CANONICAL STRUCTURE AND INTEGRABILITY OF NEW TWO - DIMENSIONAL FIELD THEORIES

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CANONICAL STRUCTURE AND INTEGRABILITY OF NEW TWO-DIMENSIONAL FIELD THEORIES

We consider two different field theories in one-space and one-time dimension. One of these is the 0(3)-invariant nonlinear sigma model whose integrability condition is the sine-Gordon equation. Using only the first derivatives of the field variable, we construct the most general conformally invariant Lagrangian for this system, the generalized sigma model (GSM). Its integrability condition is the generalized sine-Gordon equation (GSG). The other model we examine is the system of equations governing long waves on shallow water. We concentrate on the new model obtained when the effects of dispersion are included.

In these two models we analyze the properties that are related to the presence of soliton solutions to find out which properties of these systems survive when equations are modified to accomodate new effects. For the shallow water waves with dispersion we find that it is possible to generalize the symplectic structure and the canonical formulation of the original model. However the infinite sequence of conservation laws are lost. On the other hand time-independent solutions of the GSG equation still exhibit soliton-like behavior. We show that

the GSG equation can be formulated as an imbedding and an inverse scattering problem. The Gaussian curvature of the surface underlying the GSM, just like that of the original O(3) sigma model, is one. Furthermore, we find a Bäcklund transformation for the GSG equation.

YENİ İKİ BOYUTLU ALAN TEORİLERİNİN KANONİK YAPILARI VE INTEGRE EDİLEBİLME ÖZELLİKLERİ

Bir uzay ve bir zaman boyutunda iki değişik alan teorisini gözönüne aldık. Bunlardan biri, integre edilebilme şartı sinüs-Gordon denklemi olan, O(3)-değişmez, lineer olmayan sigma modelidir. Alan değişkenlerinin sadece birinci türevlerini kullanarak, en genel konform değişmez Lagranj fonksiyonunu, genelleştirilmiş sigma modelini (GSM) inşa ettik. Bu sistemin integre edilebilme şartı da genelleştirilmiş sinüs-Gordon (GSG) denklemidir. İncelediğimiz diğer model ise sığ sulardaki uzun dalgaları tarif eden denklemler sistemidir. Burada özellikle dağılma etkilerinin ilave edilmiş olduğu yeni modelin üzerinde durduk.

Her iki modelde, denklemler yeni efektleri içerecek şekilde genelleştirildikten sonra, soliton çözümü ile ilgili özelliklerden hangilerinin kalmış olduğunu inceledik. Dağılmalı sığ sularda, simplektik ve kanonik yapıyı genelleştirebileceğimizi gördük. Fakat bu durumda sistem sonsuz sayıda korunum yasası özelliğini kaybetmektedir. Öte yandan GSG denkleminin zamandan bağımsız çözümleri hala soliton gibi davranmaktadırlar. Ayrıca GSG denkleminin gömme ve ters saçılma problemi şekline sokulabildiğini gösterdik. GSM denkleminin dayandığı yüzeyin eğriliği de,

orijinal sigma modelininki gibi, bire eşittir. Sonra da GSG denklemi için Bäcklund dönüşümü bulduk.

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LIST OF SYMBOLS

A,B,C	LAKNS coefficients
C_i	Primary constraints
$C_i{}^j{}_k$	Structure constants of SL(2,R)
ds^2	Metric
f	Function modifying GSM Lagrangian
$\mathcal{H}$	Hamiltonian density
h	Height of fluid
I	Action
J	Matrix defining symplectic structure
K	Gaussian curvature
$\mathcal{L}$	Lagrangian density
M	Three-dimensional flat space
$\vec{n}$	Unit vector
P	Integral of motion
q,r	LAKNS potentials
S	Surface
t	Time coordinate
U	Ratio of eigenfunctions V_1, V_2
u	Velocity of fluid
u,v	Light-cone coordinates
V_1, V_2	Eigenfunctions
x	Space coordinate
α	Logarithm of the norms of $\vec{n}_u$ and $\vec{n}_v$
β	Invariant variable
γ	Square of the norms of $\vec{n}_u$ and $\vec{n}_v$
Γ_1	SL(2,R)-valued connection one-form
δ	Variable corresponding to an infinitesimal transformation of h

ε	Infinitesimal quantity
ζ	Eigenvalue
η	Variable corresponding to an infinitesimal transformation of u
θ	Angle between $\vec{n}_u$ and $\vec{n}_v$
Θ^i	One-form satisfying Cartan's structure equations for SL(2,R)
κ	Square-root of the norm of ρ
λ	Lagrange multiplier
Λ	Transformation matrix
μ	Lagrange multiplier
ν	Dispersive effect constant
ξ	Variable corresponding to an infinitesimal transformation of x
π_i	One forms defining the extrinsic curvature
Π_{Φ_i}	Canonical momentum of Φ_i
ρ	Sign of ν
σ	Lagrange multiplier
Σ	Scalar potential
τ	Variable corresponding to an infinitesimal transformation of t
υ	Lagrange multiplier
Υ	Scalar potential
ϕ	Function of θ
Φ	Scalar potential
χ	Constraint
ψ	Function of θ
Ψ	Scalar potential
ω^i	One-form
$\omega^i{}_j$	Connection one-form
Ω	Variable corresponding to an infinitesimal transformation of θ
$\otimes$	Tensor Product
$\wedge$	Exterior Product

I. INTRODUCTION

Wave-particle duality emerged as a revolutionary concept in the first quarter of the 20th century. When experimental techniques developed to the stage where atomic systems could be studied, difficulties appeared which could not be resolved within the framework of Newtonian Mechanics. This breakdown of classical physics finally lead to a complete formulation of Quantum Mechanics.

In this theory, a free particle was identified with a de Broglie plane wave, corresponding to a total absence of localization in space. Therefore this state function was not physically admissible. A physical state function was constructed as a superposition of these pure de Broglie waves. This was a wave packet which initially travelled undistorted with a definite group velocity but eventually began to spread out in space.

Recently another connection between waves and particles attracted the attention of physicists. This time the relation presented itself as a result of the investigation of wave equations that derive not from Quantum Mechanics but from classical physics. Unlike the wave packets of Quantum Mechanics, solutions of these classical equations retain their size and shape. They are nondissipative configurations such that their energy remains localized in a finite spatial region. In fact, these solutions are not free from dispersion. However the equations that give rise to these solutions are nonlinear and the effects of nonlinearity and of dispersion cancel each other exactly[1]. These special solutions are solitons[2,3]. Furthermore when two solitons collide, they emerge from the collision having the same shapes and velocities with which they started out. If there is a soliton and an anti-soliton in a theory, they are created and destroyed in pairs. Thus solitary waves do exhibit many properties of elementary particles of physics.

The concept of a solitary wave was first introduced, by J.S.Russell[4], to hydrodynamics. It has become a tradition to quote his famous words describing his chase on horseback behind a solitary wave in a channel 150 years ago. Later Korteweg and de Vries developed an equation for shallow water waves which exhibit solitary wave solutions[5,6,7]. Another system which possesses soliton solutions, the Boussinesq equation, was also first derived to describe shallow water waves propagating in two-directions[1]. Therefore shallow water waves have played a significant historical role in the theory of solitons.

Nonlinear sigma model and its integrability condition, the sine-Gordon equation also possess solitons[8,9,10]. These types of theories appear to be immediately relevant since most recent developments provide some confirmation of the idea that baryons are solitons in the nonlinear sigma model[11]. Two-dimensional sigma model and four dimensional Yang-Mills type field theories share such properties as asymptotic freedom, charge confinement and an infinitely degenerate vacuum[12]; furthermore pure Yang-Mills theory in four Euclidean dimensions has instanton solutions that are static soliton solutions corresponding to the tunneling probability between two vacua[13]. On the other hand Coleman[14] has shown perturbatively that Quantum Mechanical Thirring model of particles and anti-particles moving in a one-dimensional space and the sine-Gordon equation with its solitons describe the same phenomena. Hence, the special role played by the nonlinear sigma model and the shallow water wave equations in the soliton theory provide the motivation for considering generalizations of these models in two-dimensions.

Two-dimensional theories have been very useful in physics so far by providing insight into field theoretical possibilities which can then be developed for physical four-dimensional models[15]. For example, dynamical gauge symmetry breaking was first understood in two-dimensional massless electrodynamics, the Schwinger model[16]. The solution of two-dimensional quantum chromodynamics explained large-N behavior of non-Abelian gauge theories[17]. Hence it is natural to seek enlightenment in a two-dimensional setting where complete and simple solutions can be obtained; thereby mathematical concepts and techniques can easily be developed.

At present, the mathematical framework necessary for determining whether a given wave equation indeed possesses soliton solutions, without direct numerical computation, is lacking. The most useful concept so far is "complete integrability"[12]. The sine-Gordon, Korteweg-de Vries, and Boussinesq equations are all examples of types of systems that are called "completely integrable". In addition to having solitary wave solutions, these systems have an infinite sequence of conservation laws. Furthermore, they each possess a Bäcklund transformation, used to generate new solutions once a solution has been found[18,19,20]. They are amenable to the inverse scattering method[21,22], through which the problem of solving a nonlinear equation reduces to solving a coupled set of linear integral equations. Completely integrable systems are usually invariant under a one parameter Lie group of transformations. Therefore an invariant variable which reduces a partial differential equation to an ordinary one may be defined[23]. These systems can also be analyzed within the framework of differential geometry. Then the equations governing these systems can be described by an SL(2,R)-valued connection one-form with zero curvature; or they can be formulated as an imbedding problem[24,25,34]. Furthermore, it seems necessary to find a Hamiltonian density corresponding to these systems. However given a nonlinear equation, it is not known which of these properties are necessary and sufficient to insure the existence of solitons. Therefore, new models, even if they do not turn out to be physically relevant, are still important for the sake of clarifying the relation between the existence of soliton solutions and complete integrability.

To this end, in this work we first construct the most general conformal O(3) invariant nonlinear sigma model in two-dimensions, whose Lagrangian contains only the first derivatives of the field variable[26]. We find out that the integrability condition of this generalized sigma model (GSM) yields an equation which encompasses the sine-Gordon equation in addition to some special cases which are of the same form. Then we discover that the time-independent solutions of this generalized sine-Gordon equation (GSG) exhibit soliton-like behaviour. So it seems that the sine-Gordon equation does not lose its complete integrability when generalized. Therefore we look for other properties that are common to most completely integrable systems. First we point out that the GSG equation follows from the fact that the Gaussian curvature underlying the model is constant[27]. We further note that the GSG equation can be described by an SL(2,R)-valued connection one-form with zero curvature, just like the Korteweg-de Vries and the sine-Gordon equations. Also we show that the GSG can be formulated as an imbedding and an inverse scattering problem. Finally the invariance properties of the GSG equation are considered.

We investigate some other conditions for complete integrability in a hydrodynamic system, shallow water waves with dispersive effects. For this system we look for a Hamiltonian density and for an infinite sequence of conservation laws. Surface waves without dispersion manifest an interesting feature within the framework of complete integrability because even though they possess an infinite sequence of conservation laws[28], they have no proper solitons[29]. They do have a Hamiltonian density, however[30]. For the generalized version of this model which includes the effects of dispersion in water waves, we find out that we can cast the system into canonical form and obtain an explicit expression for the Hamiltonian using Dirac's theory of constraints[31]. We formulate the symplectic structure of the system and obtain through this structure, the general equation to be satisfied by all integrals of motion[32]. Furthermore, we show that the infinite sequence of conservation laws are lost.

II. GENERALIZED TWO-DIMENSIONAL O(3)-INVARIANT SIGMA MODEL

Two dimensional O(n)-invariant Lagrangian field theories whose field functions describe a homogeneous space have received a lot of attention in the literature. The simplest of these models is the O(3)/O(2)[33] model whose field variable is the three-dimensional unit vector $\vec{n}$. The Lagrangian density of this theory consists of the scalar product of the first derivatives of this vector. The corresponding action is conformally invariant. It turns out that this model has a number of interesting properties, the most important of which is that its integrability condition is the sine-Gordon equation.

In this part, we generalize this model while maintaining its conformal invariance. To this end, we insert into the Lagrangian an arbitrary function of the angle between the light-cone derivatives of $\vec{n}$. Different choices of this function will lead to different conformally invariant O(3) models.

Starting from this generalized Lagrangian we proceed as follows. The Euler-Lagrange equations of motion, together with the constraint that the norm of $\vec{n}$ is equal to unity, gives us the relation between the function f that modifies the Lagrangian and the magnitudes of the first derivatives of $\vec{n}$. Making use of this relation and employing the geometrical interpretation of the field vector $\vec{n}$ and its first derivatives, we construct the equation for the angle between the light-cone derivatives of $\vec{n}$. This integrability condition reduces to the sine-

Gordon equation when the function f is taken to be unity.

We examine the time-independent, the space-independent and the Euclidean-invariant versions of this integrability condition. The equations corresponding to the first two cases reduce to the harmonic oscillator equation. The Euclidean-invariant one is the Euclidean sinh-Gordon equation. These three cases, together with the sine-Gordon equation smoothly fit together with a variable transformation.

Finally we examine the time-independent version of this integrability condition. After studying one special case, we devise a procedure for a systematic construction of potentials corresponding to different choices of the function f.

2.1. The Integrability Condition of the Generalized Sigma Model

Our starting point is the O(3)-invariant chiral theory in one-time and one-space dimension which is described by the Lagrangian density

$$L = \vec{n}_u . \vec{n}_v + \lambda(\vec{n}^2 - 1) \ . \tag{2.1.1}$$

The interaction arises from the condition that $\vec{n}.\vec{n}$ is equal to one. u and v are the light-cone coordinates;

$$u = \frac{1}{2}(t + x)$$
$$v = \frac{1}{2}(t - x) \ . \tag{2.1.2}$$

and subscripts denote differentiation.

Noting that the angle θ between $\vec{n}_u$ and $\vec{n}_v$ is a conformally invariant quantity, we propose to generalize the O(3)-invariant theory by modifying the Lagrangian in the following way

$$L = \vec{n}_u.\vec{n}_v f(\theta) + \lambda(\vec{n}^2-1) \ . \tag{2.1.3}$$

The corresponding equation of motion is

$$\vec{n}_u.\vec{n}_v \frac{\partial f}{\partial z} \frac{\partial z}{\partial \vec{n}} + 2\lambda \vec{n} = \frac{\partial}{\partial u}(\vec{n}_v f + \vec{n}_u.\vec{n}_v \frac{\partial f}{\partial z} \frac{\partial z}{\partial \vec{n}_u}) + \frac{\partial}{\partial v}(\vec{n}_u f + \vec{n}_u.\vec{n}_v \frac{\partial f}{\partial z} \frac{\partial z}{\partial \vec{n}_v}) \ , \tag{2.1.4}$$

where z is defined as

$$z \equiv \tan\theta = \frac{\vec{n}_u x \vec{n}_v . \vec{n}}{\vec{n}_u . \vec{n}_v} \ . \tag{2.1.5}$$

Multiplying Eq. (2.1.4) by $\vec{n}_u$ and making use of the fact that $\vec{n}_u$ is orthogonal to $\vec{n}$ due to $\vec{n}^2$ equalling one, we get an equation of the form

$$\vec{n}_u \ . \left[\frac{\partial \vec{F}}{\partial v} + \frac{\partial \vec{G}}{\partial u}\right] = 0 \tag{2.1.6}$$

This can be written as

$$\frac{\partial}{\partial v}(\vec{F}.\vec{n}_u) + \frac{\partial}{\partial u}(\vec{G}.\vec{n}_u) - \vec{F} \ . \frac{\partial \vec{n}_u}{\partial v} - \vec{G} \ . \frac{\partial \vec{n}_u}{\partial u} = 0 \ . \tag{2.1.7}$$

By elementary manipulations we derive an equation for the norm of $\vec{n}_u$.

$$\frac{\partial}{\partial v}(|\vec{n}_u|^2(f - \frac{\partial f}{\partial z} z)) = 0 \ ,$$

$$|\vec{n}_u|^2(f - \frac{\partial f}{\partial z} z) = Q^2(u) \ . \tag{2.1.8}$$

Similarly, multiplying Eq. (2.1.4) by $\vec{n}_v$ leads to

$$\frac{\partial}{\partial u}(|\vec{n}_v|^2(f - \frac{\partial f}{\partial z} z)) = 0 ,$$

$$|\vec{n}_v|^2(f - \frac{\partial f}{\partial z} z) = H^2(v) \qquad (2.1.9)$$

Since our original action is form invariant under a local transformation of the form

$$(u,v) \to (u',v') ,$$
$$du' = |Q(u)|du , \qquad (2.1.10)$$
$$dv' = |H(v)|dv$$

We have

$$|\vec{n}_u|^2 = Q^2(u)|\vec{n}_{u'}|^2 ,$$

$$|\vec{n}_v|^2 = H^2(v)|\vec{n}_{v'}|^2 . \qquad (2.1.11)$$

Making use of the above transformations we may write Eqs. (2.1.8) and (2.1.9) as

$$(f - \frac{\partial f}{\partial z} z) \; |\vec{n}_{u'}|^2 = 1 ,$$

$$(f - \frac{\partial f}{\partial z} z) \; |\vec{n}_{v'}|^2 = 1 , \qquad (2.1.12)$$

or using Eq. (2.1.5)

$$|\vec{n}_{u'}|^2 = |\vec{n}_{v'}|^2 = (f - \frac{\partial f}{\partial \theta} \sin\theta\cos\theta)^{-1} \equiv \gamma(\theta) . \qquad (2.1.13)$$

Hereafter we take the transformed coordinates as the basic variables and omit the primes. This transformation yields a Hamiltonian density which vanishes when the constant part is subtracted. This difficulty of correctly defining the energy also arises in the conventional σ-model and can be dealt with using standard methods.

It has been shown by Pohlmeyer[33] that the integrability condition of the dynamical system described by Eq. (2.1.1) leads to the sine-Gordon equation. Here we proceed along similar lines for the Lagrangian (2.1.3) to find a generalization of the sine-Gordon equation. First we compute the mixed derivative $\vec{n}_{uv}$ and the second derivatives $\vec{n}_{uu}$ and $\vec{n}_{vv}$ in terms of the three basic vectors $\vec{n}_u$, $\vec{n}_v$, and $\vec{n}$ which span M. Because of the constraint $\vec{n}.\vec{n}$ equals one, these vectors are linearly independent provided $\vec{n}_u.\vec{n}_v$ does not vanish.

Then, we note the following equalities.

$$
\begin{aligned}
\vec{n}_{uv}.\vec{n} &= (\vec{n}_u.\vec{n})_v - \vec{n}_u.\vec{n}_v = -\gamma\cos\theta, \\
\vec{n}_{uv}.\vec{n}_u &= \frac{1}{2}(\vec{n}_u.\vec{n}_u)_v = \frac{1}{2}\gamma_v , \\
\vec{n}_{vv}.\vec{n}_v &= \frac{1}{2}(\vec{n}_u.\vec{n}_v)_u = \frac{1}{2}\gamma_u , \\
\vec{n}_{uu}.\vec{n} &= (\vec{n}_v.\vec{n})_u - \vec{n}_u.\vec{n}_u = -\gamma , \\
\vec{n}_{uu}.\vec{n}_u &= \frac{1}{2}(\vec{n}_u.\vec{n}_u)_u = \frac{1}{2}\gamma_u , \\
\vec{n}_{uu}.\vec{n}_v &= (\vec{n}_u.\vec{n}_v)_u - \vec{n}_u.\vec{n}_{uv} = \gamma_u\cos\theta - \gamma\sin\theta\,\theta_u - \frac{1}{2}\gamma_v , \\
\vec{n}_{vv}.\vec{n} &= (\vec{n}_v.\vec{n})_v - \vec{n}_v.\vec{n}_v = -\gamma , \\
\vec{n}_{vv}.\vec{n} &= \frac{1}{2}(\vec{n}_v.\vec{n}_v)_v = \frac{1}{2}\gamma_v , \\
\vec{n}_{vv}.\vec{n}_u &= (\vec{n}_u.\vec{n}_v)_v - \vec{n}_{uv}.\vec{n}_v = \gamma_v\cos\theta - \gamma\sin\theta\,\theta_v - \frac{1}{2}\gamma_u .
\end{aligned}
\tag{2.1.14}
$$

Making use of the above expressions we obtain

$$
\begin{aligned}
\vec{n}_{uv} &= -\gamma\cos\theta\vec{n} + \frac{\vec{n}_u}{2\gamma}(\gamma_v - \underline{\alpha}\cos\theta) + \frac{\vec{n}_v}{2\gamma}\underline{\alpha} , \\
\vec{n}_{vv} &= -\gamma\vec{n} + \frac{\vec{n}_u}{2\gamma}\underline{\beta} + \frac{\vec{n}_v}{2\gamma}(\gamma_v - \underline{\beta}\cos\theta) , \\
\vec{n}_{uu} &= -\gamma\vec{n} + \frac{\vec{n}_u}{2\gamma}(\gamma_u - \underline{\delta}\cos\theta) + \frac{\vec{n}_v}{2\gamma}\underline{\delta} ,
\end{aligned}
\tag{2.1.15}
$$

$$\underline{\alpha} = \frac{\gamma_u - \gamma_v \cos\theta}{\sin^2\theta}, \quad \underline{\beta} = \frac{\gamma_v \cos\theta - 2\gamma\sin\theta\,\theta_v - \gamma_u}{\sin^2\theta}, \quad \underline{\delta} = \underline{\beta} \text{ with } u \leftrightarrow v.$$

Next we substitute these vectors into the identity

$$\vec{n}_{uv}\cdot\vec{n}_{uv} = \frac{1}{2}(\vec{n}_u^{\,2})_{vv} + \frac{1}{2}(\vec{n}_v^{\,2})_{uu} - (\vec{n}_u\cdot\vec{n}_v)_{uv} + \vec{n}_{uu}\cdot\vec{n}_{vv} \tag{2.1.16}$$

The resulting expression yields a generalized version of the sine-Gordon equation in light-cone coordinates.

$$\begin{aligned} 2\gamma\sin\theta(\gamma\sin\theta + \theta_{uv}) &+ (\theta_u^{\,2} + \theta_v^{\,2} - 2\theta_u\theta_v\cos\theta)(\gamma'' - \frac{\gamma'^2}{\gamma}) \\ &- (\theta_u^{\,2}\cos\theta + \theta_v^{\,2}\cos\theta - 2\theta_u\theta_v)(\frac{\gamma'}{\sin\theta}) \\ &+ (\theta_{uu} + \theta_{vv} - 2\theta_{uv}\cos\theta)\gamma' = 0 . \end{aligned} \tag{2.1.17}$$

This equation is the integrability condition of the equations of motion of the Lagrangian (2.1.3). A choice of the function $\gamma=\gamma(\theta)$, through Eq. (2.1.13) determines the specific form of the Lagrangian. In contrast to the sine-Gordon case which is given by γ=const this equation is not Lorentz invariant in the general case. This is expected since a conformal transformation has already been performed in (2.1.10). It follows that the Lorentz invariance of the integrability condition for the standard σ-model is the result of the specific choice γ = const. In the next section we will show that there is a one parameter family of generalized σ-model Lagrangians for which Eq. (2.1.17) after a transformation of variables again leads to the sine-Gordon equation.

2.2. Special Cases Exhibiting Symmetries

In this section, we search for certain choices of γ which will reduce Eq. (2.1.17 to a system with some kind of additional symmetry. Therefore, we first look for an expression for γ which will render

Eq. (2.1.17) Lorentz invariant. The only choice is readily seen to be γ = const, which gives the sine-Gordon equation.

Next we try to make Eq. (2.1.17) Euclidean invariant. To this end, we separate the terms that multiply the mixed derivatives $\theta_u\theta_v$ and θ_{uv}. They are

$$\cos\theta \left(\gamma'' - \frac{\gamma'}{\sin\theta\cos\theta} - \frac{\gamma'^2}{\gamma}\right) , \tag{2.2.1}$$

$$\gamma'\cos\theta - \gamma\sin\theta , \tag{2.2.2}$$

respectively.

We note that when γ equals $c^2/|\cos\theta|$ both Eq. (2.2.1) and Eq. (2.2.2) are zero. Substituting this value for γ in Eq. (2.1.17) we get an Euclidean-invariant equation

$$2c^2\sin\theta + (\theta_u^{\,2}+\theta_v^{\,2})\tan\theta+(\theta_{uu}+\theta_{vv}) = 0 \tag{2.2.3}$$

We multiply this equation by an integrating factor ϕ'. When ϕ' equals $1/\cos\theta$, Eq. (2.2.3) reduces to

$$\phi_{uu} + \phi_{vv} + 2c^2\sinh\phi = 0, \tag{2.2.4}$$

where the "potential" is

$$v(\phi) = 2c^2\cosh\phi. \tag{2.2.5}$$

Another simple case reveals itself when we impose $x \to x' = f(x)$ symmetry. In other words, we require that the integrability condition be x-independent. Going back to one-space and one-time coordinates it is seen that Eq. (2.1.17) can be written as

$$2\gamma\sin\theta + \left[\theta_t\left(1 + \frac{\gamma'}{\gamma}\tan\frac{\theta}{2}\right)\right]_t - \left[\theta_x\left(1 - \frac{\gamma'}{\gamma}\cot\frac{\theta}{2}\right)\right]_x = 0 \tag{2.2.6}$$

When γ equals $c^2/(1+\cos\theta)$, Eq. (2.2.6) reduces to

$$c^2 \tan\frac{\theta}{2} + \left[\frac{1}{2}\,\theta_t \sec^2\frac{\theta}{2}\right]_t = 0 \qquad (2.2.7)$$

Defining $\tan(\theta/2)$ as ϕ, we get the harmonic oscillator equation

$$c^2\phi + \phi_{tt} = 0 \qquad (2.2.8)$$

For this case we note that the tangent of the angle between the vectors $\vec{n}_u$ and $\vec{n}_v$ oscillates in time with a period proportional to the norm of these vectors.

We proceed along similar lines to get the time independent version of Eq. (2.2.6). When γ is $c^2/(1-\cos\theta)$, Eq. (2.2.6) becomes

$$c^2 \cot\frac{\theta}{2} - \left[\frac{1}{2}\,\theta_x \csc^2\frac{\theta}{2}\right]_x = 0\ . \qquad (2.2.9)$$

Letting $\cot(\theta/2)$ be defined as ϕ yields the harmonic oscillator equation for ϕ,

$$c^2\phi + \phi_{xx} = 0\ . \qquad (2.2.10)$$

This time the oscillation is in space.

Using the Eq. (2.1.13) which shows the relation between γ and f, we can summarize our results as follows. When the Lagrangian density is given by Eq. (2.1.1), the integrability condition is Lorentz invariant; it yields the sine-Gordon equation, a well-known result. However, when +(-) $|\vec{n}_u||\vec{n}_v|$ is added to the Lagrangian density, the corresponding integrability condition becomes time(space) independent and reduces to the harmonic oscillator equation. Finally $\vec{n}_u\cdot\vec{n}_v$ replaced by $|\vec{n}_u||\vec{n}_v|$ gives the Euclidean-invariant integrability condition, or the Euclidean sinh-Gordon equation.

All these four cases can be unified with a variable transformation as follows. Looking back at Eq. (2.2.6) we define

$$\phi' = \frac{\gamma'}{\gamma} \tan\frac{\theta}{2} + 1 \ . \tag{2.2.11a}$$

$$\psi' = 1 - \frac{\gamma'}{\gamma} \cot\frac{\theta}{2} \ . \tag{2.2.11b}$$

Hence Eq. (2.2.6) can be written as

$$2\gamma\sin\theta + \phi_{tt} - \psi_{xx} = 0 \ . \tag{2.2.12}$$

The simplest relation between ϕ and ψ would be

$$\phi = a^2\psi \ , \tag{2.2.13}$$

where a^2 is a constant. In order to satisfy Eq. (2.2.13) γ and f have to take the special values

$$\gamma = c^2\{a^2 + 1 + \cos\theta(a^2-1)\}^{-1} \ ,$$

$$f = c^{-2}\left[a^2 + 1 + \frac{a^2-1}{\cos\theta}\right] . \tag{2.2.14}$$

Disregarding an overall constant, these determine a family of Lagrangians depending on the parameter a. Substituting this in Eq. (2.2.11a) we get

$$a \tan\frac{\phi}{2a} = \tan\frac{\theta}{2} \ . \tag{2.2.15}$$

Using this as the definition of ϕ, Eq. (2.2.12) reduces to

$$c^2\sin\frac{\phi}{a} + a\,\phi_{tt} - \frac{1}{a}\,\phi_{xx} = 0 \ . \tag{2.2.16}$$

Rescaling ϕ and our time (or x) coordinates we get the sine-Gordon equation for ϕ. Furthermore from Eq. (2.2.14) we can see that when a^2 is positive, negative, zero or infinity we get the sine-Gordon, the Euclidean sinh-Gordon, the time-independent or the x-independent equations respectively. So all four cases are unified when we impose Eq. (2.2.13) upon our integrability condition.

2.3. Behavior of Time-Independent Solutions

In this section we consider only the time independent solutions of Eq. (2.2.6) which are given by

$$2\gamma\sin\theta-\left[\theta_x(1-\frac{\gamma'}{\gamma}\cot\frac{\theta}{2})\right]_x=0\ . \tag{2.3.1}$$

Before we set up a procedure for finding solutions systematically, we consider one special case. We try γ of the form

$$\gamma = a^2(1+\cos\theta)^b\ . \tag{2.3.2}$$

Substituting this in Eq. (2.3.1) we get the following potential

$$\frac{1}{2}\theta_x^{\,2}=\frac{2}{(1+b)^2}(c^2-a^2(1+\cos\theta)^{1+b})\ , \tag{2.3.3}$$

where c^2 is a constant of integration. We can put Eq. (2.2.3) in a closed form for x by letting u equal $\tan(\theta/2)$,

$$x=(1+b)\int^{\tan\theta/2}du(c^2(1+u^2)^2-a^2 2^{b+1}(1+u^2)^{1-b})^{-1/2}\ . \tag{2.3.4}$$

We first note that b equals minus one will make x equal zero. This case corresponds to the x-independent case of Section 2.2. We recover the sine-Gordon limit when b equals zero. So the next simple case is given by b equals one. We get

$$x=\frac{2}{c}\left[\frac{1}{g}F(\alpha,q)\right]\ , \tag{2.3.5}$$

where

$$q=2\sqrt{\frac{a}{2a+c}}\ ,$$

$$g = \sqrt{\frac{2a+c}{c}} \,,$$

$$\tan\alpha = \sqrt{(c-2a)^{-1}\, c} \quad \tan\frac{\theta}{2}$$

and F is an elliptic integral of the first kind. Still another manifestly integrable case is b equals minus two. This yields x as a linear combination of elliptic integrals of the first and third kinds. These solutions can be compared to the time-independent solution of the sine-Gordon equation which is an elliptic integral of the first kind.

In order to find out which function f in the Lagrangian would give these solutions, we go back and solve Eq. (2.1.13) and get

$$f(\theta) = \frac{3}{4a^2} - (4\, a^2\cos\theta)^{-1} - \frac{(1 - \cos\theta)^2}{12a^2\cos\theta(1+\cos\theta)} \tag{2.3.6}$$

when b equals one.

Going on to the general case, we note that when we let

$$\phi' = (1 - \frac{\gamma'}{\gamma}\cot\frac{\theta}{2}) \,, \tag{2.3.7}$$

$2\gamma\sin\theta$ term in Eq. (2.3.1) can be written as

$$-2\, c\, \frac{d}{d\phi}\,(\gamma(1+\cos\theta)) \,, \tag{2.3.8}$$

Identifying $2c\gamma(1+\cos\theta)$ with the potential we write Eq. (2.3.1.) as follows

$$\frac{1}{2}\,\phi_x^{\;2} + V(\phi) = c \tag{2.3.9}$$

Going back to the Eq. (2.3.7) we note, after integration,

$$\phi = \theta - \int^{\theta} \frac{\gamma'}{\gamma}\cot\frac{\theta'}{2}\, d\theta' \,. \tag{2.3.10}$$

We can extract some information from this equation. We immediately notice that the sine-Gordon limit, where γ is a constant gives ϕ equals θ.

Furthermore, when γ equals $c/(1+\cos\theta)$, Eq. (2.3.1) becomes x-independent, as we realized before. This gives ϕ equals zero as expected.

$\gamma(\theta)$ must be an even, periodic function of θ if the Lagrangian is to be parity invariant. Hence, the integrand in Eq. (2.3.10) must be even. When integrated it will, in general, give another term proportional to θ plus an odd, periodic function of θ. Therefore, by rescaling ϕ if necessary, we find ϕ and θ differ by an odd, periodic function. In this case any periodic function of θ, when expressed as a function of ϕ, is again periodic. Hence the potential $V(\phi)$ in Eq. (2.3.9) is periodic and the time independent solutions will exhibit soliton behaviour. In cases where γ is chosen such that it cancels the term proportional to θ in Eq. (2.3.10) ϕ will be a periodic function of θ, and $V(\phi)$ is not necessarily periodic in ϕ. Then the time-independent solutions need not exhibit soliton behaviour. The Euclidean sinh-Gordon equation provides an example for this case.

III. GEOMETRICAL PROPERTIES AND INVARIANCES OF THE GENERALIZED SIGMA MODEL

We exploit the methods that have been developed for the sine-Gordon equation and other totally integrable systems to further investigate the generalized sine-Gordon equation (GSG). In Section 3.1 we utilize geometrical techniques to show that just like the sine-Gordon equation[(34)], the GSG equation can be derived from the condition that its underlying surface has a Gaussian curvature equal to one. We then consider the imbedding of this surface in a three-dimensional flat space. In Section 3.2, we formulate the GSG equation as an inverse scattering problem according to the general framework provided by Lax[(35)] and Ablowitz, Kaup, Newell, and Segur(LAKNS)[(8)]. Thereby we derive the soliton connection and obtain a self-Bäcklund transformation. In Section 3.3 we use the group theoretical approach to investigate the invariance properties of the GSG equation under a transformation of the variables. This invariance study once more singles out the special cases of the GSG equation exhibiting extra symmetries, which in Section 2.2, have been shown to be unified with a variable transformation.

3.1. Surface Theory and the Generalized Sigma Model

In this section we investigate the nature of the surface underlying the generalized sigma model. To this end, we identify the first fundamental form and the Gaussian curvature of the surface. Then we consider the imbedding of the surface in a three-dimensional flat space and obtain the appropriate second fundamental form.

The GSM is described by the Lagrangian density given in Eq. (2.1.3). The action

$$I = \int L \, du \, dv \tag{3.1.1}$$

is conformally invariant. The three-dimensional unit vector $\vec{n}$ geometrically describes the surface of a unit sphere. Since $\vec{n}$ is a function of u and v, provided that this function is nonsingular, these variables, or equivalently x and t may be used as the coordinates of this two-dimensional surface. In part II it was shown that the equations of motion and the conformal invariance of the model can be utilized to choose coordinates such that

$$\begin{aligned} \vec{n}_u^2 = \vec{n}_v^2 &\equiv e^{2\alpha} \\ &= (f - \frac{\partial f}{\partial \theta} \sin\theta\cos\theta)^{-1} \\ &= \gamma(\theta) \ , \end{aligned} \tag{3.1.2}$$

where α is a function of θ which is determined by the function f in the Lagrangian. Thus the metric on the unit sphere is given by

$$\begin{aligned} ds^2 = d\vec{n}^2 &= (\vec{n}_v dv + \vec{n}_u du)^2 \\ &= e^{2\alpha}(\cos^2 \frac{\theta}{2} \, dt^2 + \sin^2 \frac{\theta}{2} \, dx^2) \ . \end{aligned} \tag{3.1.3}$$

From this equation we immediately recognize the basis one-forms

$$\begin{aligned} \omega^1 &= e^{\alpha} \cos \frac{\theta}{2} \, dt \\ \omega^2 &= e^{\alpha} \sin \frac{\theta}{2} \, dx \end{aligned} \tag{3.1.4}$$

such that

$$ds_1^2 = (\omega^1)^2 + (\omega^2)^2 \ . \tag{3.1.5}$$

The connection one-form

$$\omega_{\mu\beta} = -\omega_{\beta\mu} \tag{3.1.6}$$

is determined from the integrability condition

$$d\omega^{\mu} + \omega^{\mu}_{\ \beta} \wedge \omega^{\beta} = 0 \tag{3.1.7}$$

and is given by

$$\omega^{1}_{\ 2} = (\alpha_x \cot\frac{\theta}{2} - \frac{1}{2}\,\theta_x)\ dt - (\alpha_t \tan\frac{\theta}{2} + \frac{1}{2}\,\theta_t)\ dx\ . \tag{3.1.8}$$

The Gaussian curvature K is defined by

$$d\omega^{\mu}_{\ \nu} = K\ \omega^{\mu} \wedge \omega^{\nu}. \tag{3.1.9}$$

Letting the Gaussian curvature equal to one we get

$$((2\alpha' \cot\frac{\theta}{2} - 1)\theta_x)_x + ((2\alpha' \tan\frac{\theta}{2} + 1)\theta_t)_t + + 2e^{2\alpha}\sin\theta = 0 \tag{3.1.10}$$

This is the integrability condition for the GSM, identical to Eq. (2.2.6).

Having identified the intrinsic geometry underlying the GSM, we consider the imbedding of the surface S in a three-dimensional flat space M. We define the second fundamental form of S through

$$-ds_2^2 = \pi^1 \otimes \pi^1 + \pi^2 \otimes \pi^2\ . \tag{3.1.11}$$

The equation governing the imbedding problem is

$$d\omega^{i}_{\ k} + \omega^{i}_{\ j} \wedge \omega^{j}_{\ k} = 0 \tag{3.1.12}$$

where the indices range over three values. This is the Gauss-Codazzi equation for the imbedding problem with the identification

$$\omega^1{}_3 = \pi^1 ,$$
$$\omega^2{}_3 = \pi^2 . \tag{3.1.13}$$

Furthermore the definition of the surface S as

$$\omega^3 = 0 \tag{3.1.14}$$

imposes another condition

$$\omega^1 \wedge \pi^1 + \omega^2 \wedge \pi^2 = 0 . \tag{3.1.15}$$

We see that Eqs. (3.1.12) - (3.1.15) are all satisfied when we let

$$\pi^1 = \omega^1 ,$$
$$\pi^2 = \omega^2 . \tag{3.1.16}$$

3.2. The Soliton Connection and the Bäcklund Transformation

Having established the geometrical framework underlying the GSM, we construct an SL(2,R) valued connection one-form with zero curvature. This is the soliton connection. We perform a gauge transformation in order to cast it into LAKNS form. Finally we find the Bäcklund transformation for the GSM. Our approach closely resembles that of Ref. 25.

The Gauss-Codazzi equations for imbedding surfaces in a three-dimensional flat space form a realization of Cartan's equations for SL(2,R)

$$d\Theta^i + \frac{1}{2} C_j{}^i{}_k \Theta^j \wedge \Theta^k = 0 , \tag{3.2.1}$$

where

$$C_{1\ 2}^{\ 0} = 1$$

$$C_{0\ 1}^{\ 1} = -\ C_{0\ 2}^{\ 2} = 2 \tag{3.2.2}$$

are the structure constants of SL(2,R), with the identification

$$\Theta^0 = \frac{i}{2}\,\omega^1{}_2 \ ,$$

$$\Theta^1 = -\,\frac{1}{2}\,(\omega^2 + i\omega^1) \ , \tag{3.2.3}$$

$$\Theta^2 = \frac{1}{2}\,(\omega^2 - i\omega^1) \ .$$

For the GSM, these are

$$\Theta^0 = \frac{i}{2}\,((\alpha' \cot\frac{\theta}{2} - \frac{1}{2})\theta_x)dt - ((\alpha' \tan\frac{\theta}{2} + \frac{1}{2})\theta_t)dx \ ,$$

$$\Theta^1 = -\,\frac{1}{2}\,(e^{\alpha} \sin\frac{\theta}{2}\,dx + ie^{\alpha} \cos\frac{\theta}{2}\,dt) \ , \tag{3.2.4}$$

$$\Theta^2 = \frac{1}{2}\,(e^{\alpha} \sin\frac{\theta}{2}\,dx - ie^{\alpha} \cos\frac{\theta}{2}\,dt) \ .$$

Now we construct a connection one-form Γ ,

$$\Gamma = \begin{bmatrix} \Theta^0 & \Theta^1 \\ \Theta^2 & -\Theta^0 \end{bmatrix} \tag{3.2.5}$$

with a vanishing curvature

$$d\Gamma + \Gamma \wedge \Gamma = 0 \ . \tag{3.2.6}$$

This can be traced back to Gauss-Codazzi equations through Eq. (3.2.1).

Now we can briefly summarize the LAKNS formalism. The integrability conditions for the systems of linear partial differential equations

$$V_{1x} - i\zeta V_1 = q\, V_2 \, ,$$
$$V_{2x} + i\zeta V_2 = r\, V_1 \, , \tag{3.2.7}$$

where the eigenfunctions V_1 and V_2 evolve in time according to

$$V_{1t} = A\, V_1 + B\, V_2 \, ,$$
$$V_{2t} = C\, V_1 - A\, V_2 \, , \tag{3.2.8}$$

are

$$A_x = q\, C - r\, B$$
$$B_x - 2i\zeta B = q_t - 2q\, A \tag{3.2.9}$$
$$C_x + 2i\zeta C = r_t + 2r\, A \, .$$

Here A, B, and C are functions of x, t and ζ, where ζ is a constant. If we construct a connection one-form with the identification

$$\Theta^0 = -\,(A\, dt + i\zeta dx) \, ,$$
$$\Theta^1 = -\,(B\, dt + q\, dx) \, , \tag{3.2.10}$$
$$\Theta^2 = -\,(C\, dt + r\, dx) \, ,$$

the condition that its curvature vanishes yields Eq. (3.2.9). We want to identify the set described in Eq. (3.2.10) with that in Eq. (3.2.4) to read off the LAKNS potentials for the GSM. However, as they stand, these two sets of equations are incompatible since ζ has to be a constant. To circumvent this problem we perform a gauge transformation

$$\Gamma' = \Lambda\Gamma\, \Lambda^{-1} + \Lambda d\Lambda^{-1} \tag{3.2.11}$$

in order to cast Eq. (3.2.4) into LAKNS form described by Eq. (3.2.10).

Here Λ has a determinant equal to unity. We find that for the GSM, Λ has a very simple form

$$\Lambda = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\zeta x} & e^{-i\zeta x} \\ -e^{i\zeta x} & e^{i\zeta x} \end{pmatrix} \tag{3.2.12}$$

Equating the transformed one-form Γ' with Eq. (3.2.10) we get

$$\begin{aligned}
A &= \frac{i}{2} e^{\alpha} \cos\frac{\theta}{2} \\
C &= \frac{i}{2} e^{2i\zeta x} (\alpha' \cot\frac{\theta}{2} - \frac{1}{2})\theta_x \\
B &= \frac{i}{2} e^{-2i\zeta x} (\alpha' \cot\frac{\theta}{2} - \frac{1}{2})\theta_x \\
q &= \frac{1}{2} e^{-2i\zeta x} (e^{\alpha} \sin\frac{\theta}{2} - i\,(\alpha' \tan\frac{\theta}{2} + \frac{1}{2})\theta_t) \\
r &= -\frac{1}{2} e^{2i\zeta x} (e^{\alpha} \sin\frac{\theta}{2} + i\,(\alpha' \tan\frac{\theta}{2} + 1)\theta_t) \;.
\end{aligned} \tag{3.2.13}$$

The eigenvalue ζ usually has a real and a imaginary part. If we restrict its value to be real for simplicity then

$$r = -\,q^* \;. \tag{3.2.14}$$

Furthermore,

$$B = -\,C^* \;. \tag{3.2.15}$$

In the literature, it has been shown that the Gelfand-Levitan-Marchenko integral equation associated with the inverse scattering problem is uniquely solvable if $r = -\,q^*$[36]. For this specific case, there is also a general method for deriving the Bäcklund transformation from the equations for the inverse problem[18]. This is as follows. To conform with the notation in Ref. 18 we let ζ go to $-\zeta$ in Eq. (3.2.7). Then, defining a quantity $U = V_1/V_2$ we get a system of equations

$$U_x = -2i\zeta U - r\,U^2 + q$$

$$U_t = 2A\,U - C\,U^2 + B\ . \tag{3.2.16}$$

For the GSM we have

$$U_x = -2i\zeta U + q^*\,U^2 + q \tag{3.2.17a}$$

$$U_t = 2A\,U + C\,U^2 - C^*\ , \tag{3.2.17b}$$

where

$$A = \frac{1}{2}\left(\frac{q_t\ e^{2i\zeta x} + cc}{q\ e^{2i\zeta x} - cc}\right)$$

$$C = \frac{i}{2}\left(\frac{1}{q + q^*\ e^{-4i\zeta x}}\right)\left(\frac{iq_t\ e^{2i\zeta x} - cc}{-q\ e^{2i\zeta x} + cc}\right)_x\ . \tag{3.2.18}$$

Using Eq. (3.2.17a) and its complex conjugate we derive an expression for q in terms of U, U^*, U_x and U^*_x. Substituting this expression in Eq. (3.2.17b), we can eliminate q and thereby get a non-linear partial differential equation for U and U^*. This equation is invariant under the transformation

$$(U,\ \zeta) \to (-U,\ \zeta) \tag{3.2.19}$$

The existence of this gaugelike invariance makes it possible to find a self-Bäcklund transformation since we know we have a second solution q' such that

$$U_x = -2i\zeta U - q'^*\,U^2 - q' \tag{3.2.20a}$$

$$U_t = 2A\,U - C\,U^2 + C^* \tag{3.2.20b}$$

Here A and C are functions of q', q'_t, q'_x and q'_{xt} and their complex conjugates. Subtracting Eq. (3.2.20a) from Eq. (3.2.17a) we obtain the following expression for U

$$U = \pm i \frac{|q+q'|}{(q+q')^*} H(x-x_0+4\zeta t) \qquad (3.2.21)$$

where H is the Heaviside step function. In order to get the spatial part of the Bäcklund transformation for the GSG equation, we add Eq. (3.2.20a) and Eq. (3.2.17a) and then substitute for U from Eq. (3.2.21). Rearranging terms we get

$$(q-q')_x = -2i\zeta(q+q')+i(q-q')|q\pm q'| H(x-x_0+4\zeta t) \qquad (3.2.22)$$

Similarly we add Eqs. (3.2.20b) and (3.2.17b) and substitute for U to get the temporal part of the Bäcklund transformation.

3.3. Invariances and Infinitesimal Properties

In this section we shall investigate the invariance properties of the GSG equation under one-parameter Lie group of transformations[23]. We shall find the invariant variable in terms of which we can reduce the partial differential equation with two independent variables and one dependent variable to an ordinary differential equation. Finally we find the explicit solution of this ODE for a special case.

Let us rewrite the GSG equation, Eq. (3.1.10).

$$2\gamma(\theta)\sin\theta + \theta_{tt}F_1(\theta) - \theta_{xx}F_2(\theta) + \theta_t^2 F_1' + \theta_x^{\ 2} F_2'$$

$$= H(\theta,\theta_{xx},\theta_{tt},\ldots) = 0 \qquad (3.3.1)$$

Here

$$F_1 = 1 + \frac{\gamma_\theta}{\gamma} \tan\frac{\theta}{2} \qquad (3.3.2a)$$

$$F_2 = 1 - \frac{\gamma_\theta}{\gamma} \cot \frac{\theta}{2} \tag{3.3.2b}$$

$$\gamma = e^{2\alpha} \tag{3.3.3}$$

and subscripts denote partial differentiation with respect to θ. If H is invariant under the one parameter (ε) group of transformations obtained from the infinitesimal transformation

$$\begin{aligned} x' &= x + \varepsilon\xi(x,t,\theta) \\ t' &= t + \varepsilon\tau(x,t,\theta) \\ \theta' &= \theta + \varepsilon\Omega(x,t,\theta) \end{aligned} \tag{3.3.4}$$

through exponentiation, then

$$XH = 0 \tag{3.3.5}$$

where X is the operator

$$X = \xi\frac{\partial}{\partial_x} + \tau\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta} + \left(\Omega_x\right)\frac{\partial}{\partial\theta_x} + \left(\Omega_{xx}\right)\frac{\partial}{\partial\theta_{xx}} \cdots \tag{3.3.6}$$

Here $\left(\Omega_x\right)$ and $\left(\Omega_{xx}\right)$ are the infinitesimals for θ_x and θ_{xx} respectively whose explicit forms in terms of ξ, Ω, and τ can be obtained from Eq. (3.3.4). Furthermore if H is invariant under the transformation defined by Eq. (3.3.4) then the following equation must hold

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{d\theta}{\Omega} \tag{3.3.7}$$

Using Eq. (3.3.5), we find the infinitesimals Ω, ξ and τ. Then substituting these values in Eq. (3.3.7) we get a solution with two arbitrary constants. One of them, β, will be the similarity variable. The other one, f(β) will be the independent variable.

For Eq. (3.3.1), when we impose the condition (3.3.5) and collect together the like-derivative terms in θ we get

$$\eta = 0$$

$$\tau = \tau(x) + b \qquad (3.3.8)$$

$$\xi = \xi(t) + b$$

restricted by the following equations

$$\tau_x F_2 = \xi_t F_1$$

$$(3.3.9)$$

$$\tau_x F_2' = \xi_t F_1'$$

Equation (3.3.9) tells us immediately that either

$$F_1 = a^2 F_2 , \qquad (3.3.10)$$

where a is just a constant, or

$$\tau_x = \xi_t = 0 . \qquad (3.3.11)$$

This equation is the same as Eq. (2.2.13) in Section II. If this relation holds, then the integrability condition of the GSM reduces to the sine-Gordon, the Euclidean sinh-Gordon, the time-independent or the space-independent equations for different values of a^2. The invariance properties of the sine-Gordon equation has been analyzed in Ref.23; the other three cases are related to the sine-Gordon equation by a variable transformation.

If we do not have the special case described by Eq. (3.3.9) then using Eqs. (3.3.11), (3.3.8) and (3.3.7) we find the invariant variables β and $f(\beta)$

$$x - \frac{a}{b} t = \beta \qquad (3.3.12)$$

$$u = f(\beta) .$$

This reduces the GSG equation to a particularly simple form,

$$\left\{(F_2 - \frac{a^2}{b^2} F_1)f'\right\}' = 2\gamma \sin f \qquad (3.3.13)$$

where primes denote differentiation with respect to β.

IV. CANONICAL STRUCTURES FOR DISPERSIVE WAVES IN SHALLOW WATER

The equations governing long waves in shallow water consist of a familiar pair of coupled first order partial differential equations which can be interpreted as a Hamiltonian system in several different ways. First, Luke's[(37)] variational principle for these equations was cast into canonical form by Zakharov[(38)], Broer[(39)], and Miles[(40)]. But with this approach it was not possible to obtain an explicit expression for the exact Hamiltonian in terms of the canonical variables. Later Manin[(41)] considered the symplectic structure of shallow water waves from a completely different point of view. Recently a new formulation of these equations by Nutku[(30)] in terms of potentials led to the construction of the requisite Hamiltonian through the use of Dirac's theory of constraints[(31)]. In addition to these Hamiltonian structures the equations for shallow water waves admit an infinite number of conservation laws which are in involution relative to Manin's symplectic form, or alternatively possess vanishing Poisson brackets with the Hamiltonian[(28)].

This variety of interesting structures makes the theory of shallow water waves a prototype of two-dimensional field theories. It will be of interest to find out which properties of this system of equations are stable in the sense that they survive in an appropriately generalized form when the equations are modified to accomodate new effects. To this end we shall now consider the theory of shallow water waves including dispersion. We shall find that it is possible to generalize both Manin's symplectic structure and the Hamiltonian obtained by applying Dirac's theory to a new variational formulation of the equations for dispersive waves. However the infinite set of conservation laws are lost.

The plan of this part is as follows: In Section 4.1 we consider the equations governing dispersive waves in shallow water and their variant, the Boussinesq equations. By introducing potentials we construct new variational principles for these equations. It is not a straight-forward task to formulate the correct initial value problem for the Boussinesq equations as they possess time derivatives of an order higher than that of the shallow water limit. We shall show in Section 4.2 that with the help of the variational principle the Cauchy problem for the Boussinesq equations can be posed correctly, and in general it entails the solution of a constraint which is given by a differential equation. We shall consider the Hamiltonian formulation of dispersive waves in Section 4.3. The Lagrangian is degenerate, as in the limit of no dispersion and once again we shall use Dirac's theory in order to cast this system into canonical form. In Section 4.4 we shall generalize Manin's symplectic structure to allow for the effects of dispersion for both types, including the Boussinesq equations. These are four conservation laws for dispersive waves corresponding to the conservation of mass, linear momentum and energy, and one further equation which follows as a consequence of these three. In Section 4.5 we shall construct these conserved quantities, first obtained by Whitham[42] in the Boussinesq case, and prove that there are no further conservation laws in either case. Finally, in Section 4.6 we discuss the invariance properties for the shallow water wave equations without dispersion.

4.1. Potentials and Variational Principles

We refer to Whitham[28] for a complete discussion of the equations of motion for long waves in shallow water including the effects of dispersion. In deriving these equations we are first led to

$$h_t + (hu)_x = 0 , \qquad (4.1.1a)$$

$$u_t + uu_x + h_x + \nu h_{3x} = 0 , \qquad (4.1.1b)$$

where u is the velocity and h the height of the fluid. Subscripts denote partial derivatives with respect to time and space coordinates, t and x. Shallow water equations without dispersion are obtained when the constant ν in Eq. (4.1.1b) is set equal to zero. In place of Eqs. (4.1.1) Boussinesq[43] has proposed

$$h_t + (hu)_x = 0 , \qquad (4.1.2a)$$

$$u_t + uu_x + h_x + \nu h_{xtt} = 0 , \qquad (4.1.2b)$$

for dealing with dispersive effects.

Here and in the following we shall first discuss the system of Eqs. (4.1.1) and then consider the Boussinesq case of Eqs. (4.1.2). In this way we hope to avoid disturbing the continuity of the discussion.

We shall start with a reformulation of these equations in terms of potentials. For this purpose note that Eqs. (4.1.1) are the conditions for the one-forms

$$\omega^1 = hdx - hudt \qquad (4.1.3a)$$

$$\omega^2 = udx - (\tfrac{1}{2} u^2 + h + \nu h_{xx})dt \qquad (4.1.3b)$$

to be closed

$$d\omega^1 = 0 \quad , \quad d\omega^2 = 0 \; . \tag{4.1.4}$$

Therefore, using Poincaré's lemma we have locally

$$\omega^1 = d\Psi \quad , \quad \omega^2 = d\Phi \tag{4.1.5}$$

where Φ and Ψ are scalar potentials. In term of components, Eqs. (4.1.5) and (4.1.3) yield the relation

$$\Phi_x = u \quad , \quad \Phi_t = -\left(\frac{1}{2}u^2 + h + vh_{xx}\right) , \tag{4.1.6a}$$

$$\Psi_x = h \quad , \quad \Psi_t = -\,uh \; , \tag{4.1.6b}$$

between the phenomenological fields u, h and the potentials Φ, Ψ. The integrability conditions of Eqs. (4.1.6) yield the original equation of motion, and their compatibility requires that

$$\Psi_t + \Phi_x \Psi_x = 0 \tag{4.1.7a}$$

$$\Phi_t + \frac{1}{2}\Phi_x^2 + \Psi_x + \nu\Psi_{xxx} = 0 \tag{4.1.7b}$$

which are non-linear partial differential equations satisfied by the potentials.

These equations can be derived from an action principle

$$\delta I = 0 \quad , \quad I = \int L_1 dx\, dt$$

where

$$L_1 = \Phi_t \Psi_x + \Phi_x \Psi_t + \Phi_x^{\;2} \Psi_x + \Psi_x^{\;2} - \nu \Psi_{xx}^{\;\;2} \tag{4.1.8}$$

is the Lagrangian density.

The introduction of potentials for the Boussinesq equations follows along similar lines. The differences consist of substituting h_{tt} in place of h_{xx} in Eqs. (4.1.3b) and (4.1.6b) which results in the equations

$$\Psi_t + \Phi_x \Psi_x = 0 \tag{4.1.9a}$$

$$\Phi_t + \frac{1}{2}\Phi_x^{\;2} + \Psi_x + \nu\Psi_{xtt} = 0 \tag{4.1.9b}$$

for the potentials. Finally

$$L_2 = \Phi_t \Psi_x + \Phi_x \Psi_t + \Phi_x^{\;2}\Psi_x - \nu\Psi_{xt}^{\;\;2} + \Psi_x^{\;2} \tag{4.1.10}$$

is the Lagrangian for the Boussinesq equations.

4.2. Cauchy Problem for the Boussinesq Equations

The difference between Eqs. (4.1.1) and (4.1.2) may at first sight appear to be slight but in fact these are two completely different sets of equations. In particular the initial value problems for these equations exhibit important differences. The Cauchy data for Eqs. (4.1.1) is essentially the same as that of the ordinary shallow water theory obtained in the limit $\nu \to 0$. On the other hand in Eqs. (4.1.2) we find time derivatives of an order higher than that of the dispersionless limit $\nu \to 0$ which changes the character of these equations drastically.

In order to isolate the constraints and the dynamical variables for the Boussinesq equations we shall start with a variational formulation of these equations. For this purpose it is necessary to modify the Lagrangian in Eq. (4.1.10) so that it will involve only the first derivatives of the potentials. We shall therefore introduce another potential T and with its help express the Lagrangian L_2 in the form

$$L_3 = \Phi_t\Psi_x + \Phi_x\Psi_t + \Phi_x{}^2\Psi_x - \kappa(T_t\Psi_x + T_x\Psi_t) - 2T\,\Psi_x + \rho T^2 + (1+\rho)\Psi_x{}^2 \tag{4.2.1}$$

where

$$\kappa = |\nu|^{1/2}\ , \quad \rho = \text{sign}\ (\nu) \tag{4.2.2}$$

Variations with respect to Φ, Ψ, T lead to

$$\{\Psi_t + \Phi_x\Psi_x\} = 0 \tag{4.2.3a}$$

$$\{\Phi_t - \kappa T_t + \frac{1}{2}\Phi_x{}^2 - T + (1+\rho)\ \Psi_x\}_x = 0\ , \tag{4.2.3b}$$

$$T - \rho\Psi_x + \rho\kappa\Psi_{tx} = 0 \tag{4.2.3c}$$

respectively. Remembering that $\Phi_x = u$, $\Psi_x = h$ we see that Eqs. (4.2.3) reduce to Eqs. (4.1.2).

We see that the time derivatives of Φ and T appear only in the combination $\Phi - \kappa T$ in the Lagrangian (4.2.1) and subsequently in Eqs. (4.2.3). This suggests that we define

$$\Sigma = \Phi - \kappa T \tag{4.2.4}$$

and eliminate T in favor of Σ from the problem. In this case we find

$$\Psi_t + \Phi_x\Psi_x = 0\ , \tag{4.2.5a}$$

$$\Sigma_t + \frac{1}{2}\Phi_x{}^2 - \frac{1}{\kappa}\ (\Phi - \Sigma) + (1+\rho)\ \Psi_x = 0\ , \tag{4.2.5b}$$

$$\Phi - \Sigma - \rho\kappa\Psi_x - \nu(\Phi_x\Psi_x)_x = 0 \tag{4.2.5c}$$

where we have omitted possible arbitrary functions of time on the right hand sides of Eqs. (4.2.5a,b).

Equation (4.2.5c) contains no time derivatives of the potentials and is therefore a constraint equation which must be satisfied at every slice t = const. Given the initial values of Φ, Ψ and Σ subject to Eq. (4.2.5c) we can solve Eqs. (4.2.5a,b) to obtain the values of Ψ and Σ at the next instant of time. But Φ itself is only a constraint variable and its time evolution will be determined by inserting the new values of Ψ and Σ into Eq. (4.2.5c) and solving this differential equation for Φ at that instant.

4.3. Hamiltonian

The Hamiltonian formulation of a system of equations is not very useful if the Hamiltonian cannot be given by a local expression in terms of the canonical variables. The existence of a constraint which is a differential equation such as the one in Eq. (4.2.5c) makes it impossible to construct a local Hamiltonian for the Boussinesq equations. So we shall not consider the Boussinesq equations in this section. On the other hand Eqs. (4.1.1) present a different case and we shall now use Dirac's theory to construct the appropriate Hamiltonian for these equations.

The variational principle in Eq. (4.1.8) is not suitable for passing to a Hamiltonian formulation because the Lagrangian contains the second derivatives of Ψ. In order to obtain a variational principle where the action functional depends only on the first derivatives, once again we shall introduce another potential T. We can readily verify that the Euler-Lagrange equations for

$$L_4 = \Phi_t \Psi_x + \Psi_t \Phi_x + \Phi_x^{\,2} \Psi_x - 2\,\kappa T_x \Psi_x - 2\,T\Psi_x + \rho T^2 + (1+\rho)\,\Psi_x^{\,2} \qquad (4.3.1)$$

yield Eqs. (4.1.1) together with

$$T = \rho\Psi_x - \rho\kappa\Psi_{xx} \tag{4.3.2}$$

which serves as the definition of T.

The Lagrangian (4.3.1) is degenerate. That is, the canonical momenta

$$\begin{aligned} \Pi_\Phi &= \Psi_x \ , \\ \Pi_\Psi &= \Phi_x \ , \\ \Pi_T &= 0 \ , \end{aligned} \tag{4.3.3}$$

cannot be inverted for the velocities and we need to use Dirac's theory of constraints in order to cast this system into canonical form. Therefore we introduce

$$\begin{aligned} C_1 &= \Pi_\Phi - \Psi_x \ , \\ C_2 &= \Pi_\Psi - \Phi_x \ , \\ C_3 &= \Pi_T \ , \end{aligned} \tag{4.3.4}$$

as primary constraints. Using the canonical Poisson brackets between the potentials and their conjugate momenta we find that

$$\{C_1(x),\ C_2(x')\} = -2\delta_x\ (x-x') \ , \tag{4.3.5}$$

is the only non-vanishing one among the Poisson brackets of the constraints. The primary constraints are therefore second class. The total Hamiltonian

$$H = \int \mathcal{H} dx \qquad , \qquad \mathcal{H} = \mathcal{H}_o + \mathcal{H}'$$

will be given by

$$H_o = \Pi_\Phi \Phi_t + \Pi_\Psi \Psi_t + \Pi_T T_t - L \;, \tag{4.3.6a}$$

$$H' = \lambda C_1 + \sigma C_2 + \upsilon C_3 \tag{4.3.6b}$$

where λ, σ, and υ are Lagrange multipliers. These multipliers will be determined from the requirement that the Poisson bracket of the Hamiltonian with each one of the constraints should vanish. But we find that

$$\{C_3, H\} = 2(\rho T - \Psi_x + \kappa\Psi_{xx}) \tag{4.3.7}$$

cannot be set equal to zero because it is independent of the multipliers. Therefore we introduce a secondary constraint

$$\chi = 2(\rho T - \Psi_x + \kappa\Psi_{xx}) \tag{4.3.8}$$

and modify Eq. (4.3.6b) as

$$H' = \lambda C_1 + \sigma C_2 + \upsilon C_3 + \mu\chi \tag{4.3.9}$$

where μ is another multiplier. There are no further constraints in this problem because with the choice

$$\sigma = -\Phi_x \Psi_x$$

$$\mu = T - \rho\Psi_x - \rho\kappa\Psi_{xx}$$

$$\upsilon = -\rho(\Phi_x \Psi_x)_x - \rho\kappa(\Phi_x \Psi_x)_{xx}$$

$$\lambda = -\frac{1}{2}\Phi_x^{\;2} - \Psi_x - \nu\Psi_{xxx} \tag{4.3.10}$$

the Poisson brackets of the Hamiltonian with C_1, C_2, C_3 and χ all vanish. From Eqs. (4.3.6a,b), (4.3.3), (4.3.8) and (4.3.10) we find the total Hamiltonian density

$$H = \frac{1}{2}\,\Phi_x{}^2\Psi_x + \rho\Psi_x{}^2 + \nu\Psi_{xx}{}^2 + \rho T^2 - 2\,T\Psi_x + 2\,\kappa T\Psi_{xx} - \Pi_\Phi(\frac{1}{2}\Phi_x{}^2 + \Psi_x + \nu\Psi_{3x})$$

$$- \Phi_x\Psi_x\Pi_\Psi - \{\rho(\Phi_x\Psi_x)_x - \rho\chi(\Phi_x\Psi_x)_{xx}\}\,\Pi_T \qquad (4.3.11)$$

where we have discarded a divergence. In terms of u, h Eq. (4.3.11) is, up to a divergence, equivalent to

$$H = \frac{1}{2}\,u^2h + \frac{1}{2}\,h^2 + \nu hh_{xx} + \frac{1}{2}\,\nu h_x{}^2 \qquad (4.3.12)$$

by virtue of Eqs. (4.1.6) and (4.3.3). As we shall reconfirm in the next section, this is the energy density for dispersive waves in shallow water.

4.4. Symplectic Structure

The symplectic geometry of the equations governing shallow water waves was first studied by Manin[41] (see also, Cavalcante and McKean[29] and the references contained therein). We shall now extend Manin's symplectic structure to include dispersive waves.

For Eqs. (4.1.1) the phase space consists of the set

$$\{u,\ h;\ h_x,\ h_{xx}\} \qquad (4.4.1)$$

of infinitely differentiable real functions of period one and

$$\nabla = (\frac{\partial}{\partial u}\ ,\ \frac{\partial}{\partial h}\ ,\ \frac{\partial}{\partial h_x}\ ,\ \frac{\partial}{\partial h_{xx}}) \qquad (4.4.2)$$

will denote the gradient in function space. If A, B are two smooth functions of these variables the Poisson bracket is defined to be

$$\{A,B\} = \int_0^1 \nabla A J \nabla B dx \qquad (4.4.3)$$

where

$$J = \begin{bmatrix} & D & & \\ D & & & \\ & & & \Delta \\ & & \Delta & \end{bmatrix} \tag{4.4.4a}$$

$$D = -\frac{\partial}{\partial x}\ , \tag{4.4.4b}$$

$$\Delta_1 = \frac{1}{\nu}\frac{\partial^2}{\partial t \partial x} \tag{4.4.4c}$$

is the Hamiltonian operator. With this definition of Poisson bracket, Jacobi's identity

$$\{A, \{B,C\}\} + \{B, \{C,A\}\} + \{C, \{A,B\}\} = 0 \tag{4.4.5}$$

is satisfied. From Eq. (4.3.12) (see also Eq. (8) in sequel) we find that the Hamiltonian is given by

$$H = \int \left(\frac{1}{2} u^2 h + \frac{1}{2} h^2 + \nu h h_{xx} + \frac{1}{2} \nu h_x^{\ 2}\right) dx \tag{4.4.6}$$

We can verify that with y running over the variables in (4.4.1) Hamilton's equations

$$y_t = \{y,H\} \tag{4.4.7}$$

reduce either to Eqs. (4.1.1) or identities and in fact it was this requirement which led us to the choice of variables in (4.4.1) and the definition of Hamiltonian operator (4.4.4), given the Hamiltonian (4.4.6). An integral of motion P will satisfy

$$\{P,H\} = 0 \tag{4.4.8}$$

and this condition amounts to

$$P_{uu} - hP_{hh} + \nu P_{uxxu} = 0 \tag{4.4.9a}$$

$$P_{h_x} - (P_{h_{xx}})_x = 0 \qquad (4.4.9b)$$

generalizing the result of Lax[44] for shallow water waves.

The Boussinesq equations (4.1.2) admit a symplectic structure with the following modifications to the above scheme. In place of the variables in (4.4.1) we consider the set

$$\{u,h;\ h_t,\ h_{tt}\} \qquad (4.4.10)$$

where the space derivatives of h are replaced by time derivatives and this change carries over into the definition of the gradient ∇. The definition of the Poisson bracket via Eqs. (4.4.3) and (4.4.4) is the same as before except that now

$$\Delta_2 = \frac{1}{\nu}\frac{\partial^2}{\partial t^2} \qquad (4.4.11)$$

in place of Eq. (4.4.4c). The Hamiltonian for the Boussinesq equations

$$H = \int \left(\frac{1}{2}\,u^2h + \frac{1}{2}\,h^2 + \nu hh_{tt} + \frac{1}{2}\,\nu h_t^{\,2}\right)\,dx \qquad (4.4.12)$$

differs from Eq. (4.4.6) according to the general rule that space derivatives of h are replaced by time derivatives. Similarly we find that for P to be an integral of motion

$$P_{uu} - hP_{hh} + P_{uttu} = 0 \qquad (4.4.13a)$$

$$P_{h_t} - (P_{h_{tt}})_t = 0\ . \qquad (4.4.13b)$$

4.5. Conservation Laws

The equations for shallow water waves admit an infinite sequence of conservation laws of the form

$$P_t + Q_x = 0 \qquad (4.5.1)$$

where P, Q are functions of u, h. The existence of infinitely many integrals of motion makes it possible to construct exact solutions. We shall now show that this property is lost when the effects of dispersion are included.

Whitham[42] has found that the Boussinesq equations (4.1.2) which are already in the form of conservation laws admit two further ones corresponding to the conservation of momentum and energy. The analogous conservation laws for Eqs. (4.1.1) are

$$(uh)_t + (u^2h + \frac{1}{2}h^2 + \nu hh_{xx} - \frac{1}{2}\nu h_x{}^2)_x = 0$$

$$(\frac{1}{2}u^2h + \frac{1}{2}h^2 + \nu hh_{xx} + \frac{1}{2}\nu h_x{}^2)_t + (\frac{1}{2}u^3h + uh^2 + \nu hh_{xx} - \nu hh_{xt})_x = 0 \qquad (4.5.2)$$

respectively. We can read off P from these equations and verify that it satisfies Eqs. (4.4.9) in each case. In particular, the conserved quantity for the latter of Eqs. (4.5.2) is the Hamiltonian (4.4.6).

There are no further conservation laws for $\nu \neq 0$. In order to see this let us consider the fifth conservation law. The requirement that it reduce to the shallow water expression leads to

$$(\frac{1}{3}u^3h + uh^2 + f)_t + (\frac{1}{3}u^4h + \frac{3}{2}u^2h^2 + \frac{1}{3}h^3 + g)_x = 0 \qquad (4.5.3)$$

where f and g depend on u, h, h_x, h_t, h_{xx}, ... and vanish in the limit $\nu \to 0$. Then we find

$$f_t + g_x = \nu\,(u^2h + h^2)h_{xxx} \qquad (4.5.4)$$

and using Eqs. (4.1.1) repeatedly this can be cast it into the form

$$f_t + g_x = \nu\,(h_x h_t)_t + \nu\,(u^2 hh_{xx} + h_x\,(uh)_t + h^2 h_{xx} - \frac{1}{2}\,h_t^{\,2})_x$$

$$- \nu hh_x h_{xx} + \nu^2 hh_{xx} h_{3x}\ . \qquad (4.5.5)$$

The first two groups of terms on the right hand side of Eq. (4.5.5) are of the desired form but it is not possible to write either one of the last two terms as a total divergence. It will be sufficient to prove this only for one of them, say $hh_x h_{xx}$. In order to express this term as a divergence we consider all possible divergences which can result in such an expression. Thus we write

$$hh_x h_{xx} = a(hh_x^{\,2})_x + b(h^2 h_{xx})_x + c(h^2 h_x)_{xx} \qquad (4.5.6)$$

where a, b and c are constants which must be chosen so as to make this an identity. Note that it is unnecessary to include $(h^3)_{3x}$ in Eq. (4.5.6) since it reduces to the last term above. From the coefficients of all linearly independent functions in Eq. (4.5.6) we obtain a system of linear equations for a, b and c. This system of equations has no solution, which makes it impossible to express $hh_x h_{xx}$ as a total divergence. Therefore there is no fifth conservation law for Eqs. (4.1.1). Similarly we can prove that the Boussinesq Eqs. (4.1.2) do not admit conservation laws beyond those already given by Whitham.

4.6. Invariances

We shall now point out an invariance of the equations for shallow water waves which appears not to have been noted before. This invariance is non-trivial only in the limit $\nu \to 0$ and, as in the case of infinitely many conservation laws, it is lost when we include the effects of dispersion.

We shall use Lie's theory of one-parameter group of transformations to analyze the invariance properties of Eqs. (4.1.1) with ν equal to zero. We shall find the invariant variables for this system and thereby reduce the partial differential equations to ordinary ones[(23)]. This will enable us to construct an exact solution of these equations.

Let us rewrite Eqs. (4.1.1) with ν equal to zero in the form

$$\begin{aligned} H_1(u,h,h_x,\ldots) &\equiv u_t + uu_x + h_x = 0 \\ H_2(u,h,h_x,\ldots) &\equiv h_t + uh_x + hu_x = 0 \end{aligned} \tag{4.6.1}$$

If these equations are invariant under the following infinitesimal transformations

$$\begin{aligned} x &= x + \varepsilon\xi\ (x,t,u,h) \\ t &= t + \varepsilon\tau\ (x,t,u,h) \\ u &= u + \varepsilon\eta\ (x,t,u,h) \\ h &= h + \varepsilon\delta\ (x,t,u,h) \end{aligned} \tag{4.6.2}$$

where ε is an infinitesimal parameter, then

$$XH_1 = XH_2 = 0 \ ,$$

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \delta\frac{\partial}{\partial h} + \{\eta_t\}\frac{\partial}{\partial u_t} + \{\delta_t\}\frac{\partial}{\partial h_t} \cdots \tag{4.6.3}$$

with $\{\eta_t\}$, $\{\delta_t\}$ ... denoting the first order changes in the derivatives of u_t, h_t, ... In terms of u, h, η, τ, δ, ξ the explicit expressions for the "higher extensions" $\{\eta_t\}$, $\{\delta_t\}$ can be obtained by using Eqs. (4.6.2) and a typical extension is

$$\{\delta_t\} = \delta_t + h_t(\delta_h - \tau_t) + u_t(\delta_u - \tau_u h_t) - \xi_t h_x - \xi_h h_t h_x - \xi_u u_t h_x - \tau_h h_t^2 \tag{4.6.4}$$

From Eqs. (4.6.4) we get

$$\{\eta_t\} + \eta_x + u\{\eta_x\} + \{\delta_x\} = 0$$
$$\{\delta_t\} + u\{\delta_x\} + \delta u_x + h\{\eta_x\} + \eta h_x = 0 \qquad (4.6.5)$$

into which we must substitute extensions. Eqs. (4.6.5) are algebraic equations for the variables u, h and their derivatives. Since these variables are linearly independent we require that their coefficients depending on $(\eta, \xi, \tau, \delta)$ vanish separately. This leads to

$$\xi = at + b$$
$$\eta = a$$
$$\tau = c \qquad (4.6.6)$$
$$\delta = 0$$

where a, b and c are arbitrary constants. Then from

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta} = \frac{dh}{\delta} \qquad (4.6.7)$$

we find

$$cdx = (at + b)\ dt \qquad (4.6.8a)$$

$$adt = cdu \qquad (4.6.8b)$$

$$adx = (at + b)\ du \qquad (4.6.8c)$$

$$\delta = const \qquad (4.6.8d)$$

with can be integrated readily.

Integrating Eq. (4.6.8a) we obtain the invariant variable

$$\left(\frac{at + b}{c}\right)^2 - \frac{2ax}{c} = const \equiv \beta \qquad (4.6.9)$$

and then from Eqs. (4.6.8b,c) we find

$$\beta - \text{const} = \{u - \frac{at + b}{c}\}^2 \tag{4.6.10}$$

which suggests that we define

$$g\ (\beta) \equiv u - (\frac{at + b}{c}) \tag{4.6.11}$$

as an invariant function. Finally we shall take the right hand side of Eq. (4.6.8d) as $f(\beta)$ and change from the variables (u,h,x,t) to $(f(\beta), g(\beta),\beta)$. In this way we can write Eqs. (4.6.1) as a pair of coupled ordinary differential equations for f and g

$$\begin{aligned} gg' + f' &= 1/2 \\ gf' + fg' &= 0 \end{aligned} \tag{4.6.12}$$

where prime denotes differentiation with respect to β. These equations can be integrated and we find a solution where f and g both satisfy the algebraic equation

$$f^3 - (\frac{1}{2}\beta + C_2)\ f^2 + \frac{1}{2} C_1{}^2 = 0 \tag{4.6.13}$$

and C_1, C_2 are constants. This solution contains five arbitrary constants.

When we apply this procedure to Eqs. (4.1.1) which include effects of dispersion, we find different results. The differences stem from the fact that when ν is different from zero Eqs. (4.6.8) are replaced by

$$\begin{aligned} \tau &= a \\ \xi &= b \\ \eta &= 0 \\ \sigma &= 0 \end{aligned} \tag{4.6.14}$$

and this leads to the standard choice

$$\beta = x - \frac{b}{a} t \tag{4.6.15}$$

as the invariant variable.

With the Ansatz (4.6.15) Eqs. (4.1.1) can be integrated to yield

$$x - \frac{b}{a} t = \frac{b}{a\sqrt{\nu}} \int U^{-1/2} \, hdh \ ,$$

where

$$U = h^4 - 2C_2 h^3 - 2C_3 h^2 - C_1{}^2 h \ ,$$

and

$$u = \frac{C_1}{h} + \frac{b}{a} \tag{4.6.16}$$

which results in elliptic functions.

V. SUMMARY

There are still no general notions of what soliton solutions really are, which classes of equations possess them, and how they are related to the presence of a canonical structure and properties assumed to be associated with complete integrability, such as Bäcklund and inverse scattering transformations, an infinite sequence of conservation laws, and an imbedding structure. In this work, we considered two different models that have been important in the theory of solitons; the O(3) σ-model, and the shallow water waves equation in two-dimensions. We have analyzed generalizations of these models to find out which properties survive modifications.

First, we have constructed a family of classical two-dimensional O(3) σ-models whose integrability condition is a generalization of the sine-Gordon equation. Searching for special cases of this equation which exhibit symmetries, we have found a one-parameter family of Lagrangians whose integrability condition is again given by the sine-Gordon equation. At special values of this parameter the integrability condition abruptly changes from the sine-Gordon equation to the Euclidean sinh-Gordon equation, whereas at precisely these values the equation becomes either the time-independent or the x-independent one-dimensional harmonic oscillator equation. Thus the physical behaviour of the system undergoes a change at these special values of the parameter. Going back to the general case, we have shown that the time-independent solutions in general, but not always, exhibit solitary waves. In some cases these solutions, just like the solitons of the sine-Gordon equation, are described by elliptic functions.

We have then exploited the methods that have been developed for totally integrable systems to further investigate the generalized sine-Gordon equation. We have utilized geometrical techniques to show that just like the sine-Gordon equation, the GSG equation can be derived from the condition that its underlying surface has a Gaussian curvature equal to one. We have then considered the imbedding of this surface in a three-dimensional flat space. We have also formulated the GSG equation as an inverse scattering problem according to the general framework provided by Lax[35] and Ablowitz, Kaup, Newell, and Segur[36] (LAKNS). Thereby we have derived the soliton connection and obtained a self-Bäcklund transformation. Furthermore, we have used the group theoretical approach to investigate the invariance properties of the GSG equation under a transformation of variables. This invariance study has singled out, once again, the special cases of the GSG equation exhibiting extra symmetries.

In the study of shallow water waves with dispersion we have emphasized those aspects of complete integrability which we had not considered in the case of the generalized σ-model, namely the existence of a canonical structure and an infinite sequence of conservation laws. As in the case of shallow water equations, we have introduced two potentials for long waves with dispersion. The integrability condition for these potentials has yielded the equations of motion. However, since the Lagrangian in terms of just these potentials included second derivatives, we had to introduce a third potential. The ensuing Lagrangian was degenerate. Therefore, we have used Dirac's theory of constraints to cast it into canonical form, which enabled us to express the Hamiltonian explicitly in terms of the three potentials and their conjugate momenta. We have also been able to generalize this Hamiltonian structure by defining the Poisson brackets for the system, whereby we have cast the system into symplectic form. Through this symplectic structure, we have derived the general equation which all the integrals of motion satisfy. However, we have also shown that it is not possible to write down explicit forms of any more than four integrals. Furthermore we have analyzed the invariance properties of shallow water waves with dispersion. In the process we have found one explicit solution of the dispersionless case and one for the case with dispersion in terms of elliptic functions.

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