THE FRICTIONLESS CONTACT PROBLEM FOR

AN ELASTIC LAYER RESTING ON

AN ELASTIC HALF-SPACE

by

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ABSTRACT

The frictionless contact problem for an elastic layer lying on an elastic half space is considered. It is assumed that, in addition to the main applied load P, the layer is subjected to a uniform vertical body force ρg because of the effect of gravity in the layer. It is also assumed that the contact between the layer and the half space is frictionless and that only compressive normal tractions can be transmitted through the interface. Thus, up to a certain critical magnitude of the applied load the contact between the layer and the foundation is continuous. For the values of the load exceeding this critical value, the layer is partially separated from the subspace. The separation area increases with the increasing magnitude of the load.

First, the problem of continuous contact is solved and the value Pcr is determined. Then the discontinuous contact problem is formulated in terms of a singular integral equation. The problem is formulated as a mixed boundary value problem and solved by using a treatment similar to the crack problems. The separation area along the interface is evaluated as a function of a dimensionless load parameter λ.

The contact stress distribution is obtained for various values of corresponding to both continuous and discontinuous contact along the interface. Numerical results for Pcr, contact stress distribution, and separation regions are given for various material combinations.

ÖZET

Bu çalışmada elastik yarım düzleme oturan bir plaktaki sürtünmesiz temas problemi göz önüne alınmıştır.Plağın uygulanan P yüküne ek olarak , plaktaki yerçekimi etkisinden dolayı oluşan kütle kuvvetine de maruz kaldığı varsayılmıştır.Ayrıca plak ile yarım düzlem arasındaki temasın sürtünmesiz olduğu ve temas yüzeyinin sadece basınç gerilmelerini aktardığı da kabul edilmiştir.Böylece uygulanan kritik bir yüke kadar plak ile yarım düzlem arasındaki temas süreklidir.Bu kritik yükü aşan yük değerinden sonra plak düzlemden kısmen ayrılır.Artan yük miktarına göre ayrılma bölgesi de artar.

Önce sürekli temas problemi çözülür ve Pcr değeri belirlenir.Daha sonra süreksiz temas problemi tekil integral denklem problemine dönüştürülür.Problem karışık sınır değer problemi olarak formule edilir ve çatlak problemlerine uygulanan yöntemlere benzer bir yöntemle çözülür.Temas yüzeyindeki ayrılma bölgesi , boyutsuz yük parametresi λ'nın bir işlevi olarak elde edilir.

Sürekli ve süreksiz temaslar için temas yüzeyindeki gerilme dağılımı değişik λ değerleri için elde edilir.Değişik malzemeler için Pcr temas gerilmesi dağılımı ve ayrılma bölgelerinin sayısal çözümleri verilmiştir.

LIST OF FIGURES

LIST OF SYMBOLS

ρ	Density of the elastic layer
g	Gravitational acceleration
F_y	Body force acting in negative y direction due to gravity
h	The thickness of the elastic layer
$k_1(x,t)$	Fredholm Kernel
$k_2(x)$	Bounded Function
P	Compressive load acting on the layer in negative y direction
P_{cr}	Critical load
u	Displacement in x direction
v	Displacement in y direction
X_{cr}	Distance of the point at which the interface separation starts
$\varkappa$	Elastic constant
ν	Poisson's ratio
μ	Shear modulus
λ	Load parameter
λ_{cr}	Critical load parameter
σ_{xx}	Stress in x direction
σ_{yy}	Stress in y direction
σ_{xy}	Shear stress in x,y direction
$\delta(x)$	Dirac delta function

1. INTRODUCTION

Because of their important applications in structural engineering in the past, the contact problems in solid mechanics have been widely studied (see, for example [1] for the pioneering work in the field and [2] for modern development). In earlier works the condition along the contacting surfaces was assumed to be one of either perfect adhesion or " frictionless contact " and to be continuous. For the frictionless case regardless of the sign of the normal stress the contact along the interface was assumed to be continuous. This unrealistic assumption was first corrected by introducing the notion of " receding contact " condition which simply states that along the interface the contact can be maintained only if the normal traction is compressive [3]. Some typical applications of this concept to the contact problems for an elastic layer on a rigid or an elastic foundation may be found in [4-9]. A major drawback of the type of solutions given in [4-9] is that as soon as the load is applied no matter how small in magnitude, the (infinite) layer " bends " and the contact area along the layer subspace interface diminishes to a finite size, remaining constant thereafter as the magnitude of the load is increased. This, of course, will not be the case if the effect of gravity is considered. Some examples taking the effect of gravity into account may be found in [10-15]. In references [10-14] the layer rests on a frictionless horizontal, rigid foundation. However in [15] the subspace is elastic.

In this work the standard frictionless contact problem for an infinite layer (i.e., a beam or a plate) lying on an elastic foundation is reconsidered (Figure 1). It will be assumed that in addition to the (compressive) external load P, the layer is subjected to a uniform body force $F_y = -\rho g$ acting in the same direction as P. The problem is identical to that considered in [15]. In this work a new bimaterial constant β is defined and β takes values in a finite range ($-1 < \beta < 1$). The calculation of the infinite integrals that appear in the kernels have been improved by extracting some integrals which can be evaluated in closed form.

In this problem when the applied load P reaches a critical value , due to the bending of the layer the separation will begin at $X=X_{cr}$ on the interface and the separation area (c-b) , ($b < X_{cr} < c$) will increase as the applied load P increases. Of course , in this case the separation area is finite.

The discontinuous contact problem is reduced to a singular integral equation which is solved numerically. The unknown function of the singular integral equation is the dislocation density of the separation area. Once this dislocation density is determined then the contact stress acting along the interface can be determined by some numerical calculations . Numerical results are obtained and plotted for various material combinations.

11. GENERAL EQUATIONS

Consider an infinite elastic layer of thickness h in smooth contact with a semi-infinite elastic foundation.The geometry and the coordinate system are shown in Figure 1.Let ρg be the body force density acting vertically down in the layer and note that the body force density acting in the foundation is neglected since it does not disturb the contact pressure.The governing equation for the elastic layer are

$$\mu_1\nabla^2 u_1 + (\lambda_1+\mu_1)\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)=0 ,$$

$$\mu_1\nabla^2 v_1 + (\lambda_1+\mu_1)\frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)=\rho g , \quad (2.1\ a,b)$$

where λ_1 and μ_1 are Lamé constants , u_1 and v_1 are the X and Y com - ponents of the displacement vector.Expressing the solution of (2.1) by

$$u_1 = u_{1p} + u_{1h} ,$$

$$v_1 = v_{1p} + v_{1h} , \quad (2.2\ a,b)$$

the particular solution of the problem corresponding to the non-homogeneous term ρg may be expressed as

$$u_{1p} = \frac{3-\kappa_1}{16\mu_1}\rho g h x , \quad 0 \le x < \infty ,$$

$$v_{1p} = \frac{\rho g}{2\mu_1} y\left[\frac{\kappa_1-1}{\kappa_1+1}(y-h) - \frac{\kappa_1+1}{8}h\right] . \quad (2.3\ a,b)$$

$$0 < y < h$$

And the related stress components are

$$\sigma^{p}_{1xx} = \rho g (y-h) \quad ,$$

$$\sigma^{p}_{1yy} = \rho g \left(y-\frac{h}{2}\right)\frac{\lambda_1}{\lambda_1+2\mu_1} \quad ,$$

$$\sigma^{p}_{1xy} = 0 \quad . \quad (2.4\ a\text{-}c)$$

For the homogeneous solution , observing

$$U_{1h}(x,y) = -U_{1h}(-x,y) \quad ,$$

$$V_{1h}(x,y) = V_{1h}(-x,y) \quad , \quad (2.5\ a,b)$$

one may express

$$U_{1h}(x,y) = \frac{2}{\pi}\int_0^\infty \phi(y,\alpha)\,\mathrm{Sin}\,\alpha x\,d\alpha \quad ,$$

$$V_{1h}(x,y) = \frac{2}{\pi}\int_0^\infty \psi(y,\alpha)\,\mathrm{Cos}\,\alpha x\,d\alpha \quad , \quad (2.6\ a,b)$$

and Inverse Fourier Transformations will be defined by

$$\phi(y,\alpha) = \int_0^\infty U_{1h}(x,y)\,\mathrm{Sin}\,\alpha x\,dx \quad ,$$

$$\psi(y,\alpha) = \int_0^\infty V_{1h}(x,y)\,\mathrm{Cos}\,\alpha x\,dx \quad , \quad (2.7\ a,b)$$

recalling the homogeneous equations

$$\mu_1\nabla^2 U_{1h} + (\lambda_1+\mu_1)\frac{\partial}{\partial x}\left(\frac{\partial U_{1h}}{\partial x} + \frac{\partial V_{1h}}{\partial y}\right) = 0 \quad ,$$

$$\mu_1\nabla^2 V_{1h} + (\lambda_1+\mu_1)\frac{\partial}{\partial y}\left(\frac{\partial U_{1h}}{\partial x} + \frac{\partial V_{1h}}{\partial y}\right) = 0 \quad , \quad (2.8\ a,b)$$

multiplying (2.8 a) by $\sin\alpha x\,dx$ and (2.8 b) by $\cos\alpha x\,dx$ and integrating from 0 to ∞ one obtains

$$-(\lambda_1+2\mu_1)\alpha^2\phi+\mu_1\phi''-(\lambda_1+\mu_1)\alpha\,\psi'=0 \quad ,$$

$$(\lambda_1+2\mu_1)\psi''-\alpha^2\mu_1\psi+(\lambda_1+\mu_1)\alpha\,\phi'=0 \quad , \quad (2.9\ \mathrm{a,b})$$

where through the integration by parts the expressions

$$\int_0^\infty \frac{\partial^2 U_{1h}}{\partial x^2}\sin\alpha x\,dx=-\alpha^2\phi \quad ,$$

$$\int_0^\infty \frac{\partial^2 U_{1h}}{\partial y^2}\sin\alpha x\,dx=\phi'' \quad ,$$

$$\int_0^\infty \frac{\partial^2 V_{1h}}{\partial x\partial y}\sin\alpha x\,dx=-\alpha\,\psi' \quad ,$$

$$\int_0^\infty \frac{\partial^2 V_{1h}}{\partial x^2}\cos\alpha x\,dx=-\alpha^2\psi \quad ,$$

$$\int_0^\infty \frac{\partial^2 V_{1h}}{\partial y^2}\cos\alpha x\,dx=\psi'' \quad ,$$

$$\int_0^\infty \frac{\partial^2 U_{1h}}{\partial x\partial y}\cos\alpha x\,dx=\alpha\,\phi' \quad , \quad (2.10\ \mathrm{a\text{-}f})$$

were obtained and the conditions

$$U_{1h}(0,y)=U_{1h}(\infty,y)=V_{1h}(\infty,y)=\left.\frac{\partial U_{1h}}{\partial x}\right|_{x=\infty}=\left.\frac{\partial V_{1h}}{\partial x}\right|_{x=\infty}=\left.\frac{\partial V_{1h}}{\partial x}\right|_{x=0}=0$$

were imposed.

Differentiating (2.9 a) twice and (2.9 b) once with respect to y gives

$$-(\lambda_1+2\mu_1)\alpha^2\phi''+\mu_1\phi^{IV}-(\lambda_1+\mu_1)\alpha\psi'''=0 ,$$

$$(\lambda_1+2\mu_1)\psi'''-\alpha^2\mu_1\psi'+(\lambda_1+\mu_1)\alpha\phi''=0 \qquad , (2.11\ a,b)$$

substituting for ψ''' from (2.11 a) and for ψ' from (2.9 a) into (2.11 b) one obtains

$$\phi^{IV}-2\alpha^2\phi''+\alpha^4\phi=0 \qquad . (2.12)$$

This fourth order ordinary differential equation has

$$m^4-2\alpha^2m^2+\alpha^4=0 \qquad , (2.13)$$

as the characteristic equation which has the roots

$$m_{1,2}=+\alpha \quad , \quad m_{3,4}=-\alpha \quad .$$

Hence one may express

$$\phi(y,\alpha)=(A_1+A_2y)e^{-\alpha y}+(A_3+A_4y)e^{\alpha y} \qquad , (2.14)$$

then ,

$$\psi(y,\alpha)=\left[A_1+\left(\frac{\chi_1}{\alpha}+y\right)A_2\right]e^{-\alpha y}+\left[-A_3+\left(\frac{\chi_1}{\alpha}-y\right)A_4\right]e^{\alpha y} \qquad . (2.15)$$

and substitution in(2.6 a,b) gives

$$U_{1h}(x,y)=\frac{2}{\pi}\int_0^\infty\left[(A_1+A_2y)e^{-\alpha y}+(A_3+A_4y)e^{\alpha y}\right]\sin\alpha x\,d\alpha ,$$

$$V_{1h}(x,y)=\frac{2}{\pi}\int_0^\infty\left\{\left[A_1+\left(\frac{\varkappa_1}{\alpha}+y\right)A_2\right]e^{-\alpha y}+\left[-A_3+\left(\frac{\varkappa_1}{\alpha}-y\right)A_4\right]e^{\alpha y}\right\}\cos\alpha x\,d\alpha .$$

(2.16 a,b)

Where the unknown functions $A_i(\alpha)$, $(i=1,\dots,4)$ are to be determined from the boundary conditions , $\varkappa=3-4\nu$ for plane strain and $\varkappa=(3-\nu)/(1+\nu)$ for generalized plane stress , ν being Poisson's ratio.

Now employing Hooke's Law

$$\sigma^h_{1xx}=(\lambda_1+2\mu_1)\frac{\partial U_1}{\partial x}+\lambda_1\frac{\partial V_1}{\partial y} ,$$

$$\sigma^h_{1yy}=(\lambda_1+2\mu_1)\frac{\partial V_1}{\partial y}+\lambda_1\frac{\partial U_1}{\partial x} ,$$

$$\sigma^h_{1xy}=\mu_1\left(\frac{\partial U_1}{\partial y}+\frac{\partial V_1}{\partial x}\right) , \quad (2.17\ a\text{-}c)$$

the stress components may be expressed as

$$\frac{1}{2\mu_1}\sigma^h_{1xx}(x,y)=\frac{2}{\pi}\int_0^\infty\Big\{\left[\alpha(A_1+A_2y)-\frac{3-\varkappa_1}{2}A_2\right]e^{-\alpha y}$$
$$+\left[\alpha(A_3+A_4y)+\frac{3-\varkappa_1}{2}A_4\right]e^{\alpha y}\Big\}\cos\alpha x\,d\alpha ,$$

$$\frac{1}{2\mu_1}\sigma^h_{1yy}(x,y)=\frac{2}{\pi}\int_0^\infty\Big\{-\left[\alpha(A_1+A_2y)+\frac{1+\varkappa_1}{2}A_2\right]e^{-\alpha y}$$
$$+\left[-\alpha(A_3+A_4y)+\frac{1+\varkappa_1}{2}A_4\right]e^{\alpha y}\Big\}\cos\alpha x\,d\alpha ,$$

$$\frac{1}{2\mu_1}\sigma_{1xy}^{h}(x,y)=\frac{2}{\pi}\int_0^\infty\left\{-\left[\alpha(A_1+A_2y)+\frac{\varkappa_1-1}{2}A_2\right]e^{-\alpha y}\right.$$

$$\left.+\left[\alpha(A_3+A_4y)-\frac{\varkappa_1-1}{2}A_4\right]e^{\alpha y}\right\}\sin\alpha x\,d\alpha \quad .$$

(2.18 a-c)

The total displacements and the total stresses can be obtained from

$$U_1(x,y)=U_{1h}(x,y)+U_{1p}(x,y) \quad ,$$

$$V_1(x,y)=V_{1h}(x,y)+V_{1p}(x,y) \quad ,$$

$$\sigma_{1xx}(x,y)=\sigma_{1xx}^{h}(x,y)+\sigma_{1xx}^{p}(x,y) \quad ,$$

$$\sigma_{1yy}(x,y)=\sigma_{1yy}^{h}(x,y)+\sigma_{1yy}^{p}(x,y) \quad ,$$

$$\sigma_{1xy}(x,y)=\sigma_{1xy}^{h}(x,y)+\sigma_{1xy}^{p}(x,y) \quad ,$$

(2.19 a-e)

as

$$U_1(x,y)=\frac{2}{\pi}\int_0^\infty\left[(A_1+A_2y)e^{-\alpha y}+(A_3+A_4y)e^{\alpha y}\right]\sin\alpha x\,d\alpha$$

$$+\frac{3-\varkappa_1}{8\mu_1}\,\frac{\rho g h}{2}\,x \quad ,$$

$$V_1(x,y)=\frac{2}{\pi}\int_0^\infty\left\{\left[A_1+\left(\frac{\varkappa_1}{\alpha}+y\right)A_2\right]e^{-\alpha y}+\left[-A_3+\left(\frac{\varkappa_1}{\alpha}-y\right)A_4\right]e^{\alpha y}\right\}\cos\alpha x\,d\alpha$$

$$+\frac{\rho g}{2\mu_1}y\left[\frac{\varkappa_1-1}{\varkappa_1+1}(y-h)-\frac{1+\varkappa_1}{8}h\right] \quad ,$$

$$\frac{1}{2\mu_1}\sigma_{1xx}(x,y)=\frac{2}{\pi}\int_0^\infty\left\{\left[\alpha(A_1+A_2y)-\frac{3-\varkappa_1}{2}A_2\right]e^{-\alpha y}\right.$$

$$\left.+\left[\alpha(A_3+A_4y)+\frac{3-\varkappa_1}{2}A_4\right]e^{\alpha y}\right\}\cos\alpha x\,d\alpha$$

$$+\frac{\rho g}{2\mu_1}\left(y-\frac{h}{2}\right)\frac{\lambda_1}{\lambda_1+2\mu_1}\qquad;$$

$$\frac{1}{2\mu_1}\sigma_{1yy}(x,y)=\frac{2}{\pi}\int_0^\infty\left\{-\left[\alpha(A_1+A_2y)+\frac{1+\varkappa_1}{2}A_2\right]e^{-\alpha y}\right.$$

$$\left.+\left[-\alpha(A_3+A_4y)+\frac{1+\varkappa_1}{2}A_4\right]e^{\alpha y}\right\}\cos\alpha x\,d\alpha$$

$$+\frac{\rho g}{2\mu_1}(y-h)\qquad,$$

$$\frac{1}{2\mu_1}\sigma_{1xy}(x,y)=\frac{2}{\pi}\int_0^\infty\left\{-\left[\alpha(A_1+A_2y)+\frac{\varkappa_1-1}{2}A_2\right]e^{-\alpha y}\right.$$

$$\left.+\left[\alpha(A_3+A_4y)-\frac{\varkappa_1-1}{2}A_4\right]e^{\alpha y}\right\}\sin\alpha x\,d\alpha.$$

(2.20 a-e)

The stress components and displacements equations for the half space may be expressed as follows

$$U_2(x,y)=\frac{2}{\pi}\int_0^\infty(B_1+B_2y)e^{\alpha y}\sin\alpha x\,d\alpha+\frac{3-\varkappa_2}{8\mu_2}\rho g h x\qquad,$$

$$V_2(x,y) = \frac{2}{\pi}\int_0^\infty \left[-B_1 + \left(\frac{\varkappa_2}{\alpha} - y\right)B_2\right]e^{\alpha y}\operatorname{Cos}\alpha x\, d\alpha - \frac{1+\varkappa_2}{8\mu_2}\rho g h y \ ,$$

$$\frac{1}{2\mu_2}\sigma_{2xx}(x,y) = \frac{2}{\pi}\int_0^\infty \left\{\left[\alpha(B_1 + B_2 y) + \frac{3-\varkappa_2}{2}B_2\right]e^{\alpha y}\right\}\operatorname{Cos}\alpha x\, d\alpha$$

$$+ \frac{\rho g h}{8\mu_2^2}\left[\lambda_2(1-\varkappa_2) + \mu_2(3-\varkappa_2)\right] \ ,$$

$$\frac{1}{2\mu_2}\sigma_{2yy}(x,y) = \frac{2}{\pi}\int_0^\infty \left[-\alpha(B_1 + B_2 y) + \frac{1+\varkappa_2}{2}B_2\right]e^{\alpha y}\operatorname{Cos}\alpha x\, d\alpha$$

$$- \rho g h / 2\mu_2 \ ,$$

$$\frac{1}{2\mu_2}\sigma_{2xy}(x,y) = \frac{2}{\pi}\int_0^\infty \left[\alpha(B_1 + B_2 y) - \frac{\varkappa_2 - 1}{2}B_2\right]e^{\alpha y}\operatorname{Sin}\alpha x\, d\alpha \ .$$

(2.21 a-e)

III. FORMULATION OF THE GENERAL PROBLEM

3.1 THE CASE OF CONTINUOUS CONTACT $(0 < P < P_{cr})$

Referring to Fig.1 , it is clear that for values of the resultant compressive force P which are less than a critical value Pcr the continuous contact along the interface $y=0$, $0 \le x < \infty$ will be maintained.The load Pcr corresponding to the initiation of interface separation will depend on the thickness h and the body force ρg .In order to determine the value of Pcr ; assuming that the contact between the layer and the subspace is frictionless, the problem must be solved under the following boundary conditions:

$$\sigma_{1xy}(x,h) = 0 \quad , \quad 0 \le x < \infty$$

$$\sigma_{1yy}(x,h) = -\frac{P}{2}\delta(x) \quad , \quad 0 \le x < \infty$$

$$\sigma_{1xy}(x,0) = 0 \quad , \quad 0 \le x < \infty$$

$$\sigma_{2xy}(x,0) = 0 \quad , \quad 0 \le x < \infty$$

$$\sigma_{1yy}(x,0) = \sigma_{2yy}(x,0) \quad , \quad 0 \le x < \infty$$

$$\frac{\partial V_1(x,0)}{\partial x} = \frac{\partial V_2(x,0)}{\partial x} \quad , \quad 0 \le x < \infty$$

(3.1 a-f)

The unknown functions $A_i(\alpha)$, (i=1,...,4) , $B_j(\alpha)$,(j=1,2) will be determined from the above boundary conditions.

From (2.20 e) and (3.1 c)

$$-\alpha A_1 - \frac{\varkappa_1 - 1}{2} A_2 + \alpha A_3 - \frac{\varkappa_1 - 1}{2} A_4 = 0 \qquad , (3.2)$$

from (2.21 e) and (3.1 d)

$$\alpha B_1 + \frac{1 - \varkappa_2}{2} B_2 = 0 \qquad , (3.3)$$

from (2.20 e) and (3.1 a)

$$-\left[\alpha (A_1 + A_2 h) + \frac{\varkappa_1 - 1}{2} A_2\right] e^{-\alpha h}$$

$$+ \left[\alpha (A_3 + A_4 h) - \frac{\varkappa_1 - 1}{2} A_4\right] e^{\alpha h} = 0$$

$$, (3.4)$$

from (2.20 d) and (3.1 b)

$$\frac{2}{\pi} \int_0^\infty \Big\{ -\left[\alpha (A_1 + A_2 h) + \frac{1 + \varkappa_1}{2} A_2\right] e^{-\alpha h}$$

$$+ \left[-\alpha (A_3 + A_4 h) + \frac{\varkappa_1 + 1}{2} A_4\right] e^{\alpha h} \Big\} \operatorname{Cos} \alpha x \, d\alpha$$

$$= -\frac{1}{2\mu_1} \frac{P}{2} \delta(x) \qquad ,$$

or multiplying both sides by $\operatorname{Cos} \alpha x dx$ and integrating from 0 to ∞ one obtains

$$\frac{2}{\pi}\int_0^\infty\Big[\int_0^\infty\Big\{-\Big[\alpha(A_1+A_2h)+\frac{1+\mathcal{X}_1}{2}A_2\Big]e^{-\alpha h}$$

$$+\Big[-\alpha(A_3+A_4h)+\frac{1+\mathcal{X}_1}{2}A_4\Big]e^{\alpha h}\Big\}\operatorname{Cos}\alpha x\,d\alpha\Big]\operatorname{Cos}\alpha x\,dx$$

$$=-\frac{1}{2\mu_1}\frac{P}{2}\int_0^\infty\delta(x)\operatorname{Cos}\alpha x\,dx \quad ,$$

which gives

$$-\Big[\alpha(A_1+A_2h)+\frac{\mathcal{X}_1+1}{2}A_2\Big]e^{-\alpha h}$$

$$+\Big[-\alpha(A_3+A_4h)+\frac{\mathcal{X}_1+1}{2}A_4\Big]e^{\alpha h}=-\frac{P}{4\mu_1} \quad , \tag{3.5}$$

From (2.20 d) , (2.21 d) and (3.1 e) one may express

$$2\mu_1\Big(-\alpha A_1-\frac{\mathcal{X}_1+1}{2}A_2-\alpha A_3+\frac{\mathcal{X}_1+1}{2}A_4\Big)$$

$$=2\mu_2\Big(-\alpha B_1+\frac{1+\mathcal{X}_2}{2}B_2\Big) \quad , \tag{3.6}$$

and from (2.20 b) , (2.21 b) and (3.1 f)

$$\alpha\Big(A_1+\frac{\mathcal{X}_1}{\alpha}A_2-A_3+\frac{\mathcal{X}_1}{\alpha}A_4\Big)=\alpha\Big(-B_1+\frac{\mathcal{X}_2}{\alpha}B_2\Big) \quad . \tag{3.7}$$

Solving (3.2 - 3.7) simultaneously it is possible to express the unknown functions

$$A_1 = \frac{Pe^{3\omega}\left[e^{-2\omega}(1-m)(-1+2\omega+\kappa_1)+(1-m)(1-\kappa_1)+2\omega(1+m\kappa_1)\right]}{8\mu_1\alpha\Delta},$$

$$A_2 = \frac{Pe^{2\omega}\left[me^{2\omega}(-1-2\omega)+e^{\omega}-e^{-\omega}(1-m)\right]}{4\mu_1\Delta},$$

$$A_3 = \frac{Pe^{\omega}\left[e^{2\omega}(1+m)(1+2\omega-\kappa_1)+(1+m)(\kappa_1-1)+2\omega(1-m\kappa_1)\right]}{8\mu_1\alpha\Delta},$$

$$A_4 = \frac{Pe^{2\omega}\left[m(1-2\omega)e^{-\omega}+e^{-\omega}-e^{\omega}(1+m)\right]}{4\mu_1\Delta},$$

$$B_1 = \frac{Pe^{2\omega}(\kappa_2-1)\left[(1-\omega)e^{-\omega}-(1+\omega)e^{\omega}\right]}{4\mu_1\alpha\Gamma\Delta},$$

$$B_2 = \frac{Pe^{2\omega}\left[(1-\omega)e^{-\omega}-(1+\omega)e^{\omega}\right]}{2\mu_1\Gamma\Delta}. \qquad (3.8\ a\text{-}f)$$

Where

$$m=\frac{\mu_1(\varkappa_2+1)}{\mu_2(\varkappa_1+1)}\,,\quad \Gamma=\frac{\mu_2}{\mu_1}\,,\quad \omega=\alpha h\,,$$

$$\Delta=(1+m)e^{4\omega}+2\left[2\omega-m(2\omega^2+1)\right]e^{2\omega}+m-1 \qquad . \text{(3.9 a-d)}$$

From equation (2.21 d)

$$\frac{1}{2\mu_2}\sigma_{2yy}(x,y)=\frac{2}{\pi}\int_0^\infty\left[-\alpha(B_1+B_2y)+\frac{1+\varkappa_2}{2}B_2\right]e^{-\alpha y}\cos\alpha x\,d\alpha-\frac{\rho g h}{2\mu_2}$$

or substituting from (3.8 e,f) the contact stress along the interface can be obtained as

$$\sigma_{2yy}(x,0)=\frac{2P}{\pi h}\int_0^\infty\frac{e^{\omega}\left[(1+\omega)e^{-2\omega}+\omega-1\right]}{\Delta}\cos\frac{\omega x}{h}\,d\omega-\rho g h\,,$$

$$0\leq x<\infty \qquad (\ 3.10\ )$$

Defining

$$p(x)=-\sigma_{2yy}(x,0)/\rho g h\,,\qquad \lambda=\frac{P}{\rho g h^2}\,,$$

$$\beta=\frac{1-m}{1+m}\,,\qquad -1\leq\beta\leq 1$$

The normalized contact pressure becomes

$$p(x)=1+\frac{2\lambda}{\pi}\int_0^\infty \frac{e^{\omega}[(1+\omega)e^{2\omega}+\omega-1]\ \mathrm{Cos}\frac{\omega x}{h}\,d\omega}{\frac{2e^{4\omega}}{1+\beta}+2\left[2\omega-\frac{(1-\beta)(1+2\omega^2)}{1+\beta}\right]e^{2\omega}-\frac{2\beta}{1+\beta}},$$

$$(0\leq x<\infty\ ,-1\leq\beta\leq 1)\quad(3.11)$$

It is seen from (3.11) p(x) remains positive up to a certain value of λ and the contact on $y=0$ plane remaines continuous.The critical distance X_{cr} at which the intirface separation begins is determined as the distance of the lowest point of the contact stress curve.The critical value of the load factor $\lambda>\lambda_{cr}$ at which the interface separation begins can also be determined from the condition

$$p(0)=0$$

giving

$$\frac{1}{\lambda_{cr}}=\frac{2}{\pi}\int_0^\infty \frac{e^{\omega}[1-\omega-(1+\omega)e^{2\omega}]\,\mathrm{Cos}\frac{\omega X_{cr}}{h}}{\Delta}\,d\omega\ .$$

$$(3.12)$$

For $\lambda>\lambda_{cr}$ the foregoing solution gives a negative pressure along part of the interface.Since it is assumed that there is no adhesion between the layer and the subspace , this is not possible and the layer will be partially separated from the subspace.In this case the solution leading to (3.10) is not valid and the problem must be treated as a mixed boundary problem.

3.2 THE CASE OF DISCONTINUOUS CONTACT ($P > P_{cr}$)

If the applied load P is greater than P_{cr} corresponding to the beginning of interface separation , the basic solutions for the layer (2.20) and for the half space (2.21) are still valid.However , in this case the unknown functions $A_i(\alpha), (i=1,\dots,4)$, $B_j(\alpha)$, $(j=1,2)$ must be determined from the following boundary conditions (Fig.2) :

$$\sigma_{1xy}(x,h) = 0 \qquad , \quad 0 \le x < \infty$$

$$\sigma_{1yy}(x,h) = -\frac{P}{2}\delta(x) \qquad , \quad 0 \le x < \infty$$

$$\sigma_{1xy}(x,0) = \sigma_{2xy}(x,0) = 0 \quad , \quad 0 \le x < \infty$$

$$\sigma_{1yy}(x,0) = \sigma_{2yy}(x,0) \qquad , \quad 0 \le x < \infty$$

$$\sigma_{yy}(x,0) = \sigma_{1yy}(x,0) = \sigma_{2yy}(x,0) = 0 \quad , \quad b \le x \le c$$

$$\frac{\partial}{\partial x}\left[v_1(x,0^+) - v_2(x,0^-)\right] = 0 \;, \; (\; 0 < x < b \,,\; c < x < \infty)$$

(3.13 a–f)

Introducing the new unknown function

$$\frac{\partial}{\partial x}\left[v_1(x,0^+) - v_2(x,0^-)\right] = f(x) \quad , \quad b < x < c$$

(3.14)

all other unknown functions $A_i(\alpha), (i=1,\dots,4)$ and $B_j(\alpha)$, $(j=1,2)$ may be expressed in terms of $f(x)$ by using equations (3.13 a–e) and (3.14).Since there is an interface separation along $b < x < c$ to have single-valued displacements the function $f(x)$ must satisfy the following condition :

$$\int_b^c f(x)\,dx = 0 \qquad (\; 3.15 \;)$$

It should also be noted that the distances b and c determining the region of interface separation are unknown.

$A_i(\alpha)$, $(i=1,\dots,4)$ and $B_j(\alpha)$, $(j=1,2)$ may be expressed in terms of f(x) as follows.

From (3.13 c) and (2.20 e) one obtaines

$$-\alpha A_1-\frac{\varkappa_1-1}{2}A_2+\alpha A_3-\frac{\varkappa_1-1}{2}A_4=0 \qquad , \quad (3.16)$$

from (3.13 d) and (2.21 e)

$$\alpha B_1-\frac{\varkappa_2-1}{2}B_2=0 \qquad , \quad (3.17)$$

from (3.13 a) and (2.20 e)

$$-\left[\alpha(A_1+A_2h)+\frac{\varkappa_1-1}{2}A_2\right]e^{-\alpha h}$$

$$+\left[\alpha(A_3+A_4h)-\frac{\varkappa_1-1}{2}A_4\right]e^{\alpha h}=0 \qquad , \quad (3.18)$$

from (3.13 b) and (2.20 d)

$$\frac{2}{\pi}\int_0^\infty\left\{-\left[\alpha(A_1+A_2h)+\frac{1+\varkappa_1}{2}A_2\right]e^{-\alpha h}\right.$$

$$\left.+\left[-\alpha(A_3+A_4h)+\frac{\varkappa_1+1}{2}A_4\right]e^{\alpha h}\right\}\operatorname{Cos}\alpha x\,d\alpha$$

$$=-\frac{1}{2\mu_1}\frac{P}{2}\delta(x) \qquad ,$$

or multiplying both sides by $\operatorname{Cos}\alpha x\,dx$ and integrating from 0 to ∞ one may express

$$\frac{2}{\pi}\int_0^\infty\left[\int_0^\infty\left\{-\left[\alpha(A_1+A_2h)+\frac{\chi_1+1}{2}A_2\right]e^{-\alpha h}\right.\right.$$

$$\left.\left.+\left[-\alpha(A_3+A_4h)+\frac{\chi_1+1}{2}A_4\right]e^{\alpha h}\right\}\cos\alpha x\,d\alpha\right]\cos\alpha x\,dx$$

$$=-\frac{P}{4\mu_1}\int_0^\infty\delta(x)\cos\alpha x\,dx \quad ,$$

which gives

$$-\left[\alpha(A_1+A_2h)+\frac{\chi_1+1}{2}A_2\right]e^{-\alpha h}$$

$$+\left[-\alpha(A_3+A_4h)+\frac{\chi_1+1}{2}A_4\right]e^{\alpha h}=-\frac{P}{4\mu_1} \quad . \ (3.19)$$

From (3.13 e) , (2.20 d) and (2.21 d) one may write

$$2\mu_1\left(-\alpha A_1-\frac{\chi_1+1}{2}A_2-\alpha A_3+\frac{\chi_1+1}{2}A_4\right)=$$

$$2\mu_2\left(-\alpha B_1+\frac{1+\chi_2}{2}B_2\right) \quad , \ (3.20)$$

and from (3.14) , (2.20 b) and (2.21 b) it is possible to express

$$-\frac{2}{\pi}\int_0^\infty\alpha\left(A_1+\frac{\chi_1}{\alpha}A_2-A_3+\frac{\chi_1}{\alpha}A_4\right)\sin\alpha x\,d\alpha$$

$$+\frac{2}{\pi}\int_0^\infty\alpha\left(-B_1+\frac{\chi_2}{\alpha}B_2\right)\sin\alpha x\,d\alpha=f(x) \quad ,$$

$$b<x<c$$

or recalling the second of (3.13 f) , multiplying both sides by $\sin\alpha x\,dx$ and integrating from 0 to ∞ one obtains

$$-\frac{2}{\pi}\int_0^\infty\left[\int_0^\infty \alpha\left(A_1+\frac{\varkappa_1}{\alpha}A_2-A_3+\frac{\varkappa_1}{\alpha}A_4\right)\sin\alpha x\,d\alpha\right]\sin\alpha x\,dx$$

$$+\frac{2}{\pi}\int_0^\infty\left[\int_0^\infty\left(-B_1+\frac{\varkappa_2}{\alpha}B_2\right)\alpha\sin\alpha x\,d\alpha\right]\sin\alpha x\,dx$$

$$=\int_b^c f(x)\sin\alpha x\,dx$$

or changing the dummy variable

$$\alpha\left(-A_1-\frac{\varkappa_1}{\alpha}A_2+A_3-\frac{\varkappa_1}{\alpha}A_4-B_1+\frac{\varkappa_2}{\alpha}B_2\right)$$

$$=\int_b^c f(t)\sin\alpha t\,dt \quad . \qquad (3.21)$$

Solving (3.16–3.21) one finds

$$A_1=\frac{Pe^{3\omega}\left[e^{-2\omega}(1-m)(-1+2\omega+\varkappa_1)+(1-m)(1-\varkappa_1)+2\omega(1+m\varkappa_1)\right]}{8\mu_1\alpha\Delta}$$

$$+\frac{Le^{3\omega}\left[2e^{\omega}(\varkappa_1-1)+e^{-\omega}(-4\omega+8\omega^2-2\varkappa_1+4\varkappa_1\omega+2)\right]}{2\alpha(\varkappa_1+1)\Delta},$$

$$A_2=\frac{Pe^{2\omega}\left[me^{\omega}(-1-2\omega)+e^{\omega}-e^{-\omega}(1-m)\right]}{4\mu_1\Delta}+\frac{Le^{2\omega}(-2e^{2\omega}+2-4\omega)}{(\varkappa_1+1)\Delta},$$

$$A_3 = \frac{Pe^{\omega}\left[e^{2\omega}(1+m)(-\varkappa_1+2\omega+1)+(1+m)(\varkappa_1-1)+2\omega(1-m\varkappa_1)\right]}{8\mu_1\alpha\,\Delta}$$

$$+\frac{Le^{\omega}\left[2e^{-\omega}(\varkappa_1-1)+e^{\omega}(4\omega+8\omega^2-2\varkappa_1-4\varkappa_1\omega+2)\right]}{2\alpha(\varkappa_1+1)\,\Delta},$$

$$A_4 = \frac{Pe^{2\omega}\left[me^{-\omega}(1-2\omega)+e^{-\omega}-e^{\omega}(1+m)\right]}{4\mu_1\Delta}+\frac{Le^{2\omega}(2e^{-2\omega}-2-4\omega)}{(\varkappa_1+1)\,\Delta},$$

$$B_1 = \frac{Pe^{2\omega}(\varkappa_2-1)}{4\mu_1\Gamma\alpha\,\Delta}\left[e^{-\omega}(1-\omega)-e^{\omega}(1+\omega)\right]$$

$$+\frac{Le^{2\omega}(\varkappa_2-1)(e^{2\omega}+e^{-2\omega}-4\omega^2-2)}{\alpha\Gamma(\varkappa_1+1)\,\Delta},$$

$$B_2 = \frac{Pe^{2\omega}\left[e^{-\omega}(1-\omega)-e^{\omega}(1+\omega)\right]}{2\mu_1\Gamma\,\Delta}$$

$$+\frac{2Le^{2\omega}(e^{2\omega}+e^{-2\omega}-4\omega^2-2)}{\Gamma(\varkappa_1+1)\,\Delta}. \qquad (3.22\ \text{a-f})$$

where ,

$$L = \int_b^c f(t) \sin \alpha t \, dt \qquad (3.23)$$

and Δ , Γ , and m were previously defined.

From the first of (3.13 f) one may express

$$\frac{1}{2\mu_2} \sigma_{2yy}(x,0) = \frac{2}{\pi} \lim_{y \to 0} \int_0^\infty \left[-\alpha (B_1 + B_2 y) + \frac{1+\chi_2}{2} B_2 \right] e^{\alpha y}$$

$$\cos \alpha x \, d\alpha - \frac{\rho g h}{2\mu_2} = 0 \quad .$$

$$b < x < c \qquad (3.24)$$

Substitution for B_1 and B_2 from (3.22 e,f) and taking the limit one obtains the Singular Integral Equation of the problem as

$$\frac{2\mu_1(1+\beta)}{(1+\chi_1)} \left\{ \frac{1}{\pi} \int_b^c \left(\frac{1}{t+x} + \frac{1}{t-x} \right) f(t) \, dt \right.$$

$$\left. + \frac{1}{\pi h} \int_b^c k_1(x,t) f(t) \, dt \right\} - \rho g h \left[1 - \frac{\lambda}{\pi} k_2(x) \right] = 0$$

$$(b < x < c) \qquad (3.25)$$

where ,

$$k_1(x,t)=\int_0^\infty \frac{2\left[1-(1+2w+2w^2)e^{2w}\right]}{\Delta}$$

$$\left[\sin\frac{w}{h}(t-x)+\sin\frac{w}{h}(t+x)\right]dw \qquad ,$$

(3.26)

$$k_2(x)=\int_0^\infty \frac{2e^{w}\left[1-w-(1+w)e^{2w}\right]}{\Delta}\cos\frac{wx}{h}\,dw \ .$$

(3.27)

In the calculation of the kernel $k_1(x,t)$ and the bounded function $k_2(x)$ which are in the form of infinite integrals , to improve the convergence the following procedure will be used.

For example if $k_1(x,t)$ is considered , it is possible to express

$$k_1(x,t)=\int_0^\infty\left\{\frac{2\left[1-(1+2w+2w^2)e^{2w}\right]}{\frac{2}{1+\beta}e^{4w}+2\left[2w-\frac{1-\beta}{1+\beta}(1+2w^2)\right]e^{2w}-\frac{2\beta}{1+\beta}}\right.$$

$$\left.+(1+\beta)(1+2w+2w^2)e^{-2w}\right\}\left[\sin\frac{w}{h}(t-x)+\sin\frac{w}{h}(t+x)\right]dw$$

$$-\int_0^\infty(1+\beta)(1+2w+2w^2)e^{-2w}\left[\sin\frac{w}{h}(t-x)+\sin\frac{w}{h}(t+x)\right]dw \ .$$

(3.28)

As one can observe when the two terms in the curly brackets are combined the degree of the leading term which appear in the denominator gets much larger than the degree of the leading term in the numerator and it is possible to evaluate the last integral in closed form.

Recalling,

$$\int_0^\infty e^{-st}\sin at\,dt = \frac{a}{s^2+a^2} \tag{3.29}$$

and

$$\int_0^\infty t e^{-st}\sin at\,dt = -\frac{d}{ds}\int_0^\infty e^{-st}\sin at\,dt$$

or

$$\int_0^\infty t e^{-st}\sin at\,dt = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{2sa}{(s^2+a^2)^2} \tag{3.30}$$

Similarly ,

$$\int_0^\infty t^2 e^{-st}\sin at\,dt = -\frac{d}{ds}\int_0^\infty t e^{-st}\sin at\,dt$$

or

$$\int_0^\infty t^2 e^{-st}\sin at\,dt = -\frac{d}{ds}\left[\frac{2sa}{(s^2+a^2)^2}\right]$$

hence

$$\int_0^{\infty} t^2 e^{-st} \sin at\, dt = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3} \tag{3.31}$$

Now substituting from (3.29-3.31) one obtains

$$\int_0^{\infty} (1+2w+2w^2)e^{-2w} \sin\frac{w}{h}(t-x)\,dw$$

$$= \frac{t-x}{h}\left\{ \frac{60+\left[6+\left(\frac{t-x}{h}\right)^2\right]^2}{\left[4+\left(\frac{t-x}{h}\right)^2\right]^3} \right\} \tag{3.32}$$

and

$$\int_0^{\infty} (1+2w+2w^2)e^{-2w} \sin\frac{w}{h}(t+x)\,dw$$

$$= \frac{t+x}{h}\left\{ \frac{60+\left[6+\left(\frac{t+x}{h}\right)^2\right]^2}{\left[4+\left(\frac{t+x}{h}\right)^2\right]^3} \right\} \tag{3.33}$$

Hence ,

$$k_1(x,t) = \int_0^{\infty} \left\{ \frac{2\left[1-(1+2w+2w^2)e^{2w}\right]}{\Delta} \right.$$

$$\left. + (1+\beta)(1+2w+2w^2)\, e^{-2w} \right\}$$

$$\times \left[\sin\frac{w}{h}(t-x) + \sin\frac{w}{h}(t+x) \right] dw$$

$$-(1+\beta)\left(\frac{t-x}{h}\right)\left\{\frac{60+\left[6+\left(\frac{t-x}{h}\right)^2\right]^2}{\left[4+\left(\frac{t-x}{h}\right)^2\right]^3}\right\}$$

$$-(1+\beta)\left(\frac{t+x}{h}\right)\left\{\frac{60+\left[6+\left(\frac{t+x}{h}\right)^2\right]^2}{\left[4+\left(\frac{t+x}{h}\right)^2\right]^3}\right\} \, .$$

(3.34)

Similarly it is possible to express $k_2(x)$ as

$$k_2(x) = \int_0^\infty \left\{ \frac{2e^{w}\left[1-w-(1+w)e^{2w}\right]}{\frac{2}{1+\beta}e^{4w}+2\left[2w-\frac{1-\beta}{1+\beta}(1+2w^2)\right]e^{2w}-\frac{2\beta}{1+\beta}} + (1+\beta)(1+w)e^{-w}\right\} \cos\frac{w}{h}x \, dw$$

$$-\int_0^\infty (1+\beta)(1+w)e^{-w}\cos\frac{w}{h}x \, dw \, .$$

(3.35)

Recalling ,

$$\int_0^\infty e^{-st}\cos at \, dt = \frac{s}{s^2+a^2}$$

(3.36)

and

$$\int_0^\infty t e^{-st}\cos at \, dt = -\frac{d}{ds}\int_0^\infty e^{-st}\cos at \, dt$$

or

$$\int_0^\infty t e^{-st} \cos at \, dt = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{s^2-a^2}{(s^2+a^2)^2} \qquad (3.37)$$

Substituting from (3.36) and (3.37) one gets

$$\int_0^\infty (1+w) e^{-w} \cos\frac{w}{h} x \, dw = \frac{2}{\left[1+\left(\frac{x}{h}\right)^2\right]^2} \qquad (3.38)$$

Hence ,

$$k_2(x) = \int_0^\infty \left\{ \frac{2e^{w}\left[1-w-(1+w)e^{2w}\right]}{\Delta} + (1+\beta)(1+w)e^{-w} \right\} \cos\frac{w}{h} x \, dw - \frac{2(1+\beta)}{\left[1+\left(\frac{x}{h}\right)^2\right]^2} \qquad (3.39)$$

It should be remembered that the singular integral equation of the problem (3.25) will be solved by imposing the single-valuedness condition

$$\int_b^c f(x) \, dx = 0 \qquad (3.40)$$

The contact stress may also be evaluated from

$$\sigma_{yy}(x,0) = \frac{2\mu_1(1+\beta)}{(1+\varkappa_1)} \left\{ \frac{1}{\pi} \int_b^c \left(\frac{1}{t-x} + \frac{1}{t+x}\right) f(t) \, dt + \frac{1}{\pi h} \int_b^c k_1(x,t) f(t) \, dt \right\} - \rho g h \left[1 - \frac{\lambda}{\pi} k_2(x)\right] \qquad (3.41)$$

$$(0 < x < b, \; c < x < \infty)$$

3.3 NUMERICAL SOLUTION OF THE SINGULAR INTEGRAL EQUATION

The index of the singular integral equation is -1 and to solve it the Gauss-Chebyshev integration formulas given in [16] have been used. First the equations (3.25) and (3.40) have been normalized by defining the following dimensionless quantities

$$s = \frac{2t}{c-b} - \frac{c+b}{c-b} \quad , \quad r = \frac{2x}{c-b} - \frac{c+b}{c-b} \quad ,$$

$$\phi(s) = \frac{2\mu_1 (1+\beta)}{\rho g h (1+\varkappa_1)} f(t) \quad . \qquad (3.42 \text{ a-c})$$

After normalizing one obtains

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{s-r} + \frac{1}{s+r+2(c+b)/(c-b)} + \frac{c-b}{h} k_1(r,s)\right]\phi(s)\,ds - \frac{\lambda}{\pi} k_2(r) = 1 \quad ,$$

$$(-1 < r < 1) \qquad (3.43)$$

and

$$\int_{-1}^{1} \phi(s)\,ds = 1 \qquad (3.44)$$

Then writing

$$\phi(s) = \frac{F(s)}{(1-s^2)^{1/2}} \quad , \qquad -1 < s < 1$$

and applying the quadrature formula to (3.43) and (3.44) one obtains

$$\sum_{k=1}^{n} W_k \left\{ F(s_k) \left[\frac{1}{s_k - r_i} + \frac{1}{s_k + r_i + 2\left(\frac{c+b}{c-b}\right)} + \frac{c-b}{h} k_1(r_i, s_k) \right] - \frac{\lambda}{\pi} k_2(r_i) \right\} = 1$$

$$, \quad i=1,\ldots,n-1 \qquad (3.45)$$

and

$$\sum_{k=1}^{n} W_k F(s_k) = 0 \qquad (3.46)$$

where

$$W_1 = W_n = \frac{1}{2(n-1)} \quad , W_k = \frac{1}{n-1}, \quad k=2,\ldots,n-1 \qquad (3.47)$$

$$s_k = \cos\left(\pi \frac{k-1}{n-1}\right), \quad k=1,\ldots,n \qquad (3.48)$$

$$r_i = \cos\left(\pi \frac{2i-1}{2n-2}\right), \quad i=1,\ldots,n-1 \qquad (3.49)$$

To insure the smooth contact at the end points of the separation area the following conditions

$$F(-1)=0 \quad , \quad F(+1)=0 \qquad (3.50\ a,b)$$

will be imposed.

The equations (3.45) , (3.46) and (3.50 a,b) form a system of $n+2$ algebraic equations to determine $n+2$ unknowns $F(s_1)$,...., $F(s_k)$, ... , $F(s_n)$, b/h and c/h where $s_k=\cos[\pi(k-1)/(n-1)]$, $k=1,\ldots,n$.

The equations are linear in $F(s_k)$ but very highly nonlinear in b/h and c/h.Therefore , an interpolation and iteration scheme had to be used to obtain these two unknowns.With the critical values of λ_{cr} and Xcr known , increasing λ gradually it was not difficult to make good initial guesses for b/h and c/h.Thus , determination of sufficiently accurate values for these unknowns did not require extensive numerical work.After determining the function $F(s)$ and the distances b , c the contact stress may be easily be evaluated from (3.41) by applying the quadrature formula again.

IV. NUMERICAL RESULTS AND DISCUSSION

Fig.3 displays the variation of the separation initiation distance Xcr with the bimaterial constant β and Fig.4 shows the variation of the critical load factor λ_{cr} with β .As it can be seen from these figures both Xcr and λ_{cr} increase when β decreases.It should also be noted that for $\beta < -0.4$ the subspace gets very soft and Xcr , values will get very large and for this reason it was't possible to obtain accurate values and the convergence of the integrals was observed to be very slow.

Fig.5 shows the important results giving the distances b and c which define the separation zone.It appears that for a fixed value of β increasing load factor λ , b/h approaches a constant asymptotic value which is equal to the extent of the contact zone in the receding contact problem.However , c/h keeps increasing with increasing λ .

Fig.6 shows some sample results for the pressure distribution along the contact area.For $\lambda = 30 < \lambda_{cr} = 44.4$ it is seen that the contact as well as the pressure distribution is continuous.For $\lambda > \lambda_{cr}$ the figure shows the discontinuous nature of the contact area and the pressure distribution.

The following major conclusions can be drawn from this study.

1. The definition of the bielastic material constant β has proved to be very appropriate since it takes values in a finite range unlike m (the constant used in Ref.[15) which varies from 0 to ∞.

2. It has been experienced that by extracting some closed form expressions from the infinite integrals , the convergence in the numerical evaluation of these infinite integrals has been improved to a great extent.

3. This solution could be used in engineering practice and can replace various beam models used for beams on elastic foundations.

V. SUGGESTIONS FOR FUTURE WORK

The representation of the subspace as a linear elastic continuum is the major obstacle which prevents one to apply these results directly to beams lying on soil subspace.It is believed that a better model to represent the soil should be used for the better understanding of the contact phenomenon.This can be achieved either by assuming that the soil is viscoelastic or poroelastic or elastic with a reduced modulus , etc.

Of course , these models require different mathematical approaches involving the Laplace transform or finite elements and extensive numerical calculations should be carried out.

Since there is no correlation between the Young's Modulus and the spring constant k of the spring model for the soil subspace , it was not possible to compare the results with the results of the spring model.This should be done in the future if the correlation can be established.

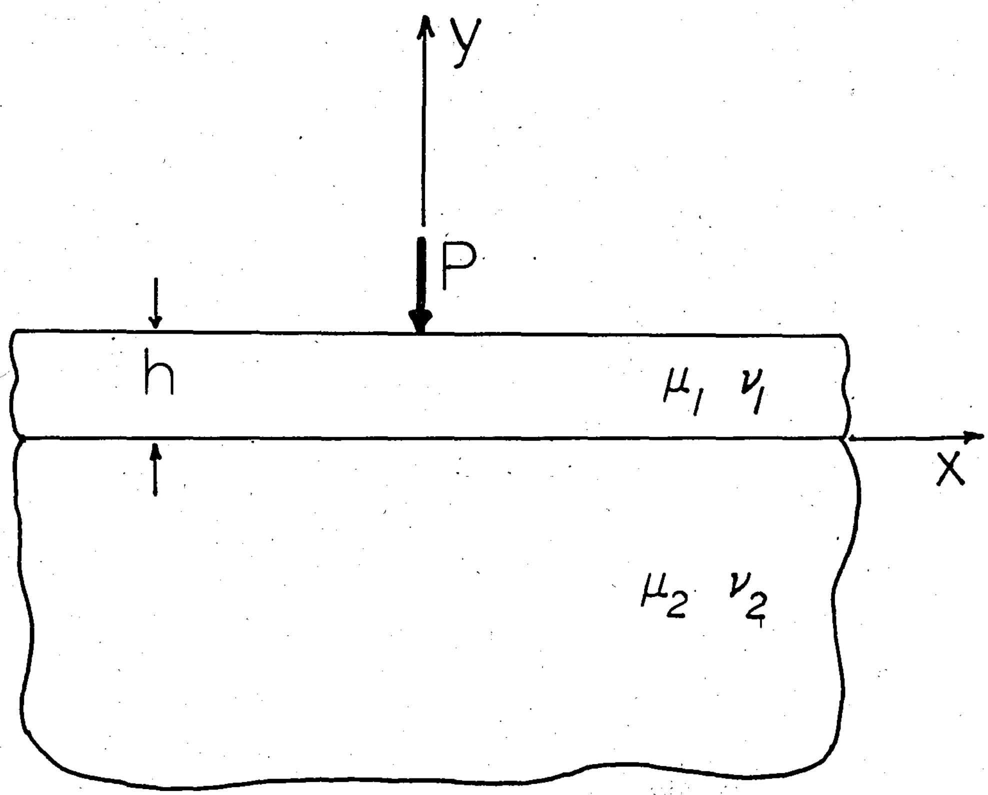

FIGURE 1. Geometry of the Elastic Layer on an Elastic Frictionless Half Space

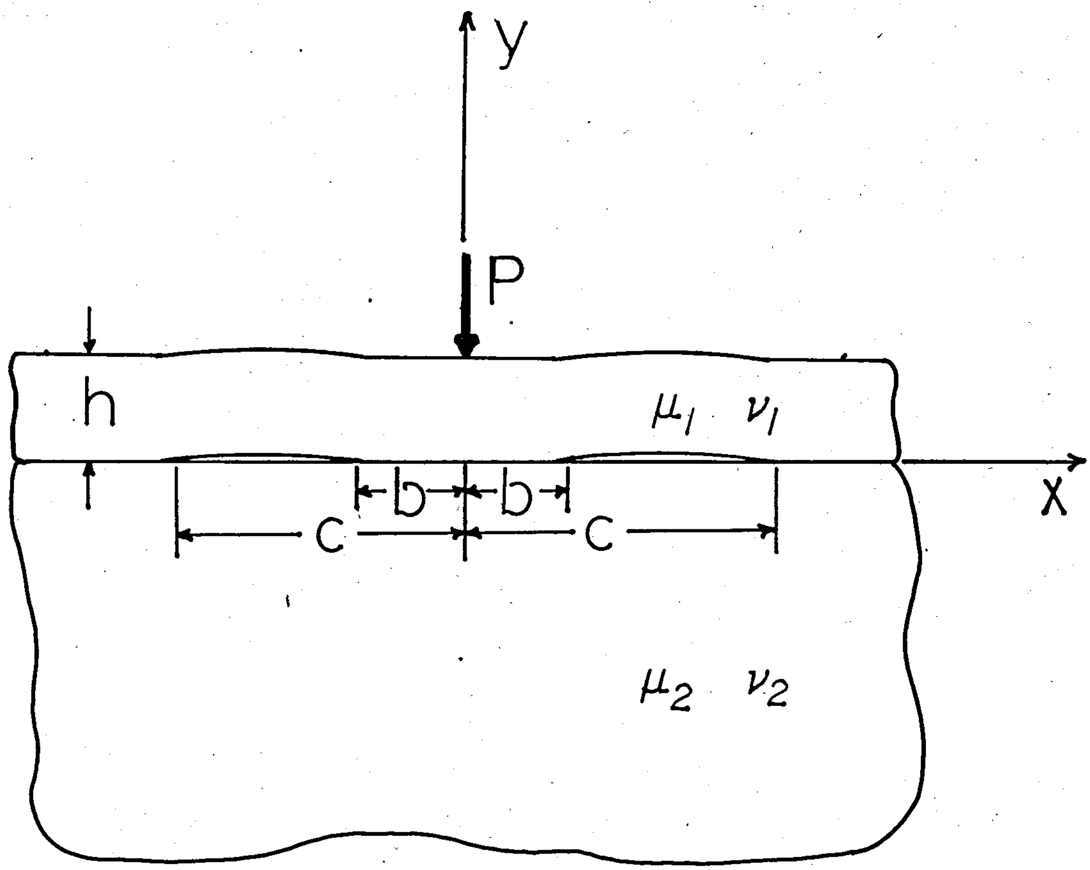

FIGURE 2. Geometry of the Elastic Layer on an Elastic Frictionless Half Space After Separation

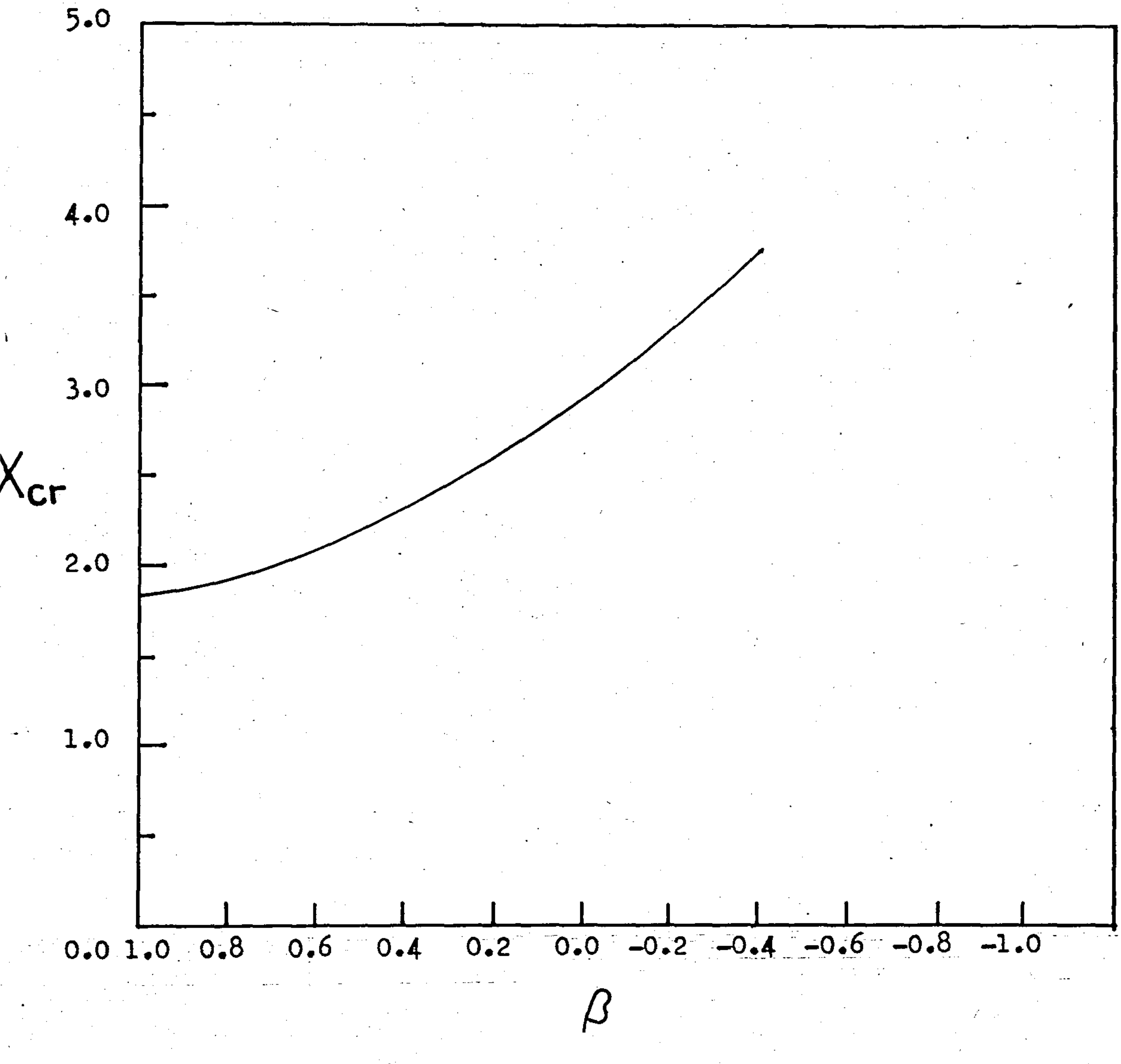

FIGURE 3. Variation of the Separation Initiation Distance with β

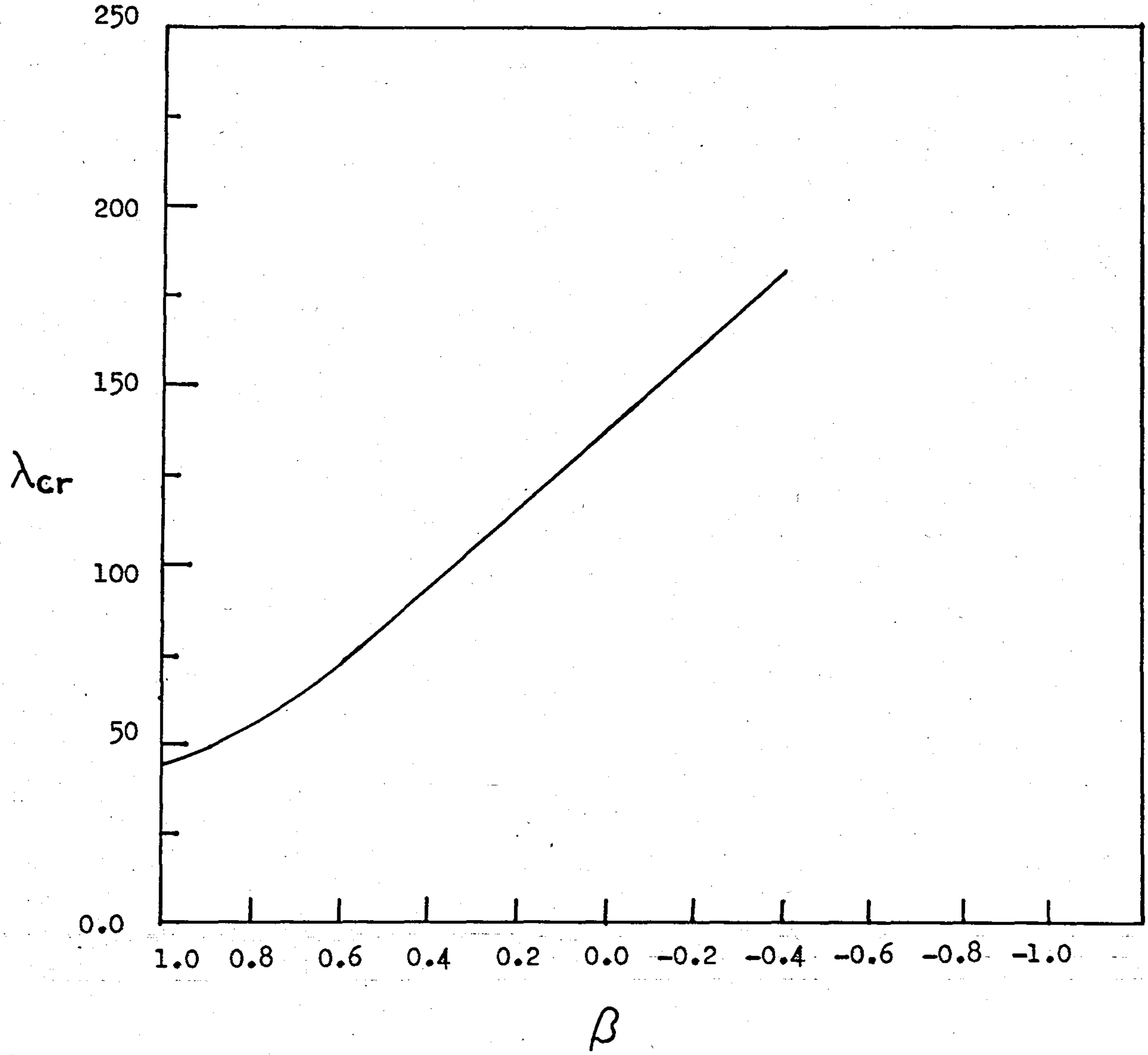

FIGURE 4. Variation of Critical Load Factor with β

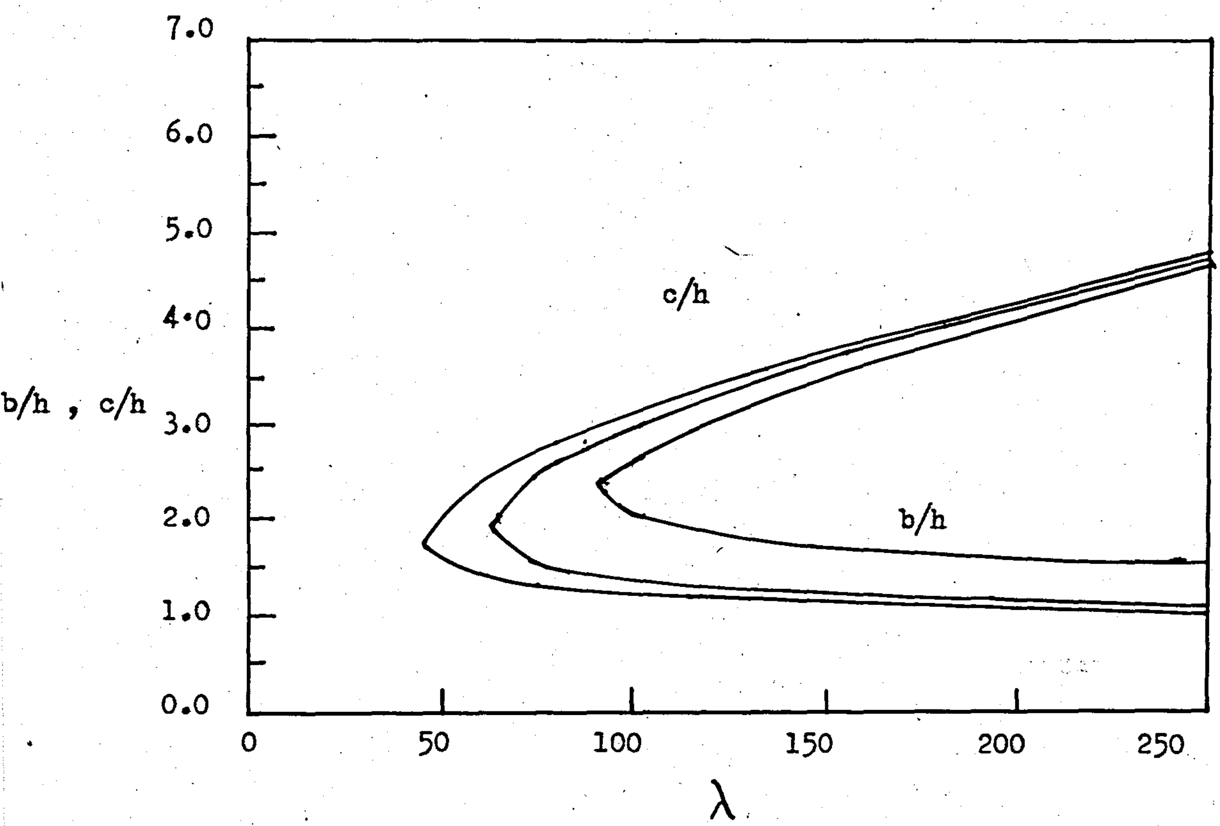

FIGURE 5. Separation Area Along the Interface Between the Layer and the Half Space

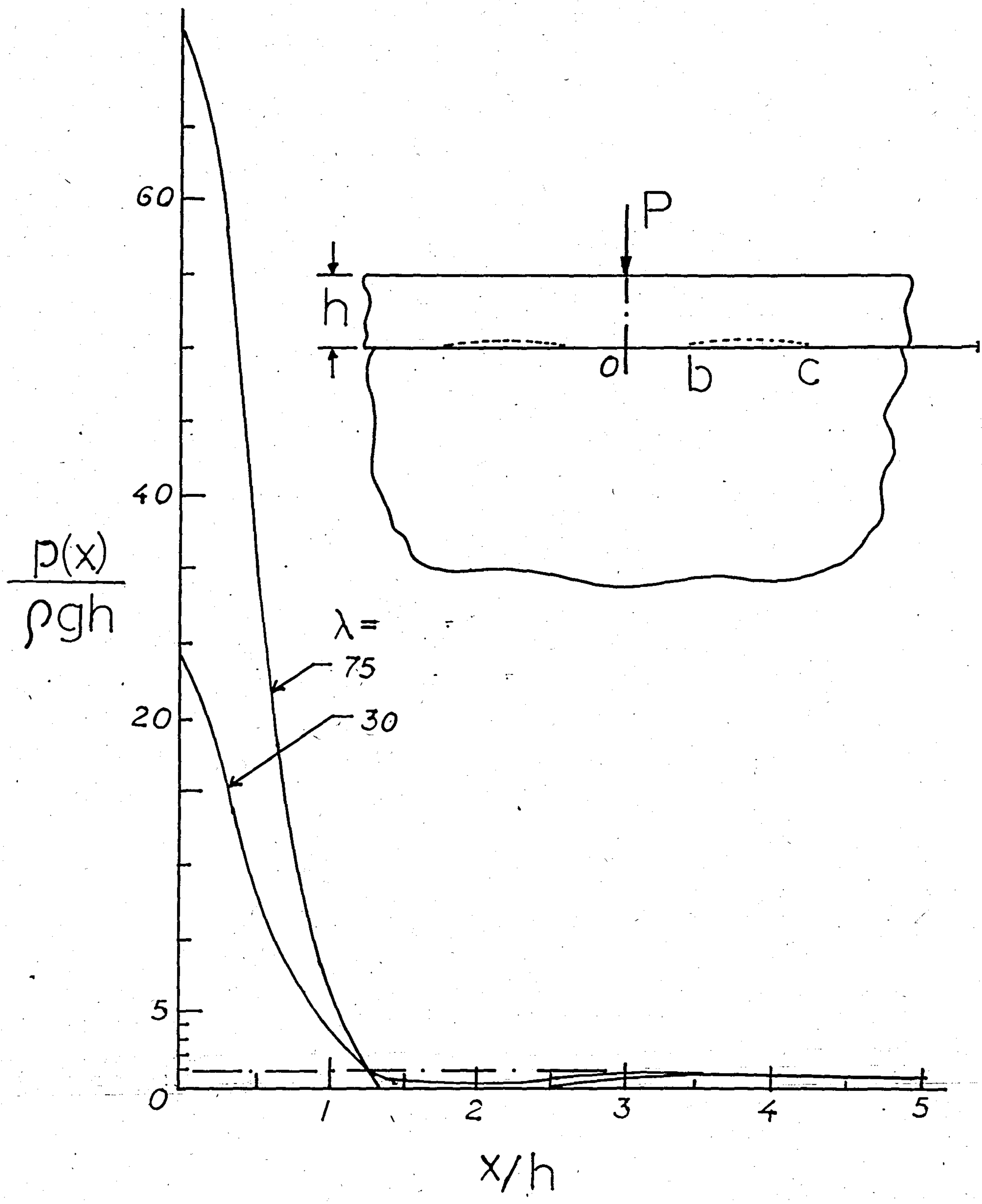

FIGURE 6. Distribution of Contact Pressure Between the Layer and the Half Space

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