ANALYSIS OF SWITCHED CAPACITOR NETWORKS

by

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im öğrencilik yaşamım boyunca bana destek olan ANNEM'e ve BABAM'a.

ABSTRACT

In this thesis, after introducing the advantages of switched capacitor (SC) networks, several time-domain analys methods of SC networks have been investigated and compared with each other. An efficient method for the solvability pro lem, which is very important for network analyzers, has also been given.

The z-domain analysis of SC networks is given by introducing z-domain equivalent circuits which are very basic tools for network designers. The z-domain analysis is built on the time-domain signal processing mechanism of SC network

Finally, frequency domain analysis of SC networks and their filtering properties have been investigated.

The examples given in the thesis can clarify all the introduced methods.

ÖZETÇE

Bu tezde, anahtarlı kapasite elemanlarından kurulu de relerin üstünlükleri sunulduktan sonra, zaman domeninde anal yöntemlerinden belli başlıları incelenmiş ve karşılaştırmala yapılmıştır. Devre analizcileri için önemli olan "çözülebili lik" sorunu da ele alınmıştır.

Devre tasarımcıları için temel araçlardan olan eşdeğe devreler yardımıyla anahtarlı kapasite devrelerinin analizi z-dönüşükleri cinsinden de sunulmuştur. Z-domen analizi anahtarlı kapasite devrelerinin zaman domeninde işaret süreç mekanizması özelliklerinden hareket edilerek incelenmiştir.

Son olarak, frekans domen analizi ele alınmış ve anahtarlı kapasite devrelerinin süzme özelliklerine değinilmiştir.

Tezde verilen örnekler sunulan yöntemleri açıklığa kavuşturabilecek niteliktedir.

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CHAPTER I

INTRODUCTION

I.1. Why SC Networks?

Rapid advances in MOS integrated circuit technology permit an ever increasing circuit density per silicon chip. So far this technology has been used mainly by the designers of digital systems, although MOS technology allows the implementation of capacitor arrays for use in analog devices as well. Consequently, interests has recently focused on the switched capacitor filters (SC) which comprise only capacitors interconnected by an array of periodically operated switches. This approach can provide the filter functions previously obtained with LC or active-RC filters. With the elimination of resistors (which require a large silicon area, have poor temperature and linearity characteristics and furthermore have temperature coefficients difficult to match with that of capacitors when realized with MOS technology), the use of this technology for the design of analog active filters may soon become feasible.

MOS switches are already available, also toggle switches can be obtained by connecting two MOSFET's as shown in Fig. I.1.

Contrary to the resistors, MOS capacitors, at this moment already have close to ideal characteristics, temperature coefficients may be as low as 10 ppm/C or less, and the loss factor can be kept sufficiently small. Furthermore, the transfer function coefficients of an SC filter are determined by a highly stable clock frequency and capacitor ratios which can be held to very tight tolerances (measured errors of less than 0.2 percent have been achieved). This process of inherent precision and quality is sufficient to meet many filter and system specifications. By applying MOS processing techniques to SC networks, the realization of the "filter on a chip" may be at hand.

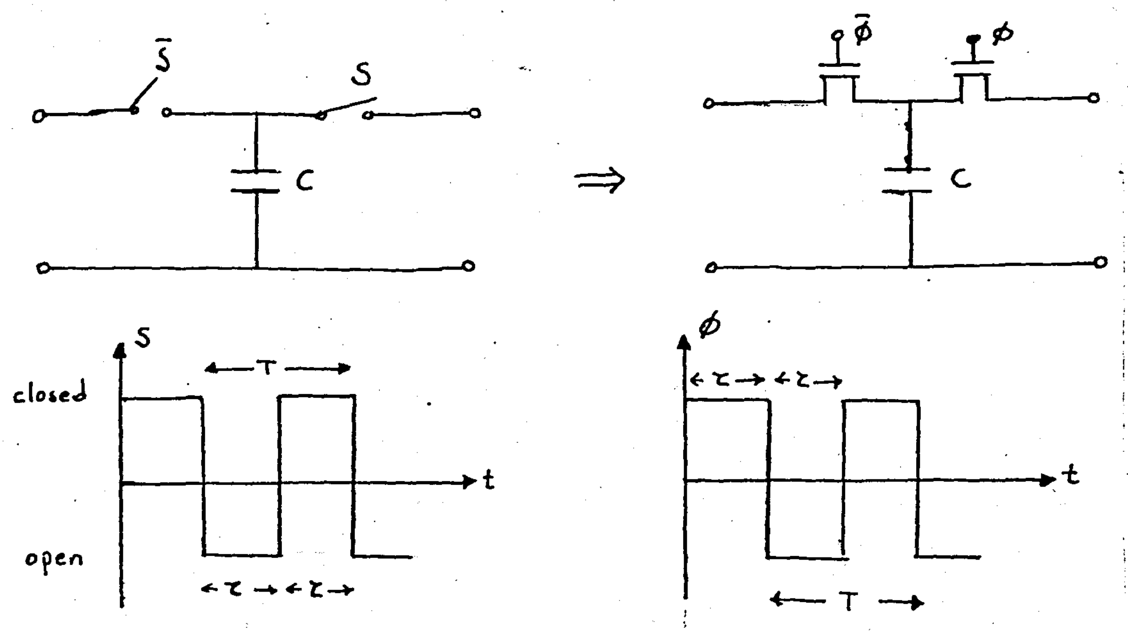


Fig.I.1.
The two gates shown are pulsed with a two phase nonoverlapping clock at a frequency $\omega_c = \frac{2\pi}{T}$.

Another technological alternative is the combination of beam leaded MOS switch arrays with thin-film capacitors deposited on a ceramic substrate to provide a hybrid integrated package resembling presently manufactured hybrid integrated RC active filters. The absence of resistors on the subtrate and the presence of only silicon chips (i.e. switches and opamps) and thin film (or chip) capacitors may permit filter packages that are both smaller and less costly than those manufactured with present techniques.

Work on the analysis of periodically switched networks was started several decades ago and general techniques have been presented for the analysis of switched RLC circuits. The formulation used in these, however is such that they can not be directly applied in the analysis of switched capacitive networks with zero resistances. The including of negligibly small "dummy" resistances to allow their use leads to unnecessarily complex formulation and time consuming calculati Thus these methods are not appropriate for analysis and design work on switched capacitor filters, where a large amount of simulation is needed to investigate new configurations, to assign elements values and to study effects like those of parasitic capacitances and operational amplifier nonidealities.

As a result, the need has emerged for efficient techniques for the analysis of these networks. But traditional two-port theory can not be directly applied to such circuits. However, by introducing some new concepts it can be shown that, ultimately, classical two-port theory can indeed be used to advantage. After the inclusion of switches, it is demonstrated that in all cases, charge equations similar to Kirchoff's current equations apply except that the storage properties of the capacitors must be taken into account.

This thesis is dealing with analysis techniques of SC networks developed so far and aims to collect all these techniques in a single text while comparing them with each other by using the same notation. In the thesis, some of the examples are from the referred papers while some are new.

I.2. Charge Conservation and Switching

Charge conservation principle on which SC networks are built can be clearly understood by the approach of Tsividis[1].

Consider a closed surface as in Fig.I.2(a) and let $q(t)$ be the time function representing the charge enclosed by it at time t. Assume there are one or more paths through which charge can be transferred to or from the outside world. Given a time reference t', $\hat{q}(t)$ is defined as the total charge that has left the surface through these paths, between times t' and t where $t > t'$. Then charge conservation dictates:

$$\hat{q}(t) \triangleq q(t') - q(t) \tag{I.1}$$

The closed surfaces considered here will include capacitor plates connected to one or more nodes. An example is shown in Fig.I.2(b) for a single node numbered (i), with the quantities q and $\hat{q}$ defined above denoted by $q^{(i)}$ and $\hat{q}^{(i)}$ respectively. $q^{(i)}$ includes the charge of all capacitor plates enclosed by the surface and will be referred to simply as "the charge on node i".

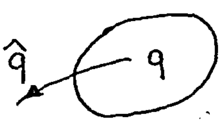


Fig.I.2(a) A closed surface containing charge q(t).

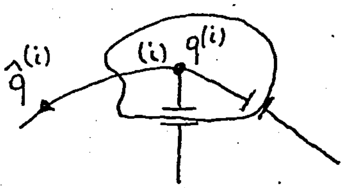


Fig.I.2(b) A closed surface containing capacitor plates connected to node (i).

Now, consider a set of m nodes $I = \{i_1, i_2 \ldots i_m\}$ as in Fig.I.3. And assume that i_1 is the smallest element in I. All nodes in the set are assumed to be simultaneously connected through switches, which are initially all open and close at time t_m.

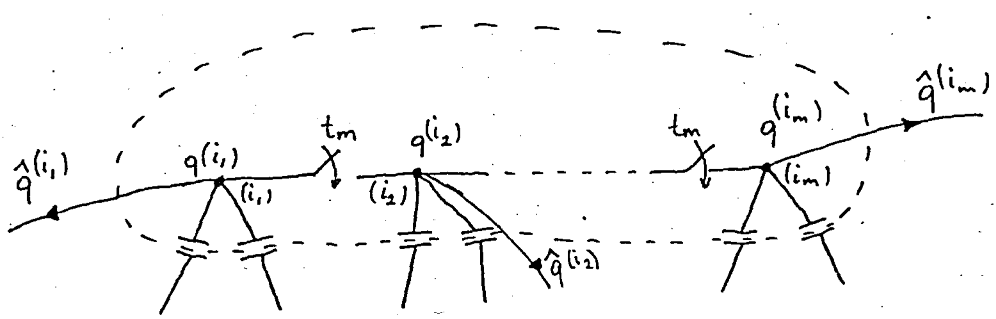


Fig.I.3. Several nodes connected through switches.

A charge conservation equation corresponding to Eq.(I.1) can be written for the greater surface enclosing all nodes, indicated by the broken line in Fig.I.3. If t_m^- indicates "the instant immediately before switching" then, using $t'=t_m^-$, we have

$$\sum_{i\in I} \hat{q}^{(i)}(t)= \sum_{i\in I} q^{(i)}(t_m^-)- \sum_{i\in I} q^{(i)}(t) \qquad (I.2)$$

The above equation is valid even if some or all of the switches in Fig.I.3 have been closed before t_m.

The formulation (I.2) is equivalent to the Kirchoff's charge law which is

$$\sum_{k\in \text{cut-set}} \hat{q}_k(t)=0 \qquad (I.3)$$

Equation (I.3) is in fact an axiom of circuit theory.

Axiom: The Kirchoff's current law (KCL) requires that the net charge transferred between t_m^- and t to any node should be zero

$$\underset{\approx}{A}\cdot\underset{\sim}{\hat{q}}(t)=0 \qquad (I.4)$$

where $\underset{\approx}{A}$ is the reduced incidence matrix of the SC network and the vector $\underset{\sim}{\hat{q}}(t)$ consists of the charges in the switches $\underset{\sim}{\hat{q}}_s$, the charges in the independent voltage sources $\underset{\sim}{\hat{q}}_u$, the charges of the charge sources $\underset{\sim}{\hat{q}}_w$, the charges of the controlled branches of VCVS, QCQS and QCVS's and the controlling branches of QCQS $\underset{\sim}{\hat{q}}_D$ and the charges of the capacitor $\underset{\sim}{\hat{q}}_c$ or explicitly

$$\underset{\sim}{\hat{q}}(t)=\begin{bmatrix} \underset{\sim}{\hat{q}}_c \\ \underset{\sim}{\hat{q}}_s \\ \underset{\sim}{\hat{q}}_u \\ \underset{\sim}{\hat{q}}_D \\ \underset{\sim}{\hat{q}}_w \end{bmatrix} \qquad (I.5)$$

where the variable $\hat{q}_k(t)$ is the net charge which has been transferred in branch k between the end t_m^- of time slot Δ_{m-1}

and the present time instance t of $\Delta_m \triangleq (t_m, t_{m+1}]$.

This choice of charge variable may not seem very straightforward in this sense that it assumes a different time reference for each time slot. Thus $\hat{q}_k(t)$ will usually have jumps at the switching instances t_m. This choice however makes the equations more transparent and does not introduce any loss of generality.

The closure of the switches will impose a set of equatio called node voltage equalities. If $v^{(i)}(t)$ denotes the voltage between node i and ground, the m-1 equations are in the form:

$$v^{(i_1)}(t)-v^{(i_j)}(t)=0, \quad i_j \in I, \quad j \neq 1 \qquad (I.6)$$

Before going into analysis techniques some preliminary assumptions and definitions are needed for the switches.

Assumption: It will be assumed that the switches of the SC networks are all of the "on-off" type. Any other type of switch can easily be represented by a combination of appropriately timed on-off switches which are closed and opened in an arbitrary fashion, not necessarily simultaneously.

Definition: A time instant t_m is a "switching time" if and only if at least one switch in the network changes state at $t=t_m$.

From the above definition it follows that the topology of the network will be fixed between any two consecutive switching times, e.g. in the intervals of the form $(t_m, t_{m+1}]$.

CHAPTER II

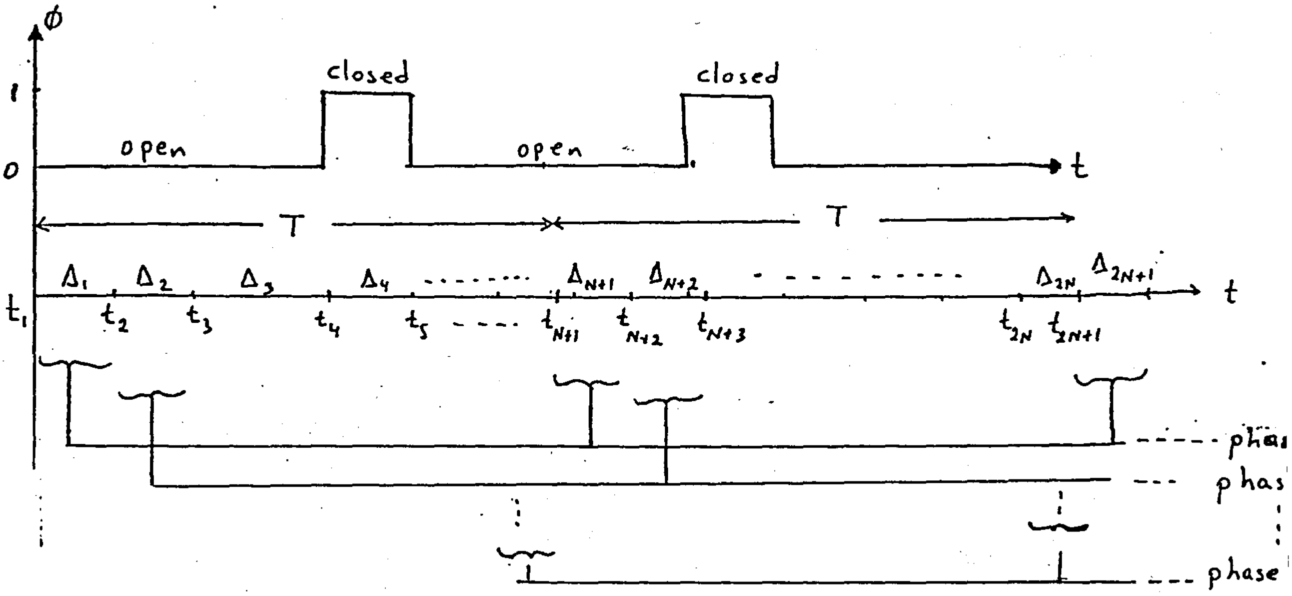
THE TIME DOMAIN ANALYSIS OF LINEAR MULTIPHASE SC NETWORKS

In this chapter, the time domain analysis methods of linear multiphase SC networks will be given. At the end of this chapter, the same example will be solved by each of these methods. First, the components of SC networks are introduced.

II.1. SC-Network Components

From a practical point of view a switched capacitor network is any network with op-amps, capacitors (C), switches (S) and voltage sources (VS) where the amplification of the op-amps is frequency independent; moreover there are no resistors nor parasitic resistances in the components Fig.II.1(b). The switches are controlled by T-periodic Boolean clock signals $\phi_i(t)$, i.e., $\phi_i(t+T)=\phi_i(t)$. If $\phi_i(t)=0$ (resp. $\phi_i(t)=1$) all the switches which are controlled by the clock ϕ_i are open (resp. closed) Fig.II.1(a). The time is partitioned into time slots $\Delta_m \triangleq (t_m, t_{m+1}]$ such that the clock signals (and hence the network) do not vary in Δ_m.

Each period has N time slots. The union of the time slots Δ_k, Δ_{k+N}, Δ_{k+2N} ..... is called phase k. All time slots of the phase k have the same duration and the clock values are the same i.e., $\phi_i(t)=\phi_{ik}$ for all t in phase k (where ϕ_{ik} is the value of the i th switch in phase k). Although a major part of the realizations only uses two phases [4,5,8], the derivations are for an arbitrary number of N phases. This is not just for mathematical generality but in order to be able to handle design techniques which use many phases.



The timing instances and phases of an N-phase SC network

Fig.II.1.a

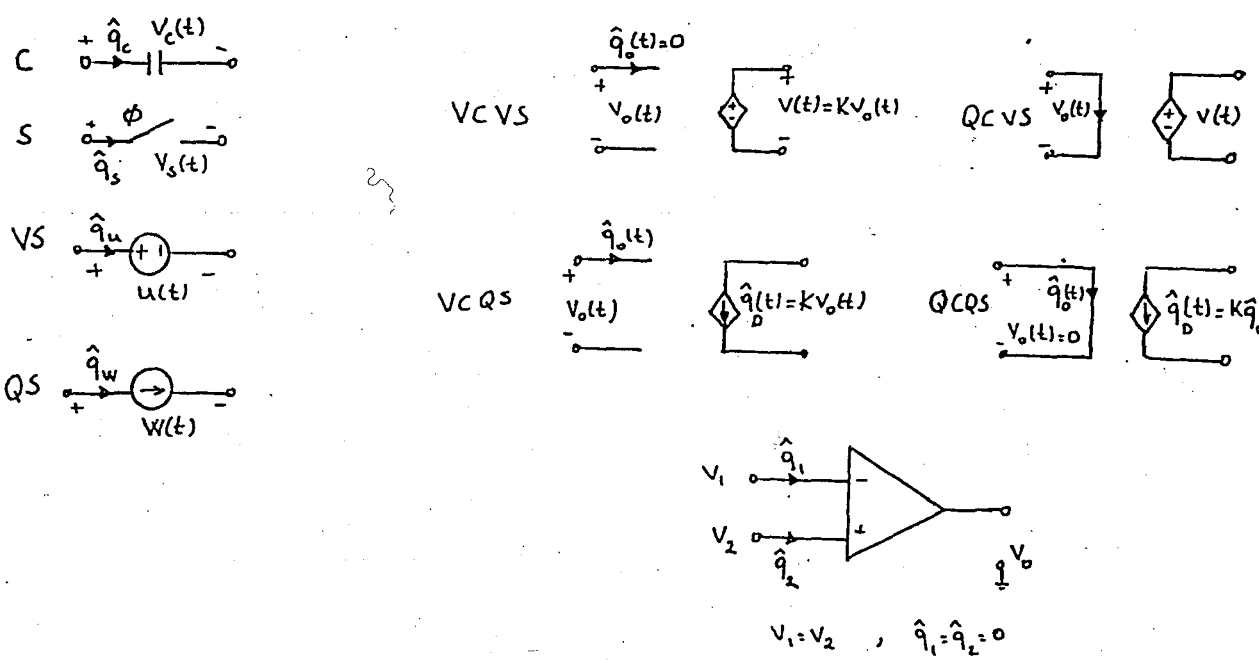


Fig.II.1.b. SC-Network Components.

II.2. An Introductory Example

In order to illustrate the signal processing mechanism which occur in SC networks and analysis techniques consider the simple "RC low pass" circuit below. (x)

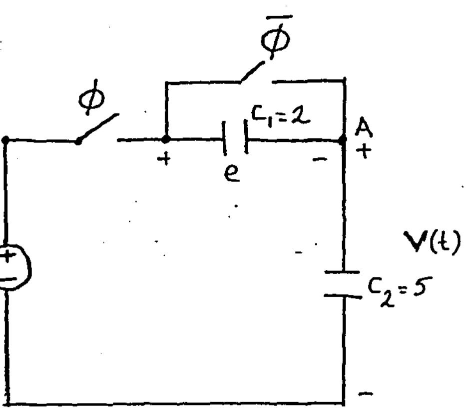


Fig.II.2(a)

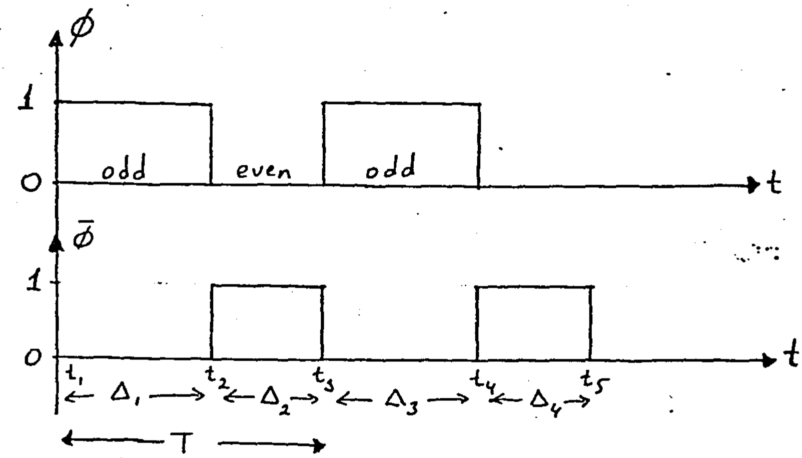


Fig.II.2(b)

During the odd phase, i.e. for all t in the odd time slots Δ_{2i+1} i=0,1,2,.... the switch controlled by Ø is closed and that controlled by $\bar{Ø}$ is open. Thus for all times t in Δ_{2i+1} the network equations are:

$$e(t)+v(t)=u(t) \quad \text{(KVL)}$$

$$C_1e(t)+C_2v(t)= -C_1e(t^-_{2i+1})+C_2v(t^-_{2i+1}) \quad \text{(KQL for node A)} \tag{II.1}$$

where t^-_m denotes the time instant just before the switching instant t_m.

During the even phase i.e. for all t in Δ_{2i}, i=1,2,... the network equations are:

$$e(t)=0 \quad \text{(KVL)}$$

$$v(t)=v(t^-_{2i}) \quad \text{(KQL for node A)} \tag{II.2}$$

(x) It will be clear in later chapters, why such a circuit is called a low pass circuit. (See page 77, chapter IV).

By eliminating e(t) in both equations

$$7v(t)=5v(t_{2i+1}^-)+2u(t) \quad t\in\Delta_{2i+1} \qquad \text{(II.3.a)}$$

$$v(t)=v(t_{2i}^-) \quad t\in\Delta_{2i} \qquad \text{(II.3.b)}$$

is obtained.

At this point some important observations can be made which extend to general SC circuits.

First the response "v(t)" satisfies different equations according to the phase. Second the equations are linear and third the response in any time slot depends on its value $v(t_m^-)$ at the end of the previous time slot and on the actual value of the input u(t).

The equations (II.3) contain the sampled data and the continous I-O (input-output) coupling effect. The sampled data effect is present in the even and odd phase but the continous I-O coupling is only present in the odd phase.

This follows immediately from the topology of the circuit since only in odd phase there is a loop of the input and output branch.

In order to decompose these effects call the value at the end t_{m+1}^- of time slot Δ_m:

$$u_m \hat{=} u(t_{m+1}^-), \quad v_m \hat{=} v(t_{m+1}^-) \qquad \text{(II.4)}$$

By substituting $t=t_{2i+2}^-$ in Eq. (II.3.a) and $t=t_{2i+1}^-$ in Eq. (II.3.b)

$$\begin{aligned} 7v_{2i+1} &= 5v_{2i}+2u_{2i+1} \\ v_{2i} &= v_{2i-1} \end{aligned} \qquad \text{(II.5)}$$

is obtained.

After elimination of the even values v_{2i}, a constant differance equation of values one full period apart results:

$$7v_{2i+1}-5v_{2i-1}=2u_{2i+1} \qquad (II.6)$$

The values of the output at other instants of the time slot Δ_m are characterized in terms of new variables as follows:

$$v^*(t)\hat{=}v(t)-v_m, \quad u^*(t)\hat{=}u(t)-u_m \quad t\in\Delta_m \qquad (II.7)$$

Substituting II.4, II.5, II.7 into II.3, the equations for the continous I-O coupling between u^* and v^* are obtained.

$$\begin{aligned} 7v^*(t)&=2u^*(t), \quad t\in\Delta_{2i+1} \\ v^*(t)&=0, \quad t\in\Delta_{2i} \end{aligned} \qquad (II.8)$$

In Fig.II.3, the decomposition (II.7) of the input and output into a part $u'(t)\hat{=}u_m$, $v'(t)\hat{=}v_m$, $t\in\Delta_m$ which is constant in each time slot and the remainder $u^*(t)$ and $v^*(t)$ which is zero at the end of each time slot, is shown.

The equations (II.5-8) give the input-output relationship for the piecewise-constant part and the input-output continous coupling. The continous I-O coupling is only a periodic scaling of the signals which can be easily computed and as in this example this contribution is usually small. The computation of the sampled data effect requires the solution of a set of difference equations (II.5) or (II.6), which can be solved in time domain via discrete time impulse responses or in z-domain. This techniques are further explained in later chapters. (See chapter III and IV)

The above equations have been obtained by using the charge conservation concept explained in chapter I and the KVL. Any SC network can be analyzied by this concept in mind. But more complicated SC networks will be solved by the methods of II.3, II.4, II.5 and II.6 built on the charge conservation principle.

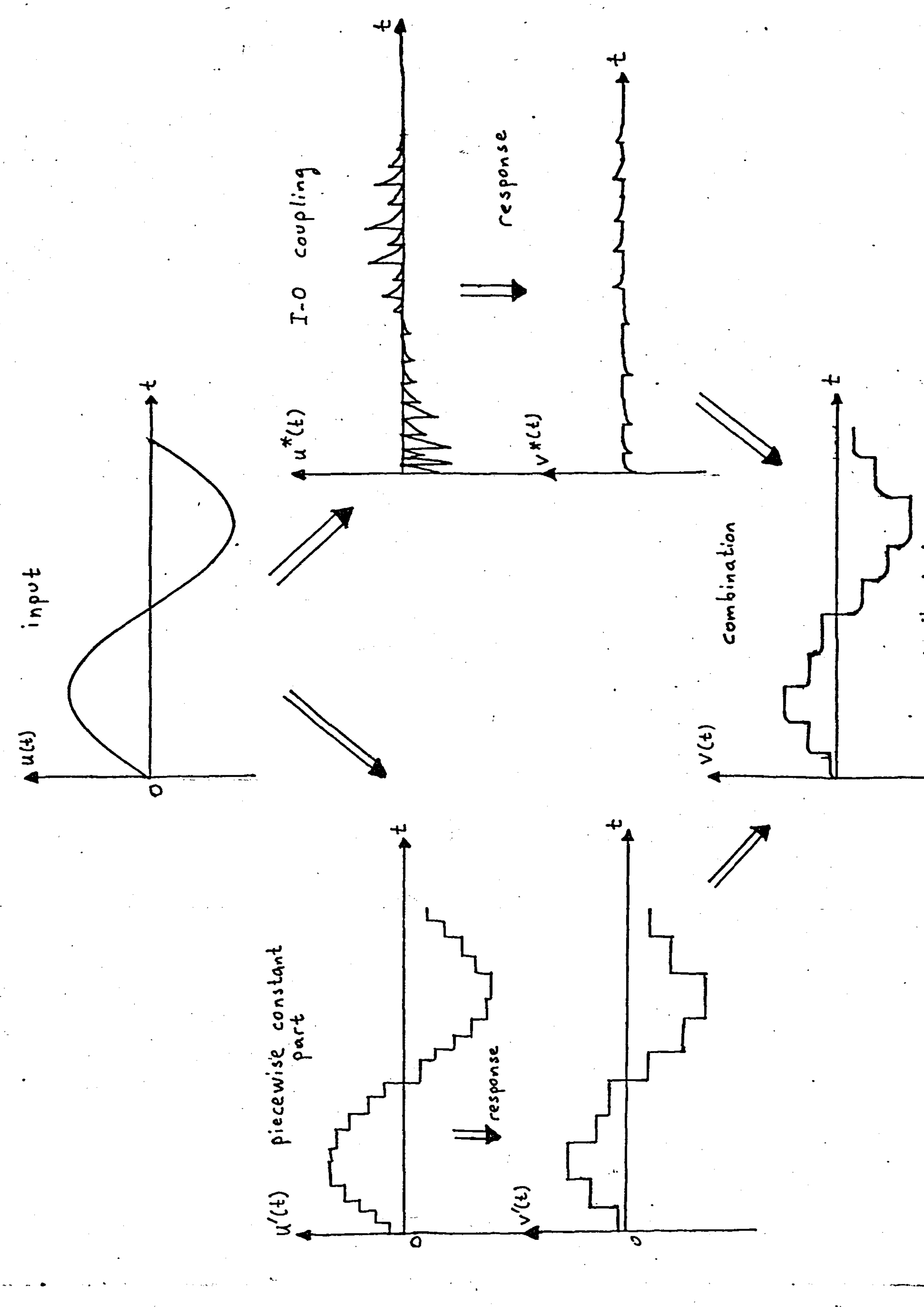

input
u(t)
0
t
I-O coupling
u*(t)
response
v*(t)
piecewise constant part
u'(t)
response
v'(t)
combination
V(t)
output

Fig. II.3.

II.3. Tableau Method

The basic signal processing mechanisms which occur in a SC network are first the charge redistribution between two consecutive phases and second the effects of the variations of inputs during the phases.

In order to describe those effects easily the variables $v_k(t)$ and $\hat{q}_k(t)$ are chosen for each branch. The variable $v_k(t)$ is the voltage in the branch k at the present time instant t of some time slot Δ_m. The variable $\hat{q}_k(t)$ has been already explained on page 5.

The constitutive equations of the uncontrolled components shown in Fig.II.1(a) at a time instant t of Δ_m are then:

$$\begin{aligned}
&C : \hat{q}_c(t)-C\left[v_c(t)-v_c(t_m^-)\right]=0 \\
&S : \bar{\emptyset}_m\hat{q}_s(t)+\emptyset_m v_s(t)=0 \qquad t\in\Delta_m \qquad \text{(II.9)} \\
&VS: v_v(t)=u(t) \\
&QS: \hat{q}_w(t)=w(t)
\end{aligned}$$

where $\bar{x}$ denotes the complement of the Boolean variable x and u(t) and w(t) are given source waveforms. By labelling the variables of the controlling port by v_o and $\hat{q}_o$ and those of the controlled port by v and $\hat{q}$ the constitutive equations of the controlled sources are:

$$\begin{aligned}
&\text{VCVS}: \quad \hat{q}_o(t)=0 \quad , \quad v(t)-Kv_o(t)=0 \\
&\text{VCQS}: \quad \hat{q}_o(t)=0 \quad , \quad \hat{q}(t)-Kv_o(t)=0 \qquad \text{(II.10)} \\
&\text{QCVS}: \quad v_o(t)=0 \quad , \quad v(t)-K\hat{q}_o(t)=0 \\
&\text{QCQS}: \quad v_o(t)=0 \quad , \quad \hat{q}(t)-K\hat{q}_o(t)=0 \\
&\text{OP-amp}: \quad v_1=v_2, \quad \hat{q}_1=\hat{q}_2=0
\end{aligned}$$

Three important observations can be made directly from the above constitutive equations:

1) all components are described by linear equations,
2) only the capacitors introduce dynamic actions in the circuit by memorizing in each time slot Δ_m the value of the voltage at the end t_m^- of the previous time slot,
3) the periodic clocking of the switches introduces periodicity in the circuit.

Now, let's define $\hat{\underset{\sim}{q}}(t)$ and $\underset{\sim}{v}(t)$ as the vectors representing the branch charges and the voltages at time instant t.

Then the constitutive equations of all the components can be brought together in the matrix form at any instant t of Δ_m as:

$$\underset{\approx}{M}_m\hat{\underset{\sim}{q}}(t)+\underset{\approx}{P}_m\underset{\sim}{v}(t)-\underset{\approx}{R}\underset{\sim}{v}(t_m^-)=\underset{\sim}{s}(t) \qquad \text{(II.11)}$$

where $\underset{\sim}{s}(t)$ contains all the input voltage or charge waveforms $\underset{\sim}{u}(t)$ and $\underset{\sim}{w}(t)$.

Each uncontrolled elements contributes one equation to (II.11) and controlled elements contributes two equations to (II.11) (II.11) is general in that it may include multi-terminal elements.

KVL requires that the voltages $\underset{\sim}{v}(t)$ in the branches are equal to the differance of the node voltages at the terminals of the branches or

$$\underset{\approx}{A}^T\underset{\sim}{v}_N(t)-\underset{\sim}{v}(t)=0 \qquad \text{(II.12)}$$

where $\underset{\approx}{A}^T$ is the transpose of the reduced incidence matrix $\underset{\approx}{A}$ and $\underset{\sim}{v}_N(t)$ is the node to ground voltages at time t of Δ_m.

Then an important theorem follows:

Theorem II.1. Given an arbitrary switched capacitor network with reduced incidence matrix $\underset{\approx}{A}$ and where $\underset{\approx}{M}_m$, $\underset{\approx}{P}_m$, $\underset{\approx}{R}$ and $\underset{\sim}{s}(t)$ characterize the components and the input waveforms in time slot Δ_m, the response $\underset{\sim}{v}_N(t)$, $\hat{\underset{\sim}{q}}(t)$ and $\underset{\sim}{v}(t)$ at time instant t of Δ_m satisfies the tableau equations:

$$\begin{bmatrix} \underset{\approx}{0} & \underset{\approx}{A} & \underset{\approx}{0} \\ \underset{\approx}{A}^T & \underset{\approx}{0} & -\underset{\approx}{I} \\ \underset{\approx}{0} & \underset{\approx}{M}_m & \underset{\approx}{P}_m \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_N(t) \\ \hat{\underset{\sim}{q}}(t) \\ \underset{\sim}{v}(t) \end{bmatrix} = \begin{bmatrix} \underset{\approx}{0} \\ \underset{\approx}{0} \\ \underset{\approx}{R} \end{bmatrix} \underset{\sim}{v}(t_m^-) + \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \\ \underset{\sim}{S}(t) \end{bmatrix} \qquad (II.13)$$

or in short

$$\underset{\approx}{F}_m \underset{\sim}{x}(t) = \underset{\approx}{G}_m \underset{\sim}{x}(t_m^-) + \underset{\sim}{r}(t) \qquad (II.14)$$

where $\underset{\sim}{r}(t) = \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \\ \underset{\sim}{S}(t) \end{bmatrix}$ is input waveform vector and

$$\underset{\approx}{G}_m = \begin{bmatrix} \underset{\approx}{0} \\ \underset{\approx}{0} \\ \underset{\approx}{R} \end{bmatrix} .$$

<u>Proof:</u> The equations (II.13) or (II.14) can be obtained by puting the KQL (I.4), the KVL (II.12) and constitutive equations (II.11) into one matrix form.

As stated by Chua and Lin [6] the importance of the tableau equations is that any general topological method of circuit analysis (loop, hybrid, modified nodal or state variable analysis) can be obtained by a preliminary Gauss elimination of certain variables in Eq.(II.13). Unlike other topological equation formulation algorithms [1] , it is a trivial matter to program the tableau equations. Although, the size of tableau matrix is considerably larger than that of the matrices with some variables eliminated, the tableau matrix is much sparser. Because of this sparsity, the tableau equations with all variables may be solved much faster than its reduced versions.

For computer implementations a good compromise has to be chosen between simplicity of formulation of the equations and the size of the resulting equations. With the mixed-nodal tableau (Modified Nodal Analysis [2,3]) formulation, it is possible to preserve sparsity while eliminating a considerable number of variables.

The other reason for deriving tableau equations for a SC network, is that the structure of these equations allows immediately to see how the signals are processed in the network. It is seen that the input $\underset{\sim}{r}(t)$ and the output $\underset{\sim}{x}(t)$ are related by linear equations. Because of the absence of derivatives of the variables in this equations (II.14), there is no continous-time dynamic action in a switched capacitor circuit. The dynamic action is however due to a discrete time memorization of the value of certain voltages at the end of the previous time slot.

Since the matrices $\underset{\approx}{M}_m$ and $\underset{\approx}{P}_m$ depend on m but also are periodic or $\underset{\approx}{M}_{m+1N}=\underset{\approx}{M}_m$ and $\underset{\approx}{P}_{m+1N}=\underset{\approx}{P}_m$ (where N is the number of phases), the waveforms in SC network can be computed by a solution of a periodic set of linear difference equations.

Before going into the domain signal processing mechanism in SC networks, the modified nodal analysis methods, the approach of Lin, will be given in the following sections.

II.4. Modified Nodal Analysis Using Composite Branches

Before going into the analysis method, it will be convenient to give the following theorem.

Theorem II.2. In general SC networks can be considered as linear resistive circuits for any subinterval of the period T.

Proof: Consider each branch "k" of the circuit as being characterized by a voltage-charge relation $\left[v_k(t),\ \hat{q}_k(t)\right]$.

With this characterization, KVL and KQL still hold, i.e.

$$\sum_{k\in loop} v_k(t)=0 \quad \text{and} \quad \sum_{k\in cut-set} \hat{q}_k(t)=0 \qquad \text{(II.16)}$$

For a capacitor, its branch characteristics is now given by

$$\hat{q}_c(t)=C\left[v_c(t)-v_c(t_m^-)\right] \qquad \text{(II.17)}$$

which is represented by an equivalent composite "resistive"(x)

(x) Here, resistive is used to mean that the charge $\hat{q}_c(t)$ is related to the voltage $\left[v_c(t)-v_c(t_m^-)\right]$ by a constant multiplier.

element with conductance equal to C as shown in Fig.II.4(b).

For an ideal switch when the switch is thrown from closed to open,

$$\hat{q}_s(t)=0 \quad \text{for } t \in \text{"open" subinterval} \qquad \text{(II.18)}$$

When the switch is thrown from open to closed $\hat{q}_s(t)$ is finite and the voltage across the switch is

$$v_s(t)=0 \quad \text{for } t \in \text{"closed" subinterval} \qquad \text{(II.19)}$$

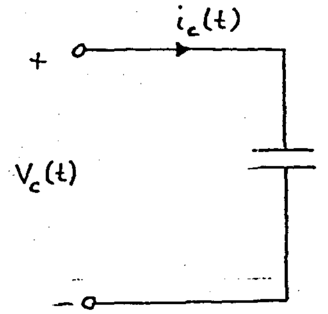
The independent current $J_s(t)$ is now represented by a "charge" source.

$$\hat{q}_W\ (t) = W\ (t) \qquad \text{(II.20)}$$

Where as the independent voltage source u(t) is the same as before.

For the controlled sources and multi-terminal elements, the new branch characteristics and the controlling relations remain the same with each branch current replaced by the branch charge.

From the above, it is concluded that the original switched capacitor circuit is transformed into a "resistive" circuit in voltage-charge domain with the same circuit topology.



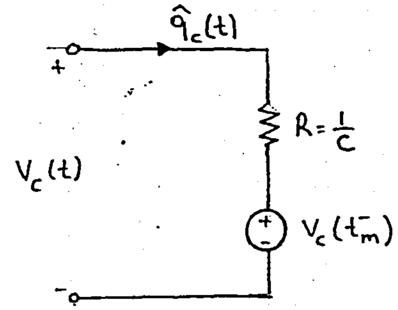


Fig.II.4(a) A capacitor. Fig.II.4(b)"Resistive" equivalent.

There exists several efficient analysis methods for resistive circuits. The classical nodal method is perhaps the simplest and easiest to implement on computers for solving linear resistive circuits containing independent current sources and VCCS as the only sources. For the "resistive" circuit under consideration, one must convert the independent voltage sources into their Norton equivalent current sources, and VCVS's and CCCS's into equivalent VCCS's before the nodal method can be applied. This conversion usually involves additional programming effort. To overcome this shortcoming, the modified nodal analysis method [7] is adapted to SC networks by Liou and Kuo [4] using the concept of composite branch [6].

Analysis Procedure:

Let the SC circuit be connected with (n+1) nodes and b composite branches as shown in Fig.II.5. Each composite branch is made of a two-terminal element b_k, a voltage source $\mathcal{E}_k(t)$ and a composite charge source $\mathcal{Z}_k(t)$. For a "resistive" branch (corresponding to a capacitor in the original circuit), $\hat{q}_k(t)=C_k\hat{v}_k(t)$ where $\hat{v}_k(t)=[v_{ck}(t)-v_{ck}(t_m^-)]$ and $\mathcal{E}_k(t)=v_{ck}(t_m^-)$.

For an independent voltage source branch $\hat{v}_k(t)=0$ and $\mathcal{E}_k(t)=E_{sk}(t)$. For a voltage-controlled voltage source. $\hat{v}_k(t)=0$ and $\mathcal{E}_k(t)$ is some linear combination of other branch voltages $v_k(t)$ and independent voltage sources $E_{sk}(t)$'s.

Independent charge sources $\hat{W}_{sk}(t)$ may be combined to form the composite charge source $Z_k(t)$. Any charge controlled charge source can be represented by the composite charge source $Z_k(t)$ and/or the two-terminal element b_k. The two-terminal element b_k can be either a linear resistor with resistance $R_k=\frac{1}{C_k}$ or a voltage controlled charge source that depends linearly on the voltage of another resistor. Observe that a charge controlled charge source which depends linearly on the charge of another resistor can be replaced by an equivalent voltage-controlled charge source. In particular, if $\hat{q}_k=\beta_{kj}\hat{q}_j$ is the terminal charge of the controlled charge source b_k, where $\hat{q}_j$ is the charge of resistor b_j with resistance $R_j=1/C_j$, then one can replace this controlled source with a voltage controlled charge source with terminal charge $\hat{q}_k=g_{kj}\hat{v}_j$, where $g_{kj}=\beta_{kj}/R_j$. Consequently, the modified nodal analysis formulation could allow both voltage-and charge-controlled charge sources.

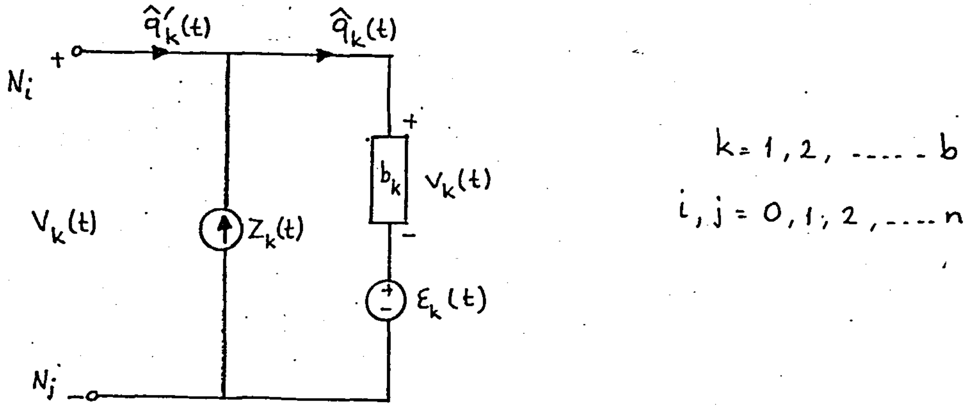


Fig.II.5. A "resistive" equivalent composite branch of a switched capacitor circuit element.

Define the charge vectors as:

$$\hat{\underset{\sim}{q}}' \triangleq \begin{bmatrix} \hat{q}'_1 \\ \hat{q}'_2 \\ \vdots \\ \hat{q}'_b \end{bmatrix} \quad \hat{\underset{\sim}{q}} \triangleq \begin{bmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \vdots \\ \hat{q}_b \end{bmatrix} \quad \underset{\sim}{Z} \triangleq \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_b \end{bmatrix} \qquad \text{(II.21)}$$

Let $\hat{\underset{\sim}{q}}$ be partitioned into two subvectors as

$$\hat{\underset{\sim}{q}} = \begin{bmatrix} \hat{\underset{\sim}{q}}_a \\ \hat{\underset{\sim}{q}}_b \end{bmatrix}$$

where $\hat{\underset{\sim}{q}}_b$ corresponds to the charge vector associated with the charges in the voltage sources, including all the independent and controlled sources and $\hat{\underset{\sim}{q}}_a$ corresponds to the remaining charges.

Define the voltage vectors as:

$$\underset{\sim}{v} \triangleq \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_b \end{bmatrix} \quad \hat{\underset{\sim}{v}} \triangleq \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \vdots \\ \hat{v}_b \end{bmatrix} \quad \underset{\sim}{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_b \end{bmatrix} \qquad \text{(II.22)}$$

Let node o be the datum node, and denote by $\underset{\sim}{v}_N=\left[v_{N1}, v_{N2},\ldots,v_{Nn}\right]^T$ the node-to-datum voltage vector and by $\underset{\approx}{A}$ the nxb reduced incidence matrix obtained from the complete incidence matrix by deleting the row corresponding to the datum node, clearly, $\underset{\sim}{v}=\underset{\approx}{A}^T \underset{\sim}{v}_N$.

Then partition all the voltage vectors in Eq.(II.22) and $\underset{\approx}{A}$ in the same manner as in $\hat{\underset{\sim}{q}}$. i.e.,

$$\underset{\sim}{v}=\begin{bmatrix}\underset{\sim}{v}_a\\ \underset{\sim}{v}_b\end{bmatrix} \qquad \hat{\underset{\sim}{v}}=\begin{bmatrix}\hat{\underset{\sim}{v}}_a\\ \hat{\underset{\sim}{v}}_b\end{bmatrix} \qquad \underset{\sim}{\mathcal{E}}=\begin{bmatrix}\underset{\sim}{\mathcal{E}}_a\\ \underset{\sim}{\mathcal{E}}_b\end{bmatrix} \qquad \underset{\approx}{A}=\left[\underset{\approx}{A}_a, \underset{\approx}{A}_b\right]$$

The KQL equation for the circuit is given by

$$\underset{\approx}{A}\hat{\underset{\sim}{q}}'=\underset{\approx}{A}(\hat{\underset{\sim}{q}}-\underset{\sim}{Z})=\underset{\sim}{0} \qquad \text{(II.23)}$$

With the above described partitioning, (II.23) can be rewritten as

$$\begin{bmatrix}\underset{\approx}{A}_a & \underset{\approx}{A}_b\end{bmatrix}\begin{bmatrix}\hat{\underset{\sim}{q}}_a\\ \hat{\underset{\sim}{q}}_b\end{bmatrix}=\underset{\approx}{A}\,\underset{\sim}{Z} \qquad \text{(II.24)}$$

Where $\hat{\underset{\sim}{q}}_a$ is related to $\hat{\underset{\sim}{v}}_a$ by the branch-admittance matrix $\underset{\approx}{Y}_a$ which always exists as the result of the partitioning procedure, i.e.,

$$\hat{\underset{\sim}{q}}_a=\underset{\approx}{Y}_a\,\hat{\underset{\sim}{v}}_a \qquad \text{(II.25)}$$

On the other hand, $\hat{\underset{\sim}{v}}_a$ can be expressed as

$$\hat{\underset{\sim}{v}}_a=\underset{\sim}{v}_a-\underset{\sim}{\mathcal{E}}_a=\underset{\approx}{A}_a^T\underset{\sim}{v}_N-\underset{\sim}{\mathcal{E}}_a \qquad \text{(II.26)}$$

Substituting (II.26) and (II.25) into (II.24)

$$\begin{bmatrix}\underset{\approx}{A}_a\underset{\approx}{Y}_a\underset{\approx}{A}_a^T & \underset{\approx}{A}_b\end{bmatrix}\begin{bmatrix}\underset{\sim}{v}_N\\ \hat{\underset{\sim}{q}}_b\end{bmatrix}=\underset{\approx}{A}_a\underset{\approx}{Y}_a\underset{\sim}{\mathcal{E}}_a+\underset{\approx}{A}\,\underset{\sim}{Z} \qquad \text{(II.27)}$$

is obtained.

Eq.II.27 is a set of n equations in n node-to-datum voltages $\underset{\sim}{v}_N$ and the charges $\hat{\underset{\sim}{q}}_b$ in all the independent and controlled voltage sources. Denote the number of these voltage sources by b_1. The branch relations of these b_1 elements are given by (x)

$$\underset{\approx}{K}\begin{bmatrix}\underset{\sim}{v}_a\\ \underset{\sim}{v}_b\end{bmatrix}=\underset{\approx}{H}\,\underset{\sim}{E}_s \quad \text{or} \quad \underset{\approx}{K}\,\underset{\approx}{A}^T\underset{\sim}{v}_N=\underset{\approx}{H}\,\underset{\sim}{E}_s \qquad \text{(II.28)}$$

Where $\underset{\sim}{E}_s(t)$ (or $\underset{\sim}{u}(t)$) is the independent voltage source vector of dimension $(\ell x1) \triangleq \left[E_{s1}(t), E_{s2}(t),\ldots,E_{s\ell}(t)\right]^T$, $\underset{\approx}{K}$ and $\underset{\approx}{H}$ are real constant matrices of dimensions $(b_1 x b)$ and $(b_1 x \ell)$ resp.

Let $\hat{\underset{\sim}{W}}_s(t)$ be the independent "charge" source vector of dimension (mx1), i.e., $\hat{\underset{\sim}{W}}_s(t)=\left[\hat{W}_{s1}(t),\ldots,\hat{W}_{sm}(t)\right]^T$ and $\underset{\sim}{v}_c(t_m^-)$ be the initial capacitor voltage vector of dimension (Mx1), i.e., $\underset{\sim}{v}_c(t_m^-)=\left[v_{c1}(t_m^-),\ldots,v_{cM}(t_m^-)\right]^T$. Without loss of generality, it is assumed that the "resistive" branches (corresponding to the M capacitors) are numbered first, then (xx)

$$\underset{\sim}{\mathcal{E}}_a=\begin{bmatrix}\underset{\approx}{I}_M\\ \underset{\approx}{0}\end{bmatrix}\underset{\sim}{v}_c(t_m^-) \qquad \text{(II.29)}$$

and

$$\underset{\approx}{A}_a\underset{\approx}{Y}_a\underset{\sim}{\mathcal{E}}_a=\underset{\approx}{A}_a\underset{\approx}{Y}'_a\underset{\sim}{v}_c(t_m^-) \qquad \text{(II.30)}$$

where $\underset{\approx}{Y}'_a$ is obtained from $\underset{\approx}{Y}_a$ by retaining only the first M columns.

Also, since $\underset{\sim}{Z}$ is a linear combination of $\hat{W}_{sk}(t)$'s, then

$$\underset{\approx}{A}\,\underset{\sim}{Z}=\underset{\approx}{A}_W\hat{\underset{\sim}{W}}_s(t) \qquad \text{(II.31)}$$

where $\underset{\approx}{A}_W$ is a real matrix of dimension (nxm) which is a sort of reduced incidence matrix involving only the independent charge source branches.

(x) Note that all the controlled sources are voltage controlled.

(xx) $\underset{\approx}{I}_M$ is the unit matrix of dimension M.

Combining (II.27) and (II.28) and using (II.30) and (II.31) one obtains (n+b) modified nodal equations:

$$\begin{bmatrix} A_a Y_a A_a^T & A_b \\ K A^T & 0 \end{bmatrix} \begin{bmatrix} v_N \\ \hat{q}_b \end{bmatrix} = \begin{bmatrix} A_a Y_a' \\ 0 \end{bmatrix} v_c(t_m^-) + \begin{bmatrix} 0 & A_w \\ H & 0 \end{bmatrix} \begin{bmatrix} E_s(t) \\ \hat{W}_s(t) \end{bmatrix} \quad \text{(II.32)}$$

In (II.32), the submatrix $A_a Y_a A_a^T$ is the node admittances matrix excluding the contributions due to all the voltage sources.

Equation (II.32) can be solved (x) to obtain

$$v_N(t) = F_m' v_c(t_m^-) + G_m' s(t) \quad \text{(II.33)}$$

where

$$s(t) = \begin{bmatrix} E_s(t) \\ \hat{W}_s(t) \end{bmatrix}$$

Then,

$$v_c(t) = A_c^T v_N(t) = F_m v_c(t_m^-) + G_m s(t), \quad t \in \Delta_m \quad \text{(II.34)}$$

where A_c is obtained from A by deleting last b-M columns which do not correspond to the M capacitor branches and $F_m \triangleq A_c^T F_m'$, $G_m \triangleq A_c^T G_m'$.

Equation (II.34) is the state equations of the SC network.

The output equation (the output voltage vector $y(t)$) is obtained in a similar manner, i.e.,

$$y(t) = A_o v_N = \hat{C}_m v_c(t_m^-) + \hat{D}_m s(t) \text{ for } t \in \Delta_m \quad \text{(II.35)}$$

where A_o is the connection matrix relating v_n to y, and $\hat{C}_m \triangleq A_o F_m$ and $\hat{D}_m \triangleq A_o G_m$.

(x) See the solvability problem section.

<u>Remarks:</u> Eq.(II.35) is the state equation of the circuit, which doesn't involve the time derivatives of the state vectors. The state vector in a subinterval depends linearly on the input source vector in the same subinterval and on the final value of the state vector (which consists of the capacitor voltages) in the preceeding subinterval.

The output formulation (II.35) is unique since $\underset{\approx}{A}_o$ is unique (if the circuit is completely solvable).

Since the circuit is "resistive" and, $\underset{\sim}{v}_c(t)$ and $\underset{\sim}{s}(t)$ completely determine any voltage in the circuit then the output voltage vector can be expressed as

$$\underset{\sim}{y}(t)=\underset{\approx}{C}_m\underset{\sim}{v}_c(t)+\underset{\approx}{D}_m\underset{\sim}{s}(t) \quad \text{for} \quad t\epsilon\Delta_m \qquad \text{(II.36)}$$

When Eq.(II.34) is substituted into Eq.(II.36) and the result is compared with Eq.(II.35), $\hat{\underset{\approx}{C}}_m=\underset{\approx}{C}_m\underset{\approx}{F}_m$ and $\hat{\underset{\approx}{D}}_m=\underset{\approx}{C}_m\underset{\approx}{G}_m+\underset{\approx}{D}_m$ are obtained.

II.5. Modified Nodal Analysis Using Stamps

As explained in page, MNA equations are intermediate to the tableau and nodal equations. They are much more compact than the tableau, but retain its properties of sparsity, generality and ease of formulation. Liou and Kuo analyzed the SC networks by adapting MNA method as explained in the previous section. Their approach requires only to convert the SC network into its resistive equivalent in the subinterval. In this case, obviously some switches are on and the remaining ones are off.

However, in [2], the switches have also been considered as elements which have constitutive equations as in Eq.(II.9). And MNA equations have been obtained by adding some constitutive equations of the switches and all the voltage sources. In [2], the stamps "contribution" in Fig.II.6 of the different components of a SC network to the MNA equations are used for the direct link with the computer implementation.

Now the method in [2] will be described.

<u>Theorem II.3.</u> A linear T-periodic SC network containing ideal switches, capacitors, independent voltage and charge sources (VC and QS) and four type of dependent sources(VCVS, VCQS, QCQS, QCVS) is described in time domain by the equations

$$\begin{bmatrix} \underset{\approx}{H}_k & \underset{\approx}{L}_k \\ \underset{\approx}{M}_k & \underset{\approx}{R}_k \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_N(t) \\ \hat{\underset{\sim}{q}}(t) \end{bmatrix} = \begin{bmatrix} \underset{\approx}{P}_k \underset{\sim}{v}_N(t^-_{k+1N}) \\ \underset{\approx}{0} \end{bmatrix} + \begin{bmatrix} \underset{\sim}{W}(t) \\ \underset{\sim}{u}(t) \end{bmatrix}, t \in \Delta_{k+1N} \qquad \text{(II.37)}$$

where $\underset{\sim}{v}_N(t)$ is the node voltage variables (except the reference voltage). $\hat{\underset{\sim}{q}}(t)$ consists of the charges in the switches $\hat{\underset{\sim}{q}}_s$, the charges in the independent voltage sources $\hat{\underset{\sim}{q}}_u$, the charges of the controlled branches of VCVS, QCQS and QCVS's and the controlling branches of QCQS $\hat{\underset{\sim}{q}}_D$. $\underset{\sim}{W}(t)$ consists of the charge sources and $\underset{\sim}{u}(t)$ is the voltage source vector.

<u>Proof:</u> The charges $\underset{\sim}{W}(t)$ injected in the nodes between t^-_{k+1N} and $t \in \Delta_{k+1N}$ is equal to the net charge flowing away from this node in the other branches.

$$\underset{\approx}{H}_k \underset{\sim}{v}_N(t) - \underset{\approx}{P}_k \underset{\sim}{v}_N(t^-_{k+1N}) + \underset{\approx}{L}_k \hat{\underset{\sim}{q}}(t) = \underset{\sim}{W}(t) \qquad \text{(II.38)}$$

The identification of $\underset{\approx}{H}_k$, $\underset{\approx}{L}_k$, $\underset{\approx}{R}_k$, $\underset{\approx}{M}_k$, $\underset{\approx}{P}_k$ matrices is given by using the method in [7] . The following procedure is given for the case in which dependent sources are only of VCVS type.

<u>Procedure:</u> Partition the KQL equations (I.4) as:

$$\begin{bmatrix} \underset{\approx}{A}_c & \underset{\approx}{A}_s & \underset{\approx}{A}_u & \underset{\approx}{A}_D & \underset{\approx}{A}_W \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{q}}_c \\ \hat{\underset{\sim}{q}}_s \\ \hat{\underset{\sim}{q}}_u \\ \hat{\underset{\sim}{q}}_D \\ \hat{\underset{\sim}{q}}_W \end{bmatrix} = \underset{\sim}{0} \qquad \text{(II.39)}$$

then

$$\underset{\approx}{A}_c \hat{\underset{\sim}{q}}_c + \underset{\approx}{A}_s \hat{\underset{\sim}{q}}_s + \underset{\approx}{A}_u \hat{\underset{\sim}{q}}_u + \underset{\approx}{A}_D \hat{\underset{\sim}{q}}_D + \underset{\approx}{A}_W \hat{\underset{\sim}{q}}_W = \underset{\sim}{0} \qquad \text{(II.40)}$$

where $\underset{\approx}{A}_k$ is obtained from the reduced incidence matrix $\underset{\approx}{A}$ by deleting the columns which do not correspond to the k branches.

With the constitutive equations (II.9) and (II.10) in mind, assume that the network contains r dependent voltage sources and the voltages of the dependent sources can be expressed as linear combinations of the controlling node voltages as

$$\underset{\sim}{v}_D(t)=\underset{\approx}{D}_e\underset{\sim}{v}_N(t) \qquad \text{(II.41)}$$

where $\underset{\sim}{v}_D(t)=\left[v_D^{(1)}(t) \quad v_D^{(2)} \ldots\ldots v_D^{(r)}(t)\right]^T$ and $\underset{\approx}{D}_e$ is rxn matrix of the controlling node voltage coefficients (n is the number of nodes except the reference node) and $v_D^{(i)}(t)$ is the voltage of the i th dependent source.

Now substitute the constitutive equation of the capacitor in Eq.(II.9) into Eq.(II.40) to obtain

$$\underset{\approx}{A}_s\hat{\underset{\sim}{q}}_s+\underset{\approx}{A}_u\hat{\underset{\sim}{q}}_u+\underset{\approx}{A}_c\underset{\approx}{C}\,'\underset{\sim}{v}_c(t)-\underset{\approx}{A}_c\underset{\approx}{C}\,\underset{\sim}{v}_c(t_m^-)+\underset{\approx}{A}_D\hat{\underset{\sim}{q}}_D(t)=-\underset{\approx}{A}_w\underset{\sim}{W}(t) \qquad \text{(II.42)}$$

KVL equations require:

$$\underset{\approx}{A}_c^T\underset{\sim}{v}_N(t)=\underset{\sim}{v}_c(t) \qquad \text{(II.43.a)}$$

$$\underset{\approx}{A}_u^T\underset{\sim}{v}_N(t)=\underset{\sim}{v}_u(t) \qquad \text{(II.43.b)}$$

$$\underset{\approx}{A}_s^T\underset{\sim}{v}_N(t)=\underset{\sim}{v}_s(t) \qquad \text{(II.43.c)}$$

$$\underset{\approx}{A}_D^T\underset{\sim}{v}_N(t)=\underset{\sim}{v}_D(t) \qquad \text{(II.43.d)}$$

Substituting Eq.(II.41) into Eq.(II.43.d) the constitutive equation of VCVS

$$(\underset{\approx}{A}_D^T-\underset{\approx}{D}_e)\underset{\sim}{v}_N(t)=\underset{\sim}{0} \qquad \text{(II.44)}$$

is obtained.

The constitutive equation of the switch holds iff $\underset{\approx}{\phi}^T\hat{\underset{\sim}{q}}_s=\hat{\underset{\sim}{q}}_s$ and $\underset{\sim}{\bar{\phi}}^T v_s=v_s$ or explicitly

$$\bar{\phi}\;\hat{q}_s+\phi\;v_s=0 \Leftrightarrow \phi^T\hat{q}_s=\hat{q}_s \text{ and } \bar{\phi}^T v_s=v_s. \qquad \text{(II.45)}$$

The validity of Eq.(II.45) is easily seen by evaluating the expressions for $\emptyset=1$ and $\emptyset=0$.

The constitutive equation of the switch can be rewritten by using Eq.(II.43.c) as

$$\bar{\emptyset}\,\hat{q}_s+\emptyset\,A_s^T v_N(t)=0 \qquad \text{(II.46)}$$

By substituing Eq.(II.45) and Eq.(II.43.a) into Eq.(II.42)

$$A_s\emptyset^T\hat{q}_s+A_u\hat{q}_u+A_cCA_c^Tv_N(t)-A_cCA_c^Tv_N(t_{k+1N}^-)+A_D\hat{q}_D(t)=-A_W\hat{W}(t) \qquad \text{(II.47)}$$

is obtained.

The constitutive equation of the independent voltage source is given by the Eq.(II.43.a) as

$$A_u^Tv_N(t)=v_u(t)=u(t) \qquad \text{(II.48)}$$

By combining all the equations (II.44), (II.46), (II.47), (II.48) in a matrix form:

$$\begin{bmatrix} A_cCA_c^T & A_s\emptyset^T & A_u & A_D \\ \emptyset A_s^T & \bar{\emptyset} & 0 & 0 \\ A_u^T & 0 & 0 & 0 \\ A_D^T-D_e & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_N \\ \hat{q}_s \\ \hat{q}_u \\ \hat{q}_D \end{bmatrix}=\begin{bmatrix} A_cCA_c^Tv_N(t_{k+1N}^-) \\ 0 \\ 0 \\ 0 \end{bmatrix}+\begin{bmatrix} -A_WW(t) \\ 0 \\ u(t) \\ 0 \end{bmatrix} \qquad \text{(II.49)}$$

By comparing Eq.(II.37) and (II.49) the following relations are obtained

$$P_k=H_k=A_cCA_c^T \qquad L_k=\begin{bmatrix} A_s\emptyset^T & A_u & A_D \end{bmatrix}$$

$$M_k=\begin{bmatrix} \emptyset A_s^T \\ A_u^T \\ A_D^T-D_e \end{bmatrix} \qquad R_k=\begin{bmatrix} \bar{\emptyset} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(II.50)}$$

As stated in [2] and seen from the above formulation

$$\underset{\approx}{M}_k \underset{\sim}{v}_N(t)+\underset{\approx}{R}_k \hat{\underset{\sim}{q}}(t)=\underset{\sim}{u}(t)$$

are precisely a vector formulation of the constitutive equations of the switches, independent voltage sources and VCVS.

Thus, the proof of the theorem is completed.

CONSTRUCTION OF THE TIME DOMAIN MNA EQUATIONS

1) Set up (II.37) with $\underset{\approx}{H}_k$, $\underset{\approx}{M}_k$, $\underset{\approx}{R}_k$, $\underset{\approx}{P}_k$, $\underset{\approx}{L}_k$, u(t) and $\underset{\sim}{W}(t)$ zero.

2) For each component of the circuit identify the stamp of figure II.6. Observe that the stamp of a switch includes the Boolean variable of the clock which controls the switch. If the component is connected to the reference node delete the corresponding row and column in the stamp. Using the indexes of the rows and columns in the stamp add the contribution to the appropriate entries in the matrices $\underset{\approx}{H}_k$, $\underset{\approx}{M}_k$, $\underset{\approx}{R}_k$, $\underset{\approx}{L}_k$, of the left-hand side of (II.37) and to $\underset{\approx}{P}_k$, u(t) and $\underset{\sim}{W}(t)$ of the right-hand side of (II.37).

Note that the stamps in Fig.II.6 are only for computer implementation and should not be taken as constitutive equations.

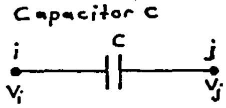
II.6. General Nodal Analysis Of Switched Capacitive Networks By Tsidivis' Topological Approach

Tsidivis, in his paper [1] specifies only one network topology and the switching schedule while opposing to specify as many topologies as there are switch position combinations. Tsidivis uses a similar idea as Kurth and Moschytz present in [5].

All derivations are based on the concepts stated in Chapter I and the networks are assumed to consists of capacitors, ideal switches, independent and dependent voltage sources.

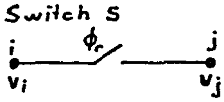
We remember from the definition on page 6 that the topology of the SC network is fixed between any two consecutive switching times e.g. in the intervals of $(t_m, t_{m+1}]$. Then the second definition immediately follows.

Capacitor C



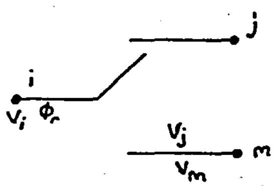
$$\begin{array}{c} \\ i \\ j \end{array}\begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{bmatrix} C & -C \\ -C & C \end{bmatrix} \end{array}\begin{bmatrix} V_i \\ V_j \end{bmatrix} = \begin{bmatrix} C & -C \\ -C & C \end{bmatrix}\begin{bmatrix} V_i(t^-_{k+lN}) \\ V_j(t^-_{k+lN}) \end{bmatrix}$$

Switch S



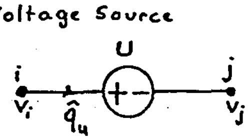
$$\begin{array}{c} \\ i \\ j \\ s \end{array}\begin{array}{c} \begin{array}{ccc} i & j & s \end{array} \\ \left[\begin{array}{cc|c} 0 & 0 & \phi_{rk} \\ 0 & 0 & \phi_{rk} \\ \hline \phi_{rk} & \phi_{rk} & \bar{\phi}_{rk} \end{array}\right] \end{array}\left[\begin{array}{c} V_i \\ V_j \\ \hline \hat{q}_s \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \end{array}\right]$$

Double throw switch



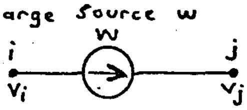
$$\begin{array}{c} \\ i \\ j \\ m \\ s \end{array}\begin{array}{c} \begin{array}{cccc} i & j & m & s \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\bar{\phi}_{rk} \\ 0 & 0 & 0 & -\phi_{rk} \\ \hline 1 & -\bar{\phi}_{rk} & -\phi_{rk} & 0 \end{array}\right] \end{array}\left[\begin{array}{c} V_i \\ V_j \\ V_m \\ \hline \hat{q}_s \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \end{array}\right]$$

Voltage Source



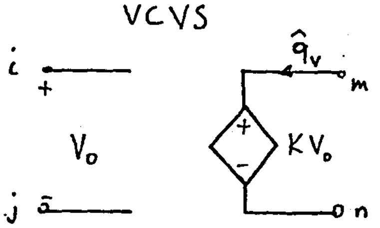
$$\begin{array}{c} \\ i \\ j \\ u \end{array}\begin{array}{c} \begin{array}{ccc} i & j & u \end{array} \\ \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \hline 1 & -1 & 0 \end{array}\right] \end{array}\left[\begin{array}{c} V_i \\ V_j \\ \hline \hat{q}_u \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline u \end{array}\right]$$

Charge source w



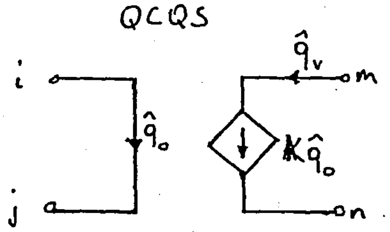
$$\begin{array}{c} \\ i \\ j \end{array}\begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}\begin{bmatrix} V_i \\ V_j \end{bmatrix} = \begin{bmatrix} w \\ -w \end{bmatrix}$$

VCVS



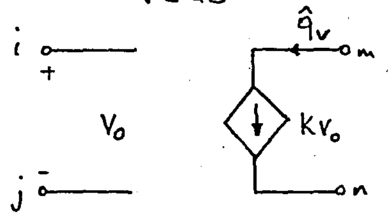
$$\begin{array}{c} \\ i \\ j \\ m \\ n \\ V_v \end{array}
\begin{array}{c}
\begin{array}{ccccc} i & j & m & n & V_v \end{array} \\
\left[\begin{array}{cccc|c}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 \\
\hline
-K & K & 1 & -1 & 0
\end{array}\right]
\end{array}
\left[\begin{array}{c} V_i \\ V_j \\ V_m \\ V_n \\ \hline \hat{q}_v \end{array}\right]
=
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \end{array}\right]$$

QCQS



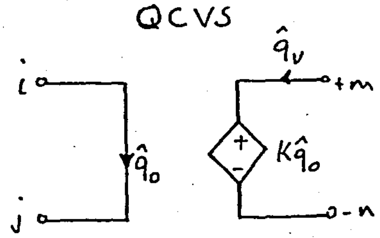
$$\begin{array}{c} \\ i \\ j \\ m \\ n \\ \hat{q}_r \end{array}
\begin{array}{c}
\begin{array}{ccccc} i & j & m & n & \hat{q}_v \end{array} \\
\left[\begin{array}{cccc|c}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & K \\
0 & 0 & 0 & 0 & -K \\
\hline
1 & -1 & 0 & 0 & 0
\end{array}\right]
\end{array}
\left[\begin{array}{c} V_i \\ V_j \\ V_m \\ V_n \\ \hline \hat{q}_o \end{array}\right]
=
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \end{array}\right]$$

VCQS



$$\begin{array}{c} \\ i \\ j \\ m \\ n \end{array}
\begin{array}{c}
\begin{array}{cccc} i & j & m & n \end{array} \\
\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
K & -K & 0 & 0 \\
-K & K & 0 & 0
\end{array}\right]
\end{array}
\left[\begin{array}{c} V_i \\ V_j \\ V_m \\ V_n \end{array}\right]
=
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

QCVS



$$\begin{array}{c} \\ i \\ j \\ m \\ n \\ V_q \\ \hat{q}_o \end{array}
\begin{array}{c}
\begin{array}{cccccc} i & j & m & n & V_q & \hat{q}_o \end{array} \\
\left[\begin{array}{cccc|cc}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
\hline
0 & 0 & 1 & -1 & 0 & -K \\
1 & -1 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\left[\begin{array}{c} V_i \\ V_j \\ V_m \\ V_n \\ \hline \hat{q}_v \\ \hat{q}_o \end{array}\right]
=
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right]$$

<u>Definition:</u> Closed switched network is the network resulting from the original one if every element, except the closed switches is removed. During each interval $(t_m, t_{m+1}]$ the closed-switch network will in general consist of several separate parts (refer to Chapter I). Each such part will consist of either a set of nodes connected together through closed switches or of an isolated node. To specify the switching pattern for each interval $(t_m, t_{m+1}]$ an nxn matrix $\underset{\approx}{S}_m$,(the switching matrix) will be defined, where n is the number of nodes except the reference node.

The entries $s_{m,ij}$ of the switching matrix are defined as follows:

$$s_{m,ij} = \begin{cases} 1 & \text{If i is the lowest numbered node of a separate part of the closed switch network, and node j belongs to that separate part.} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(II.51)}$$

Therefore, if there is a total of l_k separate parts in the closed switch network, $\underset{\approx}{S}_k$ will be a sparse matrix with l_k non-zero rows and total of n non-zero entries. Following examples will clearify the switching matrix-concept.

<u>Example II.1:</u> If all switches happen to be open, then l_k=n and $\underset{\approx}{S}_k$ will be the nxn identity matrix.

<u>Example II.2:</u> If at t the closed switches of a circuit with n=6 result in nodes 1 and 2 being connected together, nodes 4,5,6 being connected together, and non-closed switches being connected to node 3, then the switching matrix for $t \epsilon \Delta_m \hat{=} (t_m, t_{m+1}]$ will be

$$\underset{\approx}{S}_m = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where node 3 is an isolated node in Δ_m.

Definition: A sequence $\{S_k\}$, k=1,2,.....N, and(N is the number of phases)where each element of $\{S_k\}$ characterizes the corresponding intervals in one period, will be called the "switching schedule".

In the majority of cases, $\{S_k\}$ will be periodic, so only its first N values corresponding to one period, need be formed.

Network Equations In The Time Domain

For the sake of coincidence in notations, once again the definitions of some vectors are given.

The node voltage vector

$$\underset{\sim}{v}_N(t)=\left[v^{(1)}(t) \quad v^{(2)}(t)\ldots v^{(n)}(t) \right]^T$$

where $v^{(i)}(t)$ is the voltage between node i and ground at time t and n is the number of nodes excluding the ground.

The node charge vector

$$\underset{\sim}{q}_N(t)=\left[q^{(1)}(t) \quad q^{(2)}(t)\ldots q^{(n)}(t) \right]^T$$

where $q^{(i)}(t)$ is the total charge of all capacitor plates connected permanently i.e.,not through switches, to node i at time t. This is equivalent to considering the SC network with all switches removed as done by Kurth and Moschytz [5] .

Kurth and Moschytz seperate the two-phase SC network into a simple capacitor network and an array of even and odd switches as shown in Fig.II.7.below. Where S^e(respectively S^o) indicates that the corresponding switch is closed during even (resp. odd time) time intervals.

Then with the above idea, the node charges in the C-network can be axpressed as linear combinations of the node voltages as

$$\underset{\sim}{q}_N(t)=\underset{\approx}{C}_{st}\underset{\sim}{v}_N(t) \tag{II.52}$$

where $\underset{\approx}{C}_{st}$ is a nxn matrix which will be called the capacitance matrix and is defined as follows:

$$c_{ij} = \begin{cases} \text{total capacitance permanently connected to node } i;\ i=j \\ \text{negative of total capacitance permanently connected between nodes } i \text{ and } j;\ i \neq j. \end{cases} \qquad \text{(II.53)}$$

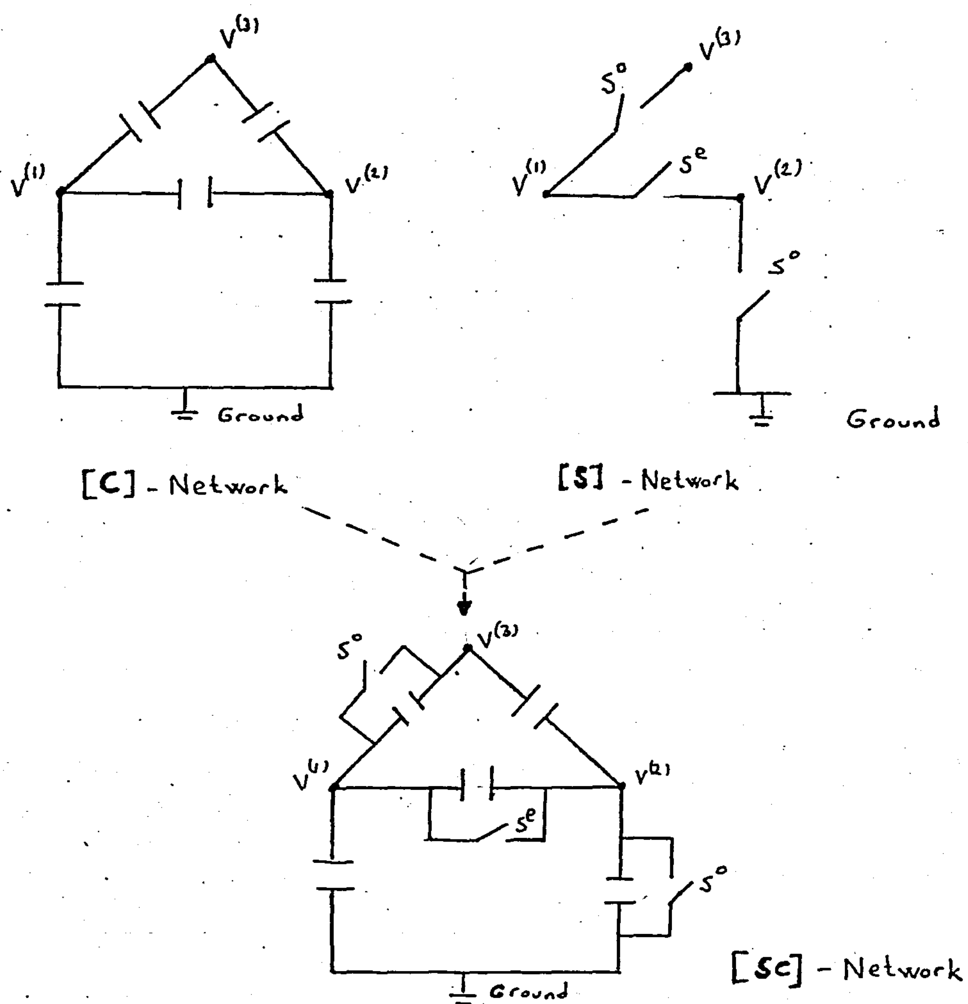


Fig.II.7. SC network as a superposition of a C-network and a switch network.

Clearly $\underset{\approx}{C}_{st}$ is independent of switching. Subscript "st" is used to remind the reader that the matrix $\underset{\approx}{C}$ in section II.5 is different than $\underset{\approx}{C}_{st}$ and $\underset{\approx}{C}_{st}$ is equal to $\underset{\approx}{A}_c \underset{\approx}{C} \underset{\approx}{A}_c^T$.

Define $\hat{q}_V^{(i)}(t)$ to be the total charge that leaves node(i) from a time reference t' to time t, through one or more voltage sources (dependent or independent) then

$$\hat{q}_v(t) = \left[\hat{q}_v^{(1)}(t) \quad \hat{q}_v^{(2)}(t) \ldots \hat{q}_v^{(n)}(t) \right]^T$$

With the previous definitions and notations in page 24

$$\hat{q}_v(t) = \left[A_u \;\vdots\; A_D \right] \begin{bmatrix} \hat{q}_u(t) \\ \hat{q}_D(t) \end{bmatrix} \qquad \text{(II.54)}$$

Once again A_u(resp.A_D) is node-to-branch incidence matrix of the network which is formed by removing all branches except the independent voltage sources (resp. dependent voltage sources).

Notice that A_u and A_D are the same matrices as in section II.5 and A_u is an nxp matrix and A_D is an nxr matrix. Equations (II.41), (II.43) and (II.44) are still valid.

Switched Network Equations

Assume that the node voltages of a network are known at t_m^- and that at t_m^- some switches change position, resulting in a topology described by a switching matrix S_m. The problem is to evaluate the node valtages $v_N^{(i)}(t)$ for all t $(t_m, t_{m+1}]$. To avoid pathological cases, it is assumed that switching is such that no loops of voltage sources and/or closed switches occur. This point will be clearified in the solvability problem section.

For each set of nodes I_g, containing n_g nodes connected together through switches closed at or before t_m, there will be one charge conservation equation in the form of Eq.(I.2) and node voltage equalities in the form of Eq.(I.6). Assume there is a total 1 such node sets (each corresponding to one of the seperate parts of the "closed switch network"). To each such node set there corresponds one non-zero row of matrix S_m, defined in Eq.(II.51). It is easily seen from Eq.(II.51) that the 1 charge conservation equations, each corresponding to one node set, will be given by the 1 non-zero rows of

$$S_m q_N(t) + S_m \hat{q}_v(t) = S_m q_N(t_m^-) \qquad \text{(II.55)}$$

Using Eq.(II.52) and Eq.(II.54) in Eq.(II.55), one obtains:

$$\underset{\approx}{S}_m \underset{\approx}{C}_{st} \underset{\sim}{v}_N(t) + \underset{\approx}{S}_m \left[\underset{\approx}{A}_u \;\vdots\; \underset{\approx}{A}_D \right] \begin{bmatrix} \hat{\underset{\sim}{q}}_u(t) \\ \hat{\underset{\sim}{q}}_D(t) \end{bmatrix} = \underset{\approx}{S}_m \underset{\approx}{C}_{st} \underset{\sim}{v}_N(t_m^-) \qquad \text{(II.56)}$$

The total number of node voltage equalities imposed by the closed switches in l sets will be $\sum_{g=1}^{\ell} (n_g - 1) = n - \ell$. If each of these equations is assumed to be in the form of Eq.(I.6), it is easily seen from Eq.(II.51) that these n-1 equations will be given by the n-1 non-zero rows of

$$\left[\underset{\approx}{S}_m^T - \underset{\approx}{I} \right] \cdot \underset{\sim}{v}_N(t) = 0 \qquad \text{(II.57)}$$

where $\underset{\approx}{I}$ is the nxn identity matrix.

The unknown variables are as follows:

a) The n node voltages

b) The p charges through the p independent voltage sources.

c) The r charges through the r dependent sources.

These unknows can be solved for, using l charge conservation equations (II.56), the (n-1) equations (II.57) imposed by the closed switches, the p KVL equations (II.48) imposed by the p independent sources and the r KVL equations (II.44) imposed by the r dependent sources.

Noticed that because of the way $\underset{\approx}{S}_m$ was defined in Eq.(II.51), Eq. (II.56) has zero rows exactly where Eq.(II.57) has non-zero rows and vice-versa. Thus, two equations can be added to yield a matrix equation with n non-zero rows:

$$\left[\underset{\approx}{S}_m \underset{\approx}{C}_{st} + \underset{\approx}{S}_m^T - \underset{\approx}{I} \right] \underset{\sim}{v}_N(t) + \underset{\approx}{S}_m \left[\underset{\approx}{A}_u \;\vdots\; \underset{\approx}{A}_D \right] \begin{bmatrix} \hat{\underset{\sim}{q}}_u(t) \\ \hat{\underset{\sim}{q}}_D(t) \end{bmatrix} = \underset{\approx}{S}_m \underset{\approx}{C}_{st} \underset{\sim}{v}_N(t_m^-) \qquad \text{(II.58)}$$

Finally combiningEq.(II.44), Eq.(II.48) and Eq.(II.58) yields:

$$\begin{matrix} p \\ r \\ n \end{matrix} \begin{bmatrix} \underset{\approx}{A}_u^T & \vdots & \\ \underset{\approx}{A}_D^T - \underset{\approx}{D}_e & \vdots & \underset{\approx}{0} \\ \underset{\approx}{S}_m \underset{\approx}{C}_{st} + \underset{\approx}{S}_m^T - \underset{\approx}{I} & \underset{\approx}{S}_m \underset{\approx}{A}_u & \underset{\approx}{S}_m \underset{\approx}{A}_D \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_N(t) \\ \hat{\underset{\sim}{q}}_u(t) \\ \hat{\underset{\sim}{q}}_D(t) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{v}_u(t) \\ \underset{\sim}{0} \\ \underset{\approx}{S}_m \underset{\approx}{C}_{st} \underset{\sim}{v}_N(t_m^-) \end{bmatrix} \begin{matrix} p \\ r \\ n \end{matrix} \qquad \text{(II.59)}$$

This equation can be obtained from Eq.(II.49) by eliminating $\hat{q}_s(t)$ for a certain phase while no charge sources exist in the circuit.

For any $t\epsilon(t_m, t_{m+1}]$ the solution of Eq.(II.59) will provide all the unknown variables. In particular the node voltages at $t=t_{m+1}^-$ can be evaluated and used as the new initial node voltages in the solution for the next interval $(t_{m+1}, t_{m+2}]$, after switching at t_{m+1} has resulted in a new topology, associated with S_{m+1}. For a complete solution of the network for all $t>0$, therefore, the following is required:

1) The waveforms of the independent voltage sources for $t>0$;
2) The initial node voltages;
3) The switching schedule $\{S_m\}$.

Let $\emptyset_m$ represent the (n+p+r)x(n+p+r) matrix which premultiplies the unknown vector in Eq.(II.59) and consider its inverse $\emptyset_m^{-1}$. Let $(\emptyset_m^{-1})_L$ be the upper left nxp submatrix of $\emptyset_m^{-1}$ and let $(\emptyset_m^{-1})_R$ be the upper right nxn submatrix of $\emptyset_m^{-1}$. (It is assumed that $\emptyset_m$ matrix is non-singular). If Eq.(II.59) is solved for $v_N(t)$, then

$$v_N(t)=A_m u(t)+B_m v_N(t_m^-) \qquad t\epsilon(t_m, t_{m+1}] \qquad \text{(II.60)}$$

where $A_m=(\emptyset_m^{-1})_L$ and $B_m=(\emptyset_m^{-1})_R S_m C$.

The above equation is already the one obtained by the method of Liou and Kuo introduced on page 22, Equation (II.33).

REMARKS

In this method, the capacitance matrix C_{st} and the incidence matrices A_u and A_D are formed only once, and are independent of switch positions. The switch position sequence is conveniently defined by the sequence $\{S_m\}$. Every time one or more switches change position, the corresponding new S_m is substituted in Eq.(II.59) and the network equations result automatically without having to respecify the topology. If the switching sequence were not done, the topology of the network would have to be respecified every time a new switch position combination occurred, the nodes possibly renumbered.

and new network matrices (in general of different dimensions from the previous ones) would have to be formed.

II.7. Illustrative Examples

In this section all methods developed so far will be applied to the circuit of Fig.II.8(a).

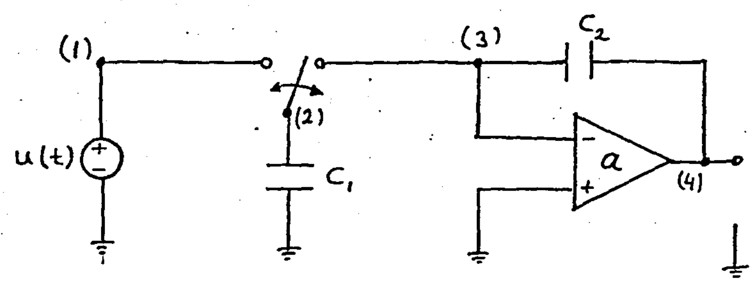


Fig.II.8(a)

The switch in Fig.II.8(a) moves periodically back and forth. The above circuit is equivalent to that in Fig.II.8(b) in which the two position switch has been replaced by two on-off switches.

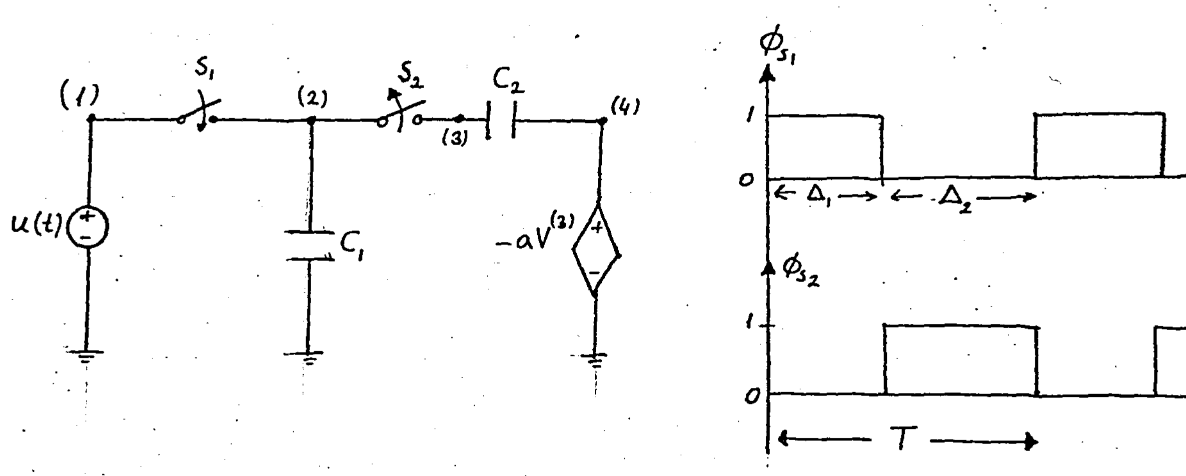
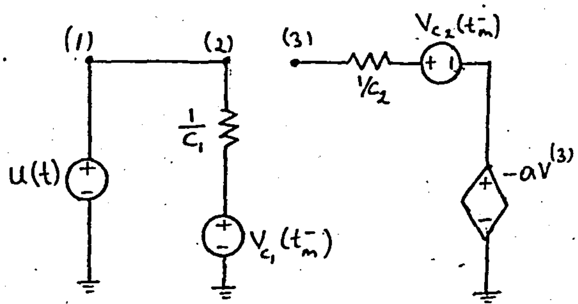


Fig.II.8(b)

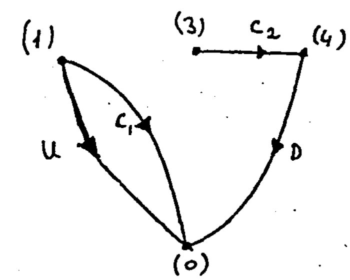
Example II.3

In this example, the method explained in section II.4 will be applied to the circuit of Fig.II.8(a).

The circuit in Fig.II.8(b) can be transformed into its resistive equivalent as in Fig.II.9(a) for $t\epsilon\Delta_m$ where m=1,3,5,.. which corresponds to phase 1.



Resistive circuit



Its corresponding graph

Fig.II.9(a)

Define the voltage vectors according to Eq.(II.22) as

$$\underset{\sim}{v} \triangleq \begin{bmatrix} v_{c1} \\ v_{c2} \\ v_u \\ v_D \end{bmatrix} \qquad \underset{\sim}{\hat{v}} \triangleq \begin{bmatrix} \hat{v}_{c1} \\ \hat{v}_{c2} \\ \hat{v}_u \\ \hat{v}_D \end{bmatrix} \qquad \underset{\sim}{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_{c1} \\ \varepsilon_{c2} \\ \varepsilon_u \\ \varepsilon_D \end{bmatrix} \qquad \text{(II.61)}$$

and the charge vectors as

$$\underset{\sim}{\hat{q}}' \triangleq \begin{bmatrix} \hat{q}'_{c1} \\ \hat{q}'_{c2} \\ \hat{q}'_u \\ \hat{q}'_D \end{bmatrix} \qquad \underset{\sim}{\hat{q}} \triangleq \begin{bmatrix} \hat{q}_{c1} \\ \hat{q}_{c2} \\ \hat{q}_u \\ \hat{q}_D \end{bmatrix} \qquad \underset{\sim}{Z} \triangleq \begin{bmatrix} z_{c1} \\ z_{c2} \\ z_u \\ z_D \end{bmatrix}$$

The explanation of the composite branch on page 18 implies the following relations:

$$\underset{\sim}{Z}=\underset{\sim}{0} \quad \text{and} \quad \hat{\underset{\sim}{v}}_b=\begin{bmatrix} \hat{v}_u \\ \hat{v}_D \end{bmatrix}=0 \quad \text{and} \quad \underset{\sim}{\mathcal{E}}=\begin{bmatrix} v_{c1}(t_m^-) \\ v_{c2}(t_m^-) \\ u(t) \\ -a(v_{c2}(t)+v_D(t)) \end{bmatrix} \tag{II.63}$$

Also,

$$\begin{bmatrix} \hat{q}_{c1} \\ \hat{q}_{c2} \end{bmatrix} = \begin{bmatrix} 1/C_1 & 0 \\ 0 & 1/C_2 \end{bmatrix} \begin{bmatrix} \hat{v}_{c1} \\ \hat{v}_{c2} \end{bmatrix} \tag{II.64}$$

From Eq.(II.64), the branch-admittance matrix is

$$\underset{\approx}{Y}_a= \begin{bmatrix} 1/C_1 & 0 \\ 0 & 1/C_2 \end{bmatrix} \tag{II.65}$$

The reduced incidence matrix of the circuit is

$$\underset{\approx}{A}= \begin{array}{c} \\ 1 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} c_1 & c_2 & u & D \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \tag{II.66}$$

Then

$$\underset{\approx}{A}_a= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} , \quad \underset{\approx}{A}_b= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{II.67}$$

The branch relations for the voltage sources are given by

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & a/1+a & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \\ v_u \\ v_D \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{II.68}$$

Then

$$\underset{\approx}{K}=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{a}{1+a} & 0 & 1 \end{bmatrix} \quad \text{and} \quad \underset{\approx}{H}=\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{II.69}$$

Since there is no independent charge sources and the voltage controlled charge sources in this circuit

$$\underset{\approx}{Y}_a=\underset{\approx}{Y}'_a \quad \text{and} \quad \underset{\approx}{A}_w=\underset{\approx}{0} \ . \tag{II.70}$$

Then

$$\underset{\approx}{A}_a\underset{\approx}{Y}_a\underset{\approx}{A}_a^T=\begin{bmatrix} 1/C_1 & 0 & 0 \\ 0 & 1/C_2 & -1/C_2 \\ 0 & -1/C_2 & 1/C_2 \end{bmatrix} \quad \text{and} \quad \underset{\approx}{K}\underset{\approx}{A}^T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{a}{1+a} & \frac{1}{1+a} \end{bmatrix} \tag{II.71}$$

Also ,

$$\underset{\approx}{A}_a\underset{\approx}{Y}'_a=\begin{bmatrix} 1/C_1 & 0 \\ 0 & 1/C_2 \\ 0 & -1/C_2 \end{bmatrix} \tag{II.72}$$

Substituting Eqs.(II.71), (II.72) and (II.67) into Eq.(II.32) yields

$$\left[\begin{array}{ccc|cc} 1/C_1 & 0 & 0 & 1 & 0 \\ 0 & 1/C_2 & -1/C_2 & 0 & 0 \\ 0 & -1/C_2 & 1/C_2 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & a/1+a & 1/1+a & 0 & 0 \end{array}\right] \left[\begin{array}{c} v^{(1)}(t) \\ v^{(3)}(t) \\ v^{(4)}(t) \\ \hline \hat{q}_u(t) \\ \hat{q}_D(t) \end{array}\right] = \left[\begin{array}{c} 1/C_1 v_{c1}(t_m^-) \\ 1/C_2 v_{c2}(t_m^-) \\ -1/C_2 v_{c2}(t_m^-) \\ \hline u(t) \\ 0 \end{array}\right] \tag{II.73}$$

Eq.(II.73) can be solved to obtain

$$\begin{bmatrix} v^{(1)}(t) \\ v^{(3)}(t) \\ v^{(4)}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1/1+a \\ 0 & -a/1+R \end{bmatrix} \begin{bmatrix} v_{c1}(t_m^-) \\ v_{c2}(t_m^-) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) \qquad (II.74)$$

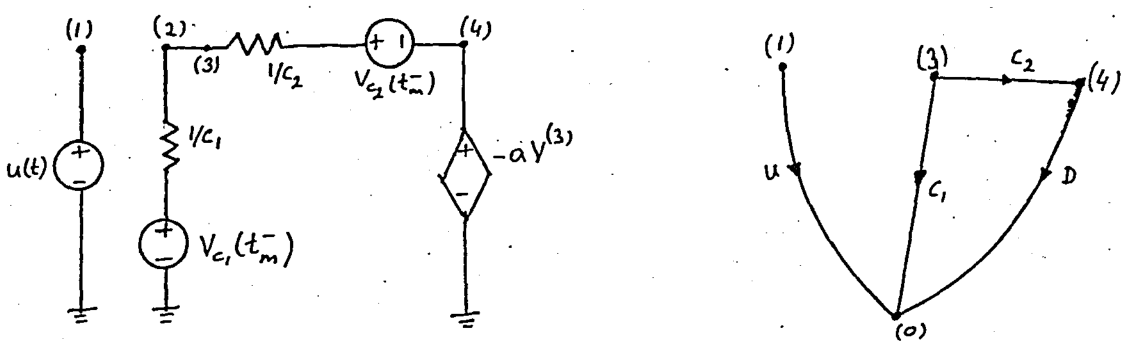
Then by Eq.(II.34)

$$\begin{bmatrix} v_{c1}(t) \\ v_{c2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{c1}(t_m^-) \\ v_{c2}(t_m^-) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \quad t\in\Delta_m \quad m=1,3,5 \qquad (II.75)$$

is obtained.

Indeed, Eq.(II.75) could be obtained by inspection from the circuit of Fig.II.9(b).

Now, the circuit in Fig.II.8(b) is solved for $t\in\Delta_m$ where m=2,4,6... which corresponds to phase 2.



Resistive circuit Its corresponding graph

Fig.II.9(b).

Eqs.(II.63), (II.65) and (II.69) are still valid for the circuit of Fig.II.9(b).

The reduced incidence matrix of the above circuit is

$$\underset{\approx}{A} = \begin{array}{c} \\ 1 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} c_1 & c_2 & u & D \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \qquad (II.76)$$

Partition $\underset{\approx}{A}$ as

$$\underset{\approx}{A}_a = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \underset{\approx}{A}_b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(II.77)}$$

Then

$$\underset{\approx}{A}_a \underset{\approx}{Y}_a \underset{\approx}{A}_a^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/C_1+1/C_2 & -1/C_1 \\ 0 & 1/C_2 & 1/C_2 \end{bmatrix} \qquad \text{(II.78)}$$

and

$$\underset{\approx}{K}\underset{\approx}{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ & & \\ 0 & a/1+a & 1/1+a \end{bmatrix}, \quad \underset{\approx}{A}_a \underset{\approx}{Y}_a = \begin{bmatrix} 0 & 0 \\ 1/C_1 & 1/C_2 \\ 0 & -1/C_2 \end{bmatrix} \qquad \text{(II.79)}$$

Substituting Eqs.(II.78), (II.79) and (II.77) into Eq.(II.32) yields

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1/C_1+1/C_2 & -1/C_2 & 0 & 0 \\ 0 & -1/C_2 & 1/C_2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & a/1+a & 1/1+a & 0 & 0 \end{bmatrix} \begin{bmatrix} v^{(1)}(t) \\ v^{(3)}(t) \\ v^{(4)}(t) \\ \hat{q}_u \\ \hat{q}_D \end{bmatrix} = \begin{bmatrix} 0 \\ 1/C_1 v_{c1}(t_m^-) + \frac{1}{C_2} v_{c2}(t_m^-) \\ -1/C_2 v_{c2}(t_m^-) \\ u(t) \\ 0 \end{bmatrix} \qquad \text{(II.80)}$$

Eq.(II.80) can be solved to obtain

$$\begin{bmatrix} v^{(1)}(t) \\ v^{(3)}(t) \\ v^{(4)}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \alpha & \beta \\ -a\alpha & -a\beta \end{bmatrix} \begin{bmatrix} v_{c1}(t_m^-) \\ v_{c2}(t_m^-) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) \qquad \text{(II.81)}$$

where $\alpha = \dfrac{C_2}{(1+a)C_1+C_2}$ and $\beta = \dfrac{C_1}{(1+a)C_1+C_2}$

Then by Eq.(II.34)

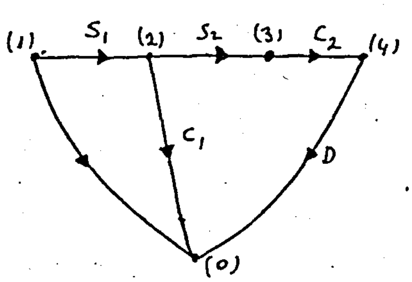
$$\begin{bmatrix} v_{c1}(t) \\ v_{c2}(t) \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \alpha(1+a) & \beta(1+a) \end{bmatrix} \begin{bmatrix} v_{c1}(t_m^-) \\ v_{c2}(t_m^-) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t) \quad t\in\Delta_m \quad m=2,4,6.. \quad (II.82)$$

is obtained.

<u>Example II.4.</u>

Now, the circuit of Fig.II.8 is solved using stamps by the method given in section II.5.

The reduced incidence matrix to Fig.II.8(b) is found from the following graph of Fig.II.8(b).



$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} u & S_1 & c_1 & S_2 & c_2 & D \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Using stamps of Fig.II.6., the following equation is immediately obtained.

$$\begin{array}{c|cccc|cc|cc|} & 1 & 2 & 3 & 4 & S_1 & S_2 & u & D \\ \hline 1 & 0 & 0 & 0 & 0 & \phi_{s1m} & 0 & 1 & 0 \\ 2 & 0 & C_1 & 0 & 0 & -\phi_{s2m} & \phi_{s2m} & 0 & 0 \\ 3 & 0 & 0 & C_2 & -C_2 & 0 & -\phi_{s2m} & 0 & 0 \\ 4 & 0 & 0 & -C_2 & C_2 & 0 & 0 & 0 & 1 \\ \hline S_1 & \phi_{s1m} & 0 & 0 & 0 & \phi_{s1m} & 0 & 0 & 0 \\ S_2 & 0 & \phi_{s2m} & -\phi_{s2m} & 0 & 0 & \bar{\phi}_{s2m} & 0 & 0 \\ \hline u & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline D & 0 & 0 & a & 1 & 0 & 0 & 0 & 0 \end{array} \begin{bmatrix} v^{(1)}(t) \\ v^{(2)}(t) \\ v^{(3)}(t) \\ v^{(4)}(t) \\ \hat{q}_{s1}(t) \\ \hat{q}_{s2}(t) \\ \hat{q}_u(t) \\ \hat{q}_D(t) \end{bmatrix} =$$

$$
= \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & -C_2 \\ 0 & 0 & -C_2 & C_2 \end{bmatrix} \begin{bmatrix} v^{(1)}(t^-_{m+1N}) \\ v^{(2)}(t^-_{m+1N}) \\ v^{(3)}(t^-_{m+1N}) \\ v^{(4)}(t^-_{m+1N}) \end{bmatrix} \\ \hline 0 \\ 0 \\ \hline u(t) \\ 0 \end{bmatrix} \qquad t \in \Delta_{m+1N} \qquad \text{(II.83)}
$$

where m=1,2 and N=2 .

The entries for the switch S_1 in the coefficient matrix of Eq.(II.83) are found using Fig.II.6 as follows:

The switch S_1 is connected between node 1 and node 2. Therefore, ϕ_{s1m} is placed to the positions $(1, S_1)$ and $(S_1, 1)$;

$-\phi_{s1m}$ is placed to the positions $(2, S_1)$ and $(S_1, 2)$;

$\bar{\phi}_{s1m}$ is placed to the position (S_1, S_1).

The capacitor C_1 is connected between datum node and and node 2. Therefore, C_1 is placed to the position (2,2).

Comparing Eq.(II.83) with Eq.(II.49) yi ds the following relations

$$\underset{\approx}{C}_{st}=\underset{\approx}{A}_c\underset{\approx}{C}\,\underset{\approx}{A}_c^T=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & C_1 & 0 & 0\\ 0 & 0 & C_2 & -C_2\\ 0 & 0 & -C_2 & C_2\end{bmatrix}$$

$$\underset{\approx}{\phi}\,\underset{\approx}{A}_s^T=\begin{bmatrix}\phi_{s1m} & 0 & 0 & 0\\ 0 & \phi_{s2m} & -\phi_{s2m} & 0\end{bmatrix}$$

$$\underset{\approx}{A}_D^T-\underset{\approx}{D}_e=\begin{bmatrix}0 & 0 & a & 1\end{bmatrix}$$

$$\underset{\approx}{A}_u^T=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}$$

where

$$\underset{\approx}{\phi}=\begin{bmatrix}\phi_{s1m} & 0\\ 0 & \phi_{s2m}\end{bmatrix}$$

<u>Example II.5:</u>

In this example, the circuit in Fig.II.8 is solved by the procedure given in section II.6.

From Eq.(II.53) the capacitance matrix is

$$\underset{\approx}{C}_{st}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & C_1 & 0 & 0\\ 0 & 0 & C_2 & -C_2\\ 0 & 0 & -C_2 & C_2\end{bmatrix}$$

We have

$$\underset{\approx}{D}_e=\begin{bmatrix}0 & 0 & -a & 0\end{bmatrix}$$

$$\underset{\approx}{A}_u=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$$

$$\underset{\approx}{A}_D=\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^T$$

If at t_1, the switch S_1 closes, then the other is open and

$$\underset{\approx}{S}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, Eq.(II.59) becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 1 & 0 & 0 \\ 0 & C_1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & -C_2 & 0 & 0 \\ 0 & 0 & -C_2 & C_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} v^{(1)}(t) \\ v^{(2)}(t) \\ v^{(3)}(t) \\ v^{(4)}(t) \\ \hat{q}_u(t) \\ \hat{q}_D(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ 0 \\ C_1 v^{(2)}(t_1^-) \\ 0 \\ C_2 v^{(3)}(t_1^-) - C_2 v^{(4)}(t_1^-) \\ -C_2 v^{(3)}(t_1^-) + C_2 v^{(4)}(t_1^-) \end{bmatrix} \quad \text{(II.84)}$$

The above algebraic equation can be verified by inspection from the circuit.

If now at t_2 the switch S_2 closes and S_1 is open then

$$\underset{\approx}{S}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and Eq.(II.59) becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & C_1 & C_2 & -C_2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & C_2 & C_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} v^{(1)}(t) \\ v^{(2)}(t) \\ v^{(3)}(t) \\ v^{(4)}(t) \\ \hat{q}_u(t) \\ \hat{q}_D(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ 0 \\ 0 \\ C_1 v^{(2)}(t_2^-) + C_2 v^{(3)}(t_2^-) - C_2 v^{(4)}(t_2^-) \\ 0 \\ -C_2 v^{(3)}(t_2^-) + C_2 v^{(4)}(t_2^-) \end{bmatrix} \quad \text{(II.85)}$$

If in Eq.(II.83) 1 is substituted for ϕ_{s1m} and 0 for ϕ_{s2m} and the charges in the switches are eliminated Eq.(II.84) is obtained.

Eq.(II.85) is obtained from Eq.(II.83) when $\phi_{s1m}=0$, $\phi_{s2m}=1$ (i.e., phase 2).

II.8. Solvability Problem

In the previous chapters, different analysis techniques have been given for the solution of the SC networks. It has been observed that it is possible to obtain these different equations from each other.

But there may be some cases that the solutions for Eqs.(II.32), (II.49) and (II.59) do not exist (i.e., those equations may not be linearly independent and the inverse of the coefficient matrices may not exist). In these cases, the unknown variables can not be found uniquely in terms of the known values or there may be inconsistencies depending upon the values of the controlling coefficients of the dependent sources.

In order to give more insight to the solvability problem, the following example will be analyzed by the method given in section II.5.

<u>Example II.6:</u>

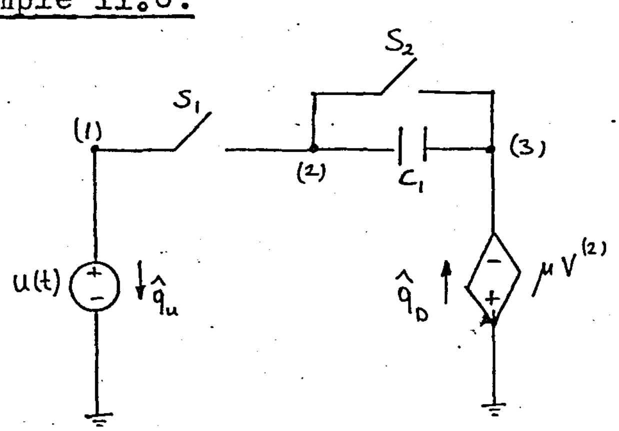


Fig.II.10.

The switch S_1 is controlled by the clock ϕ_1 and the switch S_2 is controlled by the clock ϕ_2.

The MNA equations (II.49) for the circuit in Fig.II.10 are:

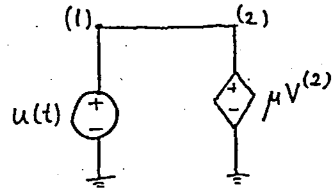
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ S_1 \\ S_2 \\ D \\ u \end{array}
\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & S_1 & S_2 & D & u \end{array} \\
\left[\begin{array}{ccccccc}
0 & 0 & 0 & \phi_{1m} & 0 & 0 & 1 \\
0 & C_1 & -C_1 & -\phi_{1m} & \phi_{2m} & 0 & 0 \\
0 & -C_1 & C_1 & 0 & -\phi_{2m} & -1 & 0 \\
\phi_{1m} & -\phi_{1m} & 0 & \bar{\phi}_{1m} & 0 & 0 & 0 \\
0 & \phi_{2m} & -\phi_{2m} & 0 & \bar{\phi}_{2m} & 0 & 0 \\
0 & & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \end{array}
\begin{bmatrix} v^{(1)} \\ v^{(2)} \\ v^{(3)} \\ \hat{q}_{s1} \\ \hat{q}_{s2} \\ \hat{q}_D \\ \hat{q}_u \end{bmatrix}
=
\begin{bmatrix} 0 \\ C_1(v^{(2)}(t^-)-v^{(3)}(t^-)) \\ -C_1(v^{(2)}(t^-)-v^{(3)}(t^-)) \\ 0 \\ 0 \\ 0 \\ u(t) \end{bmatrix} \quad (II.86)$$

$$\Delta = \det\left\{\begin{bmatrix} H_m & L_m \\ M_m & R_m \end{bmatrix}\right\} = C_1\bar{\phi}_{1m}\bar{\phi}_{2m} - \phi_{1m}\bar{\phi}_{2m} - \bar{\phi}_{1m}\phi_{2m} + C_1\mu\bar{\phi}_{1m}\bar{\phi}_{2m} - \mu\phi_{2m}\bar{\phi}_{1m}$$

Four cases can be considered for this example.

Case	ϕ_{1m}	ϕ_{2m}	Δ	
1)	1	1	0	(no solution if $\mu \neq -1$)
2)	1	0	-1	Solvable
3)	0	1	$-1-\mu$	(no unique solution if $\mu=-1$)
4)	0	0	$C_1(1+\mu)$	(no " " if $\mu=-1$)

<u>Case 1:</u> Fig.II.10 is equivalent to the following



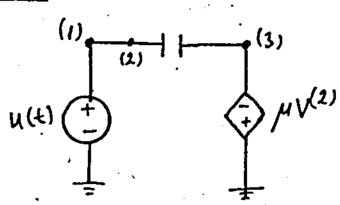
if $\mu=-1$ $\quad v^{(1)}(t)=v^{(2)}(t)=v^{(3)}(t)=u(t)$,

$$\left.\begin{array}{l} \hat{q}_{s1}=\hat{q}_{s2}=-\hat{q}_D \\ \hat{q}_u=-\hat{q}_{s1} \end{array}\right\} \Rightarrow \hat{q}_u=\hat{q}_D:$$ infinitely many solutions for the source charges.

if $\mu \neq -1$ $\quad v^{(1)}(t)=v^{(2)}(t)=v^{(3)}(t)=u(t)=-\,u(t)$.(Inconsistency)

$\therefore$ If $\mu \neq -1$, the circuit in Fig.II.10 is not solvable (no solution).

Case 2:



Eq.(II.86) yields following equations:

$v^{(1)}(t)=v^{(2)}(t)=u(t)$,

$v^{(3)}(t)=-\,u(t)$,

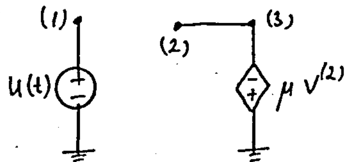
$\hat{q}_{s2}(t)=0$,

$\hat{q}_D(t)=\hat{q}_u(t)=-\hat{q}_{s1}(t)=-C_1(1+\mu)u(t)+C_1(1+\mu)u(t_m^-)$.

If $\mu=-1 \Rightarrow \hat{q}_{s1}=\hat{q}_u(t)=\hat{q}_D(t)=0$.

$\therefore$ The circuit in Fig.II.10 is solvable in case 2.

Case 3:



Again, the following equations are obtained from Eq.(II.86)

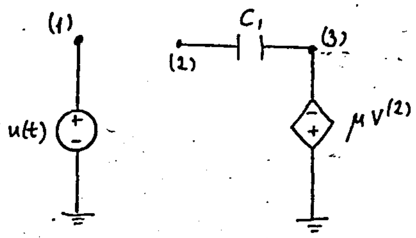
$\hat{q}_{s2}=\hat{q}_D=\hat{q}_{s1}(t)=\hat{q}_u(t)=0$, $\quad v_1(t)=u(t)$,

$v^{(2)}(t)=v^{(3)}(t)$, $\quad v^{(3)}=-\mu v^{(2)}(t)$.

If $\mu \neq -1 \Rightarrow v^{(2)}(t)=v^{(3)}(t)=0$

if $\mu=-1 \Rightarrow v^{(2)}(t)=v^{(3)}(t)$ $\therefore$ infinitely many solutions for the voltages of node 2 and 3.

Case 4:



From Eq.(II.86)

$$\hat{q}_u(t)=\hat{q}_{s1}(t)=\hat{q}_{s2}(t)=0,$$

$$v_1(t)=u(t),$$

$$v^{(3)}(t)-v^{(2)}(t)=v^{(3)}(t_m^-)-v^{(2)}(t_m^-),$$

$$v^{(3)}(t)=-\mu v^{(2)}(t),$$

if $\mu\neq-1$ $\quad v^{(2)}(t)=\frac{1}{1+\mu}(v^{(2)}(t_m^-)-v^{(3)}(t_m^-)),$

if $\mu=-1$ $\quad v^{(3)}(t)=v^{(2)}(t)$ $\quad\therefore$ infinitely many solutions for the node 2 and 3 voltages.

II.9. Hybrid Matrix Approach and Constraint Matrix:

Since any SC network with MNA equations (II.32), (II.49) or (II.59) corresponds to a linear resistive circuit in any subinterval or phase, the problem of existence and uniqueness of the solutions is exactly the same as for linear resistive circuits. For passive networks there are topologic conditions. In the case that the independent voltage and charge sources are the only active components, it can be shown that Eqs.(II.32), (II.49) and (II.59) are always solvable if in time slot k there exists no cut-set of charge sources and open switches and no loop of voltage sources and closed switches. The proof is available in [15] . Otherwise, the network is either inconsistent i.e., it has no unique solution.

This topological condition is trivially satisfied by any practical circuit since this condition is violated if the excitation is unacceptable.

It is difficult to give topological conditions for the solvability of SC circuits if all four types of controlled sources are allowed. But the hybrid matrix approach to switched capacitor circuit analysis will clearify this question.

In general, any SC circuit corresponds to a linear resistive circuit during any phase as explained section II.4. Then all the arguments on hybrid linear resistive n-port formulations (Chapter 6 of [6]) are valid for SC circuits.

A resistive equivalent n-port Fig.II.11. will consist of positive linear resistors ($R_j = \frac{1}{C_j}$), independent voltage and charge sources and all four types of controlled sources with constant real controlling coefficients.

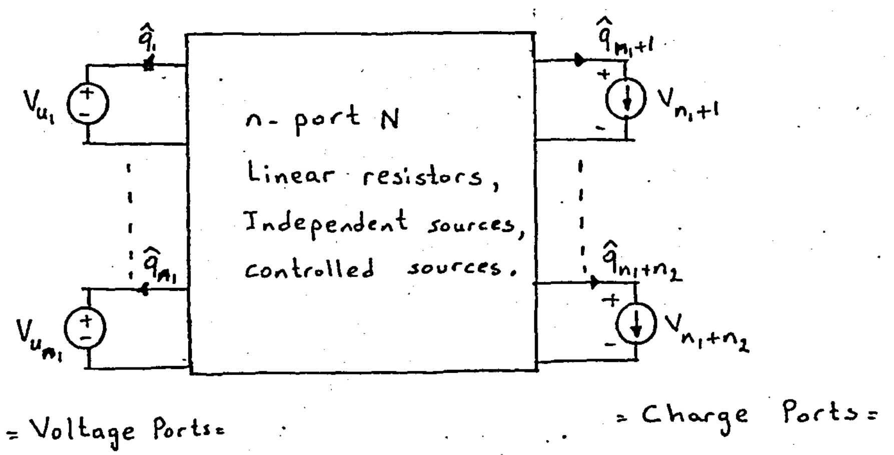


Fig.II.11.

Now, define the following port voltage and charge vectors: (Note that $n=n_1+n_2$)

Known vectors are   Unknown vectors are

$$\underset{\sim}{v}_u \triangleq \begin{bmatrix} v_{u1} \\ \vdots \\ v_{un1} \end{bmatrix} \quad \underset{\sim}{\hat{q}}_w \triangleq \begin{bmatrix} \hat{q}_{wn1+1} \\ \vdots \\ \hat{q}_{wn} \end{bmatrix} \qquad \underset{\sim}{\hat{q}}_u \triangleq \begin{bmatrix} \hat{q}_{u1} \\ \vdots \\ \hat{q}_{un1} \end{bmatrix} \quad \underset{\sim}{v}_w \triangleq \begin{bmatrix} v_{n_1+1} \\ \vdots \\ v_{wn} \end{bmatrix} \tag{II.87}$$

Let $\underset{\sim}{\hat{u}}$ be the vector representing the independent sources inside N. (Note that these sources are only the inital capacitor voltages before the switching instant t_k, hence dc sources).

Then the port voltages and charges may be related by a hybrid matrix $\underset{\approx}{H}$ and the source vector s as follows:

$$\begin{bmatrix} \hat{\underset{\sim}{q}}_u \\ \underset{\sim}{v}_w \end{bmatrix} = \underset{\approx}{H} \begin{bmatrix} \underset{\sim}{v}_u \\ \hat{\underset{\sim}{q}}_w \end{bmatrix} + \underset{\approx}{M}\, \hat{\underset{\sim}{u}} \qquad \text{(II.88)}$$

where $\underset{\sim}{s}=\underset{\approx}{M}\, \hat{\underset{\sim}{u}}$.

The matrices $\underset{\approx}{H}$, $\underset{\approx}{M}$ and $\underset{\sim}{s}$ can be partitioned according to the dimensions of $\hat{\underset{\sim}{q}}_u$ and $\underset{\sim}{v}_w$ as follows:

$$\begin{bmatrix} \hat{\underset{\sim}{q}}_u \\ \underset{\sim}{v}_w \end{bmatrix} = \left[\begin{array}{c|c} \underset{\approx}{H}_{uu} & \underset{\approx}{H}_{uw} \\ \hline \underset{\approx}{H}_{wu} & \underset{\approx}{H}_{ww} \end{array}\right] \begin{bmatrix} \underset{\sim}{v}_u \\ \hat{\underset{\sim}{q}}_w \end{bmatrix} + \begin{bmatrix} \underset{\approx}{M}_u \\ \underset{\approx}{M}_w \end{bmatrix} \hat{\underset{\sim}{u}} \qquad \text{(II.89)}$$

The elements of $\underset{\approx}{H}$ and $\underset{\approx}{M}$ are real constants and the source vector $\underset{\sim}{s}$ is constant vector. Each element of $\underset{\approx}{H}$ can be found by first setting all the dc sources inside N to zero, so that $\underset{\sim}{s}=\underset{\sim}{0}$ and then obtaining h_{jk} by the ratio

$$h_{jk} = \frac{\text{response at port } j}{\text{excitation at port } k} \qquad \text{(II.90)}$$

Under the following conditions:

1) Except for port k all voltage ports are short circuited and all charge ports are open circuited.
2) The excitation at port k is an independent
   a) Voltage source if port k is a voltage port.
   b) Charge " " " " " " charge " .
3) The response at port j is considered to be
   a) The charge of port j if port j is a voltage port.
   b) The voltage of port j if pot j is a charge port.

Such a procedure may be reasonable if n_1+n_2 is a small number. However, for $n>5$ the computational efforts become excessive. Explicit topological formulas can be found at page 240 of [6] .

Theorem II.4: The necessary and sufficient conditions for an n-ports N consisting of positive linear resistors and independent sources to posses a hybrid matrix $\underset{\approx}{H}$ as defined in Eq.(II.88) are that

1. the branches corresponding to voltage ports should form no loops and
2. the branches corresponding to charge ports should form no cut-sets (proof is available at page 239 of [6]).

When an n-port contains controlled sources, the hybrid matrix may not exist for a given port combination. In this case, the methods for formulating hybrid matrix will abort at some step where an attempt is made to invert a singular matrix. In such cases, the hybrid matrix can be made non-singular by perturbing some element parameters within the tolerance of t the element or use a different combination of voltage ports and current ports. But then the solution is only approximate and to try other port combination is clearly very inefficient, because for each port combination, a complete analysis of the network has to be done.

The above discussion was made for a certain phase (i.e. switches were either off or on). To overcome the above difficulties the following method is introduced [6] . But first the follwing definitions are needed [8] .

Definition: An uncommitted independent source is an independent source whose nature (voltage or charge source) is not specified. Similarly, an uncommitted port is a port whose nature (voltage or charge port) is not specified.

Definition: An open switch can be considered as a charge source whose value is zero (i.e. $\hat{q}(t)=0,\ t\epsilon\Delta_m$). A closed switch can be considered as a voltage source with "zero" value (i.e. $v(t)=0,\ t\epsilon\Delta_m$).

Method: Let all the p ports (switches are considered as uncommitted ports) be uncommitted ports. Instead of seeking the hybrid matrix $\underset{\approx}{H}$, which may not exist for a particular port combination, it is tried to find a maximum number m of independent equations relating the port variables in the form

$$\underset{\approx}{C}_t \begin{bmatrix} \underset{\sim}{v}_p \\ \hat{\underset{\sim}{q}}_p \end{bmatrix} = \underset{\sim}{0} \qquad \text{(II.91)}$$

Where $\underset{\approx}{C}_t$ is of dimension mx2p and is called a constraint matrix for the p-port. Normally, the number of constraint equations for a p-port as given by Eq.(91) is the same as the number of ports. However, both $m > p$ and $m < p$ are also possible. Well-known examples are nullators for the former and norators for the latter.

To facilitate the formulation of $\underset{\approx}{C}_t$. Choose an arbitrary tree T. Assume that the graph is connected and has n nodes. A tree is constructed by picking n-1 branches, paying attention to the rule that they form no loop. With respect to this tree T , the network branches can be divided into four categories distinguished by the following subscript notations:

a= port branches in the tree.

b= port branches in the co-tree.

z= nonport branches in the tree.

y= nonport branches in the co-tree.

The fundamental cut-set equations (KQL) can be written as:

$$\overset{\qquad a \qquad\quad z \qquad\quad y \qquad\quad b}{\begin{bmatrix} \underset{\approx}{I}_{aa} & \underset{\approx}{0}_{az} & \underset{\approx}{D}_{ay} & \underset{\approx}{D}_{ab} \\ \underset{\approx}{0}_{za} & \underset{\approx}{I}_{zz} & \underset{\approx}{D}_{zy} & \underset{\approx}{D}_{zb} \end{bmatrix}}_{\text{tree} \qquad\qquad \text{Cotree}} \begin{bmatrix} \hat{\underset{\sim}{q}}_a \\ \hat{\underset{\sim}{q}}_z \\ \hat{\underset{\sim}{q}}_y \\ \hat{\underset{\sim}{q}}_b \end{bmatrix} = \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \end{bmatrix} \qquad \text{(II.92)}$$

where $\underset{\approx}{D}$ is the fundamental cut-set matrix.

The fundamental-loop equations are

$$\begin{bmatrix} -\underset{\approx}{D}^T_{ay} & -\underset{\approx}{D}^T_{zy} & \underset{\approx}{I}_{yy} & \underset{\approx}{0} \\ -\underset{\approx}{D}^T_{ab} & -\underset{\approx}{D}^T_{zb} & \underset{\approx}{0} & \underset{\approx}{I}_{bb} \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_a \\ \underset{\sim}{v}_z \\ \underset{\sim}{v}_y \\ \underset{\sim}{v}_b \end{bmatrix} = \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \end{bmatrix} \qquad \text{(II.93)}$$

The nonport branches may be characterized by

$$\underset{\approx}{F}_{qz}\hat{\underset{\sim}{q}}_z+\underset{\approx}{F}_{vy}\underset{\sim}{v}_y+\underset{\approx}{F}_{qy}\hat{\underset{\sim}{q}}_y+\underset{\approx}{F}_{vz}\underset{\sim}{v}_z+\underset{\approx}{F}_{qa}\hat{\underset{\sim}{q}}_a+\underset{\approx}{F}_{vb}\underset{\sim}{v}_b+\underset{\approx}{F}_{qb}\hat{\underset{\sim}{q}}_b+\underset{\approx}{F}_{va}\underset{\sim}{v}_a=\underset{\sim}{0} \qquad \text{(II.94)}$$

Now, write Eqs.(II.70), (II.71) and (II.72) as a single matrix equation in the tableau form:

$$\begin{bmatrix} \underset{\approx}{I} & \underset{\approx}{0} & \underset{\approx}{D}_{zy} & \underset{\approx}{0} & \underset{\approx}{0} & \underset{\approx}{0} & \underset{\approx}{0} \\ \underset{\approx}{0} & \underset{\approx}{I} & \underset{\approx}{0} & -\underset{\approx}{D}_{zy}^T & \underset{\approx}{0} & \underset{\approx}{0} & -\underset{\approx}{D}_{ay}^T \\ \underset{\approx}{0} & \underset{\approx}{0} & \underset{\approx}{D}_{ay} & \underset{\approx}{0} & \underset{\approx}{I} & \underset{\approx}{0} & \underset{\approx}{0} \\ \underset{\approx}{0} & \underset{\approx}{0} & \underset{\approx}{0} & -\underset{\approx}{D}_{zb}^T & \underset{\approx}{0} & \underset{\approx}{I} & -\underset{\approx}{D}_{ab}^T \\ \underset{\approx}{F}_{qz} & \underset{\approx}{F}_{vy} & \underset{\approx}{F}_{qy} & \underset{\approx}{F}_{vz} & \underset{\approx}{F}_{qa} & \underset{\approx}{F}_{vb} & \underset{\approx}{F}_{va} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{q}}_z \\ \underset{\sim}{v}_y \\ \hat{\underset{\sim}{q}}_y \\ \underset{\sim}{v}_z \\ \hat{\underset{\sim}{q}}_a \\ \underset{\sim}{v}_b \\ \hat{\underset{\sim}{q}}_b \\ \underset{\sim}{v}_a \end{bmatrix} = \underset{\sim}{0} \qquad \text{(II.95)}$$

The vectors $\hat{\underset{\sim}{q}}_z$ and $\underset{\sim}{v}_y$ can be eliminated from Eq.(II.95) by solving from the first two equations and substituting into the last equation to yield the following:

$$\left[\underset{\approx}{F}_b \;\vdots\; \underset{\approx}{F}_p\right] \begin{bmatrix} \hat{\underset{\sim}{q}}_y \\ \underset{\sim}{v}_z \\ \hline \hat{\underset{\sim}{q}}_a \\ \underset{\sim}{v}_b \\ \hat{\underset{\sim}{q}}_b \\ \underset{\sim}{v}_a \end{bmatrix} = \underset{\sim}{0} \qquad \text{(II.96)}$$

$\hat{\underset{\sim}{q}}_y$ and $\underset{\sim}{v}_z$ may be eliminated from Eq.(II.96) by row reduction as follows. Apply elementary row operations to reduce $\left[F_b \;\vdots\; F_p\right]$ to row echelon form. In the resulting matrix, those rows in the right block (originally $\underset{\approx}{F}_p$) whose corresponding rows in the left block (originally $\underset{\approx}{F}_b$) are zero rows form the constraint matrix $\underset{\approx}{C}_t$.

Schematically, it looks as follows:

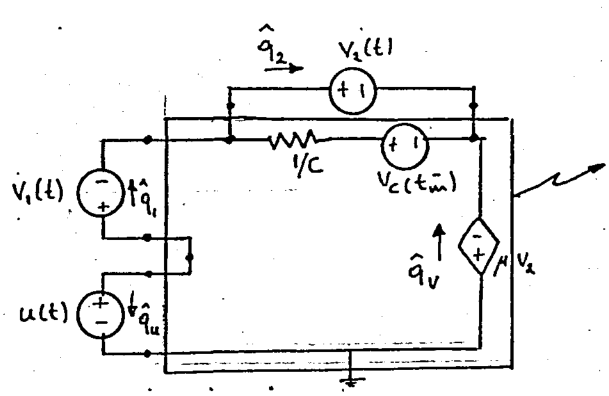
$$\text{elementary row-operations} \quad \begin{bmatrix} \underset{\approx}{F}_b & \underset{\approx}{F}_p \end{bmatrix} \quad \left[\begin{array}{c|c} \text{Row echelon form} & x \cdots\cdots x \\ 0 \;\; 0 \cdots 1 \;\; x \cdots & x \cdots\cdots x \\ \hline 0 & \text{Constraint matrix} \end{array}\right] \qquad (II.97)$$

Linearly independent columns of the constraint matrix determines the port nature (voltage port or closed switch, charge port or opened switch).

The digital computer program (e.g., page 270 of [6]) will find the constraint matrix and immediately solve the hybrid matrix from the constraint equation.

<u>Example II.7:</u> Let's return to the example given on page 46. By this example, the above algorithm will be clearified.

The switches will be considered as uncommitted ports; then Fig.(II.10) can be redrawn as follows:

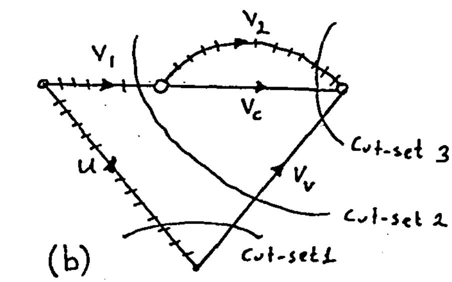


3-port network containing linear resistor $R = \frac{1}{C}$, voltage controlled voltage source and dc-voltage source.

Fig. II.11(a)

For simplicity, let's consider initial capacitor voltage to be zero and hence dc-voltage source has zero value.

The following tree is chosen with the ideas given on page 53.



(b)
Corresponding graph.

where $\hat{\underset{\sim}{q}}_a = \begin{bmatrix} \hat{q}_u \\ \hat{q}_1 \\ \hat{q}_2 \end{bmatrix} \qquad \hat{\underset{\sim}{q}}_b = \underset{\sim}{0} \qquad \hat{\underset{\sim}{q}}_z = \underset{\sim}{0} \qquad \hat{\underset{\sim}{q}}_y = \begin{bmatrix} \hat{q}_c \\ \hat{q}_v \end{bmatrix}$ (II.98)

The fundamental cut-set equations (KQL) can be written as:

$$\begin{matrix} C_1 \\ C_2 \\ C_3 \end{matrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_{aa}} \!\!\begin{matrix} | \\ | \\ | \end{matrix}\!\! \underbrace{\begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_{D_{ay}} \begin{bmatrix} q_u \\ \hat{q}_1 \\ \hat{q}_2 \\ \hat{q}_c \\ \hat{q}_v \end{bmatrix} = \underset{\sim}{0} \qquad \text{(II.99)}$$

The non-port branches may be characterized by:

$$v_c = \frac{1}{C}\,\hat{q}_c \quad \text{and} \quad v_v = (u - v_1) \qquad \text{(II.100)}$$

The fundamental-loop equations are

$$\underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix}}_{-D_{ay}^T} \!\!\begin{matrix} | \\ | \end{matrix}\!\! \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_{yy}} \begin{bmatrix} v_u \\ v_1 \\ v_2 \\ v_c \\ v_v \end{bmatrix} = \underset{\sim}{0} \qquad \text{(II.101)}$$

The above equations may be collected in a single matrix as the tableau form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1/C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\mu & \mu & 0 \end{bmatrix} \begin{bmatrix} v_c \\ v_v \\ \hat{q}_c \\ \hat{q}_v \\ \hat{q}_u \\ \hat{q}_1 \\ \hat{q}_2 \\ v_u \\ v_1 \\ v_2 \end{bmatrix} = \underset{\sim}{0} \qquad \text{(II.102)}$$

After the elimination of the vector $\underset{\sim}{v}_y = \begin{bmatrix} v_c \\ v_v \end{bmatrix}$, Eq.(II.102) is reduced to the following equation:

$$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1/C & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & (1+\mu) & -(1+\mu) & -1 \end{bmatrix} \begin{bmatrix} \hat{q}_c \\ \hat{q}_v \\ \hat{q}_u \\ \hat{q}_1 \\ \hat{q}_2 \\ v_u \\ v_1 \\ v_2 \end{bmatrix} = \underset{\sim}{0} \qquad \text{(II.103)}$$

Applying elemantary row operations Eq.(II.103) is reduced to row echelon form (see the algorithm for reducing a rectangular matrix to an echelon form on page 157 of [6]).

Corresponding row echelon form of Eq.(II.103) is:

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -C \\ 0 & 0 & 0 & 0 & 0 & (\mu+1) & -(\mu+1) & -1 \end{bmatrix} \begin{bmatrix} \hat{q}_c \\ \hat{q}_v \\ \hat{q}_u \\ \hat{q}_1 \\ \hat{q}_2 \\ v_u \\ v_1 \\ v_2 \end{bmatrix} = \underset{\sim}{0} \qquad \text{(II.104)}$$

The constraint matrix $\underset{\approx}{C}_t = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -C \\ 0 & 0 & 0 & (\mu+1) & -(\mu+1) & -1 \end{bmatrix}$ (II.105)

and the constraint equation for the three-port is;

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -C \\ 0 & 0 & 0 & (\mu+1) & -(\mu+1) & -1 \end{bmatrix} \begin{bmatrix} \hat{q}_u \\ \hat{q}_1 \\ \hat{q}_2 \\ v_u \\ v_1 \\ v_2 \end{bmatrix} = \underset{\sim}{0} \qquad \text{(II.106)}$$

From Eq.(II.106) it is immediately clear that port 1 can be considered as voltage port while port 2 as charge source since the columns corresponding to ($\hat{q}_u$, $\hat{q}_1$, v_2) are linearly independent. Therefore, $\hat{q}_u$, $\hat{q}_1$, v_2 can be found in terms of v_u, v_1 and $\hat{q}_2$ as:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -C \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{q}_u \\ \hat{q}_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ (\mu+1) & -(\mu+1) & 0 \end{bmatrix} \begin{bmatrix} v_u \\ v_1 \\ \hat{q}_2 \end{bmatrix}$$

and

$$\begin{bmatrix} \hat{q}_u \\ \hat{q}_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2C(\mu+1) & -2C(\mu+1) & 1 \\ -C(\mu+1) & C(\mu+1) & -1 \\ -(\mu+1) & (\mu+1) & 0 \end{bmatrix} \begin{bmatrix} v_u \\ v_1 \\ \hat{q}_2 \end{bmatrix} \qquad \text{(II.107)}$$

From Eq.(II.107), it may be concluded that port 1 (switch 1) can be considered as voltage source (resp. closed) and port 2 (switch 2) as charge source (resp. open). In this case hybrid matrix exists and the network is solvable which is consistent with result on page 47.

From Eq.(II.106), it can be said that port 1 (resp. switch 1) can be considered as charge source (resp. open) while port 2 (resp. switch 2) as voltage source (resp. closed) since columns corresponding to ($\hat{q}_u$, $\hat{q}_2$, v_1) are linearly independent if $\mu \neq -1$ and hence hybrid matrix exists.

The last possible port combination is that switch 1 and switch 2 are as charge sources since the columns corresponding to $\hat{q}_u$, v_1 and v_2 are linearly independent if $\mu \neq -1$ and hence the hybrid matrix exists as in the following form:

$$\begin{bmatrix} \hat{q}_u \\ v_1 \\ v_2 \end{bmatrix} = \underset{\approx}{H} \begin{bmatrix} v_u \\ \hat{q}_1 \\ \hat{q}_2 \end{bmatrix}$$

where $\hat{q}_1 = \hat{q}_2 = 0$ and $v_u = u(t)$.

<u>Conclusion:</u>

With the present method, using uncommitted ports, the constraint matrix can always be obtained. The task of determining a proper port combination (nature of the switches i.e., closed or open and nature of sources) and the corresponding hybrid matrix from the constraint equations is simple. As it is seen from the previous example, linearly independent columns will determine the port nature. Consequently, it can easily be determined in which phases an arbitrary SC network possesses a hybrid matrix and all of its variables are solvable.

CHAPTER III

SIGNAL PROCESSING MECHANISM IN TIME DOMAIN

In the time domain the signal processing effects of a switched capacitor network can be immediately understood when considering some special input waveforms. The following corollaries can be given to the theorem on page 14, in Chapter II.

<u>Corollary 1:</u> If a switched capacitor network is excited by an input $\underset{\sim}{r}(t)$ which is constant $\underset{\sim}{r}_m$ in each time slot Δ_m then the response $\underset{\sim}{x}(t)$ is also a constant $\underset{\sim}{x}_m$ in each time slot Δ_m. (Fig.III.1.a,b).

<u>Proof:</u> By setting $\underset{\sim}{r}(t)=\underset{\sim}{r}_m$ in Eq.(II.14) for all t in Δ_m, the right-handside of Eq.(II.14) is constant in Δ_m. Thus also the solution $\underset{\sim}{x}(t)$ is a constant $\underset{\sim}{x}_m$ in Δ_m or

$$\underset{\approx}{F}_m \underset{\sim}{x}_m = \underset{\approx}{G}_m \underset{\sim}{x}_{m-1} + \underset{\sim}{r}_m \tag{III.1}$$

<u>Corollary 2:</u> If a SC network has zero initial voltage at t_i and is excited by an input $\underset{\sim}{r}(t)$ which is zero at the end t_m^- of each time slot Δ_m then the response $\underset{\sim}{x}(t)$ is also zero at the end t_m^- of each time slot Δ_m (Fig.III.1.a,c).

<u>Proof:</u> Starting from $\underset{\sim}{x}(t_1^-)=0$ it follows inductively from the property of the input and Eq.(II.14) that $\underset{\sim}{0}=\underset{\sim}{x}(t_2^-)=\underset{\sim}{x}(t_3^-)=$ .... Plugging this in Eq.(II.14

$$\underset{\approx}{F}_m \underset{\sim}{x}(t) = \underset{\sim}{r}(t) \tag{III.2}$$

is obtained.

> The response of a SC network on a piecewise constant input is piecewise constant and on an input which is zero at all t_m^- is also zero at all t_m^-.

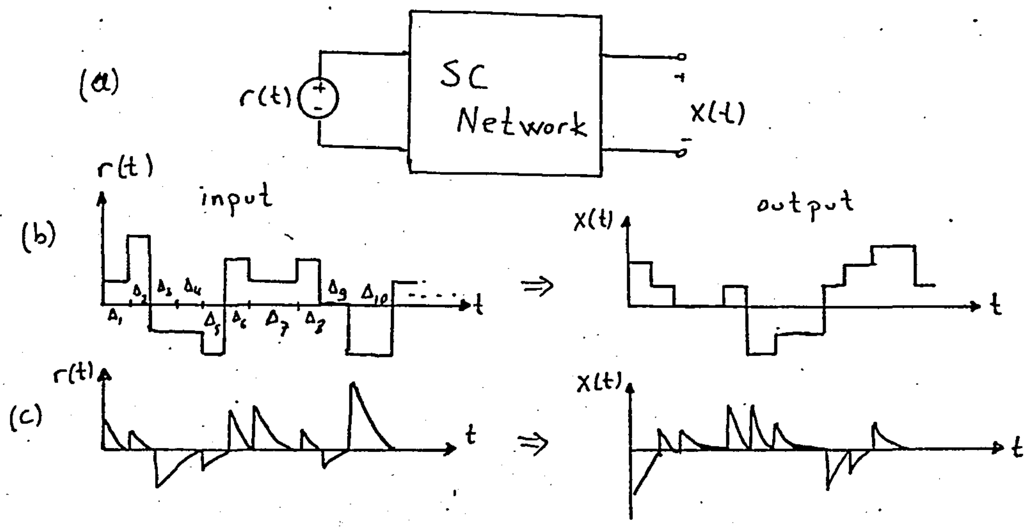


Fig. III.1.

Remarks: The underlying reason for the absence of transients in Corollary 1 is the fact that there are no poles in the op-amps and no resistors in the circuit.

Corollary 2 implies that inputs which are zero at the end of each time slot are only subject to a linear static operation which can be different in each phase (no filtering only some amplification or scaling).

These two corollaries motivate immediately a computation of the response $\underset{\sim}{x}(t)$ to an input $\underset{\sim}{r}(t)$ with zero initial condition via a decomposition of the waveforms (Fig. III.2).

Computation Procedure:

Step 1: Decompose the input $\underset{\sim}{r}(t)$ into a waveform $\underset{\sim}{r}'(t)$ which is constant in each time slot and a waveform $\underset{\sim}{r}^*(t)$ which is zero at the end of each time slot; i.e., for t in Δ_m.

$$\underset{\sim}{r}'(t) \triangleq \underset{\sim}{r}(t_{m+1}^{-}) = \underset{\sim}{r}_m$$
$$\underset{\sim}{r}^*(t) \triangleq \underset{\sim}{r}(t) - \underset{\sim}{r}'(t) \qquad t \in \Delta_m = (t_m, t_{m+1}] \quad \text{(III.3)}$$

Step 2: Compute the response of Eq.(II.14) to each input i.e., solve $\underset{\sim}{x}^*(t)$ and $\underset{\sim}{x}_m$ from

$$\underset{\approx}{F}_m \underset{\sim}{x}^*(t) = \underset{\sim}{r}^*(t) \qquad \text{(III.4.a)}$$

and

$$\underset{\approx}{F}_m \underset{\sim}{x}_m = \underset{\approx}{G}_m \underset{\sim}{x}_{m-1} + \underset{\sim}{r}_m \qquad \text{(III.4.b)}$$

Step 3: Combine the results for t in Δ_m

$$\underset{\sim}{x}(t) = \underset{\sim}{x}'(t) + \underset{\sim}{x}^*(t) = \underset{\sim}{x}_m + \underset{\sim}{x}^*(t) \qquad \text{(III.5)}$$

The linear static equation (III.4.a) which relates $\underset{\sim}{x}^*(t)$ and $\underset{\sim}{r}^*(t)$ takes into occount the effect of the continuous coupling between the input and the output. Eq.(III.4.b) is an N-periodic time-varying linear difference equation which needs only one computation for each time slot. The computation of $\underset{\sim}{x}_m$ for m=1,2,.... by Gauss elimination from Eq.(III.4.b) is the basis for the time domain analysis in the program DIANA [13,14].

An important insensitivity property which follows immediately from the theorem on page 14 is that the voltage transfer in a SC network is not modified by a scaling of all capacitors with the same factor α. This is very important since the IC technology can rarely control the absolute values of capacitors better than 20%, while ratios of capacitors can be controlled to within 1%.

Corollary 3: By multiplying all capacitances and all controlling factors of VCQS by α and by multiplying all controlling factors of QCVS by $1/\alpha$, all charges of a SC network are multiplied by α and all voltages remain unchanged.

Natural decomposition of the signals into piecewise constant part and continous I-O coupling

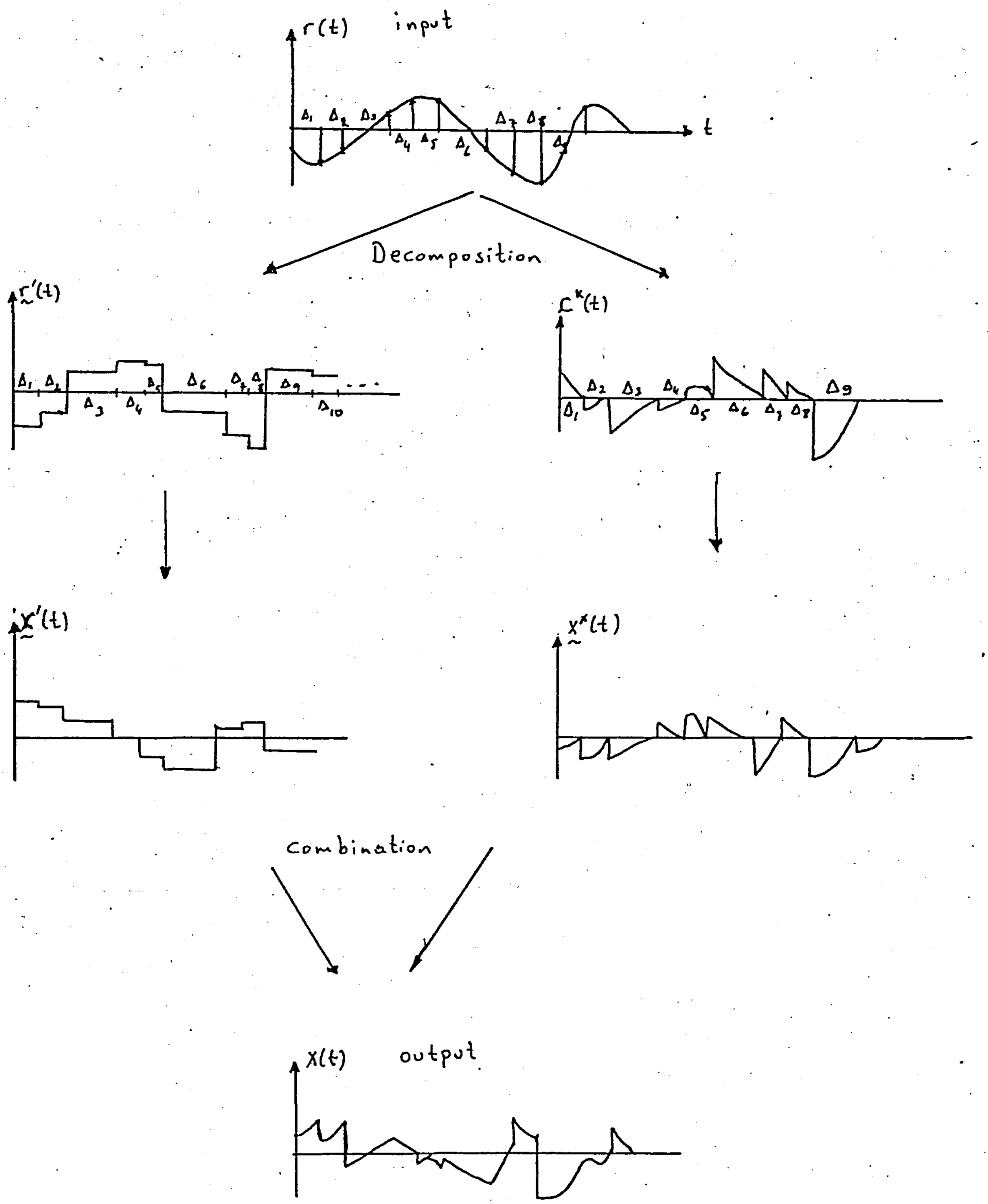


Fig.III.2.

Computation Of The Discrete Time Input-Output Effects (Fig. III.3)

Given $\underset{\sim}{r}_m$ the computation of $\underset{\sim}{x}_m$ can be done as for time-invariant systems (x). For a time-invariant discrete-time system one computes first the response to an impulse (xx) and then the response to an arbitrary input can be computed by taking its convolution with the impulse response. As for discrete-time signals, the values at the end of each time slot for the input ($\underset{\sim}{r}_m$) and for the output ($\underset{\sim}{x}_m$) are considered. The derivation makes use of the periodicity and of equations (III.4). In order to make this periodicity more explicit the following equation is written for time instances t^-_{k+lN+1} in Δ_{k+lN} which corresponds to phase k.

$$\underset{\approx}{F}_k \underset{\sim}{x}_{k+lN} = \underset{\approx}{G}_k \underset{\sim}{x}_{k+lN-1} + \underset{\sim}{r}_{k+lN} \qquad \text{(III.6)}$$

Step 1: For k=1,2,...N compute the sequence of N impulse responses $\underset{\sim}{h}_{k,m}$. In other words for zero initial condition $\underset{\sim}{x}_o=\underset{\sim}{0}$ compute the discrete time response $\underset{\sim}{x}_m=\underset{\sim}{h}_{m,k}$ on an impulse in phase k, $\underset{\sim}{r}_k=\underset{\sim}{1}$, $\underset{\sim}{r}_m=\underset{\sim}{0}$ $m\neq k$ (different impulses on all inputs if there are more inputs).

Step 2: Decompose the discrete time values of any arbitrary input $\underset{\sim}{r}_m$ into N signals $\underset{\sim}{r}_{1+lN}$, $\underset{\sim}{r}_{2+lN}$, ...$\underset{\sim}{r}_{(l+1)N}$, one for each phase.

Step 3: The response $\underset{\sim}{x}_{i+lN}$ in phase i is then the combination of the effects of the inputs in each of the N phases

$$\underset{\sim}{x}_{l+kN} = \sum_{i=1}^{N} \sum_{n=0}^{l} \underset{\sim}{h}_{l+nN;i} \underset{\sim}{r}_{i+(k-n)N} \qquad \text{(III.7)}$$

Observe that each of the square brackets is a discrete time convolution between the input values $\underset{\sim}{r}_{i+kN}$ in phase i and impulse response $\underset{\sim}{h}_{l+kN,i}$ in phase l for an impulse in phase i.

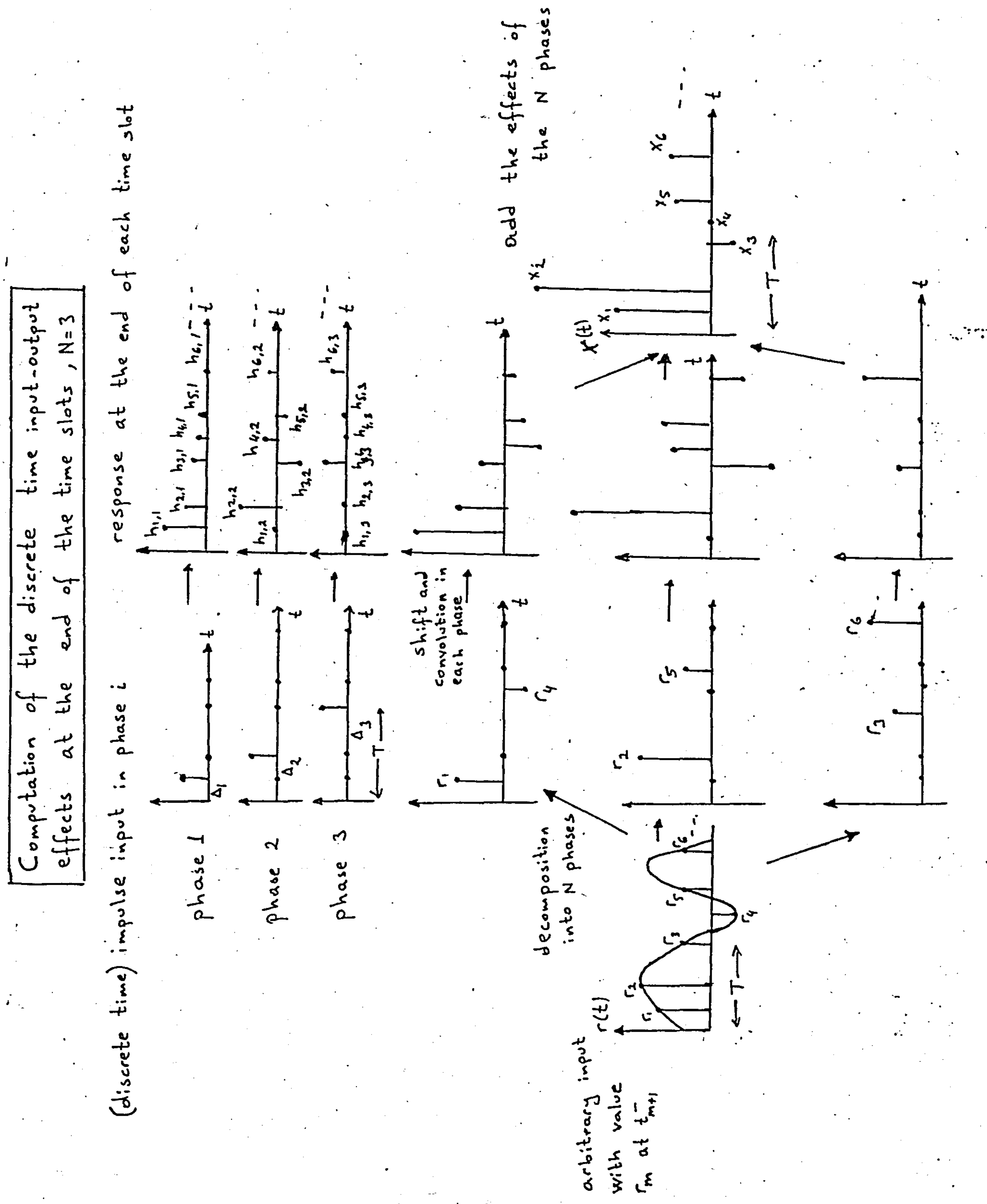
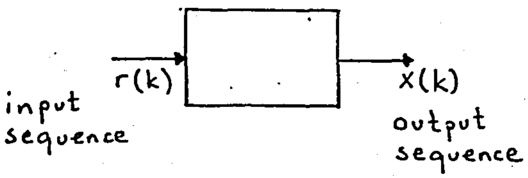

Computation of the discrete time input-output effects at the end of the time slots, N=3
(discrete time) impulse input in phase i
response at the end of each time slot
phase 1
phase 2
phase 3
shift and convolution in each phase
decomposition into N phases
add the effects of the N phases
arbitrary input with value r_m at t_{m+1}^-
r(t)
x(t)

Fig. III.3.

Additional Notes:

(x) In the following figure a discrete system governed by linear, time-invariant difference equation is shown.



$k = \ldots -2T, -T, 0, T, 2T, \ldots$

Fig.III.4.

Where h(k) is the impulse response (weighting sequence) of the discrete system [10] and

$$x(k)=\sum_{m=-\infty}^{+\infty} h(k-m)r(m)$$

which is described in the z-domain by

$$X(z)=H(z)R(z).$$

(xx) In discrete systems, sequences of numbers are considered rather than functions of continous-time; therefore the Kronecker Delta Sequence is used as the impulse function which is

$$\delta_j(k)=\begin{cases}1 & k=j\\ 0 & k\neq j.\end{cases}$$

CHAPTER IV

ANALYSIS AND SIGNAL PROCESSING MECHANISM OF LINEAR MULTIPHASE SC NETWORKS IN THE Z-DOMAIN

In this chapter, the analysis and signal processing mechanism of linear multi-phase SC networks in the z-domain will be give

Since Eq.(II.14) is time varying, the z-transform techniques are not readily applicable. Fortunately Eq.(II.14) is periodic and a method of Jury can be adapted [11, page 57].

Partition the sequence of values at the end of each time slot $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots \underset{\sim}{x}_N, \underset{\sim}{x}_{N+1}, \dots$ (resp. $\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dots \underset{\sim}{r}_N, \underset{\sim}{r}_{N+1} \dots$) into N different sequences each having the same phase:

$$\underset{\sim}{x}_1, \underset{\sim}{x}_{N+1}, \underset{\sim}{x}_{2N+1}, \dots\dots \quad \text{phase 1}$$

$$\underset{\sim}{x}_2, \underset{\sim}{x}_{N+2}, \underset{\sim}{x}_{2N+2}, \dots\dots \quad \text{phase 2}$$

$$\underset{\sim}{x}_N, \underset{\sim}{x}_{2N}, \underset{\sim}{x}_{3N}, \dots\dots \quad \text{phase N}$$

Then

$$\underset{\sim}{X}_k(z) \triangleq Z\left\{\underset{\sim}{x}_{k+lN}\right\} \triangleq \sum_{l=0}^{\infty} \underset{\sim}{x}_{k+lN}\, z^{-l} \qquad (IV.1)$$

where k=1,2,..N (Fig.IV.1) and $z=e^{j\omega T}$. (T is the sampling period).

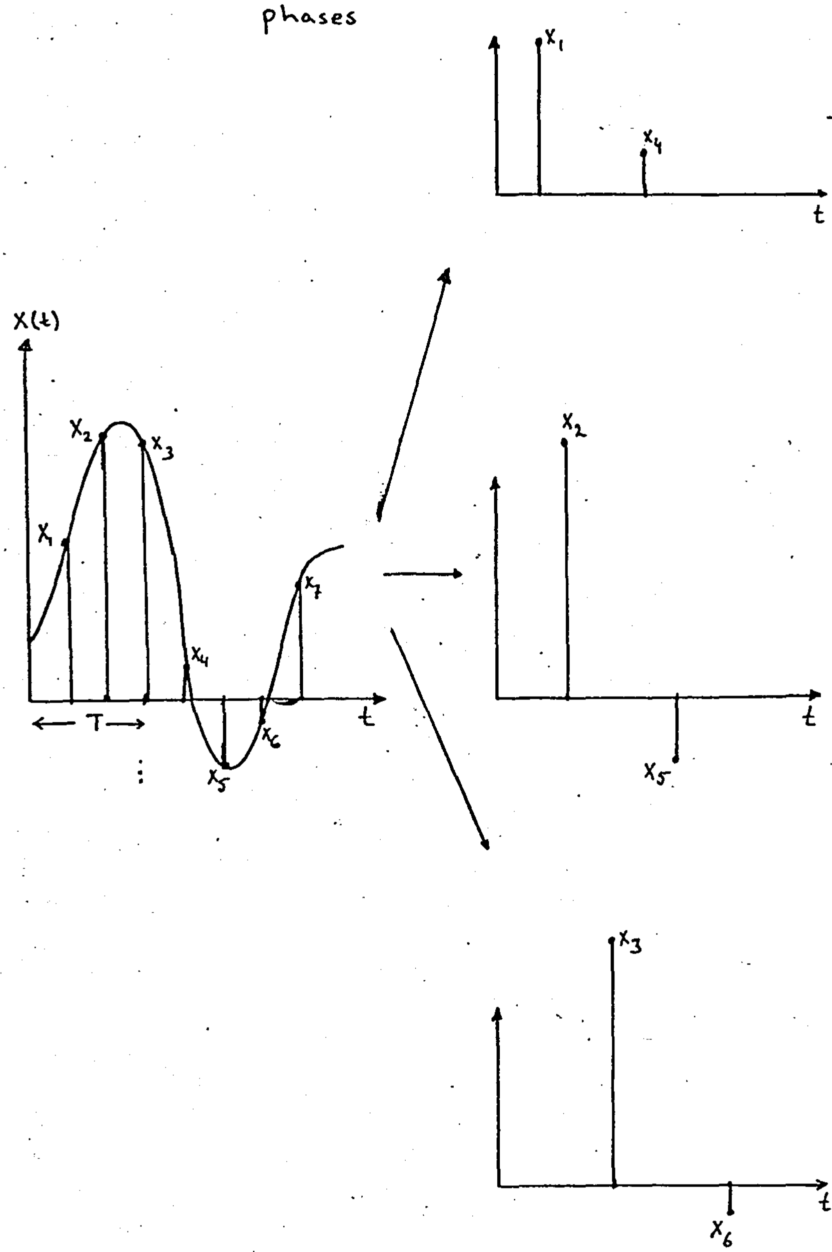
$$\underset{\sim}{R}_k(z) \triangleq Z\left\{\underset{\sim}{r}_{k+lN}\right\} \triangleq \sum_{l=0}^{\infty} \underset{\sim}{r}_{k+lN}\, z^{-l} \qquad (IV.2)$$

<u>Theorem IV.1:</u> The linear T -periodic SC network described by Eq.(II.14) is described in the z-domain by

$$\begin{bmatrix} \underset{\approx}{F}_1 & \underset{\approx}{0} & \underset{\approx}{0} & \cdots & -\underset{\approx}{G}_1 z^{-1} \\ -\underset{\approx}{G}_2 & \underset{\approx}{F}_2 & \underset{\approx}{0} & & \vdots \\ \underset{\approx}{0} & -\underset{\approx}{G}_3 & \underset{\approx}{F}_3 & & \\ \vdots & \vdots & & \underset{\approx}{F}_{N-1} & \\ \underset{\approx}{0} & \underset{\approx}{0} & & -\underset{\approx}{G}_N & \underset{\approx}{F}_N \end{bmatrix} \begin{bmatrix} \underset{\sim}{X}_1(z) \\ \underset{\sim}{X}_2(z) \\ \underset{\sim}{X}_3(z) \\ \vdots \\ \underset{\sim}{X}_N(z) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{R}_1(z) \\ \underset{\sim}{R}_2(z) \\ \underset{\sim}{R}_3(z) \\ \vdots \\ \underset{\sim}{R}_N(z) \end{bmatrix} \qquad (IV.3)$$

How to z-transform the signals of a switched capacitor circuit ? (N=3)

decompose into N=3 phases



$$X_1(z) = \sum_{l=0}^{\infty} x_{1+l3} z^{-l}$$

$$X_2(z) = \sum_{l=0}^{\infty} x_{2+l3} z^{-l}$$

$$X_3(z) = \sum_{l=0}^{\infty} x_{3+l3} z^{-l}$$

Fig.IV.1.

Proof: Multiply the time equations (III.6) by z^{-1} and take the sum for 1 with $\underset{\sim}{x}_o=\underset{\sim}{0}$

$$\underbrace{\underset{\approx}{F}_k \sum_{l=0} \underset{\sim}{x}_{k+lN}\, z^{-l}}_{\substack{\text{z-transform of}\\ \text{sequence of}\\ \text{phase } k:\underset{\sim}{X}_k(z)}} = \underset{\approx}{G}_k \underbrace{\sum_{l=0} \underset{\sim}{x}_{k-1+lN}\, z^{-l}}_{\substack{\text{if } k>1 \Rightarrow \underset{\sim}{X}_{k-1}(z)\\ \text{if } k=1 \Rightarrow z^{-1}\underset{\sim}{X}_N(z)^{(x)}}} + \underbrace{\sum_{l=0} \underset{\sim}{r}_{k+lN}\, z^{-l}}_{\substack{\text{z-transform of sequence}\\ \text{of phase } k:\underset{\sim}{R}_k(z)}} \qquad \text{(IV.4)}$$

By plugging the Eqs.(IV.4) for k=1,2,..N into one matrix equation Eq.(IV.3) is obtained.

It is seen from Eq.(IV.3) that the z-transforms of the N phases of input and output are related by linear equations with many zero submatrices and that z^{-1} only enters in the upper right submatrix.

The signal processing mechanism exhibited by the matrix in Eq.(IV.3) is a combined effect of linear combinations in each phase and a transportation from one phase to the next and so on until the last phase influences the first (circulation effect).

The N phases of the output can now be easily obtained by inverting the matrix in Eq.(IV.3). This inverse matrix is called the z-domain transfer matrix.

Corollary IV.1: The N z-transforms $\underset{\sim}{X}_k(z)$ of the outputs are given in terms of the z-transforms of the inputs

$$\begin{bmatrix} \underset{\sim}{X}_1(z) \\ \underset{\sim}{X}_2(z) \\ \vdots \\ \underset{\sim}{X}_N(z) \end{bmatrix} = \begin{bmatrix} \underset{\approx}{H}_{1,1}(z) & \underset{\approx}{H}_{1,2}(z) & \cdots & \underset{\approx}{H}_{1,N}(z) \\ \underset{\approx}{H}_{2,1}(z) & \underset{\approx}{H}_{2,2}(z) & \cdots & \underset{\approx}{H}_{2,N}(z) \\ \vdots & \vdots & & \vdots \\ \underset{\approx}{H}_{N,1}(z) & \underset{\approx}{H}_{N,2}(z) & & \underset{\approx}{H}_{N,N}(z) \end{bmatrix} \begin{bmatrix} \underset{\sim}{R}_1(z) \\ \underset{\sim}{R}_2(z) \\ \\ \underset{\sim}{R}_N(z) \end{bmatrix} \qquad \text{(IV.5)}$$

where the matrix in Eq.(IV.5) is the inverse of that in Eq.(IV.3), and is a rational matrix in z. Moreover $\underset{\approx}{H}_{i,k}(z)$ is the z-trans-

---

(x): $Z\left\{\underset{\sim}{x}_{k+(l-1)N}\right\} = z^{-1}\underset{\sim}{X}_k(z)$ if $\underset{\sim}{x}_i=0$, $i \leq 0$

form of the impulse response matrices $\underset{\approx}{h}_{i+lN,k}$.

Proof: Eq.(IV.5) follows from Eq.(IV.3). In fact, the z-transform coverts the convolution in Eq.(III.7) into a product (IV.5).

$$\underset{\sim}{x}_{k+lN} = \sum_{i=1}^{N} \left\{ \underset{\approx}{h}_{k+lN,i} * \underset{\sim}{r}_{i+lN} \right\}$$

$$Z\left\{\underset{\sim}{x}_{k+lN}\right\} = \sum_{i=1}^{N} Z\left\{ \underset{\approx}{h}_{k+lN,i} * \underset{\sim}{r}_{i+lN} \right\}$$

$$\underset{\sim}{X}_k(z) = \sum_{i=1}^{N} \underset{\approx}{H}_{k,i}(z) \cdot \underset{\sim}{R}_i(z) \qquad \text{(IV.6)}$$

The submatrices $\underset{\approx}{H}_{i,k}$ of the z-domain transfer matrix, allow a very simple interpretation. Up to this point, the SC network was considered as a discrete device which transforms the input sequences of samples at t_m^-, m=1,2,... of the voltage sources and charge sources into the output sequences of samples of voltages or charges at t_m^-, m=1,2,.....If only nonzero sources are applied during time slots k, k+N, k+2N,... and if the outputs are only observed during time slots i, i+N, i+2N,.... then $\underset{\approx}{H}_{i,k}(z)$ relates the z-transform of this input sequence to this output sequence, i.e., $\underset{\sim}{X}_i(z) = \underset{\approx}{H}_{i,k}(z)\underset{\sim}{R}_k(z)$ and it relates inputs at phase k to outputs at phase i.

In other words $\underset{\approx}{H}_{i,k}(z)$ is the z-transform of $h_{i+lN,k}$ l=0,1... i.e, the responses observed during time slots i, i+N, i+2N, ... to unit input applied during time slot k.

Corollary IV.2: The z-domain transfer matrix completely characterizes the behavior of a SC circuit. The response to the piecewise constant part of the input is given by Eq.(IV.5) in the z-domain. The continuous input-output coupling is given for t in Δ_{k+lN} by

$$\underset{\sim}{x}^*(t) = \underset{\approx}{H}_{k,k}(\infty) r^*(t) \qquad \text{(IV.7)}$$

Proof: By setting $z=\infty$ in (IV.3) $\underset{\approx}{F}_k=\underset{\approx}{H}^{-1}_{k,k}(\infty)$ is found. Then Eq. (IV.7) follows from Eq. (III.4.a).

One can wonder whether there exists a time-invariant network of impedences which is described by Eqs. (IV.3) and (IV.5) and whether such a network can be derived immediately from the SC network. The key ideal in obtaining such an equivalent network is to convert the N phases of one branch into N different branches. This converts the different instances of time into different locations in space. The following intrinsic N-port called a generalized circulator with constant G is defined by

$$\begin{bmatrix} \underset{\sim}{Q}_1(z) \\ \underset{\sim}{Q}_2(z) \\ \vdots \\ \underset{\sim}{Q}_N(z) \end{bmatrix} = \begin{bmatrix} 0 & & & & -Gz^{-1} \\ -G & 0 & & & \\ & -G & 0 & & \\ & & \ddots & \ddots & \\ & & & -G & 0 \end{bmatrix} \begin{bmatrix} v_1(z) \\ v_2(z) \\ v_3(z) \\ \vdots \\ v_N(z) \end{bmatrix} \quad \text{(IV.8)}$$

Construction of The Equivalent Circuit of a SC Network

Step 1: For each of the N phases (i.e. N time slots in one period) a network is drawn with the switches in the correct position for this time slot.

Step 2: The N-networks are interconnected by generalized circulators as follows. For each capacitor C_i in the original circuit, a circulator constant is C_i. Port 1 of this circulator is connected to the corresponding capacitor of the first circuit, port 2 to that of the second circuit and so on.

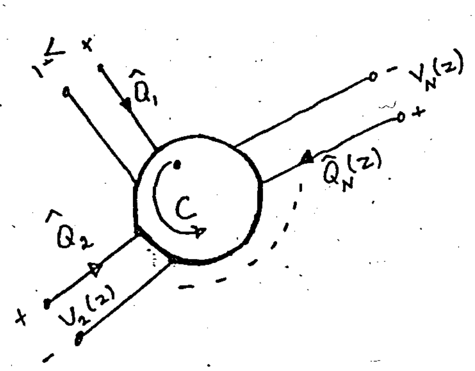


Fig.IV.2. Generalized circulator with constant G.

It is easy to check that the resulting network is described by Eq.(IV.3) (See the example (IV.1)).

For 2-phase SC networks this equivalent circuit corresponds to the gyrator of the link two port in [9] .

Example IV.1: Consider the circuit in Fig.II.8(b) on page 36. Its equivalent circuit using circulators is

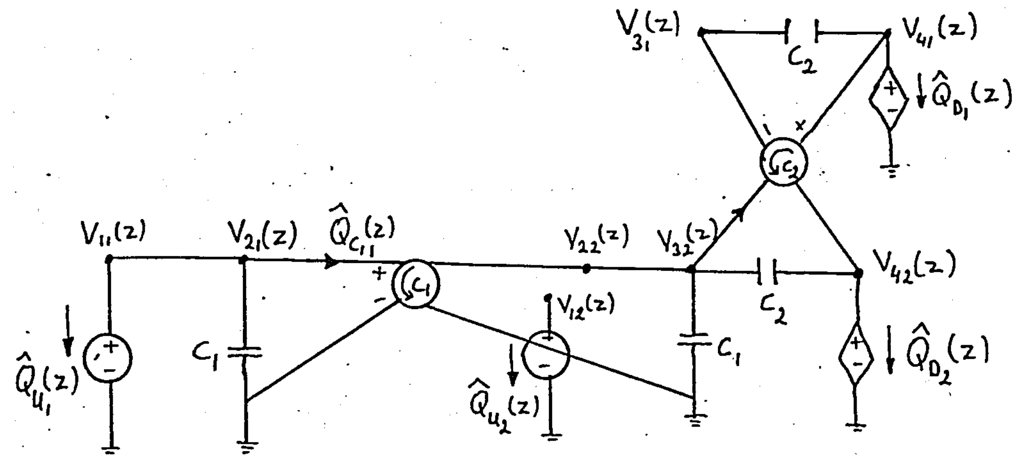


Fig.IV.1.

where $V_{ik}(z)$ is the z-transform of the i-th node voltage values at the end of each time slot of phase k according to Eq.(IV.1) which is .

$$V_{ik}(z) \triangleq \sum_{l=0}^{\infty} v_i(t^-_{k+lN+1}) z^{-l}, \quad k=1,2. \tag{IV.9}$$

and $\hat{Q}_{uk}(z)$ (resp. $\hat{Q}_{Dk}(z)$) is the z-transform of the voltage source charges (resp. dependent voltage source) for phase k.

After the above definitions, the z-domain equations can be obtained as follows.

KQL for node 1 in phase 1 and its z-transform yields:

$$\hat{Q}_{c11}(z) = -C_1 V_{21}(z) + \hat{Q}_{u1}(z). \tag{IV.10}$$

Using the definition of the circulator C_1

$$\hat{Q}_{c11}(z)=C_1z^{-1}V_{22}(z) \qquad (IV.11)$$

in Eq.(IV.10)

$$\hat{Q}_{u1}(z)=-C_1V_{21}(z)-C_1z^{-1}V_{22}(z) \qquad (IV.12)$$

is obtained.

KQL for node 4 in phase 1 and its z-transform yields:

$$\hat{Q}_{c21}(z)=C_2\left[V_{31}(z)-V_{41}(z)\right]-\hat{Q}_{D1}(z) \qquad (IV.13)$$

By the definition of the circulator C_2

$$\hat{Q}_{c21}(z)=-C_2z^{-1}\left[V_{32}(z)-V_{42}(z)\right] \qquad (IV.14)$$

Combining Eqs.(IV.13) and (IV.14)

$$-\hat{Q}_{D1}(z)+C_2V_{31}(z)-C_2V_{41}(z)+C_2z^{-1}V_{32}(z)-C_2z^{-1}V_{42}(z)=0 \qquad (IV.15)$$

results.

KQL for node 3 in phase 1 and its z-transform yields.

$$\hat{Q}_{c21}(z)=C_2\left[V_{31}(z)-V_{41}(z)\right] \qquad (IV.16)$$

Combining Eqs.(IV.16) and (IV.14) yields:

$$C_2V_{31}(z)-C_2V_{41}(z)+C_2z^{-1}V_{32}(z)-C_2z^{-1}V_{42}(z)=0 \qquad (IV.17)$$

The node voltage equality in phase 1 yields:

$$V_{11}(z)=V_{21}(z) \qquad (IV.18)$$

KVL requires that for phase 1 and 2

$$V_{11}(z)=U_1(z) \qquad (IV.19)$$

$$V_{12}(z)=U_2(z) \qquad (IV.20)$$

The constitutive equation for VCVS in phase 1 and 2 yields:

$$V_{41}(z)=-a\ V_{31}(z) \qquad (IV.21)$$

$$V_{42}(z)=-a\ V_{32}(z) \qquad (IV.22)$$

The node voltage equality for phase 2 yields

$$V_{22}(z)=V_{32}(z) \qquad (IV.23)$$

The KQL for node 2 in phase 2 yields

$$\hat{Q}_{c12}(z)=C_1V_{32}(z)+C_2\left[V_{32}(z)-V_{42}(z)\right]+\hat{Q}_{c22}(z) \qquad (IV.24)$$

From the definition of the circulator

$$\hat{Q}_{c22}(z)=C_2\left[V_{41}(z)-V_{31}(z)\right] \qquad (IV.25)$$

$$\hat{Q}_{c12}(z)=C_1V_{21}(z) \qquad (IV.26)$$

Eq.(IV.24), Eq.(IV.25) and Eq.(IV.26) yields:

$$C_1V_{21}(z)-C_1V_{22}(z)-C_2\left[V_{22}(z)-V_{42}(z)\right]-C_2\left[V_{41}(z)-V_{31}(z)\right]=0 \qquad (IV.27)$$

The KQL for node 4 in phase 2 yields

$$\hat{Q}_{c22}(z)=\hat{Q}_{D2}(z)+C_2\left[V_{42}(z)-V_{32}(z)\right] \qquad (IV.28)$$

Eq.(IV.28) and Eq.(IV.25) yields:

$$C_2V_{41}(z)-C_2V_{31}(z)-C_2V_{42}(z)+C_2V_{32}(z)-\hat{Q}_{D2}(z)=0 \qquad (IV.29)$$

KQL for node 1 in phase 2 requires

$$\hat{Q}_{u2}(z)=0 \qquad (IV.30)$$

Finally, combining Eqs.(IV.12), (IV.15), (IV.17), (IV.23), (IV.27), (IV.29) and (IV.30) properly in a single matrix yields:

$$\left[\begin{array}{cccccc|cccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_1 & 0 & 0 & 1 & 0 & 0 & -C_1z^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & C_2 & -C_2 & 0 & 0 & 0 & 0 & -C_2z^{-1} & C_2z^{-1} & 0 & 0 \\
0 & 0 & -C_2 & C_2 & 0 & 1 & 0 & 0 & C_2z^{-1} & -C_2z^{-1} & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & C_1 & C_2 & -C_2 & 0 & 0 & 0 & -(C_1+C_2) & 0 & C_2 & 0 & 0 \\
0 & 0 & -C_2 & C_2 & 0 & 0 & 0 & 0 & C_2 & -C_2 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 1 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
V_{11}(z) \\ V_{21}(z) \\ V_{31}(z) \\ V_{41}(z) \\ \hat{Q}_{u1}(z) \\ \hat{Q}_{D1}(z) \\ \hline
V_{12}(z) \\ V_{22}(z) \\ V_{32}(z) \\ V_{42}(z) \\ \hat{Q}_{u2}(z) \\ \hat{Q}_{D2}(z)
\end{array}\right]
=
\left[\begin{array}{c}
U_1(z) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline
U_2(z) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{array}\right] \qquad \text{(IV.31)}$$

<u>Example IV.2:</u> Now, consider the circuit of Fig.II.2(a) on page 9. Its equivalent circuit using circulators is

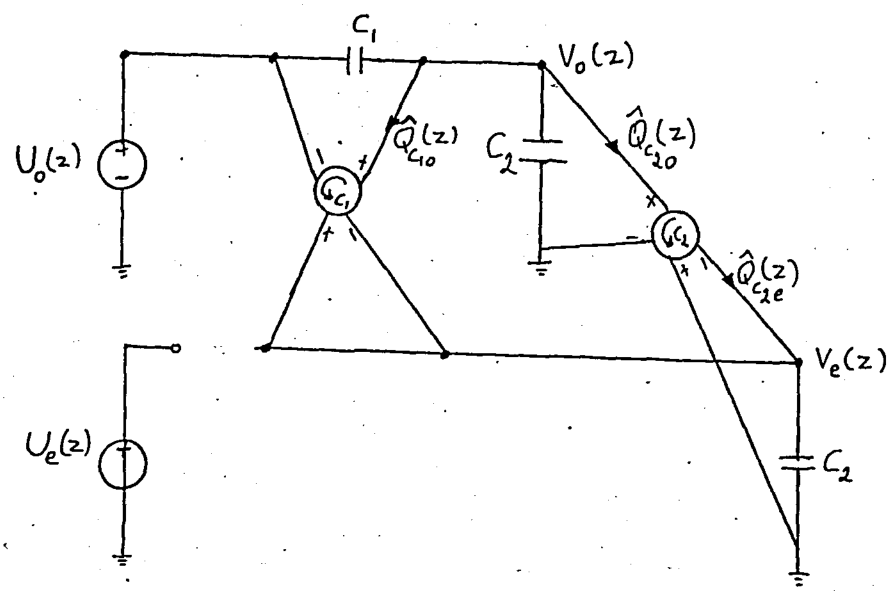


Fig.IV.2.

where superscripts o and e denote odd and even respectively. Even and odd phases have already been explained on page 9.

KQL for node A in odd phase yields

$$\hat{Q}_{c1o}(z)=C_1\left[U_o(z)-V_o(z)\right]-C_2V_o(z)-\hat{Q}_{c2o}(z) \tag{IV.32}$$

Using the definition of the circulator C_1

$$\hat{Q}_{c1o}(z)=0 \tag{IV.33}$$

KQL for node A in even phase gives

$$\hat{Q}_{c2e}(z)=C_2V_e(z) \tag{IV.34}$$

From the definition of the circulator ,C_2

$$\hat{Q}_{c2e}(z)=-C_2V_o(z) \tag{IV.35}$$

$$\hat{Q}_{c2o}(z)=C_2z^{-1}V_e(z) \tag{IV.36}$$

Eq.(IV.34) and Eq.(IV.35) yields:

$$V_e(z)=-V_o(z) \tag{IV.37}$$

Combining Eqs.(IV.32), (IV.33), (IV.36) and (IV.37) yields:

$$C_1V_o(z)=(C_1+C_2)V_o(z)+C_2z^{-1}V_e(z) \tag{IV.38}$$

Therefore

$$H_{oo}(z)\triangleq\frac{V_o(z)}{U_o(z)}=\frac{1}{(1+\frac{C_2}{C_1})-(\frac{C_2}{C_1})z^{-1}} \tag{IV.39}$$

and

$$H_{oe}(z)\triangleq\frac{V_e(z)}{U_o(z)}=H_{oo}(z) \tag{IV.40}$$

Now, it can be answered that why this circuit is called as the low pass circuit.

The Euler approximation (see page 525, [12]) implies

$z^{-1}=1-sT$ for $\omega T \ll 1$, where $s=j\omega$,

then the corresponding analog filter can be found from $H_{oo}(z)$ by the Euler approximation as:

$$H(s)=\frac{1}{(1+\frac{C_2}{C_1})-(\frac{C_2}{C_1})(1-sT)}=\frac{1}{1+s(\frac{C_2}{C_1})T}$$

which corresponds to analog low-pass filter.

<u>Remarks:</u> The low-pass digital filter obtained by the Euler mapping procedure from a low-pass analog filter will have about the same pass-band frequency characteristics as that of the original low-pass analog filter, provided that the sampling period T is sufficiently small.

For more information, see the appendix.

CHAPTER V

TWO-PORT ANALYSIS OF SWITCHED CAPACITOR NETWORKS USING FOUR-PORT EQUIVALENT CIRCUITS IN THE Z^*-DOMAIN(x)

After the z-domain analysis of multiphase SC networks and the general equivalent circuit, two port analysis of SC networks using Four Port Equivalent Circuits [9] in the z^*-domain will be given in this chapter. Throughout this chapter, the switches are assumed to change position periodically at even and odd switching times(xx). With the four port equivalent circuit representation, the traditional two port analysis tools, such as the transmission matrix and two-port transfer functions can be used conveniently. Throughout this chapter only piecewise-constant inputs are considered and it is assumed that the capacitors of the network are not charged continuously but instantaneously at the switching instants. Since no resistors are assumed in the network there is no dispersion of the charging process and the capacitor voltages can be assumed to change instantenously in steps.

V.1. Building Block Analysis of SC-Networks

Any passive SC network can be constructed with the six basic building block shown in Fig.V.1.

The nonswitched shunt capacitor and its dual are the only storage elements in SC network. Periodically switched capacitors act like resistors, since their memory is destroyed

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(x) Through out this chapter z is defined to be equal to $e^{j\omega T/2}$ (i.e, $z^*=e^{j\omega T/2}$).

(xx) The definitions for the odd and even phases have been given on page 9.

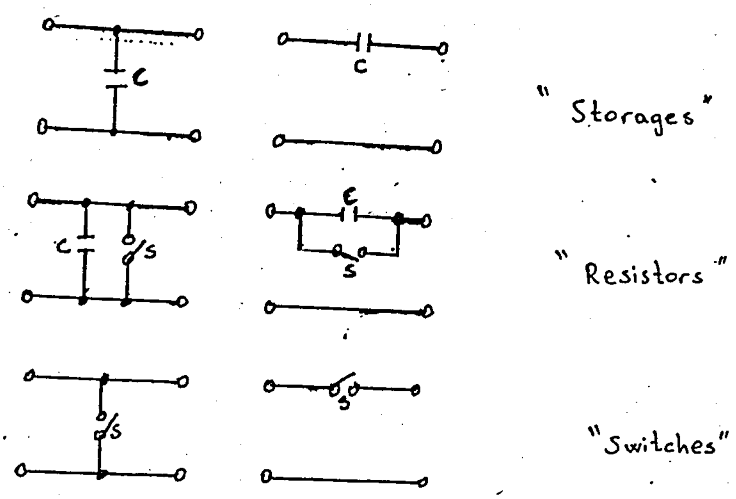
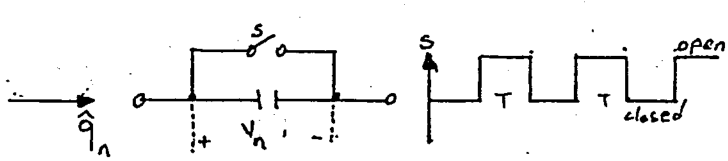


Fig.V.1. The six basic components (building blocks) in SC networs.

during the closing period of the switch. This can be demonstrated using Fig.IV.2 as follows:



(a) (b)

Fig.V.2.

$v_n = v(t\bar{n}+1)$; in $\Delta_n = (t_n, t_{n+1}]$ the switch is open.

$$\hat{q}_n = q(t\bar{n}+1) - q(t_n^-) \qquad (V.1)$$

$\hat{q}_n = C\left[v_n - v_{n-1}\right]$ where $v_{n-1} = 0$ since the switch is closed in Δ_{n-1}.

$$\hat{q}_n = Cv_n$$

It is seen that in voltage-charge domain a capacitor with a switch across it, as in Fig.V.2 behaves exactly like an equivalent resistor whose value is

$$R_{eq}[\Omega] = \frac{1}{C} \qquad (V.2)$$

The ideal switches can be considered as zero-valued capacitors with a switch in parallel. By connecting this building blocks in tandem or by combining parallel, serial and tandem connections of the building blocks, arbitrary higher order passive SC networks can be obtained.

V.2. Four-Port Equivalent Circuits of Passive SC Building Blocks

Shunt Capacitor

The shunt capacitor shown in Fig.V.3 can be described as a two-port in the time domain by applying the nodal charge equations as follows:

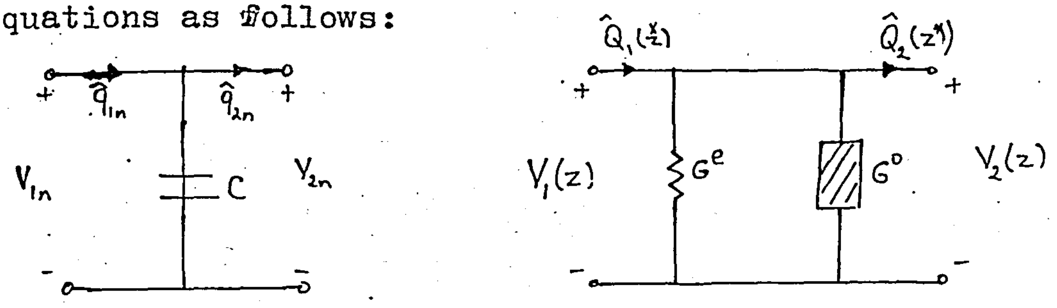


Fig.V.3. Shunt capacitor and its two-port equivalent circuit in the z-domain.

$$v_{1n}=v_{2n} \qquad n=0,1,2,\ldots \tag{V.3}$$

$$Cv_{1n}=\hat{q}_{1n}-\hat{q}_{2n}+Cv_{1(n-1)}$$

or in the z-domain in matrix form

$$\begin{bmatrix} V_1(z) \\ \hat{Q}_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ C(1-z^{-1}) & 1 \end{bmatrix} \begin{bmatrix} V_2(z) \\ \hat{Q}_2(z) \end{bmatrix} \tag{V.4}$$

The matrix in Eq.(V.4) can be interpreted as an equivalent two-port as shown in Fig.V.3. It consists of two components namely, a conductance

$$G^e=C \tag{V.5.a}$$

and a storage element

$$G^{o}=-C\,\overset{*}{z}^{-1} \qquad (V.5.b)$$

The storage element G^{o} or "storistor" has the property of delaying the charge flowing through it by one delay unit $\overset{*}{z}^{-1}$ with respect to a voltage sample applied across the element.

Eq.(V.4) can be re-written by using Eq.(V.5) as follows:

$$\begin{bmatrix} V_1(\overset{*}{z}) \\ \hat{Q}_1(\overset{*}{z}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ G^{e}+G^{o} & 1 \end{bmatrix} \begin{bmatrix} V_2(\overset{*}{z}) \\ \hat{Q}_2(\overset{*}{z}) \end{bmatrix} \qquad (V.6.a)$$

By observing that

$X^{o}(\overset{*}{z})Y^{o}(\overset{*}{z})=W^{e}(\overset{*}{z})$, $X^{e}(\overset{*}{z})Y^{e}(\overset{*}{z})=W^{e}(\overset{*}{z})$, $X^{e}(\overset{*}{z})Y^{o}(\overset{*}{z})=W^{o}(\overset{*}{z})$ and with $\hat{Q}_i(\overset{*}{z})=\hat{Q}_i^{e}(\overset{*}{z})+\hat{Q}_i^{o}(\overset{*}{z})$ and $V_i(\overset{*}{z})=V_i^{e}(\overset{*}{z})+V_i^{o}(\overset{*}{z})$ (x) the four-port transmission matrix for a shunt capacitor is obtained from Eq.(V.6.a), namely

$$\begin{bmatrix} V_{1e}(\overset{*}{z}) \\ V_{1o}(\overset{*}{z}) \\ \hat{Q}_{1e}(\overset{*}{z}) \\ \hat{Q}_{1o}(\overset{*}{z}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ G^{e} & G^{o} & 1 & 0 \\ G^{o} & G^{e} & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{2e}(\overset{*}{z}) \\ V_{2o}(\overset{*}{z}) \\ \hat{Q}_{2e}(\overset{*}{z}) \\ \hat{Q}_{2o}(\overset{*}{z}) \end{bmatrix} \qquad (V.6.b)$$

The remaining task is to find an equivalent four-port circuit for Eq.(V.6.b). The input and output voltages in the even as well as in the odd path of this four-port must be

---

(x) By definition $V_{ie}(\overset{*}{z})=v_{io}+v_{i2}\overset{*}{z}^{-2}+v_{i4}\overset{*}{z}^{-4}+\ldots$

$V_{io}(\overset{*}{z})=v_{i1}\overset{*}{z}^{-1}+v_{i3}\overset{*}{z}^{-3}+v_{i5}\overset{*}{z}^{-5}+\ldots$

and as a result of the definition of z-transform given in Eq.(IV.1), the following relations hold

$V_{ie}(\overset{*}{z})=V_{ie}(z)$ where $z=\overset{*}{z}^{2}$

$V_{io}(\overset{*}{z})=z^{-1/2}V_{io}(z)$

equal, (i.e., $v_{1e}=v_{2e}$, $v_{1o}=v_{2o}$).

Furthermore, from Eq.(V.6.b), the even and odd parts are related as follows:

$$\begin{bmatrix}\hat{Q}_{1e}\\ \hat{Q}_{1o}\end{bmatrix}=\begin{bmatrix}G^e & G^o\\ G^o & G^e\end{bmatrix}\begin{bmatrix}v_{2e}\\ v_{2o}\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\hat{Q}_{2e}\\ \hat{Q}_{2o}\end{bmatrix}$$

or (V.7)

$$\begin{bmatrix}\hat{Q}_{1e}-\hat{Q}_{2e}\\ \hat{Q}_{1o}-\hat{Q}_{2o}\end{bmatrix}=\begin{bmatrix}\hat{Q}_e\\ \hat{Q}_o\end{bmatrix}=\begin{bmatrix}G^e & G^o\\ G^o & G^e\end{bmatrix}\begin{bmatrix}v_{2e}\\ v_{2o}\end{bmatrix}$$

Eq.(V.7) can now be interpreted as the two-port shown in Fig.V.4, the π-configuration was chosen for convenience, since it yields simple expressions for the elements. By redrawing the equivalent circuit shown in Fig.V.5 one obtains the final four-port equivalent circuit for the shunt capacitor as shown in Fig.V.5, and Table 1.

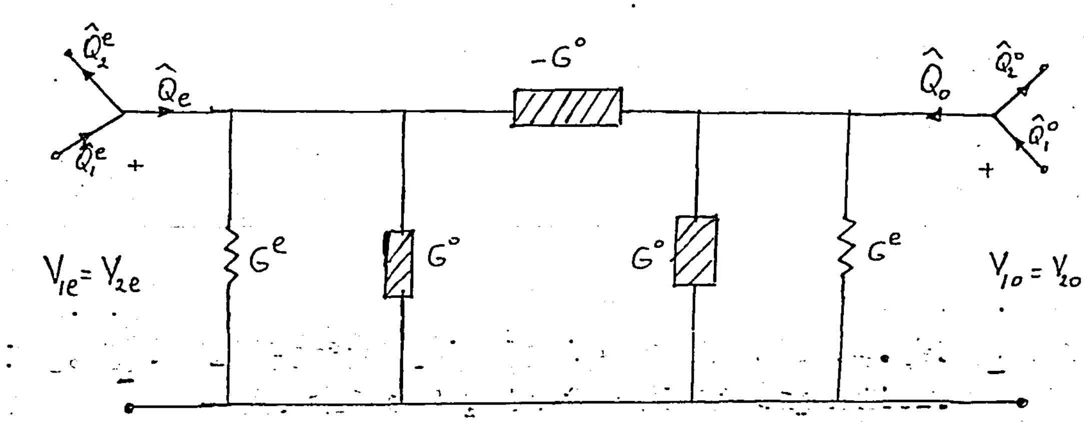


Fig.V.4. Two-port equivalent circuit for Eq.(V.6). Link between even and odd path for shunt capacitor.

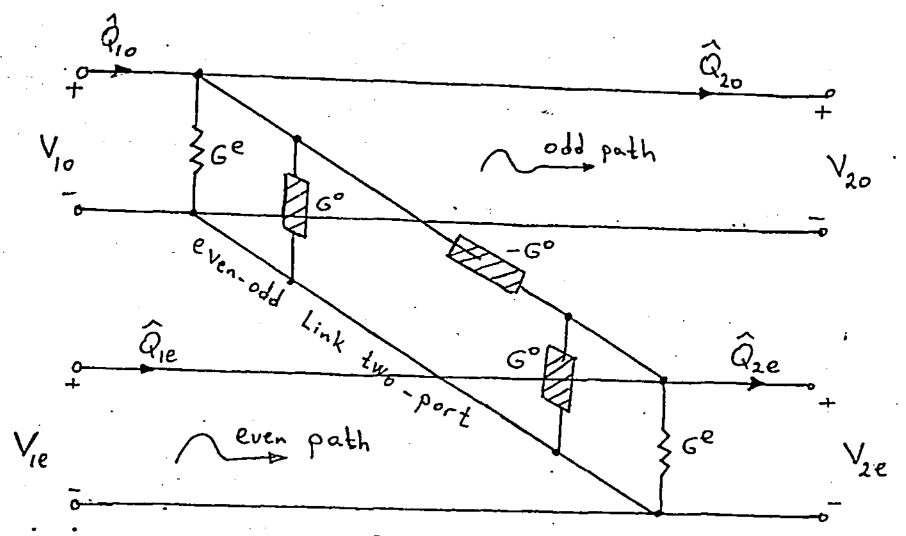


Fig.V.5. Four-port equivalent circuit for shunt capacitor.

Other equivalent circuits for the basic components in Table V.1 can be easily obtained by using the above procedure (i.e., taking z^*-transforms of the nodal charge equations).

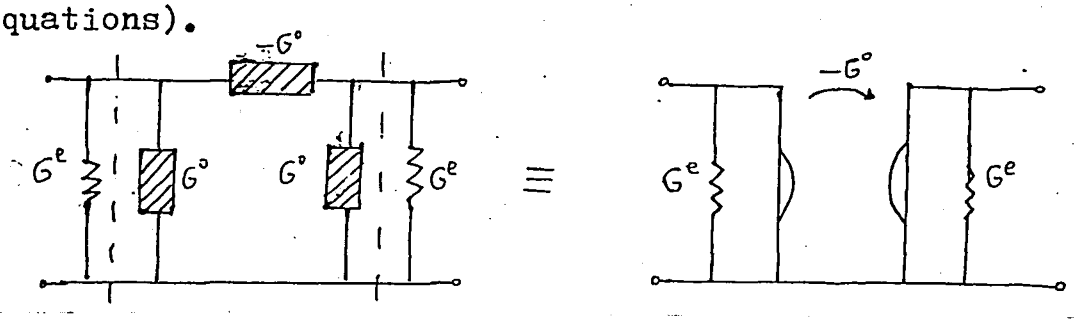


Fig.V.6. Gyrator equivalent circuit for the link two-port (LTP) in storage elements.

Active Elements and Sources:

Contralled Sources: The simplest active element in SC network is a voltage-controlled voltage source. It has no storage property. Its four-port equivalent circuit in z^*-domain is shown in Fig.V.7.

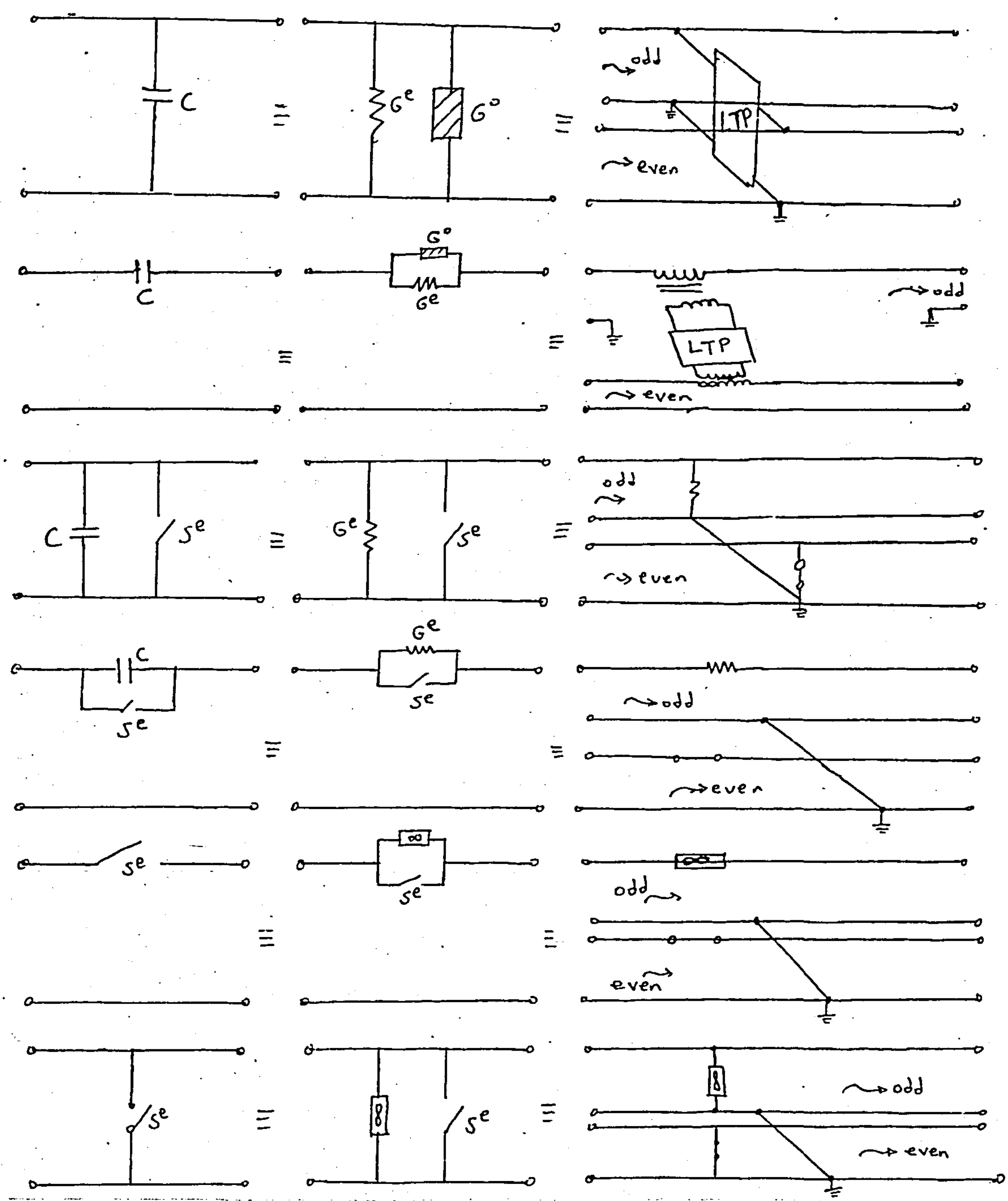


TABLE V.1.

Equivalent Circuits for Six Basic Elements In SC Filters.

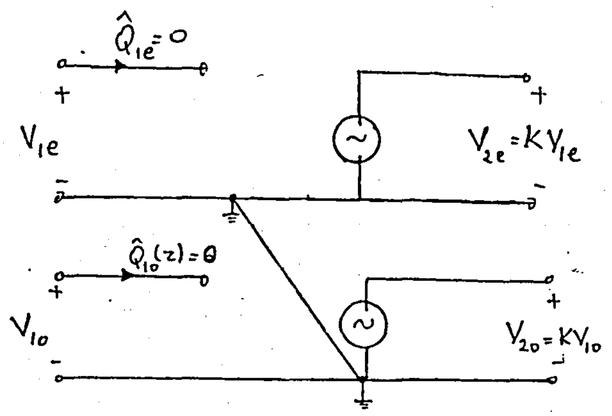


Fig. V.7. Four-port equivalent circuit of VCVS.

A charge-controlled voltage source in an SC network is more complex, since it must have the property of a capacitor, namely that of building up a voltage in response to a charge surge $\hat{q}_n$. It must therefore follow the equation

$$\frac{1}{C}\hat{q}_n = v_n - v_{n-1}$$

If the memory is periodically erased by a switch (similar to an SC) then above equation reduces to

$$v_n = \frac{1}{C}\hat{q}_n; \quad v_{n-1}=0$$

where n is either only even or only odd.

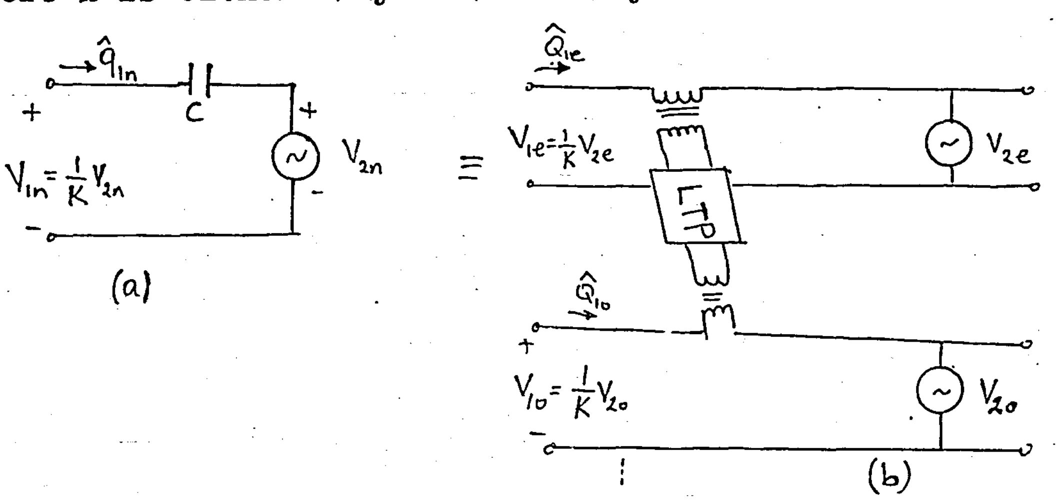


Fig. V.8(a). CCVS with storage.
(b). Its four-port equivalent circuit.

Driving Voltage Sources:

Throughout this analysis, it has been assumed that an SC network must be driven from a sampled voltage or charge source. This is achieved by sampling a continous source by a periodically operated switch.

The impedance of a voltage source must be very small, that of a charge source very large in order to guarantee an instantaneous voltage buildup across the capacitor of the SC network.

If the source is to have a finite source resistance this can be simulated by an SC combination of Table V.1 (fourth figure).

Driving Charge Sources:

Thevenin's theorem is applicable to SC networks. The charge source corresponding to $v_o(t)$ with source resistance $R_s = \frac{1}{C_s}$

$$\hat{q}_n = \frac{v_n}{R_s} = v_n \cdot C_s$$

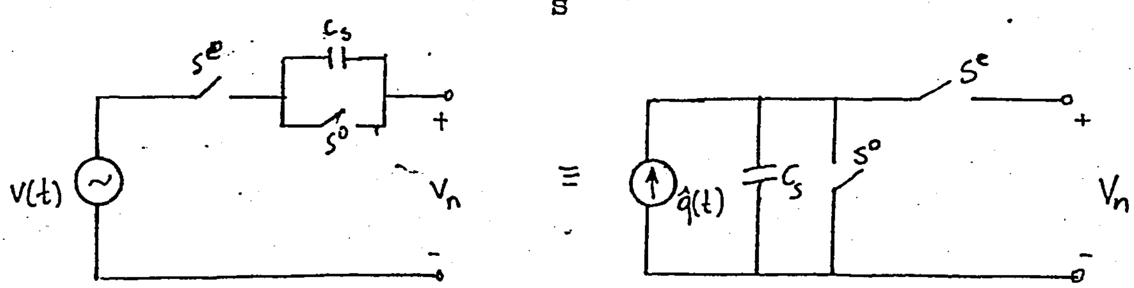


Fig.V.9. Thevenin's equivalence for SC network sources.

Fig.V.9(a) is equivalent to the Fig.V.10(a) by using Table V.1.

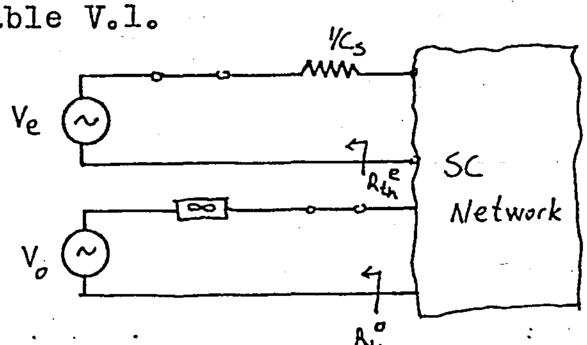


Fig.V.10.

Fig.V.9(b) is equivalent to the Fig.V.10(b)

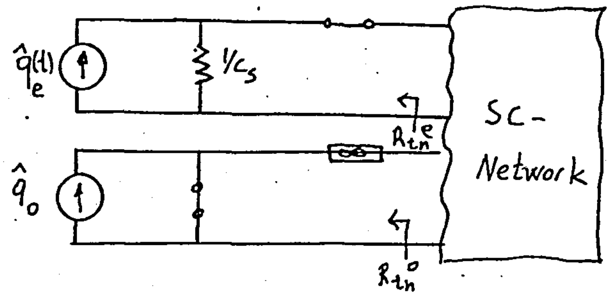


Fig.V.10(b).

Applying Thevenin theorem to both circuits it is seen that $R_{th}^e = \frac{1}{C_s}$ for the even path and $R_{th}^o = \infty$ for the odd path.

V.3. Cascading SC Building Blocks

The four-port equivalent circuits for the basic SC building blocks can be connected in cascade. A cascade of shunt or series capacitors, i.e., storage elements, merely leads to a parallel (or series) connection of LTP's without providing any filtering effect. However, alternating the tandem connection of storage elements with switched elements results in SC networks that are suitable for filtering purposes.

In Fig.V.11., the cascade connection of m alternating shunt capacitors and series switches and its four-port equivalent circuit is shown. Notice that the timing of the switches alternates along the chain. This leads to the alternating position of R_∞'s (R_∞ is an open circuit) in the even and odd path. Since, the charges, through the R_∞'s are zero the signal alternates between even and odd paths. The network therefore corresponds to a straight tandem connection of all LTP's which can be unfolded into a regular two-port network.

Other topologies can be obtained by cascading storage capacitors with SC's as shown in Figs. V.12(a) and V.13(b).

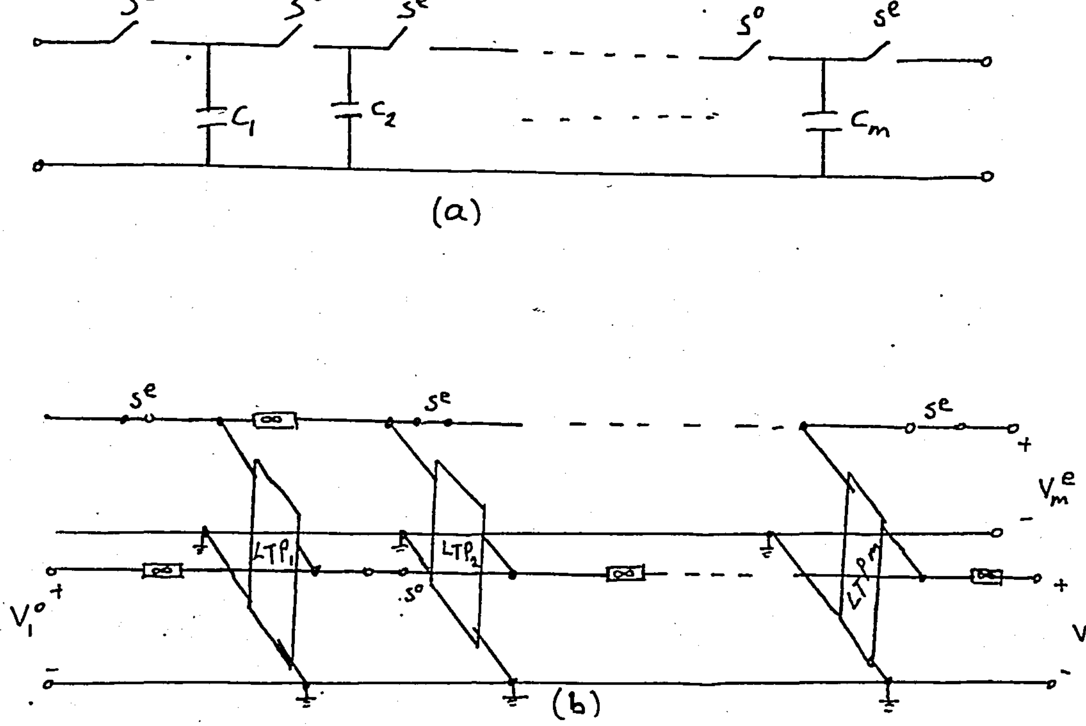


Fig.V.11(a). Cascade of alternating series switches and shunt capacitors.
(b). Four-port equivalent circuit.

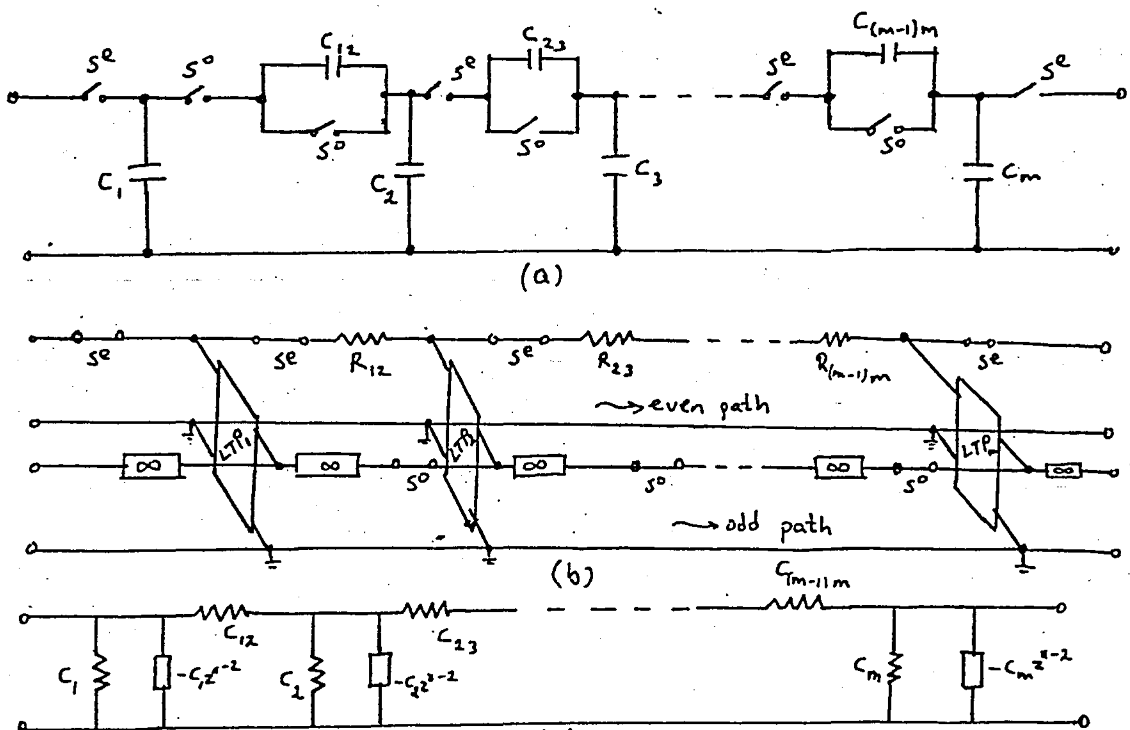


Fig.V.12(a). Cascade of shunt capacitors and switched series capacitors. (b). Four-port equivalent circuit.
(c). Final ladder equivalent circuit.

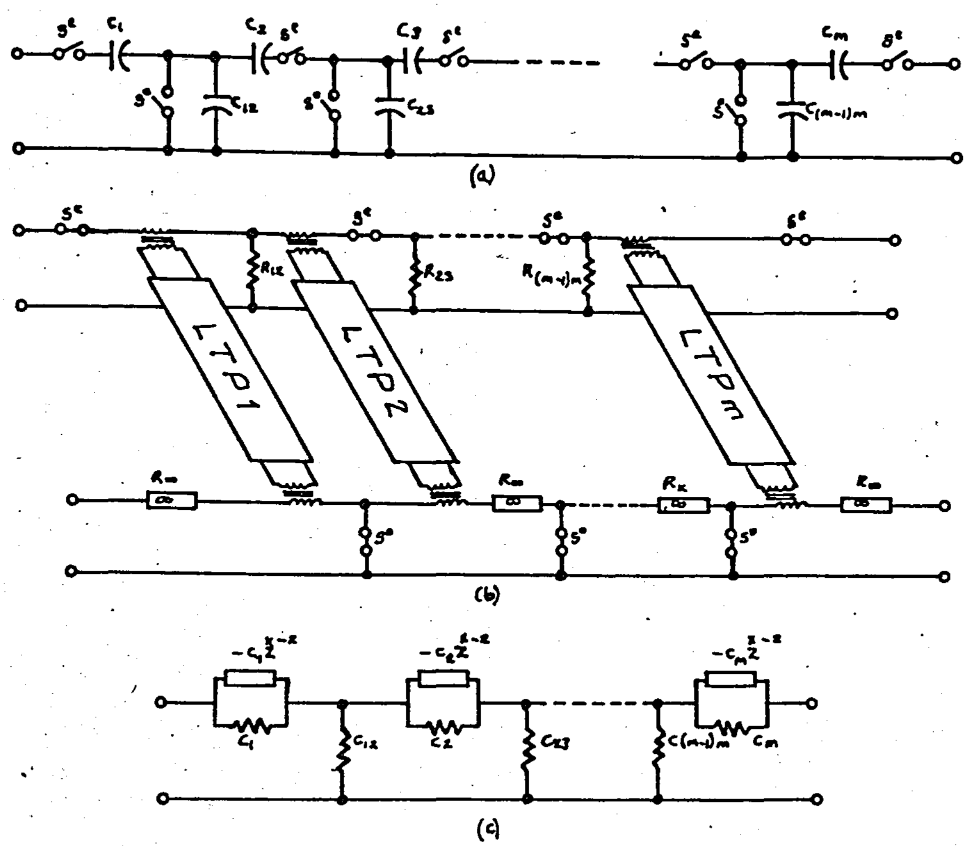


Fig.V.13.(a). Cascade of series capacitors and switched shunt capacitors.(b). Four-port equivalent circuit. (c). Final ladder equivalent circuit.

V.4. Two-Port Analysis of SC-Networks

Cascade Analysis of Building Blocks

As it has been observed that, the SC network in Fig.V.11(a) can be reduced to an equivalent two-port which resembles the tandem connection of LTP's as shown in Fig.V.14(a).

The two-port equivalent circuits for the SC networks in Fig.V.12(a) and V.13(a) can be reduced to the ones shown in Fig.V.14(b) and V.14(c) respectively. By expressing the open circuit input impedance of the LTP's in terms of their elements, the analysis of the two-ports in Fig.V.14(b) and V.14(c) reduces to that of a simple ladder structure. Using

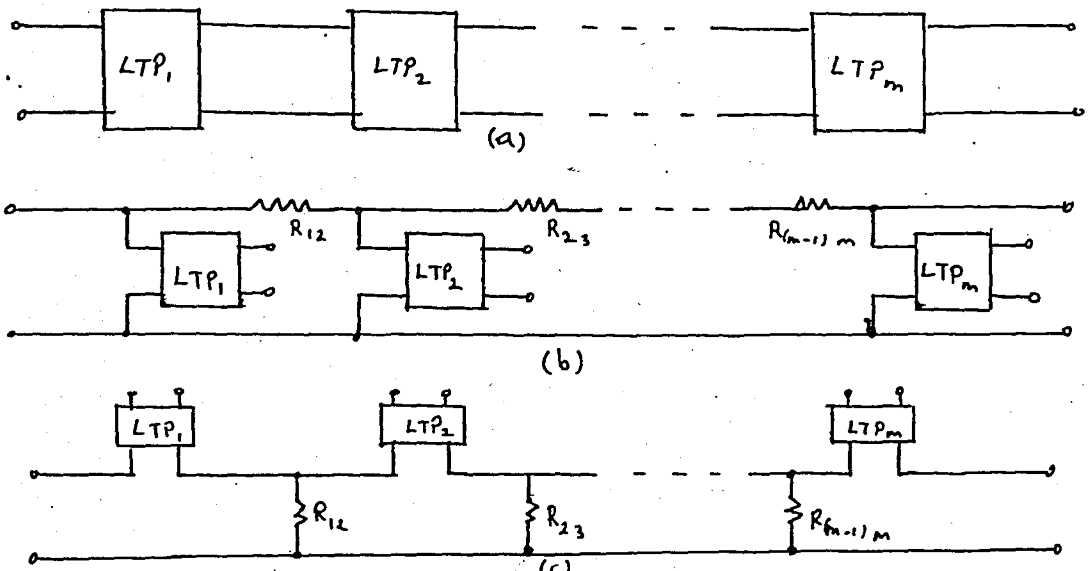


Fig.V.14. Two-port equivalent circuits of SC ladder networks (a) Corresponding to Fig.V.11. (b). Corresponding to Fig.V.12. (c). Corresponding to Fig.V.13.

the gyrator representation for the LTP as in Fig.V.6, its open circuit input impedance can be derived (see Fig.V.15).

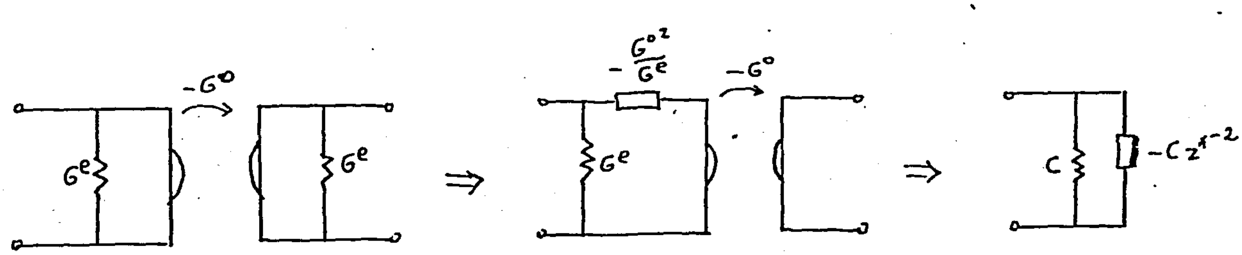


Fig.V.15. Open circuit input impedance of LTP derived via gyrator equivalent circuit.

By inspection the following is obtained

$$Z_{in} = \frac{V_1}{\hat{Q}_1} = \frac{G^e}{G^{e2}-G^{o2}} = C^{-1}\,\frac{1}{1-z^{-2}} \qquad (V.8)$$

The final two-port equivalent circuits for the SC ladder networks shown in Figs.V.12(a) and V.13(a) are shown in Figs.(V.12(c) and V.13(c)).

In order to continue the cascade analysis of LTP's as shown in Fig.V.14(a) it is necessary to derive the transmission matrix for one LTP. This can be achieved by obtaining the ABCD matrix from the y-matrix in Eq.(V.7).

$$\begin{bmatrix} V_1 \\ \hat{Q}_1 \end{bmatrix} = \begin{bmatrix} -\dfrac{G^e}{G^o} & -1/G^o \\ G^o - \dfrac{G^{e2}}{G^o} & -\dfrac{G^e}{G^o} \end{bmatrix} \begin{bmatrix} V_2 \\ \hat{Q}_2 \end{bmatrix} \qquad (V.9)$$

and with Eqs.(V.5)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{LTP} = \begin{bmatrix} z^* & z^*/C \\ z^*C(1-z^{*-2}) & z^* \end{bmatrix} \qquad (V.10)$$

The transmission matrix of an entire chain of m LTP's is now obtained by multiplying the transmission matrices of the m individual LTP's, thus

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_m = \prod_{i=1}^{m} \begin{bmatrix} z^* & z^*/C_i \\ z^*C_i(1-z^{*-2}) & z^* \end{bmatrix} \qquad (V.11)$$

For m=2 Eq.(V.11) yields

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_2 = \begin{bmatrix} z^{*2}\ 1+\dfrac{C_2}{C_1}(1-z^{*-2}) & z^{*2}\left(\dfrac{1}{C_1}+\dfrac{1}{C_2}\right) \\ (z^{*2}C_1+C_2(1-z^{*-2}) & z^{*2}\ 1+\dfrac{C_1}{C_2}(1-z^{*-2}) \end{bmatrix}$$

This matrix represents the Π-configuration shown in Fig.V.16.

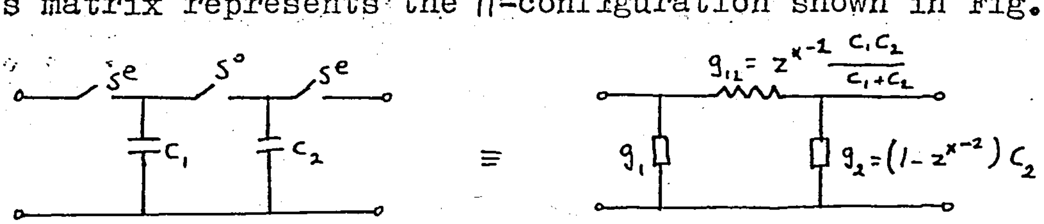


Fig.V.16. Π-equivalent circuit cascade of two LTP's.

General Two-Port Transfer Function

The previmusly described two-port analysis was restricted to cases where building blocks are cascaded by tracing the signal flow through the four-port equivalent circuit. This is possible only if the signal is transmitted along either the odd or even path, or if it alternates from one path to the other. When the signal is transmitted through the even and odd path and when LTP's are present, a more general approach should be used.

For the general case the two-phase SC network may be considered as a two-port with an input signal and an output signal in the time domain (Fig.V.17(a)).

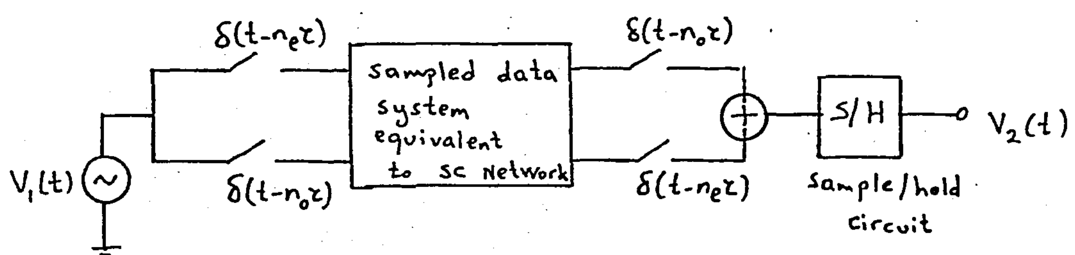


Fig.V.17(a). Time domain equivalent system.

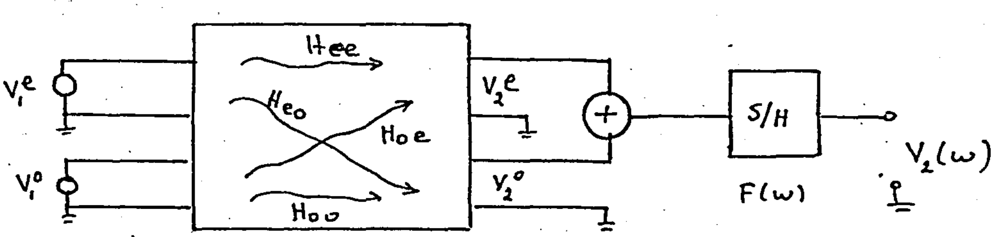


Fig.V.17(b). Frequency domain equivalent system.

The sample and hold circuit in Fig.V.17 restores the finite pulsewidth for each discrete value coming out of the sampled-data system [10] .Then by Fig.V.17, the output voltage $V_2(z^*)$ can be expressed in terms of the even and odd part of sampled input function $V_1(z^*)$ as

$$V_2(z^*)=V_2^e(z^*)+V_2(z^*)=V_1^e\left[H^{ee}+H^{eo}\right]+V_1^o\left[H^{oo}+H^{oe}\right] \quad . \qquad (V.12)$$

For a uniformly sampled sinusoidal input signal $V_1 e^{j\omega_0 n\tau}$ the z^*-transformed function is given by

$$V_1(z^*) = \sum_{n=0}^{\infty} V_1 e^{j\omega_0 n\tau} z^{*-n} = \frac{V_1 z^*}{z^* - e^{j\omega_0\tau}} \qquad (V.13)$$

The even and odd part can be determined as follows [5]

$$V_1^e(z^*) = \frac{1}{2}\left[V_1(z^*) + V_1(-z^*)\right] = \frac{V_1}{2}\left[\frac{z^*}{z^* - e^{j\omega_0\tau}} + \frac{-z^*}{-z^* - e^{j\omega_0\tau}}\right]$$

$$= V_1 \frac{z^{*2}}{z^{*2} - e^{j2\omega_0\tau}} \qquad (V.14.a)$$

and

$$V_1^o(z^*) = \frac{1}{2}\left[V_1(z^*) - V_1(-z^*)\right] = \frac{V_1}{2}\left[\frac{z^*}{z^* - e^{j\omega_0\tau}} - \frac{-z^*}{-z^* - e^{j\omega_0\tau}}\right] \qquad (V.14.b)$$

$$= V_1 \frac{z^* e^{j\omega_0\tau}}{z^{*2} - e^{2j\omega_0\tau}}$$

Substituting V_1^e and V_1^o into Eq.(V.12)

$$V_2(z^*) = V_1 \frac{z^{*2}}{z^{*2} - e^{j2\omega_0\tau}} \cdot \left\{ H_{ee}(z^*) + H_{eo}(z^*) + e^{j\omega_0\tau} z^{*-1}\left[H_{oo}(z^*) + H_{oe}(z^*)\right]\right\} \qquad (V.15)$$

is obtained.

After dividing Eq.(V.15) by (V.13), the overall transfer function of the sampled-data system in the z-domain is:

$$\frac{V_2(z^*)}{V_1(z^*)} = \frac{H_{ee} + H_{eo} + e^{j\omega_0\tau} z^{*-1}\left[H_{oo} + H_{oe}\right]}{1 + e^{j\omega_0\tau} z^{*-1}} \qquad (V.16)$$

Finally, multiplying this frequency response with, the response of the sample and hold device [9]

$$F(\omega) = \frac{\sin\frac{\omega\tau}{2}}{\omega\tau} e^{j\omega\tau/2} \qquad (V.17)$$

the overall transfer function can be established

$$H_{Total}(\omega)=\left\{\frac{H^{ee}(e^{j\omega\tau})+H^{eo}(e^{j\omega\tau})+\left[H_{oo}(e^{j\omega\tau})+\right.}{1+e^{j(\omega_o-\omega)\tau}}\right. \quad (V.18)$$

$$\left.+\frac{\left.H_{oe}(e^{j\omega\tau})\right]e^{j(\omega_o-\omega)\tau}}{1+e^{j(\omega_o-\omega)\tau}}\right\}\cdot F(\omega)$$

V. . RC Analogies of SC Networks

A frequently used SC network is a shunt capacitor with a toggle switch. It is the basic two-port associated with one LTP as can be seen from Fig.V.11. Its RC Analogy can be demonstrated by substituting the Eqs. V.5(a) and V.5(b) into the elements of the circuit in Fig.V.6. As demonstrated in Fig.V.18 the circuit can be interpreted as a capacitor for C=large and as a resistor for C=small. In both cases it is required that $\omega\tau \ll 1$. Which allows the approximation $z^{*-1}=e^{-j\omega\tau}\approx 1-j\omega\tau$ to be made .

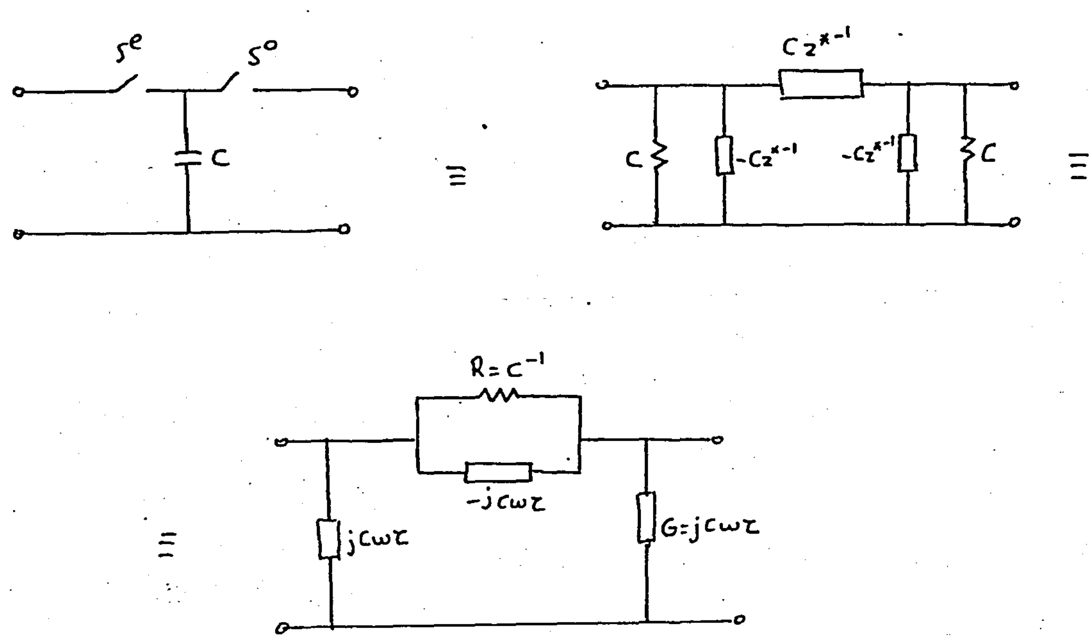


Fig. V.18(a).

if C is large $\Rightarrow C^{-1} \approx 0$ then Fig.V.18(a) is equivalent to

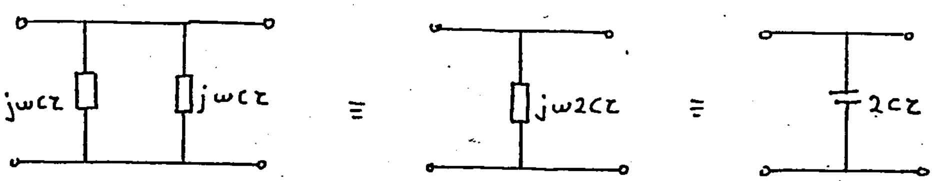


Fig.V.18(b).

if C is sufficiently small and since $\omega\tau \ll 1$ then $C\omega\tau \to 0$ and $1/j\omega c\tau \to \infty$ results

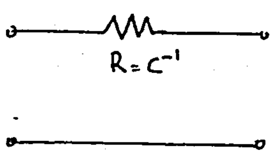


Fig.V.18(c).

A far more accurate low-pass filter approximation can be obtained by using the SC network structure shown in Fig.V.12(a) with its equivalent ladder circuit in Fig.V.12(c). This is demonstrated in Fig.V.19 for one section.

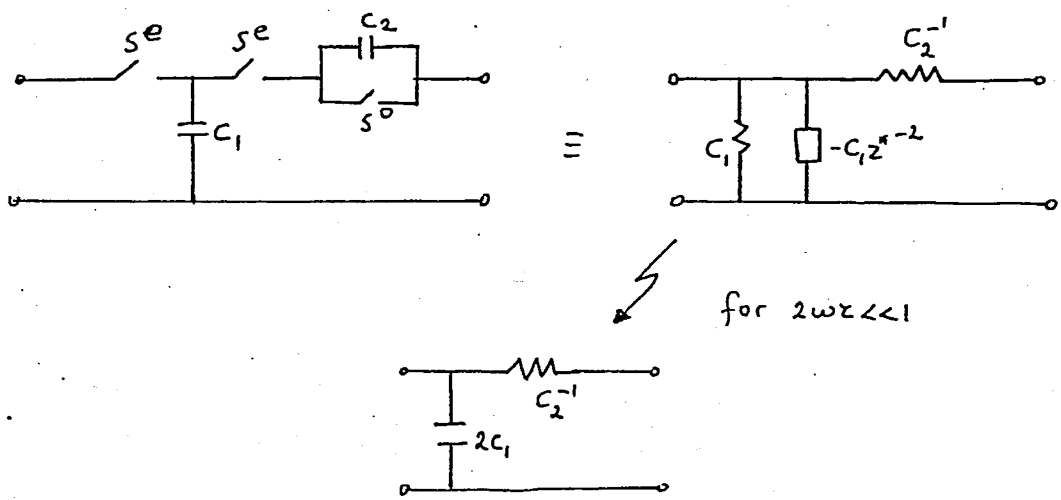


Fig.V.19. RC low-pass(x) approximation for shunt capacitor and switched series capacitor.

(x) See the appendix.

Letting

$$z^{-2} = e^{-j\omega\tau 2} \approx 1 - j2\omega\tau \;, \; 2\omega\tau \ll 1 \qquad (V.19)$$

the conductive part in the shunt branch can be eliminated and a capacitive component, related to the imaginary part in Eq.(V.19) remains. The only condition for the approximation is $2\omega\tau \ll 1$, regardless of the size of the element values.

Finally, in Fig.V.20, an RC analogy for the circuit shown in Fig.V.16 is presented. It is again based on the approximation made in Eq.(V.19) and is independent of the capacitor values. In conclusion, it can be said that passive SC networks with two-phase switches have properties similar to those of passive RC circuits.

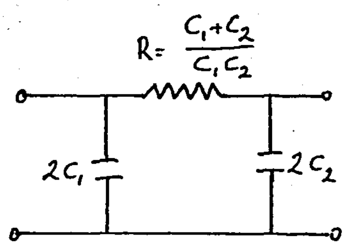
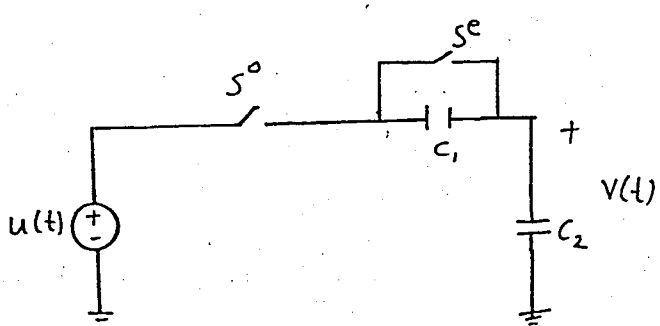


Fig.V.20, Analogy for circuit shown in Fig.V.16.

Example V.1:

Return to the example on page 9 and solve it by using LTP's and Table V.1.



The equivalent LTP circuit is

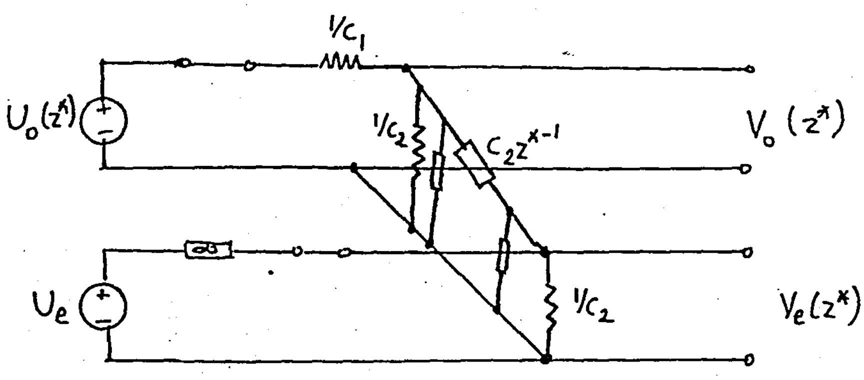


Fig. V.21(b)

Finally Fig. V.21(b) is reduced to the following figure

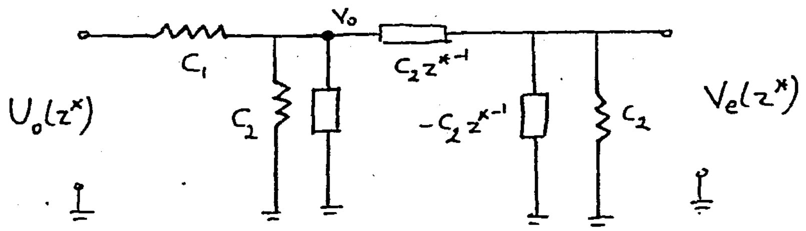


Fig. V.22.

$$H_{oo}(z^*) = \frac{V_o(z^*)}{U_o(z^*)} = \frac{1}{(1+\frac{C_2}{C_1}) - \frac{C_2}{C_1} z^{*-2}}$$

and (V.20)

$$H_{oe}(z^*) = \frac{V_e(z^*)}{U_o(z^*)} = \frac{z^{*-1}}{(1+\frac{C_2}{C_1}) - \frac{C_2}{C_1} z^{*-2}}$$

These are exactly the same results as in page 77 where

$$V_o(z) = V_o(z^*); \quad z = z^{*2}$$

$$z^{-1/2} V_e(z) = V_e(z^*)$$

Example V.2: Second-order SC Network With Operational Amplifier

The following example shown in Fig.V.23(a) was chosen to demonstrate how the described analysis method of cascaded SC networks can be used for the analysis of a second-order SC network with one active element. The purpose of this example is to illustrate how to synthesize filter networks using the four-port equivalent circuits of the building blocks introduced in the previous sections.

The first step in the analysis of the circuit shown in Fig.V.23(a) is to convert it into a four-port equivalent circuit. This is shown in Fig.V.20(b). The signal flows through the cascade of LTP's in the form of a meander. In terms of charges and the link two-ports the signal flow can be described symbolically as follows:

$$V_6^e \rightarrow \hat{Q}_F^e + \underset{\substack{\uparrow \\ V_s^e}}{\hat{Q}_s^e} =$$

$$= \hat{Q}_1^e \rightarrow LTP_1 \rightarrow \hat{Q}_2^o \rightarrow LTP_2 \rightarrow \hat{Q}_3^e \rightarrow LTP_3 \rightarrow \hat{Q}_4^o \rightarrow LTP_4 \rightarrow \hat{Q}_5^e \rightarrow V_6^e$$

The desired transfer function is

$$T(\overset{*}{z}) = \frac{V_6^e(\overset{*}{z})}{V_s^e(\overset{*}{z})} \qquad (V.21)$$

For a simple derivation of Eq.(V.21) the four-port equivalent circuit can be reduced to a two-port equivalent circuit with feedback as shown in Fig.V.23(c). The internal two-port ABCD thereby consists of the cascade of the four LTP's in Fig.V.23(b). The operational amplifier has been redrawn as a charge controlled voltage source, where $g=C_g$, thus

$$V_6^e = -\hat{Q}_5^e \frac{1}{C_g} = -\hat{Q}_5^e \cdot \frac{1}{g}$$

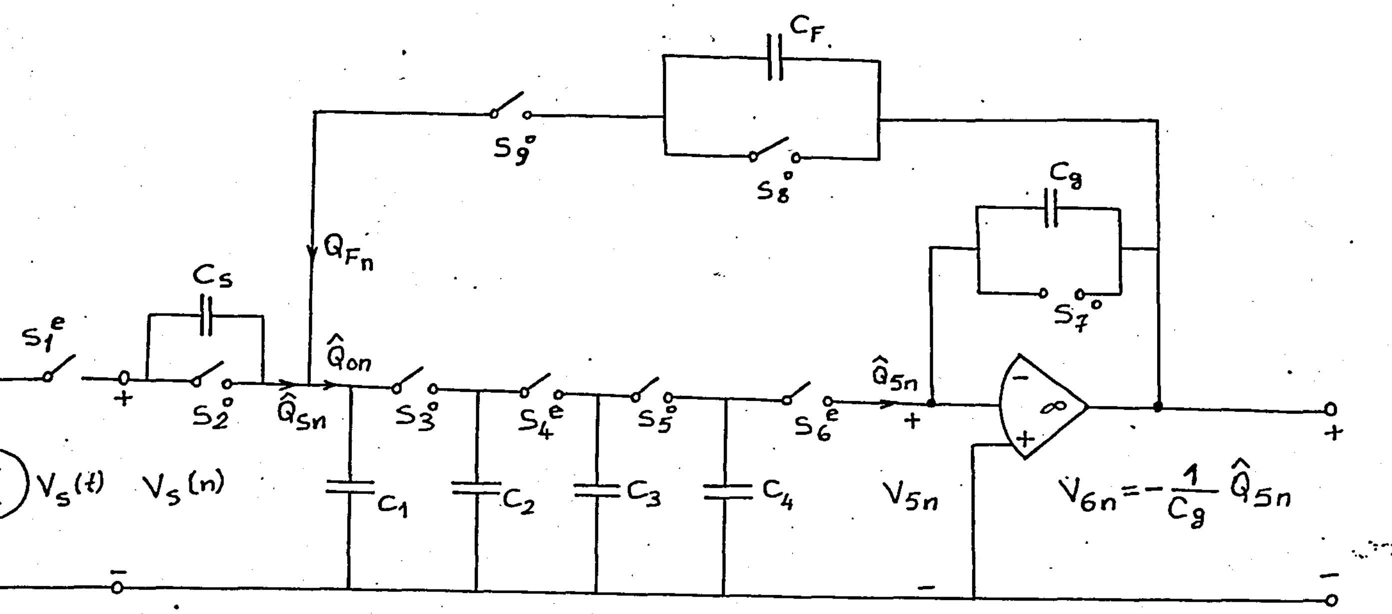


Fig.23(a) Second order SC network with operational amplifier.

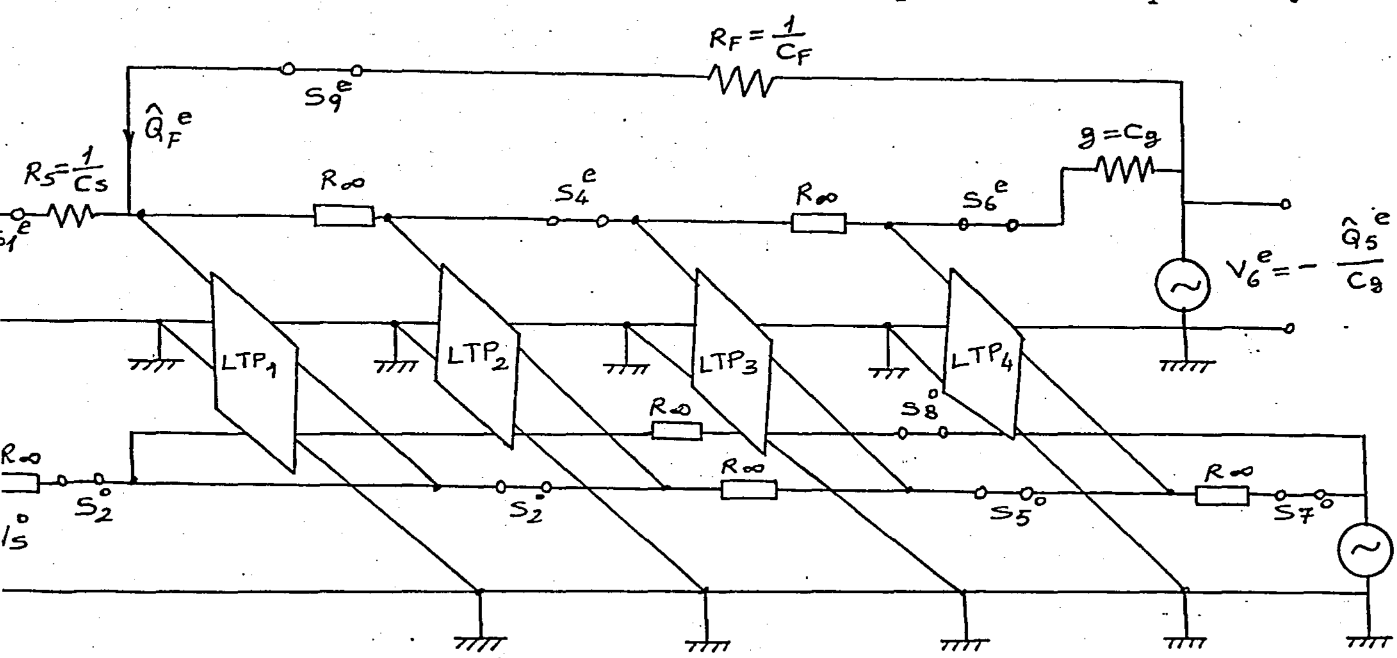


Fig.V.23(b) Four-port equivalent circuit

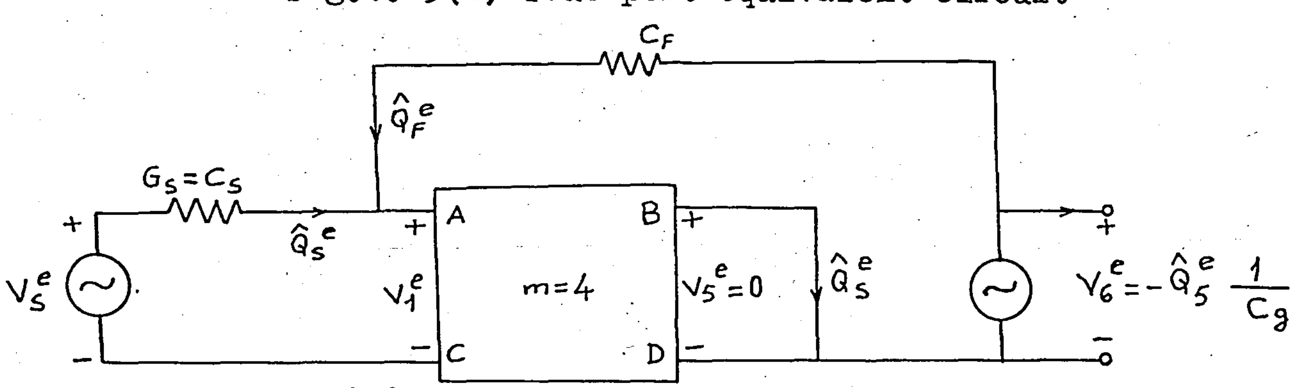


Fig.V.23(c) Two-port equivalent circuit

The transmission matrix ABCD can be obtained from Eq.(V.11) for m=2. It is more convenient to calculate the product of two matrices, therefore, with Eq.(V.11) and letting $p=1-z^{*-2}$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{m=4} = z^{*4} \begin{bmatrix} (1+\frac{C_2}{C_1}p)(1+\frac{C_4}{C_3}p)+(\frac{1}{C_2}+\frac{1}{C_1})(C_3+C_4)p & (1-\frac{C_2}{C_1}p)(\frac{1}{C_3}+\frac{1}{C_4})+(1+\frac{C_3}{C_4}p)(\frac{1}{C_1}+\frac{1}{C_2}) \\ (C_1+C_2)(1+\frac{C_4}{C_3}p)+(C_3+C_4)(1+\frac{C_1}{C_2}p) & (1+\frac{C_1}{C_2}p)(1+\frac{C_3}{C_4}p)+(\frac{1}{C_3}+\frac{1}{C_4})(C_1+C_2) \end{bmatrix} \quad (V.22)$$

Before considering the feedback loop, the following matrix relation can be derived from the circuit in Fig.V.23(c):

$$\begin{bmatrix} V_1^e \\ \hat{Q}_1^e \end{bmatrix} = \begin{bmatrix} A & & B \\ & m=4 & \\ C & & D \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -C_g & 0 \end{bmatrix} \begin{bmatrix} V_6^e \\ \hat{Q}_6^e \end{bmatrix} = \begin{bmatrix} -BC_g & 0 \\ -DC_g & 0 \end{bmatrix} \begin{bmatrix} V_6^e \\ \hat{Q}_6^e \end{bmatrix} \quad (V.23)$$

The second matrix factor multiplying the ABCD matrix represents the charge controlled voltage source. The equation above yield two simple relations:

$$V_1^e = -BC_gV_6^e$$
$$\hat{Q}_e = -DC_gV_6^e \quad (V.24)$$

With this, the overall transfer function of the network in Fig.V.23(c) results in

$$\frac{V_6^e}{V_s^e} = T(z^*) = -\frac{C_s}{C_g}\frac{1}{B(C_F+C_s)+D+\frac{C_F}{C_g}} \quad (V.25)$$

and after substituting the terms for B and D from Eq.(V.22)

$$T(z^*) = \frac{K\,z^{*-4}}{p^2+pa_1+a_o+\frac{C_F}{C_g}\alpha z^{*-4}} \quad (V.26)$$

where

$$\alpha = \frac{C_2 C_4}{C_1 C_3} \qquad K= -\alpha \frac{C_s}{C_g}$$

$$a_1 = \alpha\left[(C_F + C_s)\left\{\frac{C_2}{C_1}\left(\frac{1}{C_3} + \frac{1}{C_4}\right) + \frac{C_3}{C_4}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)\right\} + \frac{C_1}{C_2} + \frac{C_3}{C_4} + \frac{C_1}{C_3} + \frac{C_1}{C_4} + \frac{C_2}{C_3} + \frac{C}{C}\right.$$

$$a_o = \alpha\left[(C_F + C_s)\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \frac{1}{C_4}\right) + 1\right] .$$

As can be observed in Eq.(V.26), the transfer function $T(\overset{*}{z})$ is actually a function of $\overset{*}{z}^{-2}$, since no terms of $\overset{*}{z}^{-1}$ occur. The transfer function in Eq.(V.26) can be written with the relation in Eq.(V.20) $z = \overset{*}{z}^{2}$ as follows.

$$T(z) = \frac{K\, z^{-2}}{z^{-2}\left(1 - \frac{C_F}{C_g}\alpha\right) - z^{-1}(2 + a_1) + 1 + a_1 + a_o} \qquad (V.27)$$

Eq.(V.27) corresponds to the response of the sampled-data low-pass filter shown in Fig.V.23(a). This circuit has already been built in the laboratory by Kurth and Moschytz using discrete capacitors and discrete FET switches [9]. Although all elements were non-ideal (i.e., on-resistors of the switches $R_{on} \approx 300\Omega$), a relatively good match between the measured and the predicted response was achieved as shown in Fig.V.24.

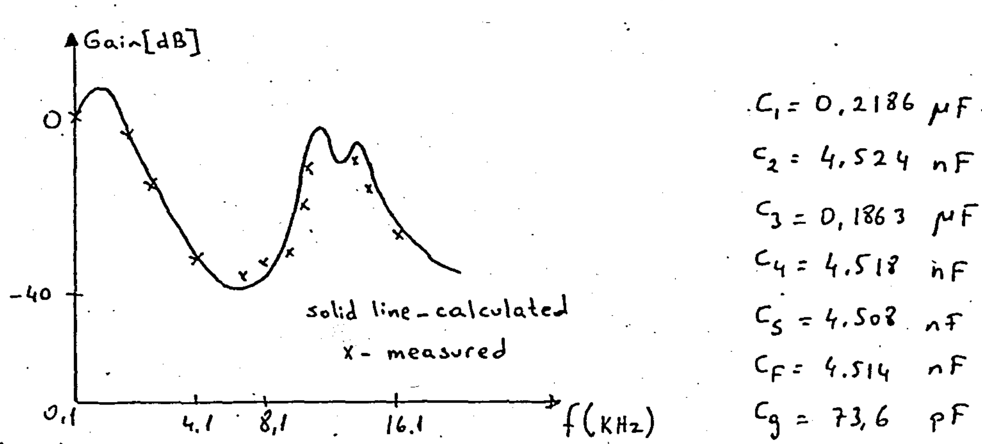


Fig.V.24. Measured and calculated response of a laboratory model.

The overall rolloff towards higher frequencies is due to a $\frac{\sin x}{x}$ response related to the finite pulsewidth at the output of the network.

Remarks: An SC network can be considered as a sampled-data system described by a set of difference equations with periodically time-varying coefficients. If the SC network has the complexity generally encountered in practice, the nodal charge equations leads to unwidely analytical expressions. A building block analysis based on six passive two-ports which are most commonly used in SC networks avoid this complexity and represents a systematic method of analyzing general SC networks. Using this analysis the performance of various practical circuits can be evaluated and a design classification may be derived.

CHAPTER VI

FREQUENCY DOMAIN SOLUTIONS OF TWO-PHASE SC NETWORKS

In this chapter, frequency domain solutions of two-phase SC networks due to arbitrary inputs will be derived and will be illustrated on the circuit of Fig. II.8.

The state equations (II.34) on page 22 can be obtained by any of the methods in Chapter II. If the capacitor voltage vectors are called as $\underset{\sim}{x}_1(t)$ and $\underset{\sim}{x}_2(t)$ (respectively for phase 1 and phase 2) then the state equations are

$$\begin{aligned} \underset{\sim}{x}_1(t) &= \underset{\approx}{F}_1 \underset{\sim}{x}_2(nT^-) + \underset{\approx}{G}_1 \underset{\sim}{s}(t) \quad \text{for} \quad t \in (nT, nT+\tau_1] \\ \underset{\sim}{x}_2(t) &= \underset{\approx}{F}_2 \underset{\sim}{x}_1(nT+\tau_1^-) + \underset{\approx}{G}_2 \underset{\sim}{s}(t) \quad \text{for} \quad t \in (nT+\tau_1, (n+1)T] \end{aligned} \qquad \text{(VI.1)}$$

Phase 1 corresponds to the intervals $(nT, nT+\tau_1]$ and phase 2 corresponds to the intervals $(nT+\tau_1, (n+1)T]$ where $n=0,1,2,\ldots$ The dimension of $\underset{\sim}{x}_1(t)$ and $\underset{\sim}{x}_2(t)$ is equal to the number of capacitors.

According to Eq. (II.35) the output equations are

$$\begin{aligned} \underset{\sim}{y}_1(t) &= \hat{\underset{\approx}{C}}_1 \underset{\sim}{x}_2(nT^-) + \hat{\underset{\approx}{D}}_1 \underset{\sim}{s}(t) \quad \text{for} \quad t \in (nT, nT+\tau_1] \\ \underset{\sim}{y}_2(t) &= \hat{\underset{\approx}{C}}_2 \underset{\sim}{x}_2(nT+\tau_1^-) + \hat{\underset{\approx}{D}}_2 \underset{\sim}{s}(t) \quad \text{for} \quad t \in (nT+\tau_1, (n+1)T] \end{aligned} \qquad \text{(VI.2)}$$

Now, define the following window functions [16] :

$$w_\tau(t) = \begin{cases} 1 & 0 < t \leq \tau \\ 0 & \text{elsewhere} \end{cases}$$

$$w_1(t) = \sum_{n=-\infty}^{+\infty} w_{\tau_1}(t-nT)$$

$$w_2(t) = \sum_{n=-\infty}^{+\infty} w_{\tau_2}(t-nT-\tau_1) \qquad \text{(VI.3)}$$

where $\tau_1+\tau_2=T$

Then from (VI.1)

$$\underset{\sim}{x}_1(t) = \underset{\approx}{F}_1 \sum_{n=-\infty}^{+\infty} \underset{\sim}{x}_2(nT^-) w_{\tau_1}(t-nT) + \underset{\approx}{G}_1 \underset{\sim}{s}(t) w_1(t) \qquad \text{(VI.4.a)}$$

for all t

$$\underset{\sim}{x}_2(t) = \underset{\approx}{F}_2 \sum_{n=-\infty}^{+\infty} \underset{\sim}{x}_1(nT+\tau_1^-) w_{\tau_2}(t-nT-\tau_1) + \underset{\approx}{G}_2 \underset{\sim}{s}(t) w_2(t) \qquad \text{(VI.4.b)}$$

The Fourier transforms of (VI.4.a) and (VI.4.b), are respectively,

$$\underset{\sim}{X}_1(\omega) = \underset{\approx}{F}_1 \hat{\underset{\sim}{X}}_2(\omega) \frac{1-e^{-j\omega\tau_1}}{j\omega} + \underset{\approx}{G}_1 \underset{\sim}{S}_1(\omega) \qquad \text{(VI.5.a)}$$

$$\underset{\sim}{X}_2(\omega) = \underset{\approx}{F}_2 \hat{\underset{\sim}{X}}_1(\omega) \frac{1-e^{-j\omega\tau_2}}{j\omega} + \underset{\approx}{G}_2 \underset{\sim}{S}_2(\omega) \qquad \text{(VI.5.b)}$$

where

$$\hat{\underset{\sim}{X}}_1(\omega) = \sum_{n=0}^{\infty} \underset{\sim}{x}_1(nT+\tau_1^-) e^{-jn\omega T} e^{-j\omega\tau_1} \qquad \text{(VI.5.c)}$$

$$\hat{\underset{\sim}{X}}_2(\omega) = \sum_{n=0}^{\infty} \underset{\sim}{x}_2(nT^-) e^{-jn\omega T} \qquad \text{(VI.5.d)}$$

$$\underset{\sim}{S}_1(\omega) = \sum_{n=0}^{\infty} \theta_{1,n} \underset{\sim}{S}(\omega-n\omega_s) \qquad \text{(VI.5.e)}$$

$$\underset{\sim}{S}_2(\omega) = \sum_{n=0}^{\infty} \theta_{2,n} \underset{\sim}{S}(\omega-n\omega_s) \qquad \text{(VI.5.f)}$$

where $\omega_s = \frac{2}{T}$ is the sampling frequency and

$$\theta_{1,n} = \frac{1-e^{-jn\omega_s\tau_1}}{jn\omega_s T} \quad n\neq 0; \quad \theta_{1,0} = \frac{\tau_1}{T} \qquad \text{(VI.5.g)}$$

$$\theta_{2,n}=\frac{e^{-jn\omega_s\tau_1}-e^{-jn\omega_s T}}{jn\omega_s T}\quad n\neq 0;\quad \theta_{2,0}=\frac{2}{T} \qquad \text{(VI.5.h)}$$

The $\hat{X}_k(\omega)$'s in the above equations may be solved from the difference equations derived from (VI.1). From (VI.1.a), leting $t=nT+\tau_1^-$

$$\underset{\sim}{x}_1(nT+\tau_1^-)=\underset{\approx}{F}_1\underset{\sim}{x}_2(nT^-)+\underset{\approx}{G}_1\underset{\sim}{s}(nT+\tau_1^-) \qquad \text{(VI.6.a)}$$

and from (VI.1.b), leting $t=nT+T^-$

$$\underset{\sim}{x}_2(nT+T^-)=\underset{\approx}{F}_2\underset{\sim}{x}_1(nT+\tau_1^-)+\underset{\approx}{G}_2\underset{\sim}{s}(nT+T^-) \qquad \text{(VI.6.b)}$$

are obtained.

Increasing the index n in (VI.6.a) by 1 and then substituting (VI.6.b) into (VI.6.a) yields

$$\underset{\sim}{x}_1(nT+T+\tau_1^-)=\underset{\approx}{F}_1\underset{\approx}{F}_2\underset{\sim}{x}_1(nT+\tau_1^-)+\underset{\approx}{F}_1\underset{\approx}{G}_2\underset{\sim}{s}(nT+T^-)+\underset{\approx}{G}_1\underset{\sim}{s}(nT+T+\tau_1^-) \qquad \text{(VI.7.a)}$$

Substituting (VI.6.a) into (VI.6.b) gives

$$\underset{\sim}{x}_2(nT+T^-)=\underset{\approx}{F}_2\underset{\approx}{F}_1\underset{\sim}{x}_2(nT^-)+\underset{\approx}{F}_2\underset{\approx}{G}_1\underset{\sim}{s}(nT+\tau_1^-)+\underset{\approx}{G}_2\underset{\sim}{s}(nT+T^-) \qquad \text{(VI.7.b)}$$

Applying the z-transform to the above equations yields:

$$z\underset{\sim}{\tilde{X}}_1(z)=\underset{\approx}{F}_1\underset{\approx}{F}_2\underset{\sim}{\tilde{X}}_1(z)+\underset{\approx}{F}_1\underset{\approx}{G}_2 z\underset{\sim}{\tilde{S}}_2(z)+\underset{\approx}{G}_1 z\underset{\sim}{\tilde{S}}_1(z)$$

$$z\underset{\sim}{\tilde{X}}_2(z)=\underset{\approx}{F}_2\underset{\approx}{F}_1\underset{\sim}{\tilde{X}}_2(z)+\underset{\approx}{F}_2\underset{\approx}{G}_1\underset{\sim}{\tilde{S}}_1(z)+\underset{\approx}{G}_2 z\underset{\sim}{\tilde{S}}_2(z)$$

where

$$\underset{\sim}{\tilde{X}}_1(z)\triangleq\sum_{n=0}^{\infty}\underset{\sim}{x}_1(nT+\tau_1^-)z^{-n} \qquad \text{(VI.8.a)}$$

$$\underset{\sim}{\tilde{X}}_2(z)\triangleq\sum_{n=0}^{\infty}\underset{\sim}{x}_2(nT^-)z^{-n} \qquad \text{(VI.8.b)}$$

$$\underset{\sim}{\tilde{S}}_1(z)\triangleq\sum_{n=0}^{\infty}\underset{\sim}{s}(nT+\tau_1^-)z^{-n} \qquad \text{(VI.8.c)}$$

$$\tilde{S}_2(z) \triangleq \sum_{n=0}^{\infty} s(nT^-) z^{-n} \tag{VI.8.d}$$

The above definitions are in coincidence with the z-transform definition on page (51).

Thus

$$\tilde{X}_1(z) = \phi_2(z) \left[F_1 G_2 z S_2(z) + G_1 z S_1(z) \right]$$

$$\tilde{X}_2(z) = \phi_1(z) \left[F_2 G_1 S_1(z) + G_2 z S_2(z) \right] \tag{VI.9}$$

where

$$\phi_1(z) = (zI - F_2 F_1)^{-1}$$

$$\phi_2(z) = (zI - F_1 F_2)^{-1} \tag{VI.10.a}$$

and I is the identity matrix.

$\phi_1(z)$ and $\phi_2(z)$ may be viewed as the characteristic matrices of a switched capacitor circuit and they are related by the following equations:

$$\phi_1(z) F_2 = F_2 \phi_2(z)$$

$$\phi_2(z) F_1 = F_1 \phi_1(z) \tag{VI.10.b}$$

Let $s(t)$ be continuous at $t=nT^-$ and $t=nT+\tau_1^-$, then with the relationship $z=e^{j\omega T}$ and using Poisson's formula [4] the $\tilde{S}_m(z)$ in (VI.8.c) and (VI.8.d) can be expressed as:

$$\tilde{S}_1(e^{j\omega T}) = \sum_{n=0}^{\infty} s(nT+\tau_1^-) e^{-jn\omega T} =$$

$$= \frac{1}{T} \sum_{n=0}^{\infty} S(\omega - n\omega_s) e^{j(\omega - n\omega_s)\tau_1} \tag{VI.11.a}$$

$$\tilde{S}_2(e^{j\omega T}) = \sum_{n=0}^{\infty} s(nT^-) e^{-jn\omega T} = \frac{1}{T} \sum_{n=0}^{\infty} S(\omega - n\omega_s) \tag{VI.11.b}$$

By comparing (VI.8.a,b) with (VI.5.a,b) the following are obtained:

$$\hat{X}_1(\omega)=\tilde{X}_1(e^{j\omega T})e^{-j\omega\tau_1} \qquad (VI.12.a)$$

$$\hat{X}_2(\omega)=\tilde{X}_2(e^{j\omega T}) \qquad (VI.12.b)$$

Substituting (VI.11.a,b) into (VI.9) then into (VI.12) we obtain:

$$\hat{X}_1(\omega)=\frac{1}{T}\sum_{n=0}^{\infty} P_{2,n}(\omega)S(\omega-n\omega_s) \qquad (VI.13.a)$$

$$\hat{X}_2(\omega)=\frac{1}{T}\sum_{n=0}^{\infty} P_{1,n}(\omega)S(\omega-n\omega_s) \qquad (VI.13.b)$$

where

$$P_{1,n}(\omega)=\emptyset_1(e^{j\omega T})(F_2G_1e^{j(\omega-n\omega_s)\tau_1}+G_2e^{j\omega T})$$

$$P_{2,n}(\omega)=\emptyset_2(e^{j\omega T})(F_1G_2e^{j\omega\tau_2}+G_1e^{j\omega\tau_2}e^{j(\omega-n\omega_s)\tau_1}) \qquad (VI.14)$$

Note that $\emptyset_1(e^{j\omega T})$ and $\emptyset_2(e^{j\omega T})$, are periodic functions of ω with period ω_s.

From the output equatoin (II.36)

$$y_m(t)=C_mx_m(t)+D_ms(t)w_m(t) \qquad m=1,2 \quad \text{for all } t$$

Its Fourier transforms is obtained as

$$Y_m(\omega)=C_mX_m(\omega)+D_mS_m(\omega) \quad m=1,2 \qquad (VI.15.a)$$

Substituting the results obtained in (VI.13.a,b) into (VI.15.a,b) and then into (VI.15.a) together with (VI.5.e) and (VI.5.f) yields.

$$Y_m(\omega)=\sum_{n=0}^{\infty} T_{m,n}(\omega)S(\omega-n\omega_s) \quad m=1,2 \qquad (VI.15.b)$$

where

$$\underset{\approx}{T}_{m,n}(\omega)=\frac{1-e^{-j\omega\tau_m}}{j\omega T}\underset{\approx}{C}_m\underset{\approx}{F}_m\underset{\approx}{P}_{m,n}(\omega)+\underset{\sim}{\theta}_{m,n}(\underset{\approx}{C}_m\underset{\approx}{G}_m+\underset{\approx}{D}_m)\qquad m=1,2 \qquad \text{(VI.15.c)}$$

or

$$\underset{\approx}{T}_{m,n}(\omega)=\frac{1-e^{-j\omega\tau_m}}{j\omega T}\hat{\underset{\approx}{C}}_m\underset{\approx}{P}_{m,n}(\omega)+\theta_{m,n}\hat{\underset{\approx}{D}}_m\qquad m=1,2 \qquad \text{(VI.15.d)}$$

The total output is

$$\underset{\sim}{Y}(\omega)=\underset{\sim}{Y}_1(\omega)+\underset{\sim}{Y}_2(\omega)=\sum_n \underset{\approx}{T}n(\omega)\underset{\sim}{S}(\omega-n\omega_s) \qquad \text{(VI.16.a)}$$

where

$$\underset{\approx}{T}_n(\omega)=\underset{\approx}{T}_{1,n}(\omega)+\underset{\approx}{T}_{2,n}(\omega) \qquad \text{(VI.16.b)}$$

The above expression (VI.16) gives the Fourier transform of the output due to an arbitrary input whose Fourier transform is $\underset{\sim}{S}(\omega)$. $\underset{\approx}{T}n(\omega)$ is the transfer function which relates the shifted input spectrum at $n\omega_s$ to the output. It is important to point out that $\underset{\approx}{T}n(\omega)$ in general consists of a constant component due to feedthrough and a component whose envelope's magnitude is inversely proportional to the frequency.

When the input is a cisoidal function

$$\underset{\sim}{s}(t)=e^{j\omega_o t}\hat{\underset{\sim}{s}}$$

where $\hat{\underset{\sim}{s}}$ is a complex vector, then

$$\underset{\sim}{S}(\omega)=2\pi\delta(\omega-\omega_s)\hat{\underset{\sim}{s}}.$$

Thus the transfer function in (VI.16) should be evaluated at $\omega=\omega_o+n\omega_s$ i.e., $\underset{\approx}{T}n(\omega)=\underset{\approx}{T}n(\omega_o+n\omega_s)$

Since an arbitrary input can be viewed as a continuum of cisoidal components in the frequency domain, using the superposition prenciple, the result for a cisoidal input can also be extended to an arbitrary input.

Fifty-Percent Duty Cycle

When a switched capacitor circuit is operated at 50-percent duty cycle more interesting results can be derived from the general expressions obtained earlier. With $\tau_1=\tau_2=T/2$ from (VI.16.b) and (VI.15)

$$\underset{\approx}{T}_n(\omega)=\frac{1-e^{-j\omega T/2}}{j\omega T}\left[\underset{\approx}{\hat{C}}_1\underset{\approx}{P}_{1,n}(\omega)+\underset{\approx}{\hat{C}}_2\underset{\approx}{P}_{2,n}(\omega)\right]+\frac{1-(-1)^n}{jn2\pi}(\underset{\approx}{\hat{D}}_1-\underset{\approx}{\hat{D}}_2),\ n\neq 0 \quad \text{(VI.17.a)}$$

$$\underset{\approx}{T}_o(\omega)=\frac{1-e^{-j\omega T/2}}{j\omega T}\left[\underset{\approx}{\hat{C}}_1\underset{\approx}{P}_{1,0}(\omega)+\underset{\approx}{\hat{C}}_2\underset{\approx}{P}_{2,0}(\omega)\right]+\frac{1}{2}(\underset{\approx}{\hat{D}}_1+\underset{\approx}{\hat{D}}_2) \quad \text{(VI.17.b)}$$

where

$$\underset{\approx}{P}_{1,n}(\omega)=\underset{\approx}{\bar{\phi}}_1(e^{j\omega T})\left[(-1)^n\underset{\approx}{F}_2\underset{\approx}{G}_1e^{j\omega T/2}+\underset{\approx}{G}_2e^{j\omega T}\right] \quad \text{(VI.17.c)}$$

$$\underset{\approx}{P}_{2,n}(\omega)=\underset{\approx}{\bar{\phi}}_2(e^{j\omega T})\left[\underset{\approx}{F}_1\underset{\approx}{G}_2e^{j\omega T/2}+(-1)^n\underset{\approx}{G}_1e^{j\omega T}\right] \quad \text{(VI.18)}$$

Notice that the $\underset{\approx}{P}_{m,n}(\omega)$'s given above are rational functions of $e^{j\omega T/2}$. If there is no feedthrough from the input to the output, i.e., $\underset{\approx}{\hat{D}}_1=\underset{\approx}{\hat{D}}_2=0$, (VI.17.a) and (VI.17.b) reduce to

$$\underset{\approx}{T}_n(\omega)=\frac{1-e^{-j\omega T/2}}{j\omega T}\,\underset{\approx}{\hat{T}}_n(z) \quad \text{(VI.19.a)}$$

$\underset{\approx}{\hat{T}}_n(z)$=a rational function of $z^{1/2}$=

$$=\underset{\approx}{\hat{C}}_1\underset{\approx}{\bar{\phi}}_1(z)\left[(-1)^n\underset{\approx}{F}_2\underset{\approx}{G}_1z^{1/2}+\underset{\approx}{G}_2z\right]+\underset{\approx}{\hat{C}}_2\underset{\approx}{\bar{\phi}}_2(z)\left[\underset{\approx}{F}_1\underset{\approx}{G}_2z^{1/2}+(-1)^n\underset{\approx}{G}_1z\right] \quad \text{(VI.19.b)}$$

The factor $(1-e^{j\omega T/2})/j\omega T$ in (VI.19.a) is a zero-order hold function of magnitude 1/T and duration T/2. In many practical applications this factor is nearly a constant of 1/2 when the switching frequency is much higher than the signal frequency. Consequently the behaviour of a switched capacitor circuit can be conveniently analyzed in the rational $z^{1/2}$ domain.

If in addition the input is sampled only once over a switching period T, say $\underset{\approx}{G}_1=\underset{\approx}{0}$, the $\underset{\approx}{\hat{T}}_n(z)$ in (VI.19.b) by using the relations $\hat{C}_m = C_m F_m$ and $\hat{D}_m = C_m G_m + D_m$ obtained in page 23,

$$\hat{T}_n(z) = (C_1 + z^{-1/2} C_2 F_2) \phi_2(z) F_1 G_2 z \qquad \text{(VI.20.a)}$$

or when $G_2 = 0$

$$\hat{T}_n(z) = (-1)^n (z^{-1/2} C_1 F_1 + C_2) \phi_1(z) F_2 G_1 z \qquad \text{(VI.20.b)}$$

It is interesting to note that the $\hat{T}_n(z)$ in (VI.20.a) is identical for all n and the $\hat{T}_n(z)$ in (VI.20.b) differs only in sign for differs only in sign for different n since the relation (VI.10.b) holds.

The $\hat{T}_n(z)$ in (VI.20.a) or (VI.20.b) is a rational function of $z^{1/2}$. The topological constraint $C_1 = C_2 F_2$ in (VI.20.a) or $C_1 F_1 = C_2$ in (VI.20.b) eliminates the dependence on $z^{1/2}$. Under these constraints the factor $(1+z^{-1/2})$ in (VI.20.a) or (VI.20.b) can be combined with the zero-order hold function $(1-e^{-j\omega T/2})/j\omega T$ when (VI.20.a) or (VI.20.b) is substituted into (VI.19.a).

Thus we have

$$T_n(\omega) = \frac{1-e^{-j\omega T}}{j\omega T} C_1 \phi_2(z) F_1 G_2 z \qquad \text{(VI.21.a)}$$

or

$$T_n(\omega) = (-1)^n \frac{1-e^{-j\omega T}}{j\omega T} C_2 \phi_1(z) F_2 G_1 z \qquad \text{(VI.21.b)}$$

The factor $1-e^{-j\omega T}/j\omega T$ which can also be expressed as $e^{-j\omega T/2}\left(\frac{\sin\omega T/2}{T/2}\right)$ in the above two equations is a zero-order hold function of magnitude 1/T and duration T. It reduces to unity when the switching frequency ω_s is much higher than the signal frequency (i.e., $\omega T \ll 1$). Under this condition the transfer function $T_n(\omega)$ can be approximated and conveniently analyzed in the rational z domain.

<u>Example VI.1:</u> Consider the circuit of Fig.II.8(a). In this example, the VCVS is taken as an operational amplifier. Therefore, node 3 is thought as virtval ground and the input charge of the op-amp is zero. Then Eqs.(II.75) and (II.82) become

$$\underset{\sim}{x}_1(t)=\begin{bmatrix} v_{c1}(t) \\ \\ v_{c2}(t) \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_{c1}(AT^-) \\ \\ v_{c2}(nT^-) \end{bmatrix}+\begin{bmatrix} 1 \\ \\ 0 \end{bmatrix}u(t) \text{ for } t\epsilon(nT,nT+\tau_1] \quad (VI.22.a)$$

$$\underset{\sim}{x}_2(t)=\begin{bmatrix} v_{c1}(t) \\ \\ v_{c2}(t) \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ \\ \frac{C_1}{C_2} & 1 \end{bmatrix}\begin{bmatrix} v_{c1}(t^-_{nT+\tau_1}) \\ \\ v_{c2}(t^-_{nT+\tau_1}) \end{bmatrix}+\begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}u(t) \text{ for } t\epsilon(nT+\tau_1,(n+1)T] \quad (VI.22.b)$$

where n=0,1,2,....

Comparison of Eqs.(VI.1) and (VI.22) yields

$$\underset{\approx}{F}_1=\begin{bmatrix} 0 & 0 \\ \\ 0 & 1 \end{bmatrix} \qquad \underset{\sim}{G}_1=\begin{bmatrix} 1 \\ \\ 0 \end{bmatrix}$$

$$\underset{\approx}{F}_2=\begin{bmatrix} 0 & 0 \\ \\ \frac{C_1}{C_2} & 1 \end{bmatrix} \qquad \underset{\sim}{G}_2=\begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \quad (VI.23)$$

The output equations are

$$\underset{\sim}{y}_1(t)=\begin{bmatrix} 0 & -1 \end{bmatrix}\begin{bmatrix} v_{c1}(t) \\ v_{c2}(t) \end{bmatrix}+\underset{\sim}{0}\ V_{in}(t)$$

$$\underset{\sim}{y}_2(t)=\begin{bmatrix} 0 & -1 \end{bmatrix}\begin{bmatrix} v_{c1}(t) \\ \\ v_{c2}(t) \end{bmatrix}+\underset{\sim}{0}\ V_{in}(t) \quad (VI.24)$$

Comparing Eq.(VI.24) with Eq.(II.36),

$$\underset{\sim}{C}_1=\underset{\sim}{C}_2=\begin{bmatrix} 0 & -1 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{D}_1=\underset{\sim}{D}_2=\underset{\sim}{0}$$

is obtained.

Then

$$\hat{C}_1 = C_1 F_1 = [0 \quad -1]$$

$$\hat{C}_2 = C_2 F_2 = \left[-\frac{C_1}{C_2} \quad -1\right] \qquad \text{(VI.25)}$$

$$\hat{D}_1 = C_1 G_1 + D_1 = 0$$

$$\hat{D}_2 = C_2 G_2 + D_2 = 0$$

From Eq.(VI.10.a)

$$\phi_1(z) = [zI - F_2 F_1]^{-1} = \left[\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]^{-1} = \begin{bmatrix} \frac{1}{z} & 0 \\ 0 & \frac{1}{z-1} \end{bmatrix} \qquad \text{(VI.26)}$$

If the switches are operated at 50-percent duty cycle and since $G_2 = 0$ then Eq.(VI.20.b) holds. Substitution of Eqs. (VI.23), (VI.25) and (VI.26) into Eq.(VI.20.b) gives

$$T_n(z) = (-1)^n (z^{-1/2} [0 \quad -1] + [0 \quad -1]) \begin{bmatrix} z^{-1} & 0 \\ 0 & \frac{1}{z-1} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{C_1}{C_2} z \end{bmatrix} =$$

$$= (-1)^n \frac{(C_1/C_2) z (1 + z^{-1/2})}{(1-z)} \qquad \text{(VI.27)}$$

Since there is no feedthrough from the input to the output, i.e., $\hat{D}_1 = \hat{D}_2 = 0$, Eq.(VI.27) can be substituted into Eq.(VI.19.a)

$$T_n(\omega) = \frac{1 - z^{-1/2}}{j\omega T} \cdot (-1)^n \cdot \frac{z (C_1/C_2)(1 + z^{-1/2})}{(1-z)} =$$

$$= \frac{(-1)^{n+1} (C_1/C_2)}{j\omega T} \quad \text{for all } n. \qquad \text{(VI.28)}$$

Thus if we make $C_1 = T/r_1$ the switched capacitor network of Fig.II.8 gives exactly the same frequency response as an analog integrator.

CHAPTER VII

CONCLUSION

As stated in chapter I, filters can be realized by using switched capacitor circuits. Analysis of switched capacitor networks is worthy of investigation since the realization of the filter on a chip can be done.

In chapter II, several time-domain methods for the analysis of SC networks have been given. Among these the approach in section II.5 is more efficient for computer implementation and does not require many extra computations as the others do. And the most important advantage of this approach is that any SC network defined by Eq.(II.14) is completely characterized by the z-domain transfer matrix of Eq.(IV.12). In this case equations for the N phases are put in one large matrix and problems of describing the relationships among different phases of the input and the output are avoided. In section II.5, equivalent circuits for the basic elements of SC networks have been derived. But when there are more than two phases, the circulator explained in Chapter IV is more convenient since z^{-1} is only used between the first and last phase instead of between any two consecutive phases.

The hybrid matrix approach described in section II.8 will yield a proper port combination such that the network has a solution.

As explained in Chapter III, discrete time effects on the values at the end of the time slots as well as continuous time effects (continuous coupling) both occur in a SC network is described by linear equations which are periodically time varying. The discrete time action is linear, periodic and dynamic and each phase at the input has an effect to each phase at the output. This effect is completely characterized by the z-domain transfer matrix of (IV.5). The continuous time action (continuous I-O coupling) is linear, periodic and not dynamic.

In chapter VI, frequency domain solution of two-phase SC network due to an arbitrary input were given. The results discussed in Chapter VI can be extended to sample-and hold inputs as well.

For some SC networks the transmission matrix cannot be established. A plausible physical explanation for this is related to the fact that at even and odd times some of the switches in the network are open, thus, no continuous transmission path exsits. One can overcome this problem, by introducing the parasitic leakage capacitors of the open switches. These parasitic leakage capacitors are generally small. The important point is, that by this "practical trick" the transmission matrix can now be derived.

APPENDIX

Design of IIR, Digital Filters

Digital filters characterized by transfer functions in the form of a rational function,

$$H(z)=\frac{\sum_{i=0}^{M} a_i z^{-i}}{\sum_{k=0}^{N} b_k z^{-k}} \triangleq \frac{A(z^{-1})}{B(z^{-1})} \qquad (A.1) \quad M \leq N$$

where $B(z^{-1})$ is not constant, are called infinite impulse response (IIR) digital filters. In the IIR filters, the filter is stable if all poles of H(z) of (1) are within the unit circle in the z-plane and causal if b_L is the first nonzero coefficient in the denominator (i.e., $b_o=b_1=\ldots b_{L-1}=0$), then $a_o=a_1=\ldots=a_{L-1}=0$ in the numerator. Because, we are concerned with causal filters (impulse response $h(n)=0$, $n<0$) only, it is convenient to assume that $b_o=1$, Hence, the general transfer functions of IIR digital filters are in the form of

$$H(z)=\frac{\sum_{i=0}^{M} a_i z^{-i}}{1+\sum_{k=1}^{N} b_k z^{-k}} \qquad (A.2)$$

The design of an IIR digital filter involves the following two steps:

Step 1: Design an analog filter by obtaining an appropriate transfer function $\hat{H}(s)$ to meet the signal-processing requirements.

Step 2: Construct a mapping procedure to transform $\hat{H}(s)$ into an appropriate transfer function H(z), thus resulting in an IIR digital filter design that will meet the specifications.

These steps are illustrated in Fig.A.1.

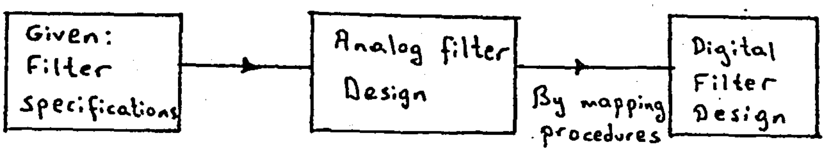


Fig.A.1.

Since IIR filters can be realized by SC networks all the discussions given for IIR filters are valid for SC networks.

When we insert e^{ST} for z in the transfer function of the digital filter, we obtain a transfer function which is a function of e^{ST}. Since an analog filter has a transfer function which is a polynomial in S not in e^{ST}, we can not realize it. Therefore we have to make some approximations for e^{ST} try to retain frequency and stability properties.

Because the analog filters in Step 1 are designed to meet the signal processing requirements, we must make sure that the resulting digital filters retain the desirable properties of the analog filters, including the frequency characteristics, the magnitude and phase behaviour of the analog filters. As a consequence, it is desirable that the imaginary axis of the s-plane ($s=j\omega$ for $-\infty<\omega<\infty$) is mapped onto the unit circle of the z-plane ($z=e^{j\theta}$ for $-\pi<\theta\leq\pi$ where θ is the digital frequency variable in rad). This condition is needed to preserve the frequency characteristics of the analog filters.

In order to preserve stability properties of analog filters, the left hang s-plane ($Re[s]<0$) is mapped into the unit circle of the z-plane ($|z|<1$).

One method to obtain a digital filter design from an analog filter design is via numerical integration techniques, where a derivative is approximated by some finite difference. This action gives rise to a mapping of the complex variable s in the transfer function of an analog filter to the complex variable z in the transfer function of a digital filter

$$s=f(z) \qquad (A.3)$$

Clearly, different numerical integration techniques will give rise to different mapping functions of (3), and, hence the resulting digital filters will be different.

The Euler approximation which is given on page 59 satisfies the stability condition but the frequency characteristics is not satisfied completely. However, for sufficiently small ωT, this mapping will give satisfactory results (i.e., for low-frequency operations and low-pass filters).

Another transformation is called the bilinear transformation defined by

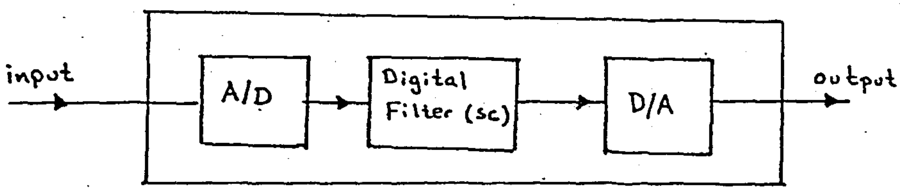
$$s=f(z)= \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}}$$

or

$$z^{-1}= \frac{2-sT}{2+sT} \qquad (A.4)$$

The frequency requirements are satisfied by the bilinear transformation. Also, the stability properties of an analog filter is preserved by this mapping. However, the frequency charactesitics of the digital filters and that of an analog filter are not identical only the shapes are identical. For detailed explainitions see the Chapter 12 of [12] .

After designing digital filter as explained above an analog filter can be obtained as in Fig.(A.2).



An analog.filter.

Fig.A.2.

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