THE FORCED VIBRATIONAL RESPONSE OF AN ELASTIC RECTANGULAR PARALLELEPIPED

by

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ABSTRACT

This work presents the analysis of the forced vibrational response of an elastic rectangular parallelepiped. Normal mode solutions were obtained for the following boundary conditions:

1. Six rigid-lubricated faces,
2. Four rigid-lubricated and two stress-free faces.

In the forced vibration analysis solutions were obtained for an impulsive and a step point load.

For the both cases of boundary conditions, computer programs were developed in order to calculate the displacements of a sample block. In the numerical calculations point of application of the force and the point at which, displacements are sensed were taken in the rectangular block.

ÖZET

Bu çalışmada elastik bir prizmanın uygulanan bir kuvvete titreşim tepkisinin analizi yapılmıştır. Normal mod çözümleri iki çeşit sınır şartları için elde edildi:

1. Altı rijid-yağlanmış yüz;
2. Dört rijid-yağlanmış ve iki gerilimsiz yüz.

Analizde impulsiv ve basamaklı nokta kuvvetler kullanılmıştır.

Her iki sınır şartları için örnek bir prizmanın deplasmanlarını hesaplayan kompüter programları yazıldı. Nümerik hesaplamalar için kuvvetin uygulandığı ve deplasmanların ölçüldüğü noktalar prizmanın içinde alınmıştır.

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LIST OF SYMBOLS

A_{iN}, $i = 1,2,\ldots,6$	Amplitude
$(A_{iN})_{\ell}$, $i = 1,2,\ldots,6$	Amplitude due to P-waves
$(A_{iN})_{t}$, $i = 1,2,\ldots,6$	Amplitude due to S-waves
a	Length
B_i, $i = 1,2,\ldots,6$	Unknown constant
b	Width
C_i, $i = 1,2,\ldots,6$	Unknown constant
c	Depth
c_{ℓ}	P-wave speed
c_t	S-wave speed
D_i, $i = 1,2,\ldots,6$	Unknown constant
E_i, $i = 1,2,\ldots,6$	Unknown constant
F	Force
F_i, $i = 1,2,\ldots,6$	Unknown constant
$\underset{\sim}{f} = (f_x, f_y, f_z)$	Body force vector
G	Scalar potential
G_{ij}	Green's Function
$\underset{\sim}{H} = (H_x, H_y, H_z)$	Vector potential
$H(\ )$	Unit step function
$\underset{\sim}{i}$	Unit vector in x-direction
$\underset{\sim}{j}$	Unit vector in y-direction
$\underset{\sim}{k}$	Unit vector in z-direction

$\mathcal{L}[\ ]$	Laplace operator
m	Integers from zero to infinity
n	Integers from zero to infinity
P	Integers from zero to infinity
S	Laplace variable
T_N	Function with time dependence
t	Time
$\underset{\sim}{u} = (u_x, u_y, u_z)$	Displacement vector
$\underset{\sim}{u}_N = (u_{xN}, u_{yN}, u_{zN})$	Normal mode displacement vector
$\dot{\underset{\sim}{u}}_o = (\dot{u}_{ox}, \dot{u}_{oy}, \dot{u}_{oz})$	Initial velocity field
$\underset{\sim}{V}$	Volume
x_o	x-coordinate of the point of application of the body force
y_o	y-coordinate of the point of application of the body force
z_o	z-coordinate of the point of application of the body force

<u>Greek Symbols</u>

α	Wave number
β	Wave number
γ	Wave number
γ_ℓ	Wave number associated with P-waves
γ_t	Wave number associated with S-waves
δ	Kronecker Delta
θ	A factor
λ	Lame constant
μ	Lame constant

ρ	Density
σ	Stress
τ	Time
$\underset{\sim}{\phi}_N = (\phi_{xN}, \phi_{yN}, \phi_{zN})$	Functions denotes spatial portion of normal modes
ϕ	Scalar Potential
$\underset{\sim}{\psi} = (\psi_x, \psi_y, \psi_y)$	Vector Potential
ω_N	Natural frequency

I. INTRODUCTION

Acoustic emissions are the transient elastic stress waves generated by a rapid release or redistribution of stored energy that accompany many deformation and fracture processes within a material. By monitoring these acoustic emissions, it is possible to trace the growth and propagation of cracks or flaws such as voids, inclusions, etc., in structures like bridges, power plant components. In addition acoustic emissions have been used for material research studies on microstructure related mechanical properties, phase transformations and fracture.

In order to deduce information from the recorded signals of acoustic emissions; it is necessary to know the frequency response of the structure. So far such analysis were done on structures such as half spaces or infinite plates [1-5]. However, many acoustic emission applications involve specimens of finite dimensions. In this respect, the vibration of a rectangular parallelepiped is of interest because many real life structures can be considered to be made up of rectangular blocks or plates, the latter being a two-dimensional version of the former.

Due to complexity of the mathematics involved, there are only a few solutions in the literature concerning the vibrations of a rectangular parallelepiped. Some of these papers involve plane strain solutions [6-8]. Fromme and Leissa [9] tried to solve the free vibration problem using associated periodicity method but their work results in an infinite set of algebraic equations which must be solved in order to obtain the natural frequencies of the body.

The free vibration problem for a rectangular parallelepiped with rigid-lubricated boundaries was first solved by Ortway [10] and then later by Nadeau [11] using normal mode technique. Then Hill and Egle [12] solved the forced vibration problem for the first time using the free vibration solution.

The free vibration solution for the case of four rigid-lubricated and two stress-free boundaries is the work of Kaliski [13]; but his work is in Polish and Malecki's text provides an English translation [14]. Kaliski's free vibration solution was then reworked by Hill [15].

It is the purpose of this work to take Hill's work as a basis and give free and forced vibration solutions of the rectangular parallelepiped. Two sets of boundary conditions considered here are (1) all six faces rigid-lubricated and (2) four rigid-lubricated and two stress-free faces. Free and forced vibration solutions for these cases are presented in Chapters III and IV respectively. In deriving the forced vibration displacement expressions, the body force is considered to be a three-dimensional concentrated force. In Chapter V, numerical results concerning the impulsive and step response of the rectangular parallelepiped with all faces rigid-lubricated and four faces rigid-lubricated and two faces stress free are given.

In the following chapter, equations of elasticity and the derivation of seperated wave equations will be given. Also in this chapter waves propagating in the bounded media and reflection phenomena from stress-free and rigid-lubricated boundaries will be presented.

II. EQUATIONS OF ELASTODYNAMICS

2.1 EQUATIONS OF ELASTICITY

The equation of motion for a linearly elastic, isotropic and homogeneous material is given by [16,17],

$$\mu\nabla^2 \underset{\sim}{u} + (\lambda + \mu)\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \rho \underset{\sim}{f} = \ddot{\underset{\sim}{u}} \tag{2.1}$$

where $\underset{\sim}{u}$ is the displacement vector, ρ is the mass density, λ and μ are the Lame constants of the material, and $\underset{\sim}{f}$ is the body force (per unit mass) vector used to represent the source of acoustic emissions. In the above equation ∇^2, $\underset{\sim}{\nabla}\cdot$, and $\underset{\sim}{\nabla}$ are the laplacian, divergence and gradient operators respectively and the superposed "dot" represents differentiation with respect to time, t. The constitutive equations for an isotropic elastic material are given by [18],

$$\underset{\approx}{\sigma} = \lambda(\underset{\sim}{\nabla} \cdot \underset{\sim}{u})\underset{\approx}{I} + 2\mu[\underset{\sim\sim}{\nabla u} + (\underset{\sim\sim}{\nabla u})^T] \tag{2.2}$$

where, $\underset{\approx}{\sigma}$ and $\underset{\approx}{I}$ are the stress and the identity tensors respectively and $(\underset{\sim\sim}{\nabla u})^T$ is the transpose of the tensor $\underset{\sim\sim}{\nabla u}$. The above stress-strain relations can be written in their explicit form in cartesian coordinate system as,

$$\sigma_x = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu \frac{\partial u_x}{\partial x} \tag{2.3}$$

$$\sigma_y = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu \frac{\partial u_y}{\partial y} \tag{2.4}$$

$$\sigma_z = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu \frac{\partial u_z}{\partial z} \tag{2.5}$$

$$\sigma_{xy} = \sigma_{yx} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{2.6}$$

$$\sigma_{yz} = \sigma_{zy} = \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \tag{2.7}$$

$$\sigma_{zx} = \sigma_{xz} = \mu\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) . \tag{2.8}$$

The coordinate system, geometry and sign convention of the stresses are given in Figure 2.1.

An alternative form of the Eq. (2.1) involving the wave speeds is

$$c_t^2 \nabla^2 \underset{\sim}{u} + (c_\ell^2 - c_t^2)\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \underset{\sim}{f} = \underset{\sim}{\ddot{u}} \tag{2.9}$$

where

$c_\ell = [(\lambda + 2\mu)/p]^{1/2}$: longitudinal wave speed

$c_t = [\mu/p]^{1/2}$: transverse wave speed.

2.2 ELASTIC WAVES IN A BOUNDED MEDIA

Within the body of a linearly elastic, isotropic and homogeneous material only two types of elastic waves can propagate. The faster of these is called the longitudinal wave which consists of compressions

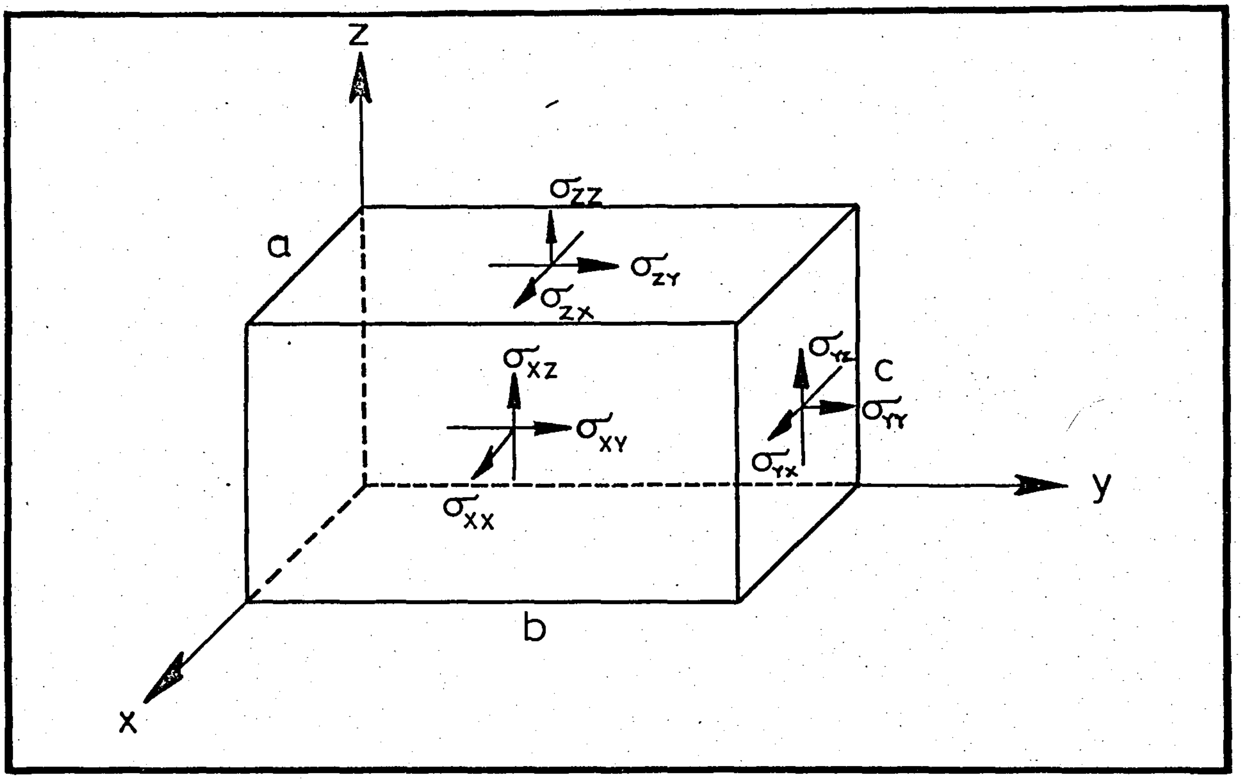


FIGURE 2.1 - Coordinate system, dimensions and stress sign convention.

(pushes) and dilatations (pulls) of the elastic material. In this case, the particle displacement is parallel to the direction of propagation. This type of wave is also known as dilatational wave or pressure wave or P-wave in short.

The slower of the two waves known as a transverse wave is of a quite different nature. The elastic body is sheared and twisted as the wave travels through it. The particle displacement lies in a plane normal to the direction of propagation thus, it can de decomposed into two orthogonal components. The one that is paralel to a given direction (usually specified by a surface in the body) is known as the SH-component while the other is the SV-component. Waves associated with these displacements are called the SH-wave (horizontally polarized) and SV-wave (vertically polarized). Transverse waves are also called equivoluminal waves, shear waves or S-waves in short. These two wave types are depicted in Figure 2.2.

When the elastic waves propagating in the bounded media reflect off the boundaries, some changes do occur in their nature, that is the reflected waves(s) need not to carry the same characteristics as the incident wave. These changes due to a reflection depend on the angle of incidence and the imposed boundary conditions. As an example, in the case of a stress-free boundary condition, an incident P-wave will give rise to both a reflected P-wave and a reflected SV-wave. Similarly an SV-wave will reflect as a SV-wave and a P-wave. This phenomena where a wave of one nature reflects as a wave of different nature is known as mode-conversion. Depending on the angle of incidence SV-waves do give rise to waves which propagate along the

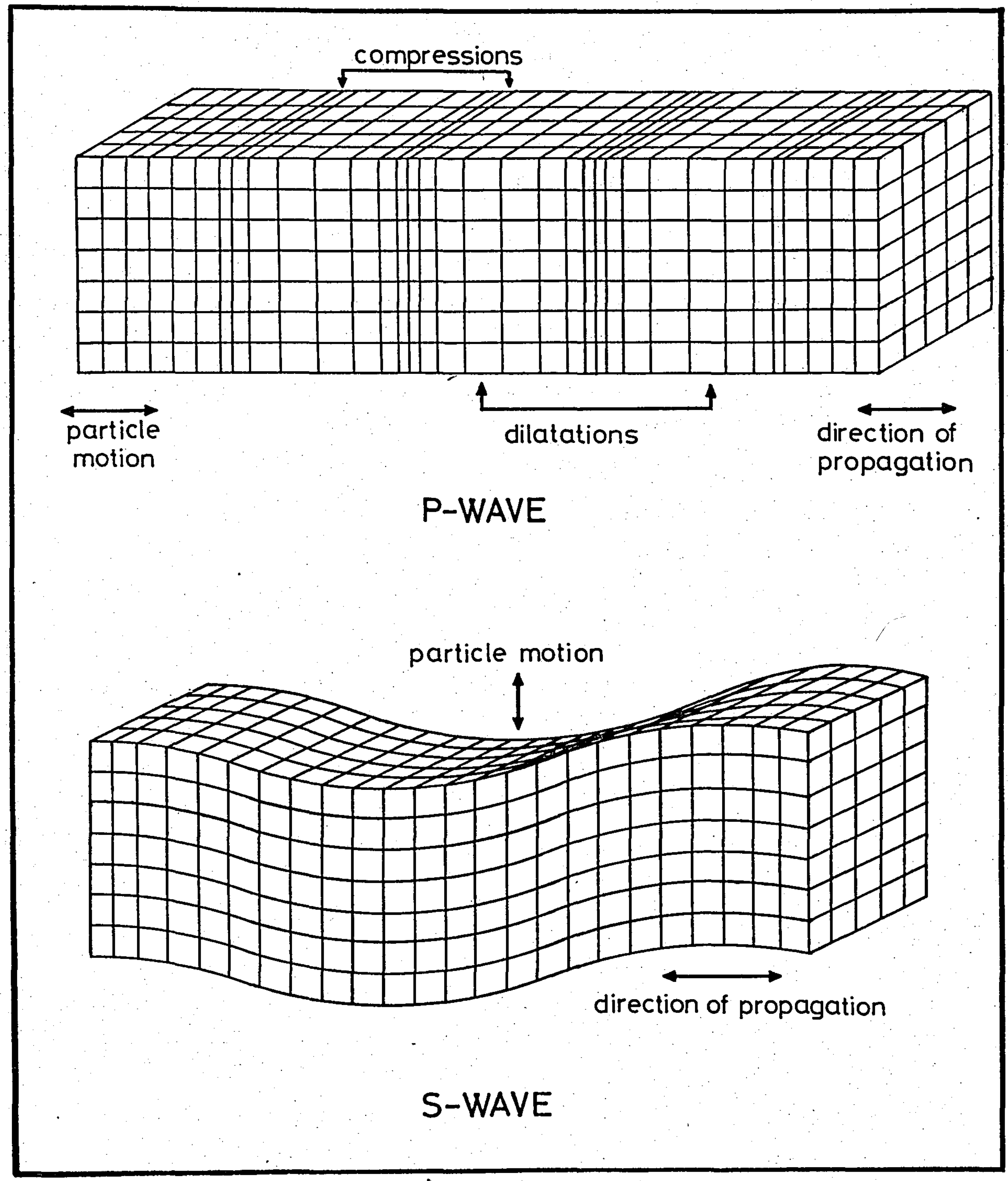


FIGURE 2.2 - Waves propagating within an elastic solid.

the boundaries known as surface waves. Such waves are confined to a small region in the neighbourhood of the surface and decay exponentially inside the media. For a free surface, these waves are often called as Rayleigh waves.

In the case of a rigid-lubricated boundary only phase changes occur and there is no mode conversions. Therefore, surface waves cannot exist in a elastic body under such boundary conditions.

2.3 DISPLACEMENT POTENTIALS

Since the equation of motion (2.9) is of a highly complex nature, one needs to transform it into a simpler form. According to Helmholtz Theorem [18,19], a vector field can be expressed as the sum of the gradient of a scalar field and the curl of a zero-divergence vector field. The vector fields of interest here, are the displacement and the body force; hence

$$\underset{\sim}{u} = \underset{\sim}{\nabla}\phi + \underset{\sim}{\nabla} \times \underset{\sim}{\psi} \quad ; \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{\psi} = 0 \tag{2.10}$$

$$\underset{\sim}{f} = \underset{\sim}{\nabla}G + \underset{\sim}{\nabla} \times \underset{\sim}{H} \quad ; \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{H} = 0 \tag{2.11}$$

where ϕ, G and $\underset{\sim}{\psi}$, $\underset{\sim}{H}$ are called scalar and vector potentials respectively. The zero divergence condition, $\underset{\sim}{\nabla}\cdot\underset{\sim}{\psi}$ provides the necessary additional condition to uniquely determine the three components of displacement from four components of ϕ and $\underset{\sim}{\psi}$. Substitution of the Eqs. (2.10) and (2.11), into the equation of motion leads to two seperated wave equations: (c.f. Appendix A)

$$c_\ell^2\nabla^2\phi + G = \ddot{\phi} \tag{2.12}$$

$$c_t^2\nabla^2\underset{\sim}{\psi} + \underset{\sim}{H} = \ddot{\underset{\sim}{\psi}} \quad . \tag{2.13}$$

From the above equations it can be seen that while the potentials ϕ, G are associated with the P-wave, $\underset{\sim}{\psi}$, $\underset{\sim}{H}$ are associated with S-wave.

Considering Eq. (2.10), it is possible to express the displacements in terms of both the scalar and the components of the vector potential as

$$u_x = \frac{\partial\phi}{\partial x} + \frac{\partial\psi_z}{\partial y} - \frac{\partial\psi_y}{\partial z}$$

$$u_y = \frac{\partial\phi}{\partial y} - \frac{\partial\psi_z}{\partial x} + \frac{\partial\psi_x}{\partial z} \qquad (2.14)$$

$$u_z = \frac{\partial\phi}{\partial z} + \frac{\partial\psi_y}{\partial x} - \frac{\partial\psi_x}{\partial y} \quad .$$

Thus, the Helmholtz Theorem mathematically uncouples the wave motion such that the displacement components due to longitudinal and transverse waves can be dealt with separately.

When a disturbance is produced at an internal point of a bounded elastic body, generally both P and S-waves will originate and propagate in all directions. A complicated nature of waves will result upon the reflections from the boundaries. As a result, a state of vibration of the whole body is reached. This state of vibration is a superposition of a number of characteristic vibrations (normal modes) of the finite body. These vibrations are represented by their distinct and discrete frequencies (natural frequencies) at which the system is capable of undergoing harmonic motion. For a continuous body there is an infinite number of natural frequencies associated with infinite number of normal modes.

Analysis of the free and forced vibrations of a rectangular parallelepiped with six faces rigid-lubricated and four faces rigid-lubricated while other faces are stress-free will be given in the following chapters.

III. RIGID-LUBRICATED BOUNDARIES

The analysis of the response of a rectangular block can be simplified by assuming rigid-lubricated boundaries. This is because of reflections from rigid-lubricated surfaces have no mode conversions but only phase changes as mentioned in the previous chapter. Although these boundary conditions are not representative of a typical acoustic emission experiment, solution of this problem provides a first step in obtaining the more difficult rigid-lubricated/stress-free solution.

3.1 FREE VIBRATION SOLUTION

The body force term in Eq. (2.9) is set equal to zero in order to obtain the equation of motion for the free vibration case.

$$c_t^2\nabla^2\underset{\sim}{u} + (c_\ell^2 - c_t^2)\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) = \underset{\sim}{\ddot{u}} \tag{3.1}$$

The rigid-lubricated boundary conditions are given as

$$
\begin{aligned}
u_x &= 0 \qquad \sigma_{xy} = \sigma_{xz} = 0 \qquad \text{at} \quad x = 0, a \quad ,\\
u_y &= 0 \qquad \sigma_{yx} = \sigma_{yz} = 0 \qquad \text{at} \quad y = 0, b \quad ,\\
u_z &= 0 \qquad \sigma_{zx} = \sigma_{zy} = 0 \qquad \text{at} \quad z = 0, c \quad .
\end{aligned}
$$

Considering stress-strain relationships given in Chapter II, boundary conditions can be expressed in terms of displacements as

$$
\begin{aligned}
&u_x = 0 \quad \partial u_x/\partial x = \partial u_z/\partial x = 0 \quad \text{at} \quad x = 0,\ a \ , \\
&u_y = 0 \quad \partial u_x/\partial y = \partial u_z/\partial y = 0 \quad \text{at} \quad y = 0,\ b \ , \\
&u_z = 0 \quad \partial u_x/\partial z = \partial u_y/\partial z = 0 \quad \text{at} \quad z = 0,\ c \ .
\end{aligned}
\tag{3.2}
$$

The problem may be solved by assuming a simple harmonic motion of the form [10,11]

$$
\begin{aligned}
u_{xN} &= A_{1N}\sin\alpha x \cos\beta y \cos\gamma z \sin\omega_N t \ , \\
u_{yN} &= A_{2N}\cos\alpha x \sin\beta y \cos\gamma z \sin\omega_N t \ , \\
u_{zN} &= A_{3N}\cos\alpha x \cos\beta y \sin\gamma z \sin\omega_N t \ ,
\end{aligned}
\tag{3.3}
$$

where ω_N are the natural frequencies or eigenvalue of the system. In order to satisfy the boundary conditions, the wave numbers α, β, and γ must be of the form $n\pi/a$, $m\pi/b$, and $p\pi/c$ respectively with n, m, p being integers 0, 1, 2, 3, ... Substituting the assumed normal modes, Eq. (3.3); into the equation of motion, Eq. (3.1), yields the following equations,

$$
\begin{aligned}
A_{1N}(\alpha^2 + \beta^2 + \gamma^2)c_t^2 + \alpha(A_{1N}\alpha + A_{2N}\beta + A_{3N}\gamma)(c_\ell^2 - c_t^2) &= A_{1N}\omega_N^2 \ , \\
A_{2N}(\alpha^2 + \beta^2 + \gamma^2)c_t^2 + \beta(A_{1N}\alpha + A_{2N}\beta + A_{3N}\gamma)(c_\ell^2 - c_t^2) &= A_{2N}\omega_N^2 \ , \\
A_{3N}(\alpha^2 + \beta^2 + \gamma^2)c_t^2 + \gamma(A_{1N}\alpha + A_{2N}\beta + A_{3N}\gamma)(c_\ell^2 - c_t^2) &= A_{3N}\omega_N^2 \ .
\end{aligned}
\tag{3.4}
$$

These equations can be written in the matrix form as

$$\begin{vmatrix} \Omega_N+\alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \Omega_N+\beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \Omega_N+\gamma^2 \end{vmatrix} \begin{matrix} A_{1N} \\ A_{2N} \\ A_{3N} \end{matrix} = 0 \tag{3.5}$$

where $\Omega_N = (c_t^2 - \Delta_N^2\omega_N^2)/(c_\ell^2 - c_t^2)$ and $\Delta_N^2 = \alpha^2 + \beta^2 + \gamma^2$. This set of equations has a nontrivial solution if and only if the determinant of this matrix is equal to zero. The resulting equation is known as the characteristic equation of motion,

$$(\Omega_N + \Delta_N^2)\Omega_N^2 = 0 \tag{3.6}$$

which has the simple root $\Omega_{1N} = -\Delta_N^2$, and the double root $\Omega_{2N} = \Omega_{3N} = 0$. The natural frequencies which correspond to these roots are

$$\omega_{1N} = c_\ell\Delta_N \tag{3.7}$$

$$\omega_{2N} = \omega_{3N} = c_t\Delta_N \ . \tag{3.8}$$

Note that while ω_{1N} is associated with the longitudinal waves, ω_{2N} and ω_{3N} are associated with the two orthogonal polarizations of the transverse waves [16]. Thus, each displacement component is made up of three contributions, one due to longitudinal wave and the other two due to the two orthogonal polarizations of transverse waves with the direction of propagation being determined by the set of integers N(n,m,p). Then the displacement components of the modes are given by

$$u_{xN} = \sin\alpha x \cos\beta y \cos\gamma z[(A_{1N})_\ell \sin\omega_{\ell N}t + (A_{1N})_t \sin\omega_{tN}t] ,$$

$$u_{yN} = \cos\alpha x \sin\beta y \cos\gamma z[(A_{2N})_\ell \sin\omega_{\ell N}t + (A_{2N})_t \sin\omega_{tN}t] , \quad (3.9)$$

$$u_{zN} = \cos\alpha x \cos\beta y \cos\gamma z[(A_{3N})_\ell \sin\omega_{\ell N}t + (A_{3N})_t \sin\omega_{tN}t] .$$

The amplitude relations associated with longitudinal waves are then obtained by subsituting the root $\Omega_{1N} = -\Delta_N^2$ into equations (3.11), yielding the relations,

$$\begin{aligned} A_{1N}\beta &= A_{2N}\alpha \\ A_{2N}\gamma &= A_{3N}\beta \\ A_{3N}\alpha &= A_{1N}\gamma . \end{aligned} \quad (3.10)$$

Note that by chosing one of the unknown amplitudes arbitrarily, the other two can be determined uniquely. A similar procedure for $\Omega_{2N} = \Omega_{3N} = 0$ results in the relation

$$A_{1N}\alpha + A_{2N}\beta + A_{3N}\gamma = 0$$

and in this case two of the three unknown amplitudes can be chosen arbitrarily. Thus the amplitude relations can be expressed as

$$\begin{aligned} (A_{1N})_\ell &= (A_{1N})_\ell \\ (A_{2N})_\ell &= (\beta/\alpha)(A_{1N})_\ell \\ (A_{3N})_\ell &= (\gamma/\alpha)(A_{1N})_\ell \end{aligned} \quad (3.11)$$

for the longitudinal waves and as

$$(A_{1N})_t = (A_{1N})_t$$

$$(A_{2N})_t = (A_{2N})_t \qquad (3.12)$$

$$(A_{3N})_t = -(\alpha/\gamma)(A_{1N})_t - (\beta/\gamma)(A_{2N})_t$$

for the transverse waves.

Hence, the normal mode displacement components take the form

$$u_{xN} = \sin\alpha x \cos\beta y \cos\gamma z\{(A_{1N})_\ell \sin\omega_{\ell N}t + (A_{1N})_t \sin\omega_{tN}t\}$$

$$u_{YN} = \cos\alpha x \sin\beta y \cos\gamma z\{(\beta/\alpha)(A_{1N})_\ell \sin\omega_{\ell N}t + (A_{2N})_t \sin\omega_{tN}t\} \quad , \qquad (3.13)$$

$$u_{zN} = \cos\alpha x \cos\beta y \sin\gamma z\{(\gamma/\alpha)(A_{1N})_\ell \sin\omega_{\ell N}t - [(\alpha/\gamma)(A_{1N})_t + (\beta/\gamma)(A_{2N})_t]\sin\omega_{tN}t\} \quad .$$

The unknown amplitudes $(A_{1N})_\ell$, $(A_{1N})_t$, $(A_{2N})_t$ are determined from the initial conditions which can be expressed generally as

$$\underset{\sim}{u}(x,y,z,0) = \underset{\sim}{u}_o(x,y,z)$$
$$\dot{\underset{\sim}{u}}(x,y,z,0) = \dot{\underset{\sim}{u}}_o(x,y,z)$$

where $\underset{\sim}{u}_o$ and $\dot{\underset{\sim}{u}}_o$ are the initial displacement and velocity fields respectively. Some of the characteristic vibration shapes associated with the displacements in the z-direction are given in Figure 3.1 through 3.4.

The general vibrational motion of the body is a superposition of inifinite number of normal modes as was mentioned previously. Thus the displacement expressions can simply be written as

$$u_x(x,y,z,t) = \sum_N u_{xN}(x,y,z,t) \quad ,$$

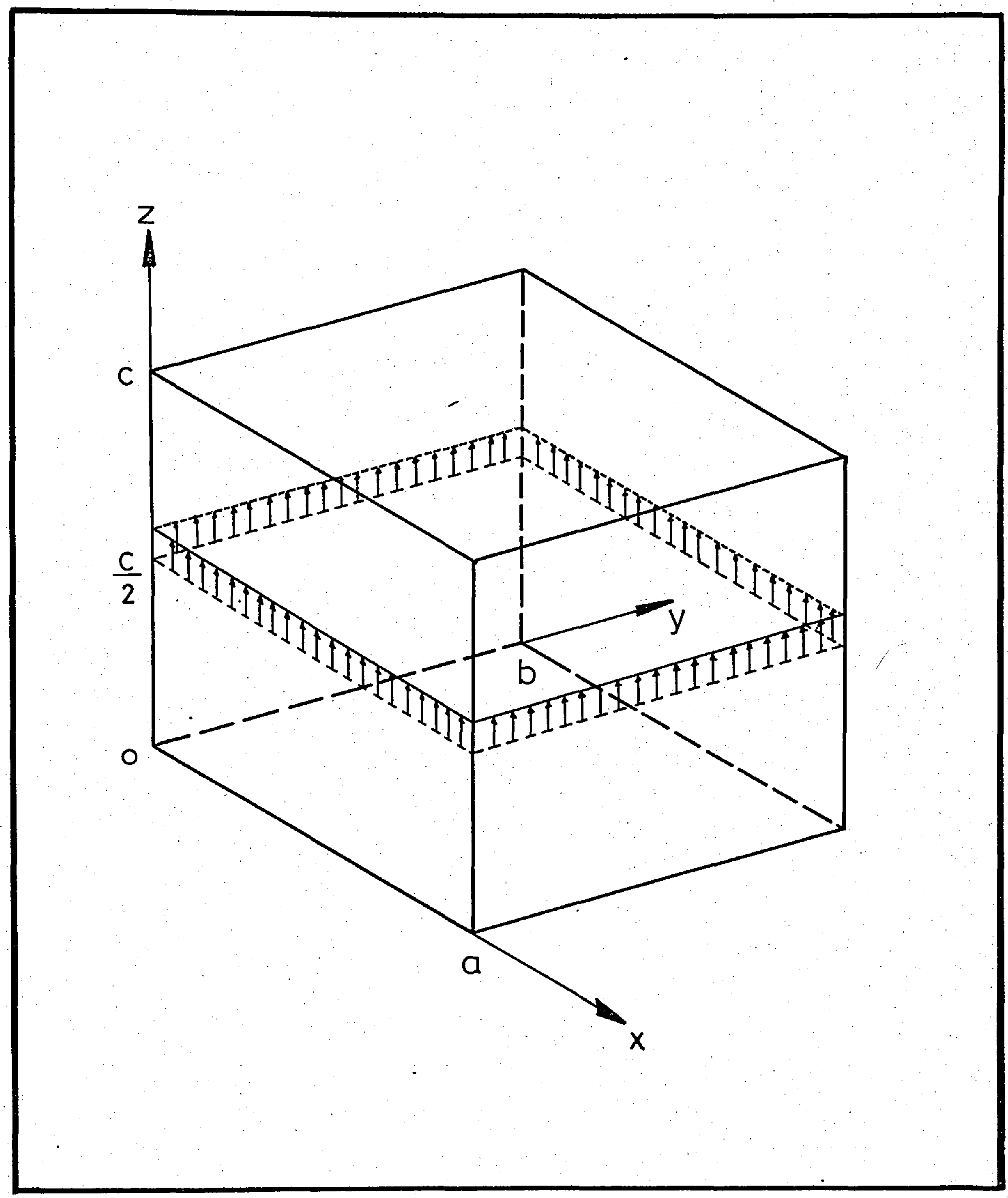


FIGURE 3.1 - (001) Mode for u_z (antisymmetric).

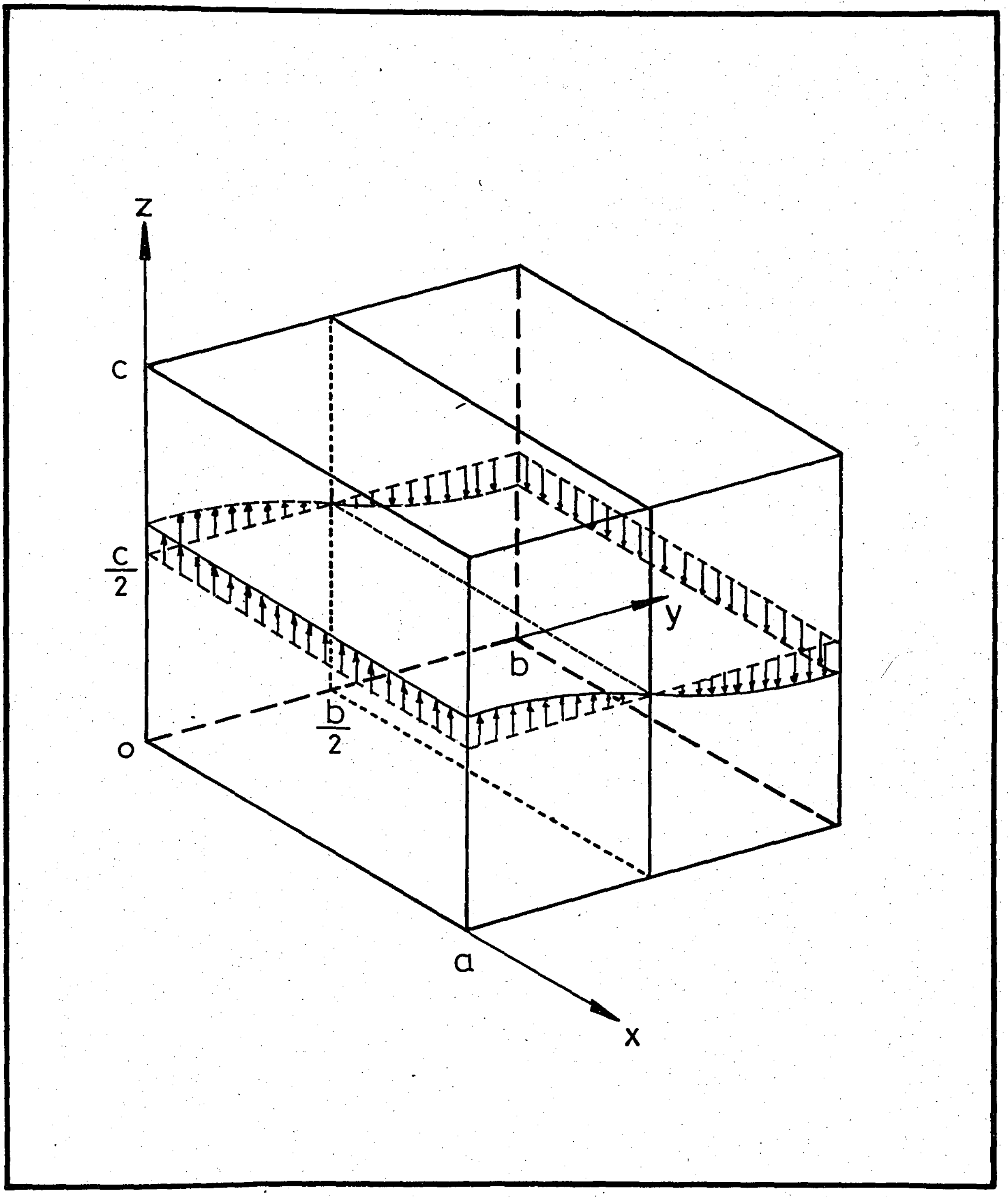


FIGURE 3.2 - (011) Mode for u_z (antisymmetric).

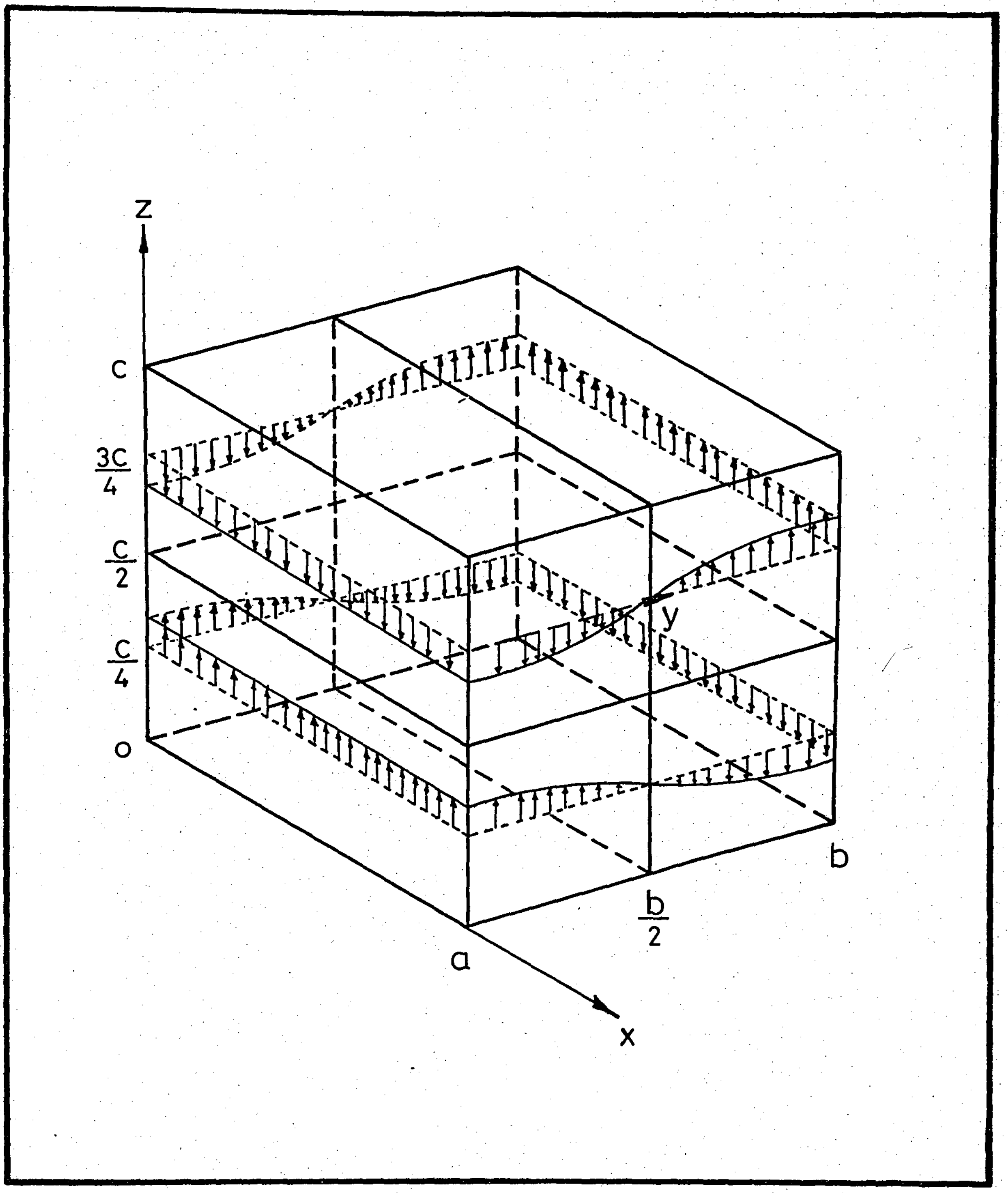


FIGURE 3.3 - (012) Mode for u_z (symmetric).

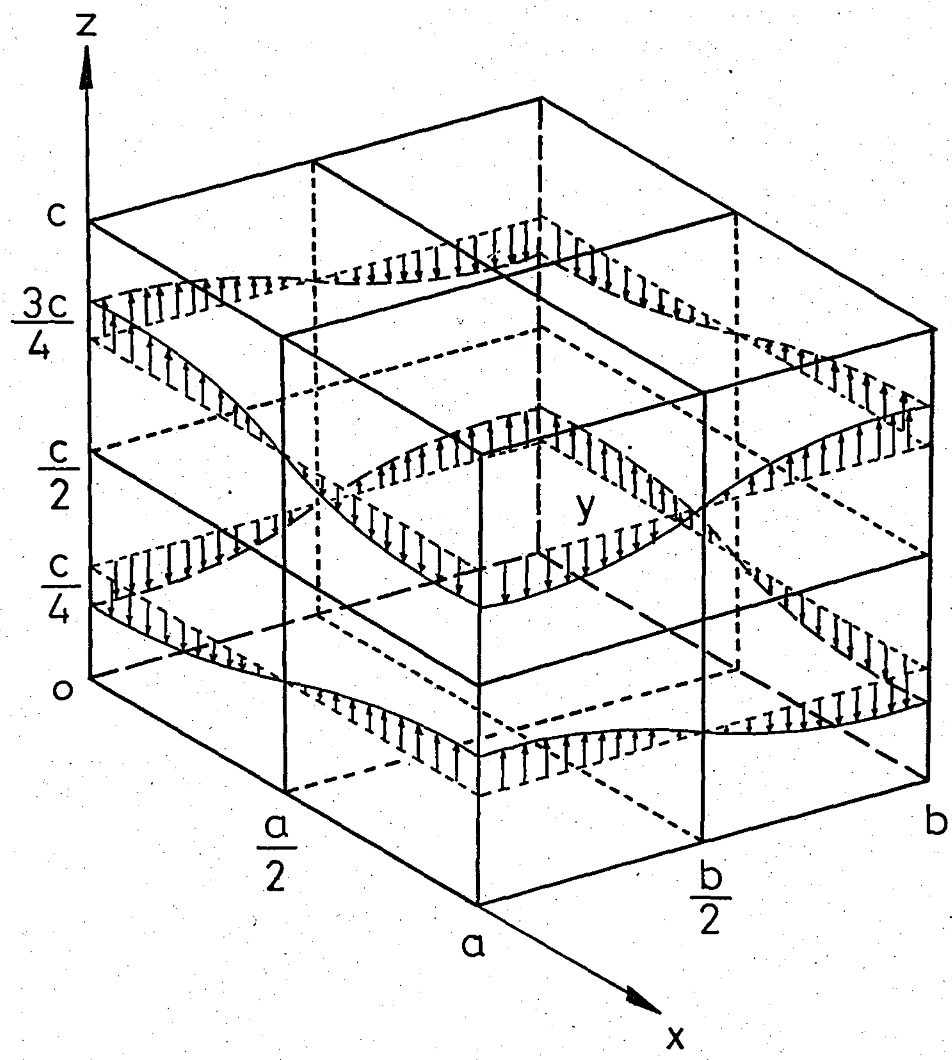


FIGURE 3.4 - (112) Mode for u_z (symmetric).

$$u_y(x,y,z,t) = \sum_N u_{yN}(x,y,z,t) \quad , \tag{3.14}$$

$$u_z(x,y,z,t) = \sum_N u_{zN}(x,y,z,t)$$

with $\sum_N = \sum_{n=0} \sum_{m=0} \sum_{p=0}$.

These equations represent the free vibration displacements of any point within or on the surface of the rectangular parallelepiped as a fuction of time.

3.2 FORCED VIBRATION SOLUTION

We will now consider the forced motion of the parallelepiped where the governing equation from Eq. (2.9) is

$$c_t^2\nabla^2\underset{\sim}{u} + (c_\ell^2 - c_t^2)\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \underset{\sim}{f} = \ddot{\underset{\sim}{u}} \quad . \tag{3.15}$$

The approach taken in forced vibration solution will be to replace the body force term, $\underset{\sim}{f}$, by an impulsive point load, solve this resulting special case to obtain the Green's function of the problem. Thus the solutions to more general problems can be then obtained through a convolution type integral. For the present three-dimensional problem, the Green's function is a tensor quantity [16,12], denoted by

$$G_{ij} = G_{ij}(x,y,z,t)/x_o,y_o,z_o,\tau) \tag{3.16}$$

where G_{ij} is the ith displacement component at position (x,y,z) and time t due to an impulsive force applied at position (x_o,y_o,z_o) and time and acting in jth direction. As stated above the solution to a general loading, $\underset{\sim}{f}$, can be obtained through the integral

$$\{u\} = \int_0^a \int_0^b \int_0^c \int_0^t [\underset{\approx}{G}]\{\underset{\sim}{f}\}\, dx_0 dy_0 dz_0 d\tau \,. \tag{3.17}$$

The above equation can be written explicitly as:

$$\begin{aligned} u_x &= \int_0^a \int_0^b \int_0^c \int_0^t (G_{xx} f_x + G_{xy} f_y + G_{xz} f_z) dx_0 dy_0 dz_0 d\tau \,, \\ u_y &= \int_0^a \int_0^b \int_0^c \int_0^t (G_{yx} f_x + G_{yy} f_y + G_{yz} f_z) dx_0 dy_0 dz_0 d\tau \,, \\ u_z &= \int_0^a \int_0^b \int_0^c \int_0^t (G_{zx} f_x + G_{zy} f_y + G_{zz} f_z) dx_0 dy_0 dz_0 d\tau \,. \end{aligned} \tag{3.18}$$

Note that, the equations governing the components of the Green's function are:

$$c_t^2\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\} G_{xx} + (c_\ell^2 - c_t^2)\{\frac{\partial^2 G_{xx}}{\partial x^2} + \frac{\partial^2 G_{yx}}{\partial y \partial x} + \frac{\partial^2 G_{zx}}{\partial z \partial x}\}$$

$$+\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(t) = \ddot{G}_{xx}$$

$$c_t^2\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\} G_{xy} + (c_\ell^2 - c_t^2)\{\frac{\partial^2 G_{xy}}{\partial x^2} + \frac{\partial^2 G_{yy}}{\partial y \partial x} + \frac{\partial^2 G_{zy}}{\partial z \partial x}\} = \ddot{G}_{xy}$$

$$c_t^2\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\} G_{xz} + (c_\ell^2 - c_t^2)\{\frac{\partial^2 G_{xz}}{\partial x^2} + \frac{\partial^2 G_{yz}}{\partial y \partial x} + \frac{\partial^2 G_{zz}}{\partial z \partial x}\} = \ddot{G}_{xz}$$

$$c_t^2\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\} G_{yx} + (c_\ell^2 - c_t^2)\{\frac{\partial^2 G_{xx}}{\partial x \partial y} + \frac{\partial^2 G_{yx}}{\partial y^2} + \frac{\partial^2 G_{zx}}{\partial z \partial y}\} = \ddot{G}_{yx}$$

$$c_t^2\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\}G_{yy} + (c_\ell^2 - c_t^2)\{\frac{\partial^2 G_{xy}}{\partial x \partial y} + \frac{\partial^2 G_{yy}}{\partial y^2} + \frac{\partial^2 G_{zy}}{\partial z \partial y}\}$$

$$+ \delta(x - x_o)\delta(y - y_o)\delta(z - z_o)\delta(t) = \ddot{G}_{yy}$$

$$c_t^2\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\} G_{yz} + (c_\ell^2 - c_t^2)\{\frac{\partial^2 G_{xz}}{\partial x \partial y} + \frac{\partial^2 G_{yz}}{\partial y^2} + \frac{\partial^2 G_{zz}}{\partial z \partial y}\} = \ddot{G}_{yz}$$

$$c_t^2\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\}G_{zx} + (c_\ell^2 - c_t^2)\{\frac{\partial^2 G_{xx}}{\partial x \partial z} + \frac{\partial^2 G_{yz}}{\partial y \partial z} + \frac{\partial^2 G_{zx}}{\partial z^2}\} = \ddot{G}_{zx}$$

$$c_t^2\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\} G_{zy} + (c_\ell^2 - c_t^2)\{\frac{\partial^2 G_{xy}}{\partial x \partial z} + \frac{\partial^2 G_{yy}}{\partial y \partial z} + \frac{\partial^2 G_{zy}}{\partial z^2}\} = \ddot{G}_{zy}$$

$$c_t^2\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\} G_{zz} + (c_\ell^2 - c_t^2)\{\frac{\partial^2 G_{xz}}{\partial x \partial z} + \frac{\partial^2 G_{yz}}{\partial y \partial z} + \frac{\partial^2 G_{zz}}{\partial z^2}\}$$

$$+ \delta(x - x_o)\delta(y - y_o)\delta(z - z_o)\delta(t) = \ddot{G}_{zz} \quad (3.19)$$

The solution of these equations can be obtained by making some assumptions. First, a factored solution with space and time dependency is assumed to represent the Green's function. Then, the spatial part of this solution is assumed to have the same form as the normal modes defined in the previous section and the time varying character of the Green's function is represented by a general function, $T_N(t)$. Therefore, the solution for the Green's function can be written as

$$G_{xxN} = \phi_{xN}(x,y,z)T_{xxN}(t) \quad ,$$

$$G_{xyN} = \phi_{xN}(x,y,z)T_{xyN}(t) \quad ,$$

$$G_{xzN} = \phi_{xN}(x,y,z)T_{xzN}(t) \quad ,$$

$$G_{yzN} = \phi_{yN}(x,y,z)T_{yxN}(t) \; ,$$

$$G_{yyN} = \phi_{yN}(x,y,z)T_{yyN}(t) \; , \qquad (3.20)$$

$$G_{yzN} = \phi_{yN}(x,y,z)T_{yzN}(t) \; ,$$

$$G_{zxN} = \phi_{zN}(x,y,z)T_{zxN}(t) \; ,$$

$$G_{zyN} = \phi_{zN}(x,y,z)T_{zyN}(t) \; ,$$

$$G_{zzN} = \phi_{zN}(x,y,z)T_{zzN}(t) \; ,$$

where ϕ_{xN}, ϕ_{yN}, ϕ_{zN} represent the spatial part of the normal modes.

Substituting Eqs. (3.20) into the Eq. (3.19a) and performing the necessary algebraic manipulations one gets the following equation:

$$\begin{aligned} & T_{xxN}[c_t^2\Delta^2 + (c_\ell^2 - c_t^2)\alpha^2]\sin\alpha x \cos\beta y \cos\gamma z \\ & + T_{yxN}[(c_\ell^2 - c_t^2)\alpha\beta]\sin\alpha x \cos\beta y \cos\gamma z \\ & + T_{zxN}[(c_\ell^2 - c_t^2)\alpha\gamma]\sin\alpha x \cos\beta y \cos\gamma z \\ & + \ddot{T}_{xxN}\sin\alpha x \cos\beta y \cos\gamma z = \delta(x - x_o)\delta(y - y_o)\delta(z - z_o)\delta(t) \end{aligned} \qquad (3.21)$$

where $\ddot{T}_{xxN}(t)$ represents the second derivative of $T_{xxN}(t)$ with respect to time. The next step is to multiply both sides of Eq. (3.21) by $\sin\alpha'x \cos\beta'y \cos\gamma'z$ and integrate over the spatial domain. Recalling the orthogonality relations of the normal modes,

$$\int_0^a \int_0^b \int_0^c \sin\alpha x \sin\alpha' x \cos\beta y \cos\beta' y \cos\gamma z \cos\gamma' z\, dx dy dz$$

$$= \begin{cases} 0 & \text{when } \alpha,\beta,\gamma \neq \alpha',\beta',\gamma' \\ \frac{\eta_1 V}{8} & \text{when } \alpha,\beta,\gamma = \alpha',\beta',\gamma' \end{cases} \tag{3.22}$$

with $\eta_1 = (1 + \delta_{\beta o})(1 + \delta_{\gamma o})$ and $V = abc$ is the volume of the parallelepiped. Therefore, performing the integrations on Eq. (3.21) gives

$$\ddot{T}_{xxN} + \{T_{xxN}[c_t^2\Delta^2 + (c_\ell^2 - c_t^2)\alpha^2] + T_{yxN}[(c_\ell^2 - c_t^2)\alpha\beta] + T_{zxN}[(c_\ell^2 - c_t^2)\alpha\gamma]\} = (8/\eta_1 V)\sin\alpha x_o \cos\beta y_o \cos\gamma z_o . \tag{3.23}$$

This expression may be solved by using method of Laplace transform. Assuming that the motion starts from rest, ($T_{xxN}(0) = \dot{T}_{xxN}(0) = \ddot{T}_{xxN}(0) = 0$) transformation yields the following expansion:

$$\bar{T}_{xxN}[s^2 + (c_\ell^2 - c_t^2)\alpha^2 + c_t^2\Delta^2] + \bar{T}_{yxN}[(c_\ell^2 - c_t^2)\alpha\beta] + \bar{T}_{zxN}[(c_\ell^2 - c_t^2)\alpha\gamma] = (8/\eta_1 V)\sin\alpha x_o \cos\beta y_o \cos\gamma z_o . \tag{3.24}$$

Application of the same procedure to the other eight equations of the set (3.19) results in the following eight expressions,

$$\bar{T}_{xyN}(s^2 + K_{xxN}) + \bar{T}_{yyN}K_{xyN} + \bar{T}_{zyN}K_{xzN} = 0$$

$$\bar{T}_{xzN}(s^2 + K_{xxN}) + \bar{T}_{yzN}K_{xyN} + \bar{T}_{zzN}K_{xzN} = 0$$

$$\bar{T}_{yxN}(s^2 + K_{yyN}) + \bar{T}_{xxN}K_{yxN} + \bar{T}_{zxN}K_{zyN} = 0$$

$$\bar{T}_{yyN}(s^2 + K_{yyN}) + \bar{T}_{xyN}K_{yxN} + \bar{T}_{zyN}K_{zyN} = (8/\eta_2 V)\cos\alpha x_o \sin\beta y_o \cos\gamma z_o \tag{3.25}$$

$$\bar{T}_{yzN}(s^2 + K_{yyN}) + \bar{T}_{xzN}K_{yxN} + \bar{T}_{zzN}K_{zyN} = 0$$

$$\bar{T}_{zxN}(s^2 + K_{zzN}) + \bar{T}_{xxN}K_{zxN} + \bar{T}_{yxN}K_{yzN} = 0$$

$$\bar{T}_{zyN}(s^2 + K_{zzN}) + \bar{T}_{xyN}K_{zxN} + \bar{T}_{yyN}K_{yzN} = 0$$

$$\bar{T}_{zzN}(s^2 + K_{zzN}) + \bar{T}_{xzN}K_{zxN} + \bar{T}_{yzN}K_{yzN} = (8/\eta_3 V)\cos\alpha x_o \cos\beta y_o \sin\gamma z_o$$

and Eq. (3.24) can be written in the same form as

$$\bar{T}_{xxN}(s^2 + K_{xxN}) + \bar{T}_{yxN}K_{xyN} + \bar{T}_{zxN}K_{xzN} = (8/\eta_1 V)\sin\alpha x_o \cos\beta y_o \cos\gamma z_o . \tag{3.26}$$

Note that,

$$K_{xxN} = (c_\ell^2 - c_t^2)\alpha^2 + c_t^2\Delta^2 \quad , \quad K_{xyN} = K_{yxN} = (c_\ell^2 - c_t^2)\alpha\beta$$
$$K_{yyN} = (c_\ell^2 - c_t^2)\beta^2 + c_t^2\Delta^2 \quad , \quad K_{yzN} = K_{zyN} = (c_\ell^2 - c_t^2)\beta\gamma$$
$$K_{zzN} = (c_\ell^2 - c_t^2)\gamma^2 + c_t^2\Delta^2 \quad , \quad K_{xzN} = K_{zxN} = (c_\ell^2 - c_t^2)\alpha\gamma$$

and $\eta_2 = (1 + \delta_{\alpha o})(1 + \delta_{\gamma o}) \quad , \quad \eta_3 = (1 + \delta_{\alpha o})(1 + \delta_{\beta o})$.

After performing the necessary algebraic manipulations, these expressions can be written in their new form as

$$\bar{T}_{xxN} = \frac{8D}{V\eta_1}[s^4 + (K_{yyN} + K_{zzN})s^2 + K_{yyN}K_{zzN} - K_{yzN}^2]\phi_{xN}(x_o,y_o,z_o)$$
$$\bar{T}_{xyN} = \frac{8D}{V\eta_2}[-s^2K_{xyN} + K_{xzN}K_{yzN} - K_{xyN}K_{zzN}]\phi_{yN}(x_o,y_o,z_o)$$
$$\bar{T}_{xzN} = \frac{8D}{V\eta_3}[-s^2K_{xzN} + K_{xyN}K_{yzN} - K_{xzN}K_{yyN}]\phi_{zN}(x_o,y_o,z_o)$$
$$\bar{T}_{yxN} = \frac{8D}{V\eta_1}[-s^2K_{xyN} + K_{xzN}K_{yzN} - K_{xyN}K_{zzN}]\phi_{xN}(x_o,y_o,z_o)$$
$$\bar{T}_{yyN} = \frac{8D}{V\eta_2}[s^4 + (K_{xxN}K_{zzN})s^2 + K_{xxN}K_{zzN} - K_{xzN}^2]\phi_{yN}(x_o,y_o,z_o)$$
$$\bar{T}_{yzN} = \frac{8D}{V\eta_3}[-s^2K_{yzN} + K_{xyN}K_{xzN} - K_{xxN}K_{yzN}]\phi_{zN}(x_o,y_o,z_o) \tag{3.26}$$

$$\overline{T}_{zxN} = \frac{8D}{V\eta_1}[-s^2K_{xzN} + K_{xyN}K_{yzN} - K_{xzN}K_{yyN}]\phi_{xN}(x_o,y_o,z_o)$$

$$\overline{T}_{zyN} = \frac{8D}{V\eta_2}[-s^2K_{yzN} + K_{xyN}K_{xzN} - K_{xxN}K_{yzN}]\phi_{yN}(x_o,y_o,z_o)$$

$$\overline{T}_{zzN} = \frac{8D}{V\eta_3}[s^4 + (K_{xxN}K_{yyN})s^2 + K_{xxN}K_{yyN} - K^2_{xyN}]\phi_{zN}(x_o,y_o,z_o) .$$

Then, inverse Laplace transforms can be obtained using partial fractions technique. Thus the results are

$$T_{xxN} = \frac{1}{\Delta_N^2}\{\frac{\alpha^2 \sin\omega_{\ell N}t}{\omega_{\ell N}} + \frac{(\Delta_N^2 - \alpha^2)\sin\omega_{tN}t}{\omega_{tN}}\}\frac{8\phi_{xN}(x_o,y_oz_o)}{V\eta_1}$$

$$T_{xyN} = \frac{\alpha\beta}{\Delta_N^2}\{\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\omega_{tN}}\}\frac{8\phi_{yN}(x_o,y_o,z_o)}{V\eta_2}$$

$$T_{xzN} = \frac{\beta\gamma}{\Delta_N^2}\{\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\omega_{tN}}\}\frac{8\phi_{zN}(x_o,y_o,z_o)}{V\eta_3}$$

$$T_{yxN} = \frac{\beta\alpha}{\Delta_N^2}\{\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\mu_{tN}}\}\frac{8\phi_{xN}(x_o,y_o,z_o)}{V\eta_1} \qquad (3.27)$$

$$T_{yyN} = \frac{1}{\Delta_N^2}\{\frac{\beta^2 \sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{(\Delta_N^2 - \beta^2)\sin\omega_{tN}t}{\omega_{tN}}\}\frac{8\phi_{yN}(x_o,y_o,z_o)}{V\eta_2}$$

$$T_{yzN} = \frac{\beta\gamma}{\Delta_N^2}\{\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\omega_{tN}}\}\frac{8\phi_{zN}(x_o,y_o,z_o)}{V\eta_3}$$

$$T_{zxN} = \frac{\gamma\alpha}{\Delta_N^2}\{\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\omega_{tN}}\}\frac{8\phi_{xN}(x_o,y_o,z_o)}{V\eta_1}$$

$$T_{zyN} = \frac{\gamma\beta}{\Delta_N^2}\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\omega_{tN}}\frac{8\phi_{yN}(x_o,y_o,z_o)}{V\eta_2}$$

$$T_{zzN} = \frac{1}{\Delta_N^2}\{\frac{\gamma^2 \sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{(\Delta_N^2 - \gamma^2)\sin\omega_{tN}t}{\omega_{tN}}\}\frac{8\phi_{zN}(x_o,y_o,z_o)}{V\eta_3} .$$

Finally these time function expressions must be substituted in the Eqs. (3.20) which give the Green's function of the system. These in turn are substituted into Eqs. (3.18) to arrive at the forced vibration displacement $\underset{\sim}{u}(x,y,z,t)$ for any generalized body force $\underset{\sim}{f}(x,y,z,t)$.

3.3 IMPULSIVE RESPONSE

The acoustic emissions generated by material flows are thought to be pulselike functions of stress (force). Much of this type of emission in solids is produced internally and can, therefore, be modelled as a body force phenomenon. Assuming a very short duration source event within the body, the Dirac Delta function provides an extremely simple mathematical approximation of the resulting impulsive body force. In general, the body force is three-dimensional and its components may be expressed mathematically as

$$\begin{aligned} f_x &= F_x\delta(x - x_o)\delta(y - y_o)\delta(z - z_o)\delta(t) \\ f_y &= F_y\delta(x - x_o)\delta(y - y_o)\delta(z - z_o)\delta(t) \\ f_z &= F_z\delta(x - x_o)\delta(y - y_o)\delta(z - z_o)\delta(t) \ . \end{aligned} \tag{3.28}$$

Note that the impulsive load is applied at the point (x_o,y_o,z_o) and at time $t = 0$ and its components have magnitudes F_x, F_y, F_z in x, y, z directions respectively, (Figure 3.5).

The first step in order to determine displacement expressions for the impulsive response is the substitution of the impulsive body force components (3.28) into Eqs. (3.18). Then the normal mode displacement expressions are obtained after performing the necessary integrals yielding

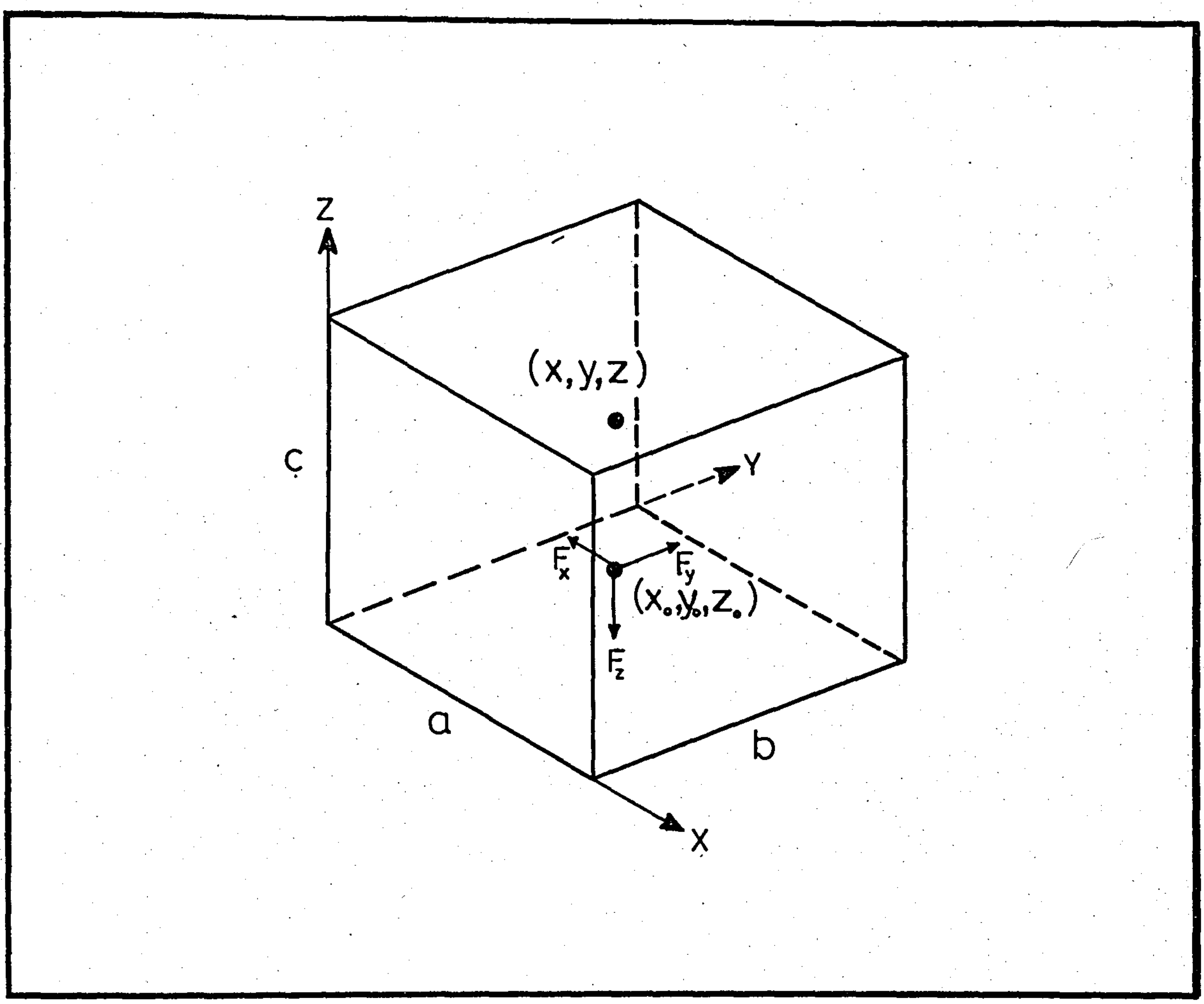


FIGURE 3.5 - Point of application of the body force and point at which displacements are sensed.

$$u_{xN} = G_{xxN}F_x + G_{xyN}F_y + G_{xzN}F_z$$
$$u_{yN} = G_{yxN}F_x + G_{yyN}F_y + G_{yzN}F_z \qquad (3.29)$$
$$u_{zN} = G_{zxN}F_x + G_{zyN}F_y + G_{zzN}F_z \quad .$$

These are then combined with Eqs. (3.20) to obtain the displacement components produced by a three-dimensional impulsive force applied at the point (x_0,y_0,z_0) and finally these can be written as

$$u_x(x,y,z,t) = \sum_N \frac{1}{\Delta_N^2}\{[\frac{\alpha^2 \sin\omega_{\ell N}t}{\omega_{\ell N}} + \frac{(\Delta_N^2 - \alpha^2)\sin\omega_{tN}t}{\omega_{tN}}] \frac{\phi_{xN}(x_0,y_0,z_0)}{\eta_1} F_x$$
$$+ \alpha\beta[\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\omega_{tN}}] \frac{\phi_{yN}(x_0,y_0,z_0)}{\eta_2} F_y$$
$$+ \alpha\gamma[\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\omega_{tN}}] \frac{\phi_{zN}(x_0,y_0,z_0)}{\eta_3} F_z\}$$
$$(8/V)\phi_{xN}(x,y,z) \qquad (3.30)$$

$$u_y(x,y,z,t) = \sum_N \frac{1}{\Delta_N^2}\{\beta\alpha[\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\omega_{tN}}] \frac{\phi_{xN}(x_0,y_0,z_0)}{\eta_1} F_x$$
$$+ [\beta^2 \frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} + (\Delta_N^2 - \beta^2) \frac{\sin\omega_{tN}t}{\omega_{tN}}] \frac{\phi_{yN}(x_0,y_0,z_0)}{\eta_2} F_y$$
$$+ \beta\gamma[\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\omega_{tN}}] \frac{\phi_{zN}(x_0,y_0,z_0)}{\eta_3} F_z\}$$
$$(8/V)\phi_{yN}(x,y,z) \quad ,$$

$$u_z(x,y,z,t) = \sum_N \frac{1}{\Delta_N^2}\{\alpha\gamma[\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\omega_{tN}}]\frac{\phi_{xN}(x_o,y_o,z_o,t)}{\eta_1} F_x$$

$$+ \beta\gamma[\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\omega_{tN}}]\frac{\phi_{yN}(x_o,y_o,z_o)}{\eta_2} F_y$$

$$+ [\gamma^2\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} + \frac{(\Delta_N^2 - \gamma^2)\sin\omega_{tN}t}{\omega_{tN}}]\frac{\phi_{zN}(x_o,y_o,z_o)}{\eta_3} F_z\}$$

$$(8/V)\phi_{zN}(x,y,z)$$

with $\sum_N = \sum_{\alpha=0}\sum_{\beta=0}\sum_{\gamma=0}$.

3.4 STEP RESPONSE

We will now obtain the solution for the case where the time dependency of the body force is a step function. The components of the body force in this case are

$$\begin{aligned} f_x &= F_x\delta(x - x_o)\delta(y - y_o)\delta(z - z_o)H(t) \\ f_y &= F_y\delta(x - x_o)\delta(y - y_o)\delta(z - z_o)H(t) \\ f_z &= F_z\delta(x - x_o)\delta(y - y_o)\delta(z - z_o)H(t) \quad . \end{aligned} \tag{3.31}$$

Note that once again, the force is assumed to be applied at the point (x_o,y_o,z_o) and at time $t = 0$ with components having magnitudes F_x, F_y, F_z in x, y, z directions respectively.

A similar procedure can be used here, in order to derive displacement expressions. Not surprisingly, these expressions have the same form as the impulsive response displacement expressions (3.30), except for the functions

$$\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} \quad , \quad \frac{\sin\omega_{tN}t}{\omega_{tN}}$$

which are

$$\frac{1 - \cos\omega_{\ell N}t}{\omega_{\ell N}^2} \quad , \quad \frac{1 - \cos\omega_{tN}t}{\omega_{tN}^2}$$

respectively.

This completes the analysis for the vibrational response of the rectangular parallelepiped with six rigid-lubricated faces. In the following chapter, analysis for the free and forced vibrations of a rectangular parallelepiped with four faces rigid-lubricated and two faces stress-free will be given.

IV. RIGID-LUBRICATED/STRESS-FREE BOUNDARIES

As a next step to the analysis of the response of a rectangular block, one can consider the case of a block with two stress-free and four rigid-lubricated faces. This system is depicted in Figure 4.1. In this figure, the z-faces (cross-hatched) are stress-free and x,y faces are rigid-lubricated. This problem is considerably more involved than the previous one due to the mode conversions on the two stress-free faces. The complexity in the wave propagation also holds true for the normal modes and the characteristic equation as well. Where in Chapter III it was possible to determine by inspection the exact form of the normal modes, however, in this case it is very difficult to do so. Hence, we will make use of the seperated wave equations in order to obtain the normal modes of the system.

4.1 FREE VIBRATION SOLUTION

The equation of interest for the free vibration solution is the Eq. (2.9) and is repeated here for convenience.

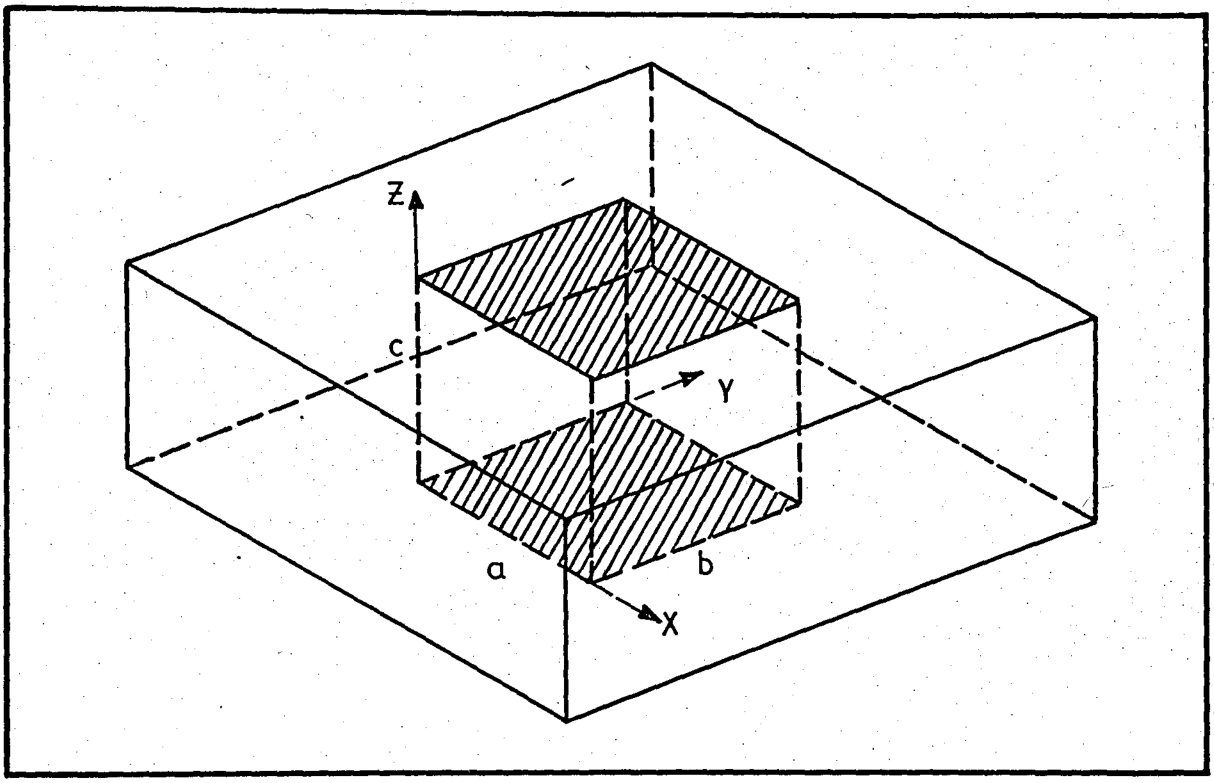


FIGURE 4.1 - Stress-free rigid-lubricated boundaries.

$$c_t^2\nabla^2 \underset{\sim}{u} + (c_\ell^2 - c_t^2)\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) = \ddot{\underset{\sim}{u}} \qquad (4.1)$$

In order to solve this equation we need the boundary conditions; and, for a block with z-faces being stress-free while x, y-faces are rigid-lubricated the boundary conditions can be expressed as

$$u_x = 0 \quad , \quad \partial u_y/\partial x = \partial u_z/\partial x = 0 \quad \text{at} \quad x = 0, a \ ;$$

$$u_y = 0 \quad , \quad \partial u_x/\partial y = \partial u_z/\partial y = 0 \quad \text{at} \quad y = 0, b \ ;$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \theta\frac{\partial u_z}{\partial z} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = 0 \qquad (4.2)$$

$$\text{at} \quad z = 0, c$$

where $\theta = 1 + (2\mu/\lambda)$.

As explained earlier, the equation of motion can not be solved by inspection so we will use the separated wave equations, (2.12) and (2.13), which were derived in Chapter II. For the free vibration case, the body force terms in these equations are neglected and the seperated wave equations can be written in their new form as

$$c_\ell^2\nabla^2\phi = \ddot{\phi} \qquad (4.3)$$

$$c_t^2\nabla^2\underset{\sim}{\psi} = \ddot{\underset{\sim}{\psi}} \ . \qquad (4.4)$$

The general solutions for these equations are (See Appendix B):

$$\phi = (C_1\cos\alpha x + C_2\sin\alpha x)(C_3\cos\beta x + C_4\sin\beta y)(C_5\cos\gamma_\ell z + C_6\sin\gamma_\ell z)\sin\omega_\ell t$$

$$\psi_1 = (D_1\cos\alpha x + D_2\sin\alpha x)(D_3\cos\beta y + D_4\sin\beta y)(D_5\cos\gamma_t z + D_6\sin\gamma_t z)\sin\omega_t t$$

$$(4.5)$$

$$\psi_2 = (E_1\cos\alpha x + E_2\sin\alpha x)(E_3\cos\beta y + E_4\sin\beta y)(E_5\cos\gamma_t z + E_6\cos\gamma_t z)\sin\omega_t t$$

$$\psi_3 = (F_1\cos\alpha x + F_2\sin\alpha x)(F_3\cos\beta y + F_4\sin\beta y)(F_5\cos\gamma_t z + F_6\sin\gamma_t z)\sin\omega_t t$$

By considering the rigid-lubricated part of the boundary conditions, some of the unknown constants can be eliminated. Then the potential expressions can be written as

$$\begin{aligned}
\phi_N &= -\cos\alpha x\cos\beta y(A_{1N}\cos\gamma_\ell z + A_{2N}\sin\gamma_\ell z)\sin\omega_N t\\
\psi_{1N} &= \cos\alpha x\sin\beta y(B_{1N}\cos\gamma_t z + B_{2N}\sin\gamma_t z)\sin\omega_N t\\
\psi_{2N} &= \sin\alpha x\cos\beta y(C_{1N}\cos\gamma_t z + C_{2N}\sin\gamma_t z)\sin\omega_N t\\
\psi_{3N} &= \sin\alpha x\sin\beta y(D_{1N}\cos\gamma_t z + D_{2N}\sin\gamma_t z)\sin\omega_N t
\end{aligned} \qquad (4.6)$$

and associated wave numbers are

$$\alpha = n\pi/a\ ,\quad \beta = m\pi/b\quad ,\quad \gamma_\ell = [(\omega_N^2/c_\ell^2) - (\alpha^2 + \beta^2)]^{1/2}\ ,$$
$$\gamma_t = [(\omega_N^2/c_t^2) - (\alpha^2 + \beta^2)]^{1/2}$$

and $n,m = 0,1,2,\ldots$.

Upon substituting the assumed potentials, Eq. (4.6), into the Eq. (2.14), one finds the normal mode displacement components related to the P and S waves seperately.

$$\begin{aligned}
u_{xN}^P &= \sin\alpha x\cos\beta y[\alpha(A_{1N}\cos\gamma_\ell z + A_{2N}\sin\gamma_\ell z)]\sin\omega_N t\\
u_{yN}^P &= \cos\alpha x\sin\beta y[\beta(A_{1N}\cos\gamma_\ell z + A_{2N}\sin\gamma_\ell z)]\sin\omega_N t\\
u_{zN}^P &= \cos\alpha x\cos\beta y[\gamma_\ell(A_{1N}\sin\gamma_\ell z - A_{2N}\cos\gamma_\ell z)]\sin\omega_N t
\end{aligned} \qquad (4.7)$$

and

$$\begin{aligned}
u_{xN}^S &= \sin\alpha x\cos\beta y[(\beta D_{1N} - \gamma_t C_{2N})\cos\gamma_t z + (\beta D_{2N} + \gamma_t C_{1N})\sin\gamma_t z]\sin\omega_N t\\
u_{yN}^S &= \cos\alpha x\sin\beta y[(\gamma_t B_{2N} - \alpha D_{1N})\cos\gamma_t z - (\alpha D_{2N} + \gamma_t B_{1N})\sin\gamma_t z]\sin\omega_N t\\
u_{zN}^S &= \cos\alpha x\cos\beta y[(\alpha C_{1N} - \beta B_{1N})\cos\gamma_t z - (\alpha C_{2N} - \beta B_{2N})\sin\gamma_t z]\sin\omega_N t\ .
\end{aligned} \qquad (4.8)$$

Then we will make use of zero divergence condition, $\underset{\sim}{\nabla}\cdot\underset{\sim}{\psi} = 0$, in order to reduce the number of unknown constants. Application of this condition to the vector potential, $\underset{\sim}{\psi}$, leads to the result that

$$D_{1N} = -(1/\gamma_t)(\alpha B_{2N} + \beta C_{2N})$$
$$D_{2N} = -(1/\gamma_t)(\alpha B_{1N} + \beta C_{1N}) \ .$$

These relations allow a simplification in Eq. (4.8) yielding

$$\begin{aligned}
u^S_{xN} &= \sin\alpha x\cos\beta y\{-(1/\gamma_t)[\alpha\beta B_{2N} + (\beta^2 + \gamma_t^2)C_{2N}]\cos\gamma_t z \\
&\quad + (1/\gamma_t)[\alpha\beta B_{1N} + (\beta^2 + \gamma_t^2)C_{1N}]\sin\gamma_t z\}\sin\omega_N t \\
u^S_{yN} &= \cos\alpha x\sin\beta y\{(1/\gamma_t)[(\alpha^2 + \gamma_t^2)B_{2N} + \alpha\beta C_{2N}]\cos\gamma_t z \\
&\quad + (1/\gamma_t)[(\alpha^2 + \gamma_t^2)B_{1N} + \alpha\beta C_{1N}]\sin\gamma_t z\}\sin\omega_N t \qquad (4.9) \\
u^S_{zN} &= \cos\alpha x\cos\beta y\{(\alpha C_{1N} - \beta B_{1N})\cos\gamma_t z + (\alpha C_{2N} - \beta B_{2N}) \\
&\quad \sin\gamma_t z\}\sin\omega_N t \ .
\end{aligned}$$

Then the displacement components due to S-waves may be expressed in a new form defining

$$\begin{aligned}
A_{3N} &= -(1/\gamma_t)[\alpha\beta B_{2N} + (\beta^2 + \gamma_t^2)C_{2N}] \\
A_{4N} &= (1/\gamma_t)[\alpha\beta B_{1N} + (\beta^2 + \gamma_t^2)C_{1N}] \\
A_{5N} &= (1/\gamma_t)[(\alpha^2 + \gamma_t^2)B_{2N} + \alpha\beta C_{2N}] \qquad (4.10) \\
A_{6N} &= -(1/\gamma_t)[(\alpha^2 + \gamma_t^2)B_{1N} + \alpha\beta C_{1N}]
\end{aligned}$$

and substituting these new expressions for the amplitudes into the Eq. (4.9) yields

$$u_{xN}^S = \sin\alpha x\cos\beta y(A_{3N}\cos\gamma_t z + A_{4N}\sin\gamma_t z)\sin\omega_N t$$

$$u_{yN}^S = \cos\alpha x\sin\beta y(A_{5N}\cos\gamma_t z + A_{6N}\sin\gamma_t z)\sin\omega_N t \tag{4.11}$$

$$u_{zN}^S = \cos\alpha x\cos\beta y\{(1/\gamma_t)[(\alpha A_{4N} + \beta A_{6N})\cos\gamma_t z - (\alpha A_{3N} + \beta A_{5N})\sin\gamma_t z]\}\sin\omega_N t$$

Finally the displacement components due to P and S-waves, Eqs. (4.7) and (4.11), are combined to generate the normal mode displacement components:

$$u_{xN} = \sin\alpha x\cos\beta y[\alpha(A_{1N}\cos\gamma_\ell z + A_{2N}\sin\gamma_\ell z) + (A_{3N}\cos\gamma_t z + A_{4N}\sin\gamma_t z)]\sin\omega_N t = \phi_{xN}\sin\omega_N t$$

$$u_{yN} = \cos\alpha x\sin\beta y[\beta(A_{1N}\cos\gamma_\ell z + A_{2N}\sin\gamma_\ell z) + (A_{5N}\cos\gamma_t z + A_{6N}\sin\gamma_t z)]\sin\omega_N t = \phi_{yN}\sin\omega_N t \tag{4.12}$$

$$u_{zN} = \cos\alpha x\cos\beta y[\gamma_\ell(A_{1N}\sin\gamma_\ell z - A_{2N}\cos\gamma_\ell z) + (1/\gamma_t)[(\alpha A_{4N} + \beta A_{6N})\cos\gamma_t z - (\alpha A_{3N} + \beta A_{5N})\sin\gamma_t z]\sin\omega_N t = \phi_{zN}\sin\omega_N t$$

The next step in the analysis of the free vibration problem is to determine the natural frequencies of the system. The above normal modes are substituted in the boundary conditions (4.2). Twelve of the eighteen boundary conditions related to the rigid-lubricated faces are satisfied exactly, while the other six stress-free boundary conditions yield six equations with six unknown constants A_{iN} $(i = 1,2,\ldots,6)$:

$$(\alpha^2 + \beta^2 + \theta\gamma_\ell^2)A_{1N} - (\theta - 1)\alpha A_{3N} - (\theta - 1)\beta A_{5N} = 0 ,$$

$$(\alpha^2 + \beta^2 + \theta\gamma_\ell^2)\cos\gamma_\ell c A_{1N} + (\alpha^2 + \beta^2 + \theta\gamma_\ell^2)\sin\gamma_\ell c A_{2N}$$
$$- (\theta - 1)\alpha\cos\gamma_t c A_{3N} - (\theta - 1)\alpha\sin\gamma_t c A_{4N} - (\theta - 1)\beta\cos\gamma_t c A_{5N}$$
$$- (\theta - 1)\beta\sin\gamma_t c A_{6N} = 0 ,$$

$$2\alpha\gamma_\ell\gamma_t A_{2N} - (\alpha^2 - \gamma_t^2)A_{4N} - \alpha\beta A_{6N} = 0 , \qquad (4.13)$$

$$-2\alpha\gamma_\ell\gamma_t\sin\gamma_\ell c A_{1N} + 2\alpha\gamma_\ell\gamma_t\cos\gamma_\ell c A_{2N} + (\alpha^2 - \gamma_t^2)\sin\gamma_t c A_{3N}$$
$$- (\alpha^2 - \gamma_t^2)\cos\gamma_t c A_{4N} + \alpha\beta\sin\gamma_t c A_{5N} - \alpha\beta\cos\gamma_t c A_{6N} = 0 ,$$

$$2\beta\gamma_\ell\gamma_t A_{2N} - \alpha\beta A_{4N} - (\beta^2 - \gamma_t^2)A_{6N} = 0$$

$$-2\beta\gamma_\ell\gamma_t\sin\gamma_\ell c A_{1N} + 2\beta\gamma_\ell\gamma_t\cos\gamma_\ell c A_{2N} + \alpha\beta\sin\gamma_t c A_{3N}$$
$$- \alpha\beta\cos\gamma_t c A_{4N} + (\beta^2 - \gamma_t^2)\sin\gamma_t c A_{5N} - (\beta^2 - \gamma_t^2)\cos\gamma_t c A_{6N} = 0 .$$

These equations can be easily put into a matrix form, where the determinant of this matrix yields the frequency equation,

$$(P^2 + R^2)\sin^2\gamma_t c \ \sin\gamma_\ell c + 2PR(1 - \cos\gamma_\ell c \ \cos\gamma_t c)\sin\gamma_t c = 0 \qquad (4.14)$$

where $P = 4(\alpha^2 + \beta^2)\gamma_\ell\gamma_t$ and $R = (\alpha^2 + \beta^2 - \gamma_t^2)^2$.

Relations between the amplitudes A_{iN} $(i = 1,2,\ldots,6)$ can be also obtained from the Eqs. (4.13) by using Gaussian elimination method. There are several combinations of frequency equations and amplitude relations depending on the values of $\sin\gamma_t c$ and the wave numbers α and β which are summarized in Table 4.1.

TABLE 4.1 - Appropriate Modal Coefficients and Frequency Equations.

Modal Coefficients	$\sin\gamma_t c = 0$	$\sin\gamma_t c \neq 0$		
	$\alpha > 0,\ \beta > 0$	$\alpha = \beta = 0$	$\alpha > 0,\ \beta = 0$	$\alpha \geq 0,\ \beta > 0$
A_{1N}	0	0	$-\dfrac{P(\alpha^2 - \gamma_t^2)}{R(2\alpha\gamma_\ell\gamma_t)} A_{3N}$	$-\dfrac{P(\alpha^2 + \beta^2 - \gamma_t^2)}{R(2\alpha\gamma_\ell\gamma_t)} A_{5N}$
A_{2N}	0	*	$\dfrac{\alpha^2 - \gamma_t^2}{2\alpha\gamma_\ell\gamma_t} A_{4N}$	$\dfrac{\alpha^2 + \beta^2 - \gamma_t^2}{2\beta\gamma_\ell\gamma_t} A_{6N}$
A_{3N}	$-(\beta/\alpha)A_{5N}$	0	$\dfrac{R(\cos\gamma_\ell c - \cos\gamma_t c)}{P\sin\gamma_\ell c + R\sin\gamma_t c}$	$(\alpha/\beta)A_{5N}$
A_{4N}	0	0	*	$(\alpha/\beta)A_{6N}$
A_{5N}	*	0	0	$-\dfrac{R(\cos\gamma_\ell c - \cos\gamma_t c}{P\sin\gamma_\ell c + R\sin\gamma_t c} A_{6N}$
A_{6N}	0 (4.15)	0 (4.17)	0 (4.19)	* (4.20)
Frequency Equations	$\omega_N = c_t\Delta$ (4.16) $\Delta^2 = \alpha^2 + \beta^2 + \gamma^2$	$\omega_N = c_\ell\Delta$ (4.18)	$(P^2 + R^2)\sin\gamma_\ell c\sin\gamma_t c + 2PR(1 - \cos\gamma_\ell c\ \cos\gamma_t c) = 0$ (4.21) $P = 4(\alpha^2 + \beta^2)\gamma_\ell\gamma_t$	$R = (\alpha^2 + \beta^2 - \gamma_t^2)^2$

$\sin\gamma_t c$ in Eq. (4.14) can be factored out and setting this term equal to zero gives the frequency equation (4.16). In this case, the only possible combination of wave numbers α, β and $\sin\gamma_t c$ are those where $\sin\gamma_t c = 0$ and $\alpha > 0$, $\beta > 0$, because if any of the two wave numbers is zero then the determinant of equations (4.13) vanishes. The amplitude relations associated with $\sin\gamma_t c = 0$ and $\alpha > 0$, $\beta > 0$ are (4.15). Considering these relations, P and S-wave displacement expressions become

$$u_{xN}^{P} = 0 \quad , \quad u_{xN}^{S} = -A_{5N}(\alpha/\beta)\sin\alpha x\cos\beta y\cos\gamma_t z\sin\omega_N t$$

$$u_{yN}^{P} = 0 \quad , \quad u_{yN}^{S} = A_{5N}\cos\alpha x\sin\beta y\cos\gamma_t z\sin\omega_N t$$

$$u_{zN}^{P} = 0 \quad , \quad u_{zN}^{S} = 0 \quad .$$

As seen in the above equations, the combination, $\sin\gamma_t c = 0$, $\alpha > 0$, $\beta > 0$ corresponds to modes in which the displacements are in x-y plane. Since displacements due to P-waves are zero, only shear waves propagate in the block. Note that, the frequency equation has only shear wave speed which means there are no mode conversions at the stress-free boundaries; therefore the waves propagating in the block are SH-waves.

Note that, Eq. (4.16) is not the only frequency equation; in the case where $\sin\gamma_t c$ is not zero it is possible to find which will cause the term given by Eq. (4.21) to vanish!. In such cases Eq. (4.21) is the frequency equation. This is a transcendental equation having both of the longitudinal and shear wave speeds which means that mode conversions at the stress-free surfaces are possible. These mode conversions are responsible for the increased complexity in the amplitude relations (4.19) and (4.20). The natural frequencies of the system can be obtained implicitly from this equation. The amplitude relations associated with the combination where $\sin\gamma_t c \neq 0$ and $\alpha = \beta = 0$ are given by Eq. (4.17). Now, P and S-wave displacement components are given as

$$u^P_{xN} = 0 \quad , \quad u^S_{xN} = 0$$

$$u^P_{yN} = 0 \quad , \quad u^S_{yN} = 0$$

$$u^P_{zN} = -A_{2N}\gamma_\ell \cos\alpha x \cos\beta y \cos\gamma_\ell z \sin\omega_N t \quad , \quad u^S_{zN} = 0 .$$

In this case the frequency equation (4.18) is a very simple form of Eq. (4.23) since the wave numbers α and β are equal to zero and the frequencies of the system can easily be calculated. This case represents P-waves propagating in z-direction only because only P-wave displacement component in the z-direction is present. Since the P-waves are normally incident to the stress-free z-faces, there are no mode conversions and they reflect back and forth between these two faces.

An alternative combination is the case $\sin\gamma_t c \neq 0$ and $\alpha > 0$, $\beta = 0$. In this case, the displacement components due to P and S-waves are

$$u^P_{xN} = \sin\alpha x[\alpha(A_{1N}\cos\gamma_\ell z + A_{2N}\sin\gamma_\ell z)]\sin\omega_N t$$

$$u^P_{yN} = 0$$

$$u^P_{zN} = \cos\alpha x[\gamma_\ell(A_{1N}\cos\gamma_\ell z - A_{2N}\sin\gamma_\ell z)]\sin\omega_N t$$

and

$$u_{xN}^S = \sin\alpha x(A_{3N}\cos\gamma_t z + A_{4N}\sin\gamma_t z)\sin\omega_N t$$

$$u_{yN}^S = 0$$

$$u_{zN}^S = \cos\alpha x((\alpha/\gamma_t)(A_{4N}\cos\gamma_t z - A_{3N}\sin\gamma_t c)]\sin\omega_N t$$

and it can easily be seen that P and S waves propagate in the block. When $\sin\gamma_t c \neq 0$ and $\alpha = 0$, $\beta > 0$, the reversed conditions exist. The displacement components are

$$u_{xN}^P = 0$$

$$u_{yN}^P = \sin\beta y[\beta(A_{1N}\cos\gamma_\ell z + A_{2N}\sin\gamma_\ell z)]\sin\omega_N t$$

$$u_{zN}^P = \cos\beta y[\gamma_\ell(A_{1N}\sin\gamma_\ell z - A_{2N}\cos\gamma_\ell z)]\sin\omega_N t$$

and

$$u_{xN}^S = 0$$

$$u_{yN}^S = \sin\beta y[A_{5N}\cos\gamma_t z + A_{6N}\sin\gamma_t z]\sin\omega_N t$$

$$u_{zN}^S = \cos\beta y[(\beta/\gamma_t)(A_{6N}\cos\gamma_t z - A_{5N}\sin\gamma_t z)]\sin\omega_N t \quad .$$

Finally for the combination $\sin\gamma_t c \neq 0$ and $\alpha > 0$, $\beta > 0$ both of the P and S-waves propagate in all directions since none of the displacement components due to these waves vanishes. The amplitude relations and the frequency equation associated with these three case are given by the equations (4.19), (4.20) and (4.21). They represent the propagation of mode converted P and SV-waves. Some of the mode shapes associated with these combinations are given in Figures (4.2) through (4.5).

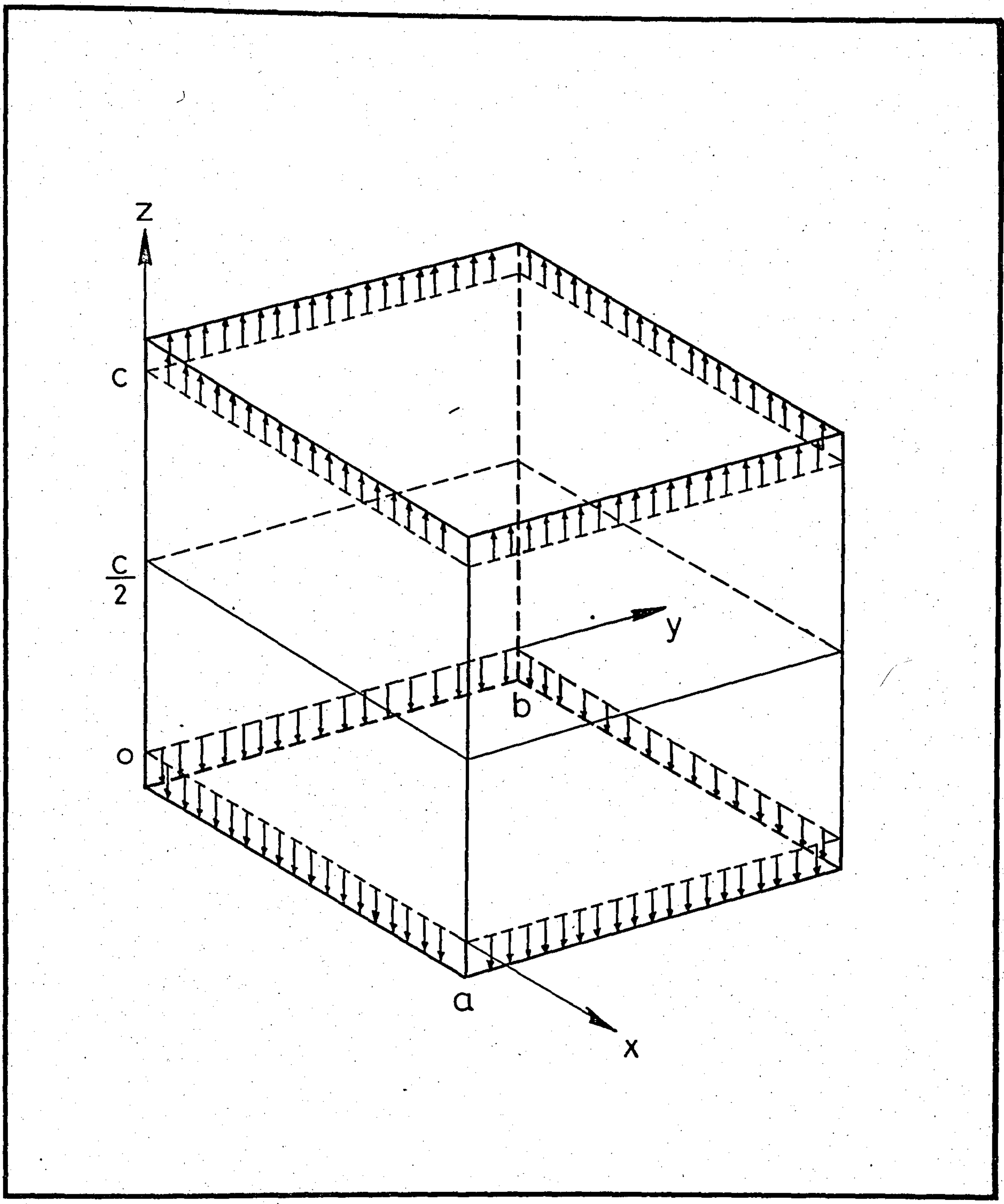


FIGURE 4.2 - (001) Mode for u_z (symmetric).

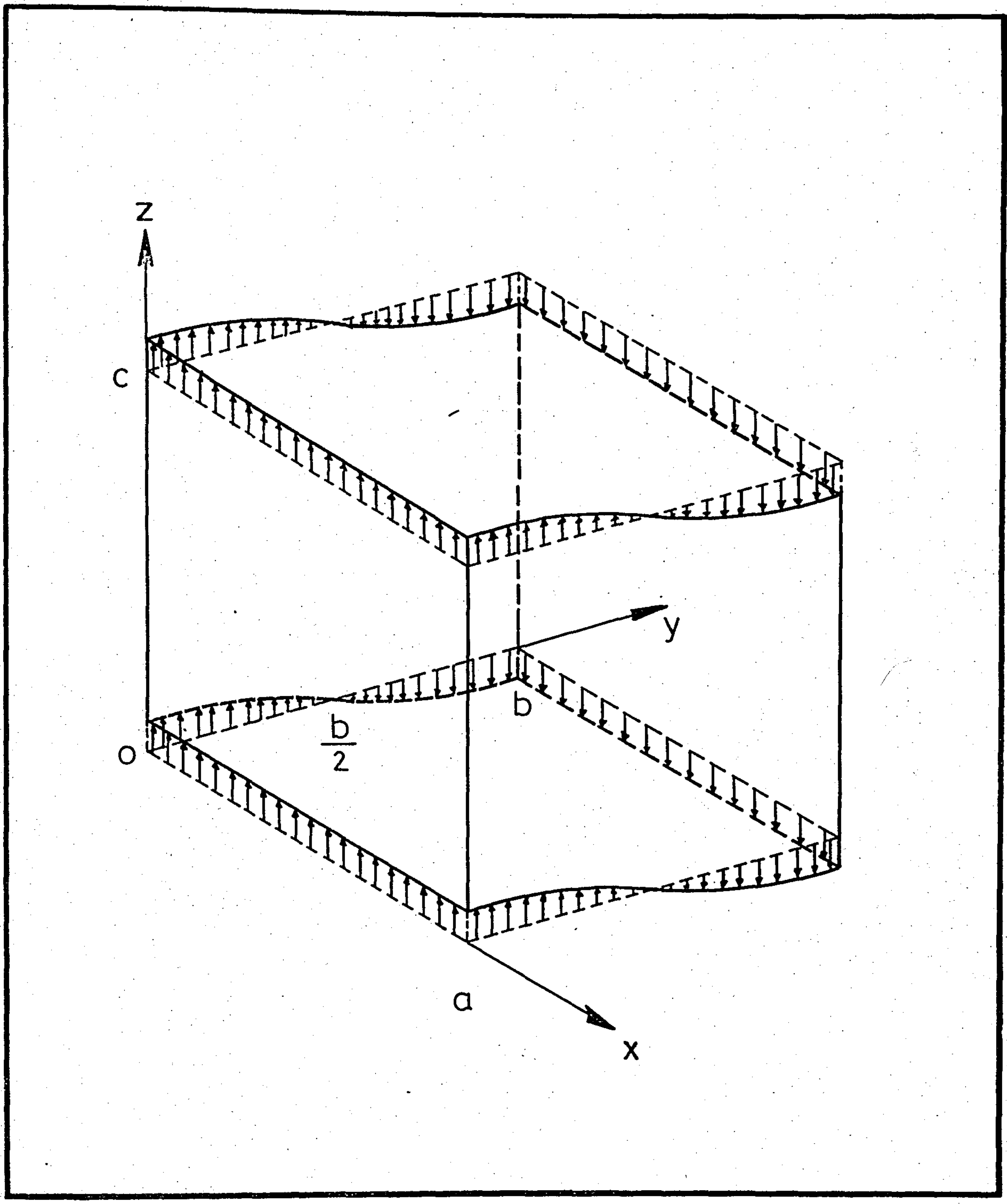


FIGURE 4.3 - (010) Mode for u_z (antisymmetric).

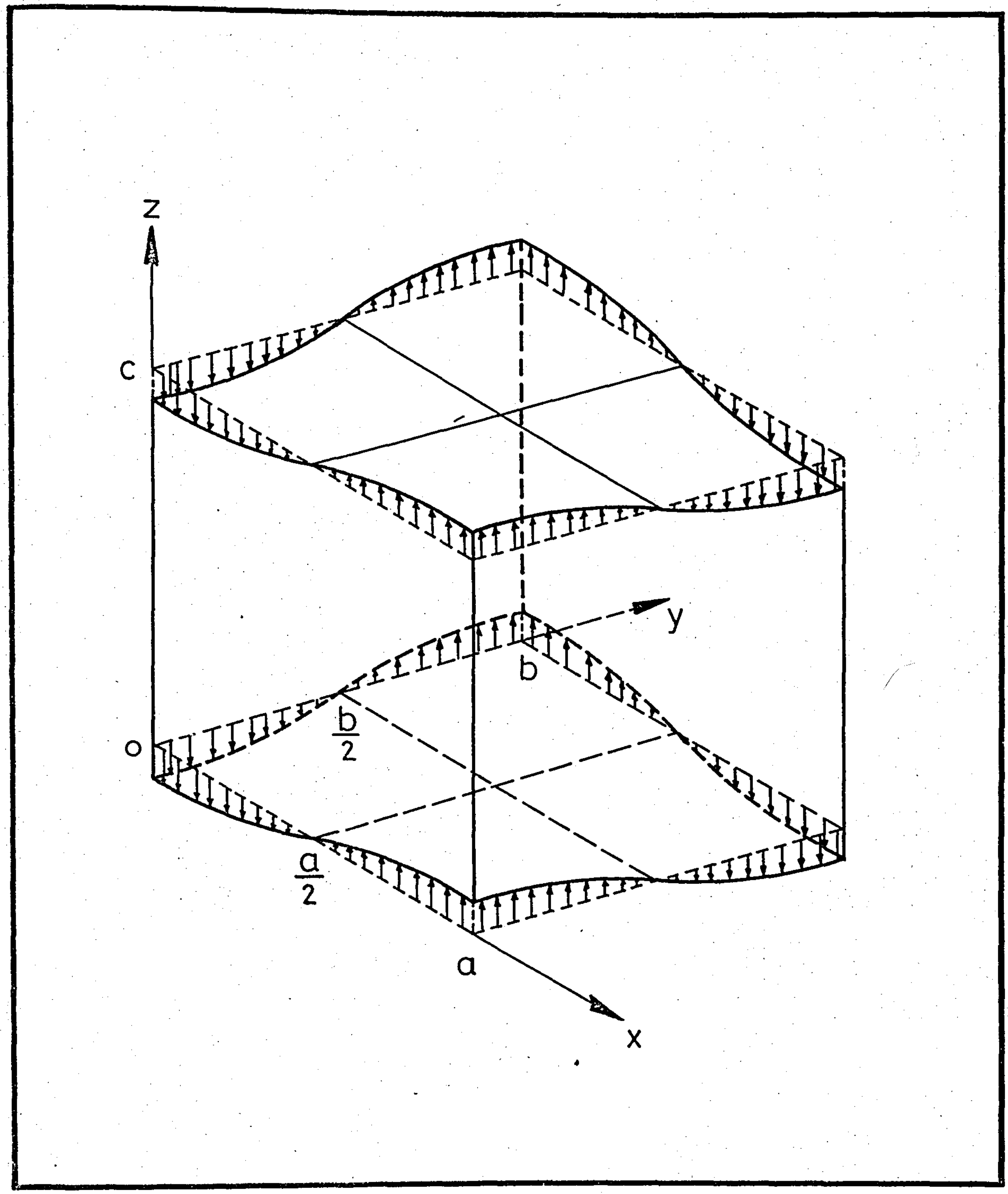


FIGURE 4.4 - (110) Mode for u_z (antisymmetric).

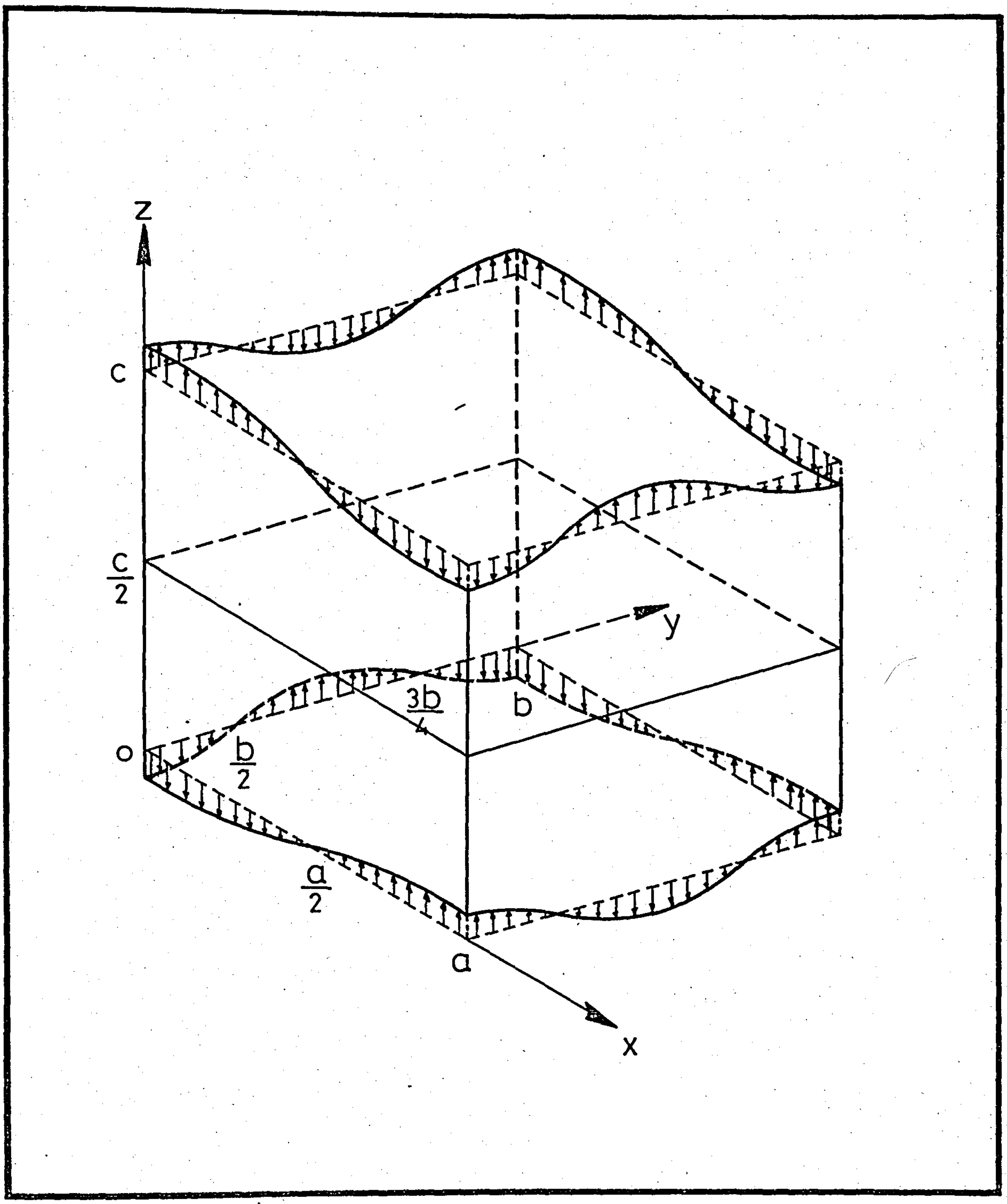


FIGURE 4.5 - (121) Mode for u_z (symmetric).

The asterisks in Table 4.1 denote the unknown amplitudes that must be determined from the initial conditions. The initial conditions are generally in the form

$$\begin{aligned} \underset{\sim}{u}(x,y,z,0) &= \underset{\sim}{u}_o(x,y,z) \\ \dot{\underset{\sim}{u}}(x,y,z,0) &= \dot{\underset{\sim}{u}}_o(x,y,z) \end{aligned} \tag{4.22}$$

where $\underset{\sim}{u}_o$ and $\dot{\underset{\sim}{u}}_o$ are the initial displacement and velocity fields respectively.

The displacement components are the infinite sum of the normal modes defined by the Eqs. (4.12) and they can simply be written as

$$u_x(x,y,z,t) = \sum_N u_{xN}(x,y,z,t)$$

$$u_y(x,y,z,t) = \sum_N u_{yN}(x,y,z,t)$$

$$u_z(x,y,z,t) = \sum_N u_{zN}(x,y,z,t)$$

with the understanding that $\sum_N = \sum_{n=o} \sum_{m=o} \sum_{p=o}$, as before. Note that n and m specify the wave numbers $\alpha = n\pi/a$, $\beta = m\pi/b$ respectively and P is used to represent infinite sets of natural frequencies corresponding to the combinations of integers n and m.

4.2 FORCED VIBRATION SOLUTION

The equation of motion for the case of forced vibration of an elastic media was given by Eq. (2.9) which we repeat here for convenience,

$$c_t^2\nabla^2\underset{\sim}{u} + (c_\ell^2 - c_t^2)\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \underset{\sim}{f} = \ddot{\underset{\sim}{u}} \quad . \tag{4.24}$$

Due to its simplicity, in our analysis of the forced vibration problem, we will utilize the normal mode approach.

It can be shown that the modal functions ϕ_{xN}, ϕ_{yN}, ϕ_{zN} are orthogonal over the domain of the block that is,

$$\int_0^a \int_0^b \int_0^c \phi_N\phi_M \, dx\, dy\, dz = 0 \qquad \text{if} \quad N(n,m) \neq M(n',m') \tag{4.25}$$

Since the normal modes (4.12) form an orthogonal set, the displacements at any point and time may be represented by the superposition of the modes [16], i.e.,

$$\underset{\sim}{u}(x,y,z,t) = \sum_N \underset{\sim}{\phi}_N(x,y,z)T_N(t) \quad , \tag{4.26}$$

where

$$\underset{\sim}{\phi}_N = \phi_{xN}\underset{\sim}{i} + \phi_{yN}\underset{\sim}{j} + \phi_{zN}\underset{\sim}{k}$$

and $T_N(t)$ represents the time varying character of the modes. Substitution of this series representation of displacement vector (4.26) into the equation of motion yields:

$$\sum_N[c_t^2\nabla^2\underset{\sim}{\phi}_N + (c_\ell^2 - c_t^2)\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{\phi}_N)]T_N(t) + \underset{\sim}{f} = \sum_N\underset{\sim}{\phi}_N\ddot{T}_N(t) \tag{4.27}$$

Recalling the free vibration displacement solution which can be expressed in the vector form as

$$\underset{\sim}{u} = \sum_N \underset{\sim}{\phi}_N \sin\omega_N t \tag{4.28}$$

and substituting it into the free vibration equation of motion, (4.1), one obtains the following relation.

$$c_t^2\nabla^2\phi_N + (c_\ell^2 - c_t^2)\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{\phi}_N) = -\omega_N^2\underset{\sim}{\phi}_N \tag{4.29}$$

Thus, utilizing Eq. (4.29) in (4.27) and rearranging the terms, we get

$$\sum_N \underset{\sim}{\phi}_N(\ddot{T}_N(t) + \omega_N^2 T_N(t)) = \underset{\sim}{f} \tag{4.30}$$

where $\ddot{T}_N(t)$ represents the second derivative of $T_N(t)$ with respect to time. Taking the scalar product of both sides of Eq. (4.32) with $\underset{\sim}{\phi}_M$ where M(n',m') denotes another modal function and integrating over the volume of the block, one gets

$$\sum_N(\ddot{T}_N(t) + \omega_N^2 T_N(t))\int_V \underset{\sim}{\phi}_N\cdot\underset{\sim}{\phi}_M\, dV = \int_V \underset{\sim}{f} \cdot \underset{\sim}{\phi}_M\, dV . \tag{4.31}$$

Recalling the orthogonality condition for the normal modes, Eq. (4.31) can be written as

$$\ddot{T}_N(t) + \omega_N^2 T_N(t) = Q_N(t) \tag{4.32}$$

where

$$Q_N(t) = \frac{1}{D_N} \int_0^a\int_0^b\int_0^c \underset{\sim}{f}(x,y,z,t)\cdot\underset{\sim}{\phi}_N(x,y,z)dx\, dy\, dz \tag{4.33}$$

and

$$D_N = \int_0^a\int_0^b\int_0^c \underset{\sim}{\phi}_N \cdot \underset{\sim}{\phi}_N\, dx\, dy\, dz \quad , \tag{4.34}$$

(See Appendix C for details).

In order to obtain the time dependency, $T_N(t)$, of the modes, we will make use of Laplace transforms. Taking Laplace transform of Eq. (4.32),

$$\mathcal{L}[\ddot{T}_N(t)] + \omega_N^2 \mathcal{L}[T_N(t)] = \mathcal{L}[Q_N(t)]$$

where

$$\mathcal{L}[\ddot{T}_N(t)] = S^2\overline{T}_N(s) - S\ T_N(0) - \dot{T}_N(0) ,$$

$$\mathcal{L}[T_N(t)] = \overline{T}_N(s) ,$$

$$\mathcal{L}[Q_N(t)] = \overline{Q}_N(s)$$

and assuming the motion starts from rest ($T_N(0) = \dot{T}_N(0) = \ddot{T}_N(0) = 0$) yields the expression

$$S^2\overline{T}_N(s) + \omega_N^2 T_N(s) = \overline{Q}_N(s) . \qquad (4.35)$$

It is possible to express the above equation in a new form as

$$\overline{T}_N(s) = \overline{V}_N(s)\overline{Q}_N(s) \qquad (4.36)$$

where

$$\overline{V}_N(s) = \frac{1}{S^2 + \omega_N^2} . \qquad (4.37)$$

The inverse transform of Eq. (4.36) may be taken by using the convolution theorem [20],

$$T_N(t) = \frac{1}{\omega_N} \int_0^t Q_N(\tau)\sin\omega_N(t - \tau)d\tau . \qquad (4.38)$$

It is possible to determine the time varying function, $T_N(t)$, for any generalized body force according to the Eqs. (4.33), (4.34) and (4.38). In the next two sections the time dependency of the normal modes for the cases where the loading has an impulsive and step like characters will be considered.

4.3 IMPULSIVE RESPONSE

A concentrated impulsive body force can be written as

$$\underset{\sim}{f}(x,y,z,t) = f_x \underset{\sim}{i} + f_y \underset{\sim}{j} + f_z \underset{\sim}{k} \quad , \tag{4.39}$$

where the components are given by

$$\begin{aligned} f_x &= F_x \delta(x - x_o)\delta(y - y_o)\delta(z - z_o)\delta(t) \\ f_y &= F_y \delta(x - x_o)\delta(y - y_o)\delta(z - z_o)\delta(t) \\ f_z &= F_z \delta(x - x_o)\delta(y - y_o)\delta(z - z_o)\delta(t) \quad . \end{aligned} \tag{4.40}$$

Upon substituting Eq. (4.39) into Eq. (4.40), one obtains

$$Q_N(t) = \frac{1}{D_N} [F_x \phi_{xN}(x_o,y_o,z_o) + F_y \phi_{yN}(x_o,y_o,z_o) + F_z \phi_{zN}(x_o,y_o,z_o)]\delta(t) \tag{4.41}$$

Thus substituting the above equation into Eq. (4.38) yileds $T_N(t)$:

$$T_N(t) = \frac{1}{D_N \omega_N}[F_x \phi_{xN}(x_o,y_o,z_o) + F_y \phi_{yN}(x_o,y_o,z_o) + F_z \phi_{zN}(x_o,y_o,z_o)]\sin\omega_N t \tag{4.42}$$

Therefore the components of the displacement vector obtained from Eq. (4.26) are

$$u_x(x,y,z,t) = \sum_N \frac{1}{D_N\omega_N} \phi_{xN}(x,y,z)[F_x\phi_{xN}(x_o,y_o,z_o) + F_y\phi_{yN}(x_o,y_o,z_o) + F_z\phi_{zN}(x_o,y_o,z_o)]\sin\omega_N t ,$$

$$u_y(x,y,z,t) = \sum_N \frac{1}{D_N\omega_N} \phi_{yN}(x,y,z)[F_x\phi_{xN}(x_o,y_o,z_o) + F_y\phi_{yN}(x_o,y_o,z_o) + F_z\phi_{zN}(x_o,y_o,z_o)]\sin\omega_N t , \quad (4.43)$$

$$u_z(x,y,z,t) = \sum_N \frac{1}{D_N\omega_N} \phi_{zN}(x,y,z)[F_x\phi_{xN}(x_o,y_o,z_o) + F_y\phi_{yN}(x_o,y_o,z_o) + F_z\phi_{zN}(x_o,y_o,z_o)]\sin\omega_N t .$$

4.4 STEP RESPONSE

The components of a concentrated force in the case where the time dependency is a step function can be written as

$$\begin{aligned} f_x &= F_x\delta(x - x_o)\delta(y - y_o)\delta(z - z_o)H(t) \\ f_y &= F_y\delta(x - x_o)\delta(y - y_o)\delta(z - z_o)H(t) \\ f_z &= F_z\delta(x - x_o)\delta(y - y_o)\delta(z - z_o)H(t) . \end{aligned} \quad (4.44)$$

The procedure for the derivation of the function $T_N(t)$ and the displacement components are same as outlined in the previous section. Thus,

$$u_x(x,y,z,t) = \sum_N \frac{1}{D_N\omega_N^2} \phi_{xN}(x,y,z)[F_x\phi_{xN}(x_o,y_o,z_o) + F_y\phi_{yN}(x_o,y_o,z_o) + F_z\phi_{zN}(x_o,y_o,z_o)](1 - \cos\omega_N t)$$

$$u_y(x,y,z,t) = \sum_N \frac{1}{D_N\omega_N^2} \phi_{yN}(x,y,z)[F_x\phi_{xN}(x_o,y_o,z_o) + F_y\phi_{yN}(x_o,y_o,z_o) + F_z\phi_{zN}(x_o,y_o,z_o)](1 - \cos\omega_N t) \qquad (4.45)$$

$$u_z(x,y,z,t) = \sum_N \frac{1}{D_N\omega_N^2} \phi_{zN}(x,y,z)[F_x\phi_{xN}(x_o,y_o,z_o) + F_y\phi_{yN}(x_o,y_o,z_o) + F_z\phi_{zN}(x_o,y_o,z_o)](1 - \cos\omega_N t).$$

This completes the analysis of the forced vibration problem of a rectangular block. In the following chapter, numerical results for two case; first for four faces rigid-lubricated, two faces stress-free, secondly for the case where all six faces are rigid-lubricated will be presented.

V. RESULTS AND CONCLUSIONS

5.1 NUMERICAL RESULTS

Numerical calculations were done for the response of a rectangular parallelepiped with two sets of boundary conditions:

1. Six faces are rigid-lubricated (6RL)
2. Four faces are rigid-lubricated and two faces are stress-free (4RL + 2SF).

The properties of the block used in the numerical calculations are given in Table 5.1.

TABLE 5.1 - Properties of the Block Used in Numerical Calculations.

Properties	Case I 6RL	Case II 4RL + 2 SF
Material	Aliminum	Aliminum
ρ	2700 Kg/m^3	2700 Kg/m^3
c_ℓ	6300 m/sec	6300 m/sec
c_t	3100 m/sec	3100 m/sec
λ	46.2 GPa	46.2 GPa
μ	25.5 GPa	25.5 GPa
a	0.1 m	0.1 m
b	0.1 m	0.1 m
c	0.1 m	0.1 m

Two computer programs have been developed in order to calculate the displacements in z-direction due to impulsive and step point loads. First program calculates the z-axis displacements as a function of time for a block with six rigid-lubricated faces and the other one does the same work for the block with four rigid-lubricated/two stress-free faces. These are given in Appendix D. Note that both of the programs are generalized so that they can be used for any block of any material and dimensions. In order to decrease the CPU time used, the frequencies and spatial part of the normal modes were generated first in a loop and the calculated values were stored. Then, by using another loop, the spatial part of the modes were combined with their time dependent part and the resulting values of the normal modes were added so that displacements were obtained for different times. In this way, the frequencies and the spatial dependent part of the modes were calculated only once instead of to calculate them for every time increment. A CDC/Cyber series, type 815 computer were used to calculate the numerical results.

The number of modes taken and the CPU times used in the runs are listed below. Note that the CPU times used in the runs for the block with four rigid-lubricated and two stress-free faces are greater than the CPU times used for the block with six rigid-lubricated faces. This is because in the second case of boundary conditions, the transcendental frequency equation is an implicit function and must be solved iteratively while for the first case, we have two simple frequency equations which can be solved explicitly thus a few CPU time was required in calculating the frequencies. On the other hand, the

TABLE 5.2 - CPU Times and Number of Modes Taken.

Number of normal modes taken	CASE I 6RL	CASE II 4RL + 2SF
1000	3.59 min	5.58 min
4096	14.27 min	-
8000	28.42 min	44.17 min
15625	55.76 min	92.76 min
21952	76.64 min	126.12 min
50653	168.17 min	276.75 min
103823	323.41 min	512.51 min

displacement expressions for the block with four rigid-lubricated and two stress-free faces are more complicated than those for the block with six rigid-lubricated faces, thus, more CPU time was required to calculate the displacement expressions in the second case. In the numerical calculations, the displacements were measured at the position (0.05; 0.05; 0.075) m. The components of the body force are acting in the x,y,z directions and each has a 0.577 N magnitude; i.e. total magnitude of the force is one Newton. The coordinates of the point of application of the body force was taken as (0.05; 0.05; 0.05) m. The location of the souce and the receiver are the same in the both cases of the block.

As a first step in this analysis, 1000 number of terms were taken in the infinite series to obtain displacements for both cases. Then the number of modes were increased and the rate of convergence in displacement values was controlled. The numbers of normal modes that were used in numerical computations are given in Table 5.2. The value of the displacement obtained by adding nearly 50,000 terms was only off by 10% from the value obtained by taking nearly 100,000 terms. Computing with larger number of terms were found to be uneconomical as would be seen from Table 5.2. Thus, the results obtained by taking nearly 100,000 terms were considered to be final results. These are given in Figures 5.1 through 5.4. The z-direction displacement v.s. time histories obtained by taking 8,000 and 50,000 terms are also given in Figures 5.5-5.8 and 5.9-5.12 respectively.

In Figures 5.1 and 5.2, the response of a block with six rigid-lubricated faces to an impulsive and a step point loads are shown respectively. In the following two figures, i.e. Figures 5.3 and 5.4, these are given for a block with four rigid-lubricated and two stress-free faces. In these figures the first peak at nearly four micro-seconds after the impulsive force is applied corresponds to the arrival of P-wave to the receiver. The ripples in the outputs before this peak are due to the fact that normal mode solution converges slowly for impulsive loads. The other peaks in these figures correspond to the arrival of various reflected waves from the boundaries. The rays associated with them are shown in Figure 5.13.

5.2 CONCLUSIONS

The normal mode solutions were presented in this thesis for the forced vibrational response of an rectangular parallelepiped with two sets of boundary conditions:

1. Completely rigid-lubricated boundaries
2. Four rigid-lubricated and two stress-free boundaries.

For these cases numerical results were obtained for the response of a sample block to an impulsive and a step point load.

In the normal mode analysis, eventhough the expressions are exact, in the numerical applications one needs to take very large number of terms (normal modes) in order to get the results within an acceptable accuracy.

Although rigid-lubricated boundaries are not representative of a typical acoustic emission experiment, solution of the problem for a block with all rigid-lubricated faces provides a first step in obtaining more difficult solution for the stress-free/rigid-lubricated case. On the other hand, the block with four rigid-lubricated and two stress-free faces is a more realistic case. Therefore, the solution of this problem can provide a better model for an acoustic emission event and can be useful in the field of nondestructive testing.

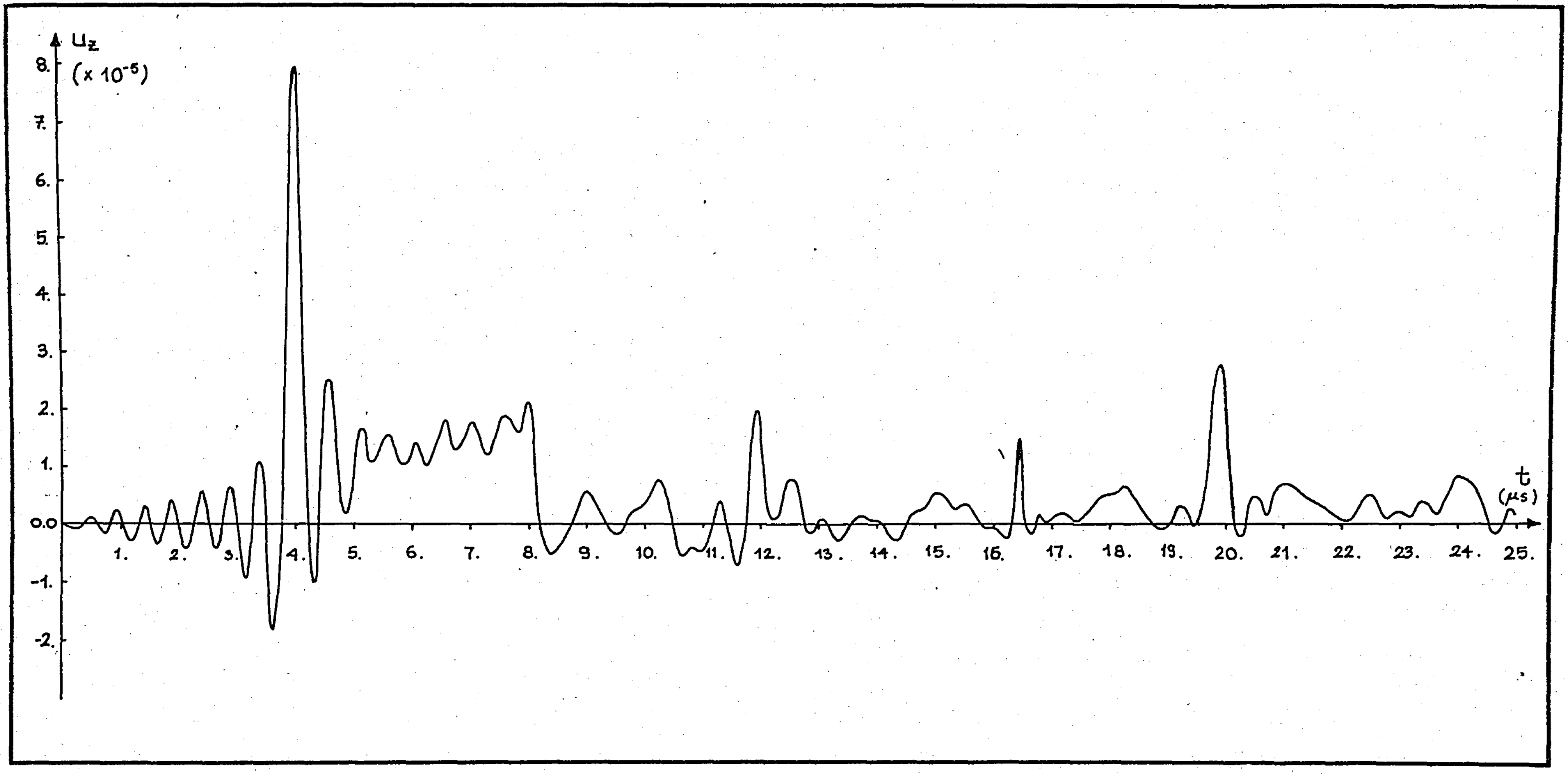


FIGURE 5.1 - Response of a rectangular parallelepiped with six rigid-lubricated faces to an impulsive point load (103823 normal modes were taken).

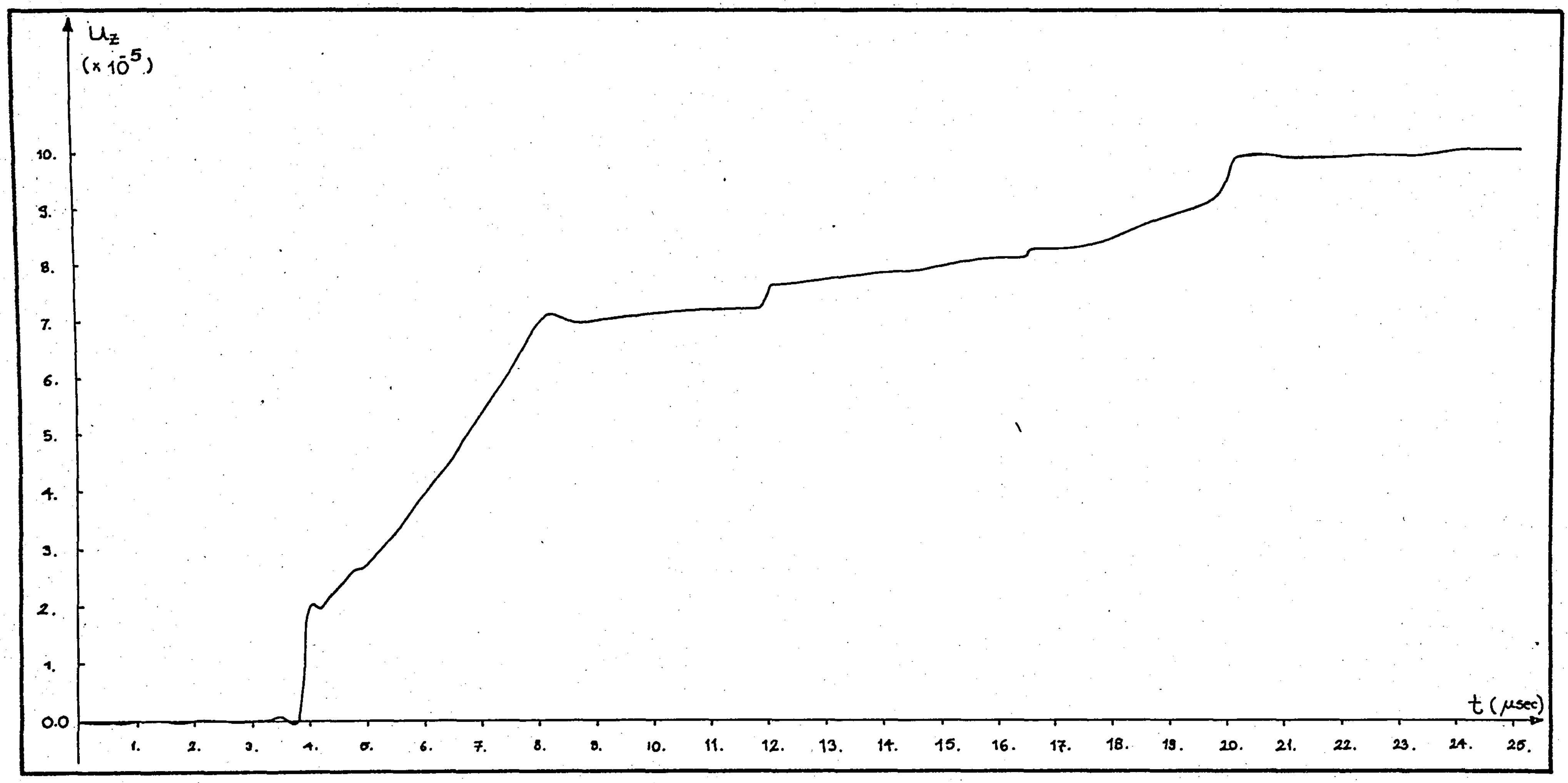


FIGURE 5.2 - Response of a rectangular parallelepiped with six rigid-lubricated faces to a step point load. (103823 normal modes were taken).

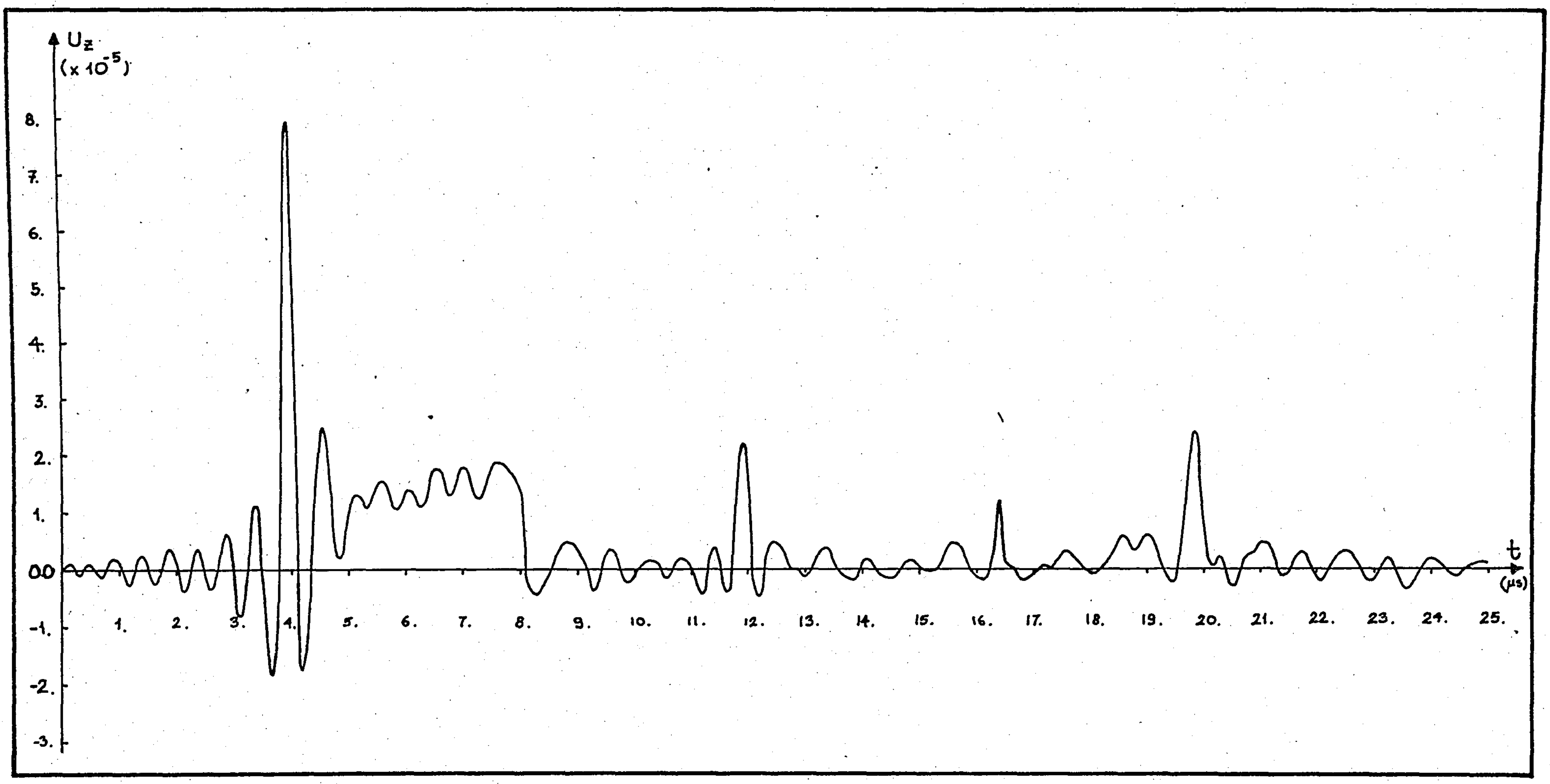


FIGURE 5.3 - Response of a rectangular parallelepiped with four rigid-lubricated and two stress-free faces to an impulsive point load (103823 normal modes were taken).

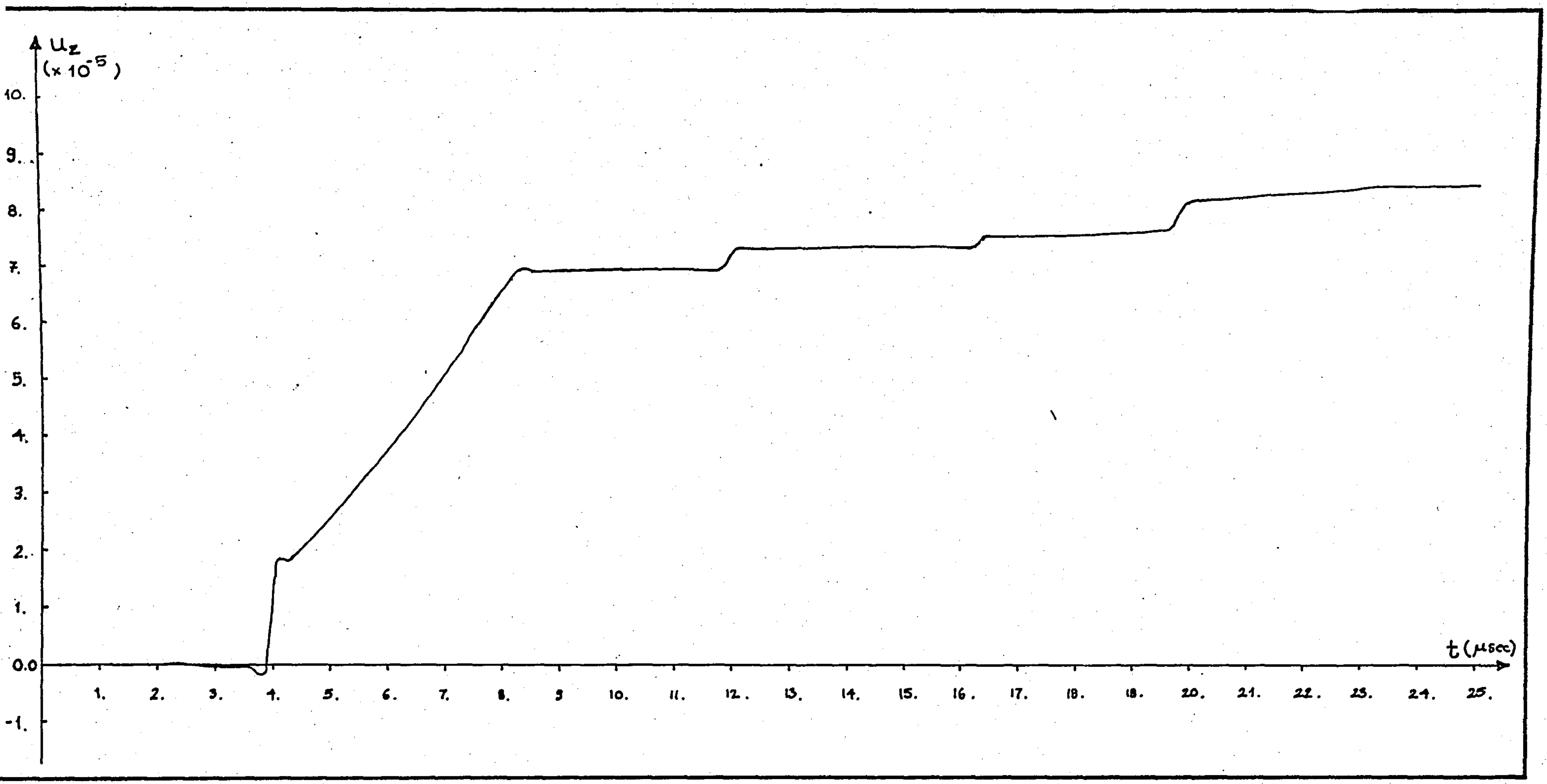


FIGURE 5.4 - Response of a rectangular parallelepiped with four rigid-lubricated and two stress-free faces to a step point load (103823 normal modes were taken).

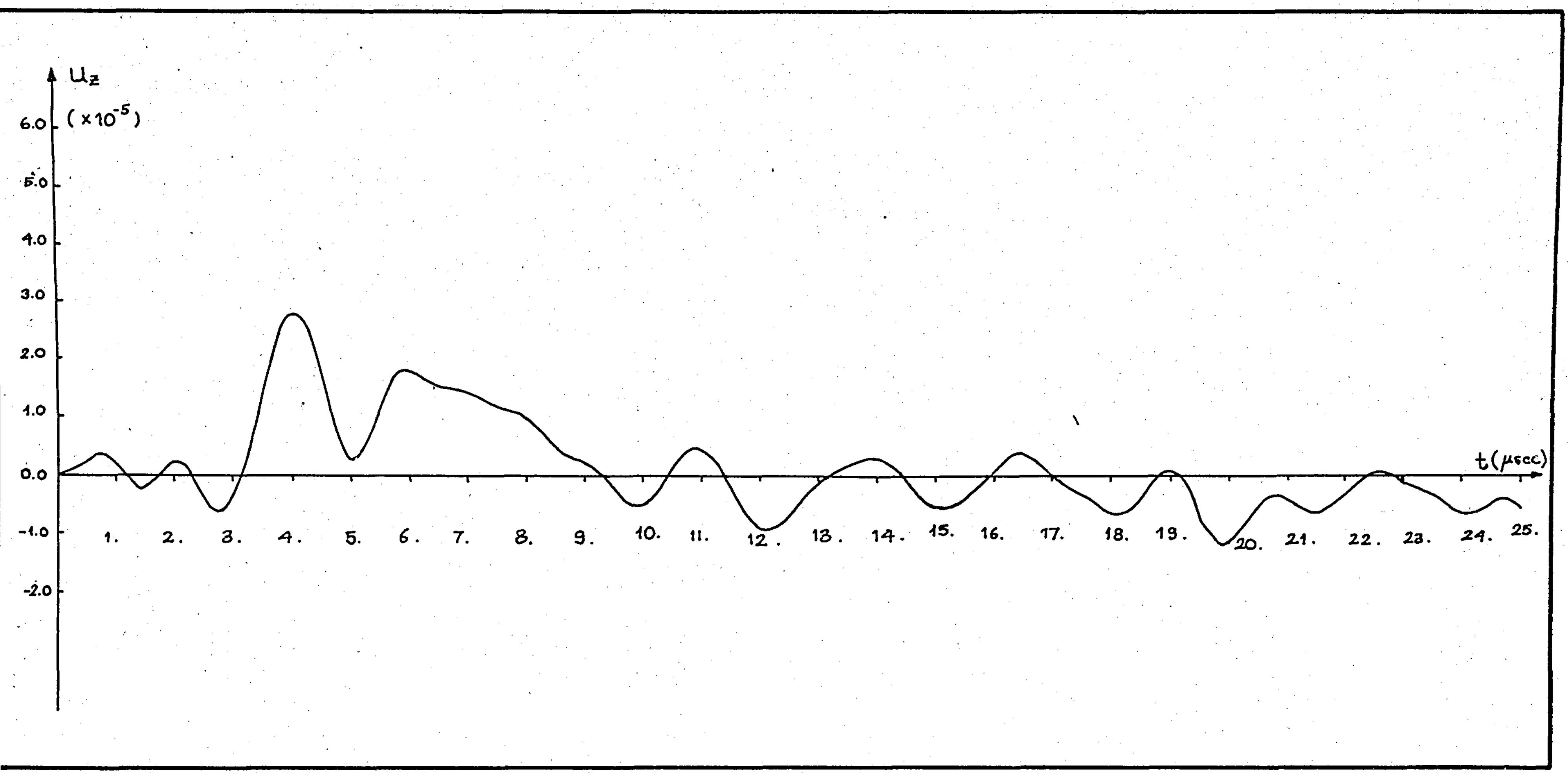


FIGURE 5.5 - Response of a rectangular parallelepiped with six rigid-lubricated faces to an impulsive point load (8000 modes were taken).

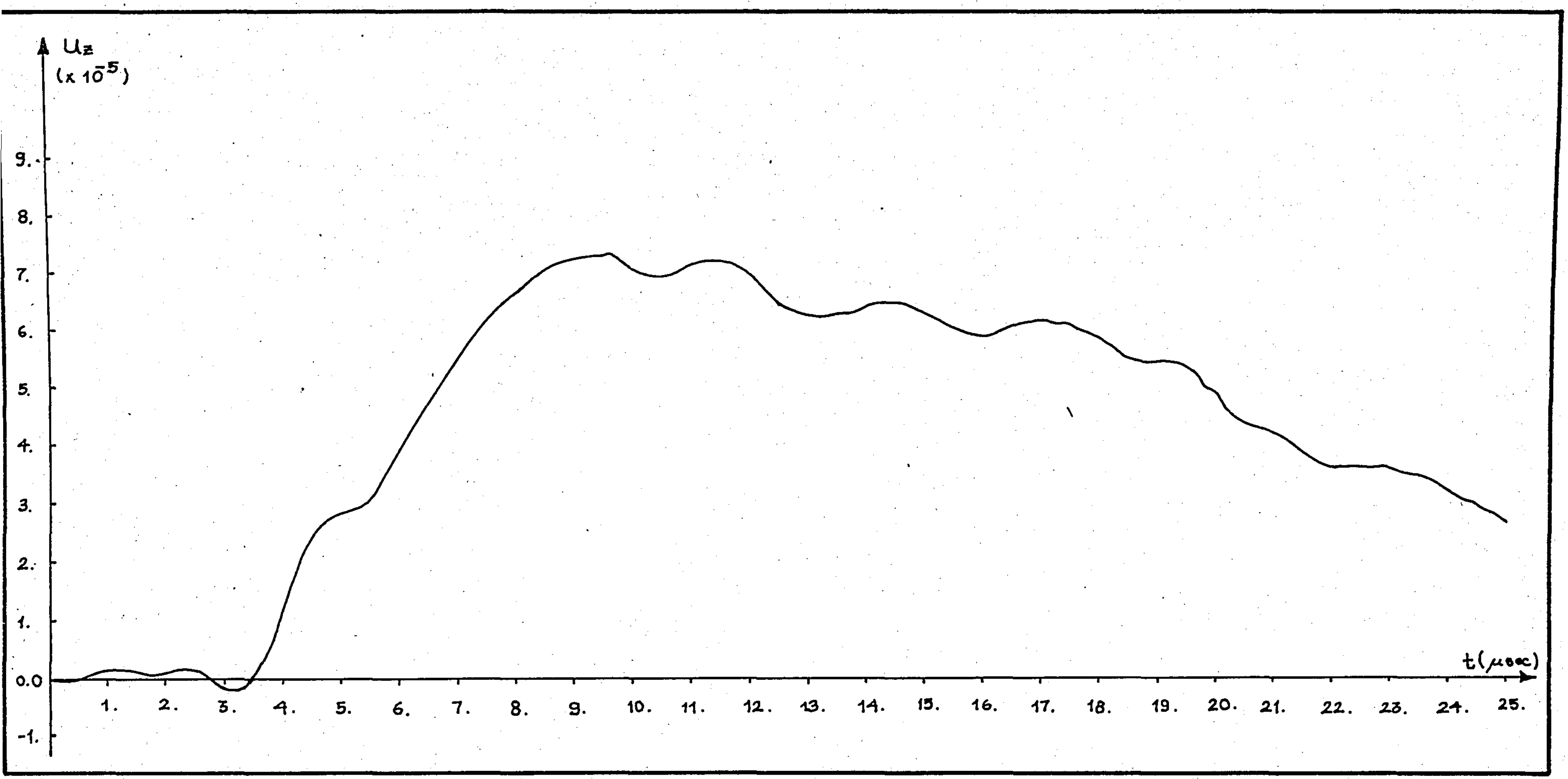


FIGURE 5.6 - Response of a rectangular parallelepiped with six rigid-lubricated faces to a step point load (8000 modes were taken).

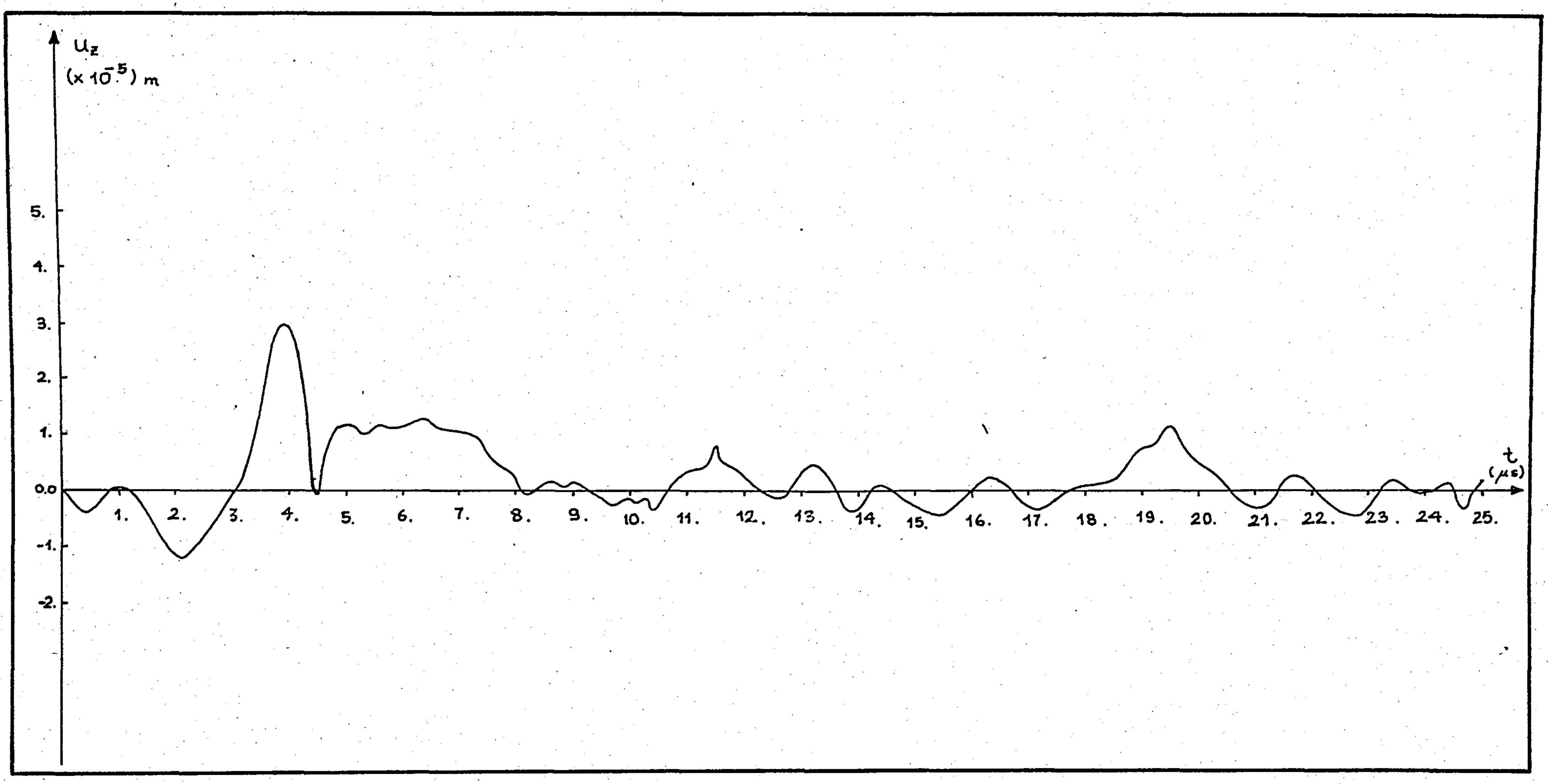


FIGURE 5.7 - Response of a rectangular parallelepiped with four rigid-lubricated and two stress-free faces to an impulsive point load (8000 modes were taken).

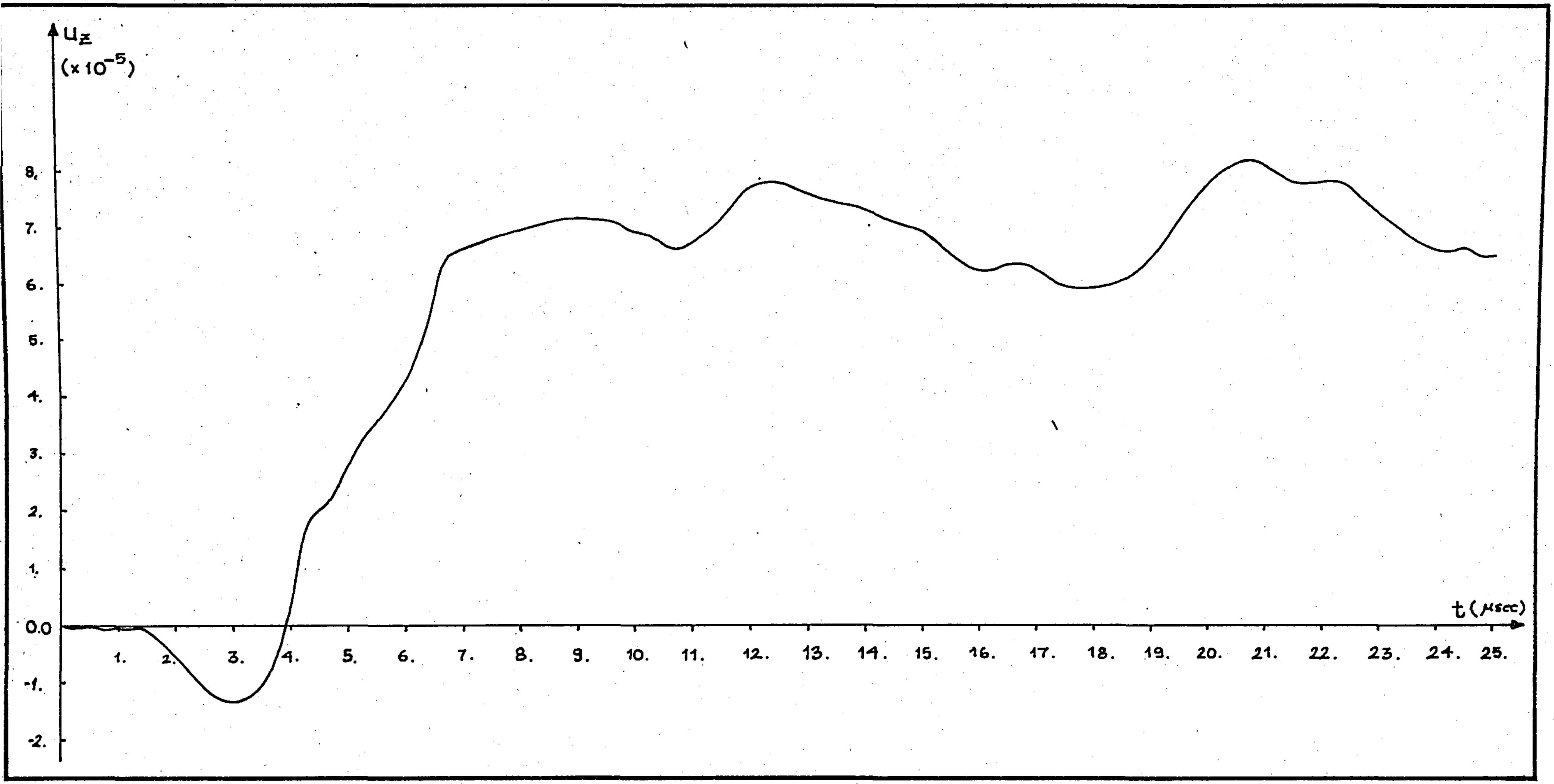


FIGURE 5.8 - Response of a rectangular parallelepiped with four rigid-lubricated and two stress-free faces to an step point load (8000 modes were taken).

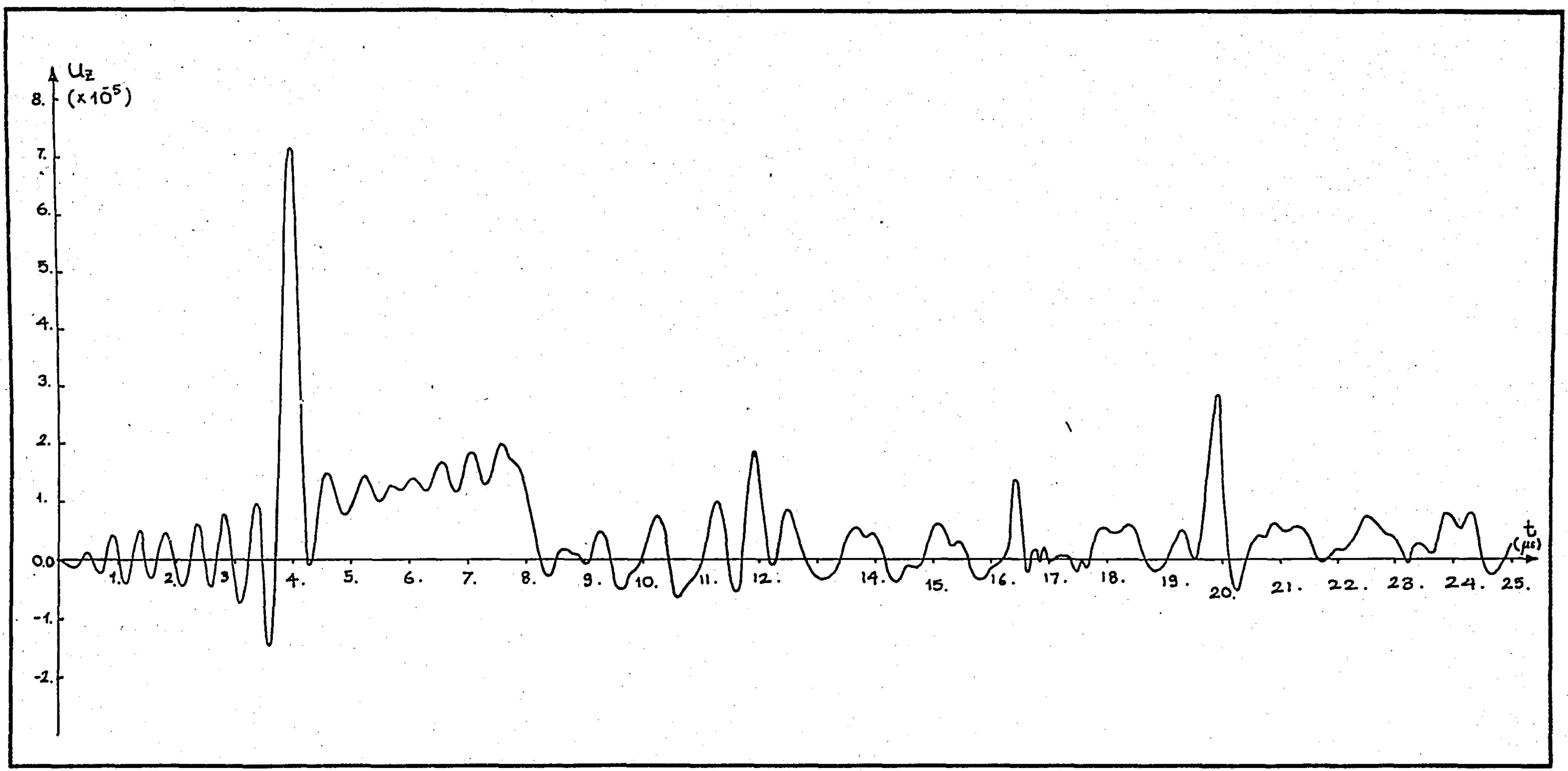


FIGURE 5.9 - Response of a rectangular parallelepiped with six rigid-lubricated faces to an impulsive point load (50653 normal modes were taken).

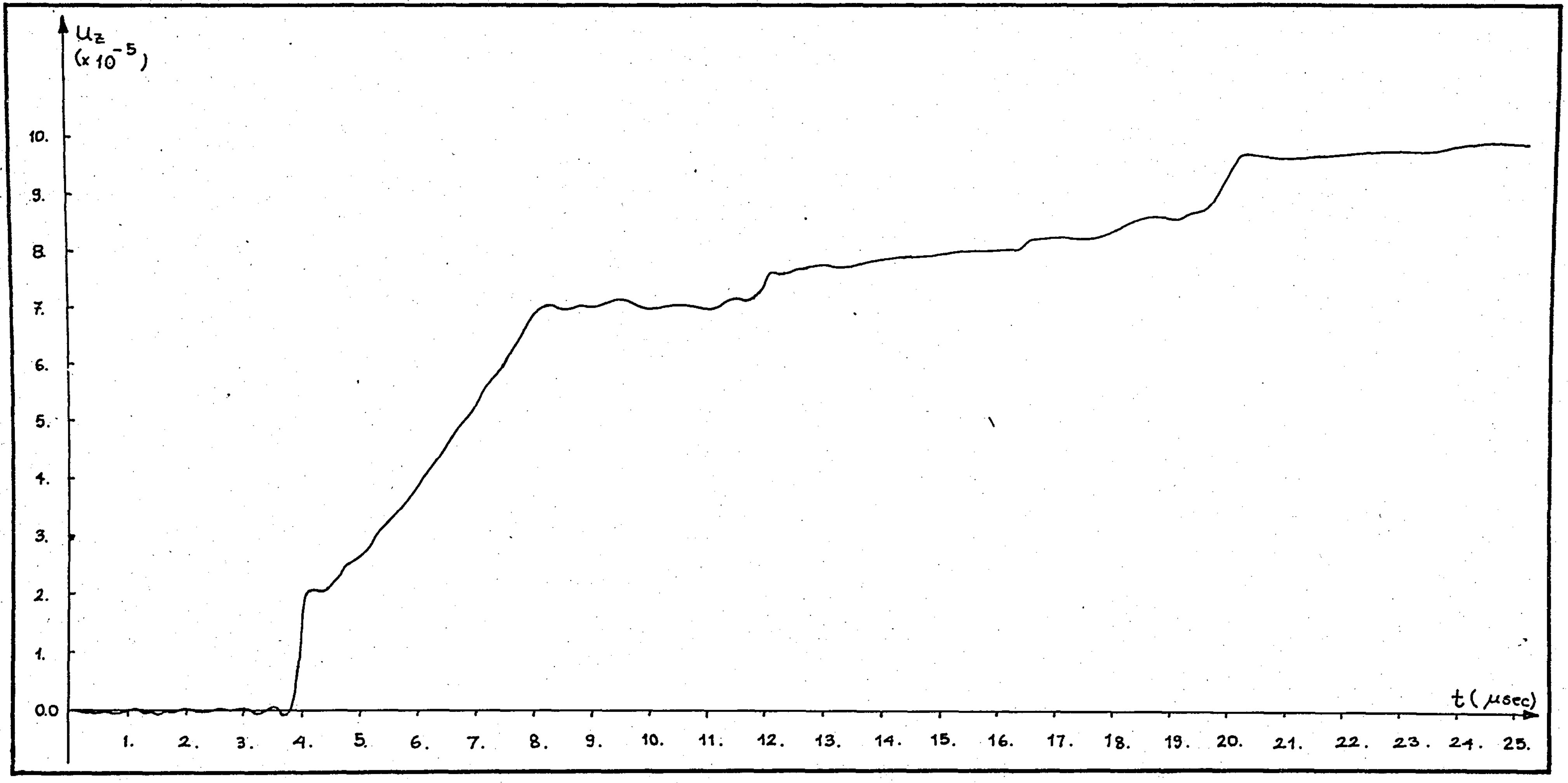


FIGURE 5.10 - Response of a rectangular parallelepiped with six rigid-lubricated faces to a step point load (50653 normal modes were taken).

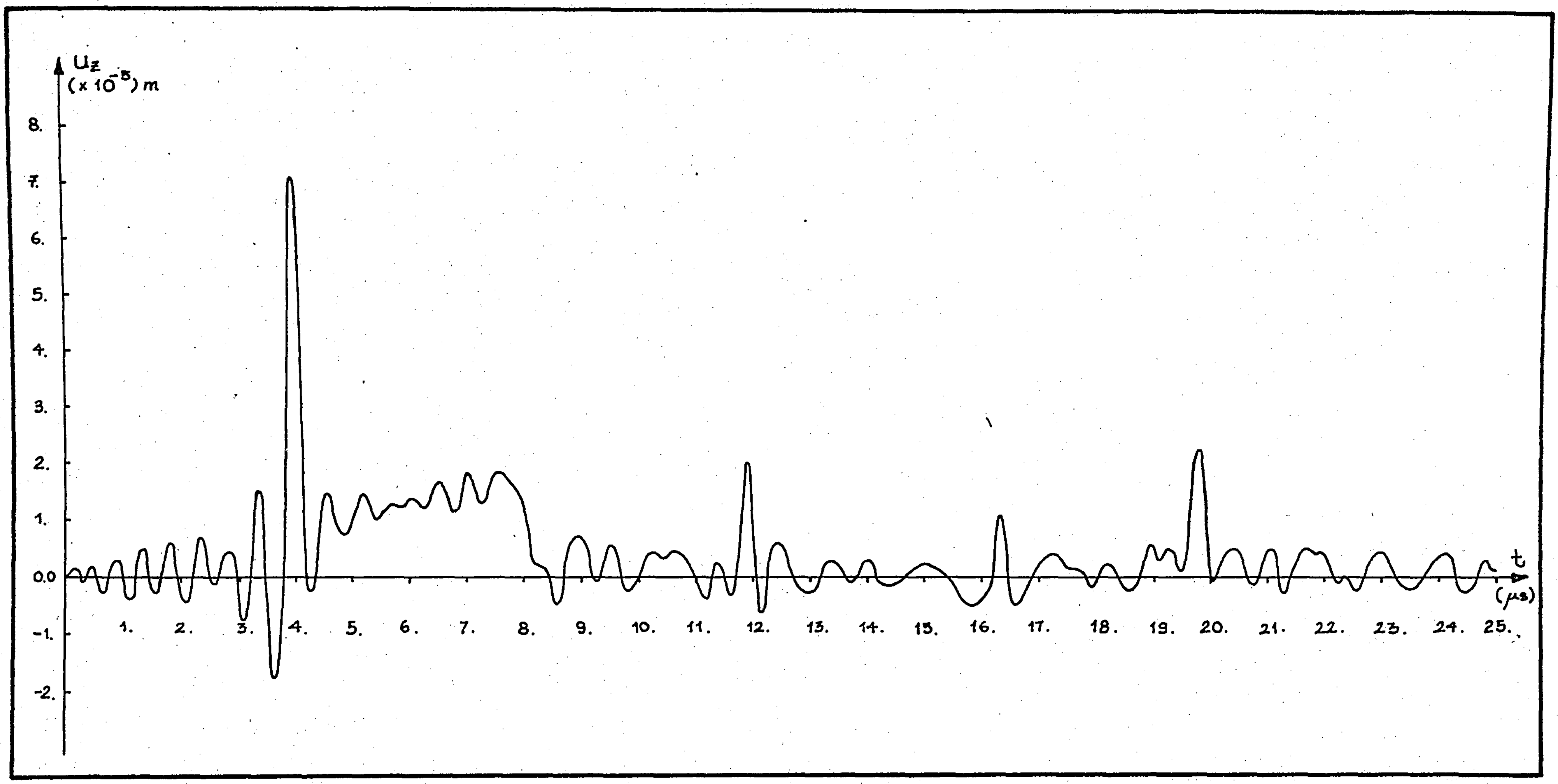


FIGURE 5.11 - Response of a rectangular parallelepiped with four rigid-lubricated and two stress-free faces to an impulsive point load (50653 normal modes were taken).

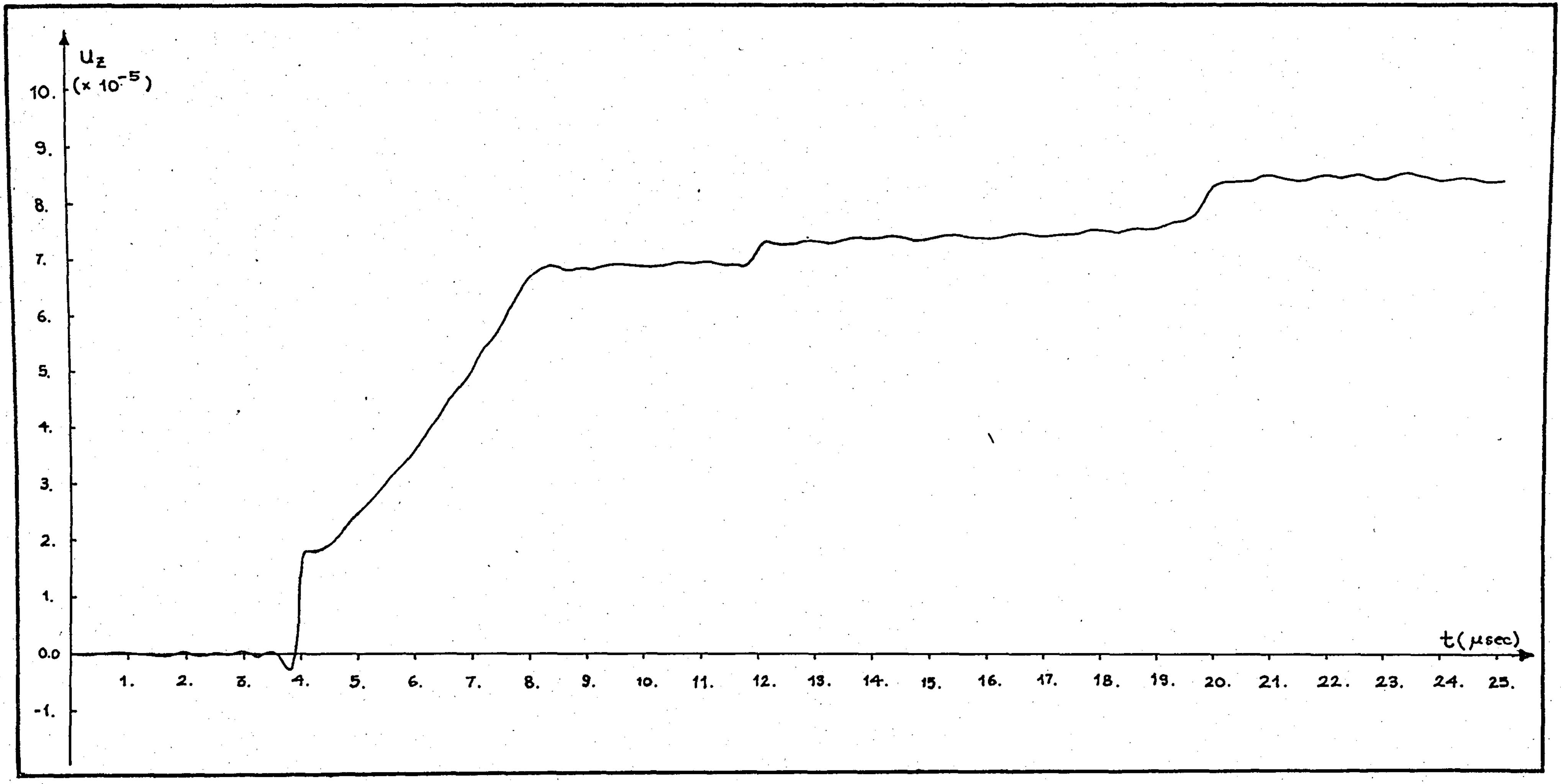
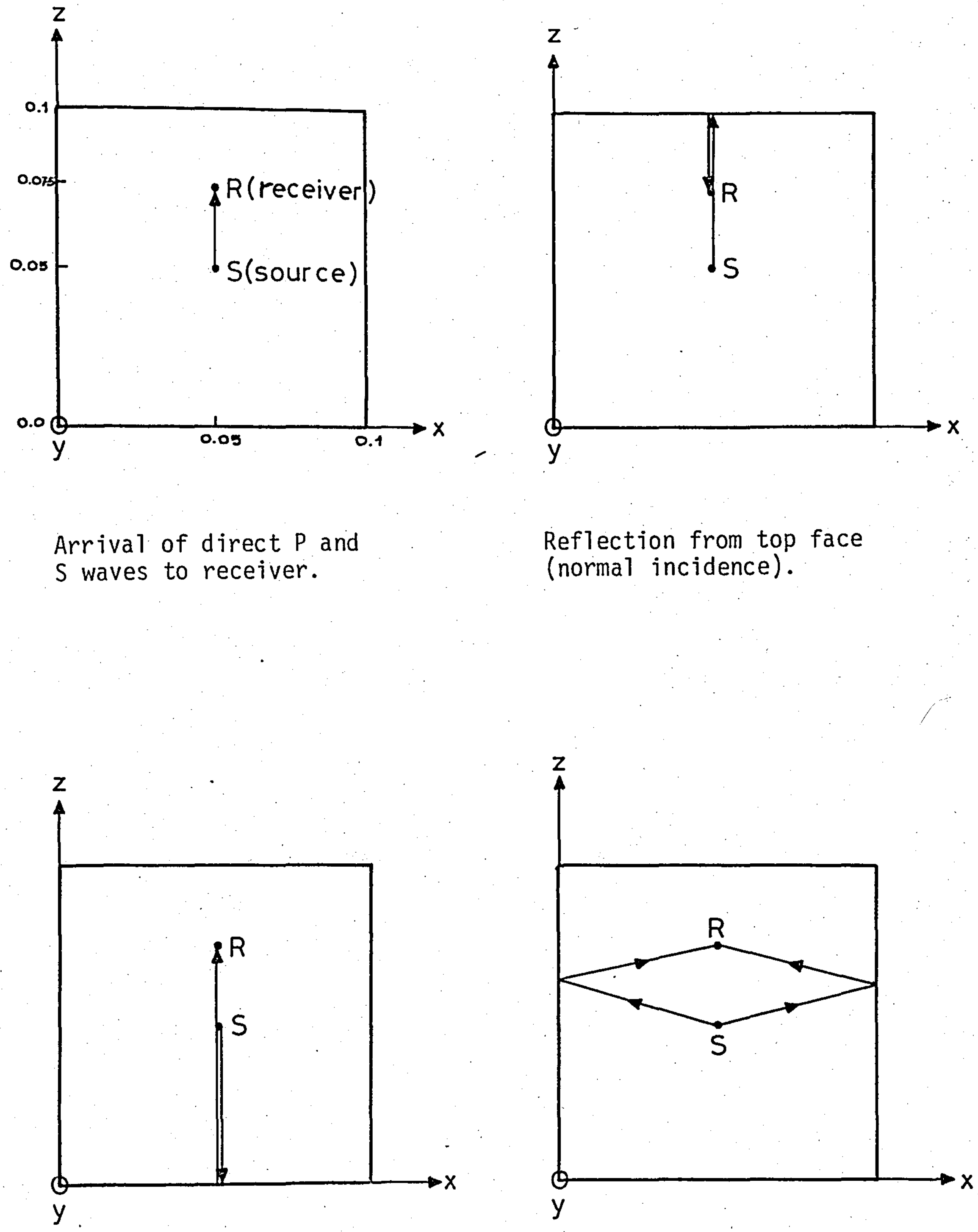


FIGURE 5.12 - Response of a rectangular parallelepiped with four rigid-lubricated and two stress-free faces to a step point load (50653 modes were taken).



Arrival of direct P and S waves to receiver.

Reflection from top face (normal incidence).

Reflection from bottom face (normal incidence).

Reflection from lateral faces.

FIGURE 5.13 - Direct waves and waves with only the reflection.

APPENDICES

APPENDIX A

SEPERATED WAVE EQUATIONS

Derivation of the decoupled wave equations will be given in this section. In deriving these equations, the equation of motion in terms of wave speeds will be used. This is Eq. (2.9)

$$c_t^2 \nabla^2 \underset{\sim}{u} + (c_\ell^2 - c_t^2)\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \underset{\sim}{f} = \underset{\sim}{\ddot{u}} \quad . \tag{A.1}$$

Substituting the Helmholtz equations of displacement

$$\begin{aligned} \underset{\sim}{u} &= \underset{\sim}{\nabla}\phi + \underset{\sim}{\nabla} \times \underset{\sim}{\psi} \\ \underset{\sim}{\nabla} \cdot \underset{\sim}{\psi} &= 0 \end{aligned} \tag{A.2}$$

and body force

$$\begin{aligned} \underset{\sim}{f} &= \underset{\sim}{\nabla}G + \underset{\sim}{\nabla} \times \underset{\sim}{H} \\ \underset{\sim}{\nabla} \cdot \underset{\sim}{H} &= 0 \end{aligned} \tag{A.3}$$

into the equation of motion (A.1) gives

$$\begin{aligned} c_t^2 \nabla^2(\underset{\sim}{\nabla}\phi + \underset{\sim}{\nabla} \times \underset{\sim}{\psi}) &+ (c_\ell^2 - c_t^2)\underset{\sim\sim}{\nabla\nabla} \cdot (\underset{\sim}{\nabla}\phi + \underset{\sim}{\nabla} \times \underset{\sim}{\psi}) \\ &+ (\underset{\sim}{\nabla}G + \underset{\sim}{\nabla} \times \underset{\sim}{H}) = (\partial^2/\partial t^2)(\underset{\sim}{\nabla}\phi + \underset{\sim}{\nabla} \times \underset{\sim}{\psi}) \quad . \end{aligned} \tag{A.4}$$

Then substituting the relations

$$\nabla^2(\underset{\sim}{\nabla}\underset{\sim}{\phi}) = \nabla(\nabla^2\underset{\sim}{\phi})$$
$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla}\underset{\sim}{\phi} = \nabla^2\underset{\sim}{\phi} \qquad (A.5)$$

and

$$\nabla^2(\underset{\sim}{\nabla} \times \underset{\sim}{\psi}) = \underset{\sim}{\nabla} \times (\nabla^2\underset{\sim}{\psi})$$
$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} \times \underset{\sim}{\psi} = 0 \qquad (A.6)$$

into the Eq. (A.6) and performing necessary algebraic manipulations, Eq. (A.6) may be rewritten as

$$\underset{\sim}{\nabla}(c_\ell^2\nabla^2 + G - \ddot{\phi}) + \underset{\sim}{\nabla} \times (c_t^2\nabla^2\underset{\sim}{\psi} + \underset{\sim}{H} - \ddot{\underset{\sim}{\psi}}) = 0 \; . \qquad (A.7)$$

Note that this equation is equal to zero if each of the terms in the parantheses vanishes and this leads to two seperated wave equations:

$$c_\ell^2\nabla^2\phi + G = \ddot{\phi} \qquad (A.8)$$

$$c_t^2\nabla^2\underset{\sim}{\psi} + \underset{\sim}{H} = \ddot{\underset{\sim}{\psi}} \qquad (A.9)$$

Now the equation of motion which includes both P and S-waves is seperated into two independent equations. The first one, Eq. (A.8), defines the P-wave motion and the second one, Eq. (A.9), the transverse motion.

APPENDIX B

SOLUTION OF SEPERATED WAVE EQUATIONS

In this section, the solution of seperated wave equations derived in the previous section, App. A, will be given.

For the free vibration case, the body force terms in the wave equations are neglected, i.e.,

$$\nabla^2\phi = \frac{1}{c_\ell^2}\ddot{\phi} \tag{B.1}$$

$$\nabla^2\underset{\sim}{\psi} = \frac{1}{c_t^2}\ddot{\underset{\sim}{\psi}} \; . \tag{B.2}$$

The seperation of variables method will be used to solve the wave equations.

Now consider the wave Eq. (B.1). The solution of this equation can be assumed to be in the form of the product of two functions, one with spatial dependency, the other with time dependency,

$$\phi(x,y,z,t) = W(x,y,z)T(t) \; . \tag{B.3}$$

Substitution of this expression into Eq. (B.1) gives

$$c_\ell^2 \frac{\nabla^2 W}{W} = \frac{\ddot{T}}{T} = -\omega_\ell^2 \qquad (B.4)$$

where ω_ℓ^2 represents the seperation of variables constant. Thus, two independent differential equations are obtained from Eq. (B.4)

$$\nabla^2 W + (\omega_\ell^2/c_\ell^2)W = 0 \qquad (B.5)$$

$$\ddot{T} + \omega_\ell^2 T = 0 \quad . \qquad (B.6)$$

The solution of the second equation is simple and is of the form

$$T(t) = A_1 \cos\omega_\ell t + A_2 \sin\omega_\ell t \qquad (B.7)$$

which represents simple harmonic motion with the frequency ω_ℓ.

The first equation, (B.5), is known as Helmholtz equation whose solution is obtained by assuming a solution of the form

$$W(x,y,z) = X(x)Y(y)Z(z) \quad . \qquad (B.8)$$

Hence, substituting this expression into the Eq. (B.5) yields the following equation

$$\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} = -\frac{\omega_\ell^2}{c_\ell^2} \quad . \qquad (B.9)$$

Setting,

$$\ddot{X}/X = -\alpha^2 \qquad (B.10)$$

$$\ddot{Y}/Y = -\beta^2 \qquad (B.11)$$

gives a third relation

$$\ddot{Z}/Z = -[(\omega_\ell^2/c_\ell^2) - (\alpha^2 + \beta^2)] = -\gamma_\ell^2 \quad . \tag{B.12}$$

These three expressions may be rearranged to give three differential equations:

$$\ddot{X} + \alpha^2 X = 0 \tag{B.13}$$

$$\ddot{Y} + \beta^2 Y = 0 \tag{B.14}$$

$$\ddot{Z} + \gamma_\ell^2 Z = 0 \quad . \tag{B.15}$$

Solutions of these equations are

$$X(x) = B_1 \cos\alpha x + B_2 \sin\alpha x \tag{B.16}$$

$$Y(y) = B_3 \cos\beta y + B_4 \sin\beta y \tag{B.17}$$

$$Z(z) = B_5 \cos\gamma_\ell z + B_6 \sin\gamma_\ell z \quad . \tag{B.18}$$

Equations (B.7), (B.16), (B.17), (B.18) may be combined according to equations (B.3) and (B.8). Assuming the initial condition T(0) = 0, the unknown constant A_1 can be eliminated. Then the final result can be written as

$$\phi(x,y,z,t) = (C_1 \cos\alpha x + C_2 \sin\beta y)(C_3 \cos\beta y + C_4 \sin\beta y)$$
$$(C_5 \cos\gamma_\ell z + C_6 \sin\gamma_\ell z) \sin\omega_\ell t \quad . \tag{B.19}$$

This is the general solution for the free vibration scalar potential.

Now consider the transverse wave equation. It can be written in component form as

$$\nabla^2 \psi_x = (1/c_t^2)\ddot{\psi}_x \tag{B.20}$$

$$\nabla^2 \psi_y = (1/c_t^2)\ddot{\psi}_y \tag{B.21}$$

$$\nabla^2 \psi_z = (1/c_t^2)\ddot{\psi}_z \quad . \tag{B.22}$$

These three equations are solved by performing similar algebraic work as done for the longitudinal wave equations. The results are

$$\psi_x(x,y,z,t) = (D_1\cos\alpha x + D_2\sin\alpha x)(D_3\cos\beta y + D_4\sin\beta y)$$
$$(D_5\cos\gamma_t z + D_6\sin\gamma_t z)\sin\omega_t t \quad , \tag{B.23}$$

$$\psi_y(x,y,z,t) = (E_1\cos\alpha x + E_2\sin\alpha x)(E_3\cos\beta y + E_4\sin\beta y)$$
$$(E_5\cos\gamma_t z + E_6\sin\gamma_t z)\sin\omega_t t \quad , \tag{B.24}$$

$$\psi_z(x,y,z,t) = (F_1\cos\alpha x + F_2\sin\alpha x)(F_3\cos\beta y + F_4\sin\beta y)$$
$$(F_5\cos\gamma_t z + F_6\sin\gamma_t z)\sin\omega_t t \quad , \tag{B.25}$$

where γ_t is transverse wave number and may be expressed as

$$\gamma_t = [(\omega_t^2/c_t^2) - (\alpha^2 + \beta^2)]^{1/2} \quad .$$

APPENDIX C

EVALUATION OF THE TERM D_N

In this part, the term D_N which was derived in Chapter IV will be evaluated. The equation

$$D_N = \int_0^a \int_0^b \int_0^c \underset{\sim}{\phi}_N \cdot \underset{\sim}{\phi}_N \, dx\, dy\, dz \tag{C.1}$$

may be written as

$$D_N = \int_0^a \int_0^b \int_0^c (\phi_x^2 + \phi_y^2 + \phi_z^2) dx\, dy\, dz \tag{C.2}$$

Since $\underset{\sim}{\phi}_N = \phi_{xN}\, \underset{\sim}{i} + \phi_{yN}\, \underset{\sim}{j} + \phi_{zN}\, \underset{\sim}{k}$. Equation (C.2) can also be written in component form as

$$D_N = D_{xN} + D_{yN} + D_{zN} \tag{C.3}$$

where

$$D_{xN} = \int_0^a \int_0^b \int_0^c \phi_{xN}^2 \, dx\, dy\, dz \tag{C.4}$$

$$D_{yN} = \int_0^a \int_0^b \int_0^c \phi_{yN}^2 \, dx\, dy\, dz \tag{C.5}$$

$$D_{zN} = \int_0^a \int_0^b \int_0^c \phi_{zN}^2 \, dx\, dy\, dz \quad . \tag{C.6}$$

Note that ϕ_{xN}, ϕ_{yN}, ϕ_{zN} are the modal functions that are derived in Chapter IV. Substituting the modal functions into Eqs. (C.4-6) and performing the necessary integrations and also algebraic manipulations, one obtains the expressions:

$$D_{xN} = \frac{\eta_1 ab}{4} \{\alpha^2(A_{1N}\Delta_1 + 2A_{1N}A_{2N}\Delta_2 + A_{2N}^2\Delta_3)$$
$$+ 2\alpha[A_{1N}(A_{3N}\Delta_4 + A_{4N}\Delta_5) + A_{2N}(A_{3N}\Delta_6 + A_{4N}\Delta_7)]$$
$$+ (A_{3N}^2\Delta_8 + 2A_{3N}A_{4N}\Delta_9 + A_{4N}^2\Delta_{10})\} \tag{C.7}$$

$$D_{yN} = \frac{\eta_2 ab}{4} \{\beta^2(A_{1N}^2\Delta_1 + 2A_{1N}A_{2N}\Delta_2 + A_{2N}^2\Delta_3)$$
$$+ 2\beta[A_{1N}(A_{5N}\Delta_4 + A_{6N}\Delta_5) + A_{2N}(A_{5N}\Delta_6 + A_{6N}\Delta_7)]$$
$$+ (A_{5N}^2\Delta_8 + 2A_{5N}A_{6N}\Delta_9 + A_{6N}^2\Delta_{10})\} \tag{C.8}$$

$$D_{zN} = \frac{\eta_3 ab}{4} \{\gamma_\ell^2(A_{1N}^2\Delta_3 + 2A_{1N}A_{2N}\Delta_2 + A_{2N}^2\Delta_1)$$
$$+ (\gamma_\ell/\gamma_t)[A_{1N}(\alpha A_{4N} + \beta A_{6N})\Delta_6 - A_{1N}(\alpha A_{3N} + \beta A_{5N})\Delta_7$$
$$- A_{2N}(\alpha A_{4N} + \beta A_{6N})\Delta_4 + A_{2N}(\alpha A_{3N} + \beta A_{5N})\Delta_5]$$
$$+ (1/\gamma_t^2)[(\alpha A_{4N} + \beta A_{6N})^2\Delta_8 - 2(\alpha A_{4N} + \beta A_{6N})$$
$$(\alpha A_{3N} + \beta A_{5N})\Delta_9 + (\alpha A_{3N} + \beta A_{5N})^2\Delta_{10}]\} \; . \tag{C.9}$$

where,

$$\eta_1 = (1 - \delta_{\alpha o})(1 + \delta_{\beta o}) \tag{C.10}$$

$$\eta_2 = (1 + \delta_{\alpha o})(1 - \delta_{\beta o}) \tag{C.11}$$

$$\eta_3 = (1 + \delta_{\alpha o})(1 + \delta_{\beta o}) \tag{C.12}$$

and

$$\Delta_1 = \frac{C}{2} + \frac{\sin 2\gamma_\ell C}{4\gamma_\ell} \tag{C.13}$$

$$\Delta_2 = \frac{\sin^2 \gamma_\ell C}{2\gamma_\ell} \tag{C.14}$$

$$\Delta_3 = \frac{C}{2} - \frac{\sin 2\gamma_\ell C}{4\gamma_\ell} \tag{C.15}$$

$$\Delta_4 = \frac{\sin(\gamma_\ell - \gamma_t)C}{2(\gamma_\ell - \gamma_t)} + \frac{\sin(\gamma_\ell + \gamma_t)C}{2(\gamma_\ell + \gamma_t)} \tag{C.16}$$

$$\Delta_5 = \frac{1 - \cos(\gamma_\ell + \gamma_t)C}{2(\gamma_\ell + \gamma_t)} - \frac{1 - \cos(\gamma_\ell - \gamma_t)C}{2(\gamma_\ell - \gamma_t)} \tag{C.17}$$

$$\Delta_6 = \frac{1 - \cos(\gamma_\ell - \gamma_t)C}{2(\gamma_\ell - \gamma_t)} + \frac{1 - \cos(\gamma_\ell + \gamma_t)C}{2(\gamma_\ell + \gamma_t)} \tag{C.18}$$

$$\Delta_7 = \frac{\sin(\gamma_\ell - \gamma_t)C}{2(\gamma_t - \gamma_t)} - \frac{\sin(\gamma_\ell + \gamma_t)}{2(\gamma_\ell + \gamma_t)} \tag{C.19}$$

$$\Delta_8 = \frac{C}{2} + \frac{\sin 2\gamma_t C}{4\gamma_t} \tag{C.20}$$

$$\Delta_9 = \frac{\sin^2 \gamma_t C}{2\gamma_t} \tag{C.21}$$

$$\Delta_{10} = \frac{C}{2} - \frac{\sin 2\gamma_t C}{4\gamma_t} . \tag{C.22}$$

Finally, these are combined according to Eq. (C.3) to obtain D_N.

APPENDIX D

COMPUTER PROGRAM LISTING

```
C
C
C     *****ALL RIGID LUBRICATED BOUNDARIES*****
C
C
      REAL L1,L2,L3,K1,K2,K3,K1X1,K2X2,K3X3,NND
      REAL K1Z1,K2Z2,K3Z3
      DIMENSION U3(250),U3S(250),TT(250),FNL(22000),FNT(22000),
     1D3N1(22000),D3N2(22000)
C
C
C     SPECIMEN DIMENSIONS
C
      L1=0.1
      L2=0.1
      L3=0.1
C
C
C     LONGITUDINAL AND TRANSVERSE WAVE SPEEDS
      CL=6300.
      CT=3100.
C
C     POINT OF APPLICATION OF THE FORCE
      Z1=L1/2.
      Z2=L2/2.
      Z3=L3/2.
C
C     POINT AT WHICH DISPLACEMENTS ARE SENSED
C
      X1=Z1
      X2=Z2
      X3=0.75*L3
C
C
      PI=3.1415926536
      N=0
      V=L1*L2*L3
C
C     INTEGER SET (N=0,1,2,...) AND WAVENUMBER ALPHA
C
      DO 10 N1=0,26
      K1=FLOAT(N1)*PI/L1
C
C     INTEGER SET (M=0,1,2,...) AND WAVENUMBER BETA
C
      DO 20 N2=0,26
      K2=FLOAT(N2)*PI/L2
C
C     INTEGER SET (P=0,1,2,...) AND WAVENUMBER GAMMA
C
      DO 30 N3=0,28
      IF(N1.EQ.0.AND.N2.EQ.0.AND.N3.EQ.0)GO TO 30
      K3=FLOAT(N3)*PI/L3
      NND=K1*K1+K2*K2+K3*K3
      A=SQRT(NND)
      N=N+1
C
C     FREQUENCY EQUATIONS
C
      WNL=CL*A
      WNT=CT*A
C
C
      FNL(N)=WNL
      FNT(N)=WNT
      IF(N1.EQ.0.AND.N2.EQ.0.AND.N3.EQ.0)GO TO 8
      IF(N1.EQ.0.AND.N2.EQ.0.AND.N3.GT.0)GO TO 9
      IF(N1.EQ.0.AND.N2.GT.0.AND.N3.EQ.0)GO TO 11
      IF(N1.EQ.0.AND.N2.GT.0.AND.N3.GT.0)GO TO 12
      IF(N1.GT.0.AND.N2.EQ.0.AND.N3.EQ.0)GO TO 13
      IF(N1.GT.0.AND.N2.EQ.0.AND.N3.GT.0)GO TO 14
      IF(N1.GT.0.AND.N2.GT.0.AND.N3.GT.0)GO TO 15
      IF(N1.GT.0.AND.N2.GT.0.AND.N3.EQ.0)GO TO 16
    5 D1=0.0
      D2=0.0
      GO TO 47
    9 ETA1=2.
      ETA2=2.
      ETA3=4.
```

```
      GO TO 17
   11 ETA1=2.
      ETA2=4.
      ETA3=2.
      GO TO 17
   12 ETA1=1.
      ETA2=2.
      ETA3=2.
      GO TO 17
   13 ETA1=4.
      ETA2=2.
      ETA3=2.
      GO TO 17
   14 ETA1=2.
      ETA2=1.
      ETA3=2.
      GO TO 17
   15 ETA1=1.
      ETA2=1.
      ETA3=1.
      GO TO 17
   16 ETA1=2.
      ETA2=2.
      ETA3=1.
   17 K1X1=K1*X1
      K2X2=K2*X2
      K3X3=K3*X3
      K1Z1=K1*Z1
      K2Z2=K2*Z2
      K3Z3=K3*Z3
      P3=COS(K1X1)*COS(K2X2)*SIN(K3X3)
      P4=SIN(K1Z1)*COS(K2Z2)*COS(K3Z3)
      P5=COS(K1Z1)*SIN(K2Z2)*COS(K3Z3)
      P6=COS(K1Z1)*COS(K2Z2)*SIN(K3Z3)
      F11=(8.*P4)/(V*ETA1)
      F22=(8.*P5)/(V*ETA2)
      F33=(8.*P6)/(V*ETA3)
      U31=K3*K1*F11
      U32=K3*K2*F22
      U3A=K3*K3*F33
      U3B=(K1*K1+K2*K2)*F33
      D1=(U31+U32+U3A)*P3/(4*A)
      D2=(U3B-U31-U32)*P3/(4*A)
   47 D3N1(N)=D1
      D3N2(N)=D2
   30 CONTINUE
   20 CONTINUE
   10 CONTINUE
C
C     SUMMING THE MODAL DISPLACEMENTS TO DETERMINE
C     THE U3 DISPLACEMENTS AS A FUNCTION OF TIME
      T=0.0
      DT=1.E-7
      FU=(0.5770)*(3.7037E-4)
      DO 61 M=1,250
      T=T+DT
      TT(M)=T
      U3N=0.0
      U3NS=0.0
      DO 60 K=1,N
      DFNL=FNL(K)
      DFNT=FNT(K)
      ARGL=DFNL*T
      ARGT=DFNT*T
      DISP1=D3N1(K)
      DISP2=D3N2(K)
      PHI1=DISP1*SIN(ARGL)/DFNL
      PHI2=DISP2*SIN(ARGT)/DFNT
      PHI=PHI1+PHI2
      PHIS1=DISP1*(1.-COS(ARGL))/(DFNL*DFNL)
      PHIS2=DISP2*(1.-COS(ARGT))/(DFNT*DFNT)
      PHIS=PHIS1+PHIS2
      U3N=U3N+FU*PHI
      U3NS=U3NS+FU*PHIS
   60 CONTINUE
      U3(M)=U3N
      U3S(M)=U3NS
   61 CONTINUE
      PRINT 1
    1 FORMAT(1H1)
```

```
      WRITE(6,40)(U3(I),I=1,250)
      PRINT 1
      WRITE(6,40)(U3S(I),I=1,250)
   40 FORMAT(11X,5E20.9)
      WRITE(7,41)(U3(I),I=1,250)
      WRITE(8,41)(U3S(I),I=1,250)
   41 FORMAT(4E20.9)
      STOP
      END
```

```
C
C
C     ****FOUR FACE RIGID-LUBRICATED AND TWO FACE STRESS-FREE****
C
C
      COMMON XL1,XL2,XL3,CL,CT,N1,N2
      COMMON XC1,XC2,XC3,WNO,IZERO,XK1,XK2,X1,X2,X3
      DIMENSION U3(250),U3S(250),FN(9000),D3N(9000),TT(250)
C
C      LONGITUDINAL AND TRANSVERSE WAVE SPEEDS
C
      CL=6300.
      CT=3100.
C
C      SPECIMEN DIMENSIONS
C
      XL1=0.1
      XL2=0.1
      XL3=0.1
C
C
C      POINT OF APPLICATION OF THE IMPULSIVE LOAD
C
C
      XC1=XL1/2.
      XC2=XL2/2.
      XC3=XL3/2.
C
C      POINT AT WHICH DISPLACEMENTS ARE SENSED
C
      X1=XC1
      X2=XC2
      X3=XL3
      PI=3.1415926536
      N=0
      K=0
C
C     NUMBER OF FREQUENCIES THAT WILL BE DETERMINED
C     FOR  THE INTEGER SET (N,M)
C
      IP=19
C
      ZERO=0.0
      IZERO=0
C
C     INTEGER SET (N=0,1,2,...) AND WAVENUMBER ALPHA
C
      DO 16 N1=0,19
      XK1=(FLOAT(N1))*PI/XL1
C
C     INTEGER SET (M=0,1,2,...) AND WAVENUMBER BETA
C
      DO 15 N2=0,19
      XK2=(FLOAT(N2))*PI/XL2
      WNO=XK1*XK1+XK2*XK2
      KK=0
      WB=1.0
      IF(WNO.GT.ZERO)GO TO 2
   39 XN3=FLOAT(K)
      IF(K.EQ.ZERO)GO TO 33
      WFN=CL*XN3*PI/XL3
      IF(K.GT.IP)GO TO 15
      GO TO 14
   33 WFN=0.0
      DR=0.0
      GO TO 34
    2 CALL FREQ(WB,WN)
      WFN=WN*CL/XL3
      WB=WN
   14 CALL MOC(WFN,DR)
   34 N=N+1
      FN(N)=WFN
      D3N(N)=DR
      K=K+1
      IF(WNO.EQ.ZERO)GO TO 39
      KK=KK+1
      IF(KK.LE.IP)GO TO 2
   15 CONTINUE
   16 CONTINUE
      WRITE(6,*) N,KK,K
```

```
      C
      C     SUMMING THE MODAL DISPLACEMENTS TO DETERMINE THE U3 DISPLACEMENTS
      C     AS A FUNCTION OF TIME
      C
            T=0.0
            DT=1.E-7
            FO=(.5773502691896)*(1./2700.)
            DO 61 M=1,250
            T=T+DT
            T1(M)=T
            U3N=0.0
            U3NS=0.0
            DO 60 K=1,N
            DFN=FN(K)
            IF(DFN.EQ.ZERO)GO TO 60
            ARG=DFN*T
            DISP=U3N(K)
            PHI=DISP*SIN(ARG)/DFN
            PHIS=DISP*(1.-COS(ARG))/(DFN*DFN)
            U3N=U3N+FO*PHI
            U3NS=U3NS+FO*PHIS
         60 CONTINUE
            U3(M)=U3N
            U3S(M)=U3NS
         61 CONTINUE
            WRITE(7,101)(U3(M),M=1,250)
            WRITE(8,101)(U3S(M),M=1,250)
        101 FORMAT(4E20.9)
            STOP
            END
      C
      C
      C
            SUBROUTINE MDC(WFN,DR)
            COMMON XL1,XL2,XL3,CL,CT,N1,N2
            COMMON XC1,XC2,XC3,WND,IZERO,XK1,XK2,X1,X2,X3
            COMPLEX A1,A2,A3,A4,A5,D,DIFL,DIFT,XKL,XKT,RS,PST,RST,T1,T2,DK,SK
            COMPLEX DKL,SKL,TT1,TT2,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,C1,C2,C3,C4
            COMPLEX AE1,AE2,AE3,BE1,BE2,BE3,Q1,Q2,CE1,CE2,CE3,E1N
            COMPLEX E2N,E3N,FN,P3NX,P3NXC,SX3,SX4,TX3,TX4
            COMPLEX CS1,CS2,CC1,CC2,DD1,DD2,DD3,P1NX,P2NX,P1NXC,P2NXC
      C
      C     DETERMINATION OF MODAL COEFFICIENTS
      C
            DIFL=(WFN*WFN/(CL*CL))-WND
            DIFT=(WFN*WFN/(CT*CT))-WND
            XKL=CSQRT(DIFL)
            XKT=CSQRT(DIFT)
            RS=WND-DIFT
            PST=-4.*WND*XKL*XKT
            RST=-RS*RS
            T1=XKL*XL3
            T2=XKT*XL3
            CS1=CSIN(T1)
            CS2=CSIN(T2)
            CC1=CCOS(T1)
            CC2=CCOS(T2)
         23 IF(N1.GT.IZERO.AND.N2.GT.IZERO) GO TO 30
            IF(N1.EQ.IZERO.AND.N2.GT.IZERO) GO TO 31
            IF(N1.GT.IZERO.AND.N2.EQ.IZERO) GO TO 32
            ETA1=0.0
            ETA2=0.0
            ETA3=4.0
            A6=0.0
            A5=CMPLX(0.,0.)
            A4=CMPLX(0.,0.)
            A3=CMPLX(0.,0.)
            A2=CMPLX(1.,0.)
            A1=CMPLX(0.,0.)
            GO TO 34
         30 ETA1=1.0
            ETA2=1.0
            ETA3=1.0
            GO TO 35
         31 ETA1=0.0
            ETA2=2.0
            ETA3=2.0
            GO TO 35
         32 ETA1=2.0
            ETA2=0.0
```

```
      ETA3=2.0
      A6=0.
      A5=CMPLX(0.,0.)
      A4=CMPLX(1.,0.)
      A3=-RST*(CC1-CC2)/(PST*CS1+RST*CS2)
      A2=(XK1*XK1-DIFT)/(2.*XK1*XKL*XKT)
      A1=-2.*XK1*A3/(XK1*XK1-DIFT)
      GO TO 34
   33 A6=1.
      A5=-RST*(CC1-CC2)/(PST*CS1+RST*CS2)

      A4=XK1/XK2
      A3=A4*A5
      A2=RS/(2.*XK2*XKL*XKT)
      A1=-A2*A5*PST/RST
C
C      CALCULATION OF GENERALIZED MASS TERM EN
C
   34 DK=XKL-XKT
      SK=XKL+XKT
      DKL=T1-T2
      SKL=T1+T2
      D1=(XL3/2.)+CS1*(CC1/(2.*XKL))
      D2=CS1*(CS1/(2.*XKL))
      D3=(XL3/2.)-CS1*(CC1/(2.*XKL))
      C1=CSIN(DKL)/(2.*DK)
      C2=CSIN(SKL)/(2.*SK)
      C3=(1.-CCOS(SKL))/(2.*SK)
      C4=(1.-CCOS(DKL))/(2.*DK)
      D4=C1+C2
      D5=C3-C4
      D6=C3+C4
      D7=C1-C2
      D8=(XL3/2.)+CS2*(CC2/(2.*XKT))
      D9=CS2*(CS2/(2.*XKT))
      D10=(XL3/2.)-CS2*(CC2/(2.*XKT))
      AE1=A1*A1*D1+2.*A1*A2*D2+A2*A2*D3
      AE2=A1*(A3*D4+A4*D5)+A2*(A3*D5+A4*D7)
      AE3=A3*A3*D8+2.*A3*A4*D9+A4*A4*D10
      BE1=AE1
      BE2=A1*(A5*D4+A6*D5)+A2*(A5*D6+A6*D7)
      BE3=A5*A5*D8+2.*A5*A6*D9+A5*A6*D10
      Q1=XK1*A4+XK2*A6
      Q2=XK1*A3+XK2*A5
      CE1=A1*A1*D3+2.*A1*A2*D2+A2*A2*D1
      CE2=Q1*(A1*D6-A2*D4)+Q2*(A2*D5-A1*D7)
      CE3=Q1*Q1*D8-2.*Q1*Q2*D9+Q2*Q2*D10
      E1N=XK1*XK1*AE1+2.*XK1*AE2+AE3
      E2N=XK2*XK2*BE1+2.*XK2*BE2+BE3
      E3N=DIFL*CE1+(XKL*CE2/XKT)+(CE3/DIFT)
      EN=(XL1*XL2/4.)*(ETA1*E1N+ETA2*E2N+ETA3*E3N)
C
C      DETERMINATION OF THE U3 MODAL DISPLACEMENT COEFFICIENTS
C
      SX1=XK1*X1
      SX2=XK2*X2
      SX3=XKL*X3
      SX4=XKT*X3
      P3NX=COS(SX1)*COS(SX2)*(XKL*(A1*CSIN(SX3)-A2*CCOS(SX3))
     1+(1./XKT)*(Q1*CCOS(SX4)-Q2*CSIN(SX4)))
      TX1=XK1*XC1
      TX2=XK2*XC2
      TX3=XKL*XC3
      TX4=XKT*XC3
      P1NXC=SIN(TX1)*COS(TX2)*(XK1*(A1*CCOS(TX3)+A2*CSIN(TX3))
     1+A3*CCOS(TX4)+A4*CSIN(TX4))
      P2NXC=COS(TX1)*SIN(TX2)*(XK2*(A1*CCOS(TX3)+A2*CSIN(TX3))
     1+A5*CCOS(TX4)+A6*CSIN(TX4))
      P3NXC=COS(TX1)*COS(TX2)*(XKL*(A1*CSIN(TX3)-A2*CCOS(TX3))
     &+(1./XKT)*(Q1*CCOS(TX4)-Q2*CSIN(TX4)))
      DD1=P1NXC*P3NX/EN
      DD2=P2NXC*P3NX/EN
      DD3=P3NXC*P3NX/EN
      D=DD1+DD2+DD3
      DR=REAL(D)
      RETURN
      END
C
C
      SUBROUTINE FREQ(W8,WN)
```

```
      COMMON XL1,XL2,XL3,CL,CT,N1,N2
      REAL L1,L2,L3,K1,K2,KL,KT
C
C     NONDIMENSIONALISING
C
      L1=XL1/XL3
      L2=XL2/XL3
      L3=XL3/XL3
      XCL=CL/CL
      XCT=CT/CL
C
C
      DELTA=0.020
      DELTA1=0.01
      NMAX=20
      CON=1.E-5
      PI=3.1415926536
      K1=(FLOAT(N1))*PI/L1
      K2=(FLOAT(N2))*PI/L2
      SWNO=SQRT(K1*K1+K2*K2)
      CON1=XCT*SWNO
      CON2=XCL*SWNO
   17 WL=WB+DELTA
      IF(WL.GT.CON1.AND.WL.LT.CON2)GO TO 5
      IF(WL.GT.CON2)GO TO 9
      CALL ALPHA(K1,K2,XCL,XCT,L3,WL,FWL)
      FWL=FWL/1.E50
      WR=WL+DELTA
      IF(WR.GT.CON1)GO TO 4
    3 WR=WR+DELTA1
      IF(WR.GT.CON1)GO TO 4
      CALL ALPHA(K1,K2,XCL,XCT,L3,WR,FWR)
      FWR=FWR/1.E50
      A2=FWL*FWR
      IF(A2.LT.0.0)GO TO 12
      GO TO 3
    4 WL=WR
    5 WR=WL+DELTA
      CALL BETA(K1,K2,XCL,XCT,L3,WL,FWL)
      FWL=FWL/1.E50
    6 WR=WR+DELTA
      IF(WR.GT.CON2)GO TO 8
      CALL BETA(K1,K2,XCL,XCT,L3,WR,FWR)
      FWR=FWR/1.E50
      A3=FWL*FWR
      IF(A3.LT.0.0)GO TO 12
      GO TO 6
    8 WL=WR
    9 WR=WL+DELTA
      CALL GAMMA(K1,K2,XCL,XCT,L3,WL,FWL)
      FWL=FWL/1.E50
   18 WR=WR+DELTA1
      CALL GAMMA(K1,K2,XCL,XCT,L3,WR,FWR)
      FWR=FWR/1.E50
      A5=FWL*FWR
      IF(A5.LT.0.0)GO TO 12
      GO TO 18
   12 N=0
      WL=WR-3*DELTA
   13 WB=(WL+WR)/2.
      IF(WB.LT.CON1)GO TO 14
      IF(WB.LT.CON2)GO TO 15
      CALL GAMMA(K1,K2,XCL,XCT,L3,WB,FWB)
      GO TO 16
   15 CALL BETA(K1,K2,XCL,XCT,L3,WB,FWB)
      GO TO 16
   14 CALL ALPHA(K1,K2,XCL,XCT,L3,WB,FWB)
   16 IF(ABS(FWB).LE.CON)GO TO 19
      FWB=FWB/1.E50
      B=FWL*FWB
      IF(B.GT.0.0)GO TO 50
      WR=WB
      WC=WB
      FWR=FWB
      GO TO 60
   50 WL=WB
      WC=WB
      FWL=FWB
   60 N=N+1
      IF(N.LE.NMAX)GO TO 13
```

```
   19 WN=W3
      RETURN
      END
      SUBROUTINE ALPHA(X,Y,CX,CY,L3,W,FW)
      REAL L3,KL,KT
      KL=SQRT((X**2.+Y**2.)-W**2./CX**2.)
      KT=SQRT((X**2.+Y**2.)-W**2./CY**2.)
      AR1=KL*L3
      AR2=KT*L3
      SIHAR1=SINH(AR1)
      SIHAR2=SINH(AR2)
      COHAR1=COSH(AR1)
      COHAR2=COSH(AR2)
      C=(16.*(X**2.+Y**2.)**2.*(KL**2.)*(KT**2.)+(X**2.+Y**2.+KT
     &**2.)**4.)
      Z=SIHAR1*SIHAR2
      ZX=C*Z
      FW=ZX+(8.*(X**2.+Y**2.)*KL*KT*(X**2.+Y**2.+KT**2.)**2.)*
     &(1.-COHAR1*COHAR2)
      RETURN
      END
      SUBROUTINE BETA(X,Y,CX,CY,L3,W,FW)
      REAL L3,KL,KT
      KL=SQRT((X**2.+Y**2.)-W**2./CX**2.)
      KT=SQRT(W**2./CY**2.-(X**2.+Y**2.))
      AR1=KL*L3
      AR2=KT*L3
      SIHAR1=SINH(AR1)
      SIAR2=SIN(AR2)
      COHAR1=COSH(AR1)
      COAR2=COS(AR2)
      AY=SIHAR1*SIAR2
      XY=((X**2.+Y**2.-KT**2.)**4.-16.*(X**2.+Y**2.)**2.*(KT**2.)*
     &(KL**2.))
      ZY=XY*AY
      FW=ZY+(8.*(X**2.+Y**2.)*KL*KT*(X**2.+Y**2.-KT**2.)**2.)*
     &(1.-COHAR1*COAR2)
      RETURN
      END
      SUBROUTINE GAMMA(X,Y,CX,CY,L3,W,FW)
      REAL L3,KL,KT
      KL=SQRT(W**2./CX**2.-(X**2.+Y**2.))
      KT=SQRT(W**2./CY**2.-(X**2.+Y**2.))
      AR1=KL*L3
      AR2=KT*L3
      SIAR1=SIN(AR1)
      SIAR2=SIN(AR2)
      COAR1=COS(AR1)
      COAR2=COS(AR2)
      ZY=SIAR1*SIAR2
      XA=(16.*(X**2.+Y**2.)**2.*(KL**2.)*(KT**2.)+(X**2.+Y**2.-KT**2.)
     &**4.)
      ZZ=XA*ZY
      FW=ZZ+(8.*(X**2.+Y**2.)*KL*KT*(X**2.+Y**2.-KT**2.)**2.)*
     &(1.-COAR1*COAR2)
      RETURN
      END
```

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