AN ANALYSIS FOR EFFECTIVE THERMAL CONDUCTIVITY ON UNIDIRECTIONAL FIBER COMPOSITES

by

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ABSTRACT

The validity of two methods of calculating effective thermal conductivity of composite materials is examined by comparing the numerical solutions for the effective thermal conductivity calculated through application of these two methods to unidirectional fiber composites.

Mathematical model and the assumptions associated with them are outlined first. Then, the approach, which is referred to as the direct approach, on which the two methods are based, is explained. In this approach, the effective thermal conductivity of a composite material is simply defined as the relation between the averaged temperature gradient and the averaged heat flux over the smallest volume representing all the characteristics of the composite material.

Both of the methods compared are based on the direct approach and differ from each other according to the boundary conditions imposed. Imposed conditions are, namely, the homogeneous boundary conditions, and the prescribed temperature and insulated surface boundary conditions over the unidirectional fiber composites in which fibers are distributed in rectangular and staggered array forms. In this work, those methods, in which homogeneous boundary conditions are incorporated are referred to as homogeneous methods, and those in which the other boundary conditions are used are referred to as flux methods.

The field equations associated with the above mentioned boundary conditions are solved by using the finite element method which has been proved to be an efficient method in the cases of heterogeneous material distribution and irregular geometry. The numerical values obtained from the two methods are also compared with bounds derived by variational analysis for effective thermal conductivities of any transversely isotropic fiber composites. To derive approximate analytic expressions representing numerical solutions obtained from the flux method, a new method, called the strip method is introduced.

It is, finally, concluded that the flux method gives more reasonable values in comparison with the homogeneous method and the numerical values obtained from both of the methods are in aggreement with the bounds.

ÖZET

İki, efektif ısı geçirgenlik katsayısı hesaplama yöntemi, tek yönlü lifler içeren karma malzemeler üzerinde denenerek elde edilen sayısal efektif ısı geçirgenlik katsayısı değerlerinin geçerliliği incelenmiştir.

İlk olarak, matematik model ve bu modelle bağlantılı kabuller anahatlarıyla anlatılmıştır. Daha sonra, bu iki yöntemin dayandığı doğrudan yaklaşım anlatılmıştır. Doğrudan yaklaşımda karma malzemelerin efektif ısı geçirgenlik katsayıları, karma malzemenin bütün özelliklerini içeren en küçük birim hacim üzerinde ortalaması alınmış ısı akısı ve sıcaklık gradyanı arasındaki ilişki olarak tanımlanmıştır.

Karşılaştırılan yöntemler, içinde tek yönlü liflerin dikdörtgen ve üçgen şekilde dağıldığı karma malzemeler üzerine "homojen" tip ile "sabit kenar sıcaklığı ve izole edilmiş kenar" tipleri gibi sınır şartları uygulayarak, doğrudan yaklaşımı kullanmaktadırlar. Bu çalışmada kullanılan yöntemler, sırasıyla homojen yöntem ve akı yöntemi olarak adlandırılmıştır.

Alan denklemleri, bahsedilen sınır şartları ile birlikte, homojen olmayan malzeme dağılımına ve düzensiz geometriye sahip durumlarda etkili olduğunu ispatlanmış olan sonlu elemanlar yöntemi kullanılarak çözülmüştür. İkinci olarak, bu iki yöntemle elde edilen sayısal değerler, varyasyon analizi kullanılarak, herhangi tip enine izotropik

karma malzemelerin efektif ısı geçirgenlik katsayıları için türetilen sınırlar ile karşılaştırılmıştır. Ek olarak, akı yöntemiyle elde edilen sayısal değerleri yaklaşık olarak veren denklemler elde etmek için bir yöntem geliştirilmiştir. Bu yönteme şerit yöntemi diyoruz.

Sonuç olarak, akı yönteminin homojen yönteme kıyasla daha sağlıklı değerler verdiği ve her iki yöntemlede elde edilen sayısal değerlerin, sınırlarla uyum içinde olduğu görülmüştür.

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LIST OF NOMENCLATURE

A	Dimensionless parameter, $2b/a(k_k/k_f - 1)$
a	Horizontal length of RVE (Figure 2.2)
b	Vertical length of RVE (Figure 2.2)
D	Domain
P_i, i =1,2,3	Component of gradient (Eq. 1.2)
k	Thermal conductivity coefficient
K	Conductivity ratio (fiber conductivity/matrix conductivity)
q	Heat flux
r	Radius of fiber
S	Surface
T	Temperature
x_i, i=1,2,3	Orthogonal coordinate system
x,y,z	Cartesian coordinate system
V	Volume
v	Volume ratio (fiber volume/total volume)
δ_{ij}	Kronecker delta
θ	Fiber packing angle (Figure 2.2)

Subscripts

e	Effective value
e-x	Effective value in the x-direction
e-y	Effective value in the y-direction
f	Fiber value
,i	Partial derivatives along x_i, i = 1,2,3
m	Matrix value
n	Normal component
x	Partial derivative along x-direction
y	Partial derivative along y-direction
1,2	Constituent phases

Superscripts

—	Volume average or Moving average
'	Value defined in RVE
*	Effective value

INTRODUCTION

A composite material is composed of two or more elemental materials joined together to form a new medium which is discontinuous in its microstructure but has combined macroscopic properties superior to those of its individual constituents. It is for this reason, intensive development efforts have been under way, for the past decades, to create literally a new class of materials for application heretofore deemed unlikely or impossible. The main competitive characteristics are, of course, light weight, high strength and dimensional stability, to name a few.

In general, there are three commonly accepted types of composite materials, which are distinguishable by their microstructures [9, p.2].

i) Fibrous composites consist of fibers in a matrix: A fiber is characterized geometrically not only by its very high length-to-diameter ratio but by its near crystal-sized diameter. The well-known example of this type is glass fibers within epoxy or a binding resin

ii) Laminated composites consist of layers of at least two different materials that are bonded together. Lamination is used to combine the best aspects of the constituents layers in order to achieve a more useful material.

iii) Particulate composites are composed of particles of one or more materials suspended in a matrix of another material. The particles can be either metallic or nonmetallic as can the matrix be. A good example for this type may be gravel concrete.

Published literature abounds with investigations on mechanical properties of composites with emphasis on stiffness, elastic moduli, fatigue limit, etc. Far fewer accounts, however, are concerned with their thermophysical properties which are now beginning to receive attention with a view to extend the application of composites to severe environments with thermal and moisture gradients, but the generality of the methods used to analyse mechanical properties has allowed the application of these available methods directly to investigate the thermophysical properties of composite materials.

Analytical determination of the properties of composite materials originates with some of the most illustrious names in science. J.C. Maxwell in 1873 and Lord Rayleigh in 1892 computed the effective conductivity of composites consisting of a matrix and certain distributions of spherical particles. Until about 1960, the primary motivation was scientific. With the advent of technology of composite materials, interest in this subject was renewed.

The investigations for fibrous composites have been along two parallel but related approaches: one is to model the fiber-in-matrix orientations by simplified geometrical equivalent, and the second approach, which relies on the results of the first for input information, is to employ a statistical technique to determine upper and lower bounds of the effective thermal conductivity. Among those employing the model approach are the often-cited works of Springer and Tsai [6], Behrens [5] and Han [1].

The publications of Hashin and Shtrikman [4], Beran and Silnutzer [7] and more recently Hashin [2] can be cited within the class of the second approach. The latter publication included a survey of the methodology and a list of related references.

In this study, the effective thermal co-ductivities of particular types of composite materials are calculated theoretically by two methods which originate from the same approach and the results obtained are compared. The approach is simply to approximate the real medium with an equivalent homogeneous medium which gives the same averaged temperature gradient and heat flux distributions. The distributed properties of the original medium are thus lumped into effective properties of an equivalent homogeneous system. In order to achieve this goal, the temperature distribution inside the composite material must be calculated.

The problem of finding the temperature distribution, however, is rather difficult to solve, if not impossible. This is due to the irregularity of the microstructure and imperfect bonding, which causes contact resistance to heat flow at the interfaces of individual phases of real composite materials. To aleviate this difficulty, idealized forms of composite materials are used, and, it is assumed that phases are dispersed in an ordered fashion and bonded perfectly with this idealization.

The problem appears to be easy to solve, however investigations have shown that even for regions having a simple geometry, it is impossible to obtain an exact analytical solution with contemporary mathematical aids. Thus a numerical technique needs to be used and the Finite Element Method (FEM) has proved to be an efficient technique in achieving this goal.

I. ANALYSIS OF COMPOSITE MATERIALS

1.1 BASIC DEFINITIONS

It is convenient at this point to give some commonly accepted assumptions of the theory of composite or heterogeneous media.

Because of the disparity between the dimensions of the overall composite material and that of each constituent phase region, rigorous evaluation of the effective thermal conductivity can be obtained only from a consideration of a representative volume element (RVE) as shown in Figure 1.1. RVE is the smallest volume element of the composite material. The volume outside the RVE is either a replica or a mirror image of the RVE. RVE, however, is large compared with typical phase region dimensions, e.g. fiber diameters and spacings.

From the practical point of view, a necessary characteristics of a composite material is the statistical homogeneity (SH). The strict definition of this is given in terms of probability functions [9]. For our purposes, it is sufficient to state that in a statistically homogeneous composite all global geometrical characteristics such as volume fractions, geometric distribution of phases, etc., are the same in any RVE.

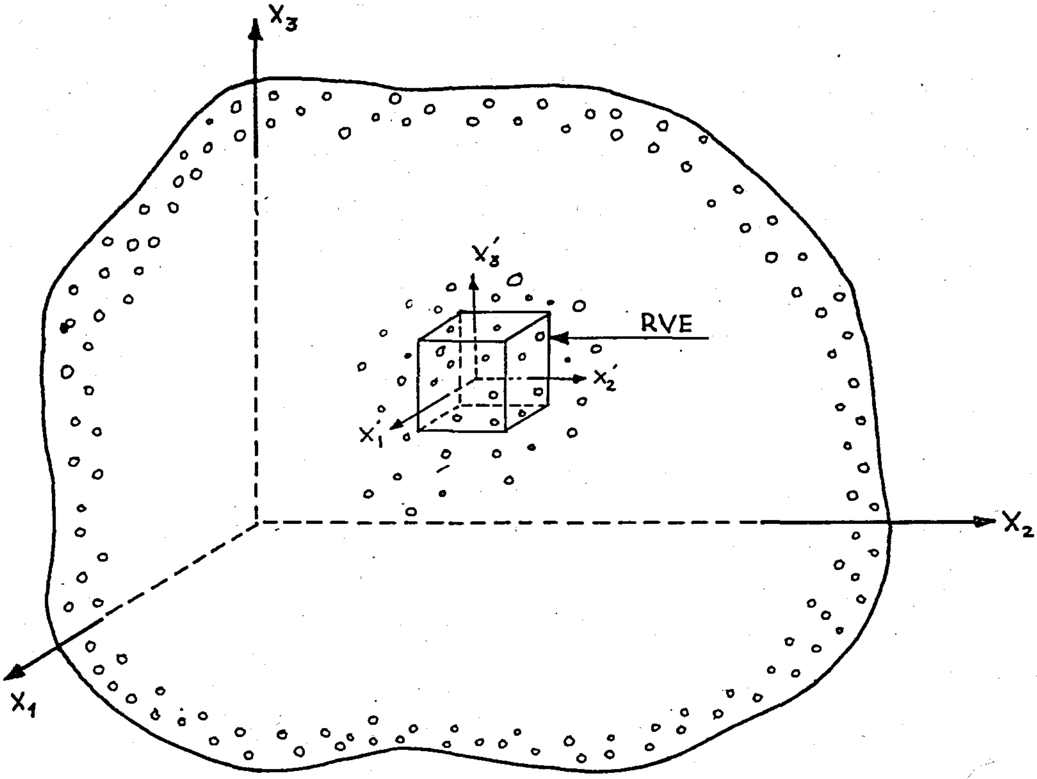


FIGURE 1.1 - Representative volume element.

In the following analysis, the RVE and the concept of statistical homogeneity respectively stand for the differential volume element and the concept of material homogeneity of the classical continua.

In the classical theory of heat conduction for homogeneous materials, the thermal conductivity tensor, k_{ij}, relates the homogeneous temperature gradient, $T,_j$, to the homogeneous heat flux, q_i, as given below

$$q_i = k_{ij}T,_j \quad , \tag{1.1}$$

In Eq. (1.1) the indicial notation, with repeated indices implying summation over these indices, is used. Unless otherwise stated, this notation will be used here on. If the fields, i.e., temperature gradient and heat flux, are not homogeneous, then they are assumed to be homogeneous within a differential element. In a heterogeneous body of statistically homogeneous phase geometry, a relation resembling Eq. (1.1) only applies for statistically homogeneous temperature gradient and heat flux fields.

Statistically homogeneous fields are statistically indistinguishable within different RVE in a heterogeneous body [9], i.e., their statistical moments such as average, variance, etc. are the same when taken over any RVE within the heterogeneous body. This implies that body averages and RVE averages of statistically homogeneous fields, where the averaging concept will be discussed in the next section, are the same. This permits to establish a relation, resembling Eq. (1.1), between averaged temperature gradient and averaged heat flux fields over RVE by considering each RVE as the differential element of classical continua of homogeneous materials.

In order to produce a SH field in a composite, it is necessary to apply boundary conditions that produce homogeneous fields in a homogeneous body. Such boundary conditions are called homogeneous boundary conditions (not to be the same as the concept of homogeneous boundary conditions in the theory of differential equations). For heat conduction, such a boundary condition is [10] given as follows.

$$T(S) = P_1x_1 + P_2x_2 + P_3x_3 \quad , \tag{1.2}$$

where T is the temperature, P_i, i = 1,2,3 are constants and S is the surface of the material body. When homogeneous boundary conditions as in Eq. (1.2) are applied, the components of averaged temperature gradient vector are constant and equal to P_i, $i = 1,2,3$ over composite material body.

Another important assumption, independent of the statistical homogeneity, is that of statistical isotropy. Essentially, this assumption implies, that macroscopic material properties of a composite body are the same in all directions or, are not effected by the orientation of the coordinate system to which the physical constants, such as thermal conductivity coefficient, elastic moduli, etc., are referred. In many cases statistical isotropy is an appropriate assumption. However, in some very important cases such as fibrous composites, the macroscopic material properties are predominantly anisotropic.

1.2 CLASSICAL APPROXIMATION

In the analysis of heterogeneous materials, the general problem is that of the suitable macrodescription of a heterogeneous material body. This problem is solved in reference [2] in analogy with to the solution of the same problem in the case of homogeneous continua. It is always assumed that such continua retain their material properties regardless of specimen size, thus also for infinitesimal elements. This permits establishment of field equations in terms of field derivatives. However, all real materials have microstructure. Metals, for example, are actually polycrystalline aggregates and are thus heterogeneous materials. Therefore, the differential element, that is considered in the derivation of field equations. Such as the classical elasticity and the heat conduction equations, is in reality a RVE which is composed of a sufficiently large number of crystalls and whose effective material properties are the material properties of the classical field theories, e.g., thermal conductivity coefficient or elastic moduli. Since, the RVE is composed of crystalls, it can not be considered as an infinitesimal differential element of the classical homogeneous continua. So, it may be concluded that the classical field theories are approximations that results in a macrodescription of a polycrystalline aggregate, i.e. heterogeneous material, when the RVE size is sufficiently small in relation to the body dimensions which is the definition of the RVE given in Section 1.1.

This leads to the idea to adopt the same approximation for a composite material body which implies that the classical field equations of heat conduction, elasticity, etc., are assumed valid for the composite material body with effective material properties replacing the usual material properties. This idea is called the classical approximation [2] and the mathematical model of the problem, stated at the beginning of this section, is based on this idea.

It is, first, necessary to define appropriate field variables for the construction of field equations, e.g., heat flux, temperature gradient, stress, etc., in terms of which a composite will be modelled as an equivalent homogeneous continuum. The appropriate choice is moving average [2]. Moving average of a vector function, f_i, is defined as

$$\overline{f}_i(x) = \frac{1}{\Delta V} \int_{\Delta V} f_i'(\underset{\sim}{x},\underset{\sim}{x}')\,dV(x') \quad , \qquad (1.3)$$

where $\underset{\sim}{x}$ is the position vector of a reference point in the RVE, e.g. the centroid, x_i' is a local coordinate system originating at $\underset{\sim}{x}$ as shown in Figure 1.1, ΔV is the volume of the RVE and the integration is over RVE. In Eq. (1.3) overbars represent averaged values and primes are used to indicate the local values inside the RVE. The moving average is simply the usual volume average associated with a position vector originating from a global coordinate system defined over the body that is divided into small volume elements (RVEs) over which volume averages are taken, and each element is defined by the position vector pointing a reference point inside the volume element. So, a different averaged value corresponds to each position vector. This causes the volume averages move together with the position vector, $\underset{\sim}{x}$.

It is shown in reference [11], that moving averaging and differentiation are commutative. This permits to establish the field equations in terms of averaged field variables. For example, by replacing the function, f_i in Eq. (1.3), with the thermal gradient vector, $T,_i$, and the heat flux vector, q_i, we obtain the important relations

$$\overline{\frac{\partial T'}{\partial x_i'}} = \frac{\partial \overline{T}}{\partial x_i} \quad , \tag{1.4}$$

and

$$\overline{\frac{\partial q_i'}{\partial x_i'}} = \frac{\partial \overline{q}_i}{\partial x_i} = 0 \quad , \tag{1.5}$$

respectively, where the use of overbars and primes are as defined in conjuction with Eq. (1.3). The Eq. (1.5) is a particular form of the general steady state heat conduction equation without any source term in terms of fluxes. This important result shows that classical field equations are also valid for averaged field variables.

The problem now reduces to obtain a relation between moving averages of field variables, e.g. temperature gradient and heat flux, when the composite body is subjected to arbitrary boundary conditions. This problem is solved by Beran in reference [12] for heat conduction through heterogeneous media, as the relation between moving average of heat flux and local definition of temperature gradient as follows

$$\bar{q}_i(\underset{\sim}{x}) = \int K^*_{ij}(\underset{\sim}{x},\underset{\sim}{x}')T'_{,j}(\underset{\sim}{x}')dV(\underset{\sim}{x}') \quad . \tag{1.6}$$

The result is, however, not a practical one since the tensor, K^*_{ij}, depends on phase properties and phase geometry in an unknown fashion for arbitrary phase distributions over RVE.

The more useful form of Eq. (1.6) is obtained by its series expansion around $\underset{\sim}{x}$ [12], which is given as follows

$$\bar{q}_i(\underset{\sim}{x}) = A^*_{ij}\bar{T}_{,j}(\underset{\sim}{x}) + B^*_{ijk}\bar{T}_{,jk}(\underset{\sim}{x}) + C^*_{ijk\ell}\bar{T}_{,jk\ell} + \ldots \tag{1.7}$$

This is a general relation between averaged temperature gradient and heat flux fields for composite materials.

In case of statistically homogeneous temperature gradient and heat flux fields, the moving averages of these fields are independent of the position vector, $\underset{\sim}{x}$, because averages of the SH fields over each RVE are the same as discussed in Section 1.1. So, for SH fields, all derivatives of temperature gradient vector with respect to $\underset{\sim}{x}$ are zero, then (1.7) reduces to the convenient form,

$$\bar{q}_i = A^*_{ij}\bar{T}_{,j} \quad , \tag{1.8}$$

which implies that averaged SH temperature gradient and heat flux fields are related by effective conductivity tensor, A^*_{ij}, as in the classical heat conduction relation, Eq. (1.1). This work is an application of this relation.

1.3 ANALYTICAL TREATMENT

Generally, the analytical treatment of the problem of calculating the effective conductivity is divided into three categories: (i) direct approach, (ii) variational approach, and (iii) approximations. In this work, we are primarily interested in the direct approach, but in some cases, we will apply the variational approach as well. In the following discussion, two phase composite materials are assumed for the sake of simplicity, the analysis, however, can be extended to multiphase composite materials without any difficulty.

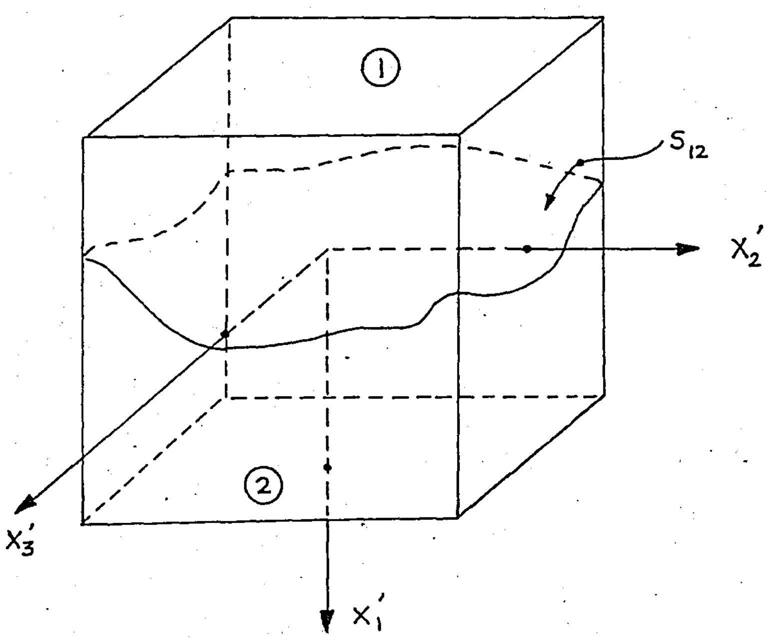


FIGURE 1.2 - A typical RVE for a two phase composite material.

The computation of the effective properties, e.g. the effective thermal conductivity, in terms of the averaged field variables, e.g. the temperature gradient and the heat flux, is called the direct approach. So, the direct approach is an application of the Eq. (1.8). In general, it, then, requires the determination of the appropriate fields in the phases satisfying the appropriate field equations subject to interface continuity conditions, and external homogeneous boundary conditions. For heat conduction, the interface continuity conditions, with no contact resistance at the interface, can be posed as

$$\begin{aligned} T^{(1)} &= T^{(2)} \\ q_n^{(1)} &= q_n^{(2)} , \end{aligned} \qquad \text{on } S_{12} \tag{1.9}$$

where superscripts 1 and 2 refer to the constituent phases, S_{12} is the contact surface of two phases as shown in Figure 1.2, q_n is the normal component of heat flux on S_{12}. For real composites determination of the field solutions is a very difficult problem not only because of the mathematical difficulties, but also because of the fact that the actual details of the interface geometry are newer known. So, the application of direct approach is restricted to types of composites with idealized phase geometries. Laminated composites or fiber composites with regularly dispersed fibers in a matrix are representative examples of such idealized composites.

As stated at the beginning of Section 1.3, direct approach deals with averaged field quantities. The averaging method used in this approach involves the volume average over the total volume of the RVE. For example, over a two phase composite, the temperature gradient vector, $T_{,i}$, is averaged as

$$\overline{T}_{,i} = \frac{1}{V}\left[\int_{V_1} T_{,i}^{(1)}(\underset{\sim}{x})dV + \int_{V_2} T_{,i}^{(2)}(\underset{\sim}{x})dV\right] \quad , \tag{1.10}$$

where 1 and 2 refer to the values of $T_{,i}$ in the constituent phases and the average is taken over the total volume, V. After calculation of the required averages, the effective conductivity tensor, k^*_{ij}, can be calculated from Eq. (1.8) by setting A^*_{ij} equal to k^*_{ij}.

Since, most of the engineering materials are anisotropic, the effective thermal conductivity tensor will have six unknown components (k^*_{ij} is a symmetric tensor according to Onsager's principle). Eq. (1.8), on the other hand, provides three equations. Thus without any further simplification, it is impossible to determine the effective thermal conductivity. Such a simplification can be achieved by assuming that the RVE has a symmetric phase distribution. With the presence of such a geometry, the conductivity tensor can be diagonalized with respect to a coordinate system defined in terms of the symmetry axes. With reference to such a coordinate system, then, there remains only three unknown components of the thermal conductivity tensor which now can be determined with the use of Eq. (1.8). At this point it should be noted that in the case of statistically isotropic composites, the thermal conductivity tensor is given as

$$k^*_{ij} = k^*\delta_{ij} \quad , \tag{1.11}$$

in which only one unknown, namely k*, is involved. k* then can easily be determined by using Eq. (1.8) alone. It then follows that, with the exception of such idealized composites, the direct approach is restricted to composites having symmetric phase distribution over RVE.

An alternative definition of effective physical properties is given in terms of energy expressions. This is known as the variational approach. Such energy expressions are bounded from above and below by extremum principles. The variational approach is, in a certain sense, more powerful that the direct approach, because it leads to bounds on effective physical properties when exact evaluation is not possible. In particular, it can give results for irregular phase geometries in terms of partial information. For the formulation of the variational approach in heat conduction, references [4] and [5] should be resorted to.

Approximations are, by their nature, of unlimited variety and is beyond the scope of this work. Therefore they will not be discussed here, but the interested reader may consult to reference [14] to see a typical application.

II. SOME APPLICATIONS

2.1 FINITE ELEMENT METHOD

As discussed in Section 1.3, the computation of effective thermal conductivity in terms of averages requires determination of the appropriate fields within the phases. This in turn requires the solution of the field equations subject to, interface continuity conditions, as given by e.g. Eq. (1.9), and the external homogeneous boundary conditions. Even for idealized composites the analytic solution of these equations is an extremely difficult problem. This is primarily due to the necessity of satisfying the interface continuity conditions. Thus, a numerical technique needs to be used and the finite element method (FEM) has proved to be an efficient tool in achieving this goal, because FEM is a powerful technique especially for the problems that involve irregular geometry and non-homogeneous material distribution over the material.

The domain of the problem is discretized by using elements of simple geometries such as rectangle or triangle. Over each element, the unknown fields are approximated by a linear combination of finite number of known basis functions with unknown expansion coefficients which are the unknown field values at some points in the element, namely the nodal points.

Finite element analysis requires a translation of the local field equations, which are subjected to certain boundary conditions, into a global model, formulated in terms of integral laws that hold over the entire region. The residual that results from the satisfaction of the integral laws over each element by approximated solution, is minimized. This leads to system of linear equations in terms of unknown coefficients over each element. By summing up these systems of equations, the solution of the governing field equation reduces to the solution of system of linear equations. Solution of these equations gives the unknown coefficients, i.e. the unknown field values at each node.

In this analysis, the governing differential equation is steady state heat conduction equation in the form of

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) = 0. \tag{2.1}$$

The flowchart and computer code for the FEM solution of this equation, subject to boundary conditions of prescribed variable temperature and/or insulated surface types is given in Appendix B. This program utilizes triangular elements with 3-nodes 6-nodes and 10-nodes [15]. The input data for the computer program used is the mesh geometry information and the thermal conductivity coefficients for each phase. The output values are temperature at each node, temperature gradient over each element, averaged temperature gradient and heat flux over the domain and the effective conductivity coefficients.

2.2 SCOPE OF ANALYSIS

In this work, we mainly compare two methods of calculating effective thermal conductivity. In both methods, direct approach is applied on a particular type of fiber composites, namely unidirectional fiber composites in which fibers are aligned in one direction.

As discussed in Section 1.3, direct approach requires determination of the appropriate fields by applying homogeneous boundary conditions over RVE. The only difference between these two methods is the types of boundary conditions applied in the calculation of the fields from the field equation. In one method, homogeneous boundary condition given by Eq. (1.2) is applied over RVE. Therefore this method is called the homogeneous method. The second method is the unidirectional heat flux method. In this method, a thermal gradient is established between two opposite faces of the RVE and the remaining faces are insulated. The validity of this method is examined on fiber composites in reference [1] and the results obtained is in agreement with the experimental data, to some extent. In the following discussion, we will call this method as flux method for simplicity.

The boundary condition applied in the flux method may be regarded as homogeneous type boundary condition. This is due to the fact that, the average temperature gradient obtained over RVE depends only on the thermal gradient established between two opposite faces of the RVE for a particular geometry when such kind of boundary condition is applied over the composite material. The thermal gradient established is, however, the same for each RVE. This is due to the similarity of RVEs in a statistically homogeneous composite materials (See Section 1.1). Thus, each RVE has the same average temperature gradient values. This makes Eq. (1.8) applicable. Accordingly, the direct approach is valid to calculate the effective thermal conductivity in both methods.

As explained in Section 1.3, in determination of the effective thermal conductivity, the variational approach is a powerful technique especially for irregular phase geometries, because it leads to bounds on effective physical properties in terms of partial information about the phase geometry. Some expressions of this type are derived by Hashin [10] for any transversely isotropic fiber composites, i.e. transverse to fiber direction, as

$$\frac{k_e^{(-)}}{k_m} = 1 + \frac{v_f/v_T}{\dfrac{1}{k_f/k_m - 1} + \dfrac{v_m/v_T}{2}}$$

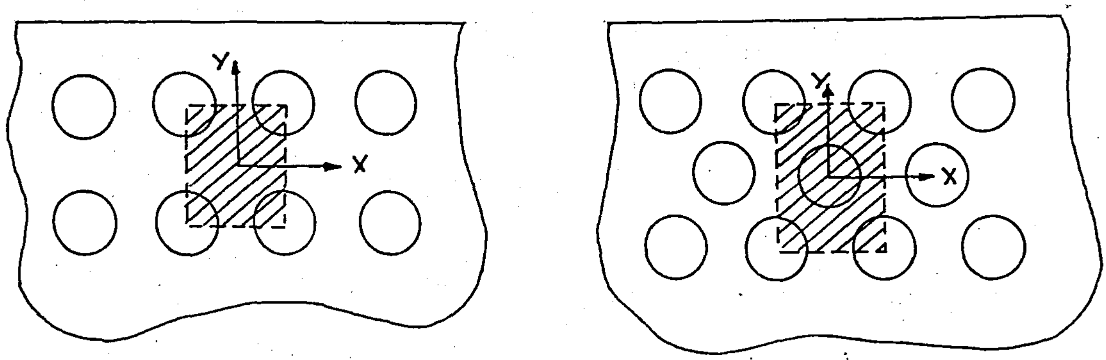
$$\frac{k_e^{(+)}}{k_m} = \frac{k_f}{k_m} + \frac{v_m/v_T}{\dfrac{1}{1 - k_f/k_m} + \dfrac{v_f/v_T}{2k_f/k_m}} , \qquad (2.2)$$

where subscripts f and m refer to fiber and matrix, respectively, v is total volume, k_e is the effective thermal conductivity and (+) and (-) refer to upper and lower bounds respectively. These bounds are compared with the numerical results obtained for effective thermal conductivity in Section 2.3.

In the flux method, it is customary to designate a principal direction of heat transfer (PDHT) and to assume that heat flux is unidirectional, that is, there is no flux components normal to PDHT. This also means that planes perpendicular to the PDHT are isothermal. That being the case, the effective conductivity of the RVE is simply the resultant of a network of smaller resistances in series and parallel over each phase.. This leads us to calculate the effective conductivity by using simple integration. We will call this new method as the strip method. In the strip method, a strip of infinitesimal width is taken along PDHT. The effective conductivity of the strip is the resultant of small resistances in series. Now, each strip has a value of effective conductivity and is connected in parallel. Resultant resistance; which defines effective thermal conductivity of RVE, is calculated by summing up the resistances of each strip by integration. Details are given in the Appendix A.

Secondly, we will investigate the effect of geometrical arrangement of the fibers on the effective thermal conductivity of the composite material, in particular, those having isotropic fibers uniformly dispersed in an isotropic matrix. Unidirectional fiber composites having two geometrical arrangements, namely, rectangular array and staggered array of fibers in a matrix, as shown in Figure 2.1, are considered specifically.

There are three principal conductivities for the composites under consideration. The effective conductivity along the z-axis, i.e.



(a) Rectangular Array (b) Staggered Array

FIGURE 2.1 - RVEs for two types of fibrous composites.

parallel to the fibers, can be simply obtained by a linear combination of the two constituent's thermal conductivities in proportion to their respective cross-sections. Hence, the only ones which need to be analyzed are those along the x and y directions. In order to calculate the effective conductivity along x and y directions, we need to obtain averaged values of temperature gradient and heat flux over the RVE. This requires the solution of the steady state heat conduction equation, Eq. (2.1), associated with boundary conditions, discussed at the beginning of Section 2.2, by using the FEM.

2.3 PROCEDURE AND RESULTS

2.3.1 <u>Rectangular Array</u> (Figure 2.1-a)

The governing heat conduction equation, Eq. (2.1), is solved by using FEM for the domains shown in Figure 2.2. Since the domain is symmetric, the effective conductivity tensor is in the diagonal form with two components in the x and y directions (See Section 1.3).

As discussed in Section 2.2, two methods will be applied in determining effective thermal conductivity, namely, homogeneous method and flux method.

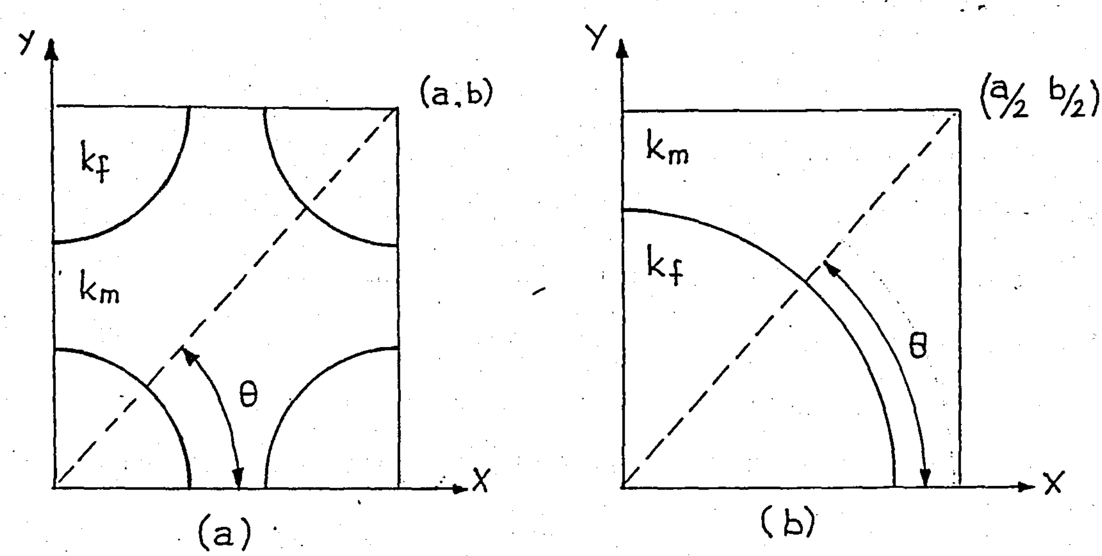


FIGURE 2.2 - Solution domains for rectangular array.

In the homogeneous method, the boundary condition associated with Eq. (2.1) is homogeneous type which is given by Eq. (1.2). Since we only deal with the determination of effective thermal conductivity, without loss of generality, we will choose simple form of Eq. (1.2) as

$$T(s) = x + y \quad , \tag{2.3}$$

where x and y are the coordinates of the points on s which is the surface of the domain shown in Figure 2.2-a. When boundary condition, Eq. (2.3), is applied over the domain, the resulting average temperature gradient vector has the components which are equal to 1. So, we can calculate the two components of the effective conductivity tensor, which are equal to components of averaged heat flux vector, from Eq. (1.8).

In the flux method, because of the symmetry of the geometry, we will consider Figure 2.2.-b, which is one quarter of the domain in Figure 2.2-a, as solution domain. As discussed in Section 2.2, in this method, a thermal gradient is established between two opposite faces of the domain and the remaining faces are insulated. Since, the value of the effective thermal conductivity is independent of the thermal gradient established, we will choose the following simple form of boundary conditions. By considering a one dimensional heat flow in the y direction, the approximate boundary conditions are

$$\begin{aligned} &T(x,b/2) = 1 \\ &T(x,0) = 0 \\ &T_x(0,y) = T_x(a/2,y) = 0 \end{aligned} \tag{2.4}$$

For one dimensional heat flow in the x direction, the boundary conditions are

$$\begin{aligned} &T(a/2,y) = 1 \\ &T(0,y) = 0 \\ &T_y(x,0) = T_y(x,b/2) = 0 \end{aligned} \tag{2.5}$$

After imposing the boundary conditions, effective conductivity tensor, which has two components for this two dimensional symmetric geometries, is obtained from Eq. (1.8) by using computer program given in Appendix B.

Additionally, strip method (See Section 2.2) is utilized to obtain an approximate analytic expression representing the numerical solution obtained from flux method. The expression for rectangular array of fibers is given by Eq. (A.7) in Appendix A. Bounds on effective thermal conductivity are also calculated from Eq. (2.2). Since these bounds are for transversely isotropic fiber composites, it corresponds to configurations with angle, θ = 45 deg.

The results obtained are expressed as the ratio of the effective conductivity to the matrix conductivity and, characteristically, this ratio depends on (i) volume ratio, v_f/v_T = volume of fiber/total volume, (ii) conductivity ratio, k_f/k_m = fiber conductivity/matrix conductivity, (iii) the angle, θ. Because of the symmetry of the geometries involved, the two principal conductivities can be handled by a complementary angle principle. In other words, solution for the effective conductivities with angles, determines the case with the complementary angle, (90-θ), e.g., the value k_{e-y} for θ = 40 deg. corresponds the value k_{e-x} for θ = 50 deg., and vice versa.

Tables 2.1-6 give tabulated numerical results for effective thermal conductivity obtained from application of homogeneous method and flux method for volume ratios, v_f/v_T, of 0.5, 0.6 and 0.7. The angle, θ, is chosen so as to avoid fibers overlapping eachother. Obviously, for θ = 45 deg, the effective thermal conductivity tensor is isotropic, i.e. $k_{e-y} = k_{e-x} = k_e$. In Figures 2.3-8, the tabulated numerical results and the expression given by Eq. (A.7) is plotted as k_f/k_m versus k_{e-x}/k_m and k_f/k_m versus k_{e-y}/k_m. Additionally, in Figures 2.6-8, the bounds obtained from Eq. (2.2) are also shown. Figures 2.9-a and 2.9-b show the trends in numerical results obtained from homogeneous method and flux method, respectively, for arbitrarily chosen k_f/k_m values of 0.4, 6, 80 and 800, with the angle θ = 45 deg. as v_f/v_T versus k_e/k_m, where v_f/v_T ratio is bounded above by limit value, v_f/v_T = 0.785. Finally, in Figure 2.10, temperature gradient distribution at some points over the domain, represented by Figure 2.2-b for one dimensional heat flow along y-direction is presented for rectangular array with volume ratio of v_f/v_T = 0.6 and the angle θ = 40 deg. case.

TABLE 2.1 - FEM Solutions for Effective Conductivity

	Rectangular Array: $v_f/v_T = 0.5$ $\theta = 35$ deg.			
	Mesh (87,136) Flux B.C.'s		*Mesh (77,118) Homogeneous B.C.'s	
k_f/k_m	k_{e-x}/k_m	k_{e-y}/k_m	k_{e-x}/k_m	k_{e-y}/k_m
0.1	0.304506	0.486725	0.409728	0.505008
0.2	0.421273	0.551816	0.503025	0.567446
0.4	0.612206	0.674741	0.657655	0.685426
0.6	0.765241	0.789569	0.786051	0.795909
0.8	0.892249	0.897664	0.898375	0.900400
1.0	1.00000	1.00000	1.00000	1.00000
2.0	1.36552	1.44594	1.42667	1.44821
4.0	1.71953	2.12380	2.13911	2.21908
6.0	1.89376	2.62691	2.79914	2.92051
8.0	1.99762	3.01862	3.44182	3.59178
10.0	2.06662	3.33319	4.07662	4.24718
20.0	2.22276	4.28741	7.21122	7.43325
40.0	2.31162	5.06561	13.4372	13.6915
60.0	2.34310	5.40381	19.6527	19.9190
80.0	2.35920	5.59299	25.8655	26.1381
100.0	2.36899	5.71387	32.0773	32.3537
200.0	2.38885	5.97423	63.1310	63.4152
400.0	2.39892	6.11469	125.233	125.522
600.0	2.40230	6.16317	187.334	187.624
800.0	2.40400	6.18774	249.435	249.726
1000.0	2.40501	6.20258	311.536	311.827

*Mesh (Number of nodes, Number of elements).

TABLE 2.2 - FEM Solutions for Effective Conductivity

	Rectangular Array: $v_f/v_T = 0.5$ $\theta = 40$ deg.			
	Mesh (99,158) Flux B.C.'s		Mesh (66,100) Homogeneous B.C.'s	
k_f/k_m	k_{e-x}/k_m	k_{e-y}/k_m	k_{e-x}/k_m	k_{e-y}/k_m
0.1	0.389031	0.457162	0.450487	0.486183
0.2	0.495229	0.526443	0.529183	0.553755
0.4	0.601108	0.675536	0.668033	0.678836
0.6	0.850014	0.799257	0.789506	0.793373
0.8	0.902129	0.877983	0.899049	0.899846
1.0	1.00000	1.00000	1.00000	1.00000
2.0	1.29666	1.59416	1.43265	1.44102
4.0	1.45880	1.97275	2.15864	2.18883
6.0	1.48519	2.21022	2.82700	2.87200
8.0	1.48040	2.35451	3.47499	3.52994
10.0	1.46806	2.44998	4.11344	4.17541
20.0	1.41165	2.66034	7.25656	7.33562
40.0	1.36129	2.77354	13.4875	13.5769
60.0	1.34022	2.81201	19.7047	19.7980
80.0	1.32878	2.83131	25.9184	26.0137
100.0	1.32161	2.84291	32.1307	32.2271
200.0	1.30656	2.86612	63.1853	63.2842
400.0	1.29867	2.87773	125.288	125.388
600.0	1.29598	2.88159	187.389	187.489
800.0	1.29462	2.88352	249.489	249.590
1000.0	1.29381	2.88468	311.590	311.690

TABLE 2.3 - FEM Solutions for Effective Conductivity

Rectangular Array: $v_f/v_T = 0.6$ $\theta = 40$ deg.				
	Mesh (78,120) Flux B.C.'s		Mesh (66,100) Homogeneous B.C.'s	
k_f/k_m	k_{e-x}/k_m	k_{e-y}/k_m	k_{e-x}/k_m	k_{e-y}/k_m
0.1	0.267747	0.379708	0.333715	0.401035
0.2	0.378200	0.459193	0.431428	0.477455
0.4	0.568713	0.608386	0.600712	0.620939
0.6	0.731223	0.747019	0.747218	0.754515
0.8	0.873506	0.877101	0.878839	0.880358
1.0	1.00000	1.00000	1.00000	1.00000
2.0	1.47502	1.53423	1.51951	1.53649
4.0	2.01869	2.34615	2.39235	2.45762
6.0	2.32436	2.95021	3.19577	3.29645
8.0	2.52077	3.42170	3.97436	4.09993
10.0	2.65772	3.80117	4.74123	4.88494
20.0	2.98871	4.95710	8.51486	8.70445
40.0	3.19142	5.90543	15.9932	16.2120
60.0	3.26594	6.31919	23.4544	23.6841
80.0	3.30465	6.55108	30.9112	31.1466
100.0	3.32837	6.69941	38.3661	38.6050
200.0	3.37694	7.01933	75.6324	75.8785
400.0	3.40182	7.19217	150.156	150.406
600.0	3.41020	7.25187	224.678	224.929
800.0	3.41441	7.28212	299.200	299.452
1000.0	3.41694	7.30041	373.721	373.973

TABLE 2.4 - FEM Solutions for Effective Conductivity

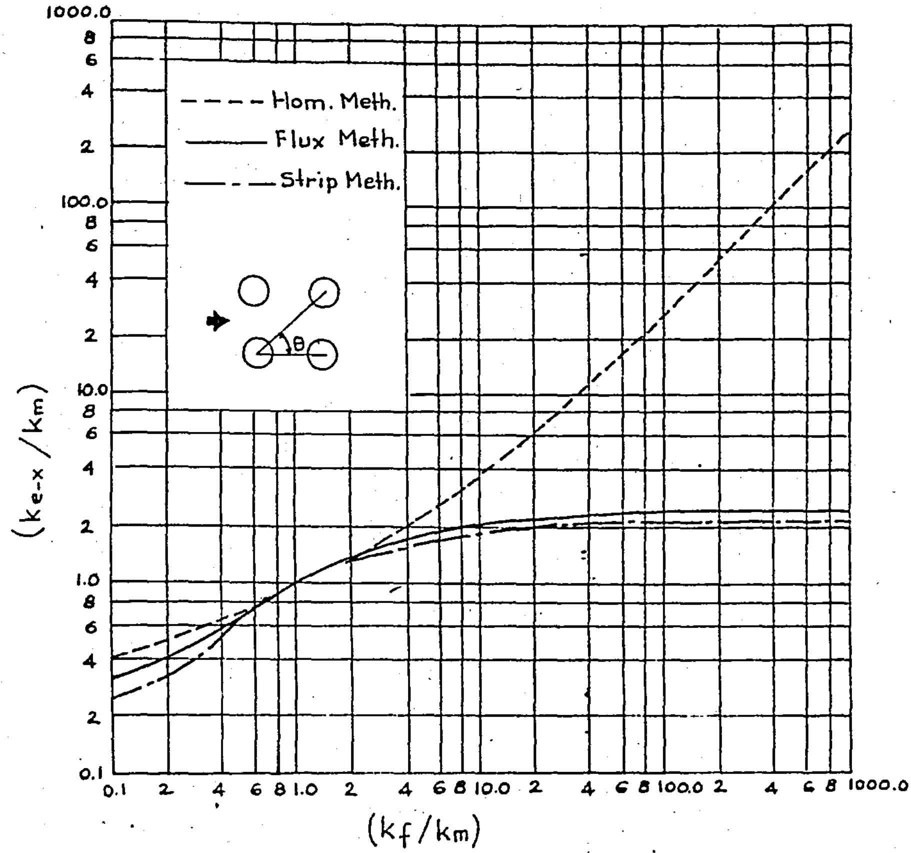
	Rectangular Array: $v_f/v_T = 0.5$ $\theta = 45$ deg.	
	Mesh (100,160) Flux B.C.'s	Mesh (68,106) Homogeneous B.C.'s
k_f/k_m	k_e/k_m	k_e/k_m
0.1	0.416827	0.468307
0.2	0.499072	0.541238
0.4	0.647472	0.673184
0.6	0.778348	0.791311
0.8	0.895053	0.899415
1.0	1.00000	1.00000
2.0	1.40044	1.43622
4.0	1.86590	2.17096
6.0	2.12969	2.84498
8.0	2.29982	3.49668
10.0	2.41870	4.13770
20.0	2.70685	7.28694
40.0	2.88388	13.5214
60.0	2.94905	19.7399
80.0	2.98293	25.9543
100.0	3.00369	32.1668
200.0	3.04623	63.2218
400.0	3.06803	125.324
600.0	3.07537	187.424
800.0	3.07906	249.524
1000.0	3.08128	311.623

TABLE 2.5 - FEM Solutions for Effective Conductivity

	Rectangular Array: $v_f/v_T = 0.6$ θ = 45 deg.	
	Mesh (79,122) Flux B.C.'s	Mesh (61,92) Homogeneous B.C.'s
k_f/k_m	k_e/k_m	k_e/k_m
0.1	0.332465	0.371999
0.2	0.423794	0.456754
0.4	0.590178	0.611273
0.6	0.739514	0.750878
0.8	0.875346	0.879578
1.0	1.00000	1.00000
2.0	1.50214	1.52683
4.0	2.15198	2.41822
6.0	2.55948	3.23406
8.0	2.83994	4.02096
10.0	3.04497	4.79368
20.0	3.57620	8.58148
40.0	3.92929	16.0683
60.0	4.06493	23.5326
80.0	4.13671	30.9909
100.0	4.18113	38.4467
200.0	4.27321	75.7145
400.0	4.32094	150.238
600.0	4.33711	224.760
800.0	4.34525	299.280
1000.0	4.35014	373.801

TABLE 2.6 - FEM Solutions for Effective Conductivity

	Rectangular Array: $v_f/v_T = 0.7$ $\theta = 45$ deg.	
	Mesh (60,88) Flux B.C.'s	Mesh (61,92) Homogeneous B.C.'s
k_f/k_m	k_e/k_m	k_e/k_m
0.1	0.250086	0.278265
0.2	0.351476	0.375022
0.4	0.535475	0.551252
0.6	0.702066	0.711312
0.8	0.856023	0.859952
1.0	1.00000	1.00000
2.0	1.61486	1.62087
4.0	2.52466	2.68140
6.0	3.18440	3.64959
8.0	3.68942	4.57955
10.0	4.08959	5.48992
20.0	5.27480	9.93192
40.0	6.21026	18.6814
60.0	6.60837	27.3953
80.0	6.82887	36.0995
100.0	6.96896	44.7997
200.0	7.26854	88.2821
400.0	7.42897	175.228
600.0	7.48414	262.169
800.0	7.51206	349.110
1000.0	7.52892	436.050



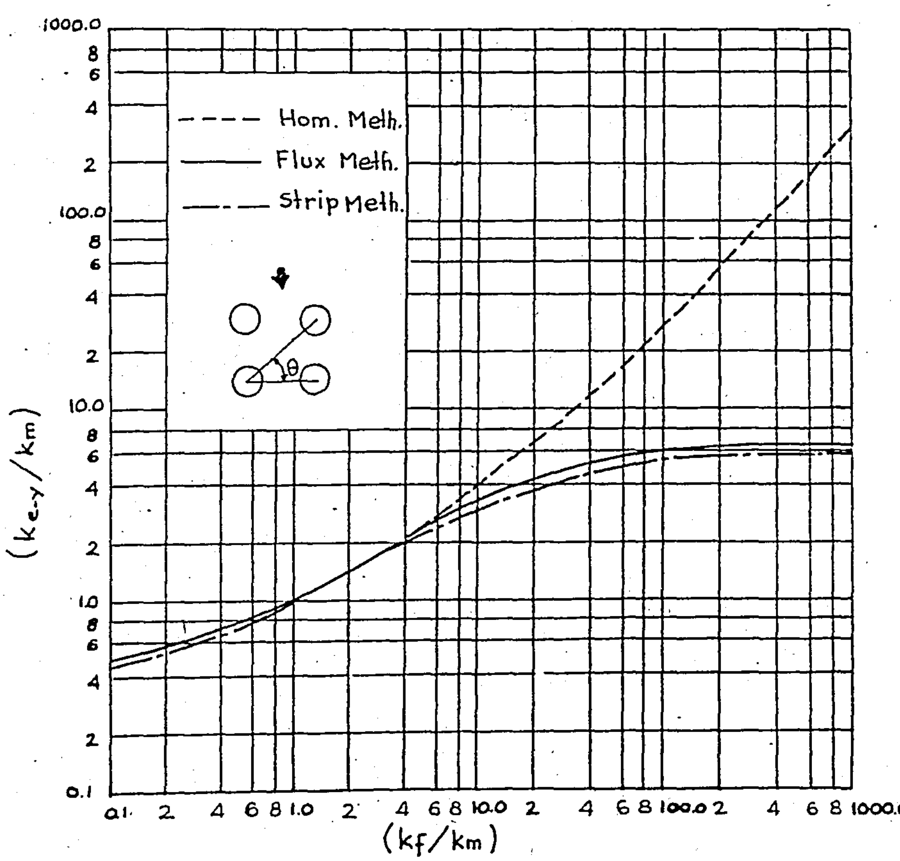


FIGURE 2.3 - Effective conductivities for rectangular array fiber composite (v_f/v_T = 0.5, θ = 35 deg.)

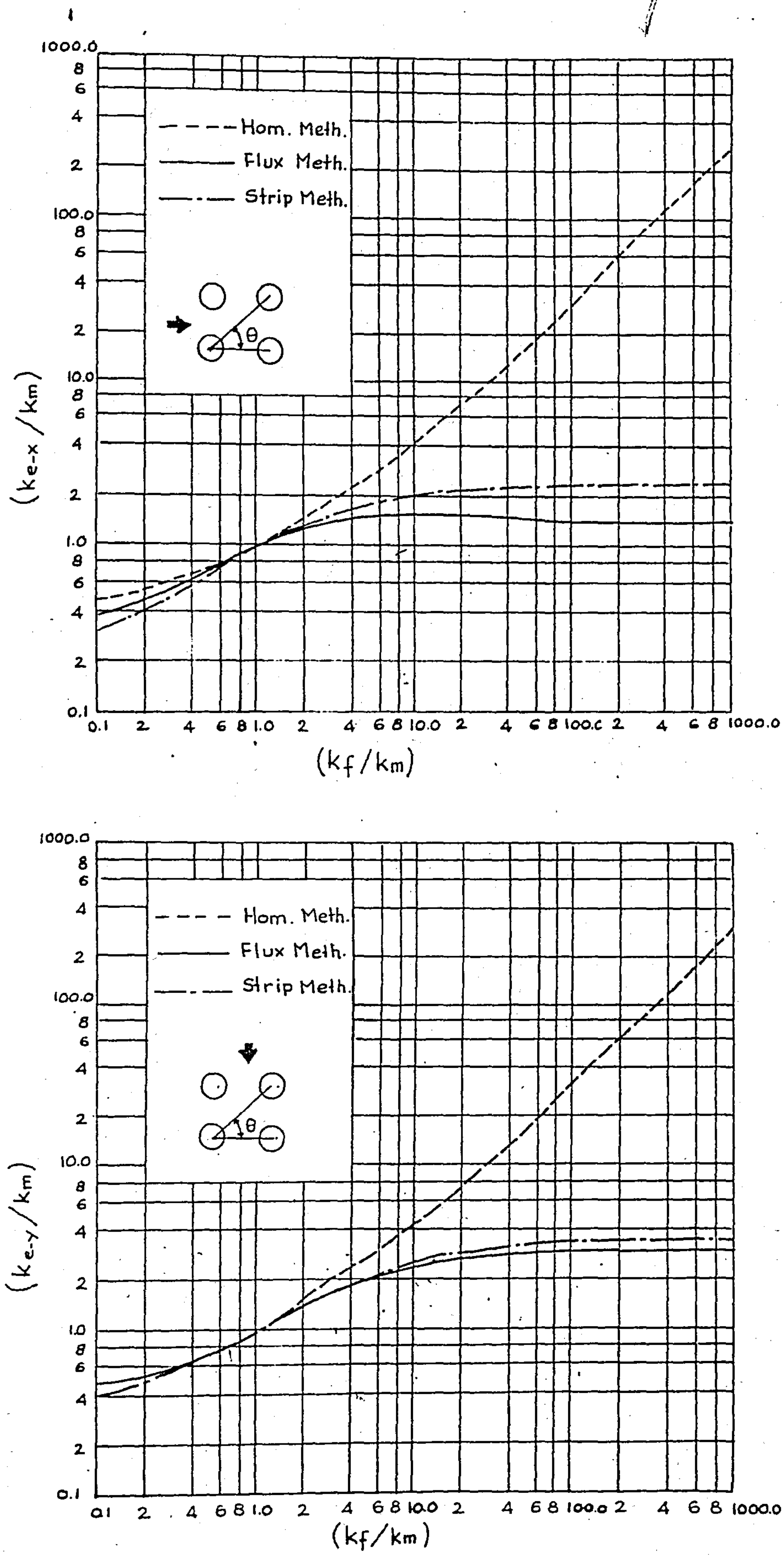
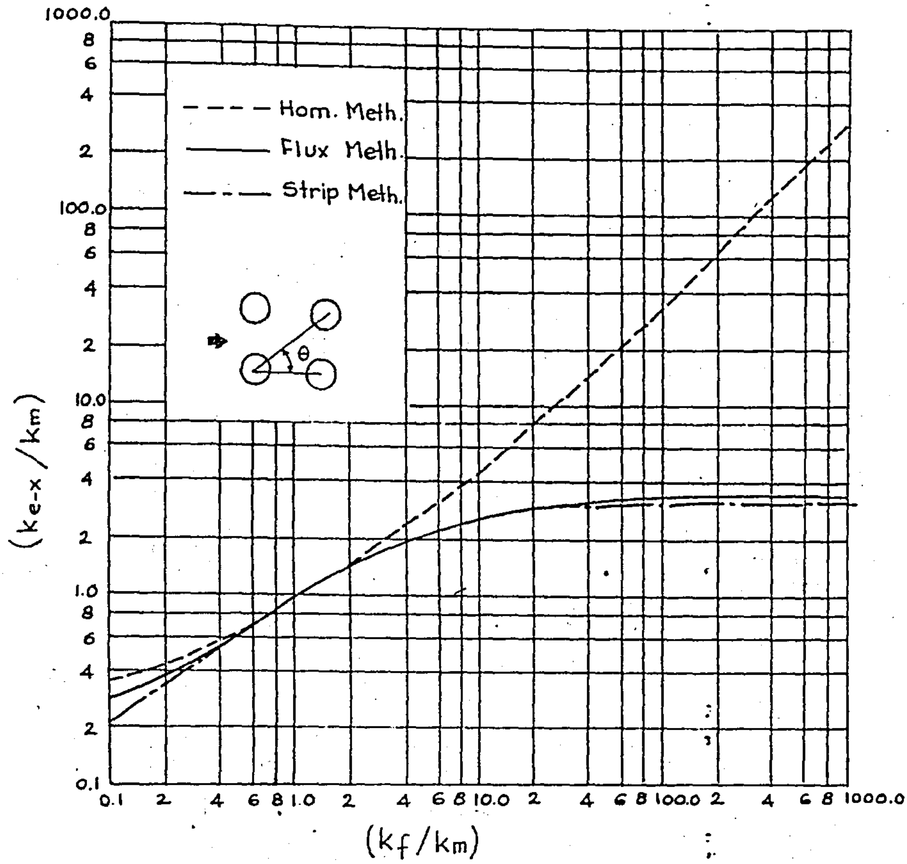


FIGURE 2.4 - Effective conductivities for rectangular array fiber composite (v_f/v_T = 0.5, θ = 40 deg.)



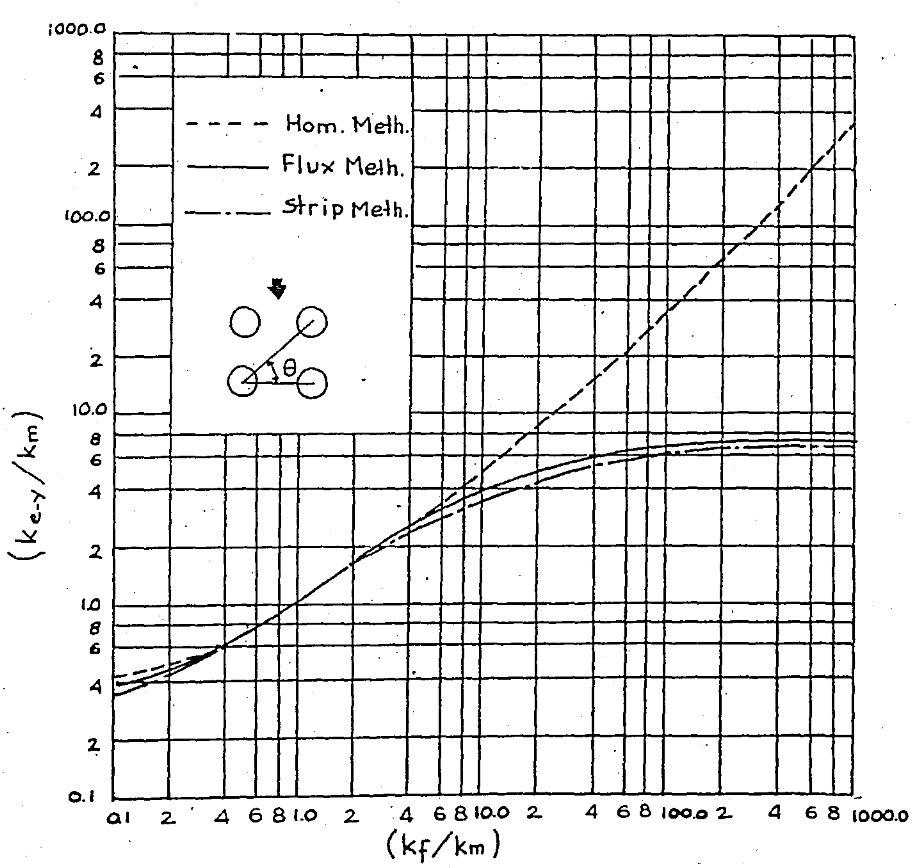


FIGURE 2.5 - Effective conductivities for rectangular array fiber composite (v_f/v_T = 0.6, θ = 40 deg.)

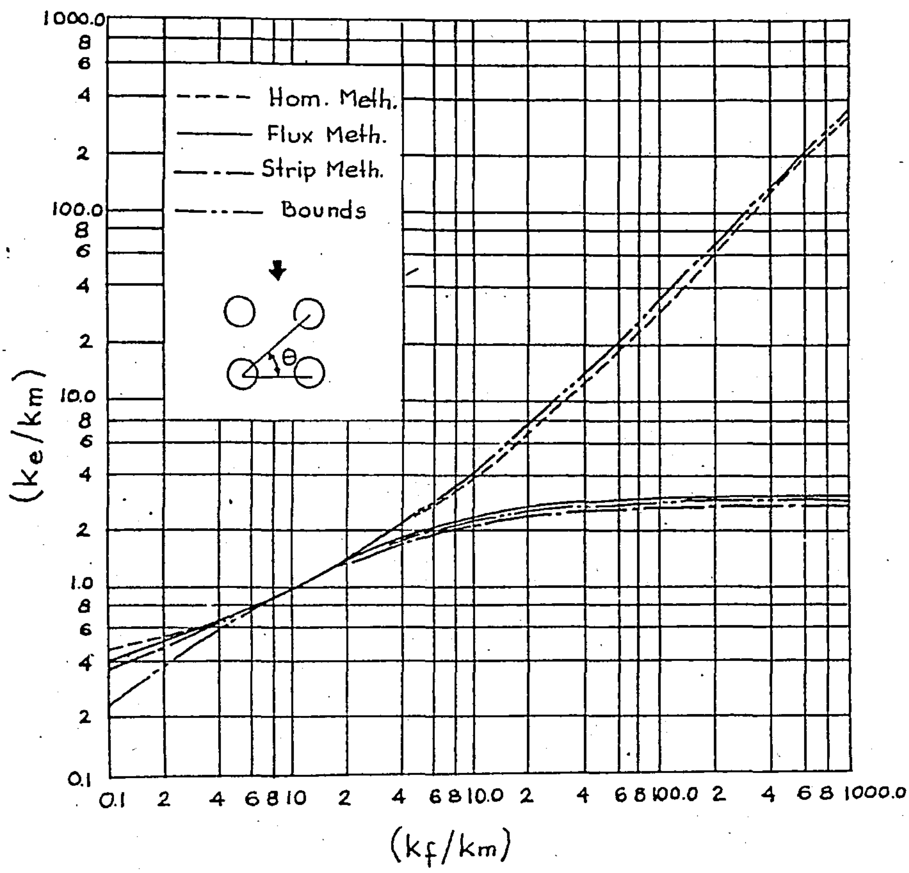


FIGURE 2.6 - Effective conductivities for rectangular array fiber composite (v_f/v_T = 0.5, θ = 45 deg.)

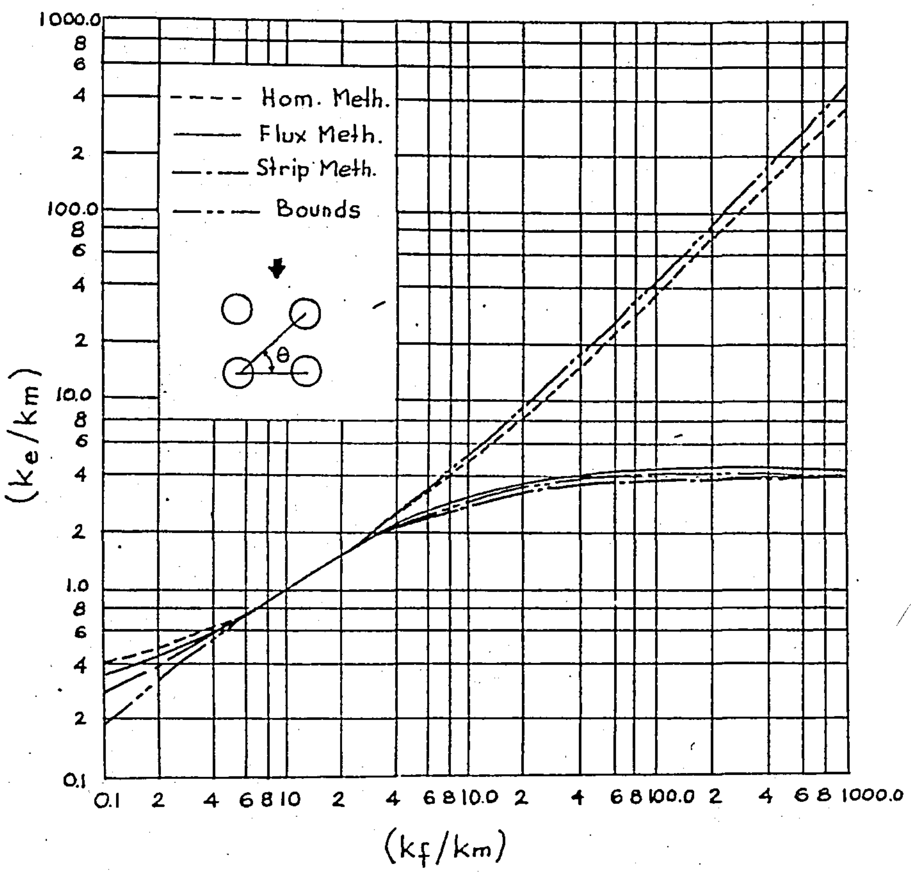


FIGURE 2.7 - Effective conductivities for rectangular array fiber composite (v_f/v_T = 0.6, θ = 45 deg.)

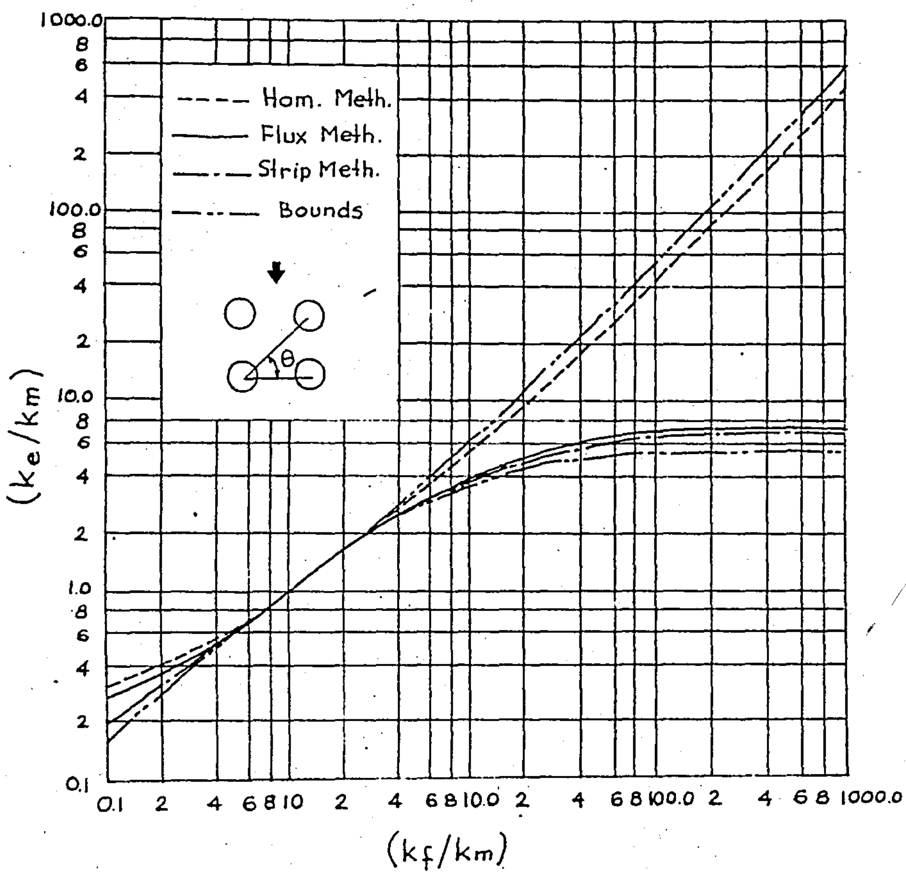
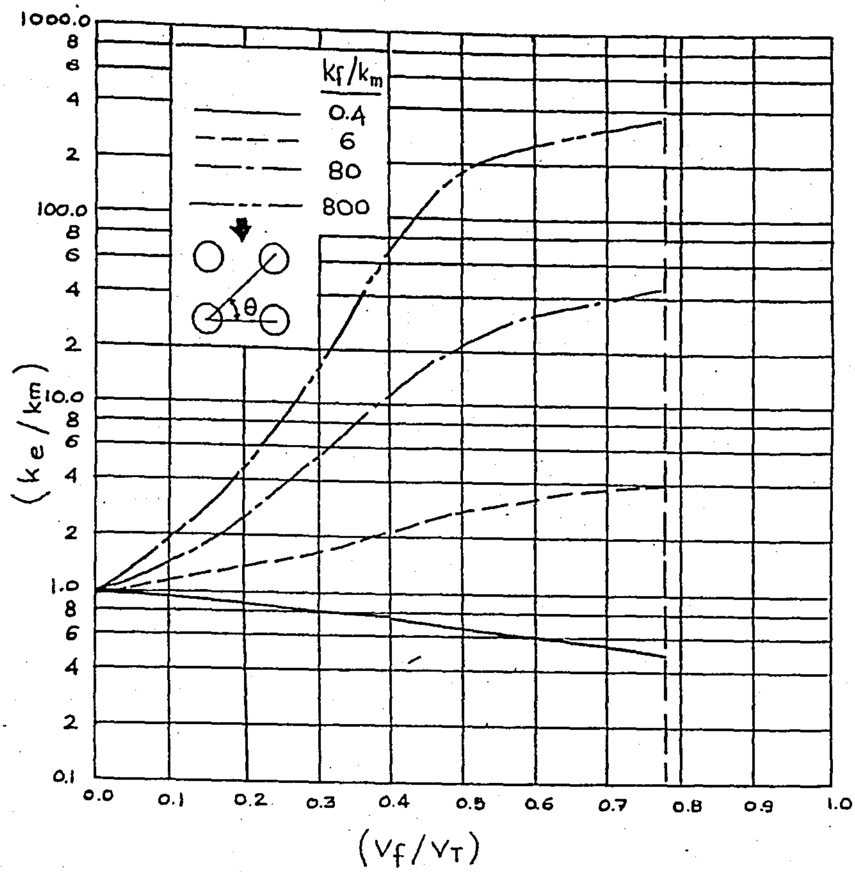
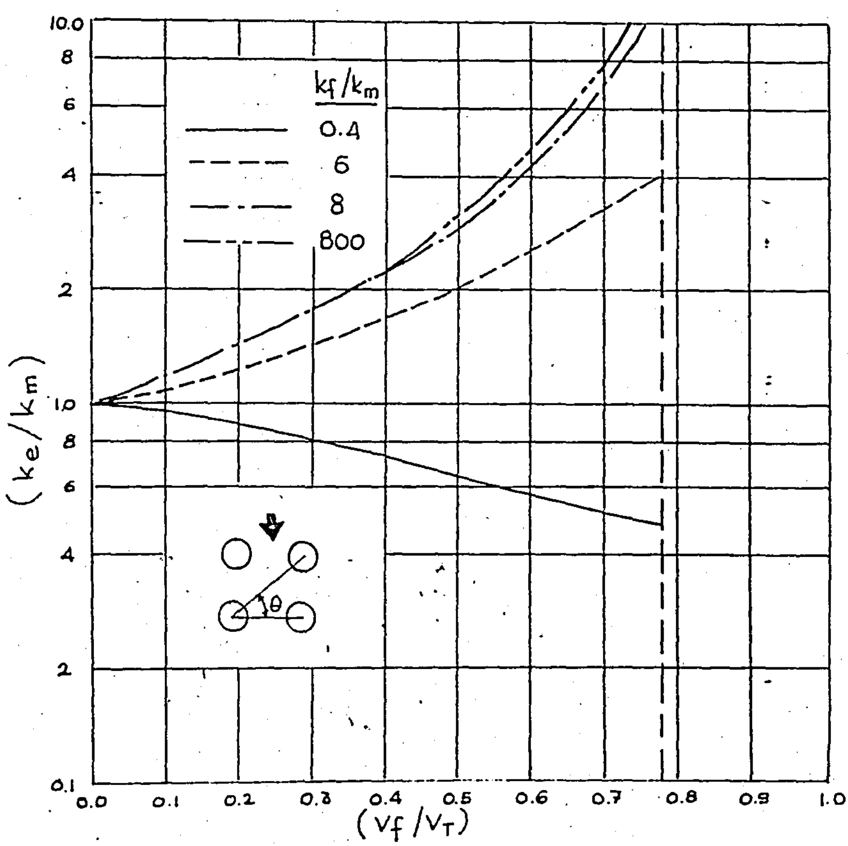


FIGURE 2.8 - Effective conductivities for rectangular array fiber composite (v_f/v_T = 0.7, θ = 45 deg.)



(a)



(b)

FIGURE 2.9 - Effective conductivities for rectangular array fiber composite (θ = 45 deg.)
a) Homogeneous Method
b) Flux Method

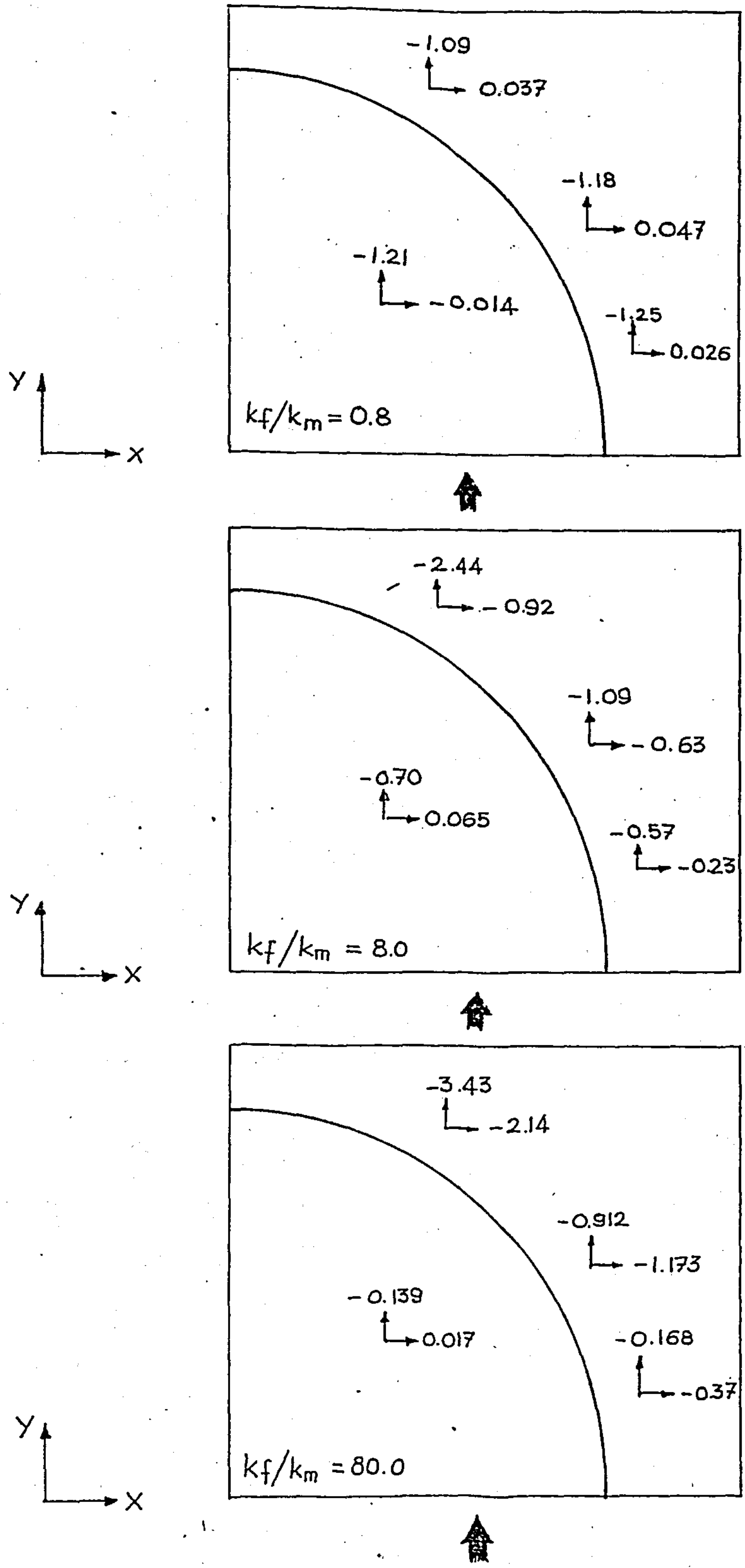


FIGURE 2.10 - Temperature gradient distribution for boundary conditions given by Eq. (2.4) for rectangular array fiber composite (v_f/v_T = 0.6, θ = 40 deg.)

2.3.2 Staggered Array (Figure 2.1-b)

The procedure applied for staggered geometry is the same as for rectangular geometry. The only difference is in the solution domains as shown in Figure 2.11, below.

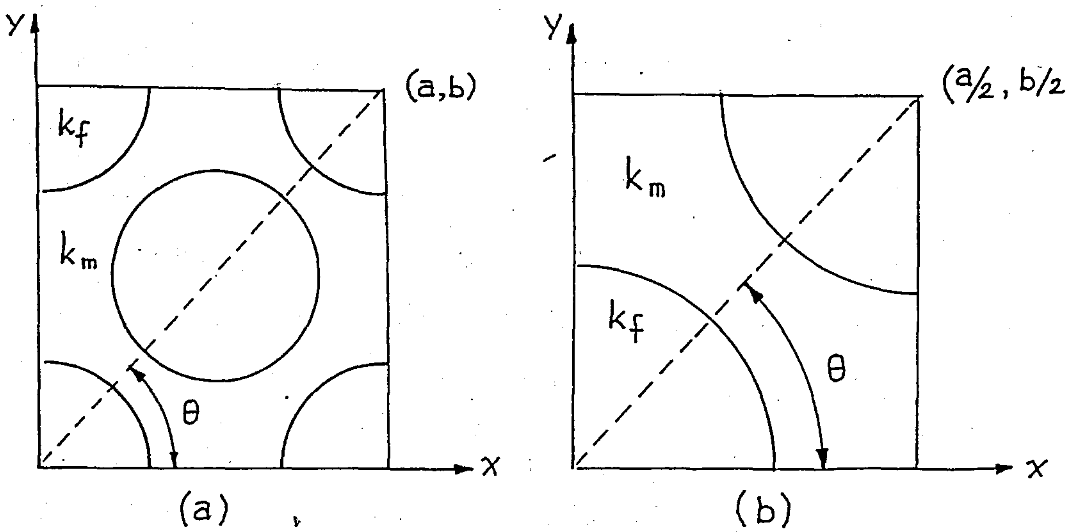


FIGURE 2.11 - Solution domains for staggered array.

Boundary condition given by Eq. (2.3) is applied over the domain shown in Figure 2.11-a. Boundary conditions given by Eqs. (2.4) and (2.5) are applied on the domain shown in Figure 2.11-b, which is one quarter of the domain shown in Figure 2.11-a. The expressions obtained by strip method for this case are given by Eqs. (A.8) and (A.9) in Appendix A. Since, bounds given by Eq. (2.2) is independent of the fiber geometry over the matrix, the values that is calculated for rectangular array fiber composites are the same for staggered array fiber composites with the same volume ratios, v_f/v_T and v_m/v_T.

The numerical results for effective thermal conductivity obtained from homogeneous method and flux method is tabulated in Tables 2.7-12 for volume ratios, v_f/v_T, of 0.5, 0.6 and 0.7 with different angles, θ. In Figures 2.12-17 the numerical results tabulated and the expressions given by Eqs. (A.8) and (A.9) are plotted as k_f/k_m versus k_{e-x}/k_m and k_f/k_m versus k_{e-y}/k_m. Additionally, in Figures 2.15-17, bounds obtained from Eq. 2.2 are also shown. Figures 2.18-a and 2.18-b show the trends in the numerical results obtained from homogeneous method and flux method, respectively, for arbitrarily chosen k_f/k_m values of 0.4, 6, 80 and 800 and the angle θ = 45 deg. as v_f/v_T ratio is bounded above by limit value, v_f/v_T = 0.785. Finally, in Figure 2.19 temperature gradient distribution at some points over the domain, which is shown in Figure 2.11-b, for one dimensional heat flow along y-direction is presented for staggered array with volume ratio of v_f/v_T = 0.6 and the angle θ = 40 deg.

TABLE 2.7 - FEM Solutions for Effective Conductivity

	Staggered Array: $v_f/v_T = 0.5$ θ= 35 deg.			
	Mesh (63,96) Flux B.C.'s		Mesh (63,96) Homogeneous B.C.'s	
k_f/k_m	k_{e-x}/k_m	k_{e-y}/k_m	k_{e-x}/k_m	k_{e-y}/k_m
0.1	0.465194	0.456237	0.488347	0.485576
0.2	0.537328	0.530650	0.555459	0.553358
0.4	0.670519	0.667161	0.680495	0.679425
0.6	0.790849	0.789508	0.795174	0.794747
0.8	0.900179	0.899876	0.901232	0.901136
1.0	1.00000	1.00000	1.00000	1.00000
2.0	1.39277	1.38828	1.41518	1.41385
4.0	1.87105	1.84982	2.03021	2.02444
6.0	2.15209	2.11578	2.51540	2.50604
8.0	2.33717	2.28897	2.94057	2.92855
10.0	2.46830	2.41077	3.33304	3.31903
20.0	2.79239	2.70869	5.10112	5.08188
40.0	2.99595	2.89365	8.38559	8.36289
60.0	3.07177	2.96212	11.6000	11.5760
80.0	3.11137	2.99779	14.7948	14.7701
100.0	3.13570	3.01969	17.9815	17.9564
200.0	3.18571	3.96460	33.8768	33.8508
400.0	3.21140	3.08765	65.6279	65.6015
600.0	3.22007	3.09542	97.3701	97.3435
800.0	3.22443	3.09933	129.110	129.083
1000.0	3.22705	3.10167	160.849	160.822

TABLE 2.8 - FEM Solutions for Effective Conductivity

	Staggered Array: $v_f/v_T = 0.5$ $\theta = 40$ deg.			
	Mesh (63,96) Flux B.C.'s		Mesh (63,96) Homogeneous B.C.'s	
k_f/k_m	k_{e-x}/k_m	k_{e-y}/k_m	k_{e-x}/k_m	k_{e-y}/k_m
0.1	0.461943	0.452765	0.488370	0.485212
0.2	0.535201	0.528334	0.555601	0.553192
0.4	0.669690	0.666220	0.680667	0.679424
0.6	0.790592	0.789201	0.795273	0.794770
0.8	0.900133	0.899818	0.901259	0.901145
1.0	1.00000	1.00000	1.00000	1.00000
2.0	1.39286	1.38812	1.41592	1.41423
4.0	1.87382	1.85116	2.03452	2.02677
6.0	2.15880	2.11973	2.52342	2.51040
8.0	2.34764	2.29550	2.95169	2.93461
10.0	2.48204	2.41956	3.34668	3.32646
20.0	2.81653	2.72472	5.12209	5.09330
40.0	3.02839	2.91544	8.41198	8.37726
60.0	3.10766	2.98631	11.6286	11.5916
80.0	3.14915	3.02329	14.8247	14.7864
100.0	3.17466	3.04600	18.0121	17.9731
200.0	3.22717	3.09265	33.9092	33.8686
400.0	3.25419	3.11661	65.6617	65.6203
600.0	3.26331	3.12469	97.4048	97.3631
800.0	3.26789	3.12875	129.149	129.104
1000.0	3.27065	3.13119	160.885	160.843

TABLE 2.9. - FEM Solutions for Effective Conductivity

	Staggered Array: $v_f/v_T = 0.6$ θ = 40 deg			
	Mesh (63,96) Flux B.C.'s		Mesh (63,96) Homogeneous B.C.'s	
k_f/k_m	k_{e-x}/k_m	k_{e-y}/k_m	k_{e-x}/k_m	k_{e-y}/k_m
0.1	0.379076	0.368329	0.405717	0.400320
0.2	0.461347	0.453224	0.481726	0.477664
0.4	0.613800	0.609633	0.624623	0.622568
0.6	0.752967	0.751277	0.757544	0.756724
0.8	0.881162	0.880775	0.882257	0.882072
1.0	1.00000	1.00000	1.00000	1.00000
2.0	1.48862	1.48255	1.51108	1.50839
4.0	2.14138	2.11079	2.30237	2.28975
6.0	2.56159	2.50693	2.93659	2.91498
8.0	2.85559	2.78069	3.48978	3.46097
10.0	3.07301	2.98146	3.99552	3.96101
20.0	3.64617	3.50431	6.21894	6.16811
40.0	4.03512	3.85413	10.2360	10.1732
60.0	4.18626	3.98903	14.1238	14.0562
80.0	4.26664	4.06054	17.9740	17.9039
100.0	4.31652	4.10483	21.8083	21.7365
200.0	4.42024	4.19675	40.9037	40.8286
400.0	4.47418	4.24445	79.0154	78.9385
600.0	4.49248	4.26062	117.109	117.031
800.0	4.50170	4.26876	155.197	155.120
1000.0	4.50725	4.27366	193.284	193.206

TABLE 2.10 - FEM Solutions for Effective Conductivity

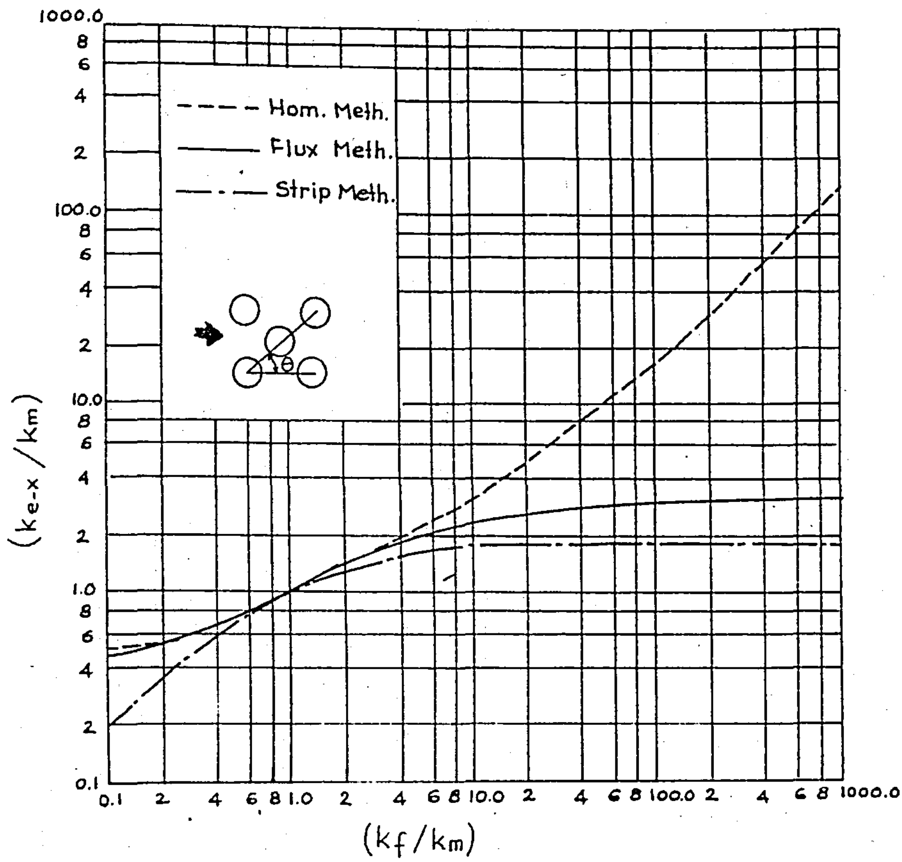
	Staggered Array: v_f/v_T = 0.5 θ = 45 deg.	
	Mesh (65,104) Flux B.C.'s	Mesh (65,104) Homogeneous B.C.'s
k_f/k_m	k_e/k_m	k_e/k_m
0.1	0.450454	0.480326
0.2	0.526679	0.549607
0.4	0.665428	0.677661
0.6	0.788893	0.794077
0.8	0.899751	0.900991
1.0	1.00000	1.00000
2.0	1.38698	1.41192
4.0	1.84518	2.01590
6.0	2.10881	2.49183
8.0	2.28038	2.90997
10.0	2.40100	3.29708
20.0	2.69589	5.05063
40.0	2.87883	8.32513
60.0	2.94654	11.5356
80.0	2.98181	14.7284
100.0	3.00345	17.9138
200.0	3.05691	33.8061
400.0	3.07063	65.5552
600.0	3.07831	97.2963
800.0	3.08216	129.035
1000.0	3.08448	160.774

TABLE 2.11 - FEM Solutions for Effective Conductivity

	Staggered Array: $v_f/v_T = 0.6$ $\theta = 45$ deg.	
	Mesh (65,104) Flux B.C.'s	Mesh (65,104) Homogeneous B.C.'s
k_f/k_m	k_e/k_m	k_e/k_m
0.1	0.365561	0.397308
0.2	0.451522	0.475565
0.4	0.609015	0.621595
0.6	0.751086	0.756348
0.8	0.880738	0.881987
1.0	1.00000	1.00000
2.0	1.48230	1.50713
4.0	2.10926	2.28298
6.0	2.50349	2.90237
8.0	2.77518	3.44328
10.0	2.97398	3.93905
20.0	3.48966	6.13300
40.0	3.83287	10.1278
60.0	3.96480	14.0066
80.0	4.03464	17.8520
100.0	4.07786	21.6833
200.0	4.18593	40.7727
400.0	4.21394	78.8820
600.0	4.22969	116.975
800.0	4.23761	155.065
1000.0	4.24237	193.152

TABLE 2.12 - FEM Solutions for Effective Conductivity

	Staggered Array: $v_f/v_T = 0.7$ $\theta = 45$ deg.	
	Mesh (125,216) Flux B.C.'s	Mesh (125,216) Homogeneous B.C.'s
k_f/k_m	k_e/k_m	k_e/k_m
0.1	0.259939	0.288549
0.2	0.359448	0.380639
0.4	0.540581	0.551350
0.6	0.705048	0.709470
0.8	0.857350	0.858386
1.0	1.00000	1.00000
2.0	1.61117	1.63116
4.0	2.52110	2.66187
6.0	3.18554	3.51664
8.0	3.69705	4.26604
10.0	4.10426	4.94590
20.0	5.32067	7.82265
40.0	6.29228	12.6947
60.0	6.70898	17.2501
80.0	6.94062	21.7035
100.0	7.08809	26.1113
200.0	7.40403	47.9252
400.0	7.57410	91.3091
600.0	7.63257	134.635
800.0	7.66217	177.946
1000.0	7.68005	221.251



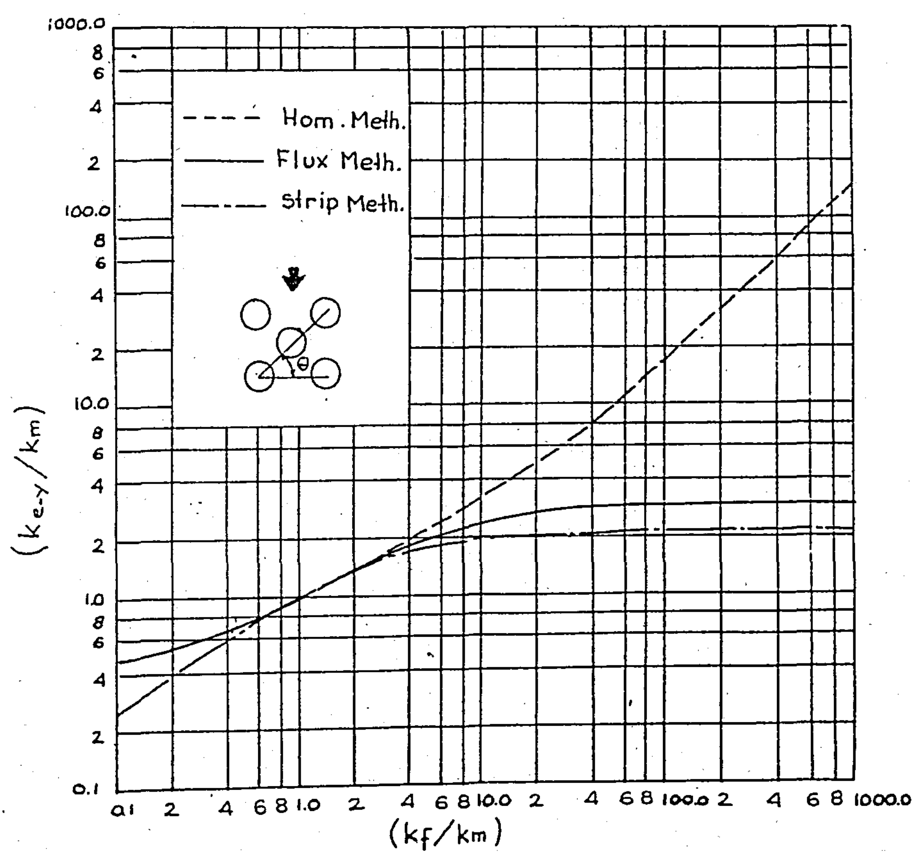


FIGURE 2.12 - Effective conductivities for staggered array fiber composite (v_f/v_T = 0.5, θ = 35 deg.)

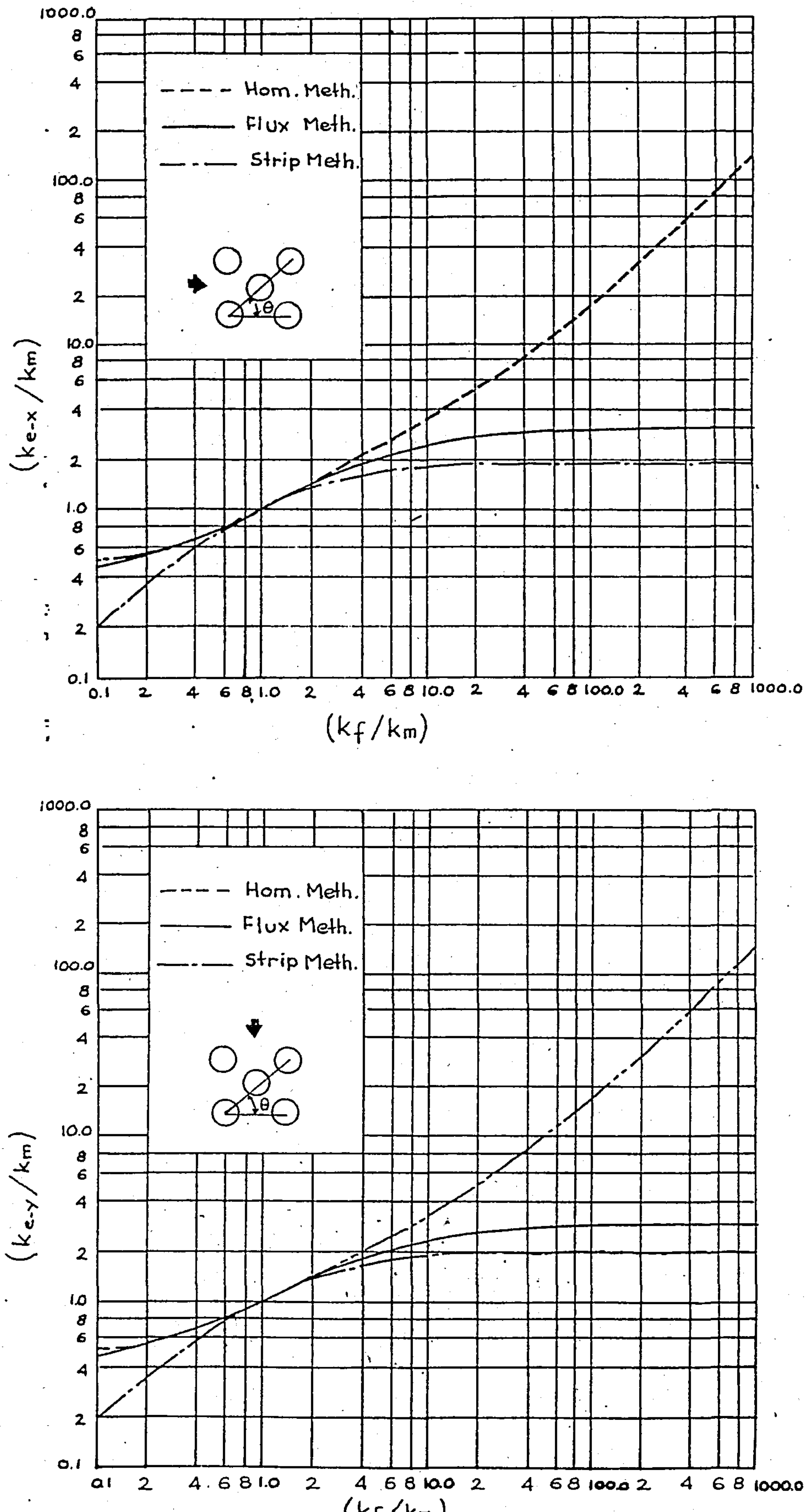


FIGURE 2.13 - Effective conductivities for staggered array fiber composite (v_f/v_T = 0.5, θ = 40 deg.)

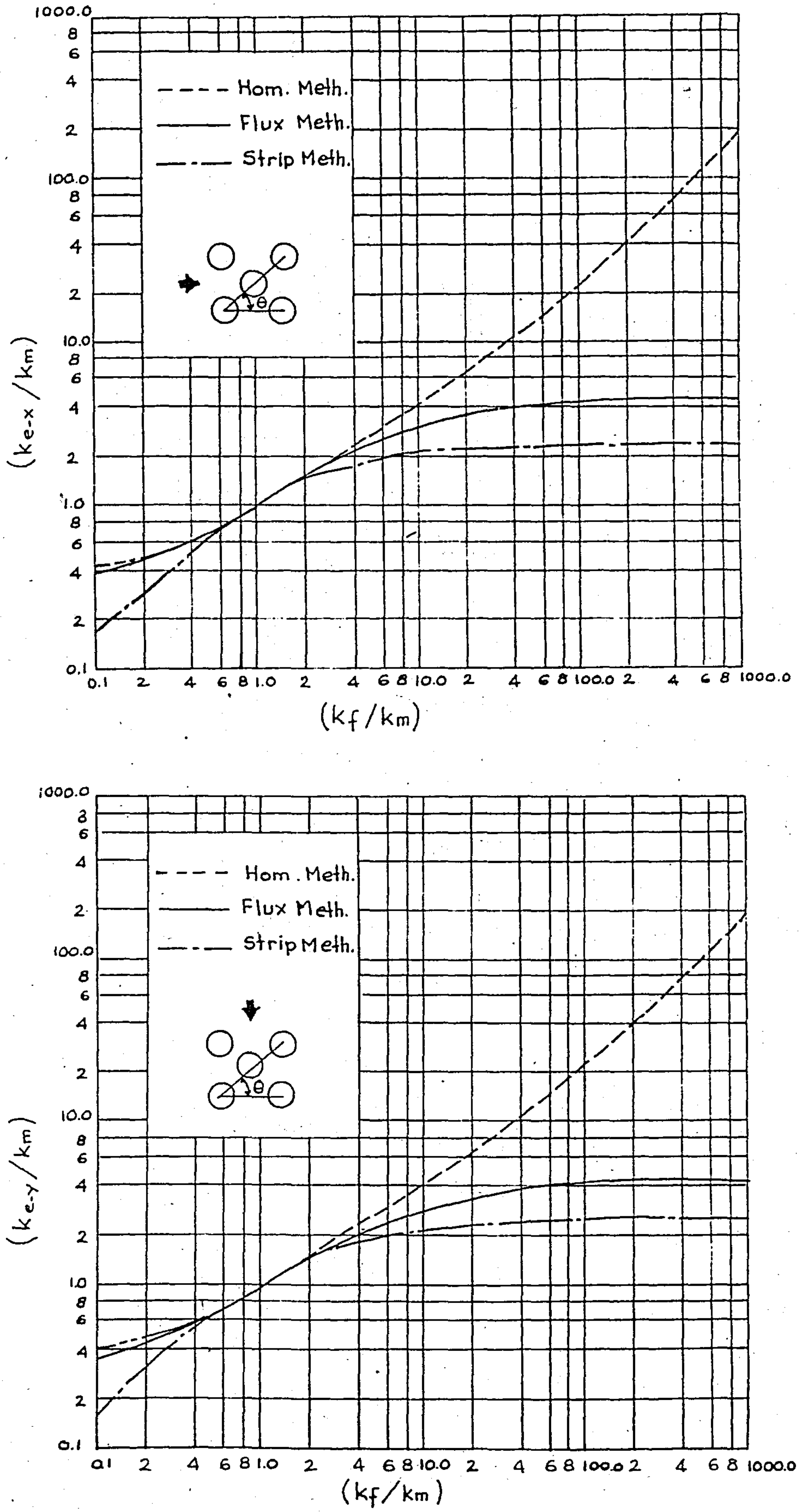


FIGURE 2.14 - Effective conductivities for staggered array fiber composite (v_f/v_T = 0.6, θ = 40 deg.).

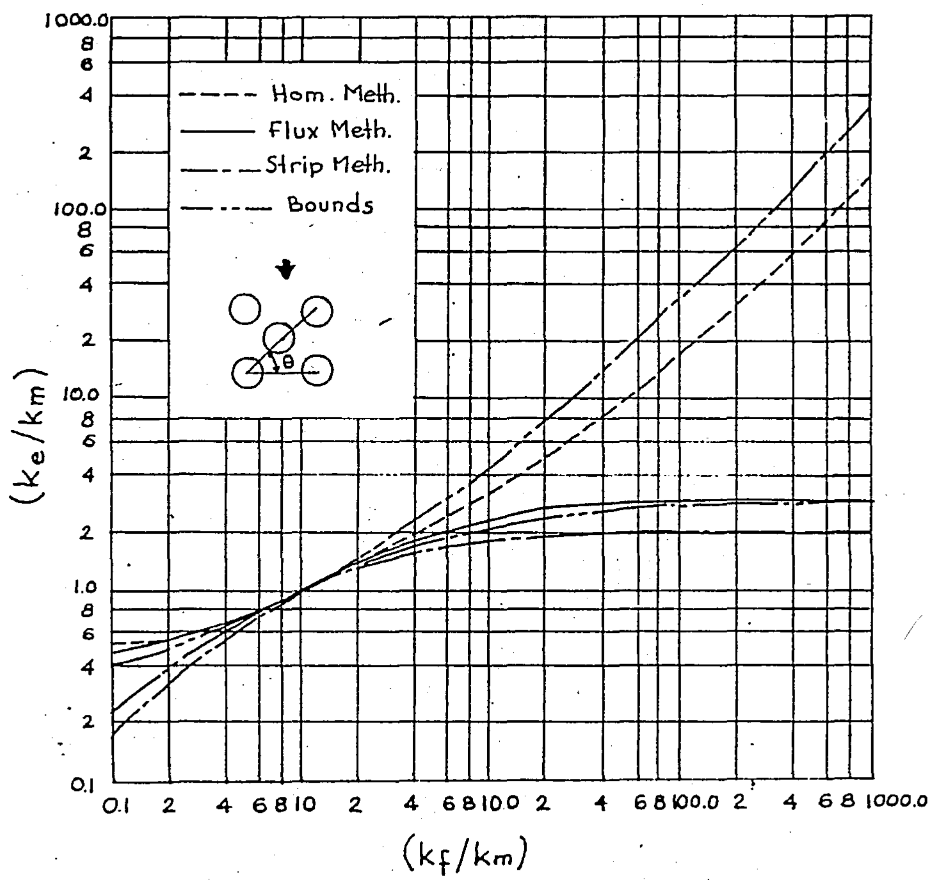


FIGURE 2.15 - Effective conductivities for staggered array fiber composite (v_f/v_T = 0.5, θ = 45 deg.)

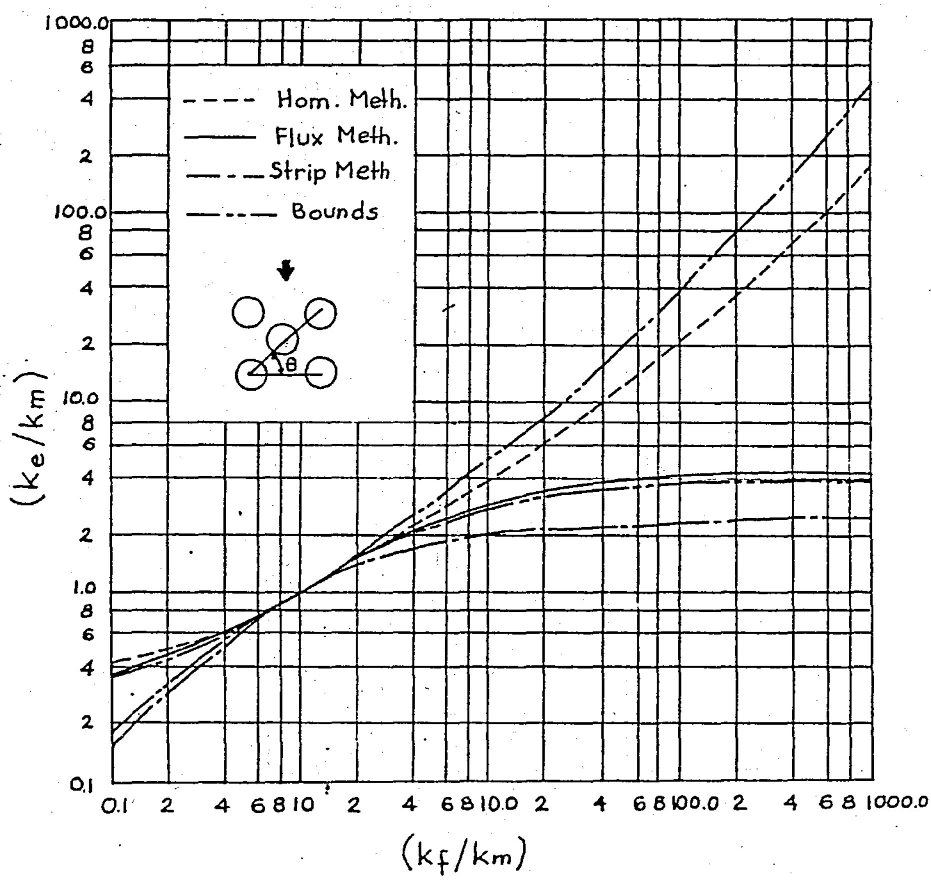


FIGURE 2.16 - Effective conductivities for staggered array fiber composite (v_f/v_T = 0.6, θ = 45 deg.)

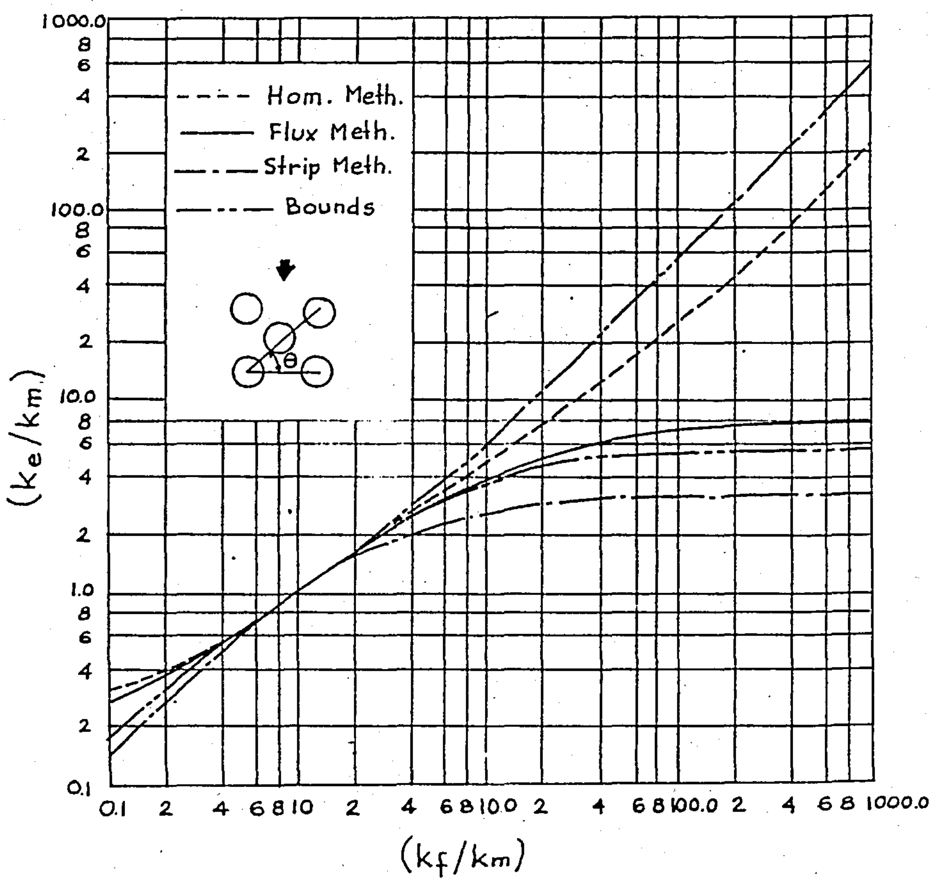


FIGURE 2.17 - Effective conductivities for staggered array fiber composite (v_f/v_T = 0.7, θ = 45 deg.)

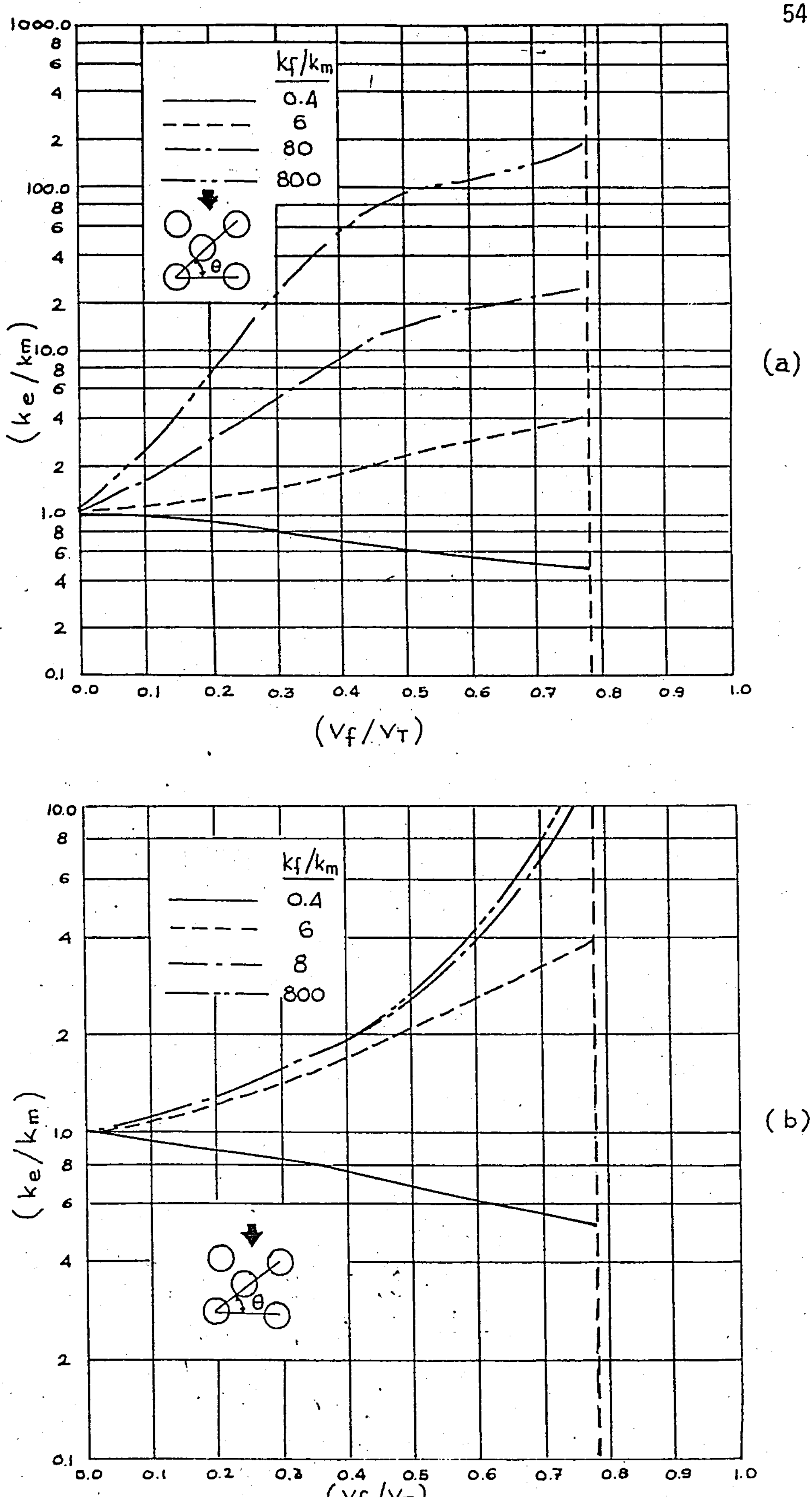


FIGURE 2.18 - Effective conductivities for staggered array fiber composite (θ = 45 deg.)
a) Homogeneous Method
b) Flux Method

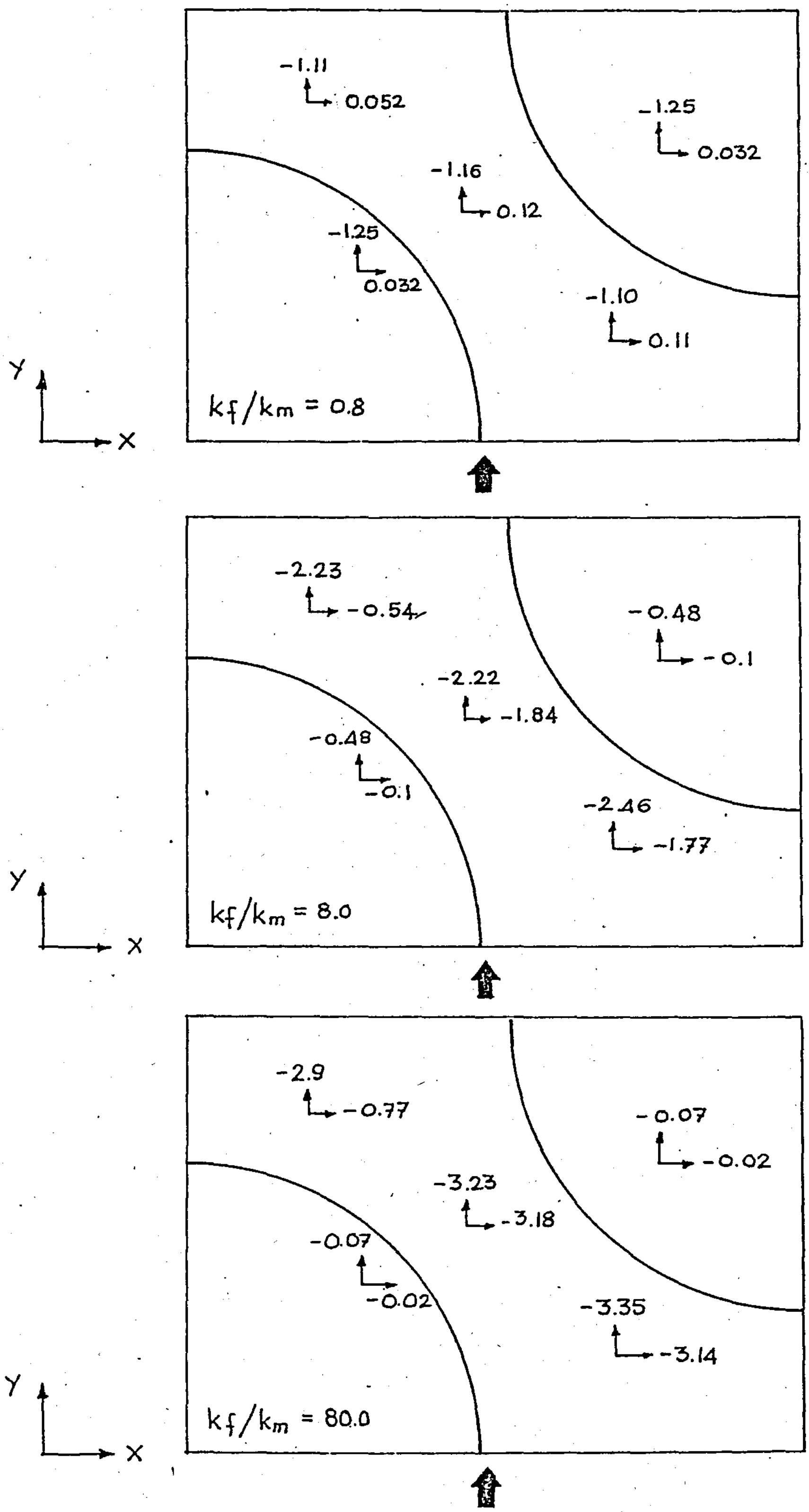


FIGURE 2.19 - Temperature gradient distribution for boundary conditions given by Eq. (2.4) for staggered array fiber composite (v_f/v_T = 0.6, θ = 40 deg.)

III. DISCUSSION OF RESULTS

As stated in Section 2.2, the primary concern in this analysis is the comparison of two methods of calculating effective thermal conductivity, namely, homogeneous method and flux method.

Inspection of the general trends of the numerical results indicates that the effective thermal conductivity values, calculated by applying two methods, differ greatly, especially for $k_f/k_m >> 1$, i.e. the fibers being much more conductive than the matrix. And, in the limit as k_f/k_m ratio goes to infinity, the effective conductivity values seem to go infinity for homogeneous method, whereas that for flux method seems to approach to a definite value. However, the methods give almost similar results for k_f/k_m ratio in the range of 0.4 to 2.0.

Again, as discussed in Section 2.2, other than numerical calculation of effective thermal conductivities, we have given some bounds (Eq. 2.2) on transversely isotropic fiber composites which correspond to geometries with angle, $\theta = 45$ deg. in our cases. These bounds are introduced in reference [10] as best possible bounds in terms of phase volume fractions and properties for effective thermal conductivity. The examination of the Figures 2.6-8 and 2.15-17 indicates that, the numerical results for effective thermal conductivity do not agree with these bounds for $k_f/k_m < 1$. However, for $k_f/k_m > 1$, the numerical results fall between these two bounds where the value obtained by homogeneous method are close to the upper bound and the values obtained by flux method are close to lower bound.

The analytic expressions obtained by strip method (See Appendix A) are plotted in Figures 2.3-8 and 2.12-17 which are introduced as a supplement to the flux method. In general, the aggrement is good as expected especially for rectangular array fiber composites, but, the important point is that the expressions obtained by strip method reflect the trend in the numerical results obtained by flux method almost exactly.

Another view to numerical results, obtained by homogeneous method and flux method is presented for the isotropic cases, i.e. the angle $\theta = 45$ deg., for randomly chosen k_f/k_m ratios of 0.4, 6, 80 and 800 in Figures 2.9 and 2.18. Although, the curves are plotted based on three points, i.e. v_f/v_T ratios of 0.5, 0.6 and 0.7, and the rest is interpolated, the trend shows that effective thermal conductivity values obtained by flux method are independent of volume ratio, v_f/v_T, as $k_f/k_m \gg 1$. However, in the case of homogeneous method the values are highly dependent on the volume ratios, v_f/v_T.

In Tables 2.1-6 and 2.7-12, we have tabulated numerical results obtained by using FEM with the mesh size information, e.g. Mesh (87,136); number of nodal points = 87, number of elements = 136. However, as can be seen, the mesh sizes are almost different for each case. In order to obtain accurate numerical results by FEM, the aspect ratio of the triangular elements, i.e. length ratio of height to base, should be of the order of one. Thus, for each geometry, a different mesh size is applied to keep the aspect ratio of one as close as possible.

IV. CONCLUSIONS AND RECOMMENDATIONS

An analysis concerning effective conductivities of composite materials has been performed for the unidirectional fiber composites, such that fibers, uniformly dispersed in the matrix, form rectangular and staggered patterns. On the basis of theoretical and numerical results presented, we reach the following conclusions concerning the applicability of the methods utilized and the effect of fiber geometry on the effective thermal conductivity.

As stated in the discussion of results, the effective conductivity values which are calculated by homogeneous method and flux method differ greatly from each other especially for large values of k_f/k_m, i.e. $k_f/k_m >> 1$. The values obtained by homogeneous method keep increasing, whereas that of flux method seems to approach to a definite value as k_f/k_m ratio goes to infinity. At first sight, the trend in numeric values obtained by homogeneous method may seem to be reasonable, because the conductivity of one of the constituent phases dominates over the other one, as k_f/k_m goes to infinity, and the value of the larger conductivity determines the value of the effective thermal conductivity as given by mixture theory which is defined by the formula

$$k_e = \frac{1}{v_T} (k_f v_f + k_m v_m)$$

where k_e is the effective conductivity, v_T is the total volume and subscripts f and m refer to fiber and matrix, respectively. But this is the case for constituent phases that are connected in parallel (See Appendix A, electrical network analogy to heat transfer problems) in the direction of heat flow. For our geometrical arrangements, we should expect the effective thermal conductivity to approach to a definite value as k_f/k_m ratio goes to infinity, because, in the geometries that we analyzed, the fibers do not touch each other. So, it may be concluded that the flux method gives reasonable values in comparison with homogeneous method.

The large difference between numerical values obtained by two methods may be a consequence of the RVE geometry that we have analyzed. The precise definition of RVE is that it should be large compared to typical phase region dimension, e.g. fiber diameters and spacings, (See Section 1.1). But, the size of the RVEs, that are analyzed in this work, are almost of the same order of magnitude as fiber diameters. This cause violation of the assumptions on which theory of heterogeneous materials (See Section 1.2) is based. So, this may be a reason for homogeneous method, that is inferred from the theory of heterogeneous materials, to give unexpected results.

For k_f/k_m ratio between 0.4 and 2.0 two methods give similar results. But, this is expected, because the methods must give 1 as k_e/k_m ratio for $k_f/k_m = 1$. Thus, this makes the methods to give similar results in the neighbourhood of $k_f/k_m = 1$, because of the continuity requirement, that is, a continuous curve maps the neighbour points to the points of the same order of magnitudes.

Strip method gives good results especially for rectangular array fiber composites. The difference between the values comes from the assumption made in the derivation of the method. We assume that there is no flux component normal to principal direction of heat transfer (PDHT). But, as seen from the Figures 2.10 and 2.19, in the flux method, the temperature gradient vector has non-zero components normal to PDHT because of the geometry. The normal component of the temperature gradient to PDHT increases with increasing k_f/k_m ratio just like the trend in the effective conductivity ratios, k_e/k_m. calculated by strip method and flux method.

The value of the effective conductivity of a composite material is bounded by the conductivities of each constituent phases. The location of the effective conductivity within this interval greatly depends on geometry of the fiber-matrix distribution. The effect of geometry on the values of effective conductivity depends on the length ratio of each phase in the strip which is taken along PDHT that effective thermal conductivity is calculated.

As it can be seen from Figures 2.6-8 and 2.15-17, bounds given by Eq. 2.2 do not agree with the numerical results for effective thermal conductivity for values of $k_f/k_m < 1$. Even though, no remark has been made on the applicability limits of these bounds in the reference [10] that these bounds are introduced, the explicitly defined upper and lower bounds are reversed for the case, $k_f/k_m < 1$. So, these bounds may be applicable only for the interval, $k_f/k_m > 1$.

Although, the finite element method is a powerful technique that gives a great freedom to users to pursue more fundamental problems, the input demanded is very difficult to prepare for realistic applications. That is why, this analysis is limited to some particular types of geometries. So, for further works on this subject, we recommend to utilize an automatic mesh generation scheme together with the computer program given in Appendix B to ease the data preparation task.

For engineering applications, the numerical results plotted in Figures 2.3-8 and Figures 2.12-17 are easy to use, because all variables are in the dimensionless form and cover a wide range of thermal conductivity ratios, k_f/k_m. The procedure to obtain effective conductivity values from these graphs is outlined below.

Step 1. Evaluate the fiber volume fraction, v_f/v_T, for the geometries given in Figures 2.2 and 2.11 with a known angle θ.

Step 2. Calculate the ratio of fiber thermal conductivity to matrix thermal conductivity ratio, k_f/k_m.

Step 3. Select the curve that corresponds to the geometrical arrangement of the case that is of interest.

Step 4. Obtain the ratio of effective thermal conductivity to matrix conductivity ratio, k_e/k_m, from the curve being selected.

APPENDICES

APPENDIX A
THE STRIP METHOD

The idea behind this method is simply the electrical network analogy to heat transfer problems that may be found in any standard heat transfer text book.

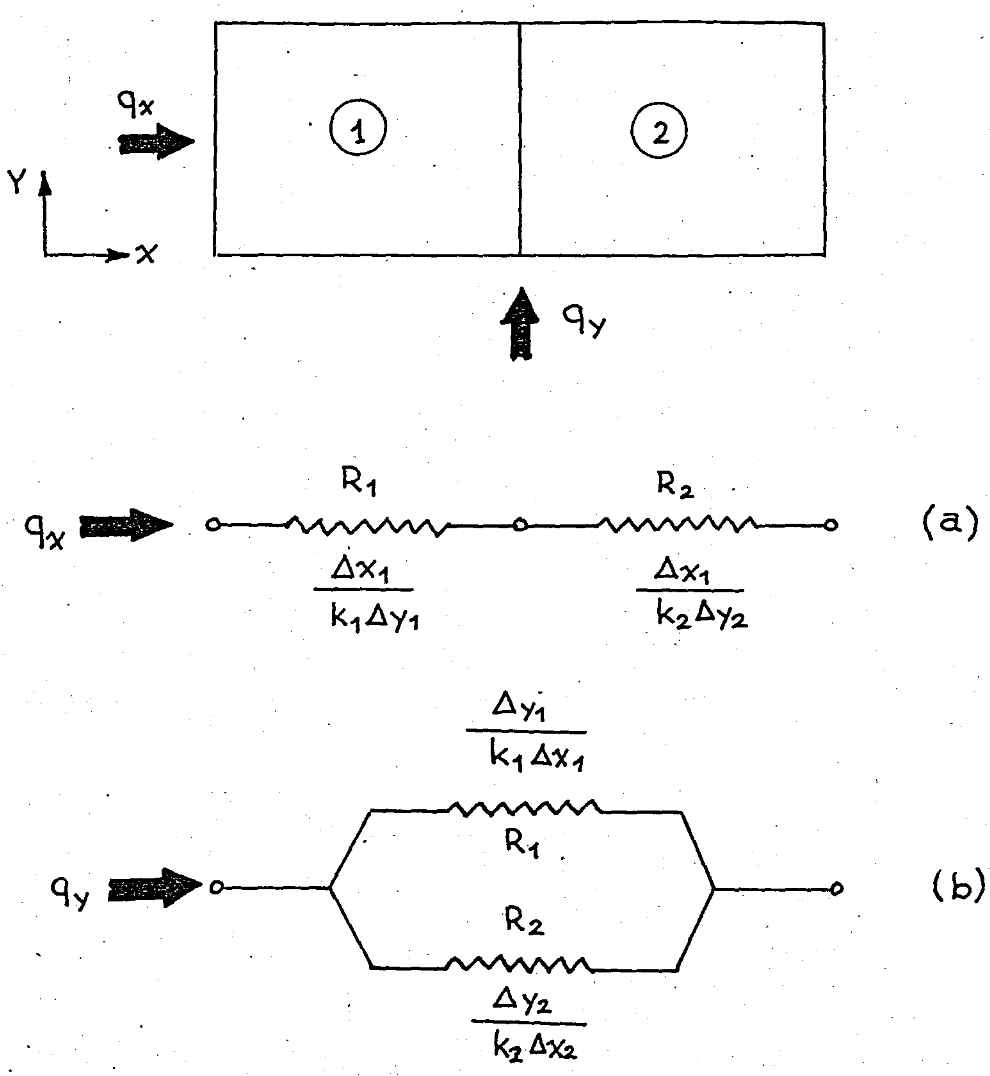


FIGURE A.1 - Series and parallel one-dimensional heat transfer through a composite wall and electrical analogy.

The equivalent resistivity on effective conductivity of the networks given by Figure (A.1) are

$$k_{e-x} = \Delta x_1 + \Delta x_2 / \frac{\Delta x_1}{k_1} + \frac{\Delta x_2}{k_2} \quad , \tag{A.1a}$$

$$k_{e-y} = \Delta x_1 k_1 + \Delta x_2 k_2 / \Delta x_1 + \Delta x_2 \quad , \tag{A.1b}$$

respectively, where Δx and Δy are thicknesses of the wall along x and y directions. Now, let us turn to our problem of evaluating effective conductivity for rectangular and staggered array fiber composites.

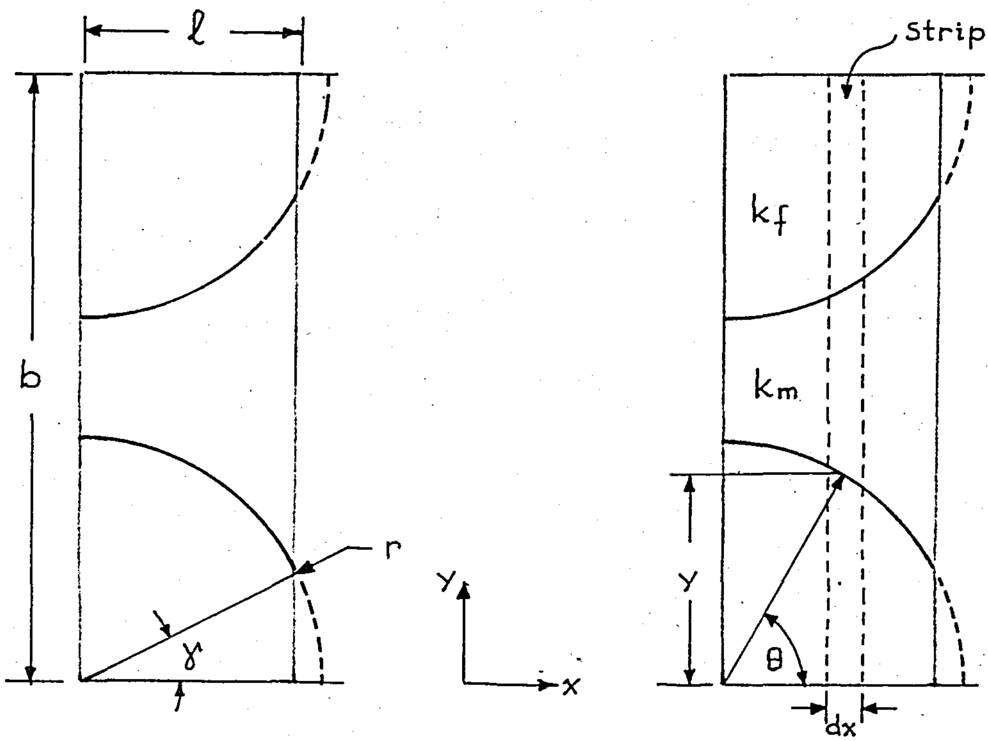


FIGURE A.2 - Model for the strip method.

Consider the strip of infinitesimal width, dx, taken along a segment of fibrous composite material as shown in Figure A.2. The effective conductivity of the strip along y direction as analogous to Eq. (A.1a) is

$$k_{e-y} = \frac{b}{\dfrac{2y}{k_f} + \dfrac{b - 2y}{k_m}} \tag{A.2}$$

Now, we have strips which are connected in parallel along x direction with conductivities k_{e-y}. Then, the effective conductivity of the system, as defined in Eq. (A.1b), is given by the integral, below.

$$I = \int \frac{(b/\ell)dx}{\dfrac{2y}{k_f} + \dfrac{b - 2y}{k_m}}$$

or in polar coordinates

$$I = \int \frac{\dfrac{-bk_f k_m \sin\theta}{\cos\gamma}\, d\theta}{2r \sin\theta(k_m - k_f) + bk_f}$$

In order to derive an expression in general form, we take the limits of integration from $\pi/2$ to a particular angle, γ. By nondimensionalizing the integral, we obtain

$$f(\gamma,A) = \frac{I}{k_m} = \int_{\gamma}^{\pi/2} \frac{\sin\theta d\theta}{(A\sin\theta + 1)\cos\gamma} \tag{A.3}$$

where

$$A = 2\,\frac{r}{b}\left(\frac{k_m}{k_f} - 1\right).$$

By integrating [10] Eq. (A.3), we get the expression,

$$f(\gamma,A) = \frac{1}{A\cos\gamma}\Big[\Big(\frac{\pi}{2} - \gamma\Big) + \frac{2}{\sqrt{1 - A^2}}\Big(\arctan \frac{\tan \gamma/2 + A}{\sqrt{1 - A^2}}$$

$$- \arctan \frac{1 + A}{\sqrt{1 - A^2}}\Big)\Big] \quad \text{for} \quad A^2 < 1 \qquad (A.4)$$

$$f(\gamma,A) = \frac{1}{A\cos\gamma}\Big[\Big(\frac{\pi}{2} - \gamma\Big) + \frac{1}{\sqrt{A^2 - 1}}\Big(\ln\left|\frac{\tan \gamma/2 + A - \sqrt{A^2 - 1}}{\tan \gamma/2 + A + \sqrt{A^2 - 1}}\right|$$

$$- \ln\left|\frac{1 + A - \sqrt{A^2 - 1}}{1 + A + \sqrt{A^2 - 1}}\right|\Big)\Big] \quad \text{for} \quad A^2 > 1 \qquad (A.5)$$

$$f(\gamma, A=1) = \frac{1}{\cos\gamma}\Big[\Big(\frac{\pi}{2} - \gamma\Big) - \tan\Big(\frac{\pi}{4} - \frac{\gamma}{2}\Big)\Big] \quad \text{for } A = 1 \qquad (A.6)$$

For limit values, as k_f/k_m ratio goes to infinity, Eq. (A.4) will be applicable for

$$A = -2(r/b).$$

Now, we are ready to evaluate an expression for effective conductivity for two types.

Rectangular Array

We will divide the RVE (Fig. 2.2-a) into three parts as shown below.

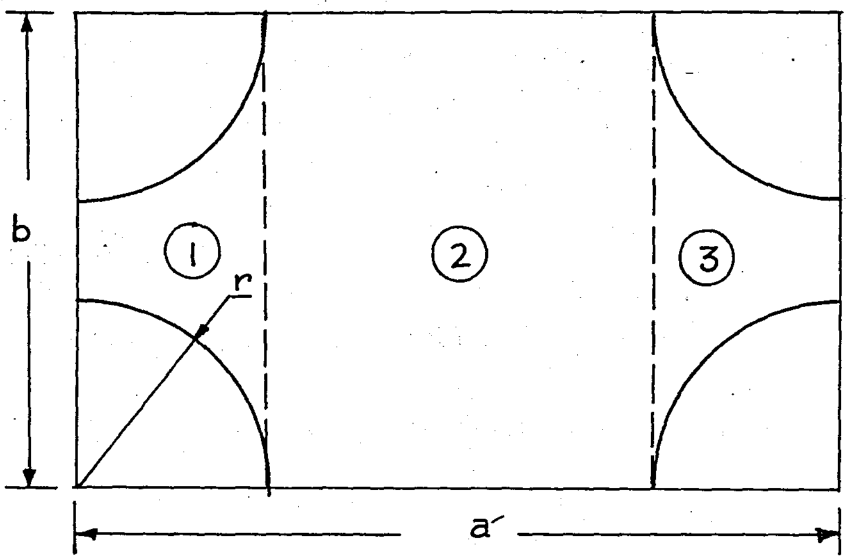


FIGURE A.3

For parts (1) and (3)

$$k_{e-y}/k_m = f(\gamma=0,A) \quad ,$$

and for part (2)

$$k_{e-y}/k_m = 1 \ ,$$

which are all connected in parallel. Then, the effective thermal conductivity for whole RVE, as defined in Eq. (A.1b), is

$$k_{e-y}/k_m = 2(r/a)(f(\gamma=0,A) - 1) + 1 \qquad \text{(A.7)}$$

Geometric parameters for rectangular array case are

$$\frac{r}{a} = \sqrt{\frac{v \tan\theta}{\pi}} \quad ; \quad \frac{r}{b} = \sqrt{\frac{v}{\pi \tan\theta}} \quad ,$$

where v and θ are defined in Section (2.3). For effective conductivity along x direction, it is enough to interchange a and b in all equations, above.

Staggered Array

We will divide the problem into two cases.

i) Overlapping occurs within the RVE. Then, we will divide the RVE (Figure 2.11-a) into six parts, as shown in Figure A.4.

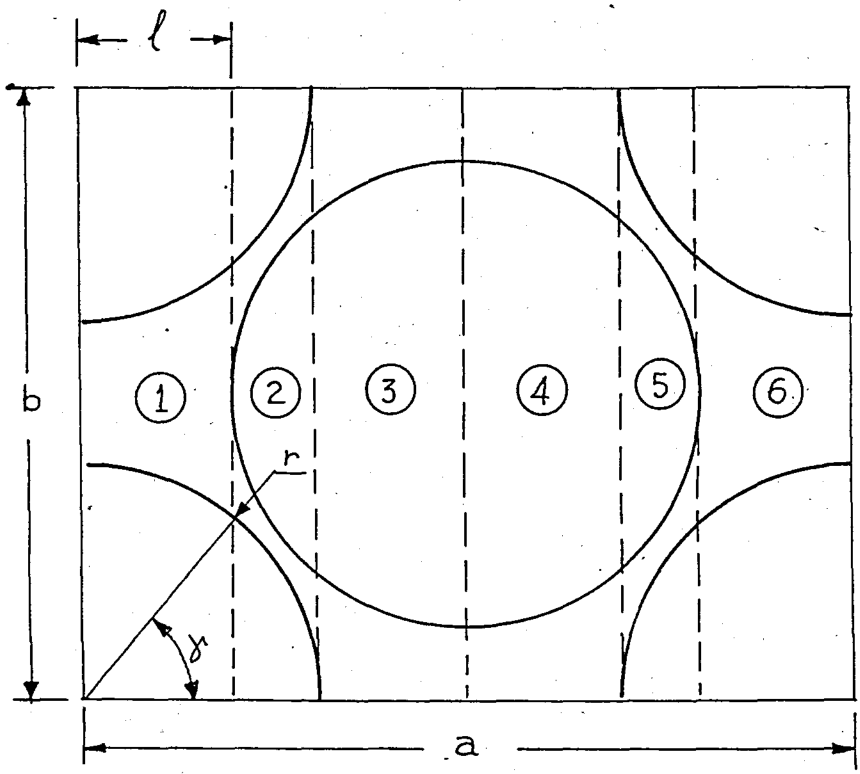


FIGURE A.4

For parts (1), (3), (4) and (6)

$$k_{e-y}/k_m = f(\gamma, A) \ .$$

But, for parts (2) and (5), the integral obtained is very difficult to evaluate. Thus, we will approximate the fiber segment as shown in Figure A.5.

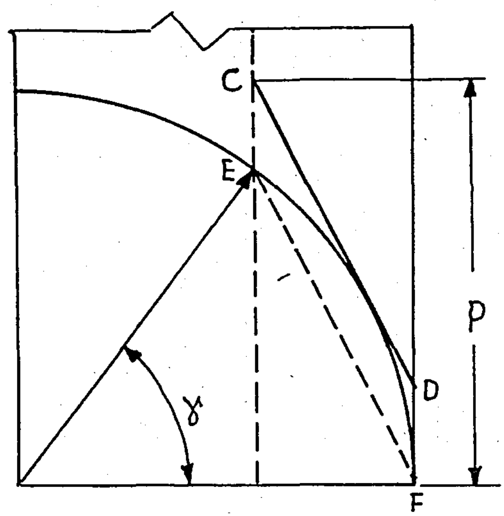


FIGURE A.5

where

$$p = r(\tan \gamma/4 + \sin\gamma)$$

we will take the line segment, $\overline{CD}$, which is parallel to the line segment, $\overline{EF}$, as interface of fiber and matrix phases. Then, effective conductivity is

$$k_{e-y}/k_m = g(\gamma,A) = 1/A(\tan \gamma/4 + \sin\gamma) + 1.$$

Now, we are ready to construct an expression for effective conductivity for whole RVE, which is

$$k_{e-y}/k_m = (r/a)(4\cos\gamma f(\gamma,A) + 2(1 - \cos\gamma)g(\gamma,A)) , \quad \text{(A.8)}$$

where

$$\gamma = \text{arc}\ \cos\left(\frac{a}{2r} - 1\right)$$

ii) No overlapping occurs within the RVE (Figure 2.11-a). Again, we will divide region into six parts as shown below.

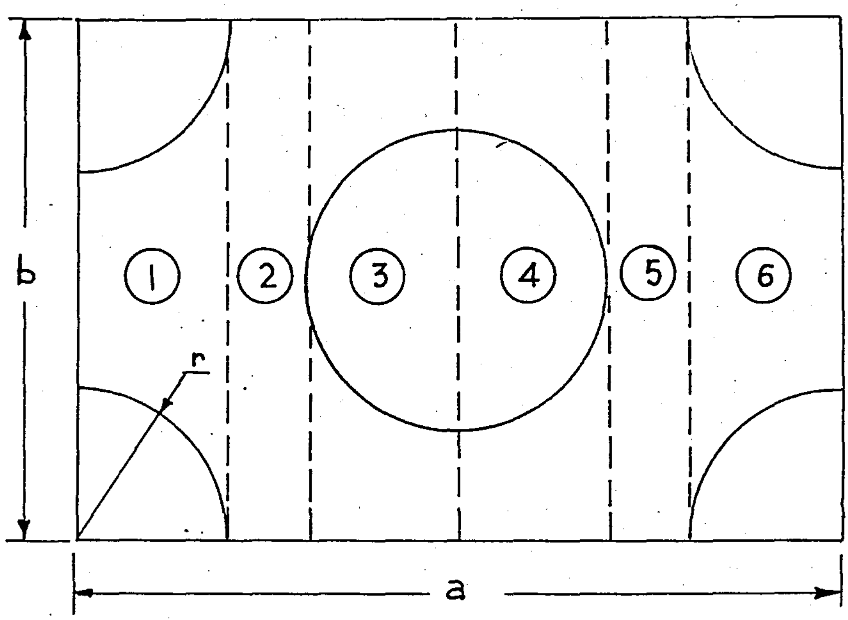


FIGURE A.6

For parts (1), (3), (4) and (6)

$$k_{e-y}/k_m = f(\gamma=0,A) ,$$

and for parts (2) and (5)

$$k_{e-y}/k_m = 1.$$

Then, effective conductivity for whole RVE is

$$k_{e-y}/k_m = 4(r/a)(f(\gamma=0,A) - 1) + 1. \qquad (A.9)$$

For staggered array case, geometric parameters are,

$$\frac{r}{a} = \sqrt{\frac{v \tan\theta}{2\pi}} \quad ; \quad \frac{r}{b} = \sqrt{\frac{v}{2\pi \tan\theta}} \quad ,$$

where v and θ are defined in Section (2.3). Similarly, for effective conductivity along x direction, we will interchange a and b in all equations, above.

APPENDIX B
COMPUTER CODE FOR FEM

In this part, we will describe a Fortran program for Finite Element Method to solve the two dimensional anisotropic steady state heat conduction equation (Eq. 2.1) with prescribed temperature and/or insulated surface boundary conditions using triangular elements of 3-noded, 6-noded and 10-noded types with first order continuity interpolation functions. CDC CYBER 170/815 CPU time of the code run with the associated sample data is 1.819 seconds.

The main program calls the seven following subroutines as shown in Figure B.1.

DATA : Reads the program input.

STIFF1 : Evaluates the element stiffness matrices and element loads by calling shape function routine SFRT, and jacobian and global derivatives routine JACOB.

ASSEMB : This routine assembles the element stiffnesses and applied loads to form the global stiffness matrix and force vector.

GREDUC : This routine reduces the global stiffness equations by direct gaussian elimination.

BAKSUB : Performs the back substitution phase.

FLUXES : Evaluates temperature gradient distribution over each element and averaged heat flux and temperature gradient over the domain by calling SFRT and JACOB.

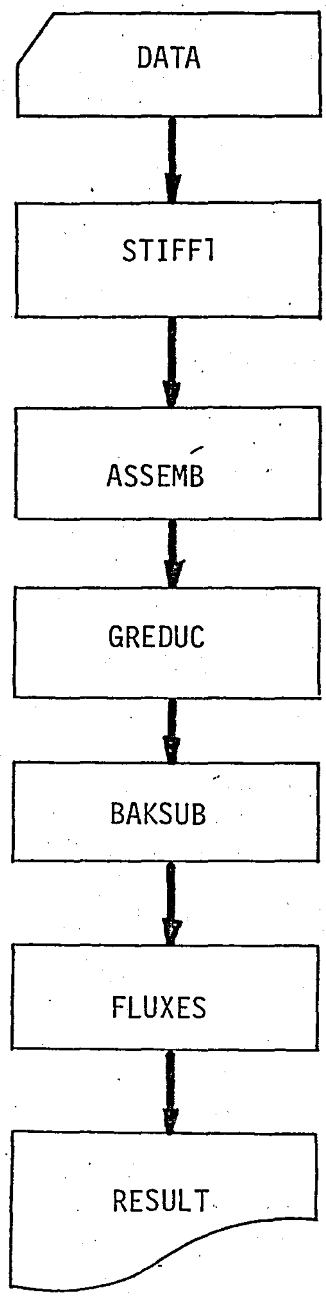


FIGURE B.1 - Flowchart for the FEM program.

RESULT : Writes the calculated results.

Computer Code for FEM is given in the following part with terminology and data preparation information.

TERMINOLOGY

AREAE	: The area of triangular element.
ASLOD(NSVAB)	: The global vector of applied loads.
ASTIF(MSTIF)	: The global stiffness matrix stored in vector form.
BVALU(INODE)	: Global derivative with respect to x.
COORD(NPOIN,NDIME)	: Coordinates of nodal points.
CVALU(INODE)	: Global derivative with respect to y.
DERIV(NDIME,INODE)	: Shape function derivatives.
ESTIF(NEVAB,NEVAB)	: The element stiffness matrix.
ETA(LINT)	: Coordinate of integration points.
EXI(LINT)	: Coordinate of integration points.
FIXED(NSVAB)	: Prescribed temperature values.
FLUX(IELEM,NDIME)	: Temperature gradient over elements.
FLUXX	: Averaged temperature gradient in x-direction.
FLUXY	: Averaged temperature gradient in y-direction.
GRADX	: Averaged heat flux in x-direction.
GRADY	: Averaged heat flux in y-direction.
ICODE(NDOFN)	: Fixity code for each degree of freedom of a restrained load-local usage 1 Temperature value corresponding to the degree of freedom is prescribed 0 Free degree of freedom.

LINT	: Number of integration points.
LNODS(NELEM,NNODE)	: Element node numbers listed for each element.
MATNO(NELEM)	: Material set number for each element.
MSTIF	: Total number of stiffness terms in the banded global stiffness matrix.
NBAND	: Maximum half-bandwidth for problem.
NBOUN	: Total number of boundary points, i.e. nodal points at which one or more degrees of freedom are restrained.
NDIME	: Number of coordinate components.
NDOFN	: Number of degrees of freedom per nodal point.
NELEM	: Total number of elements.
NEVAB	: Number of variables per element.
NPOIN	: Total number of nodal points.
NPROP	: The number of material parameters.
PROPS(NMATS,NPROP)	: Material properties for each material set.
REACT(NSVAB)	: Reactions for each degree of freedom.
DISP(NPOIN,NDOFN)	: The nodal temperature values.
TREAC(NPOIN,NDIME)	: The nodal reactions.
WEIGHT(LINT)	: Weights for numerical integration.

DATA PREPARATION INFORMATION

(For 3-noded triangular elements)

<u>CARD SET 1:</u> Title card (20A4)-one card. Title of the problem limited to 72 alphanumeric characters.

<u>CARD SET 2:</u> Control Card (16I5) - one card.

Cols. 1-5	NPOIN	Total number of nodal points.
6-10	NELEM	Total number of elements.

11-15	NBOUN	Total number of restrained boundary points at which the value of the degree of freedom is prescribed.
16-20	NMATS	Total number of different materials.
21-25	NPROP	Number of independent properties per material (=2).
26-30	NNODE	Number of nodal points per element (=3).
31-35	NDOFN	Number of degrees of freedom per node (=1).
36-40	NDIME	Number of coordinate dimensions (=2).
41-45	LINT	Number of integration points (=1).

<u>CARD SET 3</u>: Material Cards (I10, 2F15.5) - one card for each different material. Total of NMATS cards (See Card Set 2.).

Cols. 1-10	JMATS	Material identification number.
11-25	PROPS(JMATS,1)	Material coefficient, KX.
26-40	PROPS(JMATS,2)	Material coefficient, KY.

<u>CARD SET 4</u>: Element Cards (12I5) - one card for each element. Total of NELEM cards (See Card set 2.).

Cols. 1-5	JELEM	Element number.
6-10	LNODS(JELEM,1)	1st. Nodal connection number.
11-15	LNODS(JELEM,2)	2nd " " "
16-20	LNODS(JELEM,3)	3rd " " "
21-25	MATNO(JELEM)	Material Property number.

<u>Note:</u> The nodal connection numbers must be listed in an anti-clockwise sequence starting from any node.

<u>CARD SET 5</u>: Nodal Coordinate Cards (I10.2F15.5) - one card for each node. Total of NPOIN cards (See Card Set 2).

Cols. 1-10	JPOIN	Node number.
11-25	COORD(JPOIN,1)	The x coordinate of the node.
26-40	COORD(JPOIN,2)	The y coordinate of the node.

CARD SET 6: Restrained Node Cards (I10,I5,F10.5) - one card for each restrained node. Total of NBOUN cards (See Card Set 2). If NBOUN = 0 in Card Set 2, omit this card set.

Cols. 1-10	NODFX	Restrained node number.
11-15	ICODE(1)	Condition of restraint) 0 No constraint 1 Nodal degree of freedom restrained.
16-25	PRESC(1)	The prescribed value of the nodal variable (Temperature).

<u>CARD SET 7</u>: Integration Point Cards (3F11.6) Total of LINT cards (See Card Set 2)

Cols. 1-11	WEIGHT(ILINT)	Weights for numerical integration.
12-22	ETA(ILINT)	Coordinate point.
23-33	EXI(ILINT)	Coordinate point.

<u>Note:</u> The following numerical integration points are taken from reference [15].

Type of Element	Weight	Eta	Exi
3-Noded	1.0	0.333333	0.333333
6-Noded	0.333333	0.5	0.5
	0.333333	0.5	0.0
	0.333333	0.0	0.5
10-Noded	-0.5625	0.333333	0.333333
	0.520833	0.6	0.2
	0.520833	0.2	0.6
	0.520833	0.2	0.2

```
C     MAIN PROGRAM
C
      PROGRAM FEM (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      COMMON/FEM1/NPOIN,NELEM,NBOUN,NPROP,NNODE,NEVAB,NSVAB,NDOFN,
     +            LINT,NDIME,NBAND,NI,NO,AREA,GRADX,GRADY
      COMMON/FEM2/PROPS(5,3),COORD(401,2),LNODS(250,10),IFPRE(401),
     +            WEIGHT(10),FIXED(401),ETA(10),EXI(10),MATNO(250),
     +            FLUX(250,2),XDISP(401),TDISP(401,1),TREAC(401,1),
     +            ALAN(250),ASTIF(25000),ASLOD(401),REACT(401),FLUXX,
     +            FLUXY
C
C     FEM PROGRAM FOR THE SOLUTION OF TWO-DIM.
C     STEADY STATE HEAT CONDUCTION EQUATION
C
C     ASSIGN DATA SET NUMBERS FOR INPUT,NI, AND OUTPUT,NO.
C
      NI=5
      NO=6
C
C     THIS SEGMENT CONTROLS THE CALLING,IN ORDER,
C     OF ALL SUBROUTINES
C
      CALL DATA
      CALL STIFF1
      CALL ASSEMB
      CALL GREDUC
      CALL BAKSUB
      CALL FLUXES
      CALL RESULT
C
      STOP
      END
C
      SUBROUTINE DATA
      DIMENSION ICODE(2),PRESC(3),TITLE(20)
      COMMON/FEM1/NPOIN,NELEM,NBOUN,NPROP,NNODE,NEVAB,NSVAB,NDOFN,
     +            LINT,NDIME,NBAND,NI,NO,AREA,GRADX,GRADY
      COMMON/FEM2/PROPS(5,3),COORD(401,2),LNODS(250,10),IFPRE(401),
     +            WEIGHT(10),FIXED(401),ETA(10),EXI(10),MATNO(250),
     +            FLUX(250,2),XDISP(401),TDISP(401,1),TREAC(401,1),
     +            ALAN(250),ASTIF(25000),ASLOD(401),REACT(401),FLUXX,
     +            FLUXY
C
C     DATA INPUT SUBROUTINE
C
C     READ AND WRITE THE PROBLEM TITLE
C
      READ(NI,915)  TITLE
      WRITE(NO,915) TITLE
  915 FORMAT(20A4)
C
C     READ AND WRITE THE CONTROL DATA
C
      READ(NI,900) NPOIN,NELEM,NBOUN,NMATS,NPROP,NNODE,NDOFN,NDIME,
     +             LINT
  900 FORMAT(16I5)
      WRITE(NO,905) NPOIN,NELEM,NBOUN,NMATS,NPROP,NNODE,NDOFN,NDIME,
     +              LINT
  905 FORMAT(//,1X,7HNPOIN =,I5,3X,7HNELEM =,I5,3X,7HNBOUN =,I5,3X,
     + 7HNMATS =,I5,//,1X,7HNPROP =,I5,3X,7HNNODE =,I5,3X,7HNDOFN =,
     + I5,3X,7HNDIME =,I5,3X,6HLINT =,I5)
      NSVAB=NPOIN*NDOFN
      NEVAB=NNODE*NDOFN
C
C     READ AND WRITE THE MATERIAL PROPERTIES
C
      WRITE(NO,950)
  950 FORMAT(1H0,5X,19HMATERIAL PROPERTIES)
      DO 10 IMATS=1,NMATS
      READ(NI,910) JMATS,(PROPS(JMATS,IPROP),IPROP=1,NPROP)
   10 WRITE(NO,910) JMATS,(PROPS(JMATS,IPROP),IPROP=1,NPROP)
  910 FORMAT(I10,4F15.5)
C
C     READ AND WRITE THE ELEMENT NODAL
C     CONNECTIONS
C
      WRITE(NO,960)
  960 FORMAT(1H0,2X,2HEL,3X,5HNODES,3X,4HMAT.)
      DO 20 IELEM=1,NELEM
      READ(NI,920) JELEM,(LNODS(JELEM,INODE),INODE=1,NNODE),
```

```
     +               MATNO(JELEM)
   20 WRITE(NO,920) JELEM,(LNODS(JELEM,INODE),INODE=1,NNODE),
     +               MATNO(JELEM)
  920 FORMAT(12I5)
C
C     READ AND WRITE NODAL COORDINATES
C
      WRITE(NO,970)
  970 FORMAT(1H0,5X,4HNODE,5X,6HCOORD.)
      DO 30 IPOIN=1,NPOIN
      READ(NI,930) JPOIN,(COORD(JPOIN,IDIME),IDIME=1,NDIME)
   30 WRITE(NO,930) JPOIN,(COORD(JPOIN,IDIME),IDIME=1,NDIME)
  930 FORMAT(I10,2F15.5)
C
C     READ AND WRITE BOUNDARY CONDITIONS
C     AND STORE IN GLOBAL VECTORS
C
      DO 40 ISVAB=1,NSVAB
      IFPRE(ISVAB)=0
   40 FIXED(ISVAB)=0.0
      WRITE(NO,980)
  980 FORMAT(1H0,1X,28HRESTRAINED NODES,FIXITY CODE,
     + 22H AND PRESCRIBED VALUES)
      IF(NBOUN.EQ.0) RETURN
      DO 50 IBOUN=1,NBOUN
      READ(NI,940) NODFX,(ICODE(IDOFN),PRESC(IDOFN),IDOFN=1,NDOFN)
      WRITE(NO,940) NODFX,(ICODE(IDOFN),PRESC(IDOFN),IDOFN=1,NDOFN)
  940 FORMAT(I10,2(I5,F10.5))
      DO 50 IDOFN=1,NDOFN
      INDEX=(NODFX-1)*NDOFN+IDOFN
      IFPRE(INDEX)=ICODE(IDOFN)
      FIXED(INDEX)=PRESC(IDOFN)
   50 CONTINUE
C
C     READ AND WRITE INTEGRATION POINTS
C     AND CORRESPONDING WEIGHTS
C
      READ(NI,925) (WEIGHT(I),ETA(I),EXI(I),I=1,LINT)
      WRITE(NO,925) (WEIGHT(I),ETA(I),EXI(I),I=1,LINT)
  925 FORMAT(3F11.6)
C
      RETURN
      END
C
      SUBROUTINE STIFF1
      DIMENSION ESTIF(10,10),BVALU(10),CVALU(10),EJAC(2,2),DERIV(2,10)
      COMMON/FEM1/NPOIN,NELEM,NBOUN,NPROP,NNODE,NEVAB,NSVAB,NDOFN,
     +               LINT,NDIME,NBAND,NI,NO,AREA,GRADX,GRADY
      COMMON/FEM2/PROPS(5,3),COORD(401,2),LNODS(250,10),IFPRE(401),
     +               WEIGHT(10),FIXED(401),ETA(10),EXI(10),MATNO(250),
     +               FLUX(250,2),XDISP(401),TDISP(401,1),TREAC(401,1),
     +               ALAN(250),ASTIF(25000),ASLOD(401),REACT(401),FLUXX,
     +               FLUXY
C
C     EVALUATION OF ELEMENT STIFFNESS MATRICES
C     AND ELEMENT LOADS
C
      OPEN(10,FILE='TEMP')
C
C     LOOP OVER EACH ELEMENT
C
      DO 100 IELEM=1,NELEM
      DO 10 IEVAB=1,NEVAB
      DO 10 JEVAB=1,NEVAB
   10 ESTIF(IEVAB,JEVAB)=0.0
      JELEM=IELEM
      LPROP=MATNO(IELEM)
      XCOEF=PROPS(LPROP,1)
      YCOEF=PROPS(LPROP,2)
C
C     EVALUATE THE ELEMENT STIFFNESSES
C
      DO 30 IEVAB=1,NEVAB
      DO 30 JEVAB=1,NEVAB
      DO 30 K=1,LINT
      ETASP=ETA(K)
      EXISP=EXI(K)
      W=WEIGHT(K)
      CALL SFRT (DERIV,ETASP,EXISP,NNODE)
      CALL JACOB (NNODE,JELEM,DERIV,COORD,LNODS,EJAC,DET,BVALU,CVALU)
```

```
   30 ESTIF(IEVAB,JEVAB)=ESTIF(IEVAB,JEVAB)+W*(XCOEF*BVALU(IEVAB)*BVALU(
     +JEVAB)+YCOEF*CVALU(IEVAB)*CVALU(JEVAB))/DET
      WRITE(10)   ESTIF
C
  100 CONTINUE
C
      RETURN
      END
C
      SUBROUTINE JACOB(NNODE,JELEM,DERIV,COORD,LNODS,EJAC,DET,BVALU,
     +CVALU)
      DIMENSION BVALU(10),CVALU(10),EJAC(2,2),LNODS(250,10),DERIV(2,10),
     + COORD(401,2)
C
C     EVALUATION OF JACOBIAN AND
C     COMPONENTS OF GLOBAL DERIVATIVE VECTOR
C
      DO 40 I=1,2
      DO 40 J=1,2
   40 EJAC(I,J)=0.0
C
C     EVALUATE JACOBIAN
C
      DO 20 I=1,2
      DO 20 J=1,2
      DO 10 K=1,NNODE
      NODEI=LNODS(JELEM,K)
   10 EJAC(I,J)=EJAC(I,J)+DERIV(I,K)*COORD(NODEI,J)
   20 CONTINUE
C
      DET=EJAC(1,1)*EJAC(2,2)-EJAC(1,2)*EJAC(2,1)
      IF (NNODE.EQ.3) DET=2.0*DET
      DO 30 K=1,NNODE
C
C     EVALUATE COMPONENTS OF GLOBAL DERIVATIVE VECTOR
C
      BVALU(K)=EJAC(2,2)*DERIV(1,K)-EJAC(1,2)*DERIV(2,K)
   30 CVALU(K)=EJAC(1,1)*DERIV(2,K)-EJAC(2,1)*DERIV(1,K)
C
      RETURN
      END
C
      SUBROUTINE ASSEMB
      COMMON/FEM1/NPOIN,NELEM,NBOUN,NPROP,NNODE,NEVAB,NSVAB,NDOFN,
     +            LINT,NDIME,NBAND,NI,NO,AREA,GRADX,GRADY
      COMMON/FEM2/PROPS(5,3),COORD(401,2),LNODS(250,10),IFPRE(401),
     +            WEIGHT(10),FIXED(401),ETA(10),EXI(10),MATNO(250),
     +            FLUX(250,2),XDISP(401),TDISP(401,1),TREAC(401,1),
     +            ALAN(250),ASTIF(25000),ASLOD(401),REACT(401),FLUXX,
     +            FLUXY
      DIMENSION ESTIF(10,10)
      DATA MSTIF/25000/
C
C     THIS ROUTINE ASSEMBLES  THE ELEMENT
C     STIFFNESSES AND AND APPLIED LOADS TO FORM
C     THE GLOBAL STIFFNESS MATRIX AND FORCE VECTOR
C
      NFUNC(I,J)=(J-I)*(2*NSVAB+1-J+I)/2+I
      REWIND 10
      DO 5 ISTIF=1,MSTIF
    5 ASTIF(ISTIF)=0.0
      DO 10 ISVAB=1,NSVAB
      ASLOD(ISVAB)=0.0
   10 CONTINUE
      NBAND=0
      DO 30 IELEM=1,NELEM
      READ(10)   ESTIF
      DO 20 INODE=1,NNODE
      NODEI=LNODS(IELEM,INODE)
      DO 20 IDOFN=1,NDOFN
      NROWS=(NODEI-1)*NDOFN+IDOFN
      NROWE=(INODE-1)*NDOFN+IDOFN
C
C     ASSEMBLE THE ELEMENT STIFFNESS MATRICES
C
      DO 20 JNODE=1,NNODE
      NODEJ=LNODS(IELEM,JNODE)
      DO 20 JDOFN=1,NDOFN
      NCOLS=(NODEJ-1)*NDOFN+JDOFN
      NCOLE=(JNODE-1)*NDOFN+JDOFN
```

```
      IF(NCOLS.LT.NROWS) GO TO 20
      NDIFF=NCOLS-NROWS
      IF(NDIFF.GT.NBAND) NBAND=NDIFF
      NGASH=NFUNC(NROWS,NCOLS)
      ASTIF(NGASH)=ASTIF(NGASH)+ESTIF(NROWE,NCOLE)
   20 CONTINUE
   30 CONTINUE
C
      NHALF=NBAND+1
      WRITE(NO,900) NHALF
  900 FORMAT(1H0,5X,33HMAX. HALF-BANDWIDTH FOR PROBLEM =,I5)
      INDEX=NSVAB-NBAND
      NSTIF=NFUNC(INDEX,NSVAB)
      IF(NSTIF.LE.MSTIF) RETURN
      WRITE(NO,910)
  910 FORMAT(1H0,5X,24HSTIFFNESS SPACE EXCEEDED)
C
      STOP
      END
C
      SUBROUTINE GREDUC
      COMMON/FEM1/NPOIN,NELEM,NBOUN,NPROP,NNODE,NEVAB,NSVAB,NDOFN,
     +            LINT,NDIME,NBAND,NI,NO,AREA,GRADX,GRADY
      COMMON/FEM2/PROPS(5,3),COORD(401,2),LNODS(250,10),IFPRE(401),
     +            WEIGHT(10),FIXED(401),ETA(10),EXI(10),MATNO(250),
     +            FLUX(250,2),XDISP(401),TDISP(401,1),TREAC(401,1),
     +            ALAN(250),ASTIF(25000),ASLOD(401),REACT(401),FLUXX,
     +            FLUXY
C
C     THIS ROUTINE REDUCES THE GLOBAL STIFFNESS
C     EQUATIONS BY DIRECT GAUSSIAN ELIMINATION
C
      NFUNC(I,J)=(J-I)*(2*NSVAB+1-J+I)/2+I
      NEQNS=NSVAB
      DO 50 IEQNS=1,NEQNS
      NLOCA=IEQNS+NBAND
      IF(NLOCA.GT.NEQNS) NLOCA=NEQNS
      IF(IFPRE(IEQNS).EQ.1) GO TO 30
C
C     REDUCE EQUATIONS
C
      NGASH=NFUNC(IEQNS,IEQNS)
      PIVOT=ASTIF(NGASH)
      IF(ABS(PIVOT).LT.1.0E-10) GO TO 60
      IF(IEQNS.EQ.NEQNS) GO TO 50
      IEQN1=IEQNS+1
      DO 20 IROWS=IEQN1,NLOCA
      NGASH=NFUNC(IEQNS,IROWS)
      FACTR=ASTIF(NGASH)/PIVOT
      IF(FACTR.EQ.0.0) GO TO 20
      DO 10 ICOLS=IEQN1,NLOCA
      IF(IROWS.GT.ICOLS) GO TO 10
      NGESH=NFUNC(IROWS,ICOLS)
      NGISH=NFUNC(IEQNS,ICOLS)
      ASTIF(NGESH)=ASTIF(NGESH)-FACTR*ASTIF(NGISH)
   10 CONTINUE
      ASLOD(IROWS)=ASLOD(IROWS)-FACTR*ASLOD(IEQNS)
   20 CONTINUE
      GO TO 50
C
C     ADJUST  LOADS FOR PRESCRIBED DISPLACEMENTS
C
   30 DO 40 IROWS=IEQNS,NLOCA
      NGASH=NFUNC(IEQNS,IROWS)
      ASLOD(IROWS)=ASLOD(IROWS)-ASTIF(NGASH)*FIXED(IEQNS)
   40 CONTINUE
      NGISH=NFUNC(IEQNS,IEQNS)
      ASTIF(NGISH)=0.0
   50 CONTINUE
      RETURN
   60 WRITE(NO,900) PIVOT,IEQNS
  900 FORMAT(5X,17HINCORRECT PIVOT =,F20.6,5X,13HEQUATION NO.  ,I5)
C
      STOP
      END
C
      SUBROUTINE BAKSUB
      COMMON/FEM1/NPOIN,NELEM,NBOUN,NPROP,NNODE,NEVAB,NSVAB,NDOFN,
     +            LINT,NDIME,NBAND,NI,NO,AREA,GRADX,GRADY
      COMMON/FEM2/PROPS(5,3),COORD(401,2),LNODS(250,10),IFPRE(401),
```

```
     +              WEIGHT(10),FIXED(401),ETA(10),EXI(10),MATNO(250),
     +              FLUX(250,2),XDISP(401),TDISP(401,1),TREAC(401,1),
     +              ALAN(250),ASTIF(25000),ASLOD(401),REACT(401),FLUXX,
     +              FLUXY
C
C     THIS ROUTINE PERFOMS THE BACK
C     SUBSTITUTION PHASE
C
      NFUNC(I,J)=(J-I)*(2*NSVAB+1-J+I)/2+I
      NEQNS=NSVAB
      DO 5 IEQNS=1,NEQNS
      REACT(IEQNS)=0.0
    5 CONTINUE
      NEQN1=NEQNS+1
      DO 30 IEQNS=1,NEQNS
      NBACK=NEQN1-IEQNS
      NGASH=NFUNC(NBACK,NBACK)
      PIVOT=ASTIF(NGASH)
      RESID=ASLOD(NBACK)
      IF(NBACK.EQ.NEQNS) GO TO 20
      NBAC1=NBACK+1
      NLOCA=NBACK+NBAND
      IF(NLOCA.GT.NEQNS) NLOCA=NEQNS
      DO 10 ICOLS=NBAC1,NLOCA
      NGESH=NFUNC(NBACK,ICOLS)
      RESID=RESID-ASTIF(NGESH)*XDISP(ICOLS)
   10 CONTINUE
   20 IF(IFPRE(NBACK).EQ.0) XDISP(NBACK)=RESID/PIVOT
      IF(IFPRE(NBACK).EQ.1) XDISP(NBACK)=FIXED(NBACK)
      IF(IFPRE(NBACK).EQ.1) REACT(NBACK)=-RESID
   30 CONTINUE
      KOUNT=0
      DO 40 IPOIN=1,NPOIN
      DO 40 IDOFN=1,NDOFN
      KOUNT=KOUNT+1
      TDISP(IPOIN,IDOFN)=XDISP(KOUNT)
   40 TREAC(IPOIN,IDOFN)=REACT(KOUNT)
C
      RETURN
      END
C
      SUBROUTINE FLUXES
      DIMENSION BVALU(10),CVALU(10),EJAC(2,2),DERIV(2,10)
      COMMON/FEM1/NPOIN,NELEM,NBOUN,NPROP,NNODE,NEVAB,NSVAB,NDOFN,
     +              LINT,NDIME,NBAND,NI,NO,AREA,GRADX,GRADY
      COMMON/FEM2/PROPS(5,3),COORD(401,2),LNODS(250,10),IFPRE(401),
     +              WEIGHT(10),FIXED(401),ETA(10),EXI(10),MATNO(250),
     +              FLUX(250,2),XDISP(401),TDISP(401,1),TREAC(401,1),
     +              ALAN(250),ASTIF(25000),ASLOD(401),REACT(401),FLUXX,
     +              FLUXY
C
C     EVALUATION OF TEMPERATURE GRADIENT OVER EACH
C     ELEMENT AND OF AVERAGED HEAT FLUX AND AVERAGED
C     TEMPERATURE GRADIENT OVER THE DOMAIN
C
      HEATX=0.0
      HEATY=0.0
      AGX=0.0
      AGY=0.0
      AREA=0.0
C
C     LOOP OVER EACH ELEMENT
C
      DO 30 IELEM=1,NELEM
      JELEM=IELEM
      LPROP=MATNO(IELEM)
      XCOEF=PROPS(LPROP,1)
      YCOEF=PROPS(LPROP,2)
      AREAE=0.0
      QXSUM=0.0
      QYSUM=0.0
      DO 10 K=1,LINT
      ETASP=ETA(K)
      EXISP=EXI(K)
      W=WEIGHT(K)
      CALL SFRT(DERIV,ETASP,EXISP,NNODE)
      CALL JACOB(NNODE,JELEM,DERIV,COORD,LNODS,EJAC,DET,BVALU,CVALU)
      AREAE=AREAE+W*DET/2.0
C
C     EVALUATE TEMPERATURE GRADINT OVER EACH ELEMENT
```

```
C
      DO 10 INODE=1,NNODE
      NODEI=LNODS(IELEM,INODE)
      QXSUM=QXSUM+W*BVALU(INODE)*TDISP(NODEI,1)
   10 QYSUM=QYSUM+W*CVALU(INODE)*TDISP(NODEI,1)
      IF(NNODE.NE.3) GO TO 40
      AREAE=AREAE/2.0
   40 QXSUM=QXSUM/2.0
      QYSUM=QYSUM/2.0
      FLUX(IELEM,1)=QXSUM/AREAE
      FLUX(IELEM,2)=QYSUM/AREAE
C
C     EVALUATE AVERAGED TEMPERATURE GRADIENT AND
C     HEAT FLUX OVER THE DOMAIN
C
      AREA=AREA+AREAE
      ALAN(IELEM)=AREAE
      HEATX=HEATX+XCOEF*QXSUM
      HEATY=HEATY+YCOEF*QYSUM
      AGX=AGX+QXSUM
   30 AGY=AGY+QYSUM
      FLUXX=HEATX/AREA
      FLUXY=HEATY/AREA
      GRADX=AGX/AREA
      GRADY=AGY/AREA
C
      RETURN
      END
C
      SUBROUTINE SFRT(DERIV,ETASP,EXISP,NNODE)
      DIMENSION DERIV(2,10)
C
C     THIS ROUTINE CONTAINS DERIVATIVES OF SHAPE
C     FUNCTIONS FOR 3-NODED, 6-NODED AND 10-NODED
C     TRIANGULAR ELEMENTS
C
      S=EXISP
      T=ETASP
      P=1.0-S-T
      IF(NNODE.NE.3) GO TO 10
C
C     FOR 3-NODED TRIANGULAR ELEMENTS
C
      DERIV(1,1)=-1.0
      DERIV(1,2)=1.0
      DERIV(1,3)=0.0
      DERIV(2,1)=-1.0
      DERIV(2,2)=0.0
      DERIV(2,3)=1.0
   10 IF(NNODE.NE.6) GO TO 20
C
C     FOR 6-NODED TRIANGULAR ELEMENTS
C
      DERIV(1,1)=-4.0*P+1.0
      DERIV(2,1)=-4.0*P+1.0
      DERIV(1,2)=4.0*(P-S)
      DERIV(2,2)=-4.0*S
      DERIV(1,3)=-1.0+4.0*S
      DERIV(2,3)=0.0
      DERIV(1,4)=4.0*T
      DERIV(2,4)=4.0*S
      DERIV(1,5)=0.0
      DERIV(2,5)=-1.0+4.0*T
      DERIV(1,6)=-4.0*T
      DERIV(2,6)=4.0-4.0*S-8.0*T
   20 IF(NNODE.NE.10) RETURN
C
C     FOR 10-NODED TRIANGULAR ELEMENTS
C
      DERIV(1,1)=(-27.0*P**2+18.0*P-2.0)/2.0
      DERIV(2,1)=(-27.0*P**2+18.0*P-2.0)/2.0
      DERIV(1,2)=9.0*(3.0*P**2-P-6.0*S*P+S)/2.0
      DERIV(2,2)=9.0*S*(-6.0*P+1.0)/2.0
      DERIV(1,3)=9.0*(6.0*S*P-P-3.0*S**2+S)/2.0
      DERIV(2,3)=-9.0*(3.0*S**2-S)/2.0
      DERIV(1,4)=(27.0*S**2-18.0*S+2.0)/2.0
      DERIV(2,4)=0.0
      DERIV(1,5)=9.0*T*(6.0*S-1.0)/2.0
      DERIV(2,5)=9.0*(3.0*S**2-S)/2.0
      DERIV(1,6)=9.0*(3.0*T**2-T)/2.0
```

```
      DERIV(2,6)=9.0*S*(6.0*T-1.0)/2.0
      DERIV(1,7)=0.0
      DERIV(2,7)=(27.0*T**2-18.0*T+2.0)/2.0
      DERIV(1,8)=-9.0*(3.0*T**2-T)/2.0
      DERIV(2,8)=9.0*(6.0*T*P-P-3.0*T**2+T)/2.0
      DERIV(1,9)=9.0*T*(-6.0*P+1.0)/2.0
      DERIV(2,9)=9.0*(3.0*P**2-P-6.0*P*T+T)/2.0
      DERIV(1,10)=27.0*T*(P-S)
      DERIV(2,10)=27.0*S*(P-T)
C
      RETURN
      END
C
      SUBROUTINE RESULT
      COMMON/FEM1/NPOIN,NELEM,NBOUN,NPROP,NNODE,NEVAB,NSVAB,NDOFN,
     +            LINT,NDIME,NBAND,NI,NO,AREA,GRADX,GRADY
      COMMON/FEM2/PROPS(5,3),COORD(401,2),LNODS(250,10),IFPRE(401),
     +            WEIGHT(10),FIXED(401),ETA(10),EXI(10),MATNO(250),
     +            FLUX(250,2),XDISP(401),TDISP(401,1),TREAC(401,1),
     +            ALAN(250),ASTIF(25000),ASLOD(401),REACT(401),FLUXX,
     +            FLUXY
C
C     OUTPUT OF RESULTS
C
C     WRITE THE NODAL TEMPERATURE  AND REACTIONS
C
      WRITE(NO,900)
  900 FORMAT(1H0,5X,4HNODE,1X,13HDISPLACEMENTS,3X,9HREACTIONS)
      DO 10 IPOIN=1,NPOIN
   10 WRITE(NO,910) IPOIN,(TDISP(IPOIN,IDOFN),IDOFN=1,NDOFN),
     +   (TREAC(IPOIN,IDOFN),IDOFN=1,NDOFN)
  910 FORMAT(I10,3E14.6)
C
C     WRITE COMPONENTS OF TEMPERATURE GRADINENT VECTOR
C     OVER EACH ELEMENT AND AREA OF EACH ELEMENT
C
      WRITE(NO,920)
  920 FORMAT(1H0,2X,7HELEMENT,3X,9HGRAD IN X,5X,9HGRAD IN Y,5X,5HAREAE)
      DO 20 IELEM=1,NELEM
   20 WRITE(NO,910) IELEM,( FLUX(IELEM,IDIME),IDIME=1,NDIME),ALAN(IELEM)
C
C     WRITE COMPONENTS OF AVERAGED  HEAT FLUX AND AVERAGED
C     TEMPERATURE GRADIENT AND AREA OVER THE DOMAIN
C
      WRITE(NO,930) FLUXX,FLUXY,GRADX,GRADY,AREA
  930 FORMAT(1H0,1X,19HAVERAGE FLUX IN X =,E14.6,/,2X,19HAVERAGE FLUX IN
     + Y =,E14.6,/,2X,19HAVERAGE GRAD IN X =,E14.6,/,2X,19HAVERAGE GRAD
     +IN Y =,E14.6,/,2X,4HAREA,14X,1H=,E14.6)
C
C     EVALUATE AND WRITE THE COMPONENTS OF EFFECTIVE
C     CONDUCTIVITY TENSOR IN THE PRINCIPAL DIRECTIONS
C
      IF (GRADX.EQ.0.0) GO TO 11
      EKX=FLUXX/GRADX
      WRITE(NO,940)  EKX
  940 FORMAT(2X,2HKX,16X,1H=,E14.6)
   11 IF(GRADY.EQ.0.0) GO TO 12
      EKY=FLUXY/GRADY
      WRITE(NO,950)  EKY
  950 FORMAT(2X,2HKY,16X,1H=,E14.6)
C
   12 RETURN
      END

      RETURN
      END
C
C                            SAMPLE DATA
C
C            1         2         3         4         5         6
C   123456789012345678901234567890123456789012345678901234567890
C
                HEAT CONDUCTION OVER TWO PHASE COMPOSITE
   25   32   16    2    2    3    1    2    1
         1          100000          100000
         2          200000          200000
    1    1    2    7    1
    2    1    7    6    1
    3    2    3    8    1
    4    2    8    7    1
```

561	5	3	4	9	1	
562	6	3	9	8	1	
563	7	4	5	10	1	
564	8	4	10	9	1	
565	9	6	7	12	1	
566	10	6	12	11	1	
567	11	7	8	13	1	
568	12	7	13	12	1	
569	13	8	9	14	1	
570	14	8	14	13	1	
571	15	9	10	15	1	
572	16	9	15	14	1	
573	17	11	12	17	2	
574	18	11	17	16	2	
575	19	12	13	18	2	
576	20	12	18	17	2	
577	21	13	14	19	2	
578	22	13	19	18	2	
579	23	14	15	20	2	
580	24	14	20	19	2	
581	25	16	17	22	2	
582	26	16	22	21	2	
583	27	17	18	23	2	
584	28	17	23	22	2	
585	29	18	19	24	2	
586	30	18	24	23	2	
587	31	19	20	25	2	
588	32	19	25	24	2	
589		1		000000		000000
590		2		025000		000000
591		3		050000		000000
592		4		075000		000000
593		5		100000		000000
594		6		000000		025000
595		7		025000		025000
596		8		050000		025000
597		9		075000		025000
598		10		100000		025000
599		11		000000		050000
600		12		025000		050000
601		13		050000		050000
602		14		075000		050000
603		15		100000		050000
604		16		000000		075000
605		17		025000		075000
606		18		050000		075000
607		19		075000		075000
608		20		100000		075000
609		21		000000		100000
610		22		025000		100000
611		23		050000		100000
612		24		075000		100000
613		25		100000		100000
614		1	1	000000		
615		2	1	025000		
616		3	1	050000		
617		4	1	075000		
618		5	1	100000		
619		10	1	125000		
620		15	1	150000		
621		20	1	175000		
622		25	1	200000		
623		24	1	175000		
624		23	1	150000		
625		22	1	125000		
626		21	1	100000		
627		16	1	075000		
628		11	1	050000		
629		6	1	025000		
630		1000000		0333333	0333333	

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