LAMINAR FORCE CONVECTION HEAT TRANSFER IN SOME NONCIRCULAR PIPES

by

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ABSTRACT

In this study, solutions to forced convection heat transfer problems in fully developed laminar flow are obtained for Newtonian fluids with constant properties. Internal flow for square, triangular and elliptical cross-sections have been solved. In addition, an approximate solution is presented for conduits with internal flow given by $x^4 + y^4 = a^4$.
To solve the aformentioned problems, Complex Variable Techniques, Biharmonic Solutions, Variational and Finite Element Methods have been used.
Biharmonic Solutions are directly applied to square and triangular pipes using the available solutions in the plate theory.
The Variational formulation of the governing equations are obtained. Based on this formulation velocity and temperature distributions are found in square pipes using the Ritz Method. Variational formulation is further used in corporation with the Finite Element Technique to determine approximate solutions for noncircular pipes.
The Complex Variable Method is very suitable when applied to parallel plates, circular, triangular and elliptical pipes. It also gives considerable knowledge of heat transfer for the cross-section given by $x^4 + y^4 = a^4$.
Theoretical solutions for each of these geometries are then compared numerically by Finite Element Method.

ÖZET

Bu çalışmada, üniform ve laminer akışlar için zorlanmış ısı taşınımı problemleri, kare, eşkenar üçgen ve eliptik kesitli borular için çözülmüştür. Buna ek olarak kesiti $x^4+y^4=a^4$ olarak belirlenen borular için yaklaşık bir çözüm getirilmiştir.
Yukarıda sözü edilen problemlerin çözümünde Kompleks Değişkenler Metodu, Biharmonik Çözümler, Varyasyonal ve Sonlu Elemanlar Metodları kullanılmıştır.
Biharmonik Çözümler, ince plakalar teorisindeki sonuçlar kullanılmak suretiyle kare ve eşkenar üçgen borulara direkt olarak uygulanmıştır.
Problemi genel olarak tanımlayan ana differansiyel denklemlerin Varyasyonal formülasyonu yapılmış ve buna bağlı olarak kare borular için hız ve sıcaklık dağılımları Ritz Metodu kullanılarak bulunmuştur.
Kompleks Değişkenler Metodu paralel düzlemlere, dairesel, eşkenar üçgen ve eliptik kesitli borulara uygulanabilmiştir. Aynı zamanda bu metodla kesiti $x^4+y^4= a^4$ olarak tariflenen borular için yaklaşık sonuçlar elde edilmiştir.
Bulunan teorik çözümler daha sonra Sonlu Elemanlar Metodu ile elde edilen nümerik neticelerle karşılaştırılmıştır.

NOMENCLATURE

A	Cross-sectional area of the pipe
a	Side dimension, semimajor axis of an ellipse
b	Semiminor axis of an ellipse
$[B]$	Gradient matrix
c_p	Specific heat at constant pressure
C_1	A constant, $\frac{1}{\mu}\frac{dP}{dz}$
C_2	A constant, $\frac{\rho c_p}{k}\ \frac{dT}{dz}$
D_h	Hydraulic diameter, $\frac{4A}{P}$
D_f	Flexural rigidity
D	Domain cross-section
E	Elliptical integral of the second kind
e	Internal energy per unit volume
f_j	Components of body forces per unit volume
$F(z)$	Goursat function
$\overline{F}(\overline{z})$	Complex conjugate of $F(z)$
$\{f^e\}$	Element force matrix
$\{F\}$	System force matrix
G_M, G_E	Integrand of the variational integral Corresponding to momentum and energy equations
h	Heat transfer coefficient
I	Variational integral

I_M, I_E	Variational integral corresponding to momentum and energy equations
k	Thermal conductivity
$[k^e]$	Element stiffness matrix
$[K]$	System stiffness matrix
L_1, L_2, L_3	Natural coordinates of a triangular element
N_i, N_j, N_k	Shape functions for velocity and temperature
Nu	Nusselt number, $\frac{hD_h}{k}$
p	Pressure
P	Perimeter of the pipe
q	Heat transfer rate
q_0	Load intensity
s	Aspect ratio of an ellipse, $\frac{b}{a}$
t	Time
T	Temperature
T_w	Wall temperature
T_b	Bulk Temperature
$\{T\}$	Column matrix which gives the nodal values of the approximating polynomial in equation(VI.5)
u	Axial velocity
u_m	Axial avarage velocity
$\{V\}$	Column matrix which gives the nodal values of the approximating polynomial in equation(VI.4)
ω	Deflection of a plate

$X(z)$	Goursat function
$\overline{X}(\overline{z})$	Complex conjugate of $X(z)$
x, y	Cartesian coordinates
X, Y	Dimensionless coordinates, $\frac{x}{a}$, $\frac{y}{a}$
z	Axial coordinate; complex variable, $x + iy$
$\overline{z}$	Complex conjugate of z, $x - iy$
Θ	Temperature difference between fluid and wall, $T - T_w$
Θ_b	Difference between fluid bulk temperature and wall, $T_b - T_w$
Φ	Dimensionless velocity, $-\frac{\mu}{\frac{dp}{dz}a^2}u$
Ψ	Dimensionless temperature, $\frac{k\mu}{\rho c_p \frac{dT}{dz}\frac{dP}{dz}a^4}\Theta$
α	Dimensionless temperature, $\frac{8k}{q}\Theta$
α_b	Dimensionless bulk temperature, $\frac{8k}{q}\Theta_b$
ρ	Fluid density
μ	Viscosity
λ	Bulk Viscosity
∇^2	Laplacian operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
β	Beta-function
Γ	Gamma function

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I

INTRODUCTION

Forced convection heat transfer problems involving fully developed laminar flow under the conditions of constant heat flux in axial direction and uniform peripheral wall temperature have important applications in compact heat exchangers where design considerations may dictate pipes with unconventional shapes. Generally, one way to reduce heat exchanger costs is to select suitable cross-sections leading to lighter weight and higher heat transfer coefficients. It follows then that the laminar flow solutions obtained here for various cross-sections become important.

For compact heat exchangers the flow passages must have a small hydraulic radius. For the low Reynolds number design range of such heat exchangers fully developed laminar flow may prevail along most of the flow length. Thus hydraudynamically and thermally fully developed laminar flow(1)* solutions are needed.

The determination of such solutions for noncircular pipe geometries, (Fig.I.1), is the subject matter of this study. Constant property, fully developed laminar forced convection heat transfer is considered under constant heat flux and peripherally constant wall temperature boundary conditions.

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*Numbers given in brackets refer to references in text

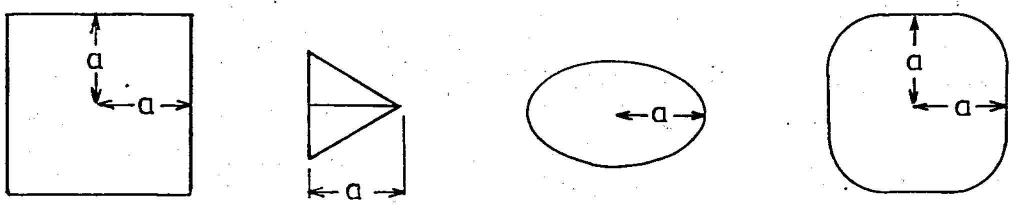

Figure I.1-Noncircular pipe geometries,
(a) Square, (b) equilateral triangular,
(c) Elliptical, (d) Shape given by $x^4 + y^4 = a^4$

A survey of the available literature shows that although considerable information exists on this problem, analytical solutions have been confined to relatively simple shapes such as circular pipes and parallel plates (2,3,4). More complicated geometries such as the square, triangular and elliptical pipes require two dimensional analysis. Profiles of flow velocity in these pipes are available in many Fluid Mechanics books (5,6,7). Existing solutions to torsional and thin plate problems (8) of elasticity can be used for square and equilateral triangular pipes due to similarity of the governing equations. The utility of Complex Variable and Variational Methods as a tool for the solutions of Poisson type equations in the theory of elasticity (9,10) is well established. The numerical solutions have been mainly obtained by the Finite Difference technique (11) for square and equilateral triangular pipes. Further results concerning these geometries can be found in(12). Solutions regarding the geometry described by $x^4 + y^4 = a^4$ has not been analyzed.

The study comprises

a) Mathematical formulation of the problem presented in Chapter II.
b) Theoretical solutions of the governing equations using the Biharmonic Solutions, Variational and Complex Variable Methods given in Chapters III, IV, V respectively.
c) Numerical solutions by Finite Element Method discussed in Chapter VI.

FORMULATION

A. Assumptions and the Governing Equations

The equations governing the motion of a Newtonian fluid are Continuity, Navier Stokes, Energy equations and the equations of state. Using indicial rotation these equations are expressed as follows:

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_k)}{\partial x_k} = 0 \qquad \text{(II.1)}$$

Navier Stokes Equation

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = -\frac{\partial P}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\left\{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\} + \rho f_j \qquad \text{(II.2)}$$

Energy Equation

$$\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = -p \frac{\partial u_k}{\partial x_k} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right) + \lambda \left(\frac{\partial u_k}{\partial x_k}\right)^2$$

$$+ \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \frac{\partial u_j}{\partial x_i} \qquad \text{(II.3)}$$

where $i,j,k = 1,2,3$

State Equations

$$P = P(\rho, T) \qquad \text{(II.4)}$$

$$e = e(\rho, T) \qquad \text{(II.5)}$$

In equation (II.1) ρ is the density, t is the time and u is the velocity.
In equation (II.2) P is the pressure, λ is the bulk viscosity, μ is the viscosity and f_j is the components of the body forces per unit volume.
In equation (II.3) e is internal energy per unit volume, T is the temperature and k is the thermal conductivity.
In the most frequently encountered situations, formulation of the problem is based on the following assumptions:

1) Flow is time independent.
2) The fluid is assumed to be viscous and its properties are constant.
3) Viscous dissipation is neglected.
4) Body forces and internal heat generation are not present.
5) Fully developed velocity profile is assumed.
6) Fully developed temperature profile is assumed.

Under these conditions, the governing equations in Cartesian Coordinates for the flow configuration shown in Figure II.1 become

$$\nabla^2 u = \frac{1}{\mu}\frac{dP}{dz} = c_1 \qquad \text{(II.6)}$$

$$\nabla^2 T = \frac{\rho c_p}{k}\frac{\partial T}{\partial z}\, u \qquad \text{(II.7)}$$

For thermally fully developed flow R.A. Seban(1) has shown that

$$\frac{\partial}{\partial z}\left(\frac{Tw-T}{Tw-Tb}\right) = 0$$

Then

$$\frac{\partial T}{\partial z} = \frac{dTw}{dz} - \frac{Tw-T}{Tw-Tb}\frac{dTw}{dz} + \frac{Tw-T}{Tw-Tb}\frac{dTb}{dz} \qquad \text{(II.8)}$$

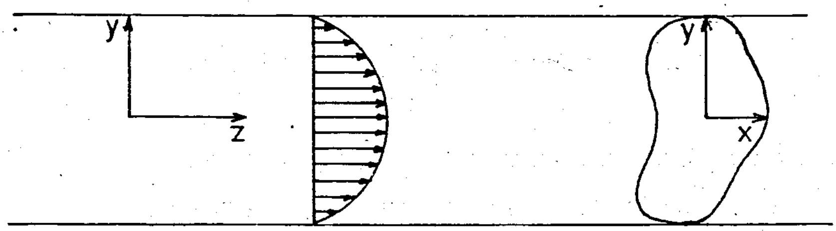

Figure II.1- Flow configuration

B. Boundary Conditions

(a) No-slip condition of the wall gives $u=o$ at the wall.

(b) In the case of constant heat flux at the wall, $q=hP(Tw-Tb) = C$ where C is constant. Since h is constant

$$\frac{dTw}{dz} - \frac{dTb}{dz} = 0$$

Thus, from equation (II.8), it follows that

$$\frac{\partial T}{\partial z} = \frac{dTw}{dz} = \frac{dTb}{dz}$$

(c) The condition of peripherally constant wall temperature gives

$T=Tw$ or $\Theta = 0$ at the wall,

where $\Theta=T-Tw$

Now summarizing the governing equations and the boundary conditions one gets

$$\nabla^2 u = C_1 \quad \text{(II.6)}$$

$$\nabla^2 \Theta = C_2 u \quad \text{(II.9)}$$

$$u = 0 \quad \text{(II.6a)}$$

and

$$\Theta = 0 \quad \text{(II.9a)}$$

at the wall.

In equation (II.6)

$$C_1 = \frac{1}{\mu} \frac{dP}{dz}$$

and in equation (II.9)

$$C_2 = \frac{\rho c_p}{k} \frac{dTb}{dz}$$

C. Determination of the Nusselt Number

Once the velocity and the temperature distribution is known, Nusselt number is calculated from its definition, that is

$$Nu = \frac{hD_h}{k} \qquad \text{(II.10)}$$

where

$$D_h = \frac{4A}{P} \qquad \text{(II.11)}$$

is the hydraulic diameter.
In equation (II.11) A is the cross-sectional area and P is the perimeter.
The heat transfer coefficient h may be evaluated from the consideration of heat balance as shown in fig.II.2

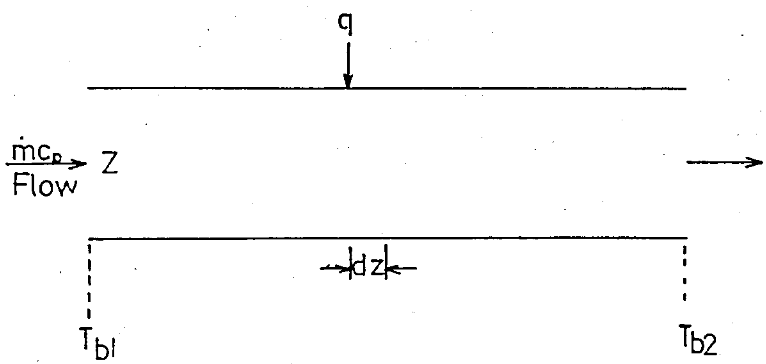

Figure II.2-Description of Heat Balance

In some differential length dz, the heat added dq can be expressed either in terms of a bulk temperature difference $(T_{b2} - T_{b1})$ or in terms of the heat transfer coefficient such that

$$dq = \dot{m}c_p dT_b = hPdz\,(T_w - T_b)$$

Noting that $\dot{m} = \rho u_m A$,

where $$u_m = \frac{1}{A} \iint_D u\,dx\,dy \qquad \text{(II.12)}$$

is the avarage velocity and

$$\Theta_b = T_b - T_w = \frac{1}{u_m A} \iint_D u\Theta\,dx\,dy \; ; \qquad \text{(II.13)}$$

is the bulk temperature, one gets

$$h = - \frac{AC_2 k u_m}{P\Theta_b} \qquad \text{(II.14)}$$

and

$$q = C_2 k u_m A \qquad \text{(II.14a)}$$

Now combining equations (II.10),(II.11)and(II.14) results in

$$Nu = -4\,\frac{A^2}{P^2} C_2 \frac{u_m}{\Theta_b} \qquad \text{(II.15)}$$

III

USE OF THE BIHARMONIC SOLUTIONS

Considering equations (II.6) and (II.9) one can eliminate u_i such that

$$\nabla^4 \Theta = C_1 C_2 \tag{III.1}$$

The corresponding boundary conditions are

$$\Theta = 0$$

and

$$\nabla^2 \Theta = 0$$

at the wall.

Equation(III.1) is similar to the governing equation for the small deflection theory of thin plates subjected to uniform load and simply supported along all edges(8). That is

$$\nabla^4 \omega = \frac{q_0}{D_f} \tag{III.2}$$

Where q_o is the load intensity, D_f is the flexural rigidity and ω is the deflection.

Now if the boundary conditions which must be satisfied by the solution of equation (III.1) are also to be satisfied by the solution of equation(III.2), then equations(III.1) and (III.2) are identical. For a simply supported plate (supported along all edges) of polygonal shape one has

$$\omega = 0$$

and

$$\nabla^2 \omega = 0$$

at the wall.

Therefore equations (III.1) and (III.2) can be made identical if ω is replaced by Θ and $\frac{q_o}{D_f}$ by C_1C_2. Above results will be applied to square and equilateral triangular pipes.

A. Pipes with Square Cross-section

For uniformly loaded square plate shown in figure III.1, deflection surface is given (8) as

$$\omega = \frac{16q_o a^4}{\pi^6 D_f} \sum_{m=1,3\ldots}^{\infty} \sum_{n=1,3\ldots}^{\infty} \frac{\sin\frac{m\pi}{a}x \ \sin\frac{n\pi}{a}y}{mn(m^2+n^2)^2} \tag{III.3}$$

Now substituting Θ for ω and C_1C_2 for $\frac{q_o}{D_f}$ one gets the corresponding solution for Θ. That is

$$\Theta = \frac{16C_1C_2a^4}{\pi^6} \sum_{m=1,3\ldots}^{\infty} \sum_{n=1,3\ldots}^{\infty} \frac{\sin\frac{m\pi}{a}x \ \sin\frac{n\pi}{a}y}{mn(m^2+n^2)^2} \tag{III.4}$$

To determine the velocity, equation (III.4) is differentiated such that

$$u = \frac{1}{C_2}\nabla^2\Theta = -\frac{16C_1a^2}{\pi^4} \sum_{m=1,3\ldots}^{\infty} \sum_{n=1,3\ldots}^{\infty} \frac{\sin\frac{m\pi}{a}x \ \sin\frac{n\pi}{a}y}{mn(m^2+n^2)} \tag{III.5}$$

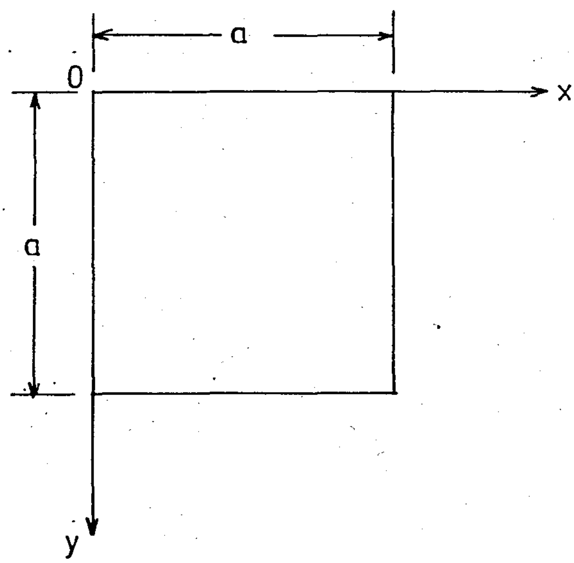

Figure III.1-Coordinate System of the Square Pipe

Now substituting equation (III.5) in equation (II.12) one gets the average velocity

$$u_m = \frac{16C_1}{\pi^4}\int_0^a\int_0^a \sum_{m=1,3\ldots}^{\infty}\sum_{n=1,3\ldots}^{\infty} \frac{\sin\frac{m\pi}{a}x \;\sin\frac{n\pi}{a}y}{mn(m^2+n^2)^2}\,dxdy$$

Integrating gives

$$u_m = -\frac{64C_1a^2}{\pi^6}\sum_{m=1,3\ldots}^{\infty}\sum_{n=1,3\ldots}^{\infty}\frac{1}{m^2n^2(m^2+n^2)} \qquad \text{(III.6)}$$

Also substituting equation (III.4) in equation(II.13), the bulk temperature is obtained

$$\Theta_b = \frac{4a^2C_1C_2}{\pi^4}\;\frac{1}{\sum\limits_{m=1,3\ldots}^{\infty}\sum\limits_{n=1,3\ldots}^{\infty}\frac{1}{m^2n^2(m^2+n^2)}}$$

$$\int_0^a\int_0^a \sum_{m=1,3\ldots}^{\infty}\sum_{n=1,3\ldots}^{\infty}\frac{\sin^2\frac{m\pi}{a}x \quad \sin^2\frac{n\pi}{a}y}{m^2n^2(m^2+n^2)^3}\,dxdy$$

Performing the integration with

$$\int_0^a \sin^2\frac{m\pi}{a}x\,dx = \frac{1}{2}a$$

$$\Theta_b = \frac{C_1C_2a^4}{\pi^4}\;\frac{\sum\limits_{m=1,3\ldots}^{\infty}\sum\limits_{n=1,3\ldots}^{\infty}\frac{1}{m^2n^2(m^2+n^2)^3}}{\sum\limits_{m=1,3\ldots}^{\infty}\sum\limits_{n=1,3\ldots}^{\infty}\frac{1}{m^2n^2(m^2+n^2)}} \qquad \text{(III.7)}$$

Finally substituting equations(III.6),(III.7) with $P = 4a$ in equation (II.15) and rearranging gives the Nusselt number as

$$Nu=\frac{16}{\pi^2}\frac{\left\{\sum_{m=1,3...}^{\infty}\sum_{n=1,3...}^{\infty}\frac{1}{m^2n^2(m^2+n^2)}\right\}^2}{\left\{\sum_{m=1,3...}^{\infty}\sum_{n=1,3...}^{\infty}\frac{1}{m^2n^2(m^2+n^2)^3}\right\}} \quad \text{(III.8)}$$

Since the series converge quite rapidly, few terms are sufficient. Performing the summation for $m=1,3,5,7,9$ and $n=1,3,5,7,9$ results in $Nu=3.600$.

B. Pipes with Equilateral Triangular Cross-section

The equations of the boundary corresponding to this pipe geometry shown in figure III.2 are

$$x+\frac{a}{3}=0$$

$$\frac{x}{\sqrt{3}}+y-\frac{2a}{3\sqrt{3}}=0$$

$$\frac{x}{\sqrt{3}}-y-\frac{2a}{3\sqrt{3}}=0$$

Multiplying these equations and rearranging gives

$$x^3-3xy^2-a(x^2+y^2)+\frac{4}{27}a^3=0 \quad \text{(III.9)}$$

For uniformly loaded equilateral triangular plate, deflection surface is(8)

$$\omega=\frac{q_0}{64\ aD_f}\left[x^3-3xy^2-a(x^2+y^2)+\frac{4}{27}a^3\right]\left[\frac{4}{9}a^2-x^2-y^2\right] \quad \text{(III.10)}$$

Multiplying out and differentiating yields

$$\nabla^2\omega=-\frac{q_0}{4aD_f}\left[x^3-a(x^2+y^2)-3xy^2+\frac{4}{27}a^3\right] \quad \text{(III.11)}$$

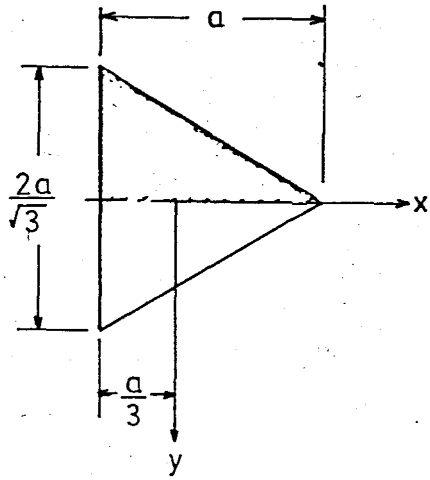

Figure III.2-Coordinate System of the Equilateral Triangular Pipe

Noting that equations (III.10) and (III.11) satisfy the boundary conditions one can make equations (III.1) and (III.2) identical by replacing ω by Θ and $\frac{q_0}{D_f}$ by C_1C_2. Then

$$\Theta = \frac{C_1C_2}{64a}\left[x^3 - 3xy^2 - a(x^2+y^2) + \frac{4}{27}a^3\right]\left[\frac{4}{9}a^2 - x^2 - y^2\right] \qquad \text{(III.12)}$$

and

$$u = \frac{1}{C_2}\nabla^2\Theta = -\frac{C_1}{4a}\left[x^3 - a(x^2+y^2) - 3xy^2 + \frac{4}{27}a^3\right] \qquad \text{(III.13)}$$

Now the average velocity and the bulk temperature can be calculated. Substituting equation (III.13), with $A = \frac{a^2}{\sqrt{3}}$ in equation (II.12) gives

$$u_m = -\frac{C_1\sqrt{3}}{2a^3}\int_{-\frac{a}{3}}^{\frac{2a}{3}}\int_0^{\frac{3x-2a}{3\sqrt{3}}}\left[x^3 - a(x^2+y^2) - 3xy^2 + \frac{4}{27}a^3\right]dx\,dy$$

Integrating results in

$$u_m = \frac{C_1 a^2}{60} \qquad \text{(III.14)}$$

Also substituting equation (III.12) in equation (II.13) one has

$$\Theta_b = -\frac{15\sqrt{3}}{32}\,\frac{C_1 C_2}{a^6}\int_{-\frac{a}{3}}^{\frac{2a}{3}}\int_{0}^{\frac{3x-2a}{3\sqrt{3}}}\left[x^3 - a(x^2+y^2) - 3xy^2 + \frac{4}{27}a^3\right]^2 \left[\frac{4}{9}a^2 - x^2 - y^2\right] dydx$$

Carrying out the multiplication and integrating results in

$$\Theta_b = -\frac{C_1 C_2 a^4}{1680} \qquad \text{(III.15)}$$

Finally substituting equations (III.14), (III.15) with $P=\frac{6a}{\sqrt{3}}$ in equation (II.15) gives the Nusselt number as

$$Nu = \frac{28}{9} = 3.11.$$

IV

VARIATIONAL METHOD

A. Description of the Method

Consider a function U which depend upon x and y and may represent the velocity or the temperature. Knowing the differential equation which U must satisfy, then according to the calculus of variations, one must first find a function G $(x,y,U,\frac{\partial U}{\partial x},\frac{\partial U}{\partial y})$ which when inserted into the Euler Equation

$$\frac{\partial}{\partial x}\left[\frac{\partial G}{\partial(\frac{\partial U}{\partial x})}\right]+\frac{\partial}{\partial y}\left[\frac{\partial G}{\partial(\frac{\partial U}{\partial y})}\right]-\frac{\partial G}{\partial U}=0 \quad \text{(IV.1)}$$

yields the differential equation for U. When G is known one can write the variational integral,

$$I=\iint G(x,y,U,\frac{\partial U}{\partial x},\frac{\partial U}{\partial y})\ dA \quad \text{(IV.2)}$$

and minimize it by the Ritz Method. According to this method, U is written as a linear combination of functions each of which satisfies the boundary conditions. With the choice of U, the function G is evaluated and the integration in equation(IV.2) is carried out. The result is an expression containing the constants of the trial function U. To achieve a minimum value of the integral I, this expression is differentiated seperately with respect to each constant and each result is set equal to zero. This provides equations which can be solved for the unknown constants.

Now considering equation(II.6) for the velocity one finds

$$G_M = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 2C_1 u$$

The variational integral corresponding to equation (II.6) is then

$$I_M = \iint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 2C_1 u\right] dx dy \qquad \text{(IV.3)}$$

Considering equation (II.9) for the temperature one finds

$$G_E = \left(\frac{\partial \Theta}{\partial x}\right)^2 + \left(\frac{\partial \Theta}{\partial y}\right)^2 + \frac{2q}{kA}\,\frac{u}{U_m}\Theta \;.$$

with $q = C_1 k A u_m$. The corresponding variational integral is

$$I_E = \iint \left[\left(\frac{\partial \Theta}{\partial x}\right)^2 + \left(\frac{\partial \Theta}{\partial y}\right)^2 + \frac{2q}{kA}\,\frac{u}{U_m}\Theta\right] dx dy \qquad \text{(IV.4)}$$

The boundary conditions that the velocity be zero at the wall and uniform peripheral wall temperature are essential. Therefore the variational integrals (III.3) and (III.4) correspond to the governing equations(II.6) and (II.9) with their associated boundary conditions without any modification.

B. Pipes with Square Cross-section

For the square pipe as shown in figure IV.1, the velocity profile chosen is

$$u = (x^2 - a^2)(y^2 - a^2)[A_0 + A_1(x^2 + y^2) + A_2 x^2 y^2] \qquad \text{(IV.5)}$$

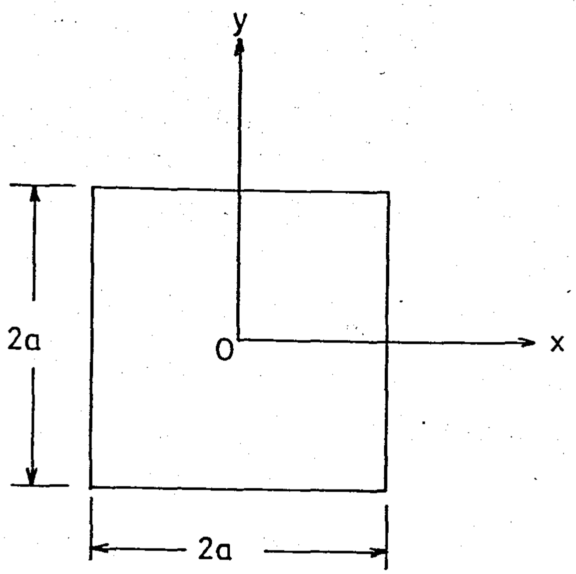

Figure IV.1-Coordinate System of the Square Pipe

Performing the differentiations and substituting equation (IV.5) in equation (IV.3) yields

$$
\begin{aligned}
I_M = 8\int_0^a\int_0^a [\, & 4A_0^2x^2y^4 - 8A_0^2x^2a^2y^2 + 4A_0^2a^4x^2 + 16A_0A_1x^4y^4 + 8A_0A_1x^2y^6 \\
& - 24A_0A_1a^2x^2y^4 - 32A_0A_1a^2x^4y^2 + 24A_0A_1a^4x^2y^2 + 16A_0A_2x^4y^6 \\
& - 32A_0A_2a^2x^4y^4 - 8A_0A_2a^2x^2y^6 + 16A_0A_2a^4x^2y^4 + 16A_0A_1a^4x^4 \\
& - 8A_0A_1a^6x^2 + 16A_0A_2a^4x^4y^2 - 8A_0A_2a^6x^2y^2 + 16A_1^2x^6y^4 + 16A_1^2x^4y^6 \\
& - 48A_1^2a^2x^4y^4 - 32A_1^2a^2x^6y^2 + 48A_1^2a^4x^4y^2 + 32A_1A_2x^6y^6 - 64A_1A_2a^2x^6y^4 \\
& - 64A_1A_2a^2x^4y^6 + 80A_1A_2a^4x^4y^4 + 4A_1^2x^2y^8 - 16A_1^2a^2x^2y^6 + 24A_1^2a^4x^2y^4 \\
& + 16A_1A_2x^4y^8 - 8A_1A_2a^2x^2y^8 + 24A_1A_2a^4x^2y^6 - 16A_1^2a^6x^2y^2 \\
& - 24A_1A_2a^6x^2y^4 + 16A_1^2a^4x^6 - 16A_1^2a^6x^4 + 32A_1A_2a^4x^6y^2 - 32A_1A_2a^6x^4y^2 \\
& + 4A_1^2a^8x^2 + 16A_2^2x^6y^8 - 32A_2^2a^2x^6y^6 + 16A_2^2a^4x^6y^4 + 8A_1A_2a^8x^2y^2 \\
& - 16A_2^2a^2x^4y^8 + 32A_2^2a^4x^4y^6 - 16A_2^2a^6x^4y^4 + 4A_2^2a^4x^2y^8 \\
& - 8A_2^2a^6x^2y^6 + 4A_2^2a^8x^2y^4\,]\,dxdy + 8C_1\int_0^a\int_0^a [A_0(x^2y^2 - a^2x^2 - a^2y^2 \\
& + a^4) + A_1(x^4y^2 + x^2y^4 - 2a^2x^2y^2 - a^2x^4 - a^2y^4 + a^4x^2 + a^4y^2) \\
& + A_2(x^4y^4 - a^2x^4y^2 - a^2x^2y^4 + a^4x^2y^2)\,]\,dxdy
\end{aligned}
$$

Integrating and rearranging gives

$$
\begin{aligned}
\frac{I_M}{8a^6} = \; & 0.71111\, A_0^2a^2 + 0.48762A_0A_1a^4 + 0.0774\, A_1A_2a^8 \\
& + 0.04064A_0A_2a^6 + 0.29799A_1^2a^6 + 0.01064A_2^2a^{10} \\
& + C_1\left(\frac{4}{9}A_0 + \frac{8}{45}A_1^2a + \frac{4}{225}A_2a^4\right) \qquad \text{(IV.6)}
\end{aligned}
$$

To determine A_0, A_1, A_2, equation (IV.6) is differentiated with respect to each A and the resulting equations are set equal to zero. This operation yields

$$1.4222\, A_0a^2 + 0.48762A_1a^4 + 0.040635A_2a^6 + \frac{4}{9}C_1 = 0$$

$$0.48762\, A_0a^2 + 0.59598A_1a^4 + 0.07740A_2a^6 + \frac{8}{45}C_1 = 0$$

$$0.040635A_0a^2 + 0.07740A_1a^4 + 0.021284A_2a^6 + \frac{4}{225}C_1 = 0$$

Solving these equations yields

$$A_0 = -\frac{0.29492}{a^2}C_1$$

$$A_1 = -\frac{0.041015}{a^4}C_1$$

$$A_2 = -\frac{0.12305}{a^6}C_1$$

After substituting these expressions in equation (IV.5), one can calculate the average velocity from equation (II.12) such that

$$u_m = -\frac{C_1}{a^8}\int_0^a\int_0^a [(x^2y^2-a^2x^2-a^2y^2+a^4)0.29492\ a^4+(x^4y^2-a^2x^4$$

$$-2a^2y^2x^2+a^4x^2)\ 0.041015a^2+(x^2y^4-a^2y^4+a^4y^2)0.041015a^2$$

$$+(x^4y^4-a^2x^4y^2-\ a^2x^2y^4+a^4x^2y^2)0.12305]dxdy$$

Carrying out the integration results in

$$u_m = -0.140555C_1a^2 \qquad \text{(IV.7)}$$

Then

$$\frac{U}{U_m} = \frac{1}{0.140555a^8}(x^2-a^2)(y^2-a^2)$$

$$[0.29492a^4+0.041015a^2(x^2+y^2)+0.12305x^2y^2] \qquad \text{(IV.8)}$$

Before proceeding to determine the temperature distribution let

$$\alpha = \frac{\Theta}{q/8k}\ , \quad X=\frac{x}{a}\ , \quad Y=\frac{y}{a}\ .$$

Then equation (IV.4) becomes

$$I_E = 4 \int_0^1\!\!\int_0^1 \left[\left(\frac{\partial\alpha}{\partial X}\right)^2 + \left(\frac{\partial\alpha}{\partial y}\right)^2 + 4\,\frac{u}{u_m}\,\alpha \right] dX\,dY \qquad \text{(IV.9)}$$

Noting that the temperature around the boundary is uniform, the temperature distribution across the cross-section would be expected to have a shape similar to the velocity. Therefore,as an approximation the following temperature distribution is selected.

$$\alpha = (X^2-1)(Y^2-1)[E_0+E_1(X^2+ Y^2)] \qquad \text{(IV.10)}$$

Substituting equations (IV.8), (IV.10) in (IV.9), differentiating and performing the integration results in

$$I_E = \frac{256}{45}\,E_0^2 + \frac{2048}{525}E_0E_1 + \frac{11264}{4725}E_1^2 + 10.01014E_0 + 2.96329E_1 \qquad \text{(IV.11)}$$

Now differentiating equation (IV.11) with respect to each E and setting the resulting equations equal to zero gives

$$\frac{512}{45}E_0 + \frac{2048}{525}E_1 = -10.01014$$

$$\frac{2048}{525}E_0 + \frac{22528}{4725}E_1 = -2.96329$$

Solving these equations yields

$E_0 = \cdot 0.1366$

$E_1 = -0.9267$

Substituting E_0 and E_1 in equation(IV.10) gives

$$\alpha = (X^2-1)(Y^2-1)[-0.9267+0.1366(X^2+Y^2)] \quad \text{(IV.12)}$$

From the usual definition(II.13), the dimensionless bulk temperature may be calculated. Then

$$\alpha_b = \frac{1}{0.140555}\int_0^1\int_0^1 \{(X^2-1)^2(Y^2-1)^2[0.29492+0.041015(X^2+Y^2)+0.12305X^2Y^2][-0.9267+0.1366(X^2+Y^2)]\}dXdY$$

Multiplying out and integrating results in

$$\alpha_b = -0.55501 \quad \text{(IV.13)}$$

Noting that $\alpha_b = \frac{\Theta_b}{\frac{q}{8k}}$ and $q = -hP\Theta_b$. Then

$$h = -\frac{8k}{P\alpha_b}$$

$$Nu = -32\frac{A}{P^2\alpha_b} \quad \text{(IV.14)}$$

Finally substituting equation (IV.13) with $A=4a^2$ and $P=8a$ in equation (IV.14) results in

$$Nu = 3.604 \quad .$$

An exact calculation by Biharmonic solutions gives a value of $Nu = 3.600$ in precise agreement (to three significant figures) with $Nu = 3.604$. Hence there is no need to use more complex approximations for u or for α .

V

COMPLEX VARIABLE METHOD

A. Use of Complex Variables

Complex Variable Method has been used succesfully in the theory of elasticity to solve problems governed by a biharmonic equation(13). The use of this method will reduce equations (II.6)and(II.9) to the determination of some analytic functions which satisfy the boundary conditions.

In terms of the variables $z=x+iy$, $\bar{z}=x-iy$, Laplace operator can be expressed as

$$\nabla^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

Applying this operator to the governing equations (II.6) and (II.9) one gets

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{C_1}{4} \qquad \text{(V.1)}$$

and

$$\frac{\partial^2 \Theta}{\partial z \partial \bar{z}} = \frac{C_2}{4} u \qquad \text{(V.2)}$$

To find the solution of Θ, u is eliminated between equations (V.1) and (V.2) such that

$$\frac{\partial^4 \Theta}{\partial z^2 \partial \bar{z}^2} = \frac{C_1 C_2}{16} \qquad \text{(V.3)}$$

Letting

$$\Theta_1 = \Theta - \frac{C_1 C_2}{64}(z\bar{z})^2 \qquad \text{(V.4)}$$

equation (V.3) becomes

$$\frac{\partial^4 \Theta_1}{\partial z^2 \partial \bar{z}^2} = 0 \qquad \text{(V.5)}$$

which admits a general solution of the form

$$\Theta_1 = \bar{z}F(z) + z\bar{F}(\bar{z}) + X(z) + \bar{X}(\bar{z}) \qquad \text{(V.6)}$$

where $F(z)$ and $X(z)$ are Goursat functions (14). The bar denotes the complex conjugate.
The corresponding boundary conditions are obtained combining equations (II.6a), (II.9a) (V.2) and (V.4). These are

$$\Theta_1 = -\frac{C_1 C_2}{64} (z\bar{z})^2 \qquad \text{(V.7)}$$

$$\frac{\partial^2 \Theta_1}{\partial z \partial \bar{z}} = -\frac{C_1 C_2}{16} z\bar{z} \qquad \text{(V.8)}$$

Now differentiating equation (V.6) gives

$$\frac{\partial^2 \Theta_1}{\partial z \partial \bar{z}} = F'(z) + \bar{F}'(\bar{z}) \qquad \text{(V.9)}$$

The equation (V.8) is comparable with equation (V.9) only if the equation of the boundary is expressible in the form of $z\bar{z} = h(z) + \bar{h}(\bar{z})$ (V.10)

Then

$$F'(z) = -\frac{C_1 C_2}{16} h(z) \qquad \text{(V.11)}$$

$$\bar{F}'(\bar{z}) = -\frac{C_1 C_2}{16} \bar{h}(\bar{z}) \qquad \text{(V.12)}$$

To obtain $\bar{z}F(z)$ and $z\bar{F}(\bar{z})$, equations (V.11), (V.12) are integrated and multiplied with $\bar{z}$ and z respectively.
Now to get $X(z)$ and $\bar{X}(z)$ equations (V.6) and (V.7) are combined such that

$$(z\bar{z})^2 = -\frac{64}{C_1 C_2}[\bar{z}F(z) = z\bar{F}(\bar{z})] - \frac{64}{C_1 C_2}[X(z) + \bar{X}(\bar{z})] \qquad \text{(V.13)}$$

The final form of the solution for velocity and temperature is obtained using equations (V.2), (V.4), (V.10) and (V.4), (V.6) respectively.
Hence

$$u=\frac{C_1}{4}\left[z\bar{z}+\frac{16}{C_1C_2}\frac{\partial^2\Theta_1}{\partial z\partial\bar{z}}\right] \tag{V.14}$$

or

$$u=\frac{C_1}{4}\left[z\bar{z}-h(z)-\bar{h}(\bar{z})\right] \tag{V.15}$$

and

$$\Theta=\frac{C_1C_2}{64}(z\bar{z})^2+\bar{z}F(z)+z\bar{F}(\bar{z})+\bar{X}(\bar{z})+X(z) \tag{V.16}$$

B. Pipes with Equilateral Triangular Cross-section

Considering figure(III.1) and using complex variables

$$z=x+iy,\quad \bar{z}=x-iy \tag{V.17}$$

equation(III.3) becomes

$$z\bar{z}=\frac{1}{2a}(z^3+\bar{z}^3)+\frac{4}{27}a^2 \tag{V.18}$$

Comparing equations (V.10) and (V.18) one gets

$$h(z)=\frac{1}{2a}z^3+\frac{2}{27}a^2 \tag{V.19}$$

$$\bar{h}(\bar{z})=\frac{1}{2a}\bar{z}^3+\frac{2}{27}a^2 \tag{V.20}$$

Using equations (V.11) and (V.12) and doing the integrations yields

$$F(z)=-\frac{C_1C_2}{16}z\left[\frac{1}{8a}z^3+\frac{2}{27}a^2\right] \tag{V.21}$$

$$\bar{F}(\bar{z})=-\frac{C_1C_2}{16}\bar{z}\left[\frac{1}{8a}\bar{z}^3+\frac{2}{27}a^2\right] \tag{V.22}$$

Then

$$\bar{z}F(z)+z\bar{F}(\bar{z})=-\frac{C_1C_2}{64}z\bar{z}\left[\frac{1}{2a}(z^3+\bar{z}^3)+\frac{16}{27}a^2\right] \tag{V.23}$$

Substituting equation(V.23) in equation(V.13) gives

$$(z\bar{z})^2 = [\frac{1}{2a}(z^3+\bar{z}^3)+\frac{16}{27}a^2]z\bar{z} - \frac{64}{C_1C_2}[X(z)+\bar{X}(\bar{z})] \quad (V.24)$$

The find $X(z)$ and $\bar{X}(\bar{z})$ both sides of the equation (V.18) is squared and rearranged such that

$$(z\bar{z})^2 = \frac{1}{2a}z\bar{z}(z^3+\bar{z}^3)+\frac{2}{27}a\ (z^3+\bar{z}^3)+\frac{16}{27}a^4$$

then when added to it the equation (V.18) in the form of

$$0=\frac{16}{27}a^2[z\bar{z}-\frac{1}{2a}(z^3+\bar{z}^3)-\frac{4}{27}a^2]$$

this will lead to

$$(z\bar{z})^2 = [\frac{1}{2a}(z^3+\bar{z}^3)+\frac{16}{27}a^2]z\bar{z} - \frac{6}{27}a(z^3+\bar{z}^3)-\frac{48}{27^2}a^4 \quad (V.25)$$

Now comparing equations (V.24) and (V.25) one gets

$$X(z)+\bar{X}(\bar{z})=\frac{C_1C_2}{64}[\frac{6a}{27}(z^3+\bar{z}^3)+\frac{48}{27^2}a^4] \quad (V.26)$$

Finally the velocity and temperature fields are obtained substituting equations (V.19), (V.20) in (V.15) and (V.23), (V.26) in (V.16) respectively.

$$u=\frac{C_1}{4}[z\bar{z}-\frac{1}{2a}(z^3+\bar{z}^3)-\frac{4}{27}a^2]$$

$$\Theta=\frac{C_1C_2}{64}(z\bar{z})^2-\frac{C_1C_2}{64}z\bar{z}[\frac{1}{2a}(z^3+\bar{z}^3)+\frac{16}{27}a^2]$$

$$+\frac{C_1C_2}{64}[\frac{2a}{9}(z^3+\bar{z}^3)+\frac{16}{243}a^4]$$

In terms of (x-y) coordinates, they are

$$u=-\frac{C_1}{4}[x^3-3xy^2-a(x^2+y^2)+\frac{4}{27}a^3] \quad (V.27)$$

$$\Theta=\frac{C_1C_2}{64a}[\frac{4}{9}a^2-x^2-y^2][x^3-3xy^2-a(x^2+y^2)+\frac{4}{27}a^3] \quad (V.28)$$

The calculation of the Nusselt number is already presented in Chapter III.

C. Pipes Described by $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1$

The expression $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1$ (V.29)

where n is an even integer, is a general equation from which some important pipe geometries can be deduced. Consider now equation (V.29) under some typical conditions.

Case I- Letting $n = 2$ provides the equation of an ellipse such that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{(V.30)}$$

with a and b being semiaxes. Also for a approaching infinity and b being finite one gets the equation of the parallel plate such that

$$y^2 = b^2 \qquad \text{(V.31)}$$

Case II- Assigning $a = b$, equation (V.29) reduces to

$$x^n + y^n = a^n \qquad \text{(V.32)}$$

For the choice of $n = 2$ this equation corresponds to that of the circle such that

$$x^2 + y^2 = a^2 \qquad \text{(V.33)}$$

Other choices (of $n = 4, 6, \ldots$) would correspond to equations of some pipe geometries which will lie between the circular and square pipes as shown in Figure V.1

Now these pipe geometries will be studied seperately.

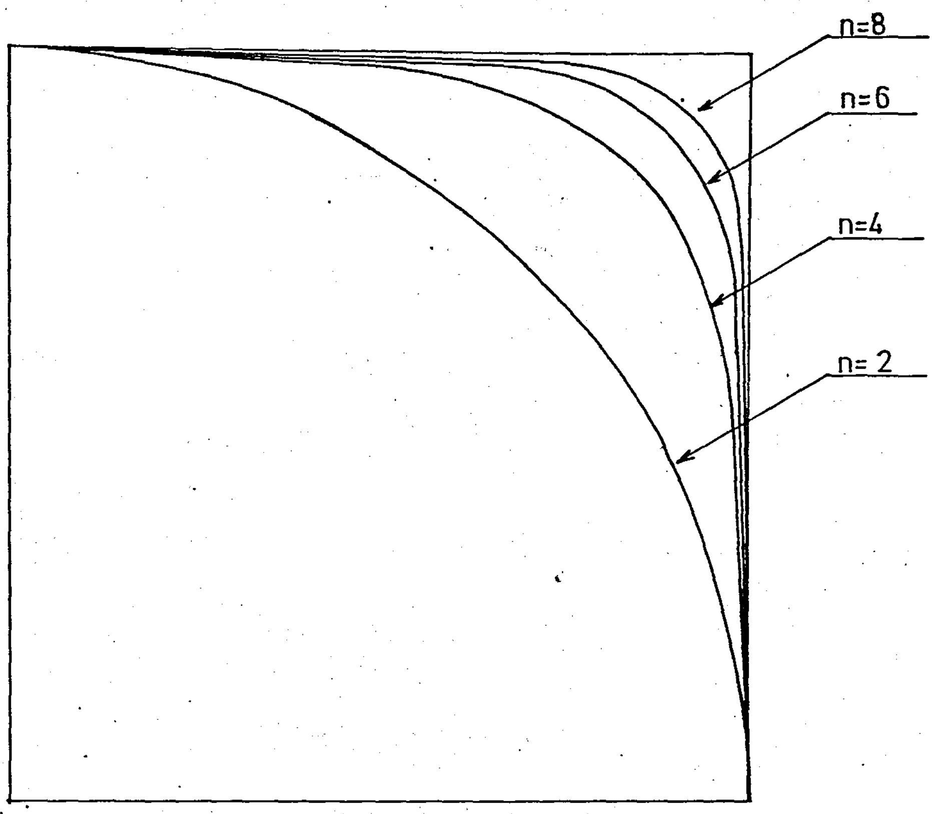

Figure V.1-Some Pipe Geometries Describey by
$x^n + y^n = a^n$

1-Elliptical Pipes

Rewrite equation (V.30) in the form of

$$b^2x^2+a^2y^2= a^2b^2 \qquad (V.34)$$

where a and b are semiaxes. In terms of complex variables it is

$$z\bar{z}= \frac{2a^2b^2}{a^2+b^2} - \frac{1}{2}\frac{b^2-a^2}{b^2+a^2}(z^2+\bar{z}^2) \qquad (V.35)$$

Comparing equation(V.35) with equation(V.10)gives

$$h(z)=\frac{a^2b^2}{a^2+ b^2} - \frac{1}{2}\frac{b^2- a^2}{b^2+ a^2} z^2 \qquad (V.36)$$

$$\bar{h}(\bar{z})=\frac{a^2b^2}{a^2+ b^2} - \frac{1}{2}\frac{b^2- a^2}{b^2+ a^2} \bar{z}^2 \qquad (V.37)$$

Using equations (V.11) and (V.12) and performing the integrations results in

$$F(z)=-\frac{C_1C_2}{96}z\left[\frac{6a^2b^2}{a^2+b^2}+\frac{a^2-b^2}{a^2+b^2}z^2\right]$$

$$\bar{F}(\bar{z})=-\frac{C_1C_2}{96}\bar{z}\left[\frac{6a^2b^2}{a^2+b^2}+\frac{a^2-b^2}{a^2+b^2}\bar{z}^2\right]$$

Then

$$\bar{z}F(z)+z\bar{F}(\bar{z})=-\frac{C_1C_2}{96}z\bar{z}\left[\frac{12a^2b^2}{a^2+b^2}+\frac{a^2-b^2}{a^2+b^2}(z^2+\bar{z}^2)\right] \qquad (V.38)$$

Substituting equation (V.38) in equation(V.13) gives

$$(z\bar{z})^2=\frac{8a^2b^2}{a^2+b^2}z\bar{z}+\frac{2}{3}\frac{a^2-b^2}{a^2+b^2}(z^2+\bar{z}^2)z\bar{z}$$
$$-\frac{64}{C_1C_2}[X(z)+\bar{X}(\bar{z})] \qquad (V.39)$$

To determine $X(z)$ and $\bar{X}(\bar{z})$ first both sides of equation (V.35) is squared and rearranged such that

$$(z\bar{z})^2=\frac{2a^2b^2}{a^2+b^2}z\bar{z}+\frac{1}{2}\frac{a^2-b^2}{a^2+b^2}z\bar{z}(z^2+\bar{z}^2) \qquad (V.40)$$

Also multiplying equation (V.35) with $(z^2+\bar{z}^2)$ and rearranging yields

$$\frac{a^2-b^2}{a^2+b^2}(z\bar{z})^2=z\bar{z}(z^2+\bar{z}^2)-\frac{1}{2}\frac{a^2-b^2}{a^2+b^2}(z^4+\bar{z}^4)$$
$$-\frac{2a^2b^2}{a^2+b^2}(z^2+\bar{z}^2) \qquad (V.41)$$

Now multiplying equation (V.35) with $\frac{8a^2b^2}{a^2+b^2}$ and rearranging gives

$$0=\frac{8a^2b^2}{a^2+b^2}z\bar{z}-\frac{16a^4b^4}{(a^2+b^2)^2}-4a^2b^2\frac{(a^2-b^2)}{(a^2+b^2)^2}(z^2+\bar{z}^2) \qquad (V.42)$$

Combining equations (V.40),(V.41) and (V.42) results in

$$(z\bar{z}) = \frac{8a^2b^2}{a^2+b^2} z\bar{z} + \frac{2}{3}\frac{a^2-b^2}{a^2+b^2} z\bar{z}(z^2+\bar{z}^2)$$

$$-\frac{8}{3}\frac{a^4b^4}{(a^2+b^2)^2}\frac{5a^4+5b^4+26a^2b^2}{a^4+b^4+6a^2b^2}$$

$$-\frac{4}{3}\frac{a^2b^2(a^2-b^2)}{(a^2+b^2)^2}\frac{3a^4+3b^4+14a^2b^2}{a^4+b^4+6a^2b^2}(z^2+\bar{z}^2)$$

$$-\frac{1}{6}\frac{(a^2-b^2)^2}{(a^4+b^4+6a^2b^2)}(z^4+\bar{z}^4) \qquad (V.43)$$

Comparing equations (V.39) and (V.43) one gets

$$X(z)+\bar{X}(\bar{z}) = \frac{C_1C_2}{64}\left[\frac{8}{3}\frac{a^4b^4}{(a^2+b^2)^2}\frac{5a^4+5b^4+26a^2b^2}{a^4+b^4+6a^2b^2}\right.$$

$$+\frac{1}{6}\frac{(a^2-b^2)^2 \quad (z^4+\bar{z}^4)}{a^4+b^4+6a^2b^2}$$

$$\left.+\frac{4}{3}\frac{a^2b^2(a^2-b^2)}{(a^2+b^2)^2}\frac{3a^4+3b^4+14a^2b^2}{a^4+b^4+6a^2b^2}(z^2+\bar{z}^2)\right] \qquad (V.44)$$

Finally the velocity and temperature fields are obtained substituting equations (V.36),(V.37) in (V.15) and (V.38),(V.44) in (V.16) respectively.

$$u=\frac{C_1}{4}\left[z\bar{z}-\frac{2a^2b^2}{a^2+b^2}-\frac{1}{2}\frac{a^2-b^2}{a^2+b^2}(z^2+\bar{z}^2)\right]$$

$$\Theta=\frac{C_1C_2}{64}[(z\bar{z})^2-\frac{8a^2b^2}{a^2+b^2}z\bar{z}-\frac{2}{3}\frac{a^2-b^2}{a^2+b^2}(z^2+\bar{z}^2)z\bar{z}$$

$$+\frac{8}{3}\frac{a^4b^4}{(a^2+b^2)^2}\frac{5a^4+5b^4+26a^2b^2}{a^4+b^4+6a^2b^2}+\frac{1}{6}\frac{(a^2-b^2)^2}{a^4+b^4+6a^2b^2}(z^4+\bar{z}^4)$$

$$+\frac{4}{3}\frac{a^2b^2(a^2-b^2)}{(a^2+b^2)^2}\frac{3a^4+3b^4+14a^2b^2}{a^4+b^4+6a^2b^2}(z^2+\bar{z}^2)]$$

In terms of (x-y) coordinates, they are

$$u=\frac{C_1}{2(1+s^2)}(s^2x^2+y^2-b^2) \qquad \text{(V.45)}$$

where $s=\frac{b}{a}$, is the aspect ratio.

$$\Theta=\frac{C_1C_2}{24(1+s^2)(1+s^4+6s^2)}(s^2x^2+y^2-b^2)$$

$$[(s^4+5s^2)x^2+(1+5s^2)y^2-\frac{a^2}{1+s^2}(5s^2+5s^6+26s^4)] \qquad \text{(V.46)}$$

To calculate the Nusselt number one needs to evaluate average velocity and bulk temperature. Substituting equation (V.45) with $A=\pi ab$ in equation (II.12) one gets

$$u_m=\frac{2C_1}{(1+s^2)\pi ab}\int_0^a\int_0^{\sqrt{b^2-s^2x^2}}(s^2x^2+y^2-b^2)\,dy\,dx$$

Before integrating one needs to transform the ellipse to circle by

$$\left.\begin{array}{l} x=\bar{x}a \\ y=\bar{y}b \end{array}\right\} \qquad \text{(V.47)}$$

Such that

$$\bar{x}^2+\bar{y}^2=1$$

Then $u_m = \frac{2C_1b^2}{(1+s^2)\pi} \int_0^1 \int_0^{\sqrt{1-\bar{x}^2}} (\bar{x}^2+\bar{y}^2-1)\,d\bar{y}\,d\bar{x}$

Integrating with respect to $\bar{y}$ yields

$$u_m = \frac{4C_1b^2}{3\pi(1+s^2)} \int_0^1 (1-\bar{x}^2)^{3/2}\,d\bar{x}$$

Letting

$$\left.\begin{aligned} \bar{x} &= \sin t \\ 1-\bar{x}^2 &= \cos^2 t \end{aligned}\right\} \tag{V.48}$$

$$u_m = -\frac{4}{3}\,\frac{C_1b^2}{\pi(1+s^2)} \int_0^{\pi/2} \cos^4 t\,d\,t$$

Performing the integration gives

$$u_m = -\frac{C_1b^2}{4(1+s^2)} \tag{V.49}$$

Substituting equations (V.45),(V.46)and (V.49) in equation (II.13) and applying the transformations (V.47),(V.48) gives

$$\Theta_b = -\frac{C_1C_2b^4}{3\pi(1+s^2)(1+s^4+6s^2)} \int_0^{\pi/2}\int_0^{\cos t} \{(s\ +5)$$

$$(\sin^6 t-2\sin^4 t+\sin^2 t)+(11s^2+7)\sin^2 t\bar{y}^4$$

$$+(11+7s^2)\sin^4 t\bar{y}^2-12(1+s^2)\sin^2 t\ \bar{y}^2$$

$$+(1+5s^2)(\bar{y}^6-2\bar{y}^4+\bar{y}^2)-\frac{1}{(1+s^2)}[(5+26s^2+5s^4)$$

$$(1+\sin^4 t+y^4)+(10+52s^2+10s^4)$$

$$(-\sin^2 t+\sin^2 t\bar{y}^2-\bar{y}^2)]\}\ \cos t\,d\bar{y}$$

Performing the integration results in

$$\Theta_{,b}=\frac{C_1C_2b^4}{144(1+s^2)^2}\ \frac{(17s^4+98s^2+17)}{(1+s^4+6s^2)} \tag{V.50}$$

Details of the integration is given in Appendix A. Also the heat flux rate from equations (II.14a) and (V.49) is

$$q=-\frac{C_1C_2k}{4(1+s^2)}\ \pi ab^3 \tag{V.51}$$

Substituting equations (V.49),(V.50) in equation (II.15) gives

$$Nu=144\frac{\pi^2a^2}{P^2}\ \frac{(1+s^2)(1+s^4+6s^2)}{(17s^4+98s^2+17)} \tag{V.52}$$

where $P=4aE(\alpha,\frac{\pi}{2})$, is the perimeter.
$E(\alpha,\frac{\pi}{2})$ is the elliptical integral of second kind. The values of P for various aspect ratios are given in Appendix A.

2-Parallel Plates

Rewrite equation (V.31) in the form of

$$y^2-b^2=0$$

Using complex variables, this equation becomes

$$z\bar{z} = 2b^2+\frac{1}{2}(z^2+\bar{z}^2) \tag{V.53}$$

which is in the form of equation (V.10).

3-Circular Pipes

Rewrite equation (V.33) in the form of

$$x^2+y^2-a^2=0$$

Using complex variables, this equation becomes

$$z\bar{z} = a^2 \tag{V.54}$$

which is also in the form of equation (V.10).

Hence Complex Variable Method is applicable to these geometries. But the solutions for these pipes will not be given in this study, since they are available in many heat transfer books.

4-Pipes given by $x^4+y^4= a^4$

Consider equation (V.32) for n=4 which will correspond to $x^4+y^4= a^4$.
Using Complex variables, this equation becomes

$$(z\bar{z})^2=\frac{4}{3} a^4 - \frac{1}{6}(z^4+\bar{z}^4) \tag{V.56}$$

$$z\bar{z} =[\frac{4}{3}a^4 - \frac{1}{6}(z^4+\bar{z}^4)]^{1/2} \tag{V.57}$$

To be able to apply the Complex Variable Method equation (V.57) is expanded by Binomial Expansion such that

$$z\bar{z} =\frac{2a^2}{\sqrt{3}} - \frac{1}{8\sqrt{3}\ a^2}(z^4+\bar{z}^4) \tag{V.58}$$

Comparing equation (V.58) with equation (V.10) one gets

$$h(z) = \frac{a^2}{\sqrt{3}} - \frac{1}{8\sqrt{3}\ a^2} z^4 \tag{V.59}$$

$$\bar{h}(\bar{z}) = \frac{a^2}{\sqrt{3}} - \frac{1}{8\sqrt{3}\ a^2} \bar{z}^4 \tag{V.60}$$

Using equations (V.11) and (V.12) and integrating one obtains

$$F(z) =-\frac{C_1C_2}{16}z\ [\frac{a^2}{\sqrt{3}} - \frac{z^4}{40\sqrt{3}a^2}]$$

$$\bar{F}(\bar{z}) =-\frac{C_1C_2}{16}\bar{z}\ [\frac{a^2}{\sqrt{3}} - \frac{\bar{z}^4}{40\sqrt{3}a^2}]$$

Then

$$\bar{z}F(z)+z\bar{F}(\bar{z})=-\frac{C_1C_2}{64}z\bar{z}\left[\frac{8a^2}{\sqrt{3}}-\frac{(z^4+z^4)}{10\sqrt{3}}\right] \qquad (V.61)$$

Substituting this equation in equation (V.13) one has

$$(z\bar{z})^2=\frac{8a^2}{\sqrt{3}}z\bar{z}-\frac{1}{10\sqrt{3}a^2}z\bar{z}(z^4+\bar{z}^4)-\frac{64}{C_1C_2}[X(z)+\bar{X}(\bar{z})] \qquad (V.62)$$

To determine $X(z)$ and $\bar{X}(\bar{z})$ first equation (V.58) is multiplied by $(-4z\bar{z})$ such that

$$4(z\bar{z})^2=\frac{8a^2}{\sqrt{3}}z\bar{z}-\frac{1}{2\sqrt{3}a^2}z\bar{z}(z^4+\bar{z}^4)$$

then adding this equation to equation (V.56) and rearranging yields

$$(z\bar{z})^2=-\frac{1}{10\sqrt{3}a^2}z\bar{z}(z^4+\bar{z}^4)+\frac{8a^2}{\sqrt{3}}z\bar{z}-4a^4+\frac{7}{30}(z^4+\bar{z}^4) \qquad (V.63)$$

Now comparing equations (V.62) and (V.63) one has

$$X(z)+\bar{X}(\bar{z})=\frac{C_1C_2}{64}\left[4a^4-\frac{7}{30}(z^4+\bar{z}^4)\right] \qquad (V.64)$$

To obtain the velocity and temperature fields one substitutes equation (V.57) in (V.14) and equations (V.61), (V.64) in (V.16) respectively.

$$u=\frac{C_1}{4}\left\{z\bar{z}-\left[\frac{4}{3}a^4-\frac{1}{6}(z^4+\bar{z}^4)\right]^{1/2}\right\}$$

$$\Theta=\frac{C_1C_2}{4}\Big[(z\bar{z})^2+\frac{1}{10\sqrt{3}a^2}z\bar{z}(z^4+\bar{z}^4)-\frac{8a^2}{\sqrt{3}}z\bar{z}$$

$$+4a^4-\frac{7}{30}(z^4+\bar{z}^4)\Big]$$

In terms of (x-y) coordinates, they are

$$u=\frac{C_1}{4}\{x^2+y^2-[\frac{4}{3}a^4-\frac{1}{3}(x^4+y^4-6x^2y^2)]^{1/2}\} \qquad (V.65)$$

$$\Theta=\frac{C_1C_2}{64}[(x^2+y^2)^2+\frac{1}{5\sqrt{3}a^2}(x^2+y^2)(x^4+y^4-6x^2y^2)$$

$$-\frac{8}{\sqrt{3}}a^2(x^2+y^2)-\frac{7}{15}(x^4+y^4-6x^2y^2)+4a^4$$

$$-\frac{1}{20\sqrt{3}a^2}(x^4+y^4-6x^2y^2)^2] \qquad (V.66)$$

The last expression in equation (V.66)

$-\frac{1}{20\sqrt{3}a^4}(x^4+y^4-6x^2y^2)^2$ is used to modify

the temperature distribution by some intuitive reasons which will be discussed in Chapter VII. For the calculation of the average velocity the expression $[\frac{4}{3}a^4-\frac{1}{3}(x^4+y^4-6x^2y^2)]^{1/2}$ in equation (V.65) is expanded by Binomial Expansion up to four terms such that when equation (V.65) is substituted in equation (II.12) one gets

$$u_m=\frac{C_1}{A}\int_0\int_0^{a\sqrt[4]{a^4-x^4}}[x^2+y^2-\frac{2a^2}{\sqrt{3}}+\frac{1}{4\sqrt{3}a^2}(x^4+y^4-6x^2y^2)$$

$$+\frac{1}{64\sqrt{3}a^6}(x^4+y^4-6x^2y^2)^2+\frac{1}{512\sqrt{3}a^{10}}(x^4+y^4-6x^2y^2)^3$$

$$+\frac{5}{16384\sqrt{3}a^{14}}(x^4+y^4-6x^2y^2)^4]dydx$$

where $A=a^2\beta(\frac{5}{4},\frac{1}{4})$ as given in Appendix B.

To perform this integration two transformations are needed such that

$$\left.\begin{array}{l} v = x^2 \\ w = y^2 \end{array}\right\} \qquad (V.67)$$

$$\left.\begin{array}{l} v = \sigma\cos\theta \\ w = \sigma\sin\theta \end{array}\right\} \qquad (V.68)$$

Using these transformations the avarage velocity becomes

$$u_m=\frac{C_1}{4A}\int_0^{\pi/2}\int_0^{a^2}\Big[\sigma(\cos\theta+\sin\theta)-\frac{2a^2}{\sqrt{3}}+\frac{\sigma^2}{4\sqrt{3}a^2}(1-6\cos\theta\ \sin\theta)$$

$$+\frac{1}{64\sqrt{3}}\ \frac{\sigma^4}{a^6}(1-6\cos\theta\sin\theta)^2+\frac{\sigma^6}{512\sqrt{3}a^{10}}(1-6\cos\theta\sin\theta)^3$$

$$+\frac{5\ \sigma^8}{16384\sqrt{3}a^{14}}(1-6\cos\theta\sin\theta\ )^4\Big]\frac{d\sigma d\theta}{\sqrt{\cos\theta\sin\theta}}$$

This integral is evaluated in terms of β functions as given in Appendix B and the following result for u_m is obtained

$$u_m= -0.1428C_1a^2=-\frac{C_1a^2}{7} \qquad (V.69)$$

Substituting the expanded velocity profile and equation (V.66) with the specified transformations (V.67) and (V.68) in equation (II.13) one obtaines the bulk temperature as

$$\Theta_b = \frac{C_1^2 C_2}{256\, u_m A} \int_0^{a^2} \int_0^{\pi/2} \{ [\sigma(\cos\theta+\sin\theta) - \frac{2}{\sqrt{3}} a^2$$

$$+ \frac{1}{4\sqrt{3}} \frac{\sigma^2}{a^2} (1-6\cos\theta\sin\theta) + \frac{1}{64\sqrt{3}} \frac{\sigma^4}{a^6} (1-6\cos\theta\sin\theta)^2$$

$$+ \frac{\sigma^6}{512\sqrt{3} a^{10}} (1-6\cos\theta\sin\theta)^3 + \frac{5\sigma^8}{16384\sqrt{3} a^{14}} (1-6\cos\theta\sin\theta)^4]$$

$$[\sigma^2 (\cos\theta+\sin\theta)^2 + 4a^4 - \frac{8}{\sqrt{3}} a^2 \sigma (\cos\theta+\sin\theta)$$

$$- \frac{7}{15} \sigma^2 (1-6\cos\theta\sin\theta) + \frac{1}{5\sqrt{3}} \frac{\sigma^3}{a^2} (\cos\theta+\sin\theta)(1-6\cos\theta\sin\theta)$$

$$- \frac{1}{20\sqrt{3}} \frac{\sigma^4}{a^4} (1-6\cos\theta\sin\theta)^2] \} \frac{d\sigma d\theta}{\sqrt{\cos\theta\sin\theta}}$$

Performing the integration the following result is obtained:

$$\Theta_b = 0.03797\, C_1 C_2 a^4 \qquad \text{(V.70)}$$

Detailes of the integration and the calculation of the perimeter is given in Appendix B.
Finally substituting equations (V.69)and(V.70) with $P = 7.0168\ a$ in equation (II.15),the Nusselt number is obtained as

$$Nu = 4.197$$

VI

FINITE ELEMENT METHOD

The governing equations are solved numerically with the Finite Element Method. Before applying this method, the following dimensionless variables are introduced

$$\Phi=-\frac{u}{C_1 a^2},\quad \Psi=\frac{\Theta}{C_1 C_2 a^4},\quad X=\frac{x}{a},\quad Y=\frac{y}{a} \tag{VI.1}$$

where a is the characteristic length of the cross-sections. Then equations (II.6) and (II.9) in dimensionless form are

$$\frac{\partial^2\Phi}{\partial X^2}+\frac{\partial^2\Phi}{\partial Y^2}+1=0 \tag{VI.2}$$

$$\frac{\partial^2\Psi}{\partial X^2}\ \frac{\partial^2\Psi}{\partial Y^2}+\Phi=0 \tag{VI.3}$$

and

$$\Phi=0$$
$$\Psi=0$$

at the wall.

The continuous quantities, velocity and temperature will be approximated by a discrete model composed of a set of piecewise continuous functions defined over a finite number of subdomains.

In this study domains used are two-dimensional pipe geometries which will be devided into a finite number of subdomains by using two-dimensional triangular elements. Details of descretization of the pipe geometries are given in Appendix C.

A. Shape Functions

In this study the velocity and temperature are approximated by

$$\Phi = [N]\{V\} \tag{VI.4}$$

and

$$\Psi = [N]\{T\} \tag{VI.5}$$

where $[N]$ is a row matrix which consists of the selected shape functions. $\{V\}$ and $\{T\}$ are the nodal values of the velocity and temperature respectively. Shape functions of a triangular element are

$$N_i = \frac{1}{2A}[a_i + b_i X + c_i Y] \tag{VI.6a}$$

$$a_i = X_j Y_k - X_k Y_j$$

$$b_i = Y_j - Y_k$$

$$c_i = X_k - X_j$$

$$N_j = \frac{1}{2A}[a_j + b_j X + c_j Y] \tag{VI.6b}$$

$$a_j = X_k Y_i - X_i Y_k$$

$$b_j = Y_k - Y_i$$

$$c_j = X_i - X_k$$

$$N_k=\frac{1}{2A}\ [a_k+b_kX+c_kY] \qquad \text{(VI.6c)}$$

$$a_k=X_iY_j-\ X_jY_i$$

$$b_k=\ Y_i-\ Y_j$$

$$c_k=\ X_j-\ X_i$$

where $2A=\begin{vmatrix} 1 & X_i & Y_i \\ 1 & X_j & Y_j \\ 1 & X_k & Y_k \end{vmatrix}$ is the area of the triangle.

(X_i,Y_i), (X_j,Y_j) and (X_k,Y_k) denote the coordinate pairs of the three nodes of the triangular element.

B.Derivation of the Element Equations

Using Euler equation (IV.1), the associated variational integrals corresponding to equations (VI.2) and (VI.3) come out to be

$$I_M=\iint \frac{1}{2}[(\frac{\partial\Phi}{\partial X})^2+(\frac{\partial\Phi}{\partial Y})^2 - 2\Phi]\ dXdY \qquad \text{(VI.7)}$$

$$I_E=\iint \frac{1}{2}[(\frac{\partial\Psi}{\partial X})^2+(\frac{\partial\Psi}{\partial Y})^2 - 2\Phi\Psi]dXdY \qquad \text{(VI.8)}$$

Since the boundary conditions are essential the variational integrals(VI.7)and (VI.8) will be not modified.

Let

$$\{g\}_M=\begin{bmatrix}\frac{\partial\Phi}{\partial X}\\ \frac{\partial\Phi}{\partial Y}\end{bmatrix} \qquad \text{(VI.9)}$$

$$[D]=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\{g\}_E=\begin{bmatrix}\frac{\partial\Psi}{\partial X}\\ \frac{\partial\Psi}{\partial Y}\end{bmatrix} \qquad \text{(VI.10)}$$

Substituting equations (VI.4), (VI.5) in equation (VI.9), (VI.10) respectively one gets

$$\{g\}_M = \begin{bmatrix} \frac{\partial N}{\partial X} \\ \frac{\partial N}{\partial Y} \end{bmatrix} \{V\} = [B]\{V\} \qquad \text{(VI.11a)}$$

and

$$\{g\}_E = [B]\ \{T\} \qquad \text{(VI.11b)}$$

where [B] is the gradient matrix.
Combining equations (VI.7), (VI.9) and (VI.11a) one has

$$I_M = \iint \frac{1}{2}\ \{V\}^T [B]^T [D][B]\{V\} dXdY - \iint [N]\{V\}\ dXdY \qquad \text{(VI.12)}$$

Using the minimization condition one obtains

$$\frac{\partial\ I_M}{\partial\{V\}} = \frac{\partial}{\partial\{V\}}\ \iint \frac{1}{2}\{V\}^T [B]^T [D][B]\{V\} dXdY$$

$$-\frac{\partial}{\partial\{V\}}\iint [N]\{V\} dXdY = 0$$

or

$$\iint [B]^T [D][B]\{V\} dXdY = \iint [N]^T dXdY \qquad \text{(VI.13)}$$

Equation (VI.13) can be written in condensed form

$$[k^e]\{V\} = \{f^e\} \qquad \text{(VI.14)}$$

$$[k^e] = \iint [B^e]^T [D][B^e] dXdY$$

$$\{f^e\} = \iint [N]^T dXdY$$

The final system is obtained by considering the whole domain

$$[K]\{V\} = \{F\} \qquad \text{(VI.15)}$$

$$[K] = \sum_{e=1}^{E}\ [k^e]$$

$$\{F\} = \sum_{e=1}^{E}\ \{f^e\}$$

Now combining equations (VI.8), (VI.10) and (VI.11a,b) one gets

$$I_E = \iint \frac{1}{2}\{T\}^T [B]^T [D][B]\{T\} dxdy$$

$$- \iint [N]\{T\}[N]\{v\} dxdy \qquad (VI.16)$$

Using the minimization condition one has

$$\frac{\partial \bar{I}_E}{\partial\{T\}} = \iint [B]^T [D][B]\{T\} dxdy$$

$$- \iint [N]^T [N]\{v\} dxdy = 0$$

or

$$\iint [B]^T [D][B]\{T\} dxdy = \iint [N]^T [N]\{v\} dxdy \qquad (VI.17)$$

Equation (VI.17) can be written in condensed form

$$[k^e]\{T\} = \{f^e\}_n \qquad (VI.18)$$

where $\{f^e\}_n = \iint [N]^T [N]\{v\} dxdy$.

The final system is obtained by considering the whole domain

$$[K]\{T\} = \{F\}_n \qquad (VI.19)$$

where $\{F\}_n = \sum_{e=1}^{E} \{f^e\}_n$.

Now the element stiffness matrix $[k^e]$ and the element force vectors $\{f^e\}, \{f^e\}_n$ are evaluated by using the shape functions.

Using equations (VI.6a,b,c) and (VI.11a,b), the gradiant matrix becomes

$$[B^e] = \frac{1}{2A} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} \qquad (VI.20)$$

Then

$$[k^e]=\iint \frac{1}{4A^2}\begin{bmatrix} b_i & c_i \\ b_j & c_j \\ b_k & c_k \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} dXdY$$

Since all the terms under the integral are constants they can be removed to yield

$$[k^e]=\frac{1}{4A}\begin{bmatrix} b_ib_i+c_ic_i & b_ib_j+c_ic_j & b_ib_k+c_ic_k \\ b_jb_i+c_jc_i & b_jb_j+c_jc_j & b_jb_k+c_jc_k \\ b_kb_i+c_kc_i & b_kb_j+c_kc_j & b_kb_k+c_kc_k \end{bmatrix} dXdY$$

To evaluate the force vectors area coordinates are employed.

Then $$\{f^e\} = \iint \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} dXdY$$

and $$\{f^e\}_n = \iint \begin{bmatrix} L_1^2 & L_1L_2 & L_1L_3 \\ L_2L_1 & L_2^2 & L_2L_3 \\ L_3L_1 & L_3L_2 & L_3^2 \end{bmatrix} \{V\}\ dXdY$$

To perform these integrals, the following formula is used:

$$\iint L_1^p L_2^q L_3^r dA = \frac{p!q!r!}{(p+q+r+2)!}2A$$

More information about the area coordinates can be found in Ref.(15).

VII

RESULTS AND DISCUSSION

The theoretical solutions of fully developed laminar forced convection under the conditions of constant heat flux in axial direction and uniform peripheral wall temperature are obtained for square, equilateral triangular and elliptical pipes. Also the geometry described by $x^4 + y^4 = a^4$ is solved approximately. To test the validity of the results obtained by Biharmonic Solutions, Variational and Complex Variable Methods, the results are compared with the solutions available in the literature and the numerical results evaluated by Finite Element Method. The flow and heat transfer characteristics are found for all cross sections. All the results are plotted in nondimensionalized form. Velocity and temperature profiles are parabolic and reach their maximum values at the center of the cross-section. The profiles decrease near the walls due to the no-slip condition and prescribed constant wall temperature at the walls respectively.

A. SQUARE PIPES

Biharmonic Solutions are directly applied to square pipes. Based on the known solutions of thin plate problem, velocity and temperature distributions are determined which are in good agreement with the solutions obtained by the Variational Method. These solutions are then compared with the numerical results of the Finite Element Method. The subdivision of the square pipe into triangular finite elements can be done in many different ways. Two possible divisions are shown in Figure C.1 (with elements of unequal size) and Figure C.2(with elements of equal size). Figure VII.1 and Figure VII.2 show the comparison of the theoretical and the numerical results of velocity and temperature fields respectively. It is observed that numerical results converge to the exact values as more nodes are used. The Nusselt number obtained both by Biharmonic Solution and Variational Method is 3.60 which agrees with the numerical result of 3.63 obtained by Clark, S.H., W.M.Kays(11) and with 3.80 given in (12).

B.EQUILATERAL TRIANGULAR PIPES

Similar to square pipes, known solutions of thin plate theory are again existing. Same solutions are obtained by Complex Variable Method. These solutions are compared with Finite Element Solutions. Two possible division of the pipe is shown in Figure C.3 and Figure C.4. The division scheme found to be most convenient to descretize the pipe in finite elements with shapes similar to the cross-section.

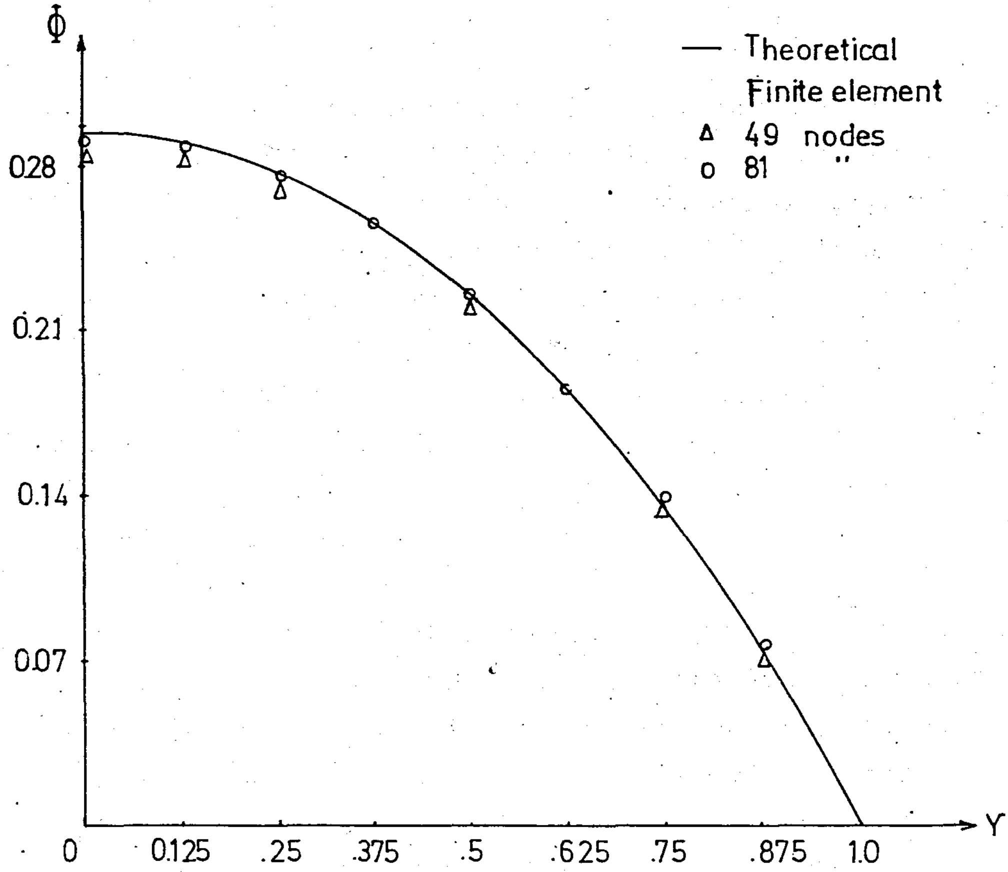

Figure VII.1 _ Velocity profiles along axis of symmetry (x=0)

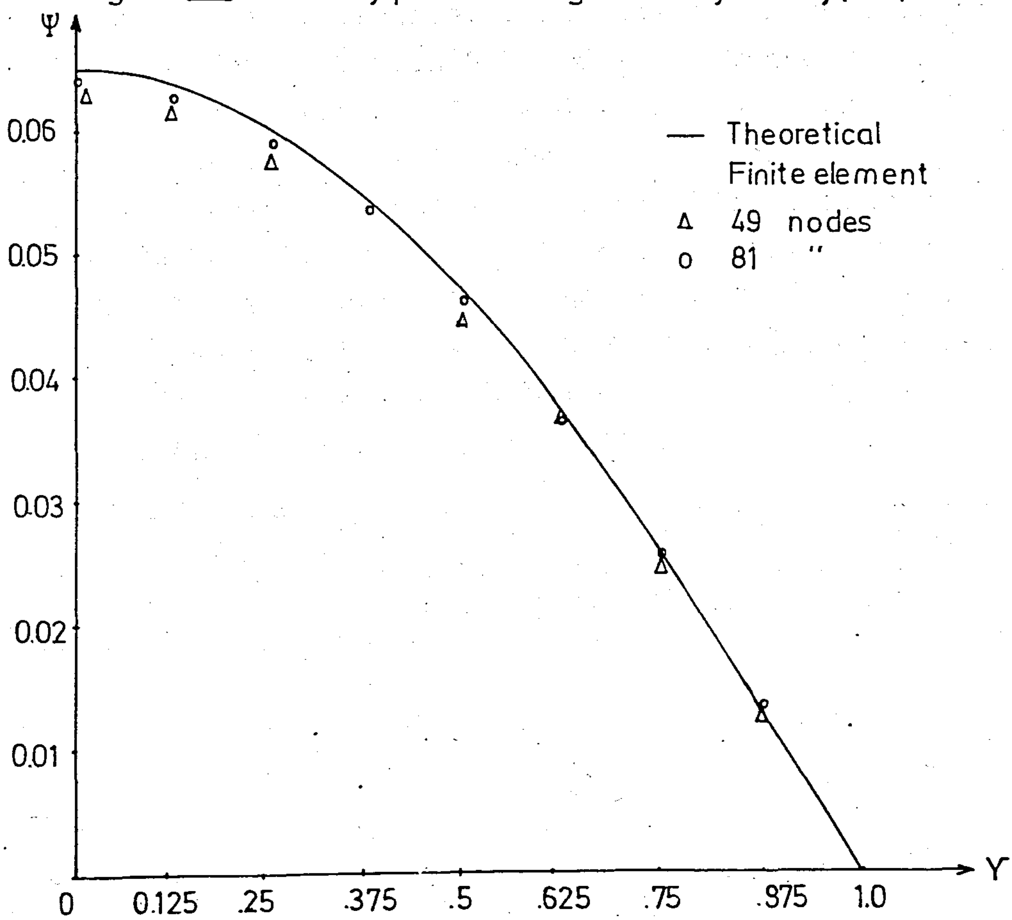

Figure VII.2 _ Temperature profiles along axis of symmetry (x=0)

Comparison of the theoretical and numerical results are shown in Figure VII.3 and Figure VII.4. As observed from the figures, convergence of the results is achieved more rapidly as more nodal points are used. The Nusselt number is calculated to be 3.11 . It is in good agreement with the numerical result of 3.00 obtained by Clark, S.H.W.M.Kays(11) and 3.1 given in (12).

C. ELLIPTICAL PIPES

The theoretical solutions for this pipe are obtained by Complex Variable Method. For numerical solution only one subdivision is presented for the aspect ratio $s=\frac{b}{a}=0.5$ as shown in Figure C.5 . Figure VII.5 and Figure VII.6 show the theoretical and numerical distributions of the velocity and the temperature respectively. The Nusselt number is calculated for s between 0.1 and 1.0 . These results are given in table VII.1 and shown in Figure VII.7. It is observed that for the limiting case s=1, Nusselt number is equal to 4.363 which corresponds to the Nusselt number of the circle. But for the other limiting case of s=0(either b=0 or $a=\infty$), the Nusselt number has no physical meaning because considering equations (V.50) and (V.51), θ_b and q are either zero or q is infinite.

s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Nu	5.123	4.962	4.802	4.666	4.558	4.477	4.421	4.387	4.370	4.363

Table VII.1

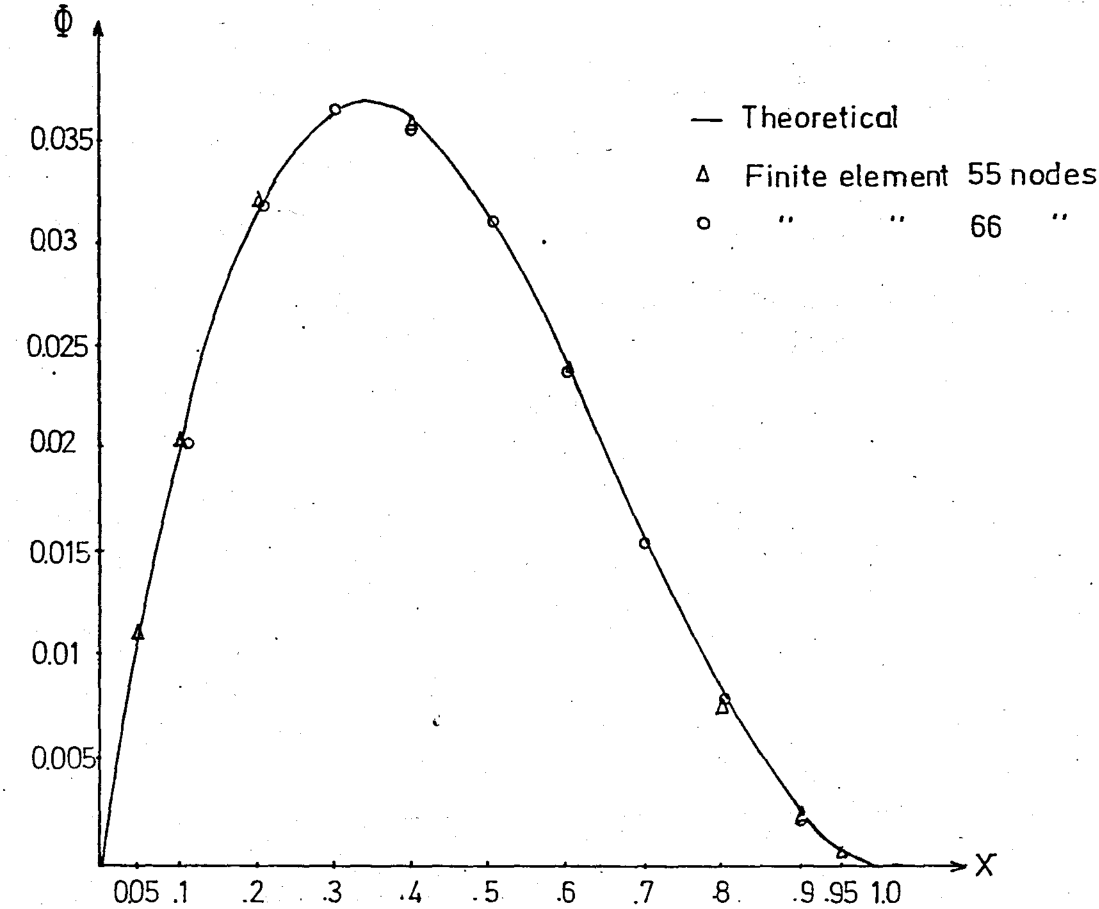

Figure VII.3 Velocity distributions along axis of symmetry (Y=0)

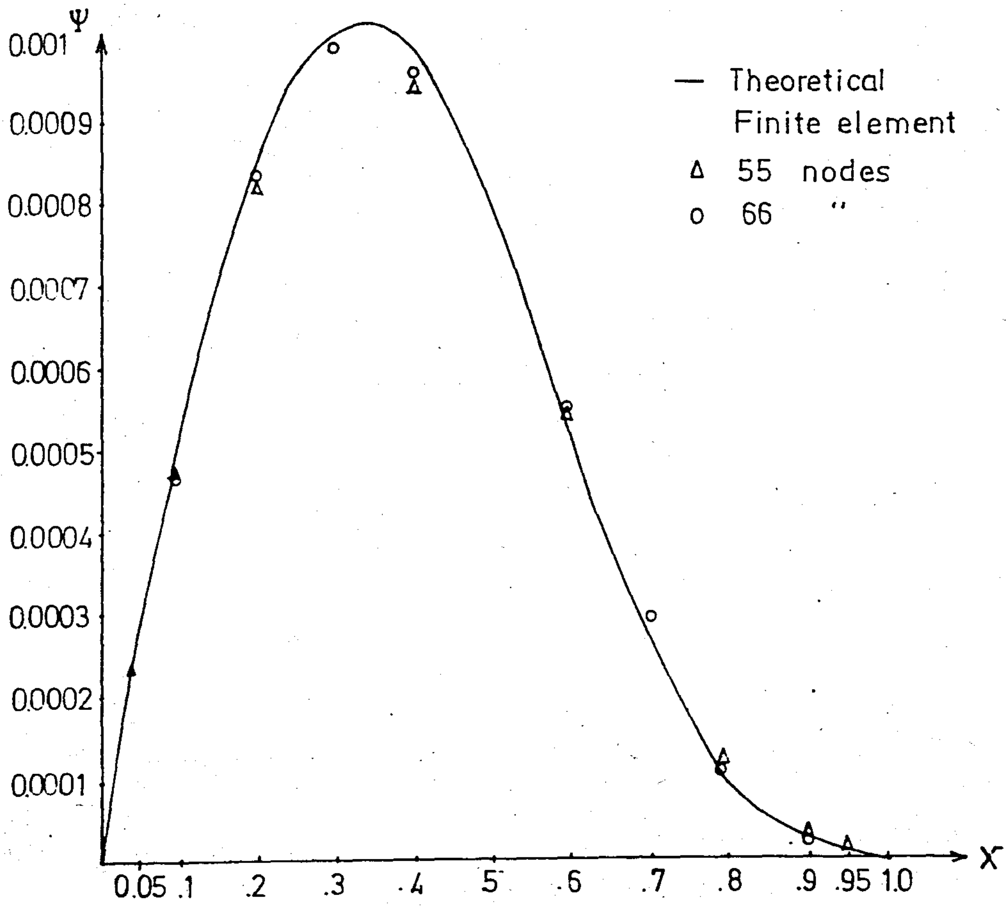

Figure VII.4 Temperature distributions along axis of symmetry (Y=0)

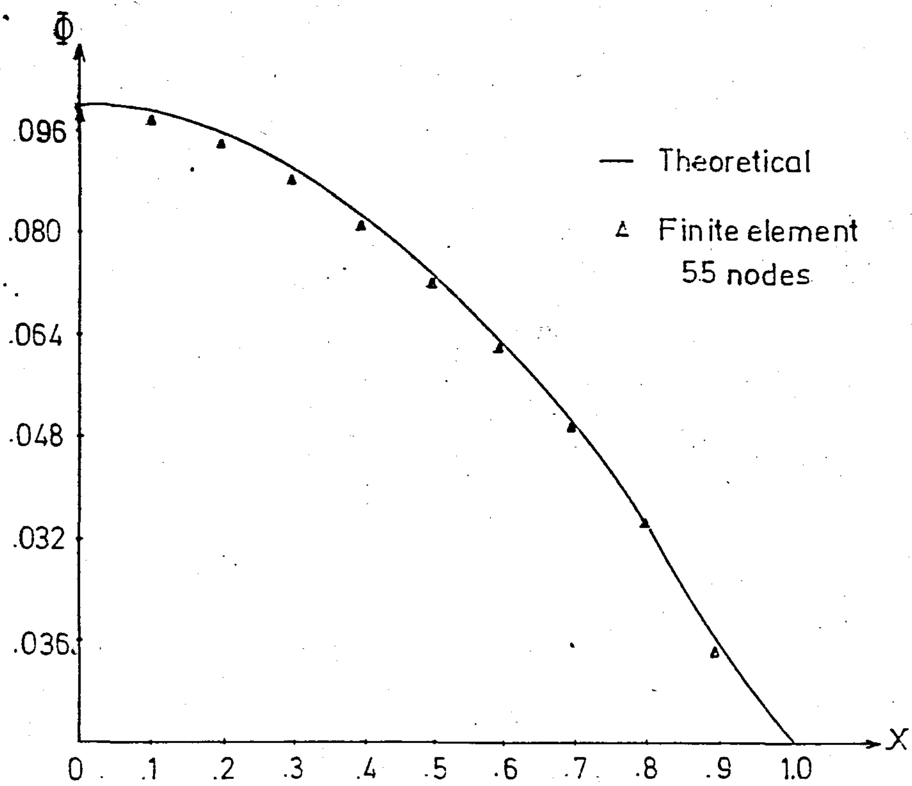

Figure VII.5-Velocity profile along axis of symmetry (Y =0) for s=0.5

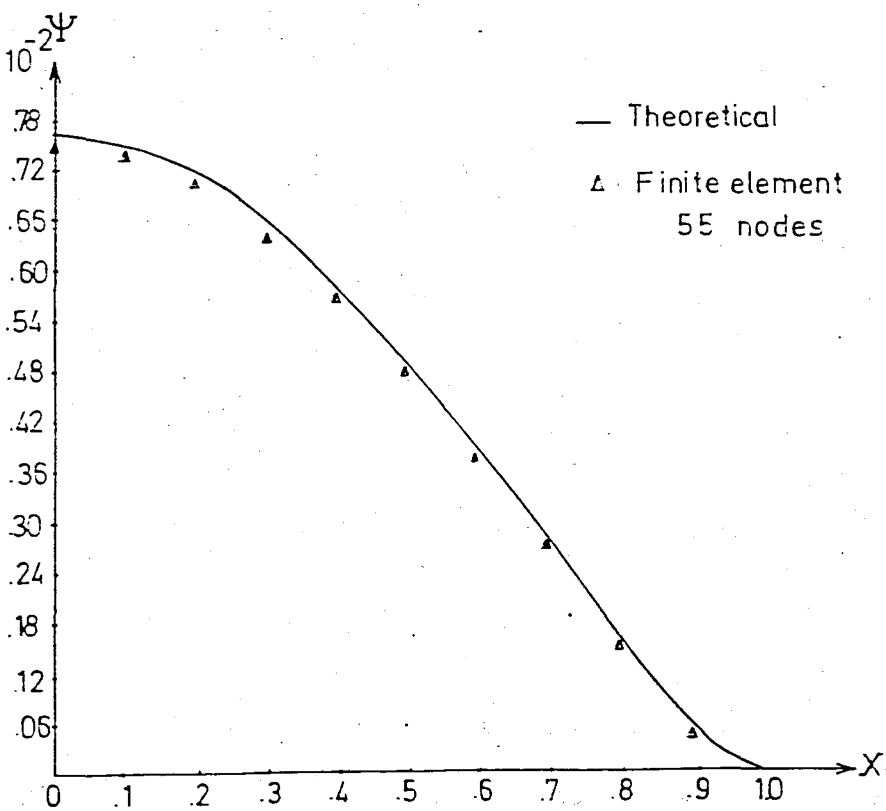

Figure VII.6- Temperature profile along axis of symmetry (Y=0) for s=0.5

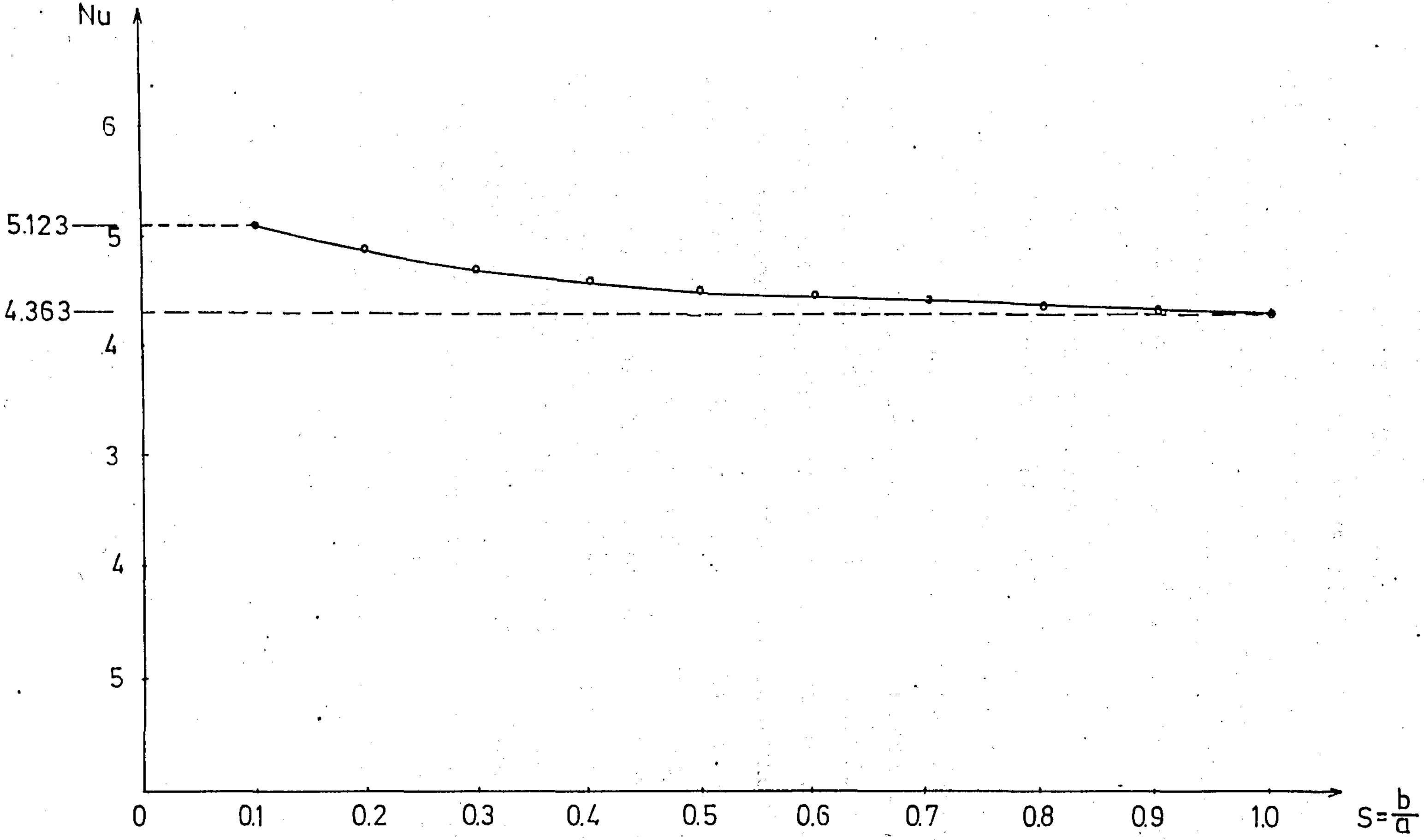

Figure VII-7 Nusselt number variation of an elliptical pipe with respect to aspect ratio S

D. PIPE DESCRIBED BY $x^4 + y^4 = a^4$

The solution for this pipe is analyzed by Complex Variable Method. This method permits exact solution for the velocity field. Expansion of the boundary equation by Binomial Expansion provides approximate solution for the temperature field. The discretization scheme of this geometry is shown in Figure C.6 . It is observed that theoretical results for the temperature field agree with the numerical results well within $|z|=\sqrt{x^2+y^2}<1$. For $|z|>1$ further terms must be included in Binomial Expansion. But in this case the method is not applicable anymore. Considering the Finite Element results to be true (with about 3% difference from actual results) we intuitively add one suitable term to the theoretical solution which makes the theoretical and numerical results comparable. Another approximation is needed to calculate the average velocity and the bulk temperature. This approximation does not disturb the accuracy much since enough terms are taken by Binomial Expansion of the velocity field. Also the expanded velocity field is compared with the exact velocity field to test the accuracy induced by this approximation. Figure VII.8 and Figure VII.9 show theoretical and numerical distributions of the velocity and temperature fields respectively. The Nusselt number is calculated to be 4.197 which lies between the value of 3.60 and 4.363 for square and circular pipes respectively.

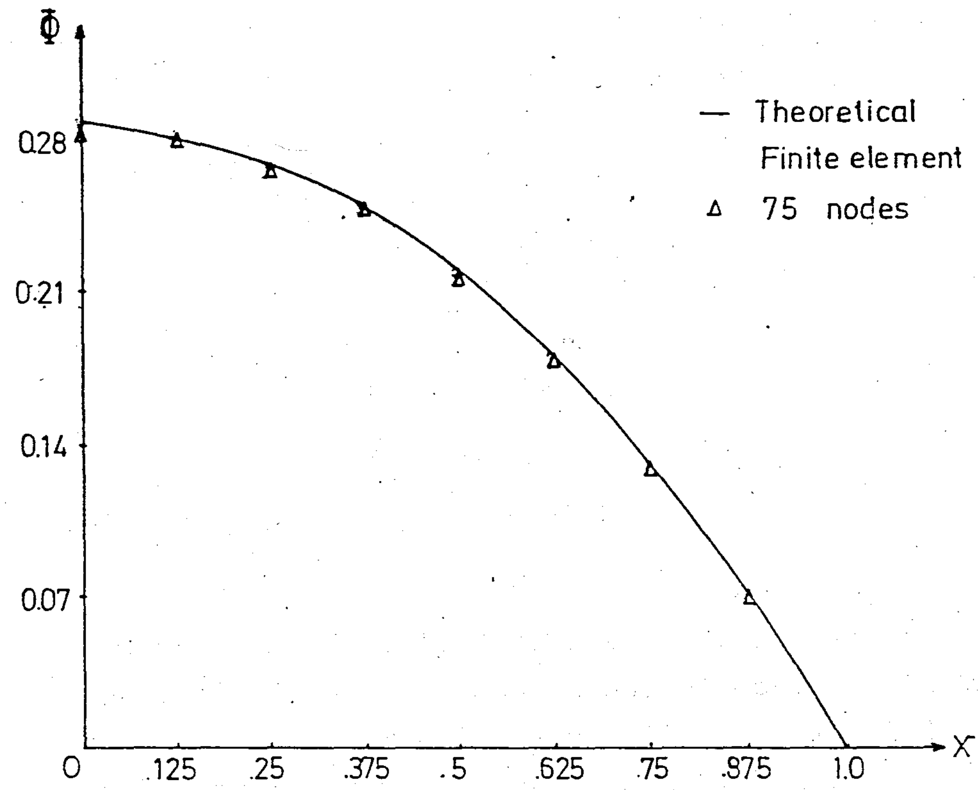

Figure VII.8- Velocity profile along axis of symmetry (Y=0)

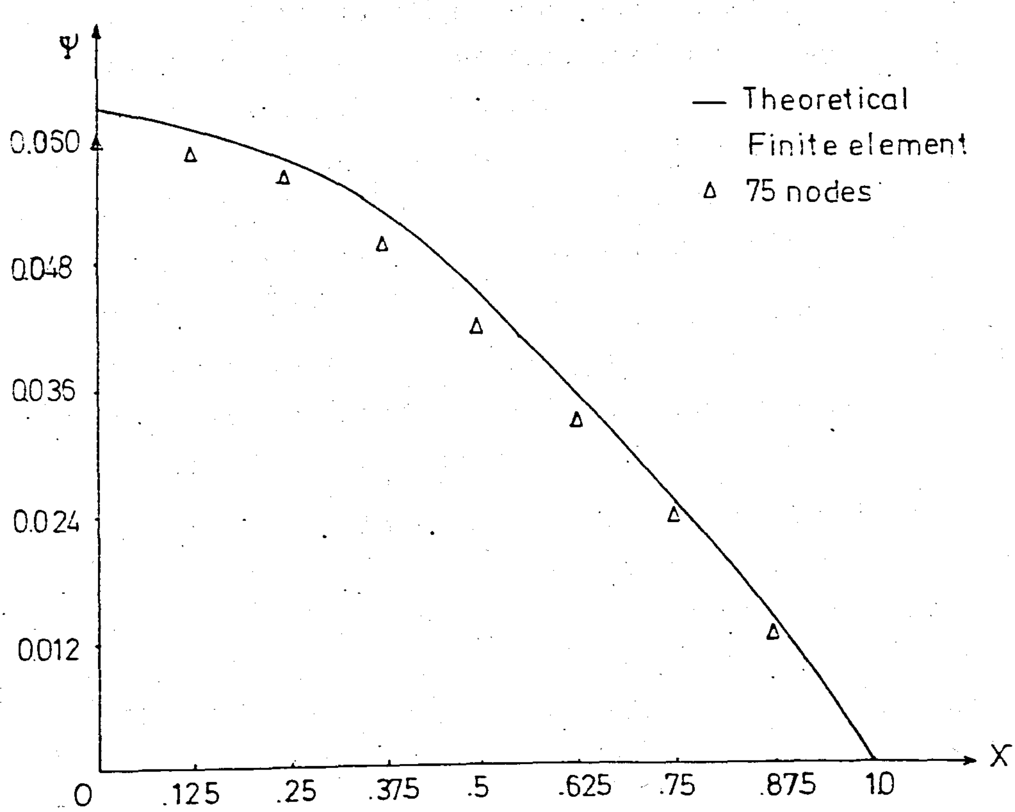

Figure VII.9- Temperature profile along axis of symmetry (Y=0)

VIII

CONCLUSION

The problem of fully developed laminar flow forced convection under the conditions of constant heat flux in axial direction and uniform peripheral wall temperature has been studied for square, triangular and elliptical pipes using Biharmonic solutions, Variational, Complex Variable and Finite Element Methods. Also approximate solutions are obtained for the pipe described by $x^4+y^4=a^4$ by Complex Variable Methods.

The following conclusions may be drawn:

1.Biharmonic solutions are convenient to use for any pipe geometry provided its solutions are known in the thin plate theory.

2. Variational Method although lengthy and approximate leads to accurate results with any desired degree of accuracy for the square pipes.

3. Complex Variable Method can be adopted to studies of certain family of pipes with uniform cross-sectional area, bounded by a closed curve such that the equation of the boundary is expressible as $z\bar{z}=h(z)+\bar{h}(\bar{z})$. The solutions for the velocity and

temperature fields are then deducible directly from the boundary conditions.

4. For laminar flow in pipes the Nusselt number is greatly influenced by the shape of the pipe. Interesting feature of the solutions for shapes with sharp corners, like the square, is the fact that the heat flux varies around the periphery and approaches zero at the corners.
Other factor which influences the Nusselt number is the hydraulic diameter.

5. The highest Nusselt number is obtained for the elliptical pipe depending on the aspect ratio. Two limiting cases are observed for this pipe, that is the Nusselt number at $s = 1$ corresponds to that of circular pipe and that at $s=0$ has no physical meaning.

6. Nusselt number for the pipe given by $x^4+y^4= a^4$ lies between the square and the circular pipe.

IX

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APPENDICES

APPENDIX A

1-Tabulated values of the Perimeter for Elliptical Pipes

The values of the Perimeter P is given in Referance(16) as follows:

$s=\frac{b}{a}$	$\frac{P}{a}$
0.1	4.0640
0.2	4.2020
0.3	4.3860
0.4	4.6026
0.5	4.8442
0.6	5.1054
0.7	5.3824
0.8	5.6723
0.9	5.9723
1.0	2π

2-Solution of some Trigonometric Integrals

General Solutions for some trigonometric integrals are given in Referance(16) as follows:

$$\int \sin^n t\,dt = \frac{\cos t \sin^{n-1} t}{n} + \frac{n-1}{n} \int \sin^{n-2} t\,dt$$

$$\int \cos^n t\,dt = \frac{\sin t \cos^{n-1} t}{n} + \frac{n-1}{n} \int \cos^{n-1} t\,dt$$

Then

$$\int_0^{\pi/2} \sin^2 t\,dt = \frac{\pi}{4} \quad , \quad \int_0^{\pi/2} \cos^2 t\,dt = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \sin^4 t\,dt = \frac{3\pi}{16} \quad , \quad \int_0^{\pi/2} \cos^4 t\,dt = \frac{3\pi}{16}$$

$$\int_0^{\pi/2} \sin^6 t\,dt = \frac{5\pi}{32} \quad , \quad \int_0^{\pi/2} \cos^6 t\,dt = \frac{5\pi}{32}$$

$$\int_0^{\pi/2} \sin^8 t\,dt = \frac{35\pi}{256} \quad , \quad \int_0^{\pi/2} \cos^8 t\,dt = \frac{35\pi}{256}$$

Also

$$\int_0^{\pi/2} \cos^2 t \sin^6 t\,dt = \frac{5\pi}{256}$$

$$\int_0^{\pi/2} \cos^2 t \sin^4 t\,dt = \frac{\pi}{32}$$

$$\int_0^{\pi/2} \sin^2 t \cos^2 t\,dt = \frac{\pi}{16}$$

$$\int_0^{\pi/2} \cos^6 t \sin^2 t\,dt = \frac{5\pi}{256}$$

$$\int_0^{\pi/2} \cos^4 t \sin^2 t\,dt = \frac{\pi}{32}$$

$$\int_0^{\pi/2} \cos^4 t \sin^4 t\,dt = \frac{3\pi}{256}$$

APPENDIX B

1-Values of some Gamma and Beta-Functions

In reference(3), the values of some Gamma-Functions are given as

$\Gamma(\frac{5}{4})=0.906$, $\Gamma(\frac{3}{2})=0.886$, $\Gamma(\frac{7}{4})=0.919$

Noting that $\Gamma(n+1)=n\Gamma(n)$

$\Gamma(\frac{1}{4})=3.6240$, $\Gamma(\frac{1}{2})=1.6120$

$\Gamma(\frac{3}{4})=1.2253$, $\Gamma(\frac{9}{4})=1.1325$

$\Gamma(\frac{11}{4})=1.6080$, $\Gamma(\frac{5}{2})=1.3290$

$\Gamma(\frac{7}{2})=3.3225$, $\Gamma(\frac{9}{2})=11.6287$

$\Gamma(\frac{11}{2})=52.3294$, $\Gamma(\frac{13}{4})=2.5481$

$\Gamma(\frac{13}{2})=287.8115$

Now using the relation $\beta(m,n)=\beta(n,m)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

one obtains

$\beta(\frac{1}{4},\frac{1}{4})=7.4116$, $\beta(\frac{3}{4},\frac{3}{4})=1.6946$

$\beta(\frac{5}{4},\frac{5}{4})=0.6176$, $\beta(\frac{7}{4},\frac{7}{4})=0.2542$

$\beta(\frac{9}{4},\frac{9}{4})=0.1362$, $\beta(\frac{11}{4},\frac{11}{4})=0.0494$

$\beta(\frac{13}{4},\frac{13}{4})=0.0225$, $\beta(\frac{1}{4},\frac{3}{4})=4.4406$

$\beta(\frac{1}{4},\frac{5}{4})=3.7058$, $\beta(\frac{3}{4},\frac{5}{4})=1.1101$

$\beta(\frac{5}{4},\frac{7}{4})=0.4163$, $\beta(\frac{7}{4},\frac{9}{4})=0.1735$

$$\beta(\frac{9}{4},\frac{11}{4})=0.0759 \quad , \quad \beta(\frac{11}{4},\frac{13}{4})=0.0341$$

2-Detailes of Integration corresponding to Chapter V

By definition

$$\beta(n,m)=\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Letting $x=\sin^2\theta$, $1-x=\cos^2\theta$

$$\int_0^{\pi/2}\cos^{-1/2}\theta\sin^{1/2}\theta d\theta=\frac{1}{2}\int_0^1(1-x)^{-3/4}x^{-1/4}dx=\frac{1}{2}\beta(\frac{3}{4},\frac{1}{4})$$

$$\int_0^{\pi/2}\cos^{1/2}\theta\sin^{-1/2}\theta d\theta=\frac{1}{2}\int_0^1 x^{-3/4}(1-x)^{-1/4}dx=\frac{1}{2}\beta(\frac{1}{4},\frac{3}{4})$$

The solution of the following integrals can be found in the same manner as

$$\int_0^{\pi/2}\sin^{1/2}\theta\cos^{1/2}\theta d\theta=\frac{1}{2}\beta(\frac{1}{4},\frac{1}{4})$$

$$\int_0^{\pi/2}\sin^{1/2}\theta\cos^{1/2}\theta d\theta=\frac{1}{2}\beta(\frac{3}{4},\frac{3}{4})$$

$$\int_0^{\pi/2}\sin^{3/2}\theta\cos^{3/2}\theta d\theta=\frac{1}{2}\beta(\frac{5}{4},\frac{5}{4})$$

$$\int_0^{\pi/2}\sin^{5/2}\theta\cos^{5/2}\theta d\theta=\frac{1}{2}\beta(\frac{7}{4},\frac{7}{4})$$

$$\int_0^{\pi/2}\sin^{7/2}\theta\cos^{7/2}\theta d\theta=\frac{1}{2}\beta(\frac{9}{4},\frac{9}{4})$$

$$\int_0^{\pi/2}\sin^{9/2}\theta\cos^{9/2}\theta d\theta=\frac{1}{2}\beta(\frac{11}{4},\frac{11}{4})$$

$$\int_0^{\pi/2}\sin^{11/2}\theta\cos^{11/2}\theta d\theta=\frac{1}{2}\beta(\frac{13}{4},\frac{13}{4})$$

$$\int_0^{\pi/2}\sin^{1/2}\theta\cos^{3/2}\theta d\theta=\frac{1}{2}\beta(\frac{3}{4},\frac{5}{4})$$

$$\int_0^{\pi/2} \sin^{3/2}\theta \cos^{5/2}\theta\, d\theta = \frac{1}{2}\beta\left(\frac{5}{4},\frac{7}{4}\right)$$

$$\int_0^{\pi/2} \sin^{5/2}\theta \cos^{7/2}\theta\, d\theta = \frac{1}{2}\beta\left(\frac{7}{4},\frac{9}{4}\right)$$

$$\int_0^{\pi/2} \sin^{7/2}\theta \cos^{9/2}\theta\, d\theta = \frac{1}{2}\beta\left(\frac{9}{4},\frac{11}{4}\right)$$

$$\int_0^{\pi/2} \sin^{9/2}\theta \cos^{11/2}\theta\, d\theta = \frac{1}{2}\beta\left(\frac{11}{4},\frac{13}{4}\right)$$

3-Calculation of the Area and the Perimeter of the Cross-section given by $x^4+y^4= a^4$

a. Calculation of the Area:

$$A=4\int_0^a \sqrt[4]{a^4-x^4}\, dx$$

Substituting $x^2=a^2\cos\theta$ one gets

$$A=2a^2\int_0^{\pi/2} \sin^{3/2}\theta\cos^{-1/2}\theta\, d\theta$$

Also letting $x= \sin^2\theta$, $1-x=\cos^2\theta$, it follows that

$$A=a^2\int_0^1 x^{1/4}(1-x)^{-3/4}\, dx=a^2\beta\left(\frac{5}{4},\frac{1}{4}\right)=3.7058a^2$$

b. Calculation of the Perimeter:

Letting $X=\frac{x}{a}$, $Y=\frac{y}{a}$, one gets by definition

$$P=4\int_0^1 \sqrt{1+\left(\frac{dY}{dX}\right)^2}\, dX$$

This integral can be approximated by

$$P=4\sum_{n=1}^{\infty} \sqrt{1+\frac{\Delta Y}{\Delta X}}\, \Delta X$$

where $\Delta X=0.01$, $\Delta Y=Y_{n-1} - Y_n$, $Y_n=\sqrt{1-X_n^4}$,

$Y_0=1$, $X_0=0$ and $X_n=X_{n-1}+\Delta X$

The calculation is done in Computer using Double Precision and the perimeter is obtained as $P= 7.0168$.

APPENDIX C

Discretization of the Domains

1. Square Pipe

The subdivision of this geometry is done in two different ways. In the first study, a grid of 72 triangular elements of unequal size with 49 nodes (Figure C.1) and in the second study, a grid of 128 triangular elements of equal size with 81 nodes (Figure C.2) is used.

2. Equilateral Triangular Pipe

The division of this pipe geometry is also done in two different ways. First it is devided into 81 triangular elements of unequal size with 55 nodes (Figure C.3) and then in 100 triangular elements of equal size with 66 nodes(Figure C.4)

3. Elliptical Pipe

The discretization of this pipe domain is done with 82 triangular elements and 55 nodes (Figure C.5)

4. Pipe Described by $x^4+y^4= a^4$

This geometry is devided into 119 triangular elements with 75 nodes (Figure C.6)

Equations of each pipe geometries are dimensionalized by $X=\frac{x}{a}$, $Y=\frac{y}{a}$ and only necessary parts of the domains are descretized due to axial symmetry.

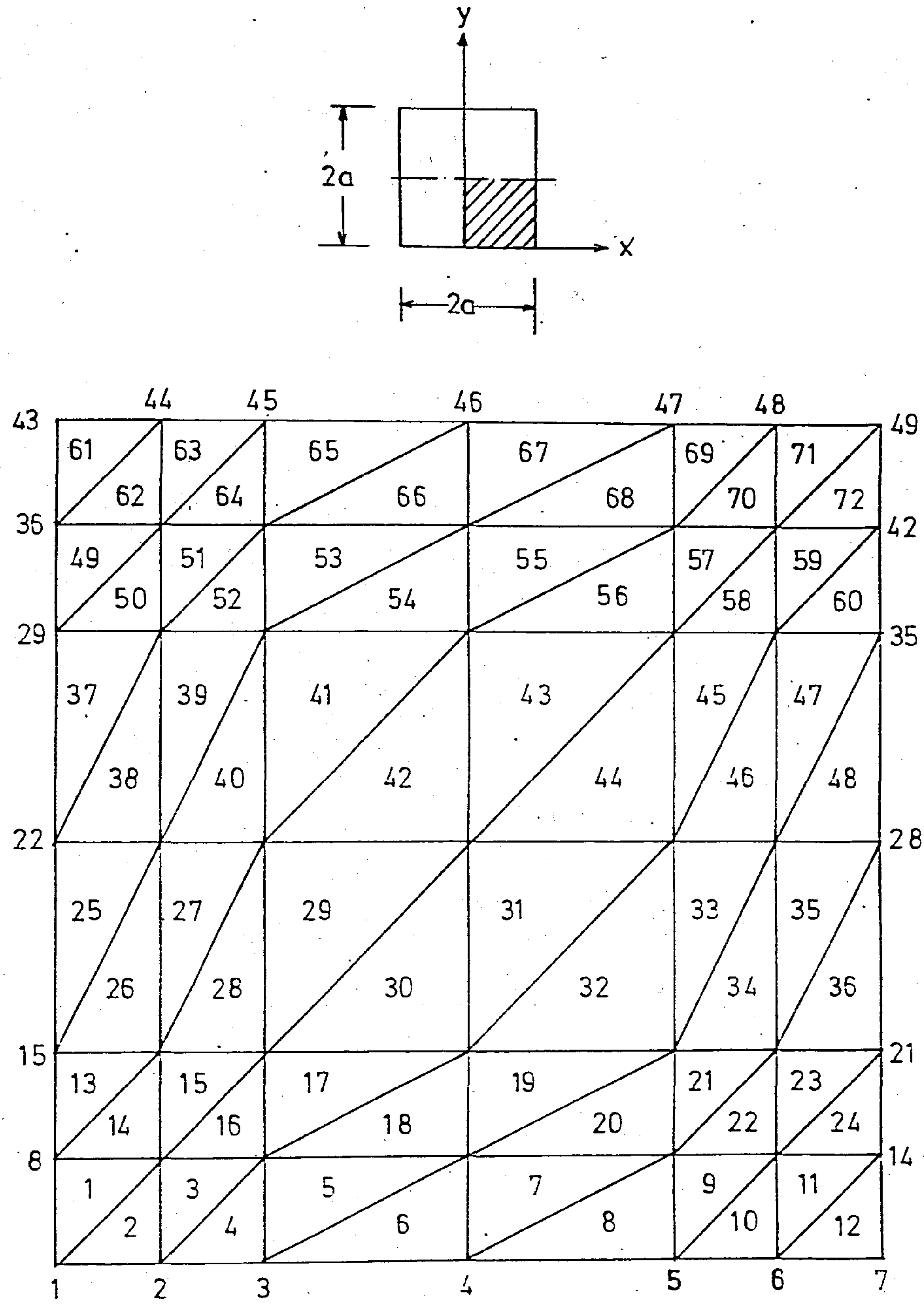
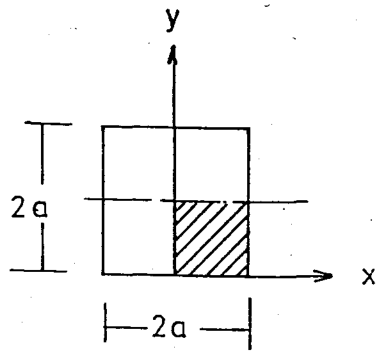


Figure C.1- Subdivision of a Square Pipe into Triangular Finite Elements of unequal Size



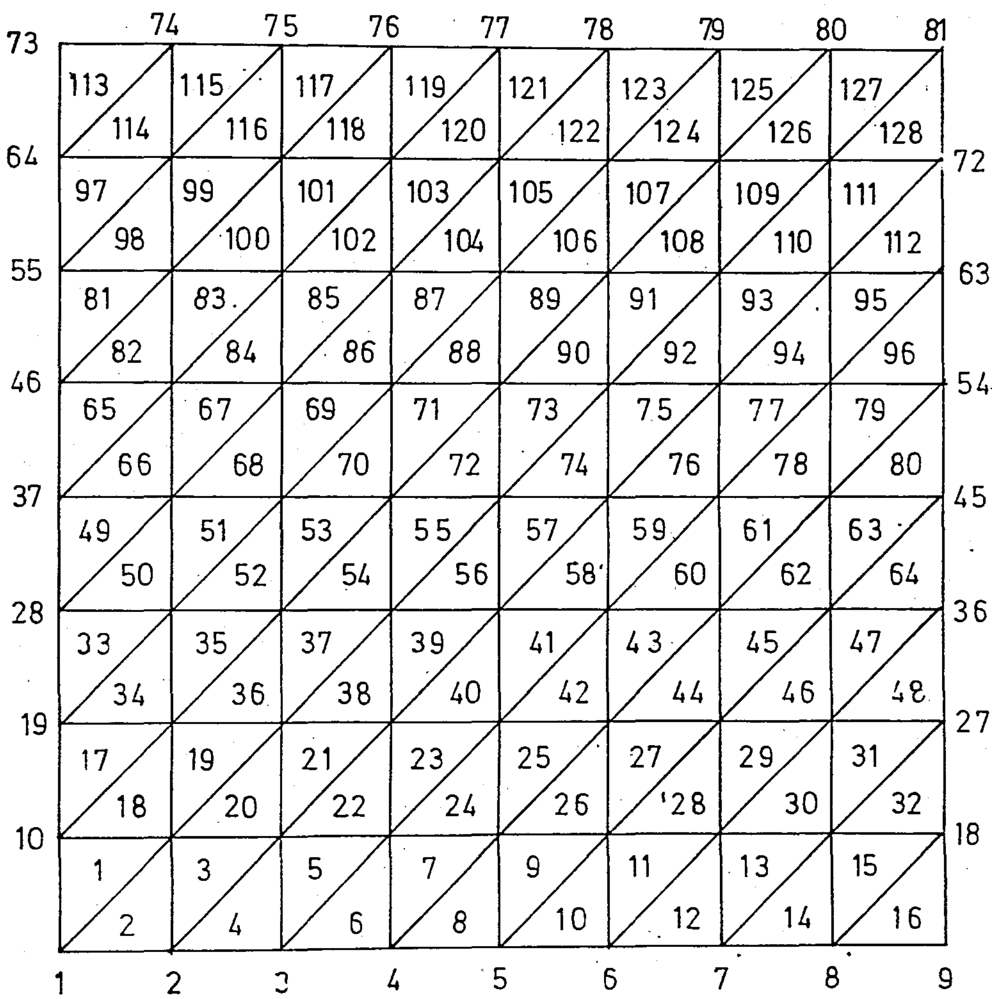
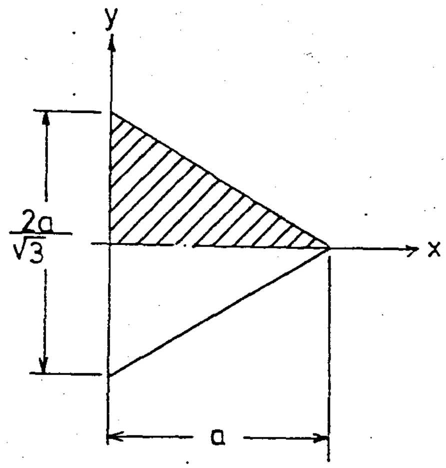


Figure C.2- Subdivision of a Square Pipe into Triangular Finite Elements of equal Size



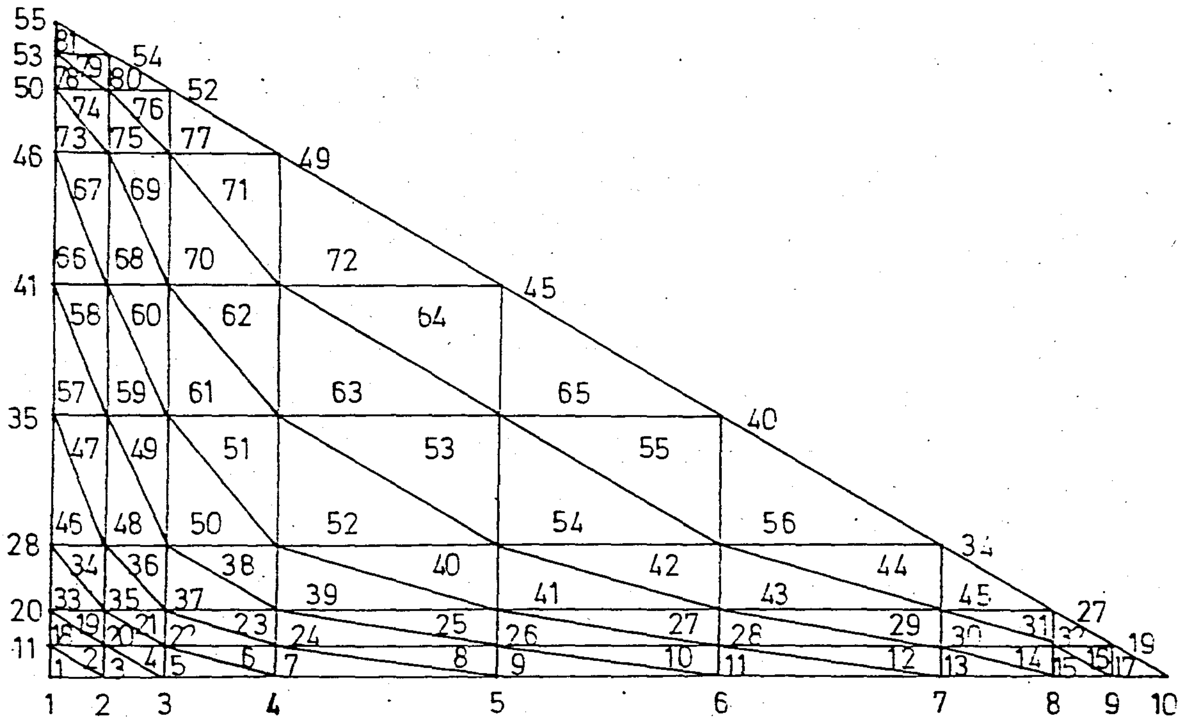
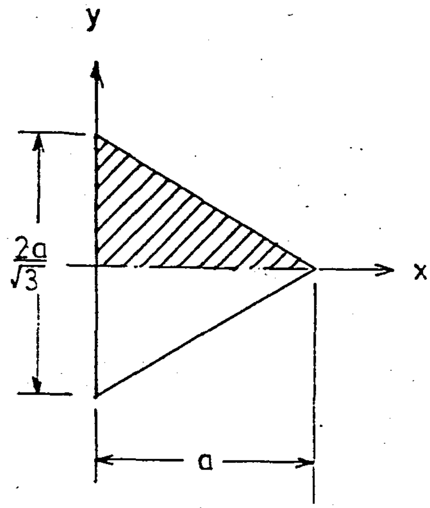


Figure C.3-Subdivision of a Equilateral Triangular Pipe into Triangular Finite Elements of unequal Size



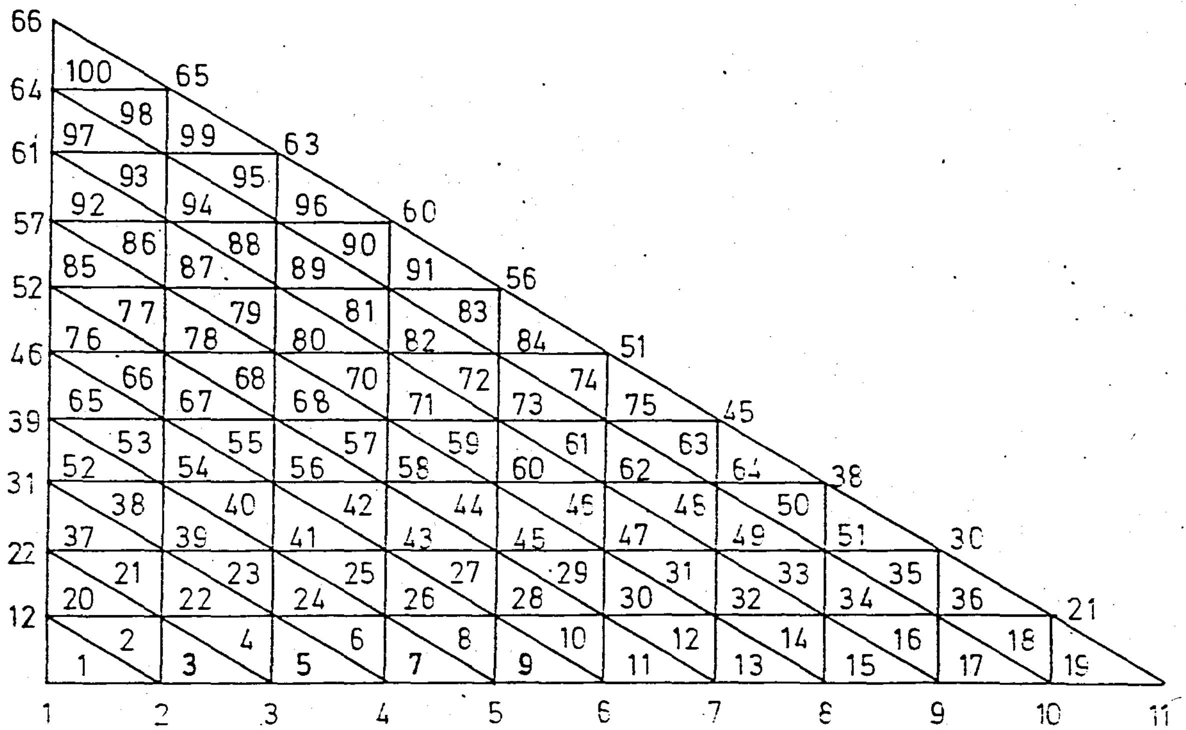


Figure C.4- Subdivision of a Equilateral Triangular Pipe into Triangular Finite Elements of equal Size

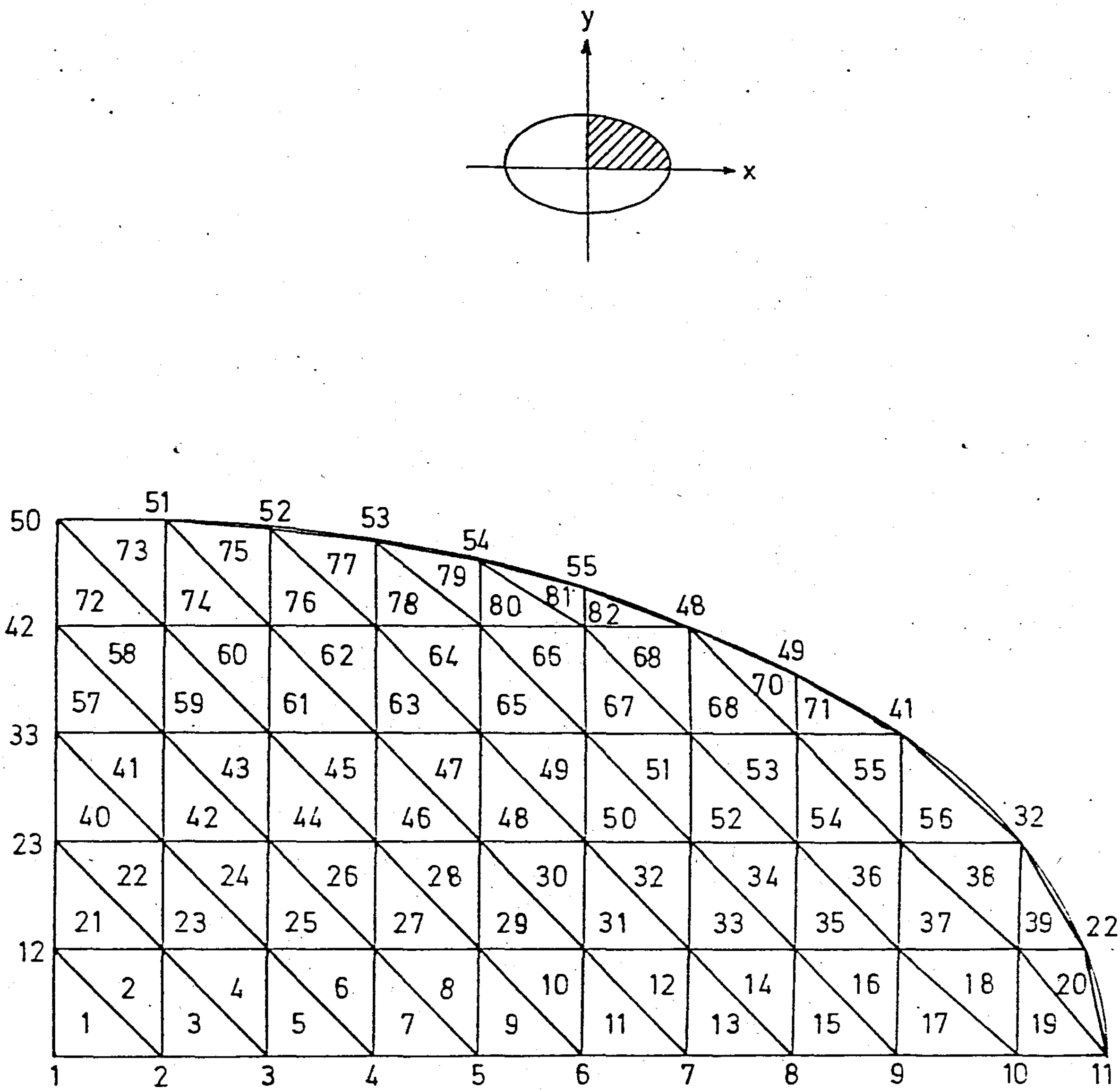


Figure C.5- Subdivision of an Elliptical Pipe into Triangular Finite Elements

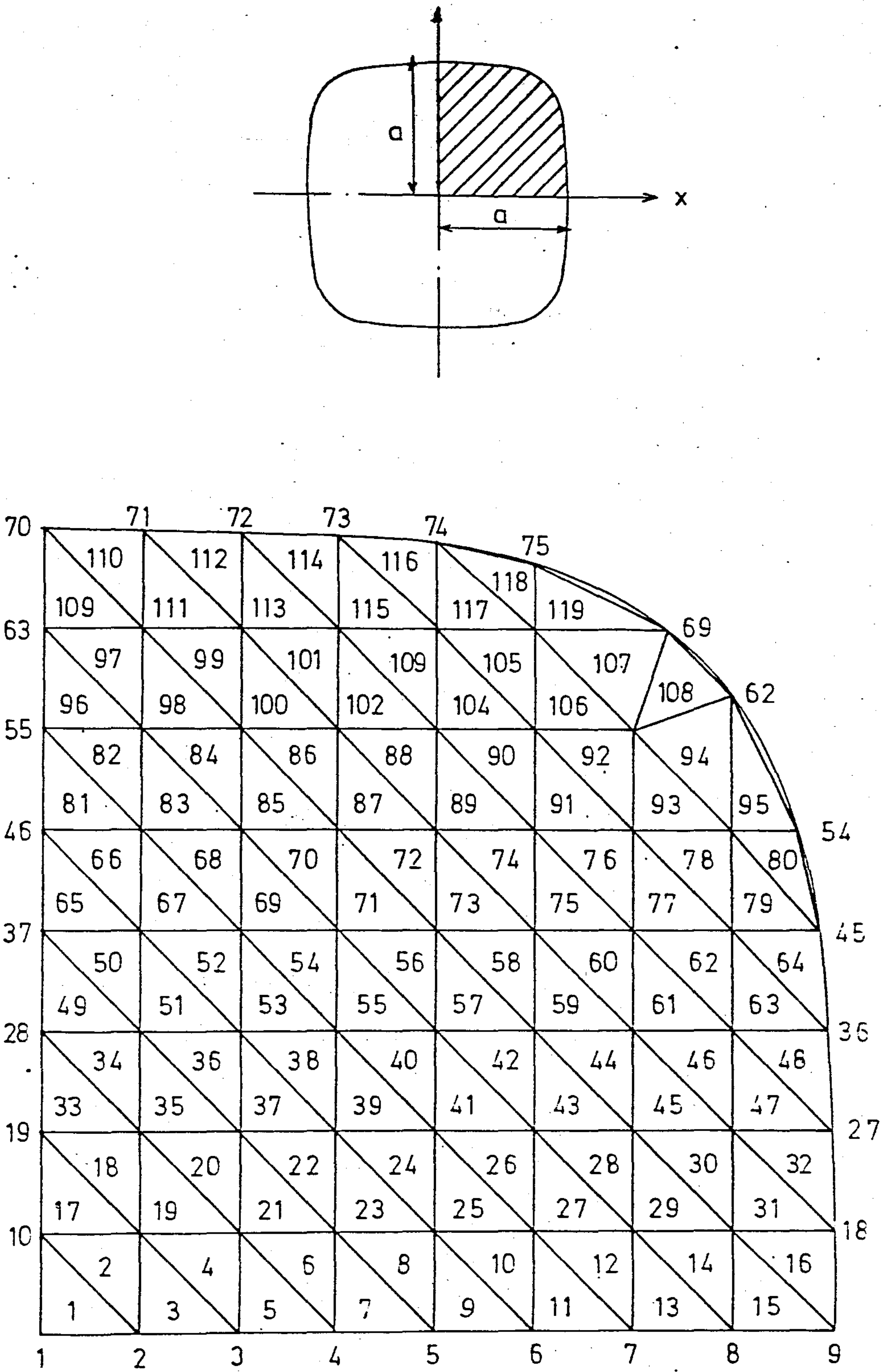
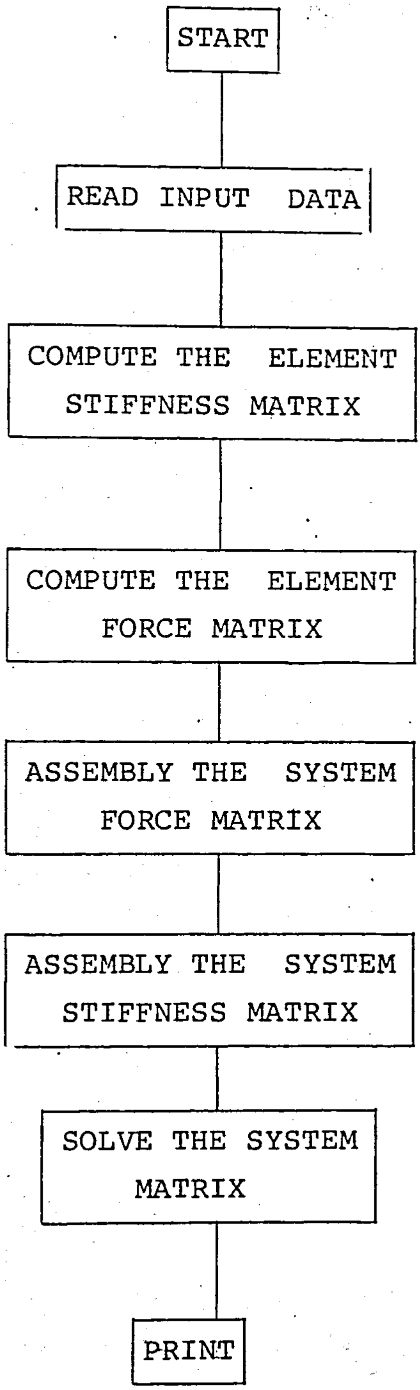


Figure C.6- Subdivision of a Pipe described by $x^4+y^4=1$ into Triangular Finite Elements

APPENDIX D

1. COMPUTER PROGRAM FLOWCHART



2. COMPUTER PROGRAM LOGIC

The computer program consists of a main program and two subroutines.

MAIN PROGRAM: Reads some parameters, necessary constants and the boundary conditions as input data, determines the element stiffness matrix and the element force matrix. The system force matrix and the system stiffness matrix are assembled by the use of code numbers. It prints out the results and stops the program.

SUBROUTINE POW: This subroutine forms the element force matrix.

SUBROUTINE GAUSS: This subroutine solves the system matrix.

Definition of the variable names used in the Program:

- MST - Degree of freedom
- NJ - Number of nodes at each element
- MS - Number of unknowns at each element
- MEN - A control parameter. It takes the values 0,1,2,3 for square, equilateral triangular, elliptical pipes and the pipe described by $x^4+y^4= a^4$ respectively.
- ME - Total number of elements
- NNO - Total number of nodes
- N - Total number of unknowns
- DEF - A control parameter. It takes the value "0" on the boundaries. Otherwise it numbers the unknowns.
- IDEF- It labelles the unknowns at the nodes
- RR - System force matrix corresponding to momentum equation

S - System Stiffness matrix
M - Specific node number
V - Reads node numbers of each element
NCODE-Code numbers
SM - Element Stiffness matrix
R PW- Element force matrix corresponding to momentum equation
THV - Theoretical velocity distribution
ERV - Absolute error
D - Numerical velocity distribution
TEMP- Assembly velocity vector for each element
ST - Convection related matrix
RS - Element force matrix corresponding to energy equation
RT - System force matrix corresponding to energy equation
THT - Theoretical temperature distribution
TT - Numerical temperature distribution

Input Description:

1. The first number given in this set is a control parameter. It takes the values 0,1,2,3 for square, equilateral triangular, elliptical pipes and the pipe described by $x^4+y^4= a^4$ respectively.
The second number gives total number of elements.
The third number gives total number of nodes.
Format:(3 I 3)

2. The first number labelles the nodes.
The second and the third numbers give the x and y coordinates of nodal points respectively.
The forth is a control parameter. It takes the value "0" on the boundaries. Otherwise it numbers the unknowns. The fifth prepares x-coordinates for graphical distribution of the results.
Format: (3F8.0, F4.0, F5.0)

3. The first number indicates specific node number.
The other numbers are node numbers of each element.
Format:(I3,3F5.0)
On the output nodal values of the theoretical and numerical velocity and temperature with the correspon-ding absolute error are listed.
Format: (14x,F4.0,7x,F12.8,7x,F12.8,10x,F12.8)
All results are plotted by Graph 4 Libary Bu*Bulib.

```
C                        .....MAIN PROGRAM.....
      PARAMETER NN=100
      DIMENSION SM(3,3),S(NN,NN),F(3),RR(NN),D(NN),YY(3)
      DIMENSION COR(100,2),JDEF(NN),V(3),NCODE(3),XX(3)
      DIMENSION XC(NN),YC(NN),C1(3),TT(NN),DEF(NN),B1(3)
      DIMENSION RT(NN),RS(3),ST(3,3),THT(NN),THV(NN)
      DIMENSION ERV(NN),TEMP(3),BON(NN),ERT(NN),CX(NN)
      DATA MST,NJ,MS/1,3,3/
      READ(5,7)MEN,ME,NNO
    7 FORMAT(3I3)
      L=0
      N=0
      WRITE(5,700)
  700 FORMAT(///////,40X,,VELOCITY DISTRIBUTION:,)
C     .....LOOP OVERALL NODAL POINTS.....
      DO 5 J=1,NNO
      READ(5,12)XN,XC(J),YC(J),DEF(J),CX(J)
   12 FORMAT(3F8.0,F4.0,F5.0)
      JN=XN
      COR(JN,1)=XC(J)
      COR(JN,2)=YC(J)
C     .....UNKNOWNS ARE NUMBERED AND STORED IN JDEF.....
      L=L+1
      IF(DEF(J).EQ.0.) GO TO 15
      N=N+1
      JDEF(L)=N
      GO TO 5
   15 JDEF(L)=0
    5 CONTINUE
      CALL GRAPH4(10.,10.,100,XC,YC)
      WRITE(5,701)
  701 FORMAT(///,10X,,NODE NUMBER:,,2X,,NUMERICAL RESULTS:,
     $,2X,,THEORETICAL RESULTS:,,2X,,PERCENTAGE ERROR %:,)
      DO 18 I=1,N
      RR(I)=0.
      DO 18 J=1,N
   18 S(I,J)=0.
C     .....LOOP OVERALL ELEMENTS.....
      DO 75 MI=1,ME
      READ(5,55)M,(V(I),I=1,NJ)
   55 FORMAT(I3,3F5.0)
C     .....DERIVATION OF CODE NUMBERS.....
      DO 60 I=1,NJ
      JN=V(I)
      IX=JN
   60 NCODE(I)=JDEF(IX)
      WRITE(7)NCODE
      DO 2 I=1,NJ
      J=V(I)
      XX(I)=COR(J,1)
    2 YY(I)=COR(J,2)
C     .....DETERMINATION OF ELEMENT STIFFNESS MATRIX.....
      B1(1)=YY(2)-YY(3)
      B1(2)=YY(3)-YY(1)
      B1(3)=YY(1)-YY(2)
      C1(1)=XX(3)-XX(2)
      C1(2)=XX(1)-XX(3)
      C1(3)=XX(2)-XX(1)
      AR=ABS(C1(1)*B1(3)-C1(3)*B1(1))
      WRITE(7) AR
      SOR=1./(2.*AR)
      DO 11 I=1,3
      DO 11 J=1,3
   11 SM(I,J)=(B1(I)*B1(J)+C1(I)*C1(J))*SOR
      CALL POW(F,AR)
C     .....ASSEMLY OF SYSTEM FORCE MATRIX
C     OF THE MOMENTUM EQUATION USING CODE NUMBERS.....
      DO 63 I=1,MS
      IF(NCODE(I).EQ.0) GO TO 63
      K=NCODE(I)
      RR(K)=RR(K)+F(I)
   63 CONTINUE
C     .....ASSEMLY SYSTEM STIFFNESS MATRIX
C     USING CODE NUMBERS.....
      DO 97 I=1,MS
      IF(NCODE(I).EQ.0) GO TO 97
      K=NCODE(I)
      DO 97 J=1,MS
      IF(NCODE(J).EQ.0) GO TO 97
      L=NCODE(J)
```

```
      S(K,L)=S(K,L)+SM(I,J)
   97 CONTINUE
C     .....CALCULATE THE THEORETICAL VELOCITY DISTRIBUTION
C     FOR THE SPECIFIED PIPE GEOMETRY.....
      IF(MI.GT.NNO)GO TO 75
      IF(DEF(MI).EQ.0.)GO TO 75
      IM=DEF(MI)
      BON(IM)=DEF(MI)
      IF(MEN.EQ.0) GO TO 830
      IF(MEN.EQ.2) GO TO 878
      IF(MEN.EQ.3) GO TO 971
      THV(IM)=XC(MI)/4.*(XC(MI)**2.-2.*XC(MI)-3.*YC(MI)**2.+1.)
      GO TO 75
  830 THV(IM)=1./2.*(XC(MI)**2.-1.)*(YC(MI)**2.-2.*YC(MI))*(0.58984
     $+0.08203*(XC(MI)**2.+YC(MI)**2.-2.*YC(MI)+1.)+0.24510*XC(MI)**2.
     $(YC(MI)**2.-2.*YC(MI)+1.))
      GO TO 75
  878 THV(IM)=-0.4*(0.25*XC(MI)**2.+YC(MI)**2.-0.25)
      GO TO 75
  971 THV(IM)=1./4.*(-XC(MI)**2.-YC(MI)**2.+(4.-XC(MI)**4.
     $-YC(MI)**4.+6.*XC(MI)**2.*YC(MI)**2.)**0.5/(3.**0.5))
   75 CONTINUE
      WRITE(7)S
C     .....THE SYSTEM IS SOLVED BY GAUSSIAN ELIMINATION.....
      CALL GAUSS(S,D,RR,N)
      DO 121 I=1,N
      ERV(I)=ABS(THV(I)-D(I))*100./THV(I)
  121 WRITE(6,200)BON(I),D(I),THV(I),ERV(I)
  200 FORMAT(14X,F4.0,7X,F12.8,7X,F12.8,10X,F12.8)
      WRITE(6,705)
  705 FORMAT(///////////,40X,'THEORETICAL VELOCITY DISTRIBUTION:')
      CALL GRAPH4(10.,10.,100,CX,THV)
      WRITE(6,706)
  706 FORMAT(///////////////,40X,'NUMERICAL VELOCITY DISTRIBUTION:')
      CALL GRAPH4(10.,10.,100,CX,D)
      WRITE(6,702)
  702 FORMAT(//////////,40X,'TEMPERATURE DISTRIBUTION:')
      CALL GRAPH4(10.,10.,100,XC,YC)
      WRITE(6,703)
  703 FORMAT(///,10X,'NODE NUMBER:',2X,'NUMERICAL RESULTS:'
     $,2X,'THEORETICAL RESULTS:',2X,'PERCENTAGE ERROR %:')
      REWIND 7
      DO 321 I=1,N
  321 RT(I)=0.
C     .....ASSEMBLY SYSTEM FORCE MATRIX
C     OF THE ENERGY EQUATION USING CODE NUMBERS.....
      DO 217 MK=1,ME
      READ(7)NCODE
      DO 10 I=1,MS
   10 TEMP(I)=0.
      DO 90 I=1,MS
      IF(NCODE(I).EQ.0) GO TO 90
      K=NCODE(I)
      TEMP(I)=D(K)
   90 CONTINUE
      READ(7) AR
      CALL POW(F,AR)
      DO 61 I=1,3
   61 ST(I,I)=AR/12.
      DO 65 J=2,3
      ST(1,J)=AR/24.
   65 ST(J,1)=AR/24.
      ST(2,3)=AR/24.
      ST(3,2)=AR/24.
      DO 322 I=1,MS
      RS(I)=0.
      DO 322 J=1,MS
  322 RS(I)=RS(I)+ST(I,J)*TEMP(J)
      DO 323 I=1,MS
      IF(NCODE(I).EQ.0.) GO TO 323
      K=NCODE(I)
      RT(K)=RT(K)+RS(I)
  323 CONTINUE
C     .....CALCULATE THE THEORETICAL TEMPERATURE DISTRIBUTION
C     FOR THE SPECIFIED PIPE GEOMETRY.....
      IF(MK.GT.NNO)GO TO 217
      NM=DEF(MK)
      IF(MEN.EQ.0) GO TO 831
      IF(MEN.EQ.2) GO TO 871
      IF(MEN.EQ.3) GO TO 992
      THT(NM)=-XC(MK)/64.*(YC(MK)**2.+YC(MK)**2.-2./3.*XC(MK)-1./3.)
     $*(XC(MK)**2.-2.*XC(MK)-3.*YC(MK)**2.+1.)
```

```
      GO TO 217
  831 THT(NM)=-0.28111/4.*(XC(MK)**2.-1.)*(YC(MK)**2.-2.*YC(MK)
     $)*(-0.9267+0.1356*(XC(MK)**2.+YC(MK)**2.-2.*YC(MK)+1.))
      GO TO 217
  871 THT(NM)=(0.25*XC(MK)**2.+YC(MK)**2.-0.25)
     $*(1.3125*XC(MK)**2.+2.25*YC(MK)**2.-2.3625)/76.875
      GO TO 217
  992 THT(NM)=1./64.*(4.+(XC(MK)**4.+YC(MK)**4.-6.*YC(MK)**2.*
     $YC(MK)**2.)*((XC(MK)**2.+YC(MK)**2.)/(5.*3.**0.5)-7./15.)+(XC(M
     $)**2.+YC(MK)**2.)*(XC(MK)**2.+YC(MK)**2.-8./(3.**0.5))-1./(20.*
     $.**0.5)*(XC(MK)**4.+YC(MK)**4.-6.*XC(MK)**2.*YC(MK)**2.)**2.)
  217 CONTINUE
      READ(7)S
C     .....THE SYSTEM IS SOLVED BY GAUSSIAN ELIMINATION.....
      CALL GAUSS(S,TT,RT,N)
      DO 341 I=1,N
      ERT(I)=ABS(THT(I)-TT(I))*100./THT(I)
  341 WRITE(6,301)BON(I),TT(I),THT(I),ERT(I)
  301 FORMAT(14X,F4.0,7X,F12.8,7X,F12.8,10X,F12.8)
      WRITE(6,708)
  708 FORMAT(///,40X,,THEORETICAL TEMPERATURE DISTRIBUTION:,)
      CALL GRAPH4(10.,10.,100,CX,THT)
      WRITE(6,709)
  709 FORMAT(///,40X,,NUMERICAL TEMPERATURE DISTRIBUTION:,)
      CALL GRAPH4(10.,10.,100,CX,TT)
      STOP
      END
```

```
      SUBROUTINE POW(PW,AR1)
      DIMENSION PW(3)
      DO 3 I=1,3
    3 PW(I)=AR1/5.
      RETURN
      END
```

```
      SUBROUTINE GAUSS(A,X,B,N)
      PARAMETER NN=100
      DIMENSION A(NN,NN),X(NN),B(NN)
      NMI=N-1
      DO 5 J=1,NMI
      JPI=J+1
      DO 5 I=JPI,N
      A(I,J)=A(I,J)/A(J,J)
      DO 5 K=JPI,N
      A(I,K)=A(I,K)-A(I,J)*A(J,K)
    5 CONTINUE
      DO 30 J=1,NMI
      JPI=J+1
      DO 30 I=JPI,N
   30 B(I)=B(I)-A(I,J)*B(J)
      X(N)=B(N)/A(N,N)
      DO 101 K=2,N
      J=N-K+1
      SUM=0.
      JJ=J+1
      DO 115 I=JJ,N
  115 SUM=SUM+A(J,I)*X(I)
  101 X(J)=(B(J)-SUM)/A(J,J)
      RETURN
      END
```