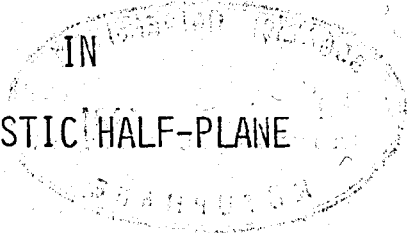# A SYSTEM OF CRACKS IN AN ELASTIC HALF-PLANE

by

M. Oktay ALNIAK

Submitted to the Faculty of Engineering
in Partial Fulfillment of the Requirements
for the Degree of
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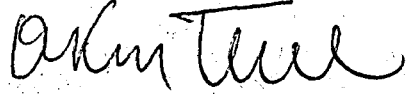
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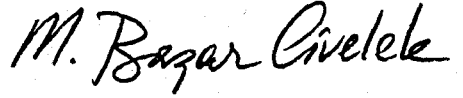
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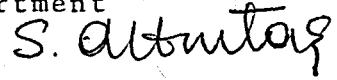Civil Engineering Department
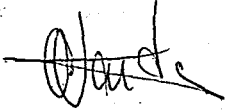(Co-Supervisor)

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M. Oktay ALNIAK

ABSTRACT

The plane elastostatic problem of an isotropic elastic half-plane containing an arbitrary system of cracks is considered. Fourier transform technique is employed to obtain the edge dislocation solutions. Using the edge dislocation solutions as Green's functions, the problem is formulated in terms of a system of singular integral equations. The system of singular integral equations is solved numerically for various crack combinations and external loads. The stress intensity factors and the probably cleavage angles are studied in detail.

ÖZET

Bu çalışmada, içinde gelişi güzel oluşmuş çatlaklar bulunan isotrop, elastik bir yarım düzlemin elastostatik problemi incelenmiştir. Fourier integral transform tekniği kullanılarak kenar dislakosyon çözümleri bulunmuştur. Kenar dislakasyon çözümleri çekirdek fonksiyonları olarak kullanılıp, problem bir dizi tekil integral denkleme indirgenmiş ve bu denklemler değişik çatlak geometrileri ve dış yükler için nümerik olarak çözülmüştür. Gerilme şiddeti faktörleri ve muhtemel ayrılma yüzeyleri detaylı olarak incelenmiştir.

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LIST OF SYMBOLS

Symbol	Definition
a_o	Horizontal distance of the edge dislocation
$(-a, a)$	End points of a single arbitrarily oriented crack
(a, b)	End points of a single radial crack
$(-a_i, a_i)$	End points of arbitrarily oriented cracks
(a_i, b_i)	End points of radial cracks
$\vec{b}_X$	Burgers vector of an edge dislocation parallel to the X axis
$\vec{b}_Y$	Burgers vector of an edge dislocation parallel to the Y axis
$\vec{b}_s, \vec{b}_q$	Burgers vectors in $\{s, q\}$ coordinate system
$k(s, t)$	
$h(s, t)$	
$m(s, t)$	Kernels of the singular integral equations
$n(s, t)$	
$k^*(s, t)$	
$n^*(s, t)$	Bounded kernels
k_1, k_2	Stress intensity factors
k'_1, k'_2	Stress intensity factor ratios

u	Displacement in s direction
v	Displacement in q direction
θ	Angle between the X axis and the axis of the crack
f(s)	
	Dislocation density functions
g(s)	
U(X,Y)	Airy stress function
$\phi(X,\alpha)$	Fourier transform of U(X,Y)
ϕ	Cleavage angle
μ	Shear modulus
ν	Poisson's ratio
κ	$3-4\nu$ for plane strain $\dfrac{3-\nu}{1+\nu}$ for generalized plane stress
{X,Y}	
{x,y}	Cartesian coordinate systems
{s,q}	
$\overset{*}{\sigma}_{XX}, \overset{*}{\sigma}_{YY}, \overset{*}{\sigma}_{XY}$	Stress components due to an edge dislocation in an infinite plane
$\sigma_{XX}^{1}, \sigma_{YY}^{1}, \sigma_{XY}^{1}$	Stress components due to an edge dislocation with $\vec{b}_X$ in a half-plane
$\sigma_{XX}^{2}, \sigma_{YY}^{2}, \sigma_{XY}^{2}$	Stress components due to an edge dislocation with $\vec{b}_Y$ in a half-plane
σ_{qq}, σ_{qs}	Normal and shear stress components in {s,q} coordinate system

CHAPTER 1

INTRODUCTION

One of the fundamental requirements of any enginee-
ring structure is that it should not fail in service.
This necessitates a sufficient knowledge about materials
containing cracks and the behavior of cracked structures.

Crack problems do not only have practical importance
in fracture mechanics but are also of mathematical inte-
rest. Many investigators have treated the problems in-
volving cracks from different points of view. In this
chapter, the practical importance of the problem is
emphasized along with the mathematical approach.

Recent publications [5,14,15] indicate that frac-
ture phenomenon which is one of the possible modes of
mechanical failure is an important problem. The fracture
behavior of materials has been considered in two groups
as ductile and brittle fracture. Ductile fracture is
characterized by appreciable plastic deformation and the
amount of gross deformation which is usually presented

at the fracture surfaces of the body. The limiting case
of flow until the cross section vanishes may be called
rupture. Griffith [11] proposed that fracture in brittle
solids arises from many fine flaws in the surface of the
material. Brittle fracture in metals is characterized by
a rapid rate of crack propagation, with no gross deforma-
tion and very little or no microdeformation. Birttle
fracture occurs without warning and usually causes disas-
trous consequences.

Fracture occurs in characteristic ways, depending
on the state of stress, the rate of application of stress,
and the temperature. In addition, more complex effects
such as torsion, fatigue, creep, low temperature, temper
embrittlement, hydrogen embrittlement and others may cause
fracture.

The most important cause of failure for all real
structures is known to be due to the material imperfec-
tions which are already present inside the body or at the
surface. This kind of imperfections may exist as flaws
or cracks, the largest of which most likely causes the
failure.

The structural strength of materials generally de-
pends on the material properties, the shape and the size
of defects as well as together with the orientation of

flaws in the medium. In the application of fracture mechanics, other kinds of imperfections due to manufacturing are particularly important, such as flat cavities which develop during casting, small cracks resulting from residual stresses in welded materials, and fatigue cracks under external loads.

Thus, in a design dealing with materials, it is necessary to have a good estimate of the disturbed stress state caused by these flaws. This general summary deals with only a minute part of the available investigations in the theory and practice of fracture mechanics from the macroscopic and microscopic points of view. In addition, to these investigations there has been a large number of theoretical and mathematical studies on the same subject.

In the last few decades a considerable progress has been made in the application of linear elastic fracture mechanics by employing plane elasticity theory. Many infinite plane problems containing cracks with arbitrary orientations and locations were solved. Various researchers have also investigated the crack problems in a semi-infinite medium. In general, the half-plane problems have been presented with a boundary which is either free or bounded to a second half-plane.

Among the recent publications dealing with a system
of cracks the paper of Savruk and Datsyshin [17] deals
with the problem on elastic, isotropic infinite medium
containing arbitrarily arranged cuts. They obtained the
solution of crack problems by means of singular integral
equations.

The problem of bonded half-planes containing an
arbitrarily oriented crack was first investigated by
Erdogan and Aksogan [8]. They reduce the problem to a
system of singular integral equation by using the Mellin
transform technique. The stress intensity factors and
the probable crack propagation angles for various crack
orientations were given. Later on Ashbaugh [1] solved
the same problem by employing a different technique.

A similar study was presented by Atkinson [2] who
used stress field for an edge dislocation in bimetallic
medium given by Head [12] to generate integral equations
for the dislocation densitites associated with cracks and
dislocation pile ups. The integral equations were solved
numerically.

The problem of interaction between a circular in-
clusion and an arbitrarily oriented crack has been solved
by Erdogan, Gupta and Ratwani [7]. They used the existing
solutions for the edge dislocations as Green's functions.

The integral equations for the line crack were obtained
as a system of singular integral equations with simple
Cauchy kernels. The problem was solved for various typi-
cal crack orientations and the corresponding stress in-
tensity factors were given. Although there is no direct
similarity between the subject of this thesis and the
paper [7], the singular integral equation formulation,
edge dislocations, Green's functions and the stress inten-
sity factors have the same important theoretical basis.

Among the papers concerning crack problems in an
elastic half-plane the paper by Ioakimidis and Theocaris
[13] presents a general formulation for a system of cur-
vilinear cracks by using complex variable methods. Even
though the formulation is quite general the numerical
examples are given only for single straight or arc-shaped
cracks.

In this study, dislocation solutions are used as
Green's functions to formulate the system of cracks prob-
lem. For this purpose first the stress field of an
elastic half-plane due to a pair of edge dislocations with
Burgers vectors parallel and perpendicular to the free
surface are determined from the infinite plane solution
by using the Fourier transform technique. The stress
field due to an edge dislocation with a Burgers vector
forming an angle with the free surface can be obtained by

resolving the Burgers vector into two components one
parallel and the other being perpendicular to the free
surface and summing the stresses set up by these compo-
nents. To simulate a line crack in a given direction
stress solutions for a pair of edge dislocations with
Burgers vectors in the specified direction and in the di-
rection perpendicular to it can be used as Green's func-
tions. Replacing Burgers vectors by dislocation densities
and integrating these density functions along the crack
yields the stresses created by introducing the crack.
Stress distributions due to several existing cracks can
be superimposed. Expressing that the total stresses on
the crack surfaces are equal to zero one obtains a sys-
tem of singular integral equations with dislocation den-
sities as unknown functions. If n is the number of cracks
then there are $2n$ unknown functions and hence $2n$ singular
integral equations can be expressed. A set of singular
integral equations for a system of radial cracks can be
readily obtained. However, for cracks whose axes do not
intersect on the free surface, a translation of axes is
necessary to be able to express the singular integral
equations. The system of singular integral equations are
solved numerically by using the Gauss-Chebyshev integra-
tion formulas. Numerical results are first obtained for
a single crack problem. It is observed that the results
match all the existing single crack solutions. Then to
characterize the crack interactions two crack problems

are solved by considering radial cracks and cracks per-
pendicular or parallel to the free surface. The effect
of variations of distances and angles between the cracks
on the stress intensity factors and cleavage angles are
investigated. The system of singular integral equations
are solved for the cases of symmetric and anti-symmetric
uniform tractions or uniform tension applied at infinity,
parallel to the free surface. In all the above-mentioned
problems, cracks are located in the vicinity of the free
surface since the solutions for cracks away from the free
surface can be approximated by infinite plane solutions.

CHAPTER 2

FORMULATION OF THE PROBLEM

First the stress field in an isotropic half-plane (which extends to the right of the free surface X=0) due to a pair of edge dislocations at the point ($X=a_0$, $Y=0$) with Burgers vectors $\vec{b}_X$ and $\vec{b}_Y$ will be determined. The desired solution can be constructed by modifying the elastic stress fields of edge dislocations in an infinite plate.

An edge dislocation, of Burgers vector $\vec{b}_X$, parallel to X axis, at the point $(a_0,0)$ of an infinite plane (see Figure 1) has the following stress field [12].

$$\overset{*}{\sigma}_{XX} = - \frac{D_X}{\Pi} \frac{Y[Y^2 + 3(X - a_0)^2]}{[(X - a_0)^2 + Y^2]^2}$$

$$\overset{*}{\sigma}_{YY} = \frac{D_X}{\Pi} \frac{Y[(X - a_0)^2 - Y^2]}{[(X - a_0)^2 + Y^2]^2}$$

$$\overset{*}{\sigma}_{XY} = \frac{D_X}{\Pi} \frac{(X - a_0)[(X - a_0)^2 - Y^2]}{[(X - a_0)^2 + Y^2]^2} \qquad (1a,b,c)$$
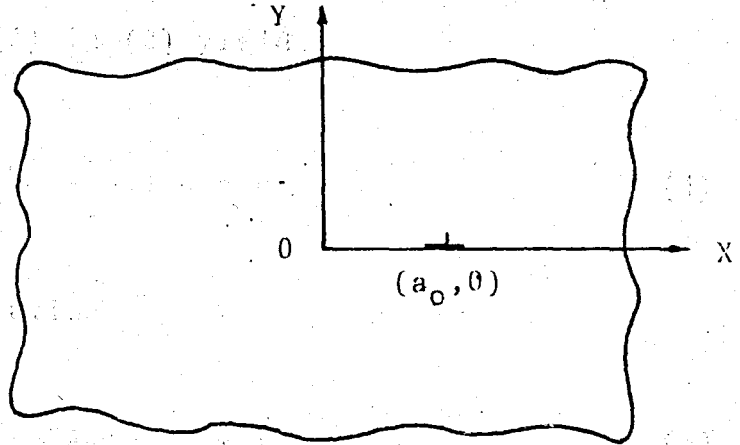
FIGURE 1: An Edge Dislocation at the Position $(a_0, 0)$ with Burgers vector $\vec{b}_X$ in an infinite plane.

where, $D_X = \dfrac{2\mu b_X}{(1+\kappa)}$

$$\kappa = \begin{cases} 3-4\nu & \text{for plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{for generalized plane stress} \end{cases}$$

μ being shear modulus, ν being Poisson's ratio.

The residual loading problem for the half-plane is formulated in terms of the biharmonic function U which satisfies

$$\nabla^4 U = 0 \tag{2}$$

It is conveninet to express

$$U(X,Y) = \frac{2}{\Pi} \int_0^\infty \phi(X,\alpha) \sin\alpha Y d\alpha \tag{3}$$

Substitution of (3) in (2) yields

$$\left(\frac{d^2}{dX^2} - \alpha^2\right)^2 \phi = 0 \tag{4}$$

which has the solution

$$\phi(X,\alpha) = (A_1 + XA_2)e^{-\alpha X} + (A_3 + XA_4)e^{\alpha X} \tag{5}$$

Since the stresses and displacements must be bounded at infinity the Airy stress function yields as

$$U(X,Y) = \frac{2}{\Pi} \int_0^\infty (A_1 + XA_2)e^{-\alpha X} \sin\alpha Y d\alpha \tag{6}$$

where $A_1(\alpha)$, $A_2(\alpha)$ are unknown functions to be determined from the boundary conditions. The stress components could be expressed as

$$\sigma_{XX} = \frac{\partial^2 U}{\partial Y^2} \quad , \quad \sigma_{YY} = \frac{\partial^2 U}{\partial X^2} \quad , \quad \sigma_{XY} = -\frac{\partial^2 U}{\partial X \partial Y} \tag{7a,b,c}$$

Upon differentiation one obtains

$$\sigma_{XX} = -\frac{2}{\Pi} \int_0^\infty \alpha^2 (A_1 + XA_2)e^{-\alpha X} \sin\alpha Y d\alpha$$

$$\sigma_{YY} = \frac{2}{\Pi} \int_0^\infty \alpha[\alpha A_1 + (\alpha X - 2)A_2]e^{-\alpha X} \sin\alpha Y d\alpha$$

$$\sigma_{XY} = \frac{2}{\Pi} \int_0^\infty \alpha[\alpha A_1 + (\alpha X - 1)A_2]e^{-\alpha X} \cos\alpha Y d\alpha \tag{8a,b,c}$$

Boundary conditions of the residual loading problem are

$$\sigma_{XX}(0,Y) = - \overset{*}{\sigma}_{XX}(0,Y) \ ,$$

$$\sigma_{XY}(0,Y) = - \overset{*}{\sigma}_{XY}(0,Y) \ .$$

From these conditions one can express

$$- \frac{2}{\Pi} \int_0^\infty \alpha^2 A_1 \sin\alpha Y d\alpha = \frac{D_X}{\Pi} \frac{Y(Y^2+3a_o^2)}{(a_o^2+Y^2)^2} \ ,$$

$$\frac{2}{\Pi} \int_0^\infty (\alpha^2 A_1 - \alpha A_2) \cos\alpha Y d\alpha = \frac{D_X}{\Pi} \frac{a_o(a_o^2-Y^2)}{(a_o^2+Y^2)^2} \ . \qquad (9a,b)$$

Multiplying Equation (9a) by $\sin\alpha Y dY$, Equation (9b) by $\cos\alpha Y dY$ and integrating from zero to infinity gives

$$-\alpha^2 A_1 = \frac{D_X}{\Pi} \int_0^\infty \frac{Y(Y^2+3a_o^2)}{(a_o^2+Y^2)^2} \sin\alpha Y dY \ ,$$

$$\alpha^2 A_1 - \alpha A_2 = \frac{D_X}{\Pi} \int_0^\infty \frac{a_o(a_o^2-Y^2)}{(a_o^2+Y^2)^2} \cos\alpha Y dY \ ,$$

evaluating the integrals appearing on the right yields

$$-\alpha^2 A_1 = \frac{D_X}{2} e^{-\alpha a_o} (1+\alpha a_o) \ ,$$

$$\alpha^2 A_1 - \alpha A_2 = \frac{D_X}{2} \alpha a_o e^{-\alpha a_o} \ ,$$

from which A_1 and A_2 can be determined as

$$A_1 = -\frac{D_X}{2\alpha^2} e^{-\alpha a_o} (1+\alpha a_o) \ ,$$

$$A_2 = -\frac{D_X}{2\alpha} e^{-\alpha a_o} (1+2\alpha a_o) \ .$$

Hence substituting A_1, A_2 in Equations (8a,b,c) results in

$$\sigma_{XX} = \frac{D_X}{\Pi} \int_o^\infty [1+\alpha a_o+\alpha X(1+2\alpha a_o)] e^{-\alpha(X+a_o)} \sin\alpha Y d\alpha \ ,$$

$$\sigma_{YY} = -\frac{D_X}{\Pi} \int_o^\infty [1+\alpha a_o+(\alpha X-2)(1+2\alpha a_o)] e^{-\alpha(X+a_o)} \sin\alpha Y d\alpha \ ,$$

$$\sigma_{XY} = -\frac{D_X}{\Pi} \int_o^\infty [1+\alpha a_o+(\alpha X-1)(1+2\alpha a_o)] e^{-\alpha(X+a_o)} \cos\alpha Y d\alpha \ ,$$

or

$$\sigma_{XX} = \frac{D_X}{\Pi} [\int_o^\infty e^{-\alpha(X+a_o)} \sin\alpha Y d\alpha+(X+a_o)\int_o^\infty \alpha e^{-\alpha(X+a_o)} \sin\alpha Y d\alpha$$
$$+ 2a_o X \int_o^\infty \alpha^2 e^{-\alpha(X+a_o)} \sin\alpha Y d\alpha] \ ,$$

$$\sigma_{YY} = \frac{D_X}{\Pi} [\int_o^\infty e^{-\alpha(X+a_o)} \sin\alpha Y d\alpha+(3a_o-X)\int_o^\infty \alpha e^{-\alpha(X+a_o)} \sin\alpha Y d\alpha$$
$$- 2a_o X \int_o^\infty \alpha^2 e^{-\alpha(X+a_o)} \sin\alpha Y d\alpha] \ ,$$

$$\sigma_{XY} = \frac{D_X}{\Pi} [(a_o-X) \int_o^\infty \alpha e^{-\alpha(X+a_o)} \cos\alpha Y d\alpha$$
$$- 2a_o X \int_o^\infty \alpha^2 e^{-\alpha(X+a_o)} \cos\alpha Y d\alpha] \ ,$$

and evaluation of infinite integrals (see Appendix) gives

$$\sigma_{XX} = \frac{D_X}{\Pi} \left\{ \frac{Y}{(X+a_0)^2+Y^2} + \frac{2Y(X+a_0)^2}{[(X+a_0)^2+Y^2]^2} + \frac{4a_0XY[3(X+a_0)^2-Y^2]}{[(X+a_0)^2+Y^2]^3} \right\} ,$$

$$\sigma_{YY} = \frac{D_X}{\Pi} \left\{ \frac{Y}{(X+a_0)^2+Y^2} - \frac{2Y(X-3a_0)(X+a_0)}{[(X+a_0)^2+Y^2]^2} \right.$$

$$\left. - \frac{4a_0XY[3(X+a_0)^2-Y^2]}{[(X+a_0)^2+Y^2]^3} \right\} ,$$

$$\sigma_{XY} = \frac{D_X}{\Pi} \left\{ (a_0-X)\frac{[(X+a_0)^2-Y^2]}{[(X+a_0)^2+Y^2]^2} - 4a_0X\frac{(X+a_0)[(X+a_0)^2-3Y^2]}{[(X+a_0)^2+Y^2]^3} \right\} .$$

$$(10a,b,c)$$

Thus the stress field of the half-plane can be obtained by superposition of expressions (1a,b,c) and (10a,b,c) as

$$\sigma^1_{XX} = \frac{D_X}{\Pi} \left\{ - \frac{Y[Y^2+3(X-a_0)^2]}{[(X-a_0)^2+Y^2]^2} + \frac{Y}{(X+a_0)^2+Y^2} \right.$$

$$\left. + \frac{2Y(X+a_0)^2}{[(X+a_0)^2+Y^2]^2} + \frac{4a_0XY[3(X+a_0)-Y^2]}{[(X+a_0)^2+Y^2]^3} \right\} ,$$

$$\sigma^1_{YY} = \frac{D_X}{\Pi} \left\{ \frac{Y[(X-a_0)^2-Y^2]}{[(X-a_0)^2+Y^2]^2} + \frac{Y}{(X+a_0)^2+Y^2} \right.$$

$$\left. - \frac{2Y(X-3a_0)(X+a_0)}{[(X+a_0)^2+Y^2]^2} - \frac{4a_0XY[3(X+a_0)^2-Y^2]}{[(X+a_0)^2+Y^2]^3} \right\} ,$$

$$\sigma^1_{XY} = \frac{D_X}{\Pi} \left\{ \frac{(X-a_0)\left[(X-a_0)^2 - Y^2\right]}{\left[(X-a_0)^2 + Y^2\right]^2} + \frac{(a_0 - X)\left[(X+a_0)^2 - Y^2\right]}{\left[(X+a_0)^2 + Y^2\right]^2} \right.$$

$$\left. - 4a_0 X \, \frac{(X+a_0)\left[(X+a_0)^2 - 3Y^2\right]}{\left[(X+a_0)^2 + Y^2\right]^3} \right\} . \qquad \text{(11a,b,c)}$$

An edge dislocation, of Burgers vector $\vec{b}_Y$, parallel to Y axis, at the point $(a_0, 0)$ of an infinite plane (see Figure 2) has the following stress field [12]
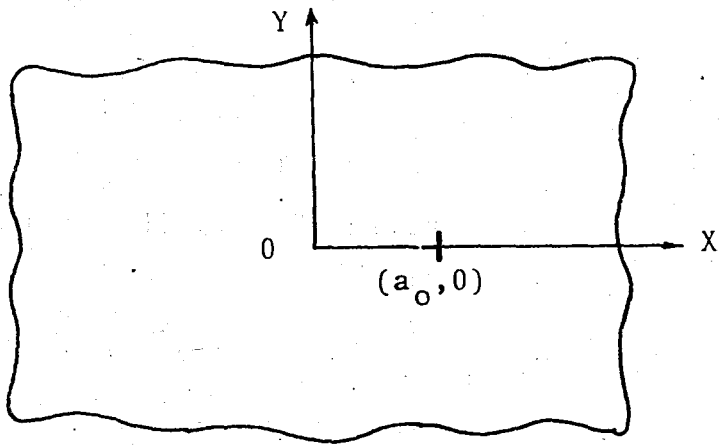
FIGURE 2. An Edge Dislocation at the Position $(a_0, 0)$ with Burgers Vector $\vec{b}_Y$ in an Infinite Plane.

$$\sigma_{XX} = \frac{D_Y}{\Pi} \frac{(X-a_0)\left[(X-a_0)^2 - Y^2\right]}{\left[(X-a_0)^2 + Y^2\right]^2} \qquad ,$$

$$\sigma_{YY} = \frac{D_Y}{\Pi} \frac{(X-a_0)\left[(X-a_0)^2 + 3Y^2\right]}{\left[(X-a_0)^2 + Y^2\right]^2} \qquad ,$$

$$\sigma_{XY} = \frac{D_Y}{\Pi} \quad \frac{Y\left[(X-a_o)^2 - Y^2\right]}{\left[(X-a_o)^2 + Y^2\right]^2} \tag{12a,b,c}$$

where,

$$D_Y = \frac{2\mu b_Y}{(1+\kappa)} \quad .$$

Again the residual loading problem will be formulated in terms of the biharmonic function $U(X,Y)$ which may be expressed as

$$U(X,Y) = \frac{2}{\Pi} \int_o^\infty \phi(X,\alpha)\cos\alpha Y d\alpha \tag{13}$$

for this case.

Substitution of (13) in (2) gives

$$(\frac{d^2}{dX^2} - \alpha^2)^2 \phi = 0 \quad .$$

A solution which is bounded at infinity can be substituted in (13). Hence one obtains

$$U(X,Y) = \frac{2}{\Pi} \int_o^\infty (A_1 + XA_2)e^{-\alpha X}\cos\alpha Y d\alpha \quad .$$

Again $A_1(\alpha)$, $A_2(\alpha)$ will be determined from the boundary conditions. The stress components are derivable from (7a,b,c) as

$$\sigma_{XX} = -\frac{2}{\Pi} \int_0^\infty \alpha^2 (A_1 + XA_2) e^{-\alpha X} \cos\alpha Y d\alpha \quad ,$$

$$\sigma_{YY} = \frac{2}{\Pi} \int_0^\infty \alpha [\alpha A_1 + (\alpha X - 2) A_2] e^{-\alpha X} \cos\alpha Y d\alpha \ ,$$

$$\sigma_{XY}' = -\frac{2}{\Pi} \int_0^\infty \alpha [\alpha A_1 + (\alpha X - 1) A_2] e^{-\alpha X} \sin\alpha Y d\alpha \quad . \qquad (14a,b,c)$$

Now expressing the boundary conditions of the residual loading problem

$$\sigma_{XX}(0,Y) = - \overset{*}{\sigma}_{XX}(0,Y) \ ,$$

$$\sigma_{XY}(0,Y) = - \overset{*}{\sigma}_{XY}(0,Y) \ ,$$

one obtains

$$-\frac{2}{\Pi} \int_0^\infty \alpha^2 A_1 \cos\alpha Y d\alpha = \frac{D_Y}{\Pi} \frac{a_0(a_0^2 - Y^2)}{(a_0^2 + Y^2)^2} \quad ,$$

$$-\frac{2}{\Pi} \int_0^\infty (\alpha^2 A_1 - \alpha A_2) \sin\alpha Y d\alpha = -\frac{D_Y}{\Pi} \frac{Y(a_0^2 - Y^2)}{(a_0^2 + Y^2)^2} \quad . \qquad (15a,b)$$

Multiplying Equation (15a) by $\cos\alpha Y dY$, Equation (15b) by $\sin\alpha Y dY$ and integrating both sides from zero to infinity gives

$$-\alpha^2 A_1 = \frac{D_Y}{\Pi} \int_0^\infty \frac{a_0(a_0^2 - Y^2)}{(a_0^2 + Y^2)^2} \cos\alpha Y dY \quad ,$$

$$-\alpha^2 A_1 + \alpha A_2 = \frac{D_Y}{\Pi} \int_0^\infty \frac{Y(Y^2 - a_0^2)}{(a_0^2 + Y^2)^2} \sin\alpha Y dY$$

evaluating the integrals appearing on the right yields

$$- \alpha^2 A_1 = \frac{D_Y}{2} \cdot \alpha a_0 e^{-\alpha a_0} \quad,$$

$$- \alpha^2 A_1 + \alpha A_2 = \frac{D_Y}{2} e^{-\alpha a_0} (1 - \alpha a_0) \quad,$$

from which A_1 and A_2 can be determined as

$$A_1 = - \frac{D_Y}{2\alpha} a_0 e^{-\alpha a_0} \quad,$$

$$A_2 = \frac{D_Y}{2\alpha} e^{-\alpha a_0} (1 - 2\alpha a_0) \quad.$$

Substitution of A_1, A_2 in Equations (14a,b,c) results in

$$\sigma_{XX} = \frac{D_Y}{\Pi} \int\limits_0^\infty [\alpha a_0 + \alpha X(2\alpha a_0 - 1)] e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha \quad,$$

$$\sigma_{YY} = \frac{D_Y}{\Pi} \int\limits_0^\infty [3\alpha a_0 - 2 + \alpha X(1 - 2\alpha a_0)] e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha \quad,$$

$$\sigma_{XY} = \frac{D_Y}{\Pi} \int\limits_0^\infty [\alpha a_0 + (\alpha X - 1)(2\alpha a_0 - 1)] e^{-\alpha(X+a_0)} \sin\alpha Y d\alpha \quad,$$

or

$$\sigma_{XX} = \frac{D_Y}{\Pi} [(a_0 - X) \int\limits_0^\infty \alpha e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha$$

$$+ 2a_0 X \int\limits_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha] \quad,$$

$$\sigma_{YY} = \frac{D_Y}{\Pi} \left[-2 \int_0^\infty e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha \right.$$

$$+ (X+3a_0) \int_0^\infty \alpha e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha$$

$$\left. - 2a_0 X \int_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha \right] \quad ,$$

$$\sigma_{XY} = \frac{D_Y}{\Pi} \left[\int_0^\infty e^{-\alpha(X+a_0)} \sin\alpha Y d\alpha \right.$$

$$- (X+a_0) \int_0^\infty \alpha e^{-\alpha(X+a_0)} \sin\alpha Y d\alpha$$

$$\left. + 2a_0 X \int_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \sin\alpha Y d\alpha \right] \quad ,$$

and evaluation of infinite integrals (see Appendix) gives

$$\sigma_{XX} = \frac{D_Y}{\Pi} \left\{ (a_0-X) \frac{[(X+a_0)^2 - Y^2]}{[(X+a_0)^2 + Y^2]^2} + 4a_0 X \frac{(X+a_0)[(X+a_0)^2 - 3Y^2]}{[(X+a_0)^2 + Y^2]^3} \right\} \quad ,$$

$$\sigma_{YY} = \frac{D_Y}{\Pi} \left\{ -2 \frac{(X+a_0)}{(X+a_0)^2 + Y^2} + (X+3a_0) \frac{[(X+a_0)^2 - Y^2]}{[(X+a_0)^2 + Y^2]^2} \right.$$

$$\left. - 4a_0 X \frac{(X+a_0)[(X+a_0)^2 - 3Y^2]}{[(X+a_0)^2 + Y^2]^3} \right\} \quad ,$$

$$\sigma_{XY} = \frac{D_Y}{\Pi} \left\{ \frac{Y}{(X+a_0)^2 + Y^2} - \frac{2Y(X+a_0)^2}{[(X+a_0)^2 + Y^2]^2} \right.$$

$$\left. + 4a_0 XY \frac{[3(X+a_0)^2 - Y^2]}{[(X+a_0)^2 + Y^2]^3} \right\} \quad . \tag{16a,b,c}$$

Consequently the stress field of the half-plane can be

obtained by superposition of expressions (12a,b,c) and (16a,b,c) as

$$\overset{2}{\sigma}_{XX} = \frac{D_Y}{\Pi} \left\{ \frac{(X-a_o)\left[(X-a_o)^2-Y^2\right]}{\left[(X-a_o)^2+Y^2\right]^2} + (a_o-X)\frac{\left[(X+a_o)^2-Y^2\right]}{\left[(X+a_o)^2+Y^2\right]^2} \right.$$

$$\left. + 4a_oX\frac{(X+a_o)\left[(X+a_o)^2-3Y^2\right]}{\left[(X+a_o)^2+Y^2\right]^3} \right\} ,$$

$$\overset{2}{\sigma}_{YY} = \frac{D_Y}{\Pi} \left\{ (X-a_o)\frac{\left[(X-a_o)^2+3Y^2\right]}{\left[(X-a_o)^2+Y^2\right]^2} - 2\frac{(X+a_o)}{(X+a_o)^2+Y^2} \right.$$

$$\left. + (X+3a_o)\frac{\left[(X+a_o)^2-Y^2\right]}{\left[(X+a_o)^2+Y^2\right]^2} - 4a_oX\frac{(X+a_o)\left[(X+a_o)^2-3Y^2\right]}{\left[(X+a_o)^2+Y^2\right]^3} \right\} ,$$

$$\overset{2}{\sigma}_{XY} = \frac{D_Y}{\Pi} \left\{ \frac{Y\left[(X-a_o)^2-Y^2\right]}{\left[(X-a_o)^2+Y^2\right]^2} + \frac{Y}{(X+a_o)^2+Y^2} \right.$$

$$\left. - \frac{2Y(X+a_o)^2}{\left[(X+a_o)^2+Y^2\right]^2} + 4a_oXY\frac{\left[3(X+a_o)^2-Y^2\right]}{\left[(X+a_o)^2+Y^2\right]^3} \right\} . \qquad (17a,b,c)$$

A powerful method of analysis is based on the representation of a crack by a continuous distribution of dislocations. To simulate a line crack in s direction (see Figure 3) stress solutions for a pair of edge dislocations with Burgers vectors $\vec{b}_s$ and $\vec{b}_q$ can be used as Green's functions. Stress distributions due to $\vec{b}_s$ and $\vec{b}_q$ can be obtained by resolving these vectors into components in X and Y directions and summing the stresses set up by

these components.
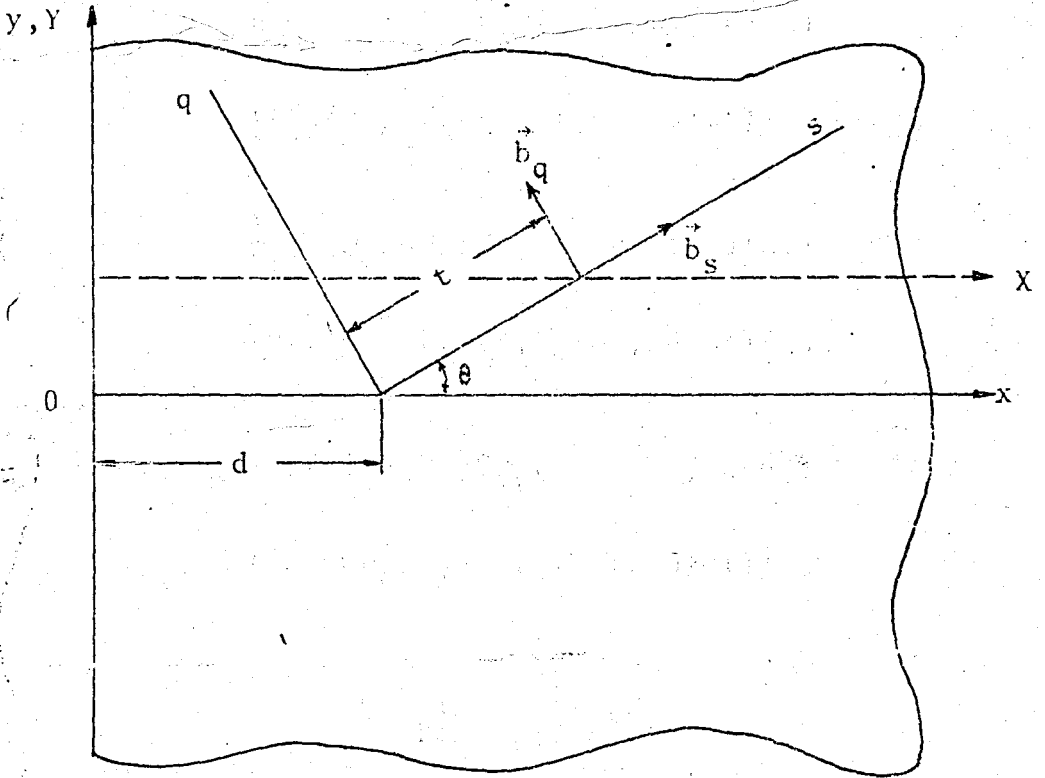
FIGURE 3. A Pair of Edge Dislocations with Burgers
Vectors $\vec{b}_s$ and $\vec{b}_q$ in a Half-Plane.

Hence introducing the notation

$$\alpha = \cos\theta \quad , \quad \beta = \sin\theta,$$

$$D_s = \frac{2\mu}{1+\kappa} b_s \quad , \quad D_q = \frac{2\mu}{1+\kappa} b_q$$

we obtain the stresses due to edge dislocations with

Burgers vectors $\vec{b}_s$ and $\vec{b}_q$ as

$$\sigma_{XX}(X,y) = \frac{1}{\Pi} \left[(\beta D_q - \alpha D_s) G_{XX}(x, y-\beta t, d+\alpha t) \right.$$
$$\left. - (\beta D_s + \alpha D_q) F_{XX}(x, y-\beta t, d+\alpha t) \right] ,$$

$$\sigma_{yy}(X,y) = \frac{1}{\Pi} \left[(\beta D_q - \alpha D_s) G_{YY}(x, y-\beta t, d+\alpha t) \right.$$
$$\left. - (\beta D_s + \alpha D_q) F_{YY}(x, y-\beta t, d+\alpha t) \right] ,$$

$$\sigma_{Xy}(X,y) = \frac{1}{\Pi} \left[(\beta D_q - \alpha D_s) G_{XY}(x, y-\beta t, d+\alpha t) \right.$$
$$\left. - (\beta D_s + \alpha D_q) F_{XY}(x, y-\beta t, d+\alpha t) \right] ,$$

where

$$G_{XX}(X,Y,a_o) = - \frac{\Pi \overset{1}{\sigma}_{XX}(X,Y,a_o)}{D_X} ,$$

$$G_{YY}(X,Y,a_o) = - \frac{\Pi \overset{1}{\sigma}_{YY}(X,Y,a_o)}{D_X} ,$$

$$G_{XY}(X,Y,a_o) = - \frac{\Pi \overset{1}{\sigma}_{XY}(X,Y,a_o)}{D_X} ,$$

$$F_{XX}(X,Y,a_o) = - \frac{\Pi \overset{2}{\sigma}_{XX}(X,Y,a_o)}{D_Y} ,$$

$$F_{YY}(X,Y,a_o) = - \frac{\Pi \overset{2}{\sigma}_{YY}(X,Y,a_o)}{D_Y} ,$$

$$F_{XY}(X,Y,a_o) = - \frac{\Pi \overset{2}{\sigma}_{XY}(X,Y,a_o)}{D_Y} . \qquad (18a\text{-}f)$$

At a point along the s-axis since $X = d + \alpha s$, $y = \beta s$ the normal and shear stress components σ_{qq} and σ_{qs} in the $\{s,q\}$ coordinate system can be expressed as

$$\sigma_{qq}(0,s) = \beta^2 \sigma_{XX}(d+\alpha s, \beta s) + \alpha^2 \sigma_{yy}(d+\alpha s, \beta s)$$

$$- 2\alpha\beta\sigma_{Xy}(d+\alpha s, \beta s) \quad,$$

$$\sigma_{qs}(0,s) = (\alpha^2-\beta^2)\sigma_{Xy}(d+\alpha s, \beta s) + \alpha\beta\left[\sigma_{yy}(d+\alpha s, \beta s)\right.$$

$$\left. - \sigma_{XX}(d+\alpha s, \beta s)\right] \quad.$$

A crack along the s axis can be represented by continuous arrays of dislocations with density functions

$$f(s) = \frac{2\mu}{1+\kappa} \frac{\partial}{\partial s} \left[v(0^+,s) - v(0^-,s)\right] \quad, \tag{19}$$

$$g(s) = \frac{2\mu}{1+\kappa} \frac{\partial}{\partial s} \left[u(0^+,s) - u(0^-,s)\right] \quad, \tag{20}$$

where u and v are the displacement components in the $\{s,q\}$ coordinate system. As the crack extends from $s=-a$ to $s=a$ we obviously have

$$f(s) = g(s) = 0 \quad \text{for} \quad |s| > a \quad.$$

To obtain the stresses due to distributions of

edge dislocations we replace D_s and D_q by $-f(t)dt$ and $-g(t)dt$ respectively and integrate from $-a$ to a. The unknown density functions $f(t)$ and $g(t)$ will be determined from the conditions that stresses are prescribed on the crack surface which yield

$$\sigma_{qq}(0,s) = \frac{1}{\Pi} \int_{-a}^{a} k(s,t)f(t)dt + \frac{1}{\Pi} \int_{-a}^{a} h(s,t)g(t)dt = p(s),$$
$$-a < s < a,$$

$$\sigma_{qs}(0,s) = \frac{1}{\Pi} \int_{-a}^{a} m(s,t)f(t)dt + \frac{1}{\Pi} \int_{-a}^{a} n(s,t)g(t)dt = q(s),$$
$$-a < s < a$$

$$(21a,b)$$

where

$$
\begin{aligned}
k(s,t) = \quad & \beta^2 \left[\alpha F_{XX}(d+\alpha s, \ \beta s - \beta t, \ d+\alpha t) \right. \\
& \left. - \beta G_{XX}(d+\alpha s, \ \beta s - \beta t, \ d+\alpha t) \right] \\
+ & \alpha^2 \left[\alpha F_{YY}(d+\alpha s, \ \beta s - \beta t, \ d+\alpha t) \right. \\
& \left. - \beta G_{YY}(d+\alpha s, \ \beta s - \beta t, \ d+\alpha t) \right] \\
- & 2\alpha\beta \left[\beta F_{XY}(d+\alpha s, \ \beta s - \beta t, \ d+\alpha t) \right. \\
& \left. + \alpha G_{XY}(d+\alpha s, \ \beta s - \beta t, \ d+\alpha t) \right] \quad ,
\end{aligned}
$$

$$
\begin{aligned}
h(s,t) = \quad & \beta^2 \left[\beta F_{XX}(d+\alpha s, \ \beta s - \beta t, \ d+\alpha t) \right. \\
& \left. + \alpha G_{XX}(d+\alpha s, \ \beta s - \beta t, \ d+\alpha t) \right] \\
+ & \alpha^2 \left[\beta F_{YY}(d+\alpha s, \ \beta s - \beta t. \ d+\alpha t) \right. \\
& \left. + \alpha G_{YY}(d+\alpha s, \ \beta s - \beta t, \ d+\alpha t) \right] \\
- & 2\alpha\beta \left[\beta F_{XY}(d+\alpha s, \ \beta s - \beta t, \ d+\alpha t) \right. \\
& \left. + \alpha G_{XY}(d+\alpha s, \ \beta s - \beta t, \ d+\alpha t) \right] \quad ,
\end{aligned}
$$

$$
\begin{aligned}
m(s,t) = \quad &(\overset{2}{\alpha}-\overset{2}{\beta}) \; [\alpha F_{XY}(d+\alpha s, \quad \beta s-\beta t, \quad d+\alpha t) \\
&\quad -\beta G_{XY}(d+\alpha s, \quad \beta s-\beta t, \quad d+\alpha t)] \\
&+ \alpha\beta \; \{ \alpha \; [\; F_{YY}(d+\alpha s, \quad \beta s-\beta t, \quad d+\beta t) \\
&\qquad - F_{XX}(d+\alpha s, \quad \beta s-\beta t, \quad d+\alpha t)] \\
&\qquad -\beta \; [G_{YY}(d+\alpha s, \quad \beta s-\beta t, \quad d+\alpha t) \\
&\qquad - G_{XX}(d+\alpha s, \quad \beta s-\beta t, \quad d+\alpha t)]\} \; ,
\end{aligned}
$$

$$
\begin{aligned}
n(s,t) = \quad &(\overset{2}{\alpha}-\overset{2}{\beta}) \; [\beta F_{XY}(d+\alpha s, \quad \beta s-\beta t, \quad d+\alpha t) \\
&\quad +\alpha G_{XY}(d+\alpha s, \quad \beta s-\beta t, \quad d+\alpha t)] \\
&+ \alpha\beta \; \{ \beta \; [\; F_{YY}(d+\alpha s, \quad \beta s-\beta t, \quad d+\alpha t) \\
&\qquad - F_{XX}(d+\alpha s, \quad \beta s-\beta t, \quad d+\alpha t)] \\
&\qquad + \alpha[G_{YY}(d+\alpha s, \quad \beta s-\beta t, \quad d+\alpha t) \\
&\qquad - G_{XX}(d+\alpha s, \quad \beta s-\beta t, \quad d+\alpha t)]\} \; .
\end{aligned}
$$

To provide the continuity of the material outside
the crack we must impose the two conditions

$$
\int_{-a}^{a} f(t)dt = 0 \; ,
$$

$$
\int_{-a}^{a} g(t)dt = 0 \; . \tag{22a,b}
$$

It can easily be shown that the kernels $k(s,t)$, $n(s,t)$
have Cauchy singularities, namely

$$
k(s,t) = \frac{1}{t-s} + \overset{*}{k}(s,t) \; ,
$$

$$n(s,t) = \frac{1}{t-s} + \overset{*}{n}(s,t) \quad,$$

where $\overset{*}{k}(s,t)$, $\overset{*}{n}(s,t)$ are bounded functions.

Thus the Equations (21a,b) form two singular integral equations of the first kind. These singular integral equations take simple forms when the crack is perpendicular or parallel to the free surface. If the crack is perpendicular to the free surface we take $\alpha=1$, $\beta=0$ in the expressions (21a,b) and we obtain.

$$\frac{1}{\Pi} \int_{-a}^{a} \left[\frac{1}{t-s} + \frac{s-t}{(t+s+2d)^2} + \frac{4(t+d)(s+d)}{(t+s+2d)^3}\right] f(t)dt = p(s), \quad -a<s<a$$

$$\frac{1}{\Pi} \int_{-a}^{a} \left[\frac{1}{t-s} + \frac{s-t}{(t+s+2d)^2} + \frac{4(t+d)(s+d)}{(t+s+2d)^3}\right] g(t)dt = q(s), \quad -a<s<a \ .$$

$$(23a,b)$$

Similarly for the case of the crack being parallel to the interface we take $\alpha=0$, $\beta=1$ and we could express

$$\frac{1}{\Pi} \int_{-a}^{a} \left\{\frac{1}{t-s} - \frac{t-s}{(t-s)^2+4d^2} - \frac{8d^2(t-s)}{[(t-s)^2+4d^2]^2} - \frac{4d^2(t-s)[12d^2-(t-s)^2]}{[(t-s)^2+4d^2]^3}\right\} f(t)dt$$

$$- \frac{1}{\Pi} \int_{-a}^{a} \left\{8d^3\frac{[4d^2-3(t-s)^2]}{[(t-s)^2+4d^2]^3}\right\} g(t)dt = p(s), \quad -a<s<a$$

$$\frac{1}{\Pi} \int_{-a}^{a} \{8d^3 \frac{[4d^2 - 3(t-s)^2]}{[(t-s)^2 + 4d^2]^3}\} \ f(t)dt + \frac{1}{\Pi} \int_{-a}^{a} \{\frac{1}{t-s} - \frac{t-s}{(t-s)^2 + 4d^2}$$

$$+ \frac{8d^2(t-s)}{[(t-s) + 4d^2]^2} - \frac{4d^2(t-s)[12d^2 - (t-s)^2]}{[(t-s)^2 + 4d^2]^3}\} \ g(t)dt = q(s),$$

$$-a < s < a$$

$$(24a,b)$$

These expressions are in agreement with those obtained by
Asbaugh [1] and also with the expressions given by
Erdoğan [9] except the sign discrepancy which was remarked
by Asbaugh [1].

Now we can easily extend the discussion to a system
of randomly oriented cracks (see Figure 4).

For this case using the same formulation described
for a single crack and using $\{s_i, q_i\}, (i=1,\ldots,n)$ rectan-
gular coordinate systems, expressions of crack surface
tractions yield the following system of singular integral
equations

$$\frac{1}{\Pi} \sum_{j=1}^{n} [\int_{-a_j}^{a_j} k_{ij}(s_i, t_j) f_j(t_j)dt_j + \int_{-a_j}^{a_j} h_{ij}(s_i, t_j) g_j(t_j)dt_j]$$

$$= p(s_i),$$

$$-a_i < s_i < a_i,$$

$$(i=1,\ldots,n),$$

$$\frac{1}{\Pi} \sum_{j=1}^{n} \left[\int_{-a_j}^{a_j} m_{ij}(s_i, t_j) f_j(t_j) dt_j + \int_{-a_j}^{a_j} n_{ij}(s_i, t_j) g_j(t_j) dt_j \right]$$

$$= q(s_i) ,$$

$$- a_i < s_i < a_i ,$$

$$(i=1,\ldots,n) ,$$

$$(25a,b)$$
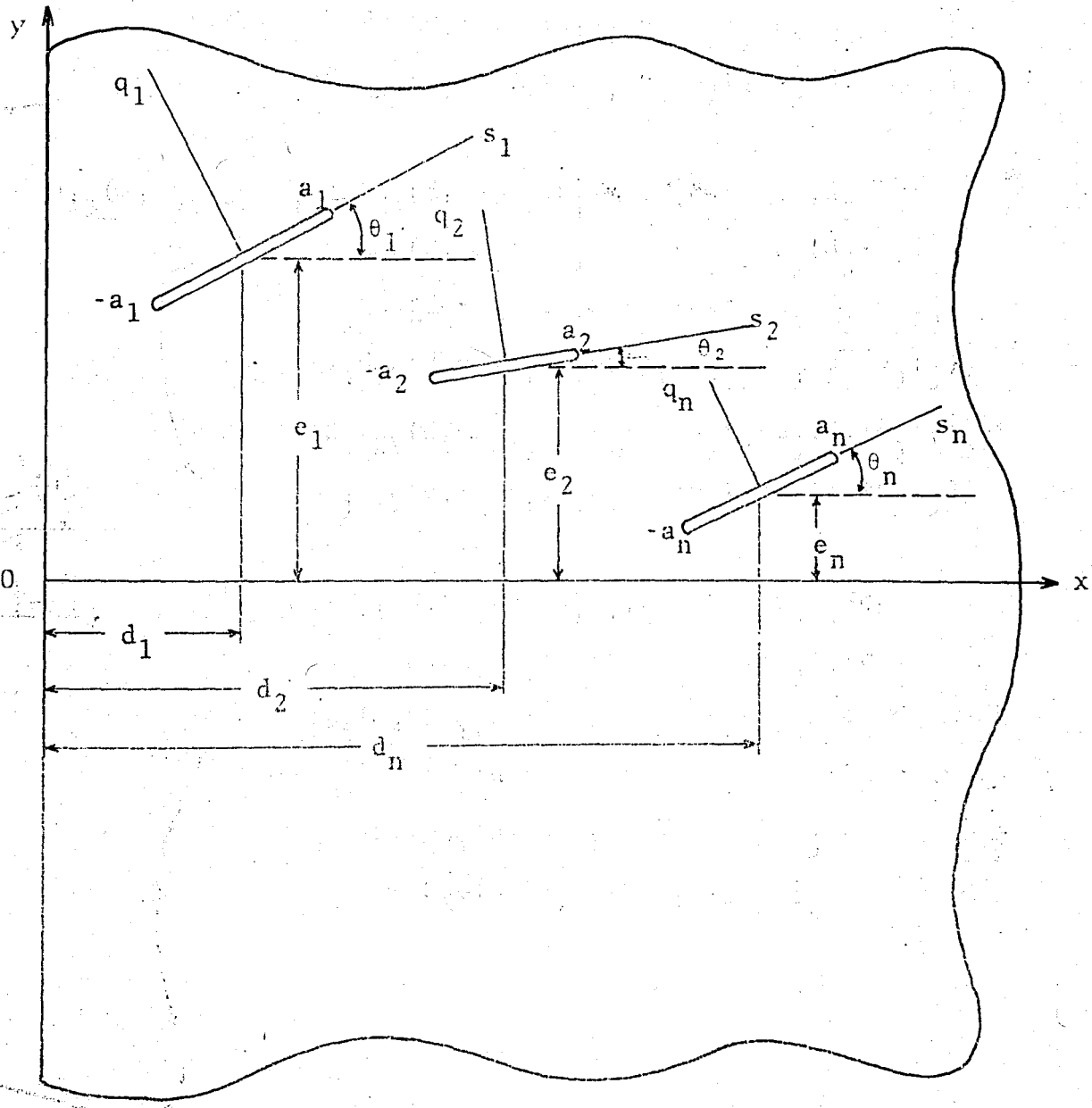
FIGURE 4. A System of Randomly Oriented Cracks in a Half-Plane.

where

$$k_{ij}(s_i,t_j) = \beta_i^2 [\alpha_j F_{XX}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)$$
$$-\beta_j G_{XX}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)]$$
$$+\alpha_i^2 [\alpha_j F_{YY}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)$$
$$-\beta_j G_{YY}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)]$$
$$- 2\alpha_i\beta_i [\alpha_j F_{XY}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)$$
$$-\beta_j G_{XY}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)] \quad,$$

$$h_{ij}(s_i,t_j) = \beta_i^2 [\beta_j F_{XX}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)$$
$$+\alpha_j G_{XX}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)]$$
$$+\alpha_i^2 [\beta_j F_{YY}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)$$
$$+\alpha_j G_{YY}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)]$$
$$- 2\alpha_i\beta_i [\beta_j F_{XY}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)$$
$$+\alpha_j G_{XY}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)] \quad,$$

$$m_{ij}(s_i,t_j) = (\alpha_i^2-\beta_i^2) [\alpha_j F_{XY}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)$$
$$-\beta_j G_{XY}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)],$$
$$+ \alpha_i\beta_i \{\alpha_j [F_{YY}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)$$
$$- F_{XX}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)],$$
$$-\beta_j [G_{YY}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)$$
$$- G_{XX}(d_i+\alpha_i s_i, \beta_i s_i+e_i-\beta_j t_j-e_j, d_j+\alpha_j t_j)]\},$$

$$n_{ij}(s_i,t_j) = (\alpha_i^2 - \beta_i^2)[\beta_j F_{XY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)$$
$$+\alpha_j G_{XY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)]$$
$$+\alpha_i \beta_i \{\beta_j [F_{YY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)$$
$$- F_{XX}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)]$$
$$+\alpha_j [G_{YY}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)$$
$$- G_{XX}(d_i + \alpha_i s_i, \beta_i s_i + e_i - \beta_j t_j - e_j, d_j + \alpha_j t_j)]$$

Again to provide the continuity of the material outside
the cracks we must impose the single-valuedness conditions

$$\int_{-a_j}^{a_j} f_j(t_j) dt_j = 0 , \quad (j=1,\ldots,n) ,$$

$$\int_{-a_j}^{a_j} g_j(t_j) dt_j = 0 , \quad (j=1,\ldots,n) . \qquad (26a,b)$$

For a system of radial cracks the singular integral
equations take simpler forms and they can be readily
expressed (see Figure 5) as

$$\frac{1}{\Pi} \sum_{j=1}^{n} [\int_{a_j}^{b_j} k_{ij}(s_i,t_j) f_j(t_j) dt_j +$$

$$\int_{a_j}^{b_j} h_{ij}(s_i,t_j) g_j(t_j) dt_j] = p(s_i) ,$$
$$a_i < s_i < b_i, \quad (i=1,\ldots,n),$$

$$\frac{1}{\Pi} \sum_{j=1}^{n} [\int_{a_j}^{b_j} m_{ij}(s_i,t_j)f_j(t_j)dt_j +$$

$$\int_{a_j}^{b_j} n_{ij}(s_i,t_j)g_j(t_j)dt_j] = q(s_i)$$

$$a_i < s_i < b_i, \quad (i=1,\ldots,n),$$

$$(27a,b)$$
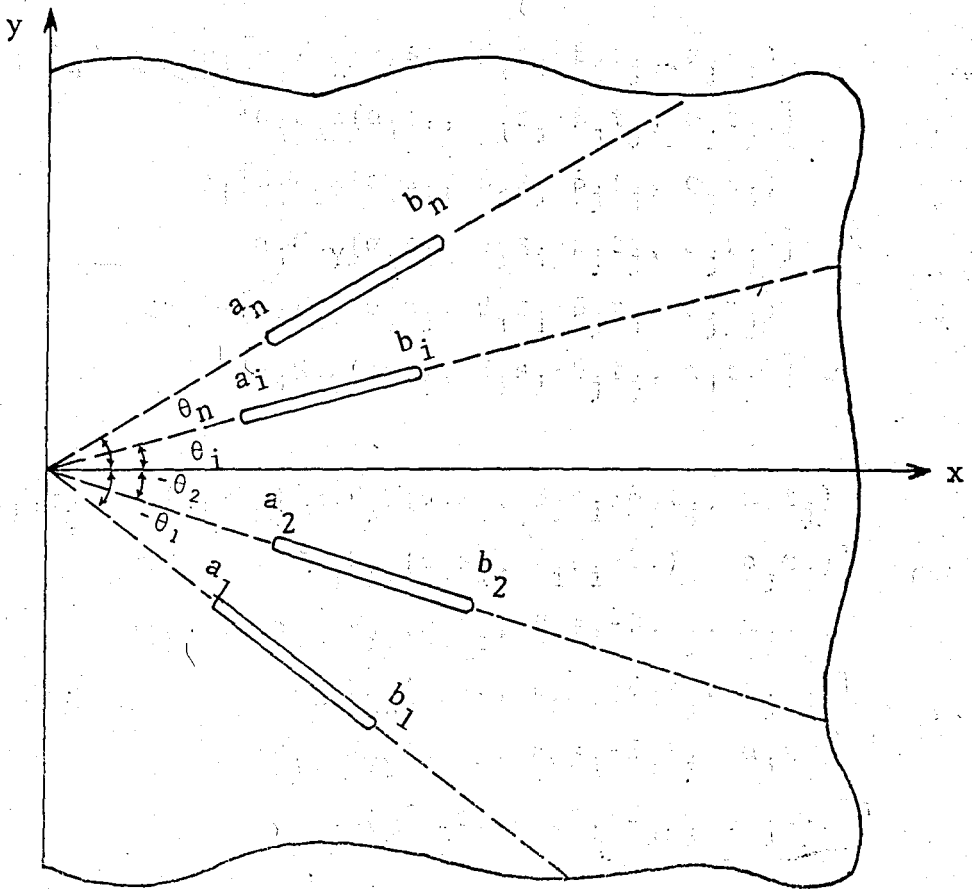
FIGURE 5. A System of Radial Cracks in a Half-Plane.

where

$$k_{ij}(s_i, t_j) = \beta_i^2 [\alpha_j F_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)$$
$$- \beta_j G_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)]$$
$$+ \alpha_i^2 [\alpha_j F_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)$$
$$- \beta_j G_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)]$$
$$- 2\alpha_i \beta_i [\alpha_j F_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)$$
$$- \beta_j G_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] \quad ,$$

$$h_{ij}(s_i, t_j) = \beta_i^2 [\beta_j F_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)$$
$$+ \alpha_j G_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)]$$
$$+ \alpha_i^2 [\beta_j F_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)$$
$$+ \alpha_j G_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)]$$
$$- 2\alpha_i \beta_i [\beta_j F_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)$$
$$+ \alpha_j G_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)] \quad ,$$

$$m_{ij}(s_i, t_j) = (\alpha_i^2 - \beta_i^2)[\alpha_j F_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)$$
$$- \beta_j G_{XY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)]$$
$$+ \alpha_i \beta_i \{ \alpha_j [F_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)$$
$$- F_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)]$$
$$- \beta_j [G_{YY}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)$$
$$- G_{XX}(\alpha_i s_i, \beta_i s_i - \beta_j t_j, \alpha_j t_j)]\} \quad ,$$

$$n_{ij}(s_i, t_j) = (\alpha_i^2 - \beta_i^2) [\beta_j \ F_{XY}(\alpha_i s_i, \ \beta_i s_i - \beta_j t_j, \ \alpha_j t_j)$$

$$+\alpha_j \ G_{XY}(\alpha_i s_i, \ \beta_i s_i - \beta_j t_j, \ \alpha_j t_j)]$$

$$+ \ \alpha_i \beta_i \ \{\beta_j [F_{YY}(\alpha_i s_i, \ \beta_i s_i - \beta_j t_j, \ \alpha_j t_j)$$

$$- F_{XX}(\alpha_i s_i, \ \beta_i s_i - \beta_j t_j, \ \alpha_j t_j)]$$

$$+ \ \alpha_j \ [\ G_{YY}(\alpha_i s_i, \ \beta_i s_u - \beta_j t_j, \ \alpha_j t_j)$$

$$- G_{XX}(\alpha_i s_i, \ \beta_i s_i - \beta_j t_j, \ \alpha_j t_j)]\} \ ,$$

Equations (27a,b) are subjected to the following single-valuedness conditions

$$\int_{a_j}^{b_j} f_j(t_j) dt_j = 0 \ , \qquad (j=1,\ldots,n) \ ,$$

$$\int_{a_j}^{b_j} g_j(t_j) dt_j = 0 \ , \qquad (j=1,\ldots,n) \ . \qquad (28a,b)$$

The system of singular integral equations (21a,b), (25a,b) and (27a,b) will be solved numerically. The numerical procedure will be described in the next chapter.

CHAPTER 3

NUMERICAL SOLUTION OF SINGULAR
INTEGRAL EQUATIONS

The quadrature method developed [10] to solve inte-
gral equations with Cauchy type kernels will be used.
Now the method will be described for equations (21a,b).

First the intervals of integration are normalized
by introducing the new variables

$$\tau = \frac{t}{a} , \qquad \xi = \frac{s}{a}$$

and the unknown functions can be expressed as

$$f(t) = F(\tau)(1-\tau^2)^{-\frac{1}{2}} ,$$
$$g(t) = G(\tau)(1-\tau^2)^{-\frac{1}{2}} .$$

Defining the new kernels by

$$K(\xi,\tau) = ak(s,t), \quad H(\xi,\tau) = ah(s,t) ,$$
$$M(\xi,\tau) = am(s,t), \quad N(\xi,\tau) = an(s,t)$$

equations (21a,b) and (22a,b) become

$$\frac{1}{\Pi} \int_{-1}^{1} K(\xi,\tau)F(\tau)(1-\tau^2)^{-\frac{1}{2}}d\tau +$$

$$\frac{1}{\Pi} \int_{-1}^{1} H(\xi,\tau)G(\tau)(1-\tau^2)^{-\frac{1}{2}}d\tau = p(\xi) \quad -1<\xi< 1,$$

$$\frac{1}{\Pi} \int_{-1}^{1} M(\xi,\tau)F(\tau)(1-\tau^2)^{-\frac{1}{2}}d\tau +$$

$$\frac{1}{\Pi} \int_{-1}^{1} N(\xi,\tau)G(\tau)(1-\tau^2)^{-\frac{1}{2}}d\tau = q(\xi) \quad -1<\xi< 1, \qquad (29a,b)$$

$$\int_{-1}^{1} F(\tau)(1-\tau^2)^{-\frac{1}{2}}d\tau = 0 ,$$

$$\int_{-1}^{1} G(\tau)(1-\tau^2)^{-\frac{1}{2}}d\tau = 0 . \qquad (30a,b)$$

Applying the Gauss-Chebyshev quadrature [10] to these equations yields the following system of linear equations for the determination of $F(\tau_k)$ and $G(\tau_k)$ $(i=1,\ldots,n)$.

$$\sum_{k=1}^{N} W_k[M(\tau_k, \ldots) \ldots \ldots , \tau_k)] = p(\xi_i) ,$$

$$\sum_{k=1}^{N} \ldots \ldots \ldots G(\tau_k)N(\ldots,\tau_k) \ldots \ldots) ,$$

$$\sum_{k=1}^{N} W_k F(\tau_k) = 0, \qquad \sum_{k=1}^{N} W_k G(\tau_k) = 0 , \qquad (31a-d)$$

where

$$W_1 = W_n = \frac{1}{2(N-1)} , \qquad W_k = \frac{1}{N-1} , \qquad k=2,\ldots,N-1,$$

$$\tau_k = \cos\left(\Pi \frac{k-1}{N-1}\right), \qquad k=1,\ldots,N,$$

$$\xi_\ell = \cos\left(\Pi \frac{2\ell-1}{2N-2}\right), \qquad \ell=1,\ldots,N-1 . \tag{32a,b,c}$$

The quantities of main importance are stress intensity factors which are defined as

$$k_1(a) = \lim_{s \to a} \sqrt{2(s-a)} \; \sigma_{qq}(0,s) ,$$

$$k_1(-a) = \lim_{s \to -a} \sqrt{2(s+a)} \; \sigma_{qq}(0,s) ,$$

$$k_2(a) = \lim_{s \to a} \sqrt{2(s-a)} \; \sigma_{qs}(0,s) ,$$

$$k_2(-a) = \lim_{s \to -a} \sqrt{2(s+a)} \; \sigma_{qs}(0,s) . \tag{33a-d}$$

The stress intensity factors are proportional to the values of $F(\tau)$ and $G(\tau)$ at the corresponding end points through the relations

$$k_1(-a) = F(-1) \sqrt{a} , \qquad k_2(-a) = G(-1)\sqrt{a} ,$$

$$k_1(a) = -F(1) \sqrt{a} , \qquad k_2(a) = -G(1)\sqrt{a} .$$

For a system of randomly oriented cracks the new variables are defined as

$$\tau_j = \frac{t_j}{a_j} \quad , \qquad \xi_i = \frac{s_i}{a_i} \quad (i,j=1,\ldots,n)$$

and the unknown functions can be expressed as

$$f_j(t_j) = F_j(\tau_j)(1-\tau_j^2)^{-\frac{1}{2}} \ ,$$

$$g_j(t_j) = G_j(\tau_j)(1-\tau_j^2)^{-\frac{1}{2}} \ . \qquad (j=1,\ldots n)$$

introducing the new kernels

$$K_{ij}(\xi_i,\tau_j) = a_j k_{ij}(s_i,t_j) \ ,$$
$$H_{ij}(\xi_i,\tau_j) = a_j h_{ij}(s_i,t_j) \ ,$$
$$M_{ij}(\xi_i,\tau_j) = a_j m_{ij}(s_i,t_j) \ ,$$
$$N_{ij}(\xi_i,\tau_j) = a_j n_{ij}(s_i,t_j) \qquad\qquad (34a\text{-}d)$$

equations (25a,b) and (26a,b) become

$$\frac{1}{\Pi} \sum_{j=1}^{n} \Big[\int_{-1}^{1} K_{ij}(\xi_i,\tau_j)F_j(\tau_j)(1-\tau_j^2)^{-\frac{1}{2}}$$

$$+ \int_{-1}^{1} H_{ij}(\xi_i,\tau_j)G_j(\tau_j)(1-\tau_j^2)^{-\frac{1}{2}}d\tau_j$$

$$\frac{1}{\pi} \sum_{j=1}^{n} \left[\int_{-1}^{1} M_{ij}(\xi_i, \tau_j) F_j(\tau_j)(1-\tau_j^2)^{-\frac{1}{2}} d\tau_j \right.$$

$$\left. + \int_{-1}^{1} N_{ij}(\xi_i, \tau_j) G_j(\tau_j)(1-\tau_j^2)^{-\frac{1}{2}} d\tau_j \right] = q_i(\xi_i) ,$$

$$-1 < \xi_i < 1 \qquad (i=1,\ldots,n)$$

$$\int_{-1}^{1} F_j(\tau_j)(1-\tau_j^2)^{-\frac{1}{2}} d\tau_j = 0 ,$$

$$\int_{-1}^{1} G_j(\tau_j)(1-\tau_j^2)^{-\frac{1}{2}} d\tau_j = 0, \qquad (j=1,\ldots,n) \quad (35a\text{-}d)$$

$$\tau_j = \frac{2t_j}{b_j-a_j} - \frac{b_j+a_j}{b_j-a_j}$$

Now applying the Gauss-Chebyshev quadrature gives the following system of linear algebraic equations for unknowns $F_j(\tau_k)$ and $G_j(\tau_k)$ $(j=1,\ldots,n)$, $(k=1,\ldots,N)$.

$$\sum_{j=1}^{n} \sum_{k=1}^{N} W_k F_j(\tau_k) K_{ij}(\xi_\ell, \tau_k) + G_j(\tau_k) H_{ij}(\xi_\ell, \tau_k) = p_i(\xi_\ell) ,$$

$$\sum_{j=1}^{n} \sum_{k=1}^{N} W_k F_j(\tau_k) M_{ij}(\xi_\ell, k) + G_j(\tau_k) N_{ij}(\xi_\ell, \tau_k) = q_i(\xi_\ell) ,$$

$$(i=1,\ldots,n), \quad (\ell=1,\ldots,$$

$$\sum_{k=1}^{N} W_k F_j(\tau_k) = 0 ,$$

$$\sum_{k=1}^{N} W_k G_j(\tau_k) = 0 , \qquad (j=1,\ldots,n), \qquad (36a\text{-}d)$$

where τ_k, ξ_ℓ, W_k $(k,\ell=1,\ldots,N)$ were previously defined.

The stress intensity factors can be defined with expressions similar to (33a-d) and they can be obtained from the values of $F_j(\tau_j)$ and $G_j(\tau_j)$ at the corresponding end points.

For a system of radial cracks the new variables are defined as

$$\tau_j = \frac{2t_j}{b_j-a_j} - \frac{b_j+a_j}{b_j-a_j} \quad ,$$

$$\xi_i = \frac{2s_i}{b_i-a_i} - \frac{b_i+a_i}{b_i-a_i} \quad , \quad (i,j=1,\ldots,n) \quad ,$$

and the unknown functions can be expressed as

$$f_j(t_j) = F_j(\tau_j)(1-\tau_j^2)^{-\frac{1}{2}} \quad ,$$
$$g_j(t_j) = G_j(\tau_j)(1-\tau_j^2)^{-\frac{1}{2}} \quad . \quad (j=1,\ldots,n)$$

Introducing the new kernels

$$K_{ij}(\xi_i,\tau_j) = \frac{b_j-a_j}{2} k_{ij}(s_i,t_j) \quad ,$$

$$H_{ij}(\xi_i,\tau_j) = \frac{b_j-a_j}{2} h_{ij}(s_i,t_j) \quad ,$$

$$M_{ij}(\xi_i,\tau_j) = \frac{b_j-a_j}{2} m_{ij}(s_i,t_j) \quad ,$$

$$N_{ij}(\xi_i,\tau_j) = \frac{b_j-a_j}{2}\, n_{ij}(s_i,t_j) \qquad (37a\text{-}d)$$

then equations (27a,b) and (28a,b) will take the forms identical with (35a-d). Again by applying the Gauss-Chebyshev quadrature formula a system of linear algebraic equations will be obtained to determine the unknowns $F_j(\tau_j)$ and $G_j(\tau_k)$ $(j=1,\ldots,n;\ k=1,\ldots,N)$.

For a single radial crack $n=1$ and the subscripts of all the quantities will be removed. Then for a crack with end points (a,b) the stress intensity factors may be defined as

$$k_1(a) = \lim_{s\to a} \sqrt{2(a-s)}\ \sigma_{qq}(0,s)\ ,$$

$$k_1(b) = \lim_{s\to b} \sqrt{2(s-b)}\ \sigma_{qq}(0,s)\ ,$$

$$k_2(a) = \lim_{s\to a} \sqrt{2(a-s)}\ \sigma_{qs}(0,s)\ ,$$

$$k_2(b) = \lim_{s\to b} \sqrt{2(s-b)}\ \sigma_{qs}(0,s)\ . \qquad (38a\text{-}d)$$

It is then observed that the stress intensity factors are proportional to the values of $F(\tau)$ and $G(\tau)$ at the end points through the relations

$$k_1(a) = F(-1)\sqrt{\frac{b-a}{2}}\ ,$$

$$k_1(b) = -F(1) \sqrt{\frac{b-a}{2}} \, ,$$

$$k_2(a) = G(-1) \sqrt{\frac{b-a}{2}} \, ,$$

$$k_2(b) = -G(1) \sqrt{\frac{b-a}{2}} \, . \qquad\qquad (39a\text{-}d)$$

The stress intensity factors of a system of radial cracks can be obtained from the values $F_j(\mp 1)$ and $G_j(\mp 1)$ with expressions similar to (39a-d).

CHAPTER 4

NUMERICAL RESULTS AND DISCUSSION

The numerical results are obtained first for a single crack. If the crack is perpendicular to the free surface for the pressurized crack we take $p(s) = -\sigma_o$ in (23a) and after normalizing the singular integral equation we solve it numerically. The calculated stress intensity factors are given in Table 1 (see Figure 7).

These results are identical with those given by Cook and Erdoğan [4]. Since the singular integral equation (23b) governing the mode II solution is the same with the equation for mode I case, the results for mode I apply to an analogous mode II problem.

Table 2 shows some numerical results for both symmetric and anti-symmetric loadings of a crack parallel to the free surface obtained by solving (24a,b) with $p(s) = -\sigma_o$, $q(s) = 0$ and $p(s) = 0$, $q(s) = -\tau_o$ (Figure 8). Thus the values given in the table

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TABLE 1

Stress Intensity Factors for a Crack Perpendicular to the Free Surface

$\dfrac{d}{a}$	$\dfrac{k_1(-a)}{\sigma_0\sqrt{a}}$	$\dfrac{k_1(a)}{\sigma_0\sqrt{a}}$
1.1	1.7585	1.2108
1.2	1.4637	1.1626
1.3	1.3316	1.1331
1.4	1.2544	1.1123
1.5	1.2035	1.0967
2.0	1.0913	1.0539
3.0	1.0345	1.0246
4.0	1.0182	1.0141
5.0	1.0112	1.0092
10.0	1.0027	1.0024
∞	1.0000	1.0000

$$k_1'(-a) = k_1(-a)/\sigma_0\sqrt{a}, \quad k_1'(a) = k_1(a)/\sigma_0\sqrt{a},$$

$$k_2'(-a) = k_2(-a)/\tau_0\sqrt{a}, \quad k_2'(a) = k_2(a)/\tau_0\sqrt{a}$$

are esentially the normalized stress intensity factors. Additionally the angles corresponding to the planes of maximum cleavage stress at the crack tips are given (Figure 6). These angles are shown to be the angles of crack propagation and may be calculated from [9]

$$k_1\sin\phi + k_2(3\sin\phi - 1) = 0 \, ,$$
$$3k_2\sin\phi - k_1\cos\phi < 0 \, .$$
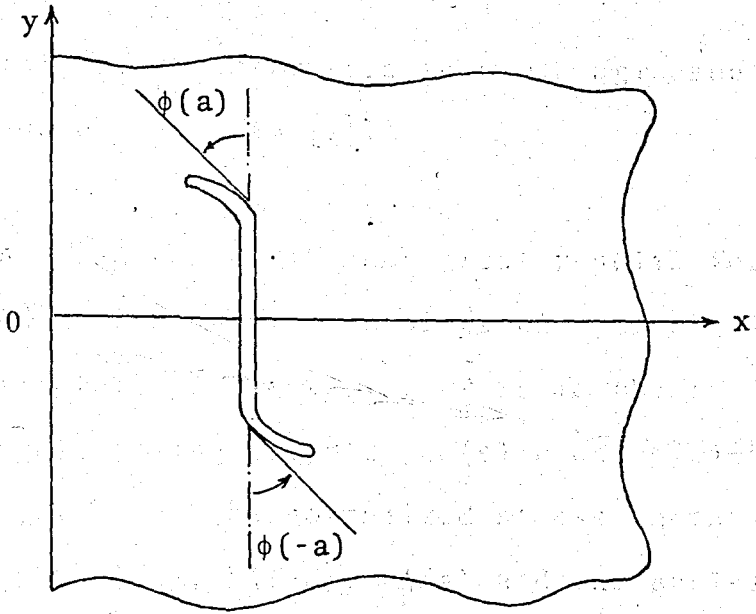
FIGURE 6. Probable Crack Propagation Angles.
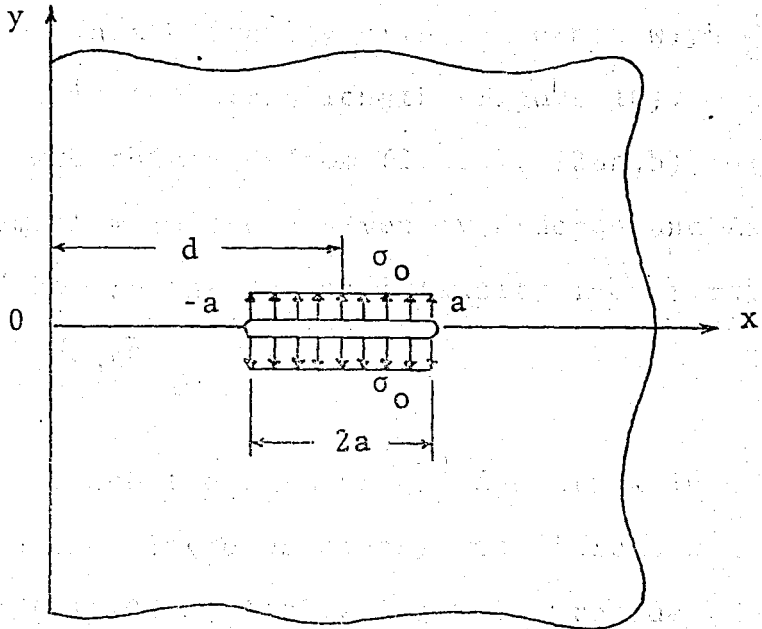
FIGURE 7. Pressurized Crack Perpendicular to the Free
Surface of a Half-Plane.

The stress intensity factors are in close agreement with those obtained by Ashbaugh [1].

Tables 3 and 4 show the numerical results for both symmetric and anti-symmetric loadings of a crack with different orientations and a fixed $\frac{d}{a}$ ratio obtained by solving (21a,b), taking $p(s)=-\sigma_0$, $q(s)=0$ and $p(s)=0$, $q(s)=-\tau_0$ (Figure 9). The normalized stress intensity factors $k_1'(-a)$, $k_1'(a)$, $k_2'(-a)$, $k_2'(a)$ and the angles for the maximum cleavage stress $\phi(-a)$, $\phi(a)$ are displayed for $\frac{d}{a} = 1.5$ or $\frac{d}{a} = 2$. ratios.

Table 5 shows the calculated stress intensity factor ratios for an arbitrarily oriented crack with $\frac{a}{c} = 1$, $\frac{b}{c} = 3$ where 2c is the crack length (Figure 10). These values which are obtained from (27a,b), (28a,b) for n = 1 are in agreement with those given by Erdogan and Aksogan [8]. k_1' and k_2' are the stress intensity factor ratios $k_1/\sigma_0\sqrt{c}$ and $k_2/\sigma_0\sqrt{c}$.

If we let a=0 the case of an edge crack is obtained. For this case the single-valuedness conditions will be replaced by F(-1)=0, G(-1)=0 which will provide the boundedness of the unknown functions (see [3]). The variations of the stress intensity factors with θ are displayed in Figure 12 for uniform loading of the half-plane at infinity

TABLE 2

Stress Intensity Factor Ratios and Cleavage Angles for a Crack Parallel to the Free Surface

$\frac{d}{a}$		0.1	0.2	0.3	0.4	0.5	1.	2.	3.	4.	5.	10.
$p=-\sigma_0$	$k_1'(-a)$	14.259	5.9499	3.8326	2.9055	2.3974	1.5110	1.1633	1.0779	1.0451	1.0292	1.0075
	$k_1'(a)$	14.259	5.9499	3.8326	2.9055	2.3974	1.5110	1.1633	1.0779	1.0451	1.0292	1.0075
	$k_2'(-a)$	8.9640	3.0301	1.5929	0.9940	0.6805	0.1849	0.0368	0.0124	0.0055	0.0029	0.0004
$q=0$	$k_2'(a)$	-8.9640	-3.0301	-1.5929	-0.9940	-0.6805	-0.1849	-0.0368	-0.0124	-0.0055	-0.0029	-0.0004
	$\phi(-a)$	-44.94	-40.60	-36.20	-31.93	-27.94	-13.55	-3.612	-1.313	0.600	0.320	0.042
	$\phi(a)$	44.94	40.60	36.20	31.93	27.94	13.56	3.612	1.313	-0.600	-0.320	-0.042
$p=0$	$k_1'(-a)$	-0.8697	-0.6151	-0.4801	-0.3876	-0.3182	-0.1331	-0.0330	-0.0117	-0.0053	-0.0028	-0.0004
	$k_1'(a)$	0.8697	0.6151	0.4801	0.3876	0.3182	0.1331	0.0330	0.0117	0.0053	0.0028	0.0004
	$k_2'(-a)$	3.0301	1.2077	1.1551	1.1329	1.1209	1.0874	1.0432	1.0233	1.0141	1.0094	1.0025
$q=-\tau_0$	$k_2'(a)$	-3.0301	1.2077	1.1551	1.1329	1.1209	1.0874	1.0432	1.0233	1.0141	1.0094	1.0025
	$\phi(-a)$	-82.79	-80.44	-78.61	-77.16	-76.02	-72.88	-71.13	-70.75	-70.63	-70.58	-70.54
	$\phi(a)$	-59.12	-61.18	-62.83	-64.15	-65.21	-68.21	-69.93	-70.31	-70.43	-70.48	-70.52
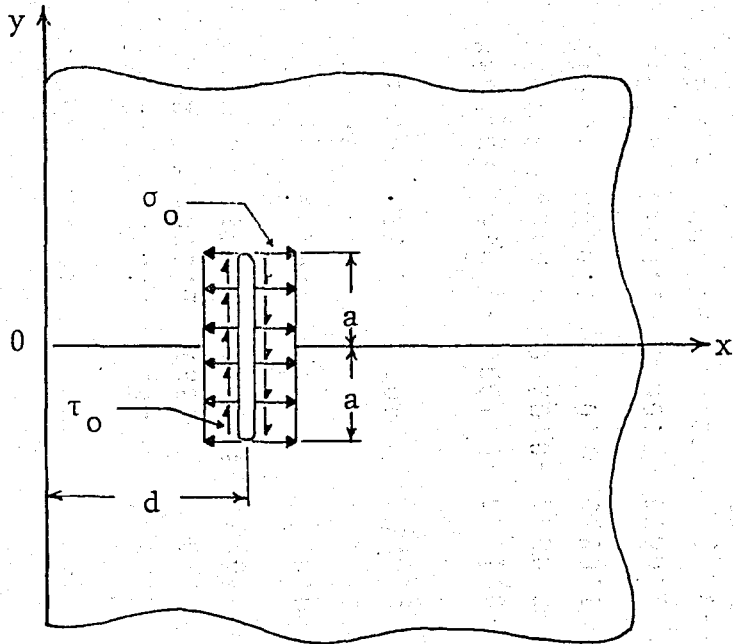
FIGURE 8. Crack Parallel to the Free Surface of a Half-Plane with Uniform Tractions Applied on the Crack Surfaces.
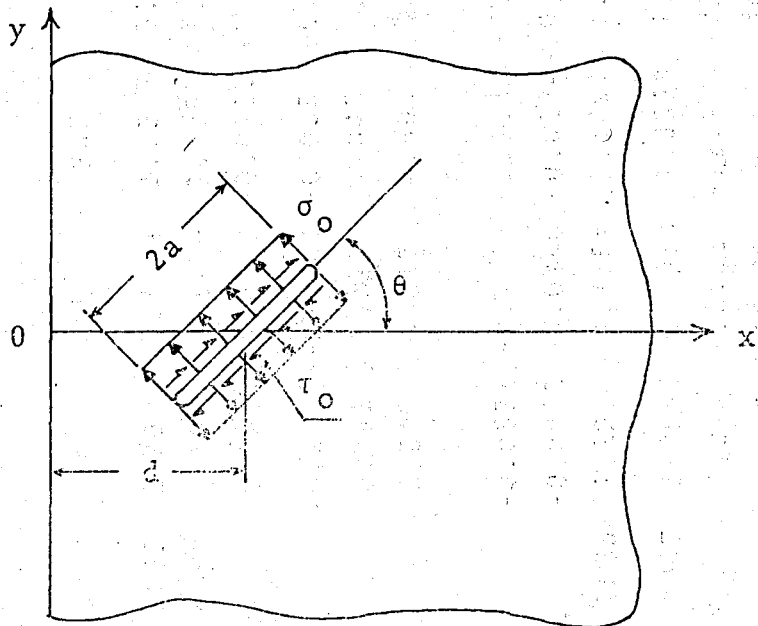
FIGURE 9. Arbitrarily Oriented Crack in a Half-Plane with Uniform Tractions Applied on the Crack Surfaces.

TABLE 3

Stress Intensity Factor Ratios and Cleavage Angles for an Arbitrarily Oriented Crack ($\frac{d}{a}=1.5$)

θ		0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
	$k_1'(-a)$	1.2035	1.2080	1.2204	1.2377	1.2558	1.2710	1.2807	1.2835	1.2790	1.2677
$p=-\sigma_o$	$k_1'(a)$	1.0967	1.1006	1.1119	1.1297	1.1523	1.1777	1.2038	1.2287	1.2505	1.2677
	$k_2'(-a)$	0.	-0.0083	-0.0129	-0.0113	-0.0031	0.0104	0.0271	0.0447	0.0611	0.0750
	$k_2'(a)$	0.	-0.0245	-0.0469	-0.0657	-0.0797	-0.0887	-0.0925	-0.0913	-0.0853	-0.0750
$q=0$	$\phi(-a)$	0.	0.791	1.212	1.045	0.281	-0.941	-2.424	-3.978	-5.449	-6.728
	$\phi(a)$	0.	2.549	4.817	6.609	7.842	8.519	8.688	8.407	7.735	6.728
	$k_1'(-a)$	0.	-0.0571	-0.1031	-0.1322	-0.1438	-0.1411	-0.1283	-0.1090	-0.0864	-0.0628
$p=0$	$k_1'(a)$	0.	-0.0083	-0.0146	-0.0174	-0.0156	-0.0086	-0.0034	-0.0200	-0.0402	0.0628
	$k_2'(-a)$	1.2035	1.1958	1.1754	1.1491	1.1229	1.1006	1.0836	1.0719	1.0648	1.0615
	$k_2'(a)$	1.0967	1.0953	1.0914	1.0858	1.0795	1.0733	1.0678	1.0637	1.0614	1.0615
$q=-\tau_o$	$\phi(-a)$	-70.53	-71.44	-72.21	-72.74	-72.99	-72.99	-72.80	-72.48	-72.09	-71.66
	$\phi(a)$	-70.53	-70.67	-70.78	-70.83	-70.80	-70.68	-70.47	-70.17	-69.81	-69.40

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Stress Intensity Factor Ratios and Cleavage Angles for an Arbitrarily Oriented Crack ($\frac{d}{a}=2$)

	θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
	$k_1'(-a)$	1.0913	1.0949	1.1049	1.1192	1.1349	1.1494	1.1604	1.1667	1.1676	1.1633
$p=-\sigma_o$	$k_1'(a)$	1.0539	1.0566	1.0644	1.0766	1.0918	1.1087	1.1257	1.1414	1.1543	1.1633
	$k_2'(-a)$	0.0	-0.0068	-0.0114	-0.0124	-0.0093	-0.0026	0.0067	0.0172	0.0276	0.0368
	$k_2'(a)$	0.0	-0.0130	-0.0251	-0.0354	-0.0431	-0.0480	-0.0497	-0.0483	-0.0439	-0.0363
$q=0$	$\phi(-a)$	0.0	0.709	1.183	1.272	0.941	0.258	-0.662	-1.689	-2.704	-3.612
	$\phi(a)$	0.0	1.414	2.700	3.755	4.512	4.941	5.041	4.830	4.340	3.612
	$k_1'(-a)$	0.0	-0.0236	-0.0440	-0.0588	-0.0669	-0.0685	-0.0646	-0.0563	-0.0453	-0.0330
$p=0$	$k_1'(a)$	0.0	-0.0053	-0.0095	-0.0116	-0.0109	-0.0073	-0.0003	0.0091	0.0205	0.0330
	$k_2'(-a)$	1.0913	1.0894	1.0842	1.0768	1.0686	1.0607	1.0540	1.0488	1.0452	1.0432
	$k_2'(a)$	1.0539	1.0535	1.0523	1.0505	1.0484	1.0463	1.0444	1.0431	1.0426	1.0432
$q=-\tau_o$	$\phi(-a)$	-70.53	-70.94	-71.31	-71.57	-71.73	-71.77	-71.70	-71.56	-71.36	-71.13
	$\phi(a)$	-70.53	-70.63	-70.70	-70.74	-70.73	-70.66	-70.54	-70.36	-70.15	-69.93

TABLE   5

Stress Intensity Factor Ratios for an

Arbitrarily Oriented Crack ($\frac{a}{c}=1$, $\frac{b}{c}=3$).

| θ°	$k_1'(a)$	$k_1'(b)$	$k_2'(a)$	$k_2'(b)$
0	1.0913	1.0539	0	0
5	1.0838	1.0468	0.0918	0.0849
10	1.0613	1.0255	0.1802	0.1674
15	1.0245	0.9908	0.2646	0.2448
20	0.9741	0.9437	0.3422	0.3150
25	0.9112	0.8857	0.4110	0.3757
30	0.8374	0.8186	0.4693	0.4253
35	0.7540	0.7444	0.5154	0.4623
40	0.6630	0.6655	0.5480	0.4854
45	0.5662	0.5842	0.5657	0.4940
50	0.4658	0.5028	0.5677	0.4879
55	0.3642	0.4238	0.5530	0.4670
60	0.2642	0.3492	0.5209	0.4320
65	0.1691	0.2806	0.4720	0.3838
70	0.0834	0.2191	0.4041	0.3241
75	0.0127	0.1647	0.3210	0.2548
80	-0.0356	0.1159	0.2254	0.1789
85	-0.0507	0.0689	0.1237	0.0996
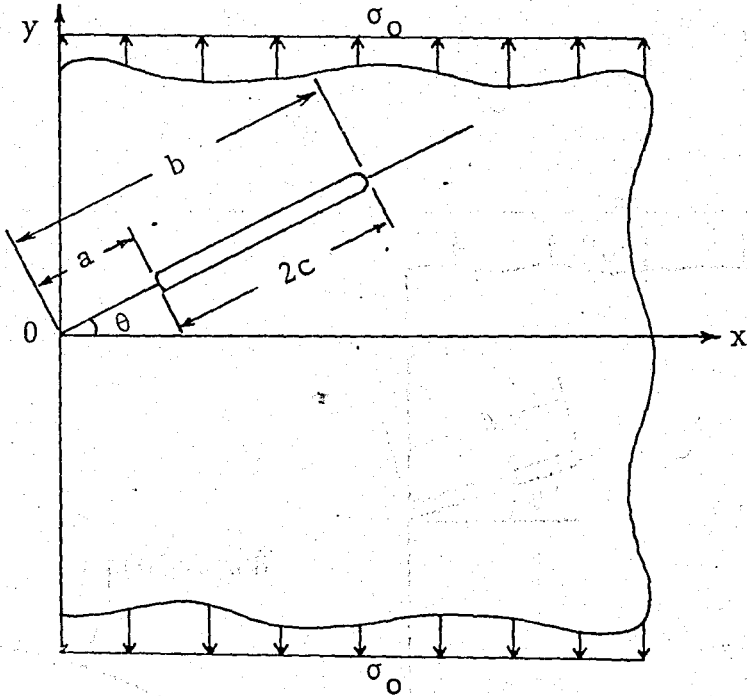
FIGURE 10. Arbitrarily Oriented Crack in a Half-Plane
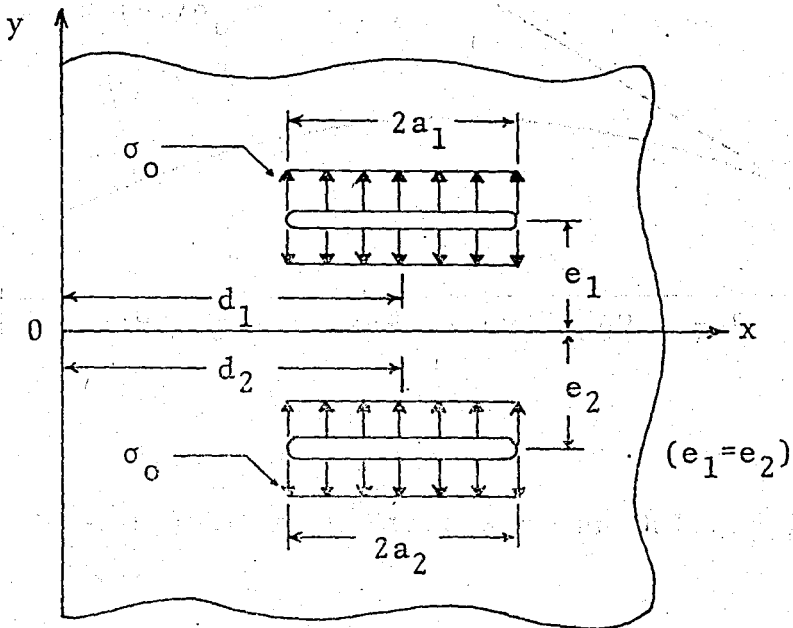with Uniform Tension Applied at Infinity.

FIGURE 11. Two Cracks Perpendicular to the Free Surface
of a Half-Plane with Uniform Pressure Acting
on the Crack Surfaces.
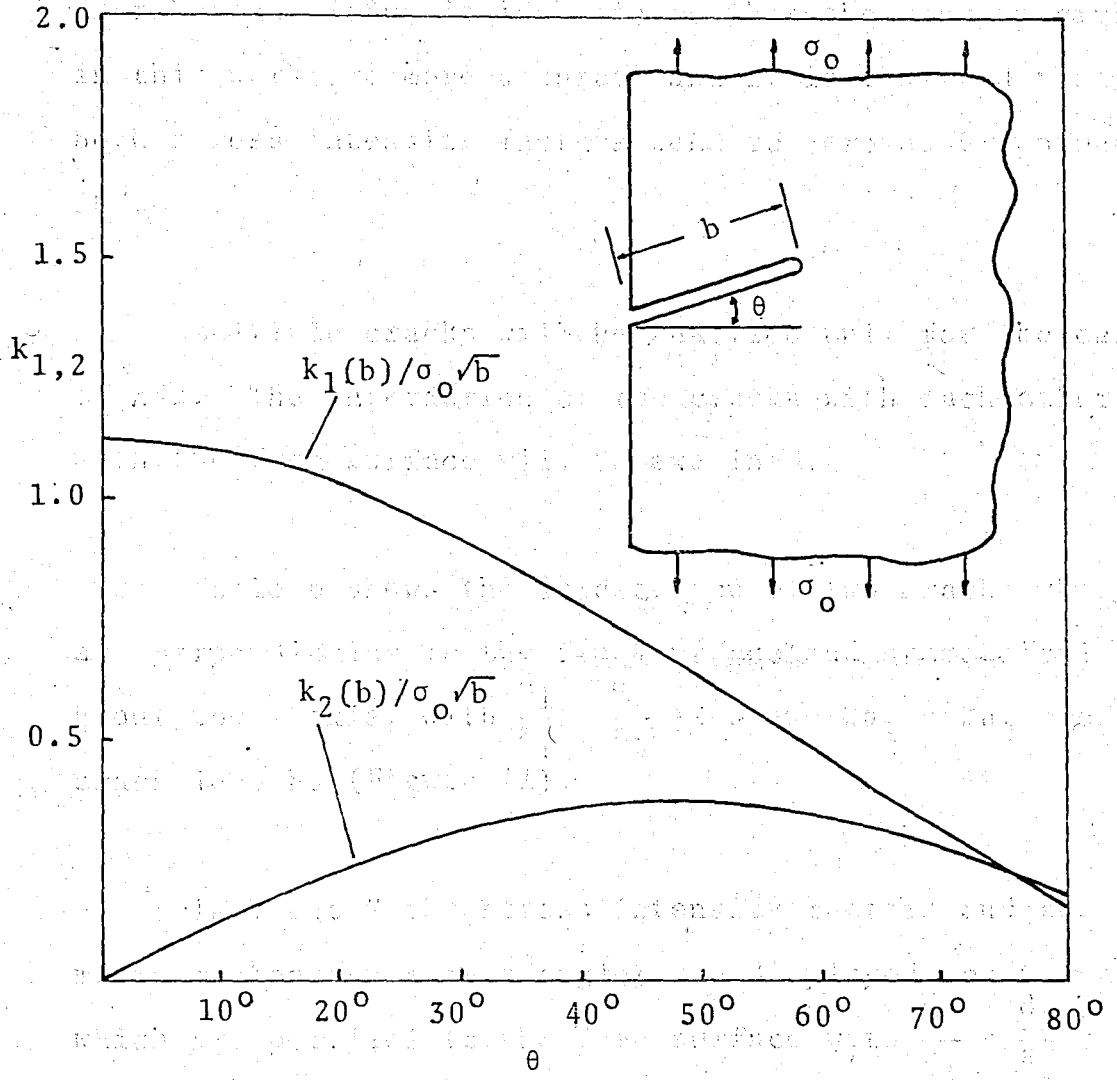
FIGURE 12. Stress Intensity Factors of an Oblique Edge Crack.

These curves are in close agreement with those given by Ioakimidis and Theocaris [13] except the results for large values of θ. Even though the convergence is very poor beyond $\theta=80^O$ it is believed that the results given in this work are more accurate and it is observed that both stress intensity factors tend to zero as θ approaches 90°.

Multiple cracks will be analyzed only for the case of $n=2$. The interaction of two cracks with each other and with the free surface will be examined.

Table 6 shows the interaction of two cracks which are perpendicular to the free surface and symmetrical about the x axis, with $\frac{d_1}{a_1} = \frac{d_2}{a_2} = 2$ where $2a_1 = 2a_2$ are the crack lengths (Figure 11).

In Table 7 the stress intensity factors and the maximum cleavage stress angles are displayed for two cracks which are parallel to the free surface with $\frac{d_1}{a_1} = \frac{d_2}{a_2} = 2$ where $2a_1 = 2a_2$ are the crack lengths (Figure 13). From both tables it is observed that when the cracks are suffi-ciently away from each other single crack solutions are obtained.

In Tables 8, 9 and 10 some radial crack interactions are shown. Table 8 shows the interaction of two radial

TABLE 6

Stress Intensity Factor Ratios and Cleavage Angles for Two Cracks Perpendicular to the Free Surface ($\frac{d_1}{a_1} = \frac{d_2}{a_2} = 2$, $p(s)=-\sigma_o$, $q(s)=0$, $k'=k/\sigma_o\sqrt{a_1}$)

$\frac{e_1}{a_1}$	$k_1'(-a_1)$	$k_1'(a_1)$	$k_2'(-a_1)$	$k_2'(a_1)$	$\phi(-a_1)$	$\phi(a_1)$
0.05	0.7562	0.7254	0.2052	-0.2094	-26.999	28.288
0.10	0.7732	0.7391	0.1899	-0.2009	-24.98	27.04
0.20	0.8012	0.7633	0.1613	-0.1818	-21.21	24.38
0.30	0.8246	0.7846	0.1382	-0.1652	-18.08	22.02
0.40	0.8449	0.8040	0.1197	-0.1498	-15.53	19.85
0.50	0.8634	0.8219	0.1048	-0.1351	-13.46	17.77
1.00	0.9466	0.9012	0.0586	-0.0692	-7.027	8.684
2.00	1.0310	0.9956	0.0092	-0.0164	-1.028	1.888
3.00	1.0486	1.0213	-0.0064	-0.0118	0.679	1.324
4.00	1.0570	1.0286	-0.0078	-0.0104	0.845	1.163
5.00	1.0638	1.0327	-0.0063	-0.0083	0.673	0.926
10.00	1.0813	1.0449	-0.0014	-0.0021	0.153	0.233
20.00	1.0885	1.0513	-0.0002	-0.0003	0.022	0.035
40.00	1.0906	1.0532	-0.0003	-0.0004	0.003	0.005

TABLE 7

Stress Intensity Factor Ratios and Cleavage Angles for Two Cracks·Parallel to the Free Surface ($\frac{d_1}{a_1} = \frac{d_2}{a_2} = 2$, $p(s)=-\sigma_o$, $q(s)=0$, $k'=k/\sigma_o\sqrt{a_1}$)

$\frac{e_1}{a_1}$	$k_1'(-a_1)$	$k_1'(a_1)$	$k_2'(-a_1)$	$k_2'(a_1)$	$\phi(-a_1)$	$\phi(a_1)$
1.1	1.9193	1.3360	-0.0253	-0.0982	1.510	8.320
1.2	1.5856	1.2679	-0.0258	-0.0819	1.867	7.327
1.3	1.4578	1.2293	-0.0224	-0.0707	1.785	6.536
1.4	1.3510	1.2042	-0.0169	-0.0619	1.436	5.857
1.5	1.2935	1.1871	-0.0104	-0.0549	0.919	5.269
2.0	1.1753	1.1551	0.0218	-0.0350	-2.123	3.464
3.0	1.1528	1.1561	0.0435	-0.0298	-4.307	2.953
4.0	1.1587	1.1606	0.0418	-0.0329	-4.125	3.238
5.0	1.1616	1.1623	0.0394	-0.0348	-3.873	3.425
10.0	1.1633	1.1633	0.0369	-0.0366	-3.625	3.601
20.0	1.1634	1.1634	0.0368	-0.0368	-3.612	3.612

TABLE 8

Stress Intensity Factor Ratios and Cleavage Angles for Two Radial Cracks in a Half-Plane, Symmetrical About x Axis Loaded at Infinity with σ_o Parallel to the Free Surface $(\frac{a_1}{c_1} = 1, \frac{b_1}{c_1} = 3, \frac{a_2}{c_2} = 1, \frac{b_2}{c_2} = 3, c_1 = c_2, k' = k/\sigma_o\sqrt{c_1})$

θ^o	$k_1'(a_1)$	$k_1'(b_1)$	$k_2'(a_1)$	$k_2'(b_1)$	$\phi(a_1)$	$\phi(b_1)$
5	0.7407	0.7831	0.2882	-0.0859	-34.75	12.24
10	0.7451	0.8158	0.3185	0.0146	-36.81	- 2.05
20	0.7285	0.8160	0.3797	0.1948	-41.09	-24.41
30	0.6785	0.7485	0.4429	0.3394	-45.68	-38.10
40	0.5799	0.6316	0.5018	0.4302	-50.81	-46.50
50	0.4316	0.4884	0.5291	0.4566	-56.02	-52.08
60	0.2545	0.3437	0.4986	0.4174	-61.16	-55.90
70	0.0825	0.2173	0.3957	0.3191	-66.60	-58.24
80	-0.0353	0.1155	0.2241	0.1781	-73.56	-58.79
85	-0.0506	0.0689	0.1234..	0.0994	-78.50	-58.05

TABLE 9

Stress Intensity Factor Ratios and Cleavage Angles for Two Symmetrically Located Cracks in a Half-Plane Forming Angles $\mp 45^{\circ}$ with x Axis, Loaded at Infinity with σ_o Parallel to the Free Surface $(k'=k/\sigma_o\sqrt{c_1})$

$\dfrac{a_1+b_1}{2c_1}$	$k_1'(a_1)$	$k_1'(b_1)$	$k_2'(a_1)$	$k_2'(b_1)$	$\phi(a_1)$	$\phi(b_1)$
1.1	0.5984	0.6818	0.8315	0.3758	-57.60	-42.26
1.2	0.5433	0.6505	0.6880	0.3886	-56.45	-43.91
1.3	0.5245	0.6297	0.6255	0.4001	-55.66	-45.15
1.4	0.5166	0.6139	0.5901	0.4103	-55.07	-46.14
1.5	0.5131	0.6013	0.5674	0.4195	-54.61	-46.96
2.0	0.5111	0.5617	0.5212	0.4518	-53.43	-49.55
3.0	0.5101	0.5290	0.5032	0.4794	-52.92	-51.58
4.0	0.5077	0.5163	0.5000	0.4893	-52.90	-52.29
5.0	0.5057	0.5104	0.4993	0.4936	-52.93	-52.62
10.0	0.5018	0.5025	0.4995	0.4987	-53.06	-53.02

cracks which are symmetrical about x axis with $\frac{a_1}{c_1} = 1$, $\frac{b_1}{c_1} = 3$ and $\frac{a_2}{c_2} = 1$, $\frac{b_2}{c_2} = 3$ where $2c_1 = 2c_2$ are the crack lengths (Figure 14). Table 9 displays the interaction of two cracks of equal length and of equal radial distance forming angles $\theta = \mp 45°$ with x axis (figure 15). In Table 10 the results are given for two radial crakcs, one on x axis and the other one forming different angles with x axis where $\frac{a_1}{c_1} = 1$, $\frac{b_1}{c_1} = 3$ and $\frac{a_2}{c_2} = 1$, $\frac{b_2}{c_2} = 3$ (Figure 16).

Figure 17 shows the stress intensity factor ratios for two collinear cracks perpendicular to the free surface when the uniform tension is applied at infinity parallel to the free surface. It is assumed that one of the cracks is an edge crack. It is observed from the figure that when the negihboring crack tips get close the stress intensity factors become very large which implies that the cracks will run into each other with a slight loading. But if the crack tips are significantly away from each other then the stress intensity factor ratios are very close and one can determine at which load level the cracks will run into each other from the highest stress intensity factor ratio, namely, $k_1(b_1)/\sigma_1\sqrt{b_1}$.

The results presented here show the effects of crack orientations on the stress intensity factors. It is seen that as the cracks move away from the free surface the results approach to the values corresponding to the infinite plane.
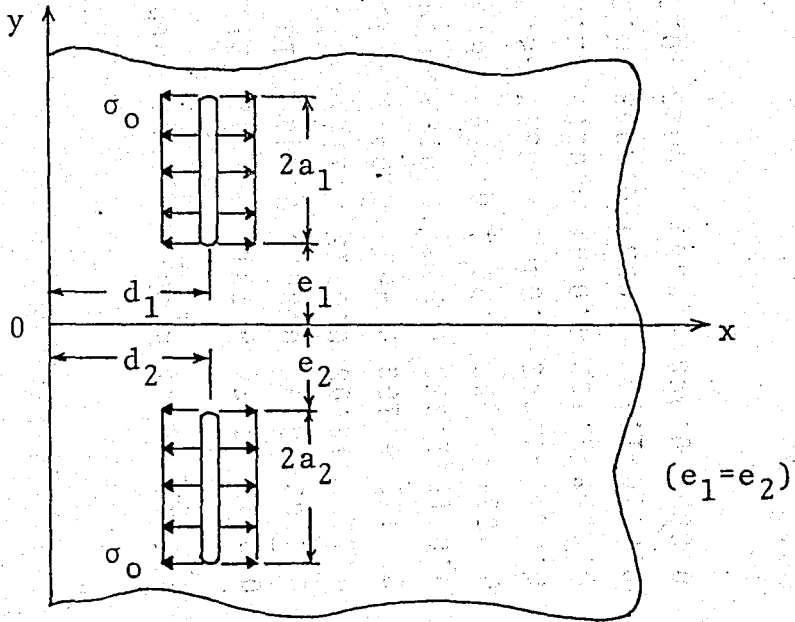
FIGURE 13. Two Cracks Parallel to the Free Surface of a
Half-Plane with Uniform Pressure Acting on
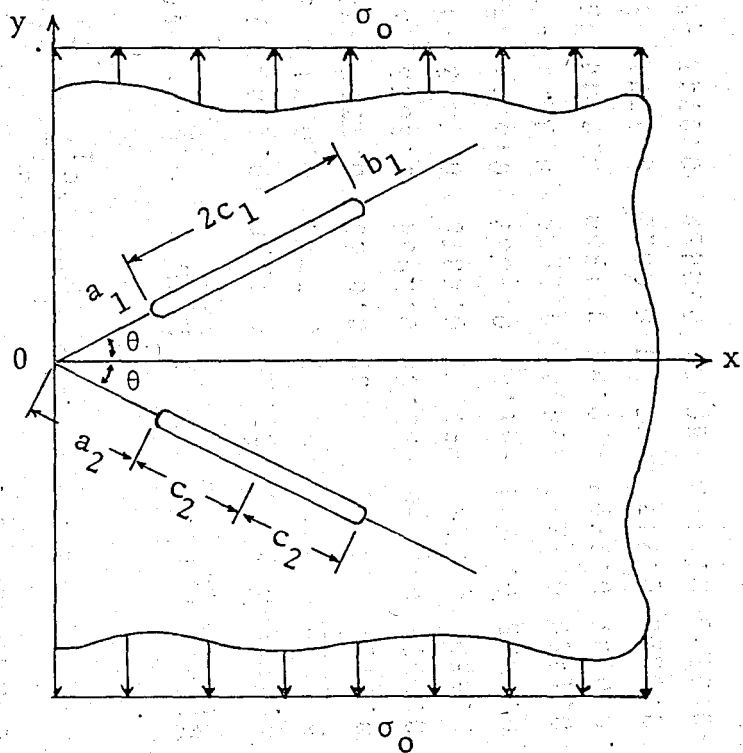Crack Surfaces.

FIGURE 14. Two Symmetrical Cracks in a Half-Plane with
Uniform Tension Applied at Infinity.

TABLE 10

Stress Intensity Factor Ratios and Cleavage Angles for Two Radial Cracks on a Half-Plane, One on x Axis and the Other One Arbitrarily Oriented, Loaded at Infinity with σ_o Parallel to the Free Surface ($\frac{a_1}{c_1}=1.$, $\frac{b_1}{c_1}=3.$, $\frac{a_2}{c_2}=1.$, $\frac{b_2}{c_2}=3.$, $k'=k/\sigma_o\sqrt{c_1}$)

| θ_1	θ_2	$k'_1(a_1)$	$k'_2(a_1)$	$\phi(a_1)$	$k'_1(b_1)$	$k'_2(b_1)$	$\phi(b_1)$	$k'_1(a_2)$	$k'_2(a_2)$	$\phi(a_2)$	$k'_1(b_2)$	$k'_2(b_2)$	$\phi(b_2)$
5.00	0.00	0.7565	0.3004	-35.20	0.7268	-0.1184	17.62	0.7125	-0.2416	31.74	0.7817	0.1642	-21.98
10.00	0.00	0.7652	0.3605	-38.93	0.7210	-0.0321	5.07	0.7058	-0.2102	28.95	0.8343	0.1404	-18.15
20.00	0.00	0.7192	0.4870	-46.39	0.6765	0.1315	-20.58	0.7301	-0.1223	18.08	0.9115	0.1151	-13.97
30.00	0.00	0.6077	0.5836	-52.50	0.5954	0.2635	-37.57	0.7878	-0.0427	6.168	0.9637	0.0975	-11.33
40.00	0.00	0.4605	0.6267	-57.35	0.4876	0.3462	-47.29	0.8587	0.0115	-1.533	0.9997	0.0789	-8.913
50.00	0.00	0.3005	0.6104	-61.48	0.3682	0.3712	-53.26	0.9297	0.0381	-4.682	1.0242	0.0587	-6.515
60.00	0.00	0.1450	0.5343	-65.44	0.2535	0.3381	-57.10	0.9922	0.0413	-4.746	1.0402	0.0387	-4.246
70.00	0.00	0.0142	0.3997	-69.85	0.1569	0.2544	-59.33	1.0417	0.0288	-3.159	1.0491	0.0207	-2.261
80.00	0.00	-0.0590	0.2172	-75.79	0.0822	0.1371	-59.62	1.0760	0.0110	-1.169	1.0530	0.0068	-0.744
85.00	0.00	-0.0576	0.1182	-80.01	0.0488	0.0743	-58.64	1.0861	0.0039	-0.408	1.0538	0.0023	-0.251
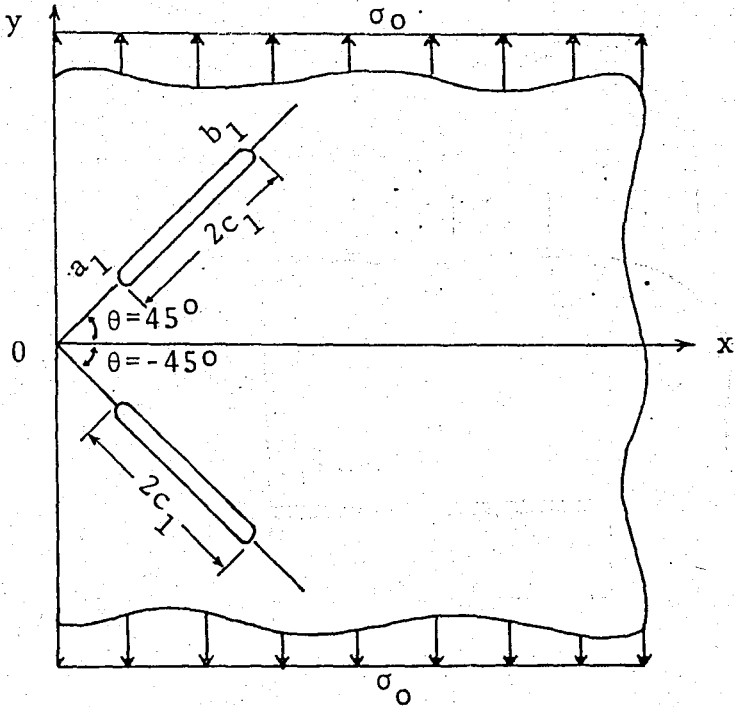
FIGURE 15. Two Symmetrical Cracks in a Half-Plane Forming
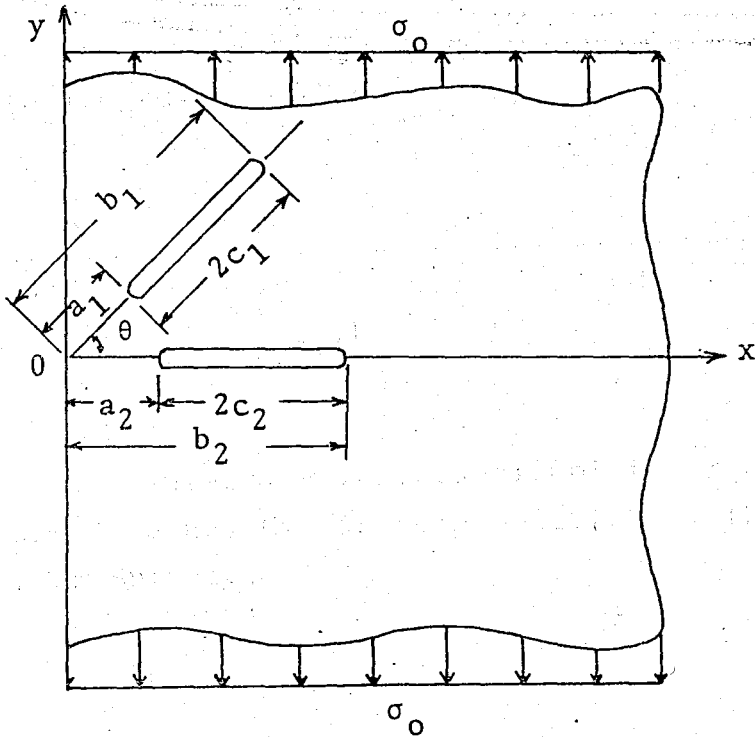Angles $\mp 45^{\circ}$ with x Axis.

FIGURE 16. Two Radial Cracks in a Half-Plane, One Located
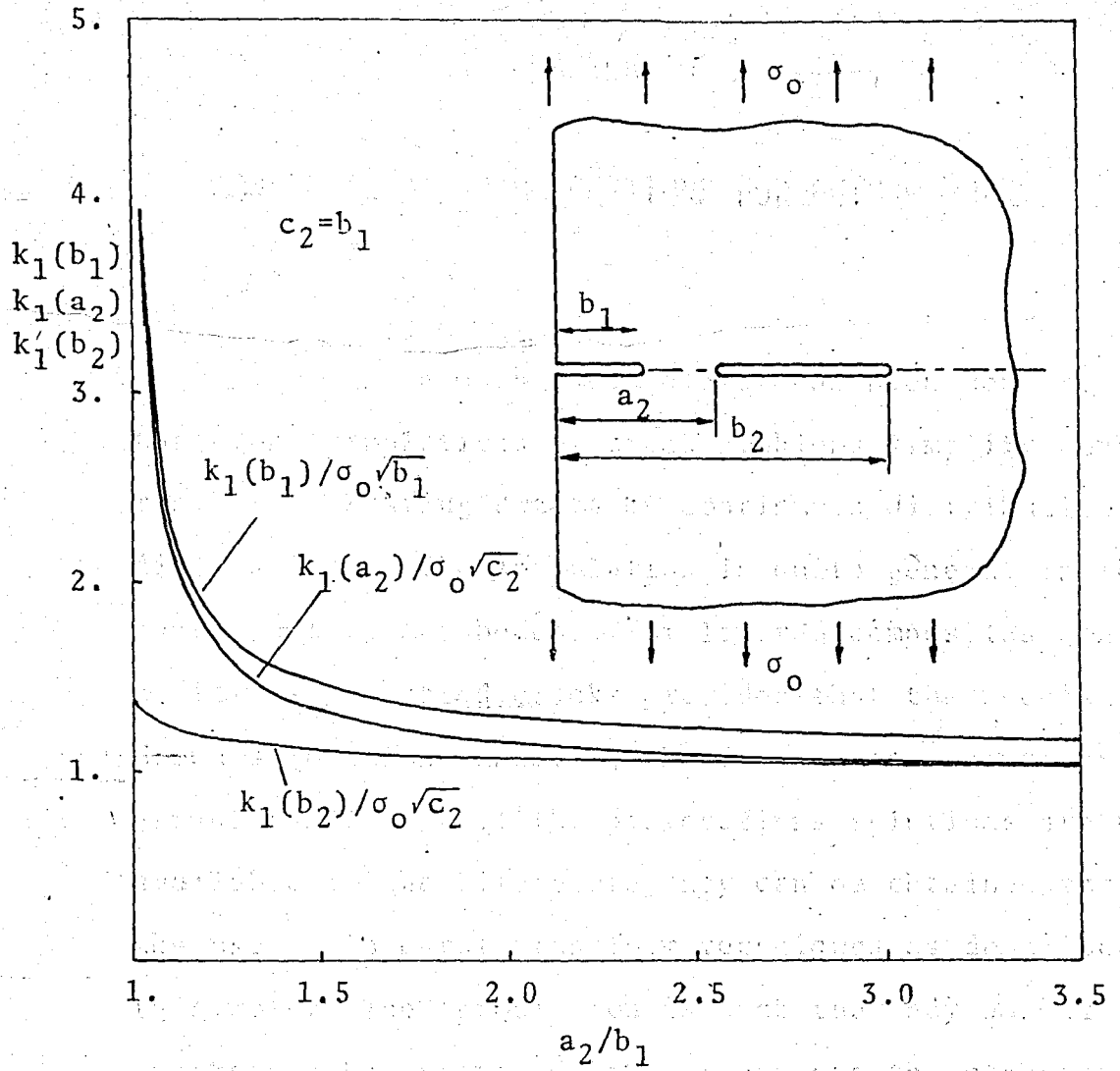on x Axis, the Other One Arbitrarily Oriented.

FIGURE 17. Stress Intensity Factors of Collinear Edge and Internal Cracks Perpendicular to the Free Surface.

CHAPTER 5

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

In the preceding sections it has been demonstrated
that the formulations of crack problems simplify conside-
rably representing cracks by continuous distributions of
dislocations. The formulation is quite general in the
sense that it may be used for layered composites containin
arbitrarily located cracks provided that the necessary
Green's functions, namely, the dislocation solutions are
established.   If the dislocations solutions are not
available in the literature they can be obtained through
the use of integral transform techniques as described in
this work. The formulation is such that any number of
cracks can be treated. Limitations are the computer's
memory and the execution time.

It is suggested that this formulation should be ap-
plied to bonded elastic half-planes and bonded elastic
strips next for a better understanding of the applications
of linear elastic fracture mechanics in layered composites

REFERENCES

1. Ashbaugh, N., "Stress Solution for a Crack at an Arbitrary Angle to an Interface", _International Journal of Fracture_, 11 (1975), pp. 205-219.

2. Atkinson, C., "On Dislocation Densities and Stress Singularities Associated with Cracks and Pile Ups in Inhomogenous Media", _Int. J. Engng. Sci._, Vol. 10, Pergamon Press, 1972, pp. 45-71.

3. Civelek, M.B., "Simetrik Tekil Kuvvetlerle Yüklü Sonsuz Şeritte Çatlak Problemi", _Doçentlik Tezi_, 1978, pp- 73-76.

4. Cook, T.S. and Erdogan, F., "Stresses in Bonded Materials with a Crack Perpendicular to the Interface", _International Journal of Engineering Science_, Vol. 11, 1973, pp. 745-766.

5. Dieter, G.E., _Mechanical Metallurgy_, McGraw-Hill Series in Materials Science, 1976, pp. 112-120.

6. Erdelyi, A., _Tables of Integral Transforms_, Vol. 1, McGraw-Hill, New York, 1954.

7. Erdogan, F., Gupta, G.D. and Ratwani, M., "Interaction Between a Circular Inclusion and an Arbitrarily Oriented Crack", _Journal of Applied Mechanics_, December 1974, pp. 1007-1013.

8. Erdogan, F. and Aksogan, O., "Bonded Half Planes Containing an Arbitrarily Oriented Crack", _Int. J. Solids Structures_, Vol. 10, 1974, pp. 569-585.

9. Erdogan, F. and Sih, G.C., "On the Crack Extension in Plates under Plane Loading and Transverse Shear", _J. Basic Engng Trans._ ASME, 85, Series D, pp. 519 (1963).

10. Erdogan, F. and Gupta, G.D., "On the Numerical Solution of Singular Integral Equations", _Q. Appl. Math._ 29, 525 (1972).

11. Griffith, A.A., "The Phenomenon of Rupture and Flow in Solids", _Phil. Trans. Royal Soc._, London, 1921.

12. Head, A.K., "Edge Dislocations in Inhomogenous Media", _Proc. Phys. Soc._, Lond. 66 B, 1953, pp. 793-805.

13. Ioakimidis, N.I. and Theocaris, P.S., "A Sytem of Curvilinear Cracks in an Isotropic Elastic Half-Plane", _International Journal of Fracture_, Vol. 15, No. 4, August 1979, pp. 299-309.

14. Knott, J.F., _Fundamentals of Fracture Mechanics_, Department of Metallurgy and Materials Science, Cambridge University, London, 1973, pp. 1-11.

15. McClintock, F.A. and Argon, A.S., _Mechanical Behavior of Materials_, Addison-Wesley Publishing Company, Inc., USA, 1966, pp. 96-106.

16. Mura, T., "Individual Dislocations and Continuum Mechanics", _Inelastic Behavior of Soilds_, McGraw-Hill Book Company, Materials Science and Engineering Series, 1970, pp. 221-230.

17. Savruk, M.P. and Datsyshin, A.P., "Interaction Between a System of Cracks and the Boundaries of an Elastic Body", Physicomechanical Institute, Academy of Sciences of the Ukrainian SSR, L'vov. Translated from _Prikladnaya Mekhanika_, Vol. 10, No. 7, 1974, pp. 84-92.

18. Singer, F.L., _Strength of Materials_, Harper, Row, Publishers, Inc., 1962, pp. 326-330.

19. Timoshenko, S.P. and Goodier, J.N., _Theory of Elasticity_, McGraw-Hill, Inc., 1970, pp. 53-55.

20. Wylie, C.R., _Advanced Engineering Mathematics_, McGraw-Hill Kogakusha Ltd., 1975, pp. 240-252.

APPENDIX

EVALUATION OF CERTAIN INFINITE INTEGRALS

$$I_1 = \int_0^\infty \frac{Y(Y^2+3a_o)^2}{(a_o^2+Y^2)^2} \sin\alpha Y dY = \int_0^\infty \frac{Y}{a_o^2+Y^2} \sin\alpha Y dY$$

$$- a_o \frac{d}{da_o} \int_0^\infty \frac{Y}{a_o^2+Y^2} \sin\alpha Y dY$$

From Tables [6]

$$\int_0^\infty \frac{Y}{a_o^2+Y^2} \sin\alpha Y dY = \frac{\Pi}{2} e^{-\alpha a_o}$$

$$I_1 = \frac{\Pi}{2} e^{-\alpha a_o} - a_o \frac{d}{da_o} \left(\frac{\Pi}{2} e^{-\alpha a_o}\right)$$

$$= \frac{\Pi}{2} e^{-\alpha a_o} + \frac{\Pi}{2} a_o e^{-\alpha a_o}$$

$$= \frac{\Pi}{2} e^{-\alpha a_o} (1+\alpha a_o) \tag{A.1}$$

$$I_2 = \int_0^\infty \frac{Y(Y^2-a_o^2)}{(a_o^2+Y^2)} \sin\alpha Y dY = \int_0^\infty \frac{Y}{a_o^2+Y^2} \sin\alpha Y dY$$

$$+ a_o \frac{d}{da_o} \int_0^\infty \frac{Y}{a_o^2+Y^2} \sin\alpha Y dY$$

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From Tables [6]

$$\int_0^\infty \frac{Y}{a_0^2+Y^2} \sin\alpha Y dY = \frac{\Pi}{2} e^{-\alpha a_0}$$

$$I_2 = \frac{\Pi}{2} e^{-\alpha a_0} + a_0 \frac{d}{da_0} \left(\frac{\Pi}{2} e^{-\alpha a_0}\right)$$

$$= \frac{\Pi}{2} e^{-\alpha a_0} - \frac{\Pi}{2} a_0 e^{-\alpha a_0}$$

$$= \frac{\Pi}{2} e^{-\alpha a_0} (1-\alpha a_0) \qquad\qquad (A.2)$$

$$I_3 = \int_0^\infty \frac{a_0(a_0^2-Y^2)}{(a_0^2+Y^2)} \cos\alpha Y dY$$

$$= \int_0^\infty a_0 \left[\frac{1}{a_0^2+Y^2} - \frac{2Y^2}{(a_0^2+Y^2)}\right] \cos\alpha Y dY$$

$$= a_0 \left[\int_0^\infty \frac{1}{a_0^2+Y^2} \cos\alpha Y dY - 2 \frac{d}{d\alpha} \int_0^\infty \frac{Y}{(a_0^2+Y^2)} \sin\alpha Y dY\right]$$

$$\int_0^\infty \frac{1}{a_0^2+Y^2} \cos\alpha Y dY = \frac{\Pi}{2a_0} e^{-\alpha a_0}$$

$$\int_0^\infty \frac{Y}{(a_0^2+Y^2)^2} \sin\alpha Y dY = \frac{\Pi}{4a_0} \alpha e^{-\alpha a_0}$$

$$I_3 = a_0 \left[\frac{\Pi}{2a_0} e^{-\alpha a_0} - 2 \frac{d}{d\alpha}\left(\frac{\Pi}{4a_0} \alpha e^{-\alpha a_0}\right)\right]$$

$$= a_0 \left[\frac{\Pi}{2a_0} e^{-\alpha a_0} - 2\left(\frac{\Pi}{4a_0} e^{-\alpha a_0} - \frac{\Pi}{4a_0} \alpha a_0 e^{-\alpha a_0}\right)\right]$$

$$= \frac{\Pi}{2} \alpha a_0 e^{-\alpha a_0} \qquad\qquad (A.3)$$

$$I_4 = \int_0^\infty e^{-\alpha(X+a_0)} \sin\alpha Y d\alpha = \frac{Y}{(X+a_0)^2 + Y^2} \qquad (A.4)$$

$$I_5 = \int_0^\infty \alpha e^{-\alpha(X+a_0)} \sin\alpha Y d\alpha = \frac{2Y(X+a_0)}{[(X+a_0)^2 + Y^2]^2} \qquad (A.5)$$

$$I_6 = \int_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \sin\alpha Y d\alpha = \frac{2Y[3(X+a_0)^2 - Y^2]}{[(X+a_0)^2 + Y^2]^3} \qquad (A.6)$$

$$I_7 = \int_0^\infty e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha = \frac{X+a_0}{(X+a_0)^2 + Y^2} \qquad (A.7)$$

$$I_8 = \int_0^\infty \alpha e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha = \frac{(X+a_0)^2 - Y^2}{[(X+a_0)^2 + Y^2]^2} \qquad (A.8)$$

$$I_9 = \int_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha = \frac{2(X+a_0)[(X+a_0)^2 - 3Y^2]}{[(X+a_0)^2 + Y^2]^3} \qquad (A.9)$$

Integrals I_4 and I_7 can be written from Tables [6] and others are obtained by differentiation.

$$I_4 = \int_0^\infty e^{-\alpha(X+a_0)} \sin\alpha Y d\alpha = \frac{Y}{(X+a_0)^2 + Y^2}$$

$$I_5 = \int_0^\infty \alpha e^{-\alpha(X+a_0)} \sin\alpha Y d\alpha = -\frac{d}{dX}\int_0^\infty e^{-\alpha(X+a_0)} \sin\alpha Y d\alpha$$

$$= \frac{2Y(X+a_0)}{\left[(X+a_0)^2+Y^2\right]^2}$$

$$I_6 = \int_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \sin\alpha Y d\alpha = \frac{d^2}{dX^2} \int_0^\infty e^{-\alpha(X+a_0)} \sin\alpha Y d\alpha$$

$$= \frac{2Y\left[3(X+a_0)^2-Y^2\right]}{\left[(X+a_0)^2+Y^2\right]^3}$$

$$I_7 = \int_0^\infty e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha = \frac{X+a_0}{(X+a_0)^2+Y^2}$$

$$I_8 = \int_0^\infty \alpha e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha = -\frac{d}{dX} \int_0^\infty e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha$$

$$= \frac{(X+a_0)^2-Y^2}{\left[(X+a_0)^2+Y^2\right]^2}$$

$$I_9 = \int_0^\infty \alpha^2 e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha = \frac{d^2}{dX^2} \int_0^\infty e^{-\alpha(X+a_0)} \cos\alpha Y d\alpha$$

$$= \frac{2(X+a_0)\left[(X+a_0)^2-3Y^2\right]}{\left[(X+a_0)^2+Y^2\right]^3}$$