being equal to zero. Therefore in the multiplication of B^T with Q these columns and the corresponding rows of Q will not have any effect on the resulting matrix, since their product will be identically equal to zero. By deleting these trivial columns of B^T, the reduced transposed input matrix $\hat{B}^T_{cm}$, which becomes a square matrix of order m, is obtained. Similarly the corresponding rows of Q must be deleted, so that it is now reduced down to a (m x n) matrix. In the same way the zero rows of B do not have any contribution to the product of the first two matrices. Therefore without any loss in information the n-m zero rows of B and the columns of Q corresponding to the deleted rows of B can be eliminated. This reductions in B and Q matrices can be done, since the only rows of the plant and the model's system matrices required in the computation of the state feedback term are the ordered $\sigma_k, k=1,\ldots,m$ rows of A_p and A_m. Let $(M)_i$ denote the i-th row of the matrix M, then using the considerations as explained in detail above, the following equality can easily be established :

$$(A_p - B(B^TQB)^{-1}B^TQ(A_p - A_m))_i =$$
$$= (A_p)_i - (\hat{B}_{cm}(\hat{B}^T_{cm}Q_r\hat{B}_{cm})^{-1}\hat{B}^T_{cm}Q_r(A_p - A_m))_i$$
$$i = \sigma_1, \ldots, \sigma_m \qquad (4\text{-}43)$$

where $\hat{B}_{cm}$ is the (m x m) matrix consisting of the nontrivial (non-zero) rows of B (see eq'n (2-8c)), and Q_r being an (m x m) matrix obtained from Q by deleting the rows and columns corresponding to the zero columns of B^T and the zero rows of B. Due to its special structure $\hat{B}_{cm}$ is nonsingular. Furthermore the nonsingularity of Q is preserved in Q_r, since the same rows and columns have been deleted in the process of obtaining Q_r. Therefore using the property $(CD)^{-1} = D^{-1}C^{-1}$, equation (4-43) can be reformulated as :

$$(A_p)_i - \hat{B}_{cm}\hat{B}_{cm}^{-1}Q_r^{-1}(\hat{B}_{cm}^T)^{-1}\hat{B}_{cm}^T Q_r(A_p - A_m)_i = (A_m)_i$$

$$i = \sigma_1, \ldots, \sigma_m \qquad (4\text{-}44)$$

Hence the ordered σ_k, $k=1,\ldots,m$ rows of the closed loop plant matrix have been exactly matched with that of the model. In other words the entries on these ordered σ_k rows which are actually the coefficients of the characteristic equations of different subblocks of the new closed loop plant matrix, have been replaced by the coefficients of model characteristic equation, in the case when the input weighting matrix R in the performance index is set equal to zero.

Note that this result is independent of the specific choice of Q. If Q is chosen as the identity matrix I, then equation (4-42) can be actually simplified to the form of the equations (2-19) and (2-18).

There are some other works [16], [17], [18], published in this field which prove the equivalency between the implicit model following and pole assignment. However their work only cover a certain class of pole assignment problems, such as all the eigenvalues of the model are restricted to be distinct, and there are some other limitations on the chosen model equations. This approach followed in this section cover all possible pole assignment problems without any restrictions.

CHAPTER 5

POLE ASSIGNMENT in DISCRETE TIME and STOCHASTIC SYSTEMS

5-1 A Recursive Pole Assignment Algorithm for Discrete Time Systems

In some cases determining the feedback gain matrix K necessary for implicit model following and therefore for eigenvalue placement is rather cumbersome, since in higher order systems solutions of the Matrix Ricatti equation cannot be easily obtained. For discrete time system a recursive algorithm, to evaluate the K matrices can be set up, so that arbitrary pole assignment can be realized. In the algorithm introduced in [21] slight modifications in the problem formulation must be introduced so that the desired results can be obtained.

The plant and the model equations are basically the same as in the previous chapter, with the only difference being the notations F_p, F_m and G, which are used to denote the plant matrix, model matrix and the input matrix, respectively. In particular the plant and model equations are

$$\begin{aligned} \text{PLANT:} \quad & x_p(k+1) = F_p\, x_p(k) + G(u(k) + r(k)) \\ \text{MODEL:} \quad & x_m(k+1) = F_m\, x_p(k) + G\, r(k) \end{aligned} \qquad (5\text{-}1)$$

where, F_p :open loop plant system matrix of order n,

F_m :model system matrix of order n, whose eigenvalues are the desired pole locations in the complex plane.

NOTE : F_p and F_m must be of the same structure.

G :(n x m) input matrix of both systems, with $\text{rank}(G) = m \leq n$.

u(k):(m x 1) control input to the plant.

r(k):(m x 1) reference input to both systems.

We also assume that all the states of the plant and the model are avaiable at their outputs.

The error e between the plant and the model system outputs at the (k+1)-st sampling period is defined as :

$$\begin{aligned} e(k+1) &= x_p(k+1) - x_m(k+1) = \\ &= (F_p - F_m)x_p(k) + G\,u(k) \end{aligned} \tag{5-2}$$

The performance index, which is a measure of the error, is given for a N-stage optimization problem as follows,

$$J = \min_{u(N-1)} \cdots \min_{u(0)} \left\{ \sum_{i=1}^{N} (e^T(i)Q(i)e(i) + u^T(i-1)R(i-1)u(i-1) \right\} \tag{5-3}$$

where Q(i) and R(i-1) are the error and control weighting matrices of appropriate dimensions. Rather than keeping the weighting matrices constant they can as well be chosen as time varying matrices, such that an additional flexibility is introduced in the control system design. The recursive equations for the control vector u are nothing but a special case of the algorithm derived in [21] for discrete time optimal control problem. The results can be easily summarized in the following theorem :

<u>THEOREM (5-1)</u> The optimal control law for the deterministic system described in (5-1) which minimizes the performance index (5-3) is :

$$u(k) = S(k)\, x_p(k) \qquad (5\text{-}4)$$

where the (m x n) feedback control matrix S(k) is to be determined recursively from the set of relations:

$$W(k+1) = M(k+1) + Q(k+1) \qquad (5\text{-}5)$$

$$S(k) = -\left[G^T W(k+1) G + R(k)\right]^{-1} G^T W(k+1)(F_p - F_m) \qquad (5\text{-}6)$$

$$M(k) = (F_p - F_m)^T W(k+1)\left[(F_p - F_m) + GS(k)\right] \qquad (5\text{-}7)$$

for k= N-1, N-2, ..., 0, where W(N)=Q(N) and the (m x m) matrix $(G^T W(k+1) G + R(k))$ is required to be positive definite for all k. Furthermore the minimum value of the performance measure for N-k stages of control is

$$V_{N-k} = x_p^T(k) M(k) x_p(k) \qquad (5\text{-}8)$$

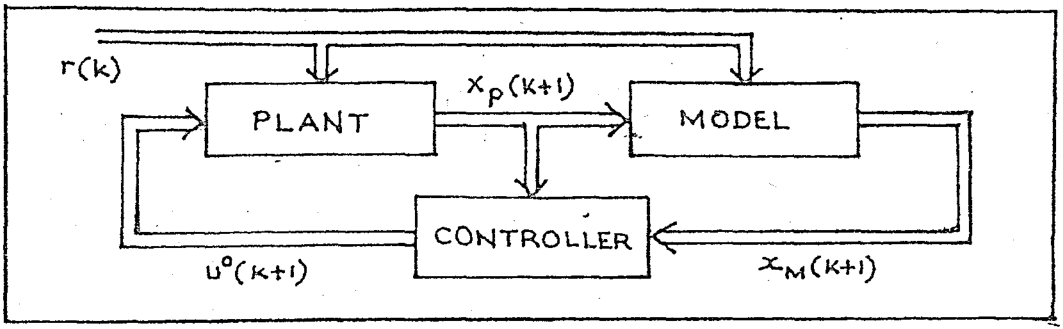


Fig.5-1 Block Diagram of the Control System of Th'm (5-1)

For this N-stage optimal control problem first N feedback control matrices S(k),k=(N-1),(N-2),...,0 must be computed recursively backwards in time and then stored for actual use.Once the plant is run forward in time,these stored feedback gain matrix element values are used one by one at the appropriate time instants.

If in the performance index (5-3) excess amount of input is not penalized,the sequence of control weighting matrices R(0),...,R(N-1) must be all set identically equal to zero matrix.In this case only a single stage control will give us the required pole configuration for the closed loop system.As a further remark we also note that,in this special case any other algorithm to evaluate the feedback gain matrix can be equally well employed-such as the Ackermann's procedure outlined in Chapter 3.

Choosing the correct Q and R matrices is again a problem which can only be answered according to the design criteria of a particular problem. Since usually we wish exact matching between the closed loop plant's and the model poles for all time the plant is under operation, we should either let N-number of optimization stages- approach to infinity or use a steady-state feedback gain after a finite-stage optimization procedure has been applied to the system. If the error and control weighting matrices Q(k) and R(k) are constant for all $k=0,\ldots,(N-1)$, then it has been noted that the eigenvalue configuration of the closed loop plant in the first few steps of a finite-stage optimization procedure is closer to the model's eigenvalues. Then how can we choose a reasonable steady-state feedback gain? If we insist on using the last feedback matrix S(N-1) as the steady-state gain matrix, then certain modifications should be made on the performance index (5-3). Either we can introduce a terminal error term into (5-3), so that it becomes :

$$J^{*} = \min_{u(N-1)} \ldots \min_{u(0)} \left\{ e^T(N)Qe(N) + \sum_{i=1}^{N} \left(e^T(i)Qe(i) + u^T(i-1)Ru(i-1) \right) \right\} \qquad (5\text{-}9)$$

Or, we can choose the error weighting matrices to be time varying with the constraint that $Q(k) > Q(j)$, if $k > j$, It must be again noted that if the control weighting matrix R is chosen, such that it is equal to zero for an exact model matching, then none of the above modifications is necessary since a single-stage optimization will give the correct feedback gain which in turn can be used for the rest of the time,the system is under operation.

5-2 Pole Assignment in Stochastic Case

In all the previous chapters and sections our main concern has primarily been the deterministic type of systems.We have throughly discussed the pole assignment problem in this specific class of systems.A natural extension of deterministic pole assignment problem will certainly be the pole placement in stochastic systems.In this section we want to discuss this problem, if the system under consideration is operating in a noisy environment.

An optimum multivariable control system operating under noisy conditions is generally equipped with an optimum controller and a filter.The controller is used to generate the optimum control law,and the filter is employed to filter out the uncertainties created by the input and output noises and to obtain the best estimates of all the state variables from the measurable outputs of the process.One may immediately ask the question whether the separate optimization and statistical estimation yields a system which is optimal in the over-all sense.The answer to this question is an affirmative one and has been first proved by Kalman and Koepcke [22],and then by [23], [24].This idea of determining the parameters of the filter and the controller separately but still obtaining an optimum system performance in the overall sense is known as the "Principle of Separation".

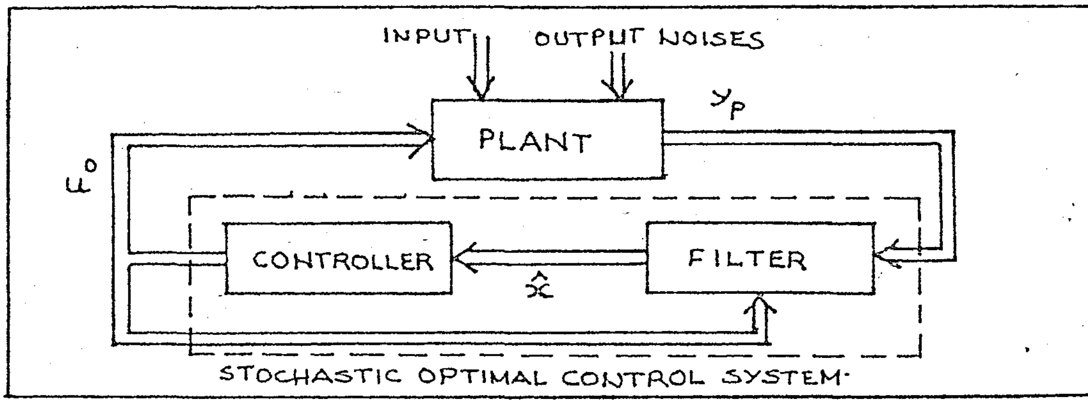


Fig.5-2 Block Diagram Illustrating the Principle of Separation

The most striking feature of the separation principle is that the feedback control gain matrix is independent of all the statistical parameters in the problem, whereas the optimal filter is independent of the matrices in the performance index.

In this section about the stochastic systems we are going to make use of the recursive discrete-time control algorithm described in in the previous section. It is clear from the previous results that the best or optimum control is the one which produces the minimum performance index of (5-3).However, the random variables in the system cause unknowns that preclude the possibility of finding an input which will be optimum in every case. A reasonable course of action is to choose a sequence of inputs that will minimize the expected value of the performance index. Hence equation (5-3) in this modified form becomes :

$$J = \min_{u(N-1)} \cdots \min_{u(0)} E\left\{ \sum_{i=1}^{N} (e^T(i)Q(i)e(i) + u^T(i-1)R(i-1)u(i-1)) \right\} \quad (5\text{-}10)$$

The proof of the separation principle is a constructive one and is based primarily on showing that the feedback gain matrices of the control law which minimize the performance index of the deterministic and stochastic systems are identically the same. In the case of dealing with a stochastic system the only change in the controller equations is the fact that the term $x_p(k)$, which appears in the optimal control law $u^o = S(k)x_p(k)$, must be replaced by $\hat{x}_p(k|k)$, where $\hat{x}_p(k|k)$ is defined as the optimal filtered estimate of the plant state vector at the k-th sampling period, obtained using the k avaiable measurements.

The revised form of Theorem (5-1) will then be as follows :

<u>THEOREM (5-2)</u> For a stochastic system the optimal control law which minimizes the performance index of (5-10) is given by :

$$u^{o} = S(k)\ \hat{x}_p(k|k) \tag{5-11}$$

where $\hat{x}_p(k|k)$ is the optimal filtered estimate of the plant state vector. The (m x n) feedback control matrices $S(k), k = 0,\ldots,(N-1)$ is to be determined recursively from the set of relations (5-5), (5-6) and (5-7).

Now let us illustrate the effects of cascading a filter with a controller, on the closed loop system dynamics. We assume that there are Gaussian white input and output disturbances present. The plant equations of a completely controllable and observable system are:

$$\begin{aligned} x_p(k+1) &= F_p\ x_p(k) + G(u(k) + r(k)) + \Gamma w(k) \\ y_p(k+1) &= C_p\ x_p(k+1) + v(k+1) \end{aligned} \tag{5-12}$$

where $w(k)$ and $v(k+1)$ are the zero mean Gaussian white input and measurement disturbances, respectively. C_p is the (p x n) output matrix and $y_p(k+1)$ is the p-dimensional output vector of the stochastic system given in (5-12). If we use a Kalman-Bucy filter to obtain the state vector estimates $\hat{x}_p$, necessary to implement the control law of (5-11). We can write the filter dynamics making use of the equations derived in [21].

$$\begin{aligned} \hat{x}_p(k+1|k+1) = {} & (F_p - K(k+1)C_pF_p)\hat{x}_p(k|k) + Gu(k) + \\ & + (K(k+1)C_pF_p)\hat{x}_p(k) + (K(k+1)C_p\Gamma)w(k) + \\ & + K(k+1)v(k+1) \end{aligned} \tag{5-13}$$

Using (5-13) and (5-12) we can write the augmented state equations, where the (2n x 1) dimensional augmented state vector x^a is equal to,

$$x^a = \left[\begin{array}{c} x_p(k+1) \\ \hline \hat{x}_p(k+1|k+1) \end{array}\right] = \left[\begin{array}{c|c} F_p & 0 \\ \hline K(k+1)C_pF_p & F_p - K(k+1)C_pF_p \end{array}\right]\left[\begin{array}{c} x_p(k) \\ \hline \hat{x}_p(k|k) \end{array}\right] +$$

$$+ \begin{bmatrix} G \\ G \end{bmatrix} u(k) + \left[\begin{array}{c|c} \Gamma & 0 \\ \hline K(k-1)C_p\Gamma & K(k+1) \end{array}\right]\left[\begin{array}{c} w(k) \\ \hline v(k+1) \end{array}\right] \qquad (5\text{-}14)$$

$$\left[\begin{array}{c} y_p(k+1) \\ \hline z_{filter}(k+1) \end{array}\right] = \left[\begin{array}{c|c} C_p & 0 \\ \hline 0 & I \end{array}\right]\left[\begin{array}{c} x_p(k+1) \\ \hline \hat{x}_p(k+1|k+1) \end{array}\right] + \left[\begin{array}{c} I \\ \hline 0 \end{array}\right] v(k+1) \qquad (5\text{-}15)$$

At this point using the fact that $u^o = S(k)\hat{x}_p(k\ k)$ and the equivalence transformation,

$T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = T^{-1}$, we can transform the equation (5-14) and (5-15) into the form,

$$\left[\begin{array}{c} x_p(k+1) \\ \hline x_p(k+1) - \hat{x}_p(k+1\ k+1) \end{array}\right] = \left[\begin{array}{c} x_p(k+1) \\ \hline \tilde{x}(k+1|k+1) \end{array}\right] =$$

$$= \left[\begin{array}{c|c} F_p + GS(k) & GS(k) \\ \hline 0 & F_p - K(k+1)C_pF_p \end{array}\right]\left[\begin{array}{c} x_p(k+1) \\ \hline \hat{x}(k|k) \end{array}\right] +$$

$$+ \left[\begin{array}{c|c} \Gamma & 0 \\ \hline \Gamma - K(k-1)C_p\Gamma & -K(k+1) \end{array}\right]\left[\begin{array}{c} w(k) \\ \hline v(k+1) \end{array}\right] \qquad (5\text{-}16)$$

It is now immediately apparent that the plant dynamics can be altered by a suitable choice of the feedback matrices $S(k), k=0,\ldots,(N-1)$, i.e. according to the recursive algorithm described by (5-5),(5-6) and(5-7). The entire n-dimensional filtering error , $\tilde{x}(k+1|k+1) = x_p(k+1) - \hat{x}_p(k+1|k+1)$ is proved [21] to be a zero-mean Gauss-Markov sequence.Hence with enough time elapsed since the system has been in operation, the filtering error state will approach the origin of state space which in turn will make the closed loop plant equations and closed loop eigenvalues identically equal to that of the model.

EXAMPLE (5-1) To illustrate the use of the control procedure outlined in the previous pages we consider a first order system described by the equations,

$$x_p(k+1) = (2)x_p(k) + (1)u(k) + (1)r(k)$$

The equation of the model is given as :

$$x_m(k+1) = (1/2)x_p(k) + (1)r(k)$$

We have chosen a deterministic system,since uncertainties in the system will only affect the overall response time of the system and not the control sequence.

(a.) Set Q = 10 and R = 0 and let k_o denote the first sampling period when the control is applied,then :

$$S(k_o) = -(G^TQG + R)^{-1}G^TQ(F_p-F_m) = -1.5$$

and $x_p(k_o+1) = (2)x_p(k_o) + (1)(-1.5)x_p(k_o) + (1)r(k_o)$

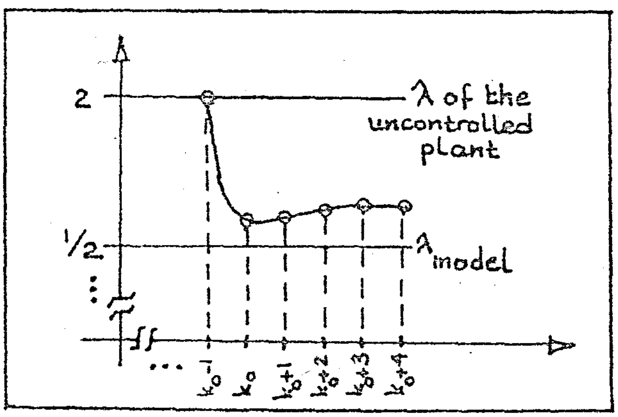
$$= (1/2)x_p(k_o) + (1)r(k_o)$$

It can be seen that only a single-stage procedure is necessary to achieve a perfect match between the poles of the plant and the model. Using $S(k_o)$ as the steady state feedback gain we obtain the desired pole configuration for the rest of the operational time.

(b.) Set Q = 20 and R = 2 and solve for the required feedback gains for a 5-stage optimization procedure:

k	S(k)	λ_{plant}
⋮	⋮	⋮
k_o-1	0	2
k_o	-1.385	0.6146
k_o+1	-1.385	0.6146
k_o+2	-1.386	0.6147
k_o+3	-1.387	0.6149
k_o+4	-1.364	0.6364

$\lambda_{model} = 0.5$



(c.) Let us also illustrate the effect of various Q/R ratios on the best obtainable value when the last feedback gain, $S(k_o+4)$, is taken as the steady-state feedback gain.

Q/R	0	...	2	4	10	20	40	100	200	...	∞
λ_{best}	2	...	1	0.8	0.64	0.57	0.54	0.51	0.51	...	0.5

(d.) This time to obtain a better steady-state performance we modify our performance index as in (5-9)

$$J = \min_{u(k_o)} \cdots \min_{u(k_o+4)} \Big\{ (e^T(k_o+5)Qe(k_o+5) + \\ + \sum_{i=0}^{4} (e^T(k_o+1+i)Qe(k_o+1+i)+u^T(k_o+i)Ru(k_o+i) \Big\}$$

If we again use the last feedback gain $S(k_o+4)$ as the steady state value, we obtain the following table :

Q/R	0	...	2	4	10	20	40	100	200	...	∞
λ_{best}	2	...	0.8	0.67	0.57	0.54	0.518	0.507	0.504	...	0.5

CHAPTER 6

CONCLUSIONS

In this study various approaches to pole assignment problem have been discussed in detail. Unfortunately most of the previously appeared control schemes used in the pole assignment problem lack the flexibility of adopting themselves to different types of pole assignment problems, such as in the case of multiple open loop or multiple closed loop poles. Furthermore most of the avaiable algorithms proceed by first transforming the system equations into a canonical form in the interest of computational tractability.

The most attention deserving part of this study is the generalization of Ackermann's [9] procedure to multivariable systems. A trick is used to generalize Ackermann's procedure to multivariable systems, namely by first transforming the system of interest into an "equivalent" single input system. Ackermann's procedure is extremely convenient to use with multivariable systems, since it requires no explicit transformation of the system equations into a canonical form and it considerably reduces the number of computations required in determining the feedback gain matrix K. As explained in detail in the third chapter, this transformation into an equivalent single input system is established by choosing the feedback matrix K to be of unity rank,namely by setting $K = qp^T$.p^T is determined for a particular closed loop pole configuration,where q is arbitrarily chosen,with the only restriction of preserving the system's controllability characteristics.However additional flexibility can be introduced into the control system design if q can be chosen appropriately.Therefore an interesting point which still deserves special attention is the way in which q must be chosen.If a two-stage control algorithm can be established such that first q is selected to minimize a certain performance index and then p^T is subsequently calculated to obtain the desired closed loop pole configuration.

Another point which is still open for further research is the modification of Ackermann's original procedure such that it will also cover pole assignment through only output feedback.Use of the output controllability matrix to derive a formula similar to Ackermann's original one will be a logical step to start this further research.

The second important result established in the remaining part of this study is the proof of equivalency between implicit model following control scheme and eigenvalue placement.This particular topic has been recently considered by some authors [16] , [17] and they established the same equivalency for only a restricted number of cases,i.e. when the closed loop poles are all distinct and only for a specific model type.However the proof presented in this study makes use of the canonical system equations and is general enough to include all possible cases. Hence it is a big improvement over what has appeared in this field before.

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