NOVEL RESULTS ON FREQUENCY ESTIMATION AND STATISTICAL CHARACTERIZATION OF THREE SPECTRAL ESTIMATION TECHNIQUES

by

Emin Anarın

B.Sc. in Electrical Eng., Boğaziçi University, 1981

M.Sc. in Electrical Eng., Bogaziçi University, 1983

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NOVEL RESULTS ON FREQUENCY ESTIMATON
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APPROVED BY

Doç. Dr. Bülent SANKUR
(Thesis Supervisor)

Prof. Dr. Nezihi CANITEZ

Doç. Dr. Yorgo ISTEFANOPULOS

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ABSTRACT

The problem of detecting a sinusoidal signal in white noise and estimation of its parameters is essentially a problem in signal processing such as radar, sonar, biomedical, etc. In this dissertation, various modern spectrum estimation approaches, Maximum Entropy Spectral Analysis (MESA) spectral moments and analytic signal techniques and their statistical characterization have been investigated and formulated in detail.

The development of modern spectrum estimation known as parametric techniques for estimating parameters of sinusoïdal signals in white noise is important. Therefore the parameter estimation technique based on previously appeared and as well as some other newly developed modern spectrum estimation procedures have been presented in this dissertation. Comparative performances and drawbacks of most of the parametric techniques known as ; Maximum Likelihood (ML), Maximum Entropy (ME), Pisarenko, Kumerason, Prony methods used in frequency estimation have been summarized.

The analytic signal model called as Argument method to estimate frequency and bandwidth of a sinusoidal signal is studied in detail. New expressions related to the expected value , variance and probability density function of estimate are derived analytically.

The problem of the estimation of spectral moments and their applications are also studied. Special attention is given to the case of sinusoids in white noise. In fact, for this case, properties, statistics and asymptotic behaviour of the spectral moments are investigated. The moment estimates are also used for the hypothesis testing problem where the alternative and null hypotheses represent the tone in noise and noise only situations, respectively. The potential use of the moments for the tone frequency estimation is also considered.

Finally this dissertation deals with the effectiveness of the Autoregressive method in frequency estimation problem. In this sense, the coefficient deviation and root displacement of AR polynomial are formulated for different structured matrices such as circulant, transformed circulant, complex Toeplitz and real symmetric. The frequency analysis of AR method based on Sakai's and Taylor series approximation methods is studied. The analysis establishes the relationship between frequency error and coefficient deviation. The statistical characterization of radius of signal poles is formulated as a final study.

ÖZETÇE

Beyaz gürültüye bulanmış sinusoidal imlerin sezimi ve parametrelerinin kestirimi ; radar, sonar, biomedical ve benzeri problemlerde esaslı bir sorun teşkil etmektedir. Bu tezde çeşitli çağdaş izge kestirimleri, en büyük entropy izge momentleri ve analitik im teknikleri ve bunların istatistiksel karekteristikleri incelenip formülize edilmiştir.

Sinusoidal imlerin parametrelerinin kestirimi için kullanılan, parametrik yöntemler olarak da bilinen çağdaş izge kestirimlerindeki gelişmeler önemlidir. Bu sebeble; daha önce kullanılan yöntemlerle birlikte, zamanımızda kullanılan yöntemler bu tezde sunulmuştur. Çağdaş izge kestirim yöntemleri olarak bilinen; en büyük olabilirlik, en büyük entropi, Kumerason, Prony gibi yöntemlerin başarımları ve sorunları özetlenmiştir.

Beyaz gürültüdeki sinüsoidal imin sıklık ve bant genişliğinin kestiriminde kullanılan ve argument yöntemi oyarak da bilinen analitik im yöntemi detaylı olarak incelenmiştir. Kestiricinin; beklenti, değişinti ve olasılık dağılım islevine ait yeni ifadeler elde edilmiştir.

İzge momentlerini kestirimi ve uygulamaları üzerinde de çalışılmıştır. Özel olarak sorun, beyaz gürültü içindeki sinüsoidal im durumu için irdelenmiştir. Bu durum için izge momentlerinin; özellikleri, istatistiksel ve sonuşur davranışları araştırılmıştır. Kestirilen momentlerin hipotez sınavı için kullanılması ve

diğer önemli uygulamalarda ele alınmıştır.

Otoregresif modelin sıklık kestirimindeki etkinliği çalışmanın son bölümünü kapsamaktadır. Çeşitli dizey yapılarına göre, otoregresif çokterimlisinin katsayı sapması ve kök değişimi analitik olarak elde edilmiştir. Sıklık hata analizi Sakai ve Taylor seri açılımı anlamında irdelenmiştir. Analiz sıklık hataları ile katsayı sapmaları ve kök değişimleri ile ilgili bağlantılar içermektedir. Son bölümde bu köklerin yarıçaplarının istatistiksel davranışları türetiltiştir.

LIST OF FIGURES

NOMENCLATURE

Abbreviations

F.F.T.	Fast Fourier Transform.
D.F.T.	Discrete Fourier Transform.
M.E.M.	Maximum Entropy Method.
C.R.L.B.	Cramer Rao Lower Bound.
P.M.	Pisarenko Method.
SNR	Signal to Noise Ratio.
N (m,)	White Gaussian noise, mean=m , variance=
Y.W.E.	Yule-Walker Equations.
H.O.Y.W.E.	Higher Order Yule-Walker Equations.
A.R.	Autoregressive
A.R.M.A.	Autoregressive-Moving Average.
R.M.L.	Recursive Maximum Likelihood.
A.C.	Autocorrelation Function.
S.D.	Steepest Descent
C.G.M.	Conjugate Gradient Method.
I.P.M.	Inverse Power Method.
F.B.L.P.	Forward-Backward Linear Prediction.
S.V.D.	Singular Value Decomposition.
A/D	Analog to Digital Convresion.
L.P.F.	Low Pass Filter.
BW	Bandwidth.
M.E.S.A.	Maximum Entropy Spectral Analysis.
P.E.F.	Prediction Error Filter.

List of symbols

N	Number of data.
M	Model order.
L	Number of sinusoids.
λ_i	i´th eigenvalue.
t	Time index.
T	Sampling period.
w	Frequency index.
x	Discrete time samples.
r	Autocorrelation function for lag k.
a_i	i´th coefficient of AR process.
A	Power of a sinusoid.
R_x^c	Complex Toeplitz matrix.
R_x^r	Real symmetric matrix.
$P_x(.)$	Probability density function of x.
σ_n^2	Noise power.
ϕ	Phase of a sinusoid.
n_k	k´th noise sample.
$\sigma_{\hat{f}}^2$	Variance of the estimated frequency.
S(w)	Estimated power spectrum.
Δa_i	i´th coefficient deviation of AR polynomial.
Δw	Frequency error.
r_k'	Autocorrelation function (circular) for lag k.

Special symbols

$\wedge$	Estimated value of (.)
T	Transpose of (.).
E [.]	Expected value of (.).
Var [.]	Variance of (.).
Cov[.]	Covariance of (.).
|matrix|	Determinant of the matrix.
|.|	Absolute value of (.).
$\_$	Vector.
$=$	Matrix.
*	Complex conjugate of (.).
$[\text{matrix}]^{-1}$	Inverse of the matrix.
Re[.]	Real part of (.).
Im[.]	Imaginary part of (.)
arg [.]	Argument of (.).

TABLE OF CONTENTS

CHAPTER I : INTRODUCTION

A summary of many of the new techniques developed in the last years for spectrum estimation of discrete time analysis is presented in this section.

1. 1. LITERATURE SURVEY

Power spectrum estimation has progressed through several stages for at least 18th century. In evaluating spectral estimation methods, three criteria are usually used. The first resolution: the ability of an estimate to reveal the presence of two equal power tone (sinusoidal signal) whose frequency are close to each other. When two sinusoidal signals are resolved, there are two peaks in the spectrum. If not resolved, only one peak is present. Also for good resolution, the peaks must be narrower and sharper. Note that resolved peaks do not necessarily imply that the peaks are located at the proper frequency. The second criterion is therefore the bias of the estimate. When one source is present, the bias (the error in the location of the spectral peak or the difference between tone frequency and the location of the spectral peak) is rarely zero or usually non zero. These two criteria of the "goodness" of a spectrum may conflict: good resolution is often obtained at the expense of a biased estimate. The third criterion is variability: the range of the frequency over which the location of a spectral peak can be expected to vary. Analytic evaluation of the variability for a given spectral estimate is usually difficult. The Cramer Rao lower bound on the variance is usually used as a benchmark to evaluate the measured performance of a given method. Therefore a major problem in the time series analysis is choosing an algorithm to estimate the spectrum from a finite observation of the process in such way that the estimate is not dominated by bias, is consistent and statistically meaningful and maintains these properties in the presence of some variations of assumptions.

Before going ahead, let us give the paragraph from [1] which can be expressed originally as follows:

Let us say a few words about the terms "spectrum" and "spectral". Sir Isaac Newton introduced the scientific term "spectrum" using the Latin word for an image. Today in English, we have the word spectre meaning ghost or appariation and the corresponding adjective spectral. We also have the scientific word spectrum and the dictionary list the word spectral as the corresponding adjective. Thus "spectral" has two mewanings. Many feel that we should be creful to use "spectrum" in place of "spectral" **a)** whenever the reference is to data or physical phenomena and **b)** whenever the word "estimation".

The modern history of spectrum estimation begins with Blackman and Tukey [2] in 1949 which is actually the statistical counterpart of the classical Fourier transform. Their method for spectrum estimation can be summarized as follows:

-Estimating the autocorrelation function from the discrete observations.

-Windowing the autocorrelation function in an appropriate manner.

-Fourier transforming the windowed autocorrelation function to finally obtain the estimated spectrum.

This method made possible an active development of spectral estimation by researchers. However, it was computationally expensive. A significant decrease in computational complexity was achieved with the publication in 1965 of the Fast Fourier Transform (FFT) algorithm by Cooley and Tukey [3] and this method became popular and is still used today.

Since Blackman-Tukey Spectral Estimate is the Fourier Transform of the windowed autocorrelation estimate, various procedures are used to estimate the autocorrelation function. The objective is usually to obtain a minimum bias and minimum variance estimate of the true autocorrelation function. Similarly the estimate of the autocorrelation function is windowed to reduce the bias and variance of the spectral estimate thus increasing its statistical stability. Various window functions are used which generally unrelated to the data or random process being analysed. But the finite record length of the autocorrelation function estimate and the windowing process applied to the autocorrelation function decrease the resolution of the spectrum. Another approach for reducing the variance of the estimate is known as segmental averaging [4]. In this case the data record is decimated into independent segments and the autocorrelation function of each segment is estimated from which the average is calculated. Then the power spectrum estimate is the Fourier transform of the average of the autocorrelation functions. Variations on this task include windowing each data segment prior to estimating the segmental autocorrelation functions and/or windowing each segmental autocorrelation function. The expected value of these power spectral estimate is biased which is easily seen in the frequency domain. The expected value of the estimate is the convolution of the spectrum of the segmental window with the true or actual power spectrum. The bias of these estimates exceeds that of the unsegmented or complete data record since the data segments are shorter than the complete data record, thus implying that the main lobe of the segmental spectral window is broader than the window used for the previous case. However the variance of the estimate is less than the variance of the unsegmented estimate by a factor equal to the number of segments which are assumed uncorrelated. Also segmental averaging procedure decrease the resolution with respect to previous one. An alternative method used in spectrum estimation for reducing the variance of the estimate is the Barlett method. It is based on calculating the periodogram of individual segments of data and averaging. This is similar to averaging the autocorrelation function of each segment and then transforming.

Further important contribution in spectrum estimation were the introduction of Maximum Entropy developed by Burg [5] and Maximum Likelihood by Capon [6]. The major attraction of these two methods is that they show considerable promise for estimating spectrum when the length of available data is short.

Maximum Entropy method attempts to fit, in a Least Squares sense, an Autoregressive (AR) model to an input time series [7]. There are two general techniques for estimating the filter coefficents $\left[a_1, a_2, \ldots a_p\right]$ One first proposed by Yule [8] and by Walker [9], involves the solution of the normal equations and necessarily requires explicit knowledge of the autocorrelation function of the input data. The other , associated with Burg estimates the AR parameters without prior knowledge of the autocorrelation funtion.

Maximum Likelihood method spectral estimate may be derived by solving a classical optimal filtering problem. The filter is designed to pass the power in a narrow band about the signal frequency of interest and minimize the power due to undesired spectral components such as noise, or interferance. The Maximum Likelihood method may be considered a minimum variance, unbiased estimator of the spectral components [10].

Edward and Fitelson [11] formally proved Burg´s observation that the Maximum Entropy method provides a spectrum which maximizes the entropy of a stationary random process consistent with the first M lags of the autocorrelation function. The Maximum Entropy method suggests instead of appending zeroes to increase the length of the estimated autocorrelation function that they should be extrapolated or predicted beyond the data limited range. The objective is to add no information as a result of the extrapolation process. In contrast to Maximum Entropy method, the Maximum Likelihood method does not provide an extension of the autocorrelation function and the inverse Fourier transform of the Maximum Likelihood spectral estimation does not in general agree with the measured autocorrelation values.

Pusey has shown that both methods can resolve tones which are close together for any nonzero timebandwidth product if the signal to noise ratio is sufficently high. This can not be achieved with the conventional Fourier methods unless of course, the sampling rate and the number of data are increased [12].

The Maximum Entropy method and Maximum Likelihood method are related to each other. If the MEM spectrum are calculated for k=1,2.......M and the average of the reciprocal of these spectrum determined then it is equal to the reciprocal of the Maximum Likelihood spectrum [11]. This result implies that the Maximum Likelihood spectrum is more stable statistically but has less resolution than the Maximum Entropy spectrum. Lacoss [13] has compared the Maximum Entropy with Maximum Likelihood and conventional Fourier methods and demonstrated the superiority of the Maximum Entropy spectrum in terms of spectral resolution. Furthermore Burg [14] showed the superior resolving power of Maximum Entropy compared with the Maximum Likelihood approach.

Burg method has the drawbacks of line splitting in low noise case and frequency shifting in the high noise case. Spectral line splitting is the resolution of two or more closely spectral peaks when only one is truly present. The problem of line splitting in Maximum Entropy spectrum was first documented by Fougere et. al. [15]. They noted that the spectral line splitting was most likely to occur when

i. The signal to noise ratio is high

ii. The initial phase of sinusoidal is some odd multiple of 45 degrees

iii. Time duration of the data sequence is such that sinusoidal components have an odd number of quarter cycles.

iv. Number of filter parameters estimated is a large percentage of the number of data values used for estimation.

The connection between line splitting and the number of filter parameters estimated i.e., model order highlights a problem area common to all of non-linear spectrum methods how to select the model order. Akaike [16] has suggested two popular criteria for order determination. However this author´s experience has shown that most order selection, including Akaike´s are not effective against the line splitting phenomenom. Therefore in the method proposed by Fougere [15], the filter coefficents are redetermined iteratively by starting with the Burg´s filter coefficents. The iterative procedure involves nonlinear optimization with respect to the filter coefficients with the constraint that the filter must be stable by requiring the reflection coefficents to have magnitudes les than one. The constraint is necessary to provide a filter (prediction error filter) which yields the lowest possible error power. The Fougere´s method not only removes the drawback of the Burg´s method but also provides a much better spectral resolution. In [17], the variance of the frequency estimation of the Fougere´s method is compared with the Cramer-Rao bound and the Burg´s method. For the same filter length which is 10, the Fougere´s method has a threshold signal to noise ratio of 8.5 db which is much lower compared to threshold at 15 db for the Burg´s method.

A second problem with the Burg algorithm as with Yule-Walker case is the bias in the position of spectrum peaks with respect to the true frequency location of those peaks. If one defines the sampling frequency f=1/2T where T is the sampling rate, then the bias changes with the true fractional frequency location of the spectral peaks. The spectral peaks with fractional frequencies from zero to 0.5 f tend to be biased more than the spectral peaks with fractional frequencies from 0.5 f to f than their actual values. Swingler [18] has shown that the bias can pull the peak off frequency by as much as 16% of a resolution cell when using the Burg algorithm. Marple [19] has proposed a new algorithm hsa proposed a new algorithm for filter parameter estimation that yields the spectrum with no apparent line splitting and reduced spectral peak frequency estimation biases. This method has the same order of computational complexity as the burg algorithm. Frequency bias in processing sinusoidal signals with the Burg algorithm has been experimentally investigated by Chen [20] and Stegen for the noisy case and theoretically analyzed by Swingler [18] for noiseless case.

Another modification to Burg algorithm is proposed by Nikias and Scott [21] to obtain high resolution in the persence of noise. Their energy-weighted method introduces a weight whichis a function of the power of the received data in the prediction error calculations. They also analyzed the frequency bias of the energy-weighted method in processing sinusoidal signals, for the noiseless case [22]. Consequently the energy-weighted method produces frequency bias is generally less than the Burg technique bias.

In an attempt to improve the Maximum Entropy method with respect to bias, Nutall [23] and Ulrych and Clayton [24] proposed the Least-Squares fitting of an AR model based on a criterion involving both forward and backward prediction errors but unlike Burg method, without using a lattice filter model. Marple [25] derived a recursive form of the above method and has shown that the bias in the spectral estimates can be reduced significantly compared to the Burg technique. It should be noted that this is true only for short data lenghts, since all technique give identical results for large data lenghts.

Another all poles or AR spectrum estimation is the Yule-Walker (Y.W) or autocorrelation technique [26]. This method substitutes biased estimate of the autocorrelation lags generated form available data into the Yule-Walker normal equations. AR spectrum estimation method sometimes termed Maximum Entropy spectrum method also has become popular alternative to the periodogram as an estimate of the power spectrum for a sampled process. For signal to noise ratios (SNR's) greater than zero decibel, the AR power spectrum estimate has higher frequency resolution than the conventional Fourier type estimate [27]. The lower the signal to noise ratio, the more all pole assumption is violated and poorer the spectral estimate obtained [28]. Or we can state simply when noise is added to the time series under analysis, the resolution of the spectral estimator decreases rapidly as signal to noise ratio decreases.

Since an AR process with an additive noise becomes an Autoregresive-Movingaverage (ARMA) process; the usual approach to this problem is to model the time series by the more proper Autoregresive-Movingaverage process rather than AR and use standard time series analysis technique to identify AR parameters. This standard technique however does not yield a positive definite autocorrelation matrix. Also, it is shown that the resulting spectral estimator may have a large variance [28]. An alternative approach termed the noise compensation technique is proposed by Kay [29]. In [29] Kay compensated for the noise effects by subtracting a noise term from equation that contains the reflection coefficients. Cadzow [30] recognizing that the Autoregresive-Movingaverage model is more general and realistic for spectrum estimation. He developed an algorithm for computing the ARMA coefficients and showed that the ARMA model is capable of providing high resolution estimates. The main drawback of the ARMA model is the nonlinear equation we must deal with. It is also difficult to determine the adequate model order (number of poles and zeros). Also, the use of higher order AR models yields another noise compensation technique, since it is known that an ARMA model can be considered as an AR of infinite order. A practical problem is the limited number of AR parameters that can be reliably estimated from available data length (A practical limit is not to use a model order above N/2 and below N/3).

If (M+1) lags of the autocorrelation function for a time series are known or estimated from the data samples, the M autoregressive parameters are obtained by solving the Yule-Walker equations, using the Levinson-Durbin algorithm (recursion algorithm). If unbiased autocorrelation estimates are used, one may also have numerical ill-condition case during the solution of the normal equations. Biased autocorrelation estimates reduce the risk of ill-conditioning but at the expense of a degradation of the AR spectral resolution and a shifting of spectral peaks from their true locations i.e., yield the frequency bias. Also it was shown that the spectral estimator based on the unbiased autocorrelation estimate displays a large increase in the sharpness of the spectral estimate as compared to that based on the biased autocorrelation estimate [31]. Since the location of the spectral peak is used as an estimate of frequency of sinusoidal signal,in order to obtain frequency estimation accuracies the statistical fluctuation of a peak frequency due to the different method must be analyzed. In [31] Sakai has been made such an analysis for AR spectrum estimation,by using the periodogram technique and assuming that the deviation from the true tone frequency is small. He found that the variance of the main peak frequency is inversely proportional to number of data and SNR as expected. One remarkable point in [32] ,if the condition P M $\gg$ 1 is met (M: Model order, P $\ll$ 1) as shown in Lacoss [13], the estimated spectrum has a mean peak at true frequency location. Hence the AR spectrum estimator becomes unbiased. Also Lang [33] found the expressions for the variance of the spectral estimate peak position at high SNR condition. His expressions are for both Covariance and Modified Covariance methods.

Spectral line splitting event is an important problem in AR spectral estimate as in Burg technique. The reasons for spectral line splitting in the Yule-Walker technique has been documented by Kay and Marple [34]. As a comparison, the Levinson-Durbin recursion algorithm requires a number of computational operations proportional to M^2 whereas Burg technique requires NM [35].

Another AR method is the High-Order Yule-Walker Equation (HOYWE) were discussed in [36]. The main advantage of HOYWE is that their covariance matrix does not contain the autocorrelation function at zero lag so that an unbiased estimate of the AR coefficients of a process can be obtained in the presence of the white noise. The drawback is that the some covariance matrix can become singular under certain conditions, whereas the normal covariance matrix which contains the noise variance terms in the main diagonal is always nonsingular. For this reason the Higher-Order Yule Walker Equations were not considered as a promising alternative for spectrum estimation. In [36], Y.T.Chang prevents the singularity by computing the matrix pseudo-inverse instead of the ordinary inverse whenever ill-conditioning is encountered. Then an AR spectrum estimator incorporating HOYWE and matrix pseudo inverse is able to provide high resolution and stable estimate. Also it permits the use of a high order model necessary for resolving closely spaced spectral peaks and does not require an estimate of the noise power. As a result two drawbacks are that we have to estimate autocorrelations until a lag higher than the model order used and the ill-conditioned matrices to be inverted.

Recently proposed method is Recursive Maximum Likelihood estimation of AR process by Kay [37]. It is closer approximation to the true Maximum Likelihood estimator that obtained using Linear Prediction techniques. In fact all of these methods may be viewed as attemps to approximate the M.L. estimators of the AR parameters with each method adapting varying degree of approximation. An improvement in statistical accuracy is obtained for short data records or sharply peaked spectrum with Recursive M.L. method. It operates in a recursive fashion in that it allows one to successively fit higher order AR models to the data. In addition the estimated all pole filter is guarenteed to be stable. But its performance is not good as the Forward-Backward method for spectral peak estimation.

Let´s say a few words about the remaining techniques. In Pisarenko method [38], the spectral estimate is formed as the sum of line components in a background of noise. This technique requires an eigen analysis of the covariance matrix. The smallest eigenvalue λ_{min} is the noise power. When this smallest eigen value has multiplicity one, the location of the spectral lines can be determined by finding the zeros of a polynomial whose coefficients are the elements of the eigenvectors corresponding to λ_{min} When the multiplicity of is greater than one the number of spectral lines is reduced order problem. Variations of Pisarenko´s eigenvector method have been published in [39]. In [39], the technique called MUSIC is formed from the eigenvectors of the estimated covariance matrix. Corresponding to each eigenvalue (ordered by increasing magnitude) is a polynomial $A_i(z)$ whose coefficients are the elements of the eigenvectors for λ_i . The MUSIC estimate is obtained by forming;

where the number of terms L is determined by subtracting the number of sinusoidal signals thought to be present from the dimension of the covariance matrix. The location of the lines are obtained by picking the largest peak of Eq.(1.1).

Thompson [40] suggested a constrained gradient search procedure for obtaining an adaptive version of eigen analysis. However the main drawback of this technique was that the initial convergence rate could be very slow for certain poor initial conditions. Reddy [41] derived an alternative Gauss-Newton type recursive algorithm which also used the second derivative matrix (Hessian). Their technique may also viewed as an approximate Least Squares algorithm and has faster convergence in the beginning while its convergence rate close to the true parameters depends on signal to noise ratio of the input signal. This technique also resolves the sinusoidal signals much faster than the Gradient version. Finally since their technique is adaptive implementation of Pisarenko's method in which the estimation can be updated as new data is observed, it has the ability to track slowly time-varying processes. One has to note that since Thompson and Reddy methods are Gradient technique, its performance can be seriously degraded by the presence of unstable stationary points on the error surface. Further simulation results for the Reddy method can be found in [48]. Also if the initial guess is not close to the solution, Gauss-Newton does not converge, whereas the method of Steepest Descent of Thompson converges for any initial guess [49]. Larimore [42] studied the convergence behaviour of adaptive Pisarenko method and shown that ; a) Once convergence occurs tracking of time-varying lines proceeds in much the same manner as adaptive linear prediction. b) Global convergence may be nonmonotonic and require long windows of data to isolate spectral lines. c) Low frequency lines may present severe problems, at very low frequencies convergence speed may be significantly slower than adaptive linear prediction. d) Furthermore high parameter variance may likewise destroy accuracy of frequencies estimates at very low frequencies.

Durrani [43] proposed an efficient algorithm for extracting an orthogonal eigen vector-oriented spectrum directly from Maximum Entropy Spectral Analysis coefficients. In effect, the algorithm estimates the eigenvector corresponding to the smallest eigen value of the data covariance matrix and the oriented spectrum results are better than MESA. The main cost of this technique is that it requires N(N+1)-1 complex operations per step and the convergence rate depends upon the noise power and the frequency difference between signals. Also in [44] Gueguen suggested various algorithms including sequential estimation procedure for determining eigen vectors of covariance matrix. Barabell [45] presented several methods for reducing the SNR required for resolution such as a)The first examining the roots of spectrum polynomial. b) The second method uses the properties of so called signal space eigen vectors to define a rational (pole-zero) spectrum function with improved capabilities. The statistical analysis of Pisarenko's method has been investigated by Sakai in [46], using the periodogram technique as in [32]. Under the assumption that the input samples consist of multiple sinusoidal signals plus white noise ,he found the asymptotic expression of the error variance of the frequency estimator and showed that the variance of the estimate of one frequency is independent of the SNR of the sinusoid for the case of two sinusoid case. He suggested also a modified Pisarenko method by making use of certain symmetry of the prediction error filter presented in [71]. This technique is computationally efficient but it does not gives improvements over the original method. Aktar [47] determined the variance of the spectral peak for case of a sinusoid in white noise as a function of number of samples, SNR and frequency of sinusoidal signal. Sarkar [50] suggested to use the Conjugate Gradient Method (CGM) to obtain the eigen vector corresponding to the minimum eigen value. The advantages of this technique over the method of Steepest Descent (SD) is that it is a finite step iterative method and secondly there is no arbitrary constant in the expression which governs the rate of convergence. In this proposed method, the spread of the eigenvalues has no significant effect on the overall rate of convergence. The disadvantage is that one has to store a matrix of data instead of one row as is conventionally done. In [51] the use of the Inverse Power Method (IPM) is investigated and since a single iteration of the IPM requires the solving a system of linear equations, The Cholesky modification is proposed to reduce the amount of computation. Another study can be found in [52].

The comparison of the Pisarenko and Prony approaches to spectral line analysis is presented in [53].

Kumerason and Tufts [54,55] compare their realization of ML estimate of multiple frequency with improved version of Owslew's method and with Forward-Backword Linear Prediction (FBLP) method. They had shown that no significant bias in frequency estimates was noticable in all methods above the threshold points. The threshold difference between ML and LP method can be greater than 15 desibel. The threshold of their improved LP method is moved much closer to that of the ML method. That is the performance is close to the Cramaer Rao Lower Bound (CRLB) even for closely spaced frequencies at much lower values of SNR than other LP methods [56]. In [57] they have improved FBLP estimation of frequencies at low SNR by using Singular Value Decomposition (SVD) of the covariance matrix. The improved performance at low SNR is also better than Pisarenko method and its variants. They demonstrated that when model order is chosen equal to (N-L/2) where L being number of sinusoidal signals and N being number of data samples, the principal eigenvector and FBLP methods (when the minimum norm prediction filter coefficients are found) are the same and this situation does not require SVD calculations. SVD is useful also for finding number of sinusoids and estimating accurate frequency of sinusoid, alleviating much of the ill-conditioned nature of FBLP by removing the noise subspace perturbation effects.

They suggest to use for a given data number N, a linear predictor of an order M equal to (N-L) [58]. In this case the covariance matrix of the process has a rank equal to 2L. The reason for this suggestion is the remark that eigen vectors of the covariance matrix when its order is larger than 2L, can be grouped into two sets. The first one corresponds to 2L large and generally well seperated (in magnitude) eigenvalues. These eigen values are less perturbed from their noiseless directions, whereas the second set corresponding to the originally (in the noiseless case) zero eigen values could chance directions abrubtly depend on the noise perturbation. On the basis of this remark, minimum number of sample necessary for estimating L real sinusoids is 3L taking account the minimum order for predictor should be 2L.

They also proposed a method to find a vector d spanning the whole noise subspace of the covariance matrix. The effect of using such a coefficient vector [59],

i) Frequency estimate is accurate for low SNR.

ii) Less spurious estimates.

iii) Extranous zeros are uniformly distributed.

They had shown that if one can estimate the covariance matrix from the data the property of the noise subspace eigenvectors is approximately true, that is L zeros of the predictor fall near their noiseless locations for moderate SNR values [59].

As a final remark, we present some relations about the Kumaresan and Pisarenko methods. Pisarenko used as data the autocorrelation values of stationary sinusoids or the asymptotic case of $N \rightarrow \infty$ and used some special properties of the resulting Toeplitz correlation matrix. In practice since N is finite, the signal is deterministic and covariance matrix is not Toeplitz in general. For long observation times $N \rightarrow \infty$ and $L=M$ the Kumaresan's method coincides with Pisarenko's.

For a more detailed description of the modern spectrum estimation, methods the reader is referred to reference [60].

1.2. OUTLINE OF THE THESIS

The problem in the dissertation is to investigate and analyze the use of the existing modern spectrum estimation methods, for frequency estimation problem. The problem is complicated by tye fact that only relatively short discrete time samples of data are available. Actually the main structure of this dissertation details the development and simulation results of applying modern spectrum techniques to the problem defined above.

In Chapter II, we investigate the use of an analytic signal model and its complex autocorrelation function for frequency estimation. Section 1 of Chapter II describes the methods alreadily used for frequency estimation problem. The formulation of the parametric methods is explained in detail in Section 2. Section 3 gives the background on CRLB and ML estimation. In Section 4, a statistical analysis of the proposed technique is carried out for both single tone and two tones cases and the effect of autocorrelation lag on the estimation performance is examined. In this sense the variance of the estimated frequency is derived analytically in terms of data samples, SNR, and autocorrelation lag in order to measure the accuracy of this technique. Since CRLB is usually used as a benchmark to evaluate the performance of a given method, the calculated variance is compared to the CRLB. In Section 5, the comparative variance of the frequency estimate behaviour using Modified Covariance, Covariance and the proposed Argument Methods is presented.

In order to determine the detection performance of the method, it is desirable to know the PDF of $\hat{f}$ under the tone present (H_1) and noise only (H_0) hypotheses. In this regard, the expressions for the PDF are derived for both cases in Section 6. Final study of Chapter II is the bandwidth estimation with a simple formula.

Chapter III presents the results of a study to determine the asymptotic behaviour and statistical properties of MESA moments. This chapter is composed of three sections. In the first section, we investigate the use of Maximum Entropy spectrum in the estimation of spectral moments and consider some of their elementary properties such as filtering, windowing and shifting. In the second section, the asymptotic behaviour of MESA moment technique for the case of sinusoid embedded in additive white noise is analyzed- The asymptotic formula for the spectral mean frequency and mean square bandwidth are derived by assuming known autocorrelation function. Also we compare the asymptotic behaviour of the estimator with the case when only a few lag terms are involved. As a bonus of the analysis the same moment expressions for the case of two sinusoids with equal or non-equal power are also determined analytically. In the last section, the analysis of statistical properties of MESA moments is given. It is shown that the probability of detection and false alarm can be written as functions of the expected value and the variance of the n´th moment for tone detection problem. Finally, the variance and PDF of the estimated mean frequency are derived analytically in this section.

Chapter IV deals with the effectiveness of AR method in frequency estimation problem. In Section 1, we obtain the exact solution of AR parameters by employing standard Matrix Inversion Lemma for different structured matrices such as Complex Toeplitz and Real Symmetric matrices. It follows that the solution of AR parameters is expressed for certain classes of matrices as a function of SNR, frequency of sinusoid and the AR model order if the true autocorrelation function is known. Section 2 gives the formulation of coefficient deviation of AR polynomial due to the noisy observation. Section 3 deals with the effect of the inexact AR coefficient on the roots of the AR polynomial analytically for the case of various matrices mentioned above. In Section 4, as frequency estimation accuracies, the statistical fluctuation of a peak frequency is investigated since the location of a spectral peak is used as an estimate of tone frequency. The analysis of the performance of frequency estimation is accordingly based on Sakai´s and Taylor series approximation methods. The analysis establishes the relationship between $\Delta\omega$ and coefficient deviation so that one can determine the frequency error for AR process using the results of Section 2 In Section 5, an analytical expression is derived for the probability density function of the radius of root of AR polynomial. Based on this observation, probability of error is obtained after some simple assumptions with zero (pole) method. AR spectrum is expressed in terms of radius of root for both H_0 and H_1 cases. Final study of this section is to illustrate the several possible positions of signal zero in z-plane and give some analytic expressions for these cases.

Chapter V gives a brief summary of what has been presented in this dissertation and suggests topics for further research.

CHAPTER II : FREQUENCY ESTIMATION PROBLEM

2.1 DESCRIPTION OF FREQUENCY ESTIMATION METHODS.

The estimation of frequency of a sinusoidal signal in white noise has many applications such as radar, sonar [61-63], speech [66] and communication technology. In many cases the presence of noise troubles the estimation of frequency determination. Recently many methods have been proposed based upon signal processing which try to overcome this difficulty. The most classical and widely used approaches; the zero crossing method and secondly involve Discrete Fourier Transform (DFT) i.e., a transformation of the input samples into frequency domain via a DFT and searching a peak.

Mathematically, the number of zero crossing is closely related to the phase change of the signal during the measuring time i.e., if N_Z denotes the number of zero crossing in the time interval (0,T), the frequency estimate is given by [67]

$$\hat{f}_e = \frac{N_Z}{2T_i} \qquad 2.1$$

The implementation of the conventional zero crossing technique is shown in Fig.II.1. Details are found in [68]. However the zero crossing estimator which can be implemented more exactly than ideal frequency demodulator has a discontinuous output signal which introduces a quantization error. At short time measurement in particular this effect greatly reduces the accuracy of the method. Also it breaks down when the signal to noise ratio falls momentarily too low and gives a biased estimate [68].

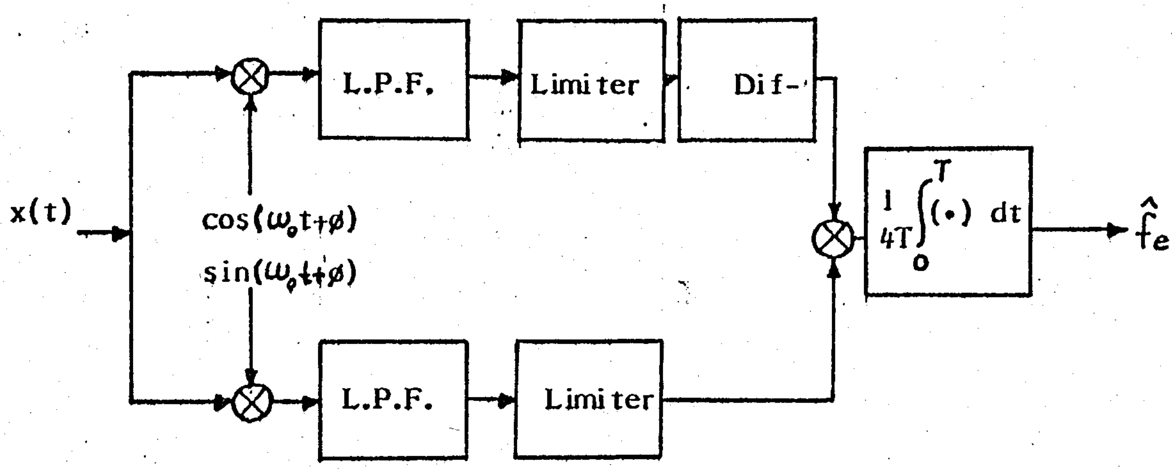


Fig.II.1. Pawula's implementation of frequency estimator.

A general model of the frequency estimation process is shown in Fig.II.2 [69]. The continuous input signal x(t) can be written as

$$x(t)=A\ \sin(\omega_1 t+\phi)+n(t) \qquad 2.2$$

where A is signal amplitude, f_1 and ϕ are frequency and phase of the signal which are unknown with uniform prior densities over (π, π) and $(f_o-w,\ f_o+w)$ respectively. Assume that the expected value of the input frequency f_o is also known exactly and n(t) is white noise with power σ_n^2. The input signal is multiplied with unity power inphase and quadrature local oscillators at frequency f_o and the output of each channel is filtered and sampled and A/D converted. A block of N complex $Z=\left\{z_o, z_1, z_2, \ldots\ldots, z_{N-1}\right\}$ is collected for processing. Sampling occurs at a frequency f_S so that k'th sample is given by

$$Z_k = \frac{A}{2} \exp \left[j2\pi \frac{(f-f_o)}{f_s} k + j\,\emptyset \right] + n_k \qquad 2.3$$

where n_k is complex zero mean random variable both real and imaginary parts have variance $\sigma_n^2/2$. The complex samples Z are weighted by the window function and the output is processed by the Fast Fourier Transform (F.F.T.) algorithm. The frequency location of the largest magnitude square of DFT coefficient is taken as a frequency estimate. Recently several data windows were considered in addition to no weighting [69] and it was shown that there was an advantage in using weighting to estimate the frequency of a sinusoidal signal [69-70]. By examining the Fig.II.2, one can conclude that the DFT forms the main part of the estimation procedure. Therefore the performance of the mentioned method will depend on the performance of DFT completely. These two methods explained so far are known non-parametric methods in frequency estimation problem.

Several parametric frequency estimators proposed recently are based on Maximum Likelihood, Maximum Entropy, Pisarenko, Prony ...etc methods. The most efficient method known as Maximum Likelihood frequency estimator in the sense of minimum variance of the estimate consists of a parellel bank of narrowband filters [6]. The center frequency of the filter with largest output is the estimate that maximize the likelihood function. Also with present day technology, the filter bank can be easily implemented digitally as a FFT machine. Maximum Likelihood estimate of frequency of sinusoid tend to be more accurate than other methods especially at low signal to noise ratio [6]. This is to be expected because the prior information about the sinusoidal form of signal is used in a statistically appropriate way in the Maximum Likelihood method. Although the variance of the Maximum Likelihood estimate asymptotically approaches to the Cramer-Rao Lower Bound (CRLB), it may be unattractive in multi tone applications because of computationally burden. Except the Maximum Likelihood method, other techniques are computationally attractive but they are suboptimal.

The MEM uses the spectral maxima and/or pole method to estimate the

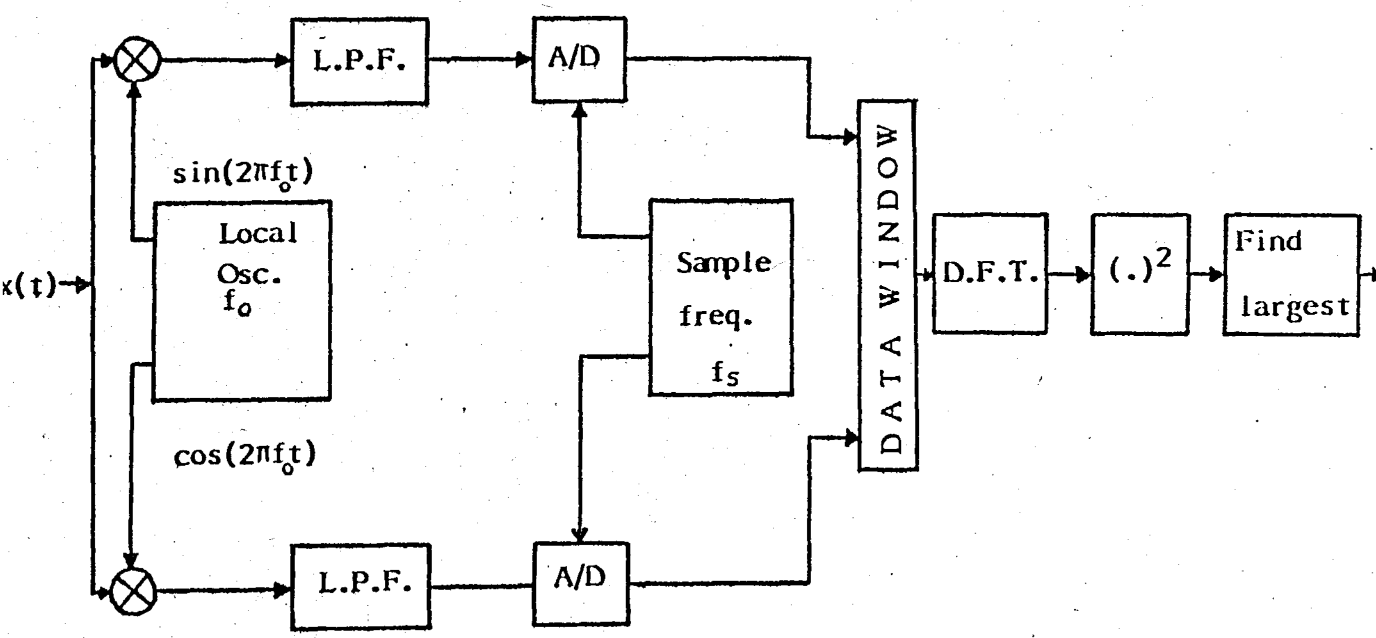


Fig.II.2. General model of discrete frequency estimation.

frequency. The spectral maxima method is based on the estimation of the power spectrum density and on testing for location and energy of the spectral peak. In the pole method, the angles of two poles of the prediction error filter (PEF) closest to the unit circle (measured counterclockwise) are found and the smaller of the angles is taken as the estimate of the frequency. Some difficulties were encountoured in using MEM especially with short data record length for frequency estimation due to the so called line splitting problem [34] and the phase dependence of the estimate [20].

Pisarenko's method requires an eigen analysis of the autocorrelation matrix of observation. The smallest eigenvalue is the noise power. When this smallest eigenvalue has multiplicity one, thefrequency of sinusoid can be determined by finding the zeros of a polynomial whose coefficients are the elements of the eigenvector corresponding to minimum eigenvalue [38]. If one is interested in only estimating the frequency of the sinusoid; finding the spectrum estimate which requires a large of computation is superfluous. In this regard, we suggest a simple method to estimate frequency of sinusoid.

In this chapter, a method of estimating the frequency of a sinusoid in white noise based on analytic signal is investigated [72]. The complex autocorrelation function of an analytic signal model is used to estimate the frequency i.e.,

$$\hat{f} = \frac{1}{2\pi k} \arg\left[r_k\right]$$

where r_k denotes the the k'th lag of the autocorrelation function. This technique will be called Argument Method. The properties of this frequency estimation method have been studied. It is shown that it is an unbiased estimator. Analytic expression for the variance of the estimate is derived to compare with optimum estimator. An expression for the PDF of the estimated frequency $\hat{f}$ is derived for both the null hypothesis and the alternative. For the case of two sinusoids; the problem is studied to arrive the reasonable situation. Bandwidth estimation problem is the final original study of this chapter.

2.2. FORMULATION OF THE PARAMETRIC METHODS

Consider L sinusoids of amplitudes A_i, phase $\emptyset_i$ and distinct frequencies w_i, i=1,2,.......L. The sum of these sinusoids at any sampling instant n is given by

$$S_n = \sum_{k=1}^{L} A_k \text{Sin}\left[w_k n + \emptyset_k\right] \qquad 2.4$$

It was shown that a unique real 2L-vector [61]

$$a^T = \left[a_1, a_2, \ldots\ldots\ldots\ldots a_{2L}\right] \qquad 2.5$$

exists such that for any n

$$S_n = \sum_{\ell=1}^{2L} a_\ell \sum_{k=1}^{L} A_i \sin\left[w_k (n-\ell) + \emptyset_k\right] \qquad 2.6$$

$$= \sum_{\ell=1}^{2L} a_\ell S_{n-\ell}$$

Expanding the sine function in (2.5) and reordering the sums gives

$$S_n = A_k \sin\left(w_k n + \emptyset_k\right) \sum_{\ell=1}^{2L} a_1 \text{Cos} w_k \ell$$

$$- \sum_{k=1}^{L} A_k \cos\left(w_k n + \emptyset_k\right) \sum_{\ell=1}^{2L} a_\ell \sin w_k \ell \qquad 2.7$$

Since (2.4) and (2.7) are equal for all A_k, $\emptyset_k$ and w_k, it follows that for $k=1,2,\ldots..L$

$$\sum_{\ell=1}^{2L} a_\ell \cos w_k \ell = 1 \qquad 2.8$$

and

$$\sum_{\ell=1}^{2L} a_\ell \sin w_k \ell = 0 \qquad 2.9$$

which can be put in complex form as

$$\sum_{\ell=1}^{2L} a_\ell \exp -jw_k \ell = 1 \qquad 2.10$$

and

$$\sum_{\ell=1}^{2L} a_\ell \exp jw_k \ell = 1 \qquad 2.11$$

Combining (2.8) and (2.9) in matrix form yields

$$F \alpha = q \qquad 2.12$$

where

$$F = \begin{bmatrix} P \\ \cdots\cdots \\ U \end{bmatrix}$$

$$P = \begin{bmatrix} \text{Cos } w_1 & \cdots\cdots\cdots & \text{Cos } 2Lw_1 \\ \vdots & & \vdots \\ \text{Cos } w_L & \cdots\cdots\cdots & \text{Cos } 2Lw_L \end{bmatrix}$$

$$U = \begin{bmatrix} \text{Sin } w_1 & \cdots\cdots\cdots & \text{Sin } 2Lw_1 \\ \vdots & & \vdots \\ \text{Sin } w_L & \cdots\cdots\cdots & \text{Sin } 2Lw_L \end{bmatrix}$$

The paremetric approach to frequency estimation is to first estimate the a_l and then find the roots $\left\{\exp\left(\mp jw_k\right)\right\}$ of the polynomial

$$\sum_{\ell=1}^{2L} a_\ell \lambda^\ell = 1 \qquad 2.13$$

Alternatively, with $\lambda = \exp(-jw)$,solving for the roots of (2.13) is equivalent to taking the Fourier Transform of the sequence $\left[a_o, a_1, \ldots a_{2L}\right]$ with $a_o=-1$ and noting that the w at which the valley of the spectrum occur.

With the addition of zero-mean white noise in (2.4), the data sequence is

$$X_n = S_n + W_n \qquad 2.14$$

Substitution of (2.6) into (2.14) yields

$$X_n = \sum_{\ell=1}^{2L} a_\ell X_{n-\ell} - \sum_{\ell=1}^{2L} a_\ell W_{n-\ell} + W_n \qquad 2.15$$

Hence the model for the sinusoids in white noise problem contains poles and zeros. This one cause of the difficulties reported [20], [72] in using all pole methods.

2.3 BACKGROUND ON CRLB AND ML ESTIMATION

The CRLB provides a useful tool for evaluating the performance of the parameter estimation techniques. Comparison of the error variance of a given parameter estimation technique to the CRLB provides a reliable measure of the estimation accuracy of that technique, in other words, the CRLB can be evaluated to determine whether the estimator has performance sufficiently near the optimum.

Let $\underline{\theta} = \left[\theta_1, \theta_2, ..\theta_p\right]^T$ be a set of parameters to be estimated. It is assumed that the underlying PDF is $P(\underline{X}|\underline{\theta})$ where $\underline{X} = [\ x_o, x_1, \ ....x_{N-1}\ ]^T$ is a sample of size N, Let $\underline{\hat{\theta}}$ be an unbiased estimator $\underline{\theta}$, i.e., $E[\underline{\hat{\theta}}] = \underline{\theta}$ where

E (.)

denotes the expectation operator. Then the covariance matrix of $\underline{\hat{\theta}}$ given by $\underline{\underline{C}}_{\hat{\theta}}$ satisfies the CRLB [63].

$$\underline{\underline{C}}_{\hat{\theta}} = E\left[(\hat{\theta}-\hat{\theta})\ (\hat{\theta}-\hat{\theta})^T\right] \geq \left[\underline{\underline{I}}(\hat{\theta})\right]^{-1} \qquad 2.16$$

where $\underline{\underline{I}}(\theta)$ is the Fischer Information matrix whose (i,j)'th element is

$$\left[\underline{\underline{I}}(\theta)\right]_{ij} = E\left[\frac{\partial \ln P(x|\theta)}{\partial \theta_i} \frac{\partial \ln P(X|\theta)}{\partial \theta_j}\right] \qquad 2.17$$

It should be noted that (2.16) implies

$$\mathrm{Var}(\theta_i) \geq \left[\underline{\underline{I}}^{-1}(\theta)\right]_{ii} \qquad i=1,2,\ldots,p \qquad 2.18$$

Now let $\hat{\underline{\theta}}_{ML}$ be the Maximum Likelihood estimator of $\underline{\theta}$, i.e., $\hat{\underline{\theta}}_{ML}$ is found from

$$\max_{\theta} P(\underline{X}|\underline{\theta}) = P(\underline{X}|\hat{\underline{\theta}}_{ML}) \qquad 2.19$$

or we can state that if an unbiased estimator with error variance as small as the CRLB exists, it is said to be efficient [63] and it is the Maximum Likelihood estimator defined by

$$P(\underline{X}|\hat{\underline{\theta}}) \geq P(\underline{X}|\underline{\theta}) \qquad \text{for all } \theta \qquad 2.20$$

2.3.1. PROPERTIES OF ML ESTIMATOR:

The ML estimator has the following properties [63].

1) $E\left[\underline{\theta}_{ML}\right] \rightarrow \underline{\theta}$ as $N \rightarrow \infty$.

2) $\underline{\underline{C}}_{\hat{\theta}_{ML}} \rightarrow \underline{\underline{I}}^{-1}(\theta)$ as $N \rightarrow \infty$

3. The ML estimator PDF is multivariate Gaussian with mean $\underline{\theta}$ and covariance $\underline{\underline{I}}^{-1}(\theta)$ as $N \rightarrow \infty$

4) $\hat{\underline{\theta}}_{ML} \rightarrow \underline{\theta}$ as $\rightarrow N$

The condition $N \rightarrow \infty$ is termed the asymptotic case. Thus for sufficiently large data records , $\hat{\underline{\theta}}_{ML}$ is Gaussian distributed with a mean θ and a covariance matrix given by the CRLB. Also $\hat{\underline{\theta}}_{ML}$ is a consistent estimator.

5. If $\underline{\theta}' = g(\underline{\theta})$ is a one to one transformation from $\underline{\theta}$ to $\underline{\theta}'$,then $\hat{\theta}'_{ML} = g(\hat{\theta}_{ML})$ is an ML estimator of $\underline{\theta}'$. Furthermore the asymptotic covariance matrix of $\hat{\underline{\theta}}'_{ML}$ is

$$\underline{\underline{C}}_{\hat{\theta}'_{ML}} = \underline{\underline{L}}\, \underline{\underline{C}}_{\hat{\theta}_{ML}}\, \underline{\underline{L}}^{T} \qquad 2.21$$

where

$$\left[L\right]_{ij} = \frac{\partial g_i(\theta)}{\partial \theta_j}$$

L is just a linearization of the function $g(\theta)$ about the true parameters.

6. The incremental sensitivity of the $P(X|\theta)$, to changes in the value of θ is defined by

$$S_\theta(x) = \left[\frac{\Delta P(X|\theta)}{P(X|\theta)}\right]\left[\frac{\Delta\theta}{\theta}\right]^{-1} \qquad 2.22$$

and is the ratio of the resultant percentage change in $P(X|\theta)$ to the percentage change in θ ,evaluated at θ. The limiting value as $\Delta\theta \to 0$ of $S_\theta(X)$ is given by

$$\tilde{S}_\theta(X) = \lim_{\Delta\theta \to 0} \left[\frac{\Delta P(X|\theta)}{\Delta\theta}\right]\left[\frac{\theta}{P(X|\theta)}\right] \qquad 2.23$$

$$= \left[\frac{d}{d\theta} P(X|\theta)\right]\left[\frac{\theta}{P(X|\theta)}\right]$$

Finally we have

$$\tilde{S}_{\theta}(X) = \theta \frac{d}{d\theta} \ln P(X|\theta) \qquad 2.24$$

Therefore the bound (2.18) can be characterized by the sensitivity of the PDF as

$$\text{Var} \ \theta_i \geq \frac{1}{\theta_i^2} \ E\left[\tilde{S}^2_{\theta_i}(x)\right] \qquad 2.25$$

This yields the pleasing interpretation that the attainable mean square accuracy of an unbiased estimator is lower bounded by the inverse mean square sensitivity of the PDF. Thus if the sensitivity of the PDF is high, the error variance of an efficient estimator (Maximum Likelihood) is low and vice versa.

2.3.2. ACCURACY OF MULTI-TONE PARAMETER ESTIMATION:

Let us consider the multi-tone signal in additive white noise and define for convenience the parameter of signal as

$$\theta_{3k} = w_k \qquad 2.26$$

$$\theta_{3k+1} = \phi_k$$

$$\theta_{3k+2} = A_k \qquad \text{for } k=1,2,\ldots.L$$

If **X** is the noisy observation of **S** whose typical element is of the form (2.14), then PDF of **X** given **S** is

$$P(X|S) = \frac{|R|^{1/2}}{(2\pi)^{N/2}} \exp\left[-\frac{1}{2}(X-S)^T R^{-1}(X-S)\right] \qquad 2.27$$

where **R** and N are the correlation matrix of noise and number of data samples respectively. $|\cdot|$ denotes the determinant of (.). It is also shown that in [70]

$$\left[I(\theta)\right]_{ij} = \frac{\partial S^T}{\partial \theta_i} R \frac{\partial S}{\partial \theta_j} \qquad 2.28$$

or more tractable form

$$I(\theta) = \frac{1}{\sigma_n^2} D J D \qquad 2.29$$

$$I(\theta) = \frac{1}{\sigma_n^2} D Q D \qquad 2.30$$

where σ_n^2 is the noise power and

$$J_{ii} = M(w_i - w_j, \not{0}_i - \not{0}_j)$$

$$Q_{ij} = \frac{1}{2} M(w_i - w_j, \not{0} - \not{0}) - M(w_i + w_j, \not{0}_i + \not{0}_j) B$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D_i = \begin{bmatrix} A_i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A_i \end{bmatrix}$$

$$M(w, \ ) = \begin{bmatrix} T^2 \sum_n n^2 \cos \alpha n & -T \sum_n n \sin \alpha n & T \sum_n n \cos \alpha n \\ T \sum_n n \sin \alpha n & \sum_n \cos \alpha n & \sum_n \sin \alpha n \\ T \sum_n n \cos \alpha n & -\sum_n \sin \alpha n & \sum_n \cos \alpha n \end{bmatrix}$$

$$\alpha_n = nwT + \not{0}$$

$$Q = \begin{bmatrix} Q_{11} & \cdots & Q_{1L} \\ \vdots & & \vdots \\ Q_{L1} & & Q_{LL} \end{bmatrix} \qquad D = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & & & D_L \end{bmatrix}$$

$$J = \begin{bmatrix} J_{11} & \cdots & J_{1L} \\ \vdots & & \vdots \\ J & & J \end{bmatrix}$$

An interesting situation arises from the condition at $|I(\cdot)|$ which becomes singular for the case of a single tone. Using (2.29), we have

$$|I(\theta)| = \frac{A_1^2}{\sigma_n^4} T^2 N^3 (N-1) \left(\frac{2N-1}{6} - \frac{(N-1)}{4} \right) \qquad 2.31$$

One can easily find that the value of N makes $I(\theta)$ equal to zero is minus one which is impossible. Therefore $I()$ is always non-singular.

We now present a few theorems that charecterize the CRLB for multiple tone case [70] :

THEOREM 1 : The CRLB to unbiased estimation of the parameters w_i and ϕ_i of S are functions of A_i but are independent of other levels, A_j, $j \neq i$. The bound to unbiased estimation of a level, A_i is independent of all levels.

THEOREM 2 : The bounds associated with the parameters of the first L_1 tones, when there are L_1+L_2 tones, are not less than the bounds when there are only L_1 tones.

THEOREM 3 : When the signal consists of two equal level complex tones (equal power), the CRLB for the same parameters (i.e., the two frequencies) are equal. In other words, the mutual interference is reciprocal.

THEOREM 4 ; The bounds for two tones, real or complex, are periodic in ϕ_1 and ϕ_2 with period π . This theorem follows from the easily checked fact that $M(w, \phi+\pi) = -M(w, \phi)$

THEOREM 5 : The bounds for real or complex tones are periodic in each frequency with period $2\pi / T$.

THEOREM 6 : The bounds associated with complex tones depend upon the difference frequencies and phases, but not upon the absolute values.

2.4. FREQUENCY ESTIMATION VIA ANALYTIC SIGNAL MODEL

The analytic signal formulation proceeds as follows [72]

i. Form the analytic signal z_n from the input samples x_n. The analytic signal is defined as

$$z_n = x_n + j \tilde{x}_n \qquad 2.32$$

where denotes the Hilbert transform.

ii. Decrease the sampling rate by two, in order to make the resulting complex noise white.

If the input samples consist of a sinusoid in white noise, i.e.,

$$x_n = \sin (w_1 n + \emptyset) \qquad 2.33$$

then the resulting complex data are

$$z_n = -j\, e^{j\, 2w_1 n + \emptyset} + W_n^c \qquad 2.34$$

where W_n^c is complex white noise, $\emptyset$ is its arbitrary initial phase at n=0, and w_1 is its angular frequency normalized so that $\pi < w_1 < \pi$.

The frequency is estimated as follows

$$\hat{f}_1 = \frac{1}{4\pi k} \arg (\hat{r}_k) \triangleq \frac{1}{r\pi k} \tan^{-1} \left(\frac{y}{x}\right) \qquad 2.35$$

where $\hat{r}_k$ denotes the k'th complex autocorrelation coefficient of the analytic signal model and x , y denote the real and imaginary part of $\hat{r}_k$ respectively. An alternate form for f is obtained after expressing $\hat{r}_k$ in terms of spectral components, i.e.,

$$\hat{f}_1 = \frac{1}{4\pi k} \tan^{-1} \left[\frac{\sum_{\ell=0}^{N-1} \hat{C} \sin(\frac{2\pi k \ell}{N})}{\sum_{\ell=0}^{N-1} \hat{C}_\ell \cos \frac{(2\pi k \ell)}{N}} \right] \qquad 2.36$$

where

$$\hat{C}_\ell = \left| \frac{1}{N} \sum_{i=0}^{N-1} Z_i \exp\left(-j \frac{2\pi i \ell}{N}\right) \right|^2$$

This method can also be used to estimate the frequency when input samples x_n is a discrete complex time series, i.e.,

$$x_n = A e^{j \; w_1 n + \emptyset} + W_n^c \qquad 2.37$$

where W_n^c is complex white Gaussian noise with variance σ_n^2, A is amplitude of sinusoid, $\emptyset$ is its arbitrary initial phase at n=0 and w_1 is its angular frequency, normalized so that $-\pi < w_1 < \pi$

2.4.2. EVALUATION OF THE EXPECTED VALUE OF ESTIMATE:

After expanding (2.36) into a series about the point [E(x), E(y)] and using second order approximation technique in [74], the expected value of $\hat{f}$ is given

$$E\left[\hat{f}_1\right] = \frac{1}{4\pi k} \left[\hat{f}_1(\bar{x}, \bar{y}) + \frac{1}{2}\left(\frac{\partial^2 \hat{f}_1}{\partial x^2} \partial_x^2 + 2\mathrm{Cov}(x,y) \frac{\partial^2 \hat{f}_1}{\partial x \partial y} + \partial_y^2 \frac{\partial^2 \hat{f}_1}{\partial y^2} \right) \right] \qquad 2.38$$

where σ_x^2 and σ_y^2 are the variances of x and y respectively. Both x and y are Gaussian random variables with mean and variance as [77].

$$\bar{x} = E\left[x\right] = \frac{N-k}{N} \quad Cosw_1 k$$

$$\bar{y} = E\left[y\right] = \frac{N-k}{N} \quad Sinw_1 k$$

$$\sigma_x^2 = \frac{\sigma_n^4}{N}\left[\frac{1}{2} + \mu\left(1 + \frac{(N-2k)}{N} \quad Cos2w_1 k\right)\right]$$

$$\sigma_y^2 = \frac{\sigma_n^4}{N}\left[\frac{1}{2} + \mu\left(1 - \frac{(N-2k)}{N} \quad Cos2w_1 k\right)\right] \qquad 2.39$$

$$Cov(x,y) = \sigma_n^4 \mu \frac{(N-2k)}{N^2} \quad Sin2w_1 k$$

where $\mu = \frac{1}{\sigma_n^2}$ is the signal to noise ratio. In (2.38), all derivatives are evaluated at the means of x and y respectively. Inserting (2.39) into (2.38) one obtains the exact value of f, i.e., $E(\hat{f})=f$, which implies that the frequency estimator is unbiased.

2.4.3. VARIANCE OF THE FREQUENCY ESTIMATE:

A measure of the accuracy of the frequency estimate of the analytic signal is given by the variance analysis. Again using second order approximation technique in [74] which is formulated as in general

$$Var(\hat{f}) = \sigma_{\hat{f}}^2 = \left[\frac{\partial f}{\partial y}\right]^2 \sigma_y^2 + \left[\frac{\partial f}{\partial x}\right]^2 \sigma_x^2 + 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) Cov(x,y) \qquad 2.40$$

and also equivalent of the Taylor series approximation of the variance of a function of two random variables x and y. After some work, one can obtain

$$\sigma_{\hat{f}}^2 = \frac{N + 4 k \mu}{\mu^2 (N-k)^2 \; 8\pi^2 k^2} \qquad 2.41$$

Eq. (2.41) gives the variance of frequency estimate based on analytic signal in terms number of data, signal to noise ratio, autocorrelation lag, and shows that the variance is inversely proportional to the square of signal to noise ratio, AC lag and (N-k).

The CRLB for the case of single tone is given as [75]

$$\sigma^2_{CR} = \frac{6}{4\pi^2 N.(N^2-1)\mu} \qquad 2.42$$

σ^2_{CR} is the lower bound on the variance of unbiased frequency estimation variance computed as in [75] for given signal and noise parameters.

It is clear to see that Eq.(2.41) has a broad minimum when k is 1/3 the data length at which the theoretical variance is $(2.25+(1/0.6\mu))$ times the CRLB. As a comparison Modified Covariance and Covariance method are 9/8 and 1.5 times the CRLB respectively when the model order is 1/3 the data length [33]. This situation will be discussed later.

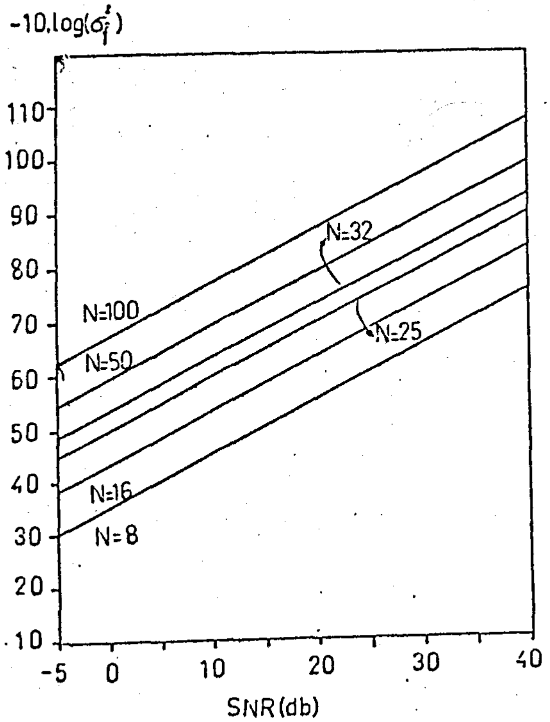


Fig.II.3. CRLB of the frequency estimation accuracy vs. SNR

The quantity $-10\log(\sigma_{CR}^2)$ is plotted in Fig.II.3 against the SNR for different values of N. Fig.II.4 shows the performance of the analytic signal method more quantitatively. The SNR values used in the simulation are in the range 0-40 dB. The quantity $-10\log(\sigma_f^2)$ is plotted by dashed lines in Fig.II.4 against the SNR with different values of k. The line labeled C.R. bound corresponds to the plot of $-10\log(\sigma_{CR}^2)$. Here we see that the performance of the analytic signal based frequency estimation can be greatly improved by using optimal value of k which is an integer number close to N/3 bringing the performance close to that of ML frequency estimator. In other words the variance of frequency estimate obtained by k=N/3 is a few decibels poorer than the C.R. bound which was attained by ML method. Note that the threshold occurs at about 10 dB. By the threshold we mean the value of SNR at which the accuracy of the frequency estimate begins to depart very rapidly from the C.R. bound as SNR is lowered. Fig.II.5 shows the variance of frequency estimate versus autocorrelation lag with different data length. Similarly, the best performance is obtained when k is approximately 1/3 data length. The same situation can be observed in Fig.II.6, i.e., the autocorrelation lag at k=5 yields very satisfactory results (N=16). In fact one has, the variance of the frequency estimate at k=5 is less than 5 or 6 dB Cramer-Rao bound; and can observe the threshold behaviour of the estimator. As SNR value is decreased, the difference between the variance of estimate and Cramer-Rao bound is also increased. In Fig.II.5, the variance of estimate at k=N/3 becomes much greater than at k=1, as N is increased.

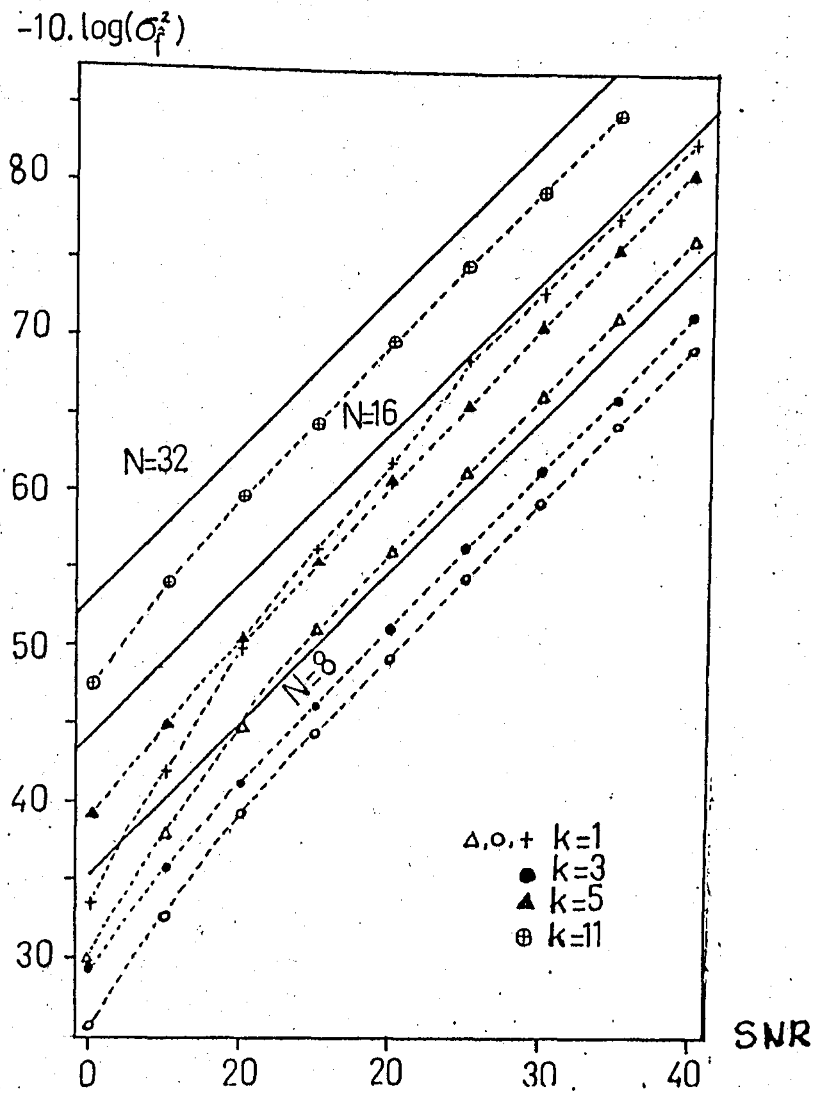


Fig.II.4. The variance of the frequency estimate versus SNR (dashed), CRLB versus SNR (solid)

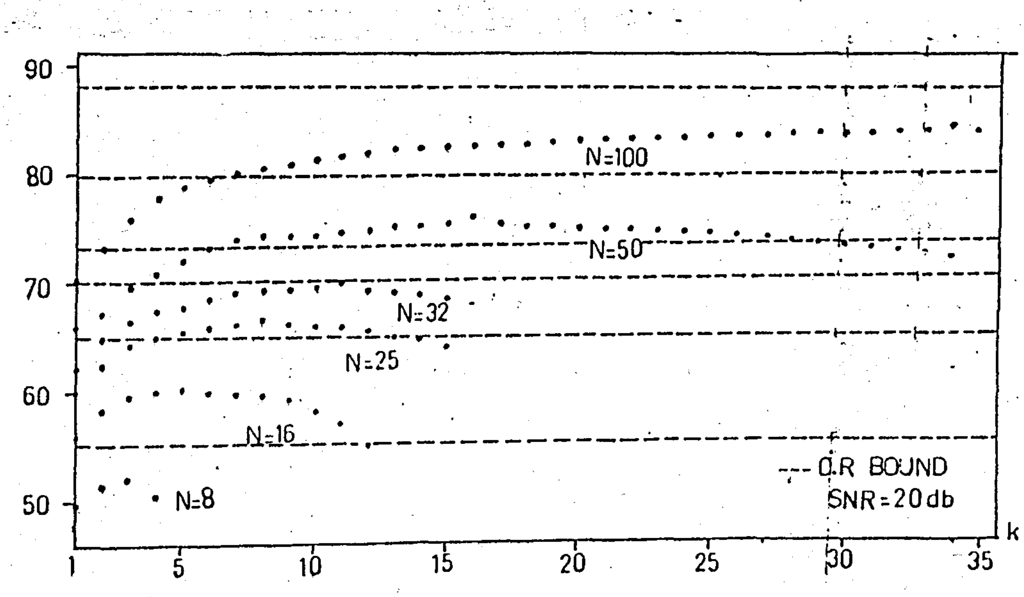


Fig.II.5. The variance of the frequency estimate versus AC lag

2.5 THE ARGUMENT METHOD AND MODERN SPECTRUM ESTIMATION METHODS

An M´th order autoregressive process is represented by

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_{n-M} + W_n^c \qquad 2.43$$

which yields power spectrum estimate of the form

$$S(f)^2 = \frac{\sigma_n^2}{\left| 1 - \sum_{k=0}^{M} a_k Z^{-k} \right|^2_{Z=e^{jw}}} \qquad 2.44$$

where $\{a_1, a_2, .. a_m\}$ are the parameters of autoregressive (AR) system determined through various algorithms [60] and S(f) is the Discrete Fourier Transform (DFT) of the signal, In this sense it is convenient to think of x_i´s as being the output of the system with gains a_i´s and input w_i^c´s . The first order AR process is then

$$x_n = a_1 x_{n-1} + W_n^c \qquad 2.45$$

where $|a_1| < 1$. It follows from [26] the AR parameter a_1 is given as

$$a_1 = r_1 / r_o \qquad 2.46$$

In order to estimate the frequency of a sinusoidal signal, the AR techniques use the spectral maxima or pole methods. The spectral maxima method is based on the estimation of the power spectrum and searching a peak whose frequency location corresponds to the frequency of the tone. Pole method is based on searching the roots near to unit circle of Z-transform of a_i´s parameters. It is easily shown that first order AR process is identical to the preceding method with k=1 for pole method, i.e.

$$\tilde{f} = \frac{1}{2\pi} \tan^{-1}(a_1) \qquad 2.47$$

Also in [33] the expressions for the variance of the spectral estimate peak position at high SNR are given for another techniques known as modified covariance (MC) and covariance (C) respectively as

$$\sigma^2_{\hat{f}MC} = \frac{1}{4\pi^2 \ \mu (N-M)^2 \ M}$$

$$\sigma^2_{\hat{f}_C} = \frac{(4M + 2)}{12\pi^2 \ M(M+1) \ (N-M)^2 \mu}$$

These expressions have again a broad minimum when M (model order) is 1/3 the data length. The comparative variance of frequency estimate behaviour using MC, C, and proposed methods is shown in Fig.II.7. The MC technique performs considerably better than other techniques. It is of interest to note that in Fig.II.7, how fastly the performances of all three methods improve by increasing the model order (M) for both MC and C and AC lag k up to N/3. [Both it is compulsary to find r_M solving the normal equation to obtain a_i's or use the proposed method]. Although the MC is the best technique, but the proposed method is much simpler frequency estimation technique which does not require the filter parameter estimate. Also Fig.II.8 shows the performances of three methods for k=1 i.e., first order MC, C, and AR process. The variance of frequency estimate using MC technique is also identical $\sigma^2_{\hat{f}c}$ for the first order case. Therefore the comparison of the lower bound performance of the mentioned three frequency estimate methods can be more easily seen from Fig.II.8. The comparative performance of the methods of frequency estimators becomes quite poor at low signal to noise ratios and is approximately close to CRLB between 6 and 10 dB at high signal to noise ratios.

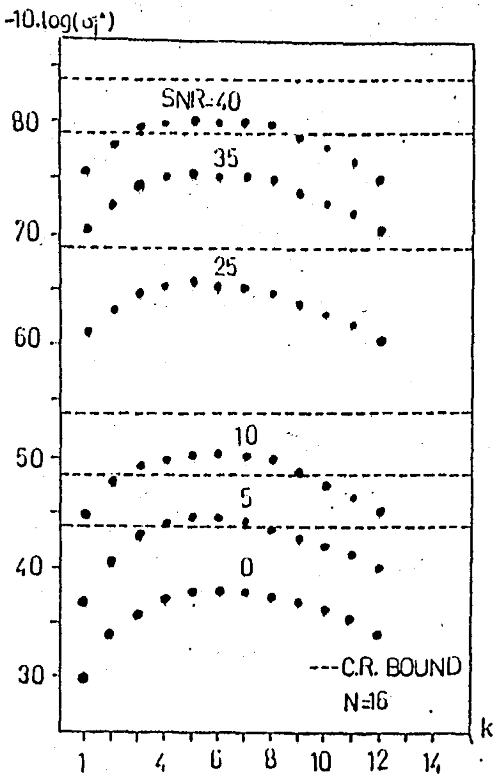


Fig.II.6. Variance of the estimate versus AC lags.

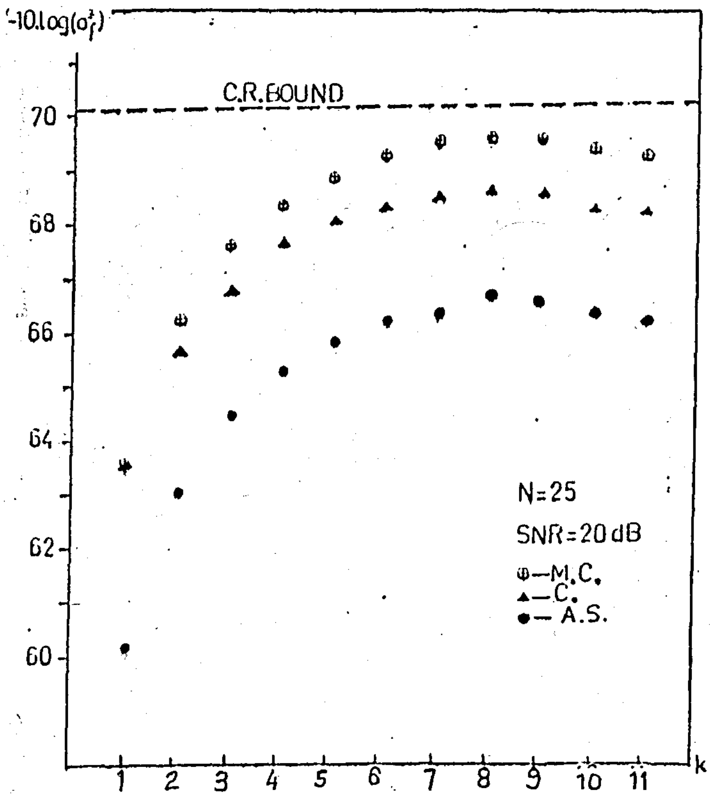


Fig.II.7. Comparison of Modified Covariance, Covariance and the proposed method.

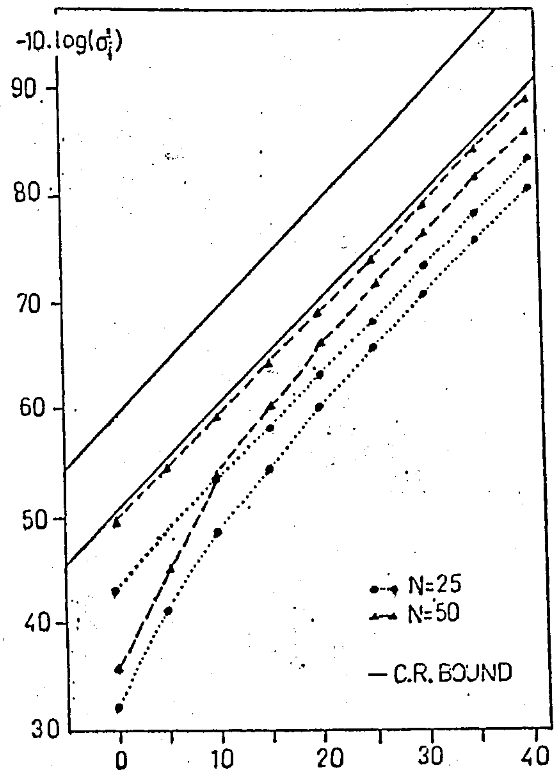


Fig.II.8. Performances of the first order AR, Covariance, Modified Covariance methods versus SNR.

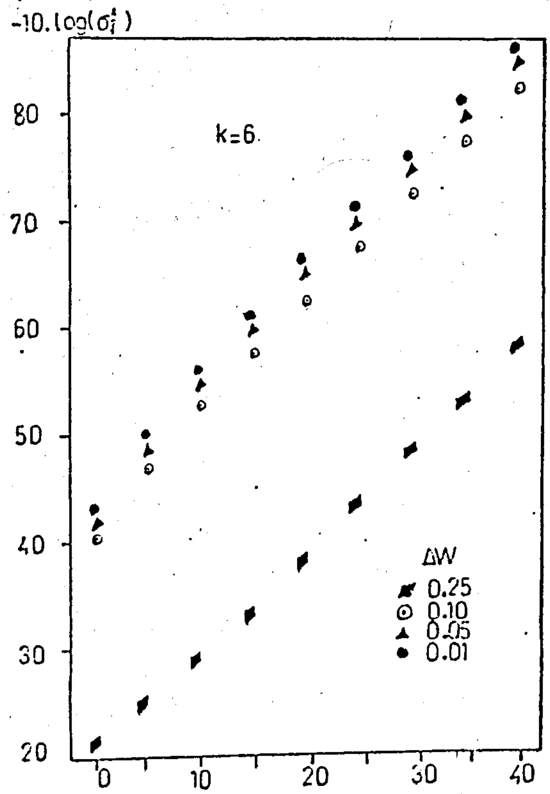


Fig.II.10. Mean frequency estimation accuracy versus SNR

2.6 TWO TONES CASE

The sampled signal x(n) consisting of two complex sinusoids in the presence of additive white noise is given by

$$X_n = A_1 \exp(jnw_1) + A_2 \exp(jnw_2) + w_n^c \qquad 2.48$$

where $A_k = A_k \exp(j\phi_k)$ is the complex amplitude of the k-th sinusoid, ϕ_k is its arbitrary initial phase at n=0 and w_k is its angular frequency, normalized so that $-\pi < w_k < \pi$, for k=1,2.,W_n^c is additive white noise. It is apparent that if $A_2 = A_1^*$ and $w_2 = -w_1$ then x(n) is a sampled real sinusoid in additive white (complex) noise.

$$x_n = A_o \exp j(nw_o + \phi_o) \left\{ \alpha \exp j(n\Delta w + \Delta\phi) + \alpha^{-1} \exp -j(n\Delta w + \Delta\phi) \right\} + w_n^c \qquad 2.49$$

Equation (2.48) can be written in a more tractable form [76]

where $A_o = A_1 A_2$ 1/2is the geometric mean of the magnitudes of the two amplitudes, $\alpha = A_1 / A_2$ is the arithmetic mean of the initial phase,$\Delta\phi = (\phi - \phi_2)2$ is one half the difference between the two phases and $\Delta w = (w_1 - w_2)/2$ is one half the difference between the two frequencies.

The autocorrelation function r_k of the complex sinusoids defined by (2.48) is given as

$$r_k = A_o^2 \left[\alpha^2 + \alpha^{-2}\right]\left[\exp(jkw_o)\right]\{\text{Cos } k\Delta w + j\, g(\alpha) \text{ Sin } k\Delta w\} + \sigma_n^2 \delta_k \qquad 2.50$$

where

$$g(\alpha) = \frac{\alpha^{-2} - \alpha^{-2}}{\alpha^2 + \alpha^{-2}}$$

If the input samples consist of two sinusoids in white noise, the resulting analytic signal is

$$Z_n = -j\,A_1 \exp(j2nw_1) - j\,A_2 \exp(j2nw_2) + w_n^c \qquad 2.51$$

In this we are going to try to estimate the mean value of the frequencies of the two tones, i.e., $(f_1+f_2)/2$ using the formula ($\alpha = 1$)

$$\hat{f} = \frac{1}{4\pi k} \arg(\hat{r}_k) = \frac{1}{4\pi k} \tan^{-1} \frac{y}{x} \qquad 2.52$$

where $y=\mathrm{Im}[r_k]$ and $x=\mathrm{Re}[r_k]$.

2.6.1. EVALUATION OF $E[\hat{f}]$

Again using the second order approximation technique, the expected value of f is found as

$$E\left[\hat{f}\right] = \frac{1}{4k\pi}\left[\hat{f}(\bar{x},\bar{y}) + \frac{1}{2}\left(\frac{2\bar{y}\bar{x}\left(\sigma_x^2 - \sigma_y^2\right) + 2\left(\bar{y}^2-\bar{x}^2\right)\mathrm{Cov}(x,y)}{(\bar{x}^2+\bar{y}^2)^2}\right)\right] \qquad 2.53$$

where

$$\bar{x} = E[x] = A_o^2\left[\alpha^2+\alpha^{-2}\right]\frac{N-k}{N}\,T_k \triangleq \beta T_k$$

$$\bar{y} = E[y] = A_o^2\left[\alpha^2+\alpha^{-2}\right]\frac{(N-k)}{N}\,C_k$$

$$\sigma_x^2 = \frac{\sigma_n^4}{N}\left[\frac{1}{2} + \mu\left(1+\frac{(N-2k)}{N}\,T_{2k}\right)\right]$$

$$\sigma_y^2 = \frac{\sigma_n^4}{N}\left[\frac{1}{2} + \mu\left(1-\frac{(N-2k)}{N}\,T_{2k}\right)\right]$$

$$\mathrm{Cov}(x,y) = \sigma_n^4\mu\frac{(N-2k)}{N^2}\,C_{2k} \triangleq C_{2k}$$

$$C_k = g(\alpha)\,\mathrm{Sin}k\Delta w\;\mathrm{Cos}k\,w_o + \mathrm{Sin}k\,w_o\,\mathrm{Cos}k\Delta w$$

$$T_k = \mathrm{Cos}k\,w_o\,\mathrm{Cos}k\Delta w - g(\alpha)\,\mathrm{Sin}k\,w_o\,\mathrm{Sin}k\Delta w$$

$$\mu = A_o^2\,(\alpha^2+\alpha^{-2})/\sigma_n^2$$

Eq.(2.53) is used to calculate the expected value of f when the theoretical autocorrelation function is given. Putting this expression into a more convenient form, we have

$$E\ \hat{f} = \frac{1}{4\pi k}\ \tan^{-1}\left(\frac{C_k}{T_k}\right) + \frac{T_k C_k (T_{2k}+T_{2k}) + (C_k^2 - T_k^2)\ C_{2k}}{\beta^2 (C_k^2 + T_k^2)^2} \qquad 2.54$$

For $\alpha = 1$, $g(\alpha)=0$, it can be easily shown that 2.55

$$E[f]=(f_1+f_2)/2$$

i.e., this method correctly estimates the mean frequency of two sinusoids. As α deviates from unity E[f] takes the limiting values of f_1 and f_2. It means that $E[f]=f_1$ as $\alpha \rightarrow \infty$ and $g(\alpha)=1$ or $E[f]=f_2$ as $\alpha \rightarrow 0$ and $g(\alpha) \rightarrow -1$ and single complex sinusoid case is approached.

2.6.2. VARIANCE OF THE ESTIMATE

Following the previous step similarly, the variance of the estimate is found as

$$\sigma_{\hat{f}}^2 = \frac{\frac{\sigma_n^4}{N}\left(\frac{1}{2}+\mu\right)(C_k^2+T_k^2) + (C_k^2-T_k^2)T_{2k} - 2T_k C_k\ C_{2k}}{\beta^2 (C_k^2 + T_k^2)^2} \qquad 2.56$$

For $\alpha = 1$, $g(\alpha)=0$

$$\sigma_{\hat{f}}^2 = \frac{N + 4\,N\mu \operatorname{Sin}^2 k\Delta w + 4\mu k \operatorname{Cos} 2k\Delta w}{\left[\sqrt{2}\ \mu (N-k)k \operatorname{Cos} k\Delta w\right]^2} \qquad 2.57$$

Similarly as in the single tone case, an important question arises apart from the selection of the AC lag in order to minimize the frequency error arriving the Cramer Rao lower bound. Certainly the number of data samples N, and SNR play key roles in this situation with the desirable situation being SNR$\gg$1 and large N. For $\Delta W=0$, Eq(2.57) is similar to Eq(2.41) i.e.

single complex tone case. The Eq.(2.57) is plotted parametrically in Fig.II.9 against the AC lag k, and different values of ΔW . Again it can be seen

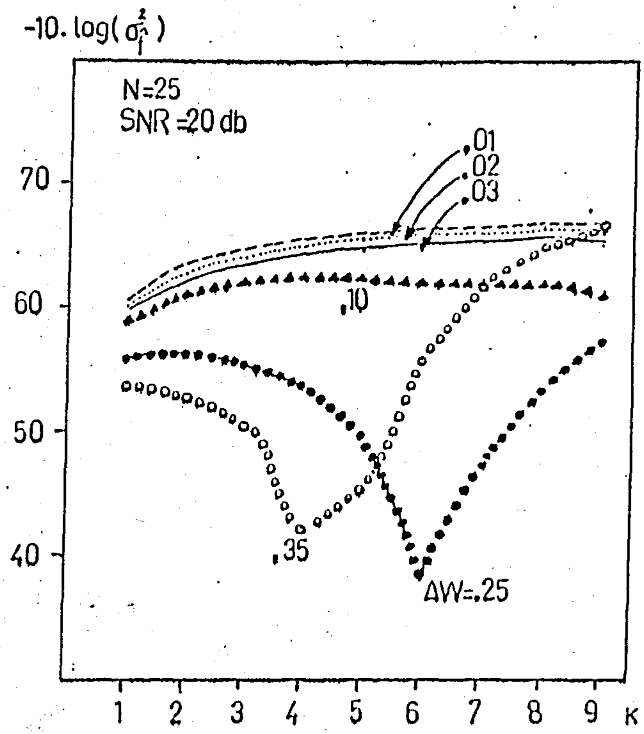


Fig.II.9. Variance of the estimate versus autocorrelation lag.

that, the performance of frequency estimation can be improved by using optimal value of k which is different than one although it breaks down after ΔW is greater than 0.10. If ΔW is close to zero i.e., we have two closely resolving sinusoidal signals, the analysis yields the proposed result. This situation can be understood clearly if one examines Eq.(2.52). Also Fig.II.10 illustrates this behaviour versus signal to noise ratio with k=6.

2.7. P.D.F. EXPRESSION OF ESTIMATED MEAN FREQUENCY

In order to determine the detection performance of the method, it is desirable to know the probability density function of $\hat{f}$ under the tone present (H_1) and noise only (H_o) hypotheses. We assume again the number of samples N is large enough to use Gaussian statistics.

— For H_o

The statistical properties of X and Y found in Appendix , can be rewritten for H_o

$$X \sim N(0, \sigma_n^4/2N) \tag{2.58}$$

$$Y \sim N(0, \sigma_n^4/2N)$$

These expressions are now used to derive the probabilty density function of $\hat{f}$. It is more convenient first to consider the dummy variable γ which is defined

$$\gamma = y/x \tag{2.59}$$

This is in turn used to derive the P.D.F. of $\hat{f}$ using the formula

$$2\pi k f = \arctan \gamma \tag{2.60}$$

From [74, pp.198], the joint P.D.F. of x and y is

$$P_{xy}(x,y) = \frac{N}{\pi \sigma_n^4} \exp\left[-N\,(x^2+y^2)/\sigma_n^4\right] \tag{2.61}$$

then

$$P_\gamma(\gamma) = \frac{1}{(\gamma^2+1)} \tag{2.62}$$

Now using Eq.(2.60), we have finally [74]

$$P_{\bar{W}}(w) = \begin{cases} \frac{1}{\pi} & 0 < w < \pi \\ 0 & 0 > w \end{cases} \tag{2.63}$$

Note that the probability density function under H_o does not depend upon the noise power σ_n^2 .

— For H_1

For signal plus noise case, the statistical properties of X and Y follow from Appendix A, as .

$$X \sim N \left(A^2 \frac{(N-k)}{N} \cos w_1 k, \; \frac{\sigma_n^4}{N} \; \frac{1}{2} + \mu \frac{(N-2k)}{N} \cos 2w_1 k \right)$$

$$Y \sim N \left(A^2 \frac{(N-k)}{N} \sin w_1 k, \; \frac{\sigma_n^4}{N} \; \frac{1}{2} + \mu \left(1- \frac{(N-2k)}{N} \cos 2w_1 k\right) \right) \qquad 2.64$$

To determine the P.D.F. expression, one can use Eq.(A-29) by substituting the statistics of both X and Y to find

$$A_1 = \frac{N}{2\,\sigma_n^4\,(a^2-b^2)^{1/2}}$$

$$A_2 = \frac{a^2-b^2 \cos^2 2w_1 k}{2(a^2-b^2)}$$

$$A_3 = \frac{N \quad a-b \cos 2(w-w_1)k}{\sigma_n^4 \cos^2 wk \left[a^2-b^2\cos^2 2w_1 k\right]}$$

$$A_4 = \frac{2\,CN[b-a] \cos (w-w_1)k}{\sigma_n^4 (a^2-b^2 \cos^2 2w_1 k) \cos wk}$$

$$A_5 = \frac{NC^2 \;(a-b)}{\sigma_n^4 (a^2-b^2 \cos^2 2w_1 k)}$$

where

$$C = \frac{A^2 (N-k)}{N}, \quad a = \frac{1}{2} + \mu \quad \text{and} \quad b = \frac{\mu\,(N-2k)}{N} \qquad 2.70$$

finally from (C-11), we have

$$P_{\hat{W}}(w/H_1) = \left[\frac{(a^2-b^2)^{1/2}}{\pi\ \Gamma\ (w,w_1)} - \frac{\sqrt{N}\ c\,(b-a)}{\sqrt{2\pi}\ \sigma_n^2\ \Gamma^{3/2}(w,w_1)} \right]$$

$$\exp\left[\frac{-NC^2}{2\ \sigma_n^4\ (a+b)} + \frac{NC^2(a-b)\ \ \text{Cos}^2(w-w_1)k}{2\sigma_n^4\ (a+b)\ \ \Gamma\ (w,w_1)} \right] \qquad 2.71$$

or

$$P_{\hat{W}}(w) = \left[\frac{(a^2-b^2)^{1/2}}{\pi\ \Gamma\ (w,w_1)} - \frac{\sqrt{N}\ c(b-a)\ \text{Cos}\ (w-w_1)}{\sqrt{2\pi}\ \sigma_n^2\ \Gamma^{3/2}(w,w_1)} \right]$$

$$\exp\left[\frac{NC^2}{2\sigma_n^4(a+b)} \cdot \left(\frac{(a-b)\text{Cos}^2(w-w_1)\ k}{\Gamma\ (w,w_1)} - 1 \right) \right] \qquad 2.72$$

where

$$\Gamma\ (w,w_1) = a-b\ \text{Cos}\ 2(w-w_1)k$$

The P.D.F. for H_1 is expressed in terms of the mean frequency, number of data samples and signal to noise ratio. Fig.II.11 and Fig.II.12 show this function for some different values of SNR and number of data samples. Considering the P.D.F. of $(\hat{W}_1 - W_1)$ as $P(\Delta W|H_1)$ it can be concluded that the mean frequency estimate is effectively unbiased since $E(\Delta W)$ is zero. In other words, since the density of mean frequency displays arithmetic symmetry around W_1, the expected value of the mean frequency equals to true one. Comparing Figs.II.11 and II.12, one can see that the maximum value of P.D.F. and variance of $\hat{W}_1$ are related to both N and SNR as expected. In these figures $W_1=72$, N=50 and N=100 respectively.

Also a different approach can be used to prove that the estimate is unbiased under a particular interpretation which is derivative of P.D.F. with respect to W. The interpretation is explained as follows :

The derivative of $P_W(w \mid H_1)$ is given

$$\frac{d}{dw} P_{\hat{W}}(w|H_1) = \left\{ \frac{d}{dw} \left[\frac{(a^2-b^2)^{1/2}}{\pi\Gamma(w_1 w_1)} - \frac{\sqrt{N}\, c(b-a)\ \mathrm{Cos}(w-w_1)k}{\sqrt{2\pi}\ \sigma_n^2\ \Gamma^{3/2}_{(w,w_1)}} \right] \right\}$$

$$\exp\left[\frac{NC^2}{2\sigma_n^4\ (a+b)} \left(\frac{(a-b)\ \mathrm{Cos}^2(w-w_1)k}{\Gamma(w,w_1)} - 1 \right) \right] + \left[\frac{(a^2-b^2)^{1/2}}{\pi\Gamma(w,w_1)} - \frac{\sqrt{N}\, c(b-a)\mathrm{Cos}(w-w_1)k}{\sqrt{2\pi}\ \sigma_n^2\ \Gamma^{3/2}_{(w,w_1)}} \right]$$

$$\frac{d}{dw}\left[\exp\left\{ \frac{NC^2}{2\sigma_n^4\ (a+b)} \left(\frac{(a-b)\mathrm{Cos}^2(w-w_1)k}{\Gamma(w,w_1)} - 1 \right) \right\} \right] \qquad 2.73$$

Since the derivatives of both $\mathrm{Cos}(w - w_1)k$ and $\Gamma(w,w_1)$ with respect to evaluated at $w = w_1$ are zero, the derivative of $P_W(w \mid H_1)$ at $w = w_1$ becomes zero. This result and symmetrical property of P.D.F. imply that the expected value of $\hat{\omega}$ is equal to ω_1. This method can be seen to be a useful technique for estimating the frequency of sinusoidal signal according to this conclusion.

2.8. BANDWIDTH ESTIMATION

In this section the problem we set ourselves is that of estimating the spectral bandwidth (damping factor) of the sinusoidal signal by analytic signal model. Let us assume that the N samples of the observed

observed data sequence z_n or analytic signal model consists of samples of an exponentially damped sinusoidal signal in complex valued white Gaussian noise W_n

$$z_n = -j\sqrt{2}\, A\, e^{-\alpha n}\, e^{i[w_1 n + \phi]} + W_n^c \tag{2.74}$$

where α is the damping factor. We set up the following equation to estimate α (BW^2) as

$$BW^2 = \frac{1}{2k}\left[1 - \frac{|r_k|}{r_o}\right] \tag{2.75}$$

by using the fact $e^x = 1+x$ and $r_k = A^2 e^{-\alpha k} e^{-jw_1 k} + \sigma_n^2 \delta(k)$ for $k=0,1,2,\ldots$ Therefore BW^2 can be simply estimated by a knowledge of the autocorrelation function at one particular appropriately small nonzero value of k. Actually the validity of the approximation of series expansion is directly related to the α for constant k. (It can be seen that for noise only case BW^2 becomes infinite as expected since the sampling time is not changed.) In this regard, thus there is no need for determining the entire power spectrum or the entire autocorrelation function as in the frequency estimation problem.

2.8.2. VARIANCE OF THE ESTIMATE

In order to find the variance expression for bandwidth estimate, first of all one has to calculate variance of r_k. Using second order approximation technique, Var(r_k) is obtained as in the previous part

$$\mathrm{Var}\ |r_k| = 4\left[\bar{x}^2 \sigma_x^2 + \bar{y}^2 \sigma_y^2 + 2\bar{x}\bar{y}\ \mathrm{Cov}\ (\bar{x},\bar{y})\right] \tag{2.76}$$

The statistics for an exponentially damped sinusoidal signal are given by

$$\bar{x} = A^2 e^{-\alpha k} \frac{(N-k)}{N} \operatorname{Cos} w_1 k$$

$$\bar{y} = A^2 e^{-\alpha k} \frac{(N-k)}{N} \operatorname{Sin} w_1 k$$

$$\sigma_x^2 = e^{-\alpha 2k} \left[\frac{\sigma_n^4}{N} \quad \frac{1}{2} + \mu (1 + \frac{(N-2k)}{N} \operatorname{Cos} 2w_1 k) \right]$$

$$\sigma_y^2 = e^{-2\alpha k} \left[\frac{\sigma_n^4}{N} \quad \frac{1}{2} + \mu \; (1 - \frac{(N-2k)}{N} \operatorname{Cos} 2w_1 k \right]$$

$$\operatorname{Cov}(x,y) = \sigma_n^4 \, e^{-2 k} \, \mu \, \frac{(N-2k)}{N^2} \operatorname{Sin} w_1 k \qquad 2.77$$

Substituting Eq.(2.77) into Eq.(2.76) we have

$$\operatorname{Var} \; |r_k| = \frac{4 \, C_k^2 \, \sigma_n^4}{N} \left[(\frac{1}{2} + \mu) - \frac{\mu (N-2k)}{N} \operatorname{Cos} 4w_1 k \right] \qquad 2.78$$

where

$$C_k^2 = \frac{A^2 (N-k)^2}{N^2} e^{-2\alpha k}$$

Finally, the variance of the bandwidth estimate is obtained by second order technique [74] as

$$\operatorname{Var}\left[BW^2 \right] = \frac{1}{4k^2} \left[\frac{\operatorname{Var} |r_k|}{r_o^2} + \frac{|r_k|^2}{r_o^4} \operatorname{Var}\left(r_o\right) - 2 \left(\frac{|r_k|}{r_o^3}\right) \operatorname{Cov} |r_k|, r_o \right] \qquad 2.79$$

Similarly this expression is a function of autocorrelation lag and it requires the simulation to determine estimation performance.

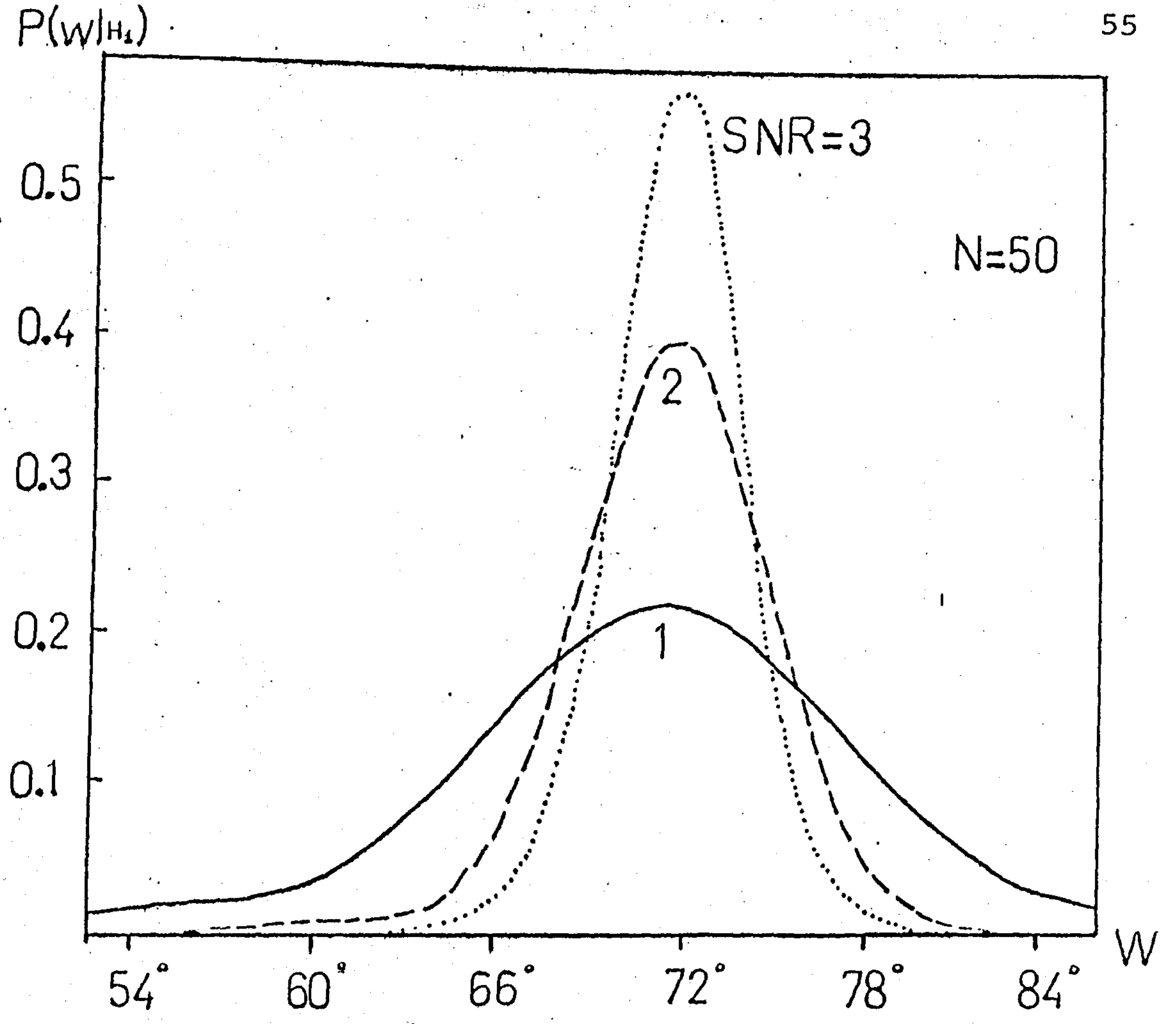


Fig.II.11. PDF of the mean frequency estimate for various values of SNR.

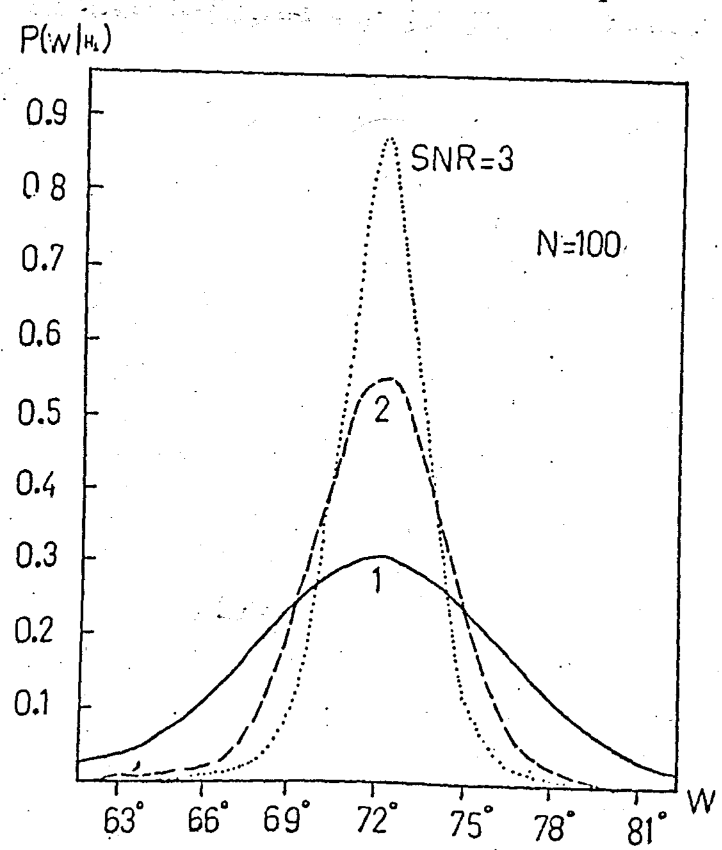


Fig.II.12. PDF of the mean frequency estimate for various values of

CHAPTER III ASYMPTOTIC BEHAVIOUR AND STATISTICS OF SPECTRAL MOMENTS

3.1. INTRODUCTION

Spectral moments have been used in applications including radar,sonar problems [78], tone detection [79], mean frequency estimation [80] and convolution, deconvolution problems [81]. As an example, in radar applications the first three moments correspond respectively, to the volume, mean velocity and range of velocities of the scatterers. Also the mean frequency is used to estimate the frequency of sinusoidal signal in white noise.

In the past, spectral moments have been estimated by first estimating the spectrum S(f) itself and then using this estimate into the definition of the spectral moments. However estimation of the spectrum is a very important problem which has led to the development of several estimation techniques. By the choice of one existing methods, one can have some advantages depending on aim and with respect to specific conditions. One possible approach to the estimation of spectral moments is to use a spectrum analyzer which is often like that a filter bank or Fast Fourier Transform (FFT) processor etc. However the specific charecteristics of this method make it improper to do it. (Conventional Fourier type estimator is known to yield poor resolution when applied to short length data records and computational problem even using FFT processor.) In addition, computing the spectrum at every point provides more information than necessary in many applications.

(For example, complete information about the spectral shape is not required in tone detection problem [79]). However one can estimate the spectral moments using time domain information as well as, i.e., by using the inphase and quadrature signals [79].

More recently an algorithm has been proposed which estimates the spectral moments without performing a Discrete Fourier Transform (D.F.T.) by using Maximum Entropy Spectral Analysis (MESA) moment matching technique [82]. In this method, spectral moments are estimated by using a series expansion in terms of the autocorrelation lags. Use can made of the MESA to extrapolate the autocorrelation lags from a few known or estimated lags.

In this chapter some of the properties of MESA moments are investigated. The MESA moment paper is composed of three section. In the first section, we investigate the use of the Maximum Entropy spectrum in the estimation of spectral moments and consider some of their elementary properties such as filtering, windowing and shifting. In the second section, the asymptotic behaviour of MESA moment technique for the case of sinusoid embedded in additive white noise is analyzed. The asymptotic formula for the spectral mean frequency and mean square bandwidth are derived by assuming known autocorrelation function. Also we compare the asymptotic behaviour of the estimator with the case when only a few terms are involved. As a bonus of the analysis the same moment expressions for the case of two sinusoids with equal or non-equal power are also determined. In the last section the analysis of statistical properties of MESA moment is given. It is shown that the probability of detection (P_D) and probability of false alarm (P_F) can be written as functions of the expected value and the variance of n´th moment for tone detection problem. In addition to these the variance and P.D.F. of the estimated mean frequency are derived analytically in this section.

3.2. MESA MOMENTS :

Let us consider as an approximation the moment integrals the sum in (3.1) with the N-point Discrete Fourier transform:

$$Mn = \frac{2}{N} \sum_{\ell=o}^{N-1} S(\ell) \left[\frac{\ell T}{N} \right]^{n} \qquad 3.1.$$

which is valid when T/N is small relative to variation in the spectrum and where m and T denote respectively, the n´th moment and the sampling period. Expanding the factor (lT/N) in terms of a cosine series [82], one obtains a moment expression in terms of the autocorrelation (AC) lags, i.e.,

$$Mn \doteq \frac{C_o^n \, r_o}{2} + \sum_{k=1}^{\infty} C_k^n \, r_k \qquad 3.2.$$

where

$$=4T \left[(-1)^k \sum_{r=o}^{(n-1)2} (-1)^r \frac{n!}{(n-2r-1)!} \frac{(2T)^{-n+2r+1}}{(2kT)^{2r+2}} - \begin{cases} (-1)^{(n-1)/2} \dfrac{n!}{(2kT)^{n+1}} & n=1,3,5,\ldots \\ o & n=0,2,4,\ldots \end{cases} \right]$$

for k=1,2,3,.........

and

$$r_k = \frac{2}{N} \sum_{\ell=0}^{N-1} S(\ell) \cos\left(\frac{2\pi k\ell}{N}\right) \qquad k=0,1,\ldots\ldots\ldots\ldots(N-1)$$

In practice only the first M AC terms may be available; however higher order lags can be found by extrapolation using the equation :

$$r_k = -a_1 r_{k-1} - \cdots - a_M r_{k-M} \qquad k > M \qquad 3.3$$

where the a , j=1,2,3,....M are the linear prediction coefficients found by one of the least squares (LS) techniques such as, PARCOR, sequential LS, lattice, covariance, autocorrelation.....algorithms. Because of the potential implication of the Maximum Entropy techniques in (3.3), we will call the estimates in (3.1) the MESA moments.

The usefulness of this method can be attributed to several factors. By the choice of MESA moment technique and for a given number of data samples, one can properly estimate the spectral moments by means of the extrapolation of AC for low order autoregressive (AR) model. (Since c_k coefficients decay as k^{-2} with increasing number of AC terms, the first a few terms are the most important. This means that only a few AR parameters are required.)

3.2.1. ELEMENTARY PROPERTIES

Certain properties of spectral moments obtained through the expansion in (3.2) are listed below.

Shifting : The shifted moments

$\mu n=\int_0^\infty (f-f_0)^n \, S(f)df$ can be found as

$$\mu n=\sum_{i=0}^{n} (-1)^i \binom{n}{i} M_{n-i} \, f_0^i \qquad 3.4$$

Filtering : A process filtered through a system with power transfer function P(f) has the moments

$$\mu n = \int_0^\infty f^n S(f)P(f)df = \frac{Co^n}{2} r'_0 + \sum_{k=1}^{\infty} C_k^n \, r'_k \qquad 3.5$$

where

$$r'_k = \sum_{i=0}^{N-1} r_i \lambda_{k-i} \quad \text{and} \quad \lambda_k = \sum_{\ell=0}^{N-1} P(\ell) \exp\left(j \frac{2\pi k\ell}{N}\right)$$

Windowing : If the process is windowed with the window function w(n), the moments take the form

$$\mu n = \frac{Co^n \, r'_0}{2} + \sum_{k=1}^{\infty} C_k^n \, r'_k \qquad 3.6$$

with

$$r_k = [w(k)*w(k)]r_k$$

where ´*´ denotes the convolution operation.

3.3. ASYMPTOTIC BEHAVIOUR

It is of interest to evaluate and compare the moment estimates in (3.1) increasing number of AC terms. Since the cases of tones in the noise background is analytically tractable, let us in fact consider their asymptotic behaviour. The AC sequence for L tones in white noise is given by

$$r_k = \sigma_n^2 . \delta(k) + \sum_{i=1}^{L} A_i^2 \, \mathrm{Cos} W_i k \qquad 3.7$$

where σ_n^2 is the noise power, $\delta(k)$ is the Kronecker delta and A_i^2, f_i are signal power, frequency of i´th sinusoid respectively.

The first step of our analysis is to determine the asymptotic behaviour of the MESA moment technique for the case of single tone case.

3.3.1. SINGLE TONE CASE

For a single tone (L=1), the first moment can be given as

$$M_1 = \frac{Co^1 ro}{2} + \sum_{k=1}^{\infty} C_k^1 r_k \qquad 3.8$$

where

$$C_k^1 = \begin{cases} \dfrac{-2}{(\Pi k)^2} & k=1,3,5,\ldots \\ 0 & k=2,4,6,\ldots \end{cases} \qquad \text{and } C_o^1 = \frac{1}{2T}$$

To obtain the mean frequency we divide the Eq.(3.8) by r_o

$$\hat{f} = \frac{1}{4T} - \frac{2}{T} \sum_{\substack{k=1 \\ k:odd}}^{\infty} \frac{r_k}{\Pi^2 k^2 r_o} \qquad 3.9$$

Since $c_k^1 0$, $\forall$ k which are even we rewrite the expression as

$$\hat{f} = \frac{1}{4T} - \frac{2}{T\Pi^2} \sum_{k=1}^{\infty} \frac{r_{2k-1}}{(2k-1)^2 r_o} \qquad 3.10$$

Now let us determine the second moment expression.The coefficients for k 0, n=2 and the second moment are given

$$c_o^2 \quad \frac{1}{6T^2} \qquad\qquad c_k^2 \quad \frac{(-1)^k}{T^2 \; \Pi^2 k^2} \qquad 3.11.a$$

and

$$M_2 = \frac{A_1^2}{12T^2} + \sum_{k=1}^{\infty} \frac{(-1)^k r_k}{T^2 \; \Pi^2 k^2} \qquad 3.11b$$

respectively.

Noise free situation

For the noise free case ($\sigma n^2=0$) and a single tone (L=1), the mean frequency and normalized second moment expression are

$$\hat{f} = \frac{1}{4T} - \frac{2}{T^2 \; ^2} \sum_{k=1}^{\infty} \frac{\text{Cos}\, 2\Pi f_1 (2k-1)}{(2k-1)^2} \qquad 3.12a$$

$$\frac{M_2}{r_o} = \frac{1}{12T^2} + \sum_{k=1}^{\infty} (-1)^k \frac{\text{Cos}\Pi 2 \; f_1 Tk}{\Pi^2 T^2 k^2} \qquad 3.12b$$

From [83,pp39] using

$$\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} = \frac{\Pi}{4}\left(\frac{\Pi}{2} - |x|\right) \text{ and } \sum_{k=1}^{\infty} (-1)^k \frac{\cos kx}{k^2} = \frac{\Pi^2}{12} - \frac{x^2}{4}$$

and considering the fact that m_1/m_0 and $[(m_2-m_1^2)/m_0]^{1/2}$ are estimates respectively, of the mean frequency and mean square bandwidth (BW) one obtains the exact values $\hat{f}=f_1$ and BW=0

Noisy situation

By following the similar step as in the previous section, for the case of $\sigma n^2 \neq 0$, the mean frequency and bandwidth become respectively,

$$\hat{f} = \frac{\mu}{\mu+1} \frac{1}{4\mu T} + f_1 \qquad 3.13.a$$

$$BW = \frac{1}{\mu+1}\left(f_1^2 + \frac{4\mu+1}{48\mu T^2} - \frac{f_1}{2T}\right)^{1/2} \qquad 3.13.b$$

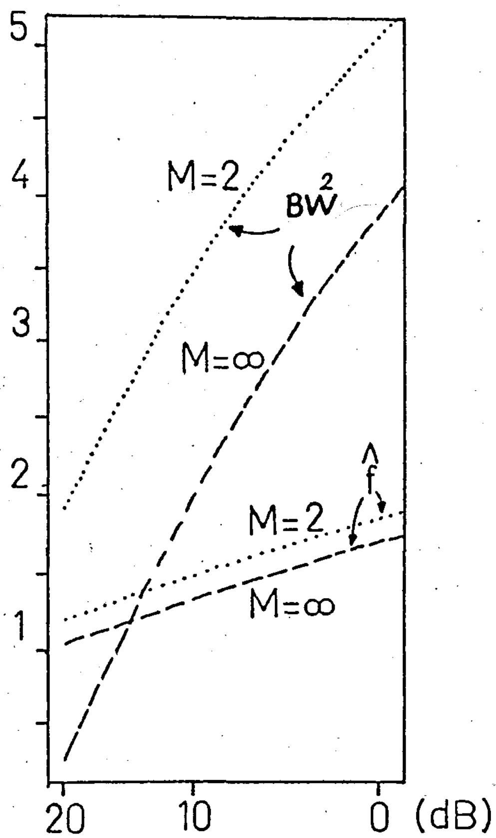


Fig.III.1. Variations of $\hat{f}$ and BW with 10.log where $\mu = A_1^2/\sigma n^2$ is the signal to noise ratio.

In Equation (3.13-a) the term 1/4T is due to the white noise mass; this bias however be eliminated and this equation can be used as a tone frequency estimator. The convergence properties of the series in (2) is shown in Fig.III.1 where both f and BW are plotted as functions of the signal to noise ratio (SNR) and parametrically dependent upon the number of lag term M. In this example one has T=0.05 sec, hence the Nyquist frequency is 10 Hz and the tone frequency is 1 Hz as in [20]. For SNR→0, one excepts $\hat{f}=1/4T$ and BW=1/2T while for SNR→∞ one should have $\hat{f}=1$ Hz. Also Equations (3.13-a,b) can be rewritten for high SNR condition as follows i.e., $\mu >> 1$

$$\hat{f} = f_1 + \frac{1}{4\mu T} \qquad 3.14.a$$

$$BW = \frac{1}{\mu}\left(f_1^2 + \frac{1}{12T^2} - \frac{f_1}{2T}\right)^{1/2} \qquad 3.14.b$$

One observes also that a good estimate of the moments can be obtained by using only a few AC lags, i.e., 5 M 10. The first and second moments can be reliably estimated using two lags, however higher order moments require more than two lags.

3.3.2. TWO TONES CASE

In this case we are going to use the first moment for estimating the mean value of the frequencies of the two tones, i.e., $(f_1 + f_2)/2$.

Noise free situation

By using the previous expressions for the first and second moment we have

$$\hat{f} = \frac{A_1^2 f_1 + A_2^2 f_2}{A_1^2 + A_2^2} \qquad 3.15.a$$

$$BW^2 = \frac{A_1^2 A_1^2}{A_1^2 + A_2^2} (f_1 - f_2)^2 \qquad 3.15.b$$

Special situation which needs mentioning is the case where $A_1 = A_2$. The expressions become

$$\hat{f} = (f_1 + f_2)/2 \qquad 3.16.a$$

$$BW^2 = (f_1 - f_2)^2/4 \qquad 3.16.b$$

One can see that from the above expressions, the depth null in the spectrum is weighted by the ratio of the power of the tones and if two tones are closely spaced then BW will be smaller as expected. The results of this section also coincide with a single tone case i.e., $f_2 = f_1$.

Noisy situation

Similarly, $\hat{f}$ and BW^2 expressions for equal power become respectively

$$\hat{f} = \frac{\mu}{\mu + 1} \quad \frac{1}{4\mu T} + \frac{f_1 + f_2}{2} \qquad 3.17.a$$

$$BW^2 = \frac{1}{(1+\mu)^2} \left[\frac{\mu(4+\mu)}{48T^2} \quad \frac{(2+\mu)}{4}(f_1^2 + f_2^2) - \frac{1}{2} \left(f_1 f_2 + \frac{\mu}{2T}(f_1 + f_2)\right) \right] \qquad 3.17.b$$

where

$$\mu = \frac{2A_1^2}{\sigma n^2}$$

3.4. STATISTICAL PROPERTIES :

In order to determine the detection performance of the moments described here, it is desirable to know the probability density function of m_n under the tone present (H_1) and noise only (H_o) hypotheses. We assume that the number of samples used N is large enough to assume Gaussian statistics. Let us consider again a two term approximation, for the moments i.e.,

$$Mn = \frac{C_o^n\, r_o}{2} + C_1^n\, r_1 + C_2^n\, r_2 \qquad 3.18$$

Statistics for m_n involving an arbitrary number of autocorrelation terms can be obtained in a straightforward but tedious manner.

Since r_k is normal random variables as shown in the Appendix (A), for both the null and alternative hypotheses, m_n has a Gaussian distribution with mean and variance respectively as

$$\lambda_{o,n} = E\left(Mn \mid Ho\right) = \frac{a_n\, \sigma n^2}{2} \qquad 3.19.a$$

$$\lambda_{1,n} = E\left(Mn \mid H_1\right) = \frac{a_n}{2}\left(\sigma n^2 + A_1^2\right) + bnA_1^2 \cos \omega_1 \qquad 3.19.b$$

$$\sigma_{o,n}^2 = E\left[(Mn - \lambda_{o,n})^2\right] = \frac{a^2 n\, \sigma n^4}{4N} + \frac{bn^2 \sigma n^4}{N} \qquad 3.19.c$$

$$\sigma_{1,n}^2 = E\left[(Mn - \lambda_{1,n})^2\right] = \frac{\sigma n^2}{N}\left[\beta \left(an^2 + \frac{bn^2}{2}\right) + bn^2 A_1^2 \cos^2\omega_1 - dn\, A_1^2 \cos\omega_1\right] \qquad 3.19.d$$

where

$$\beta = \sigma_n^2 + 2A_1^2$$

$$d_n = 2C_o^n C_1^n$$

$$a_n = C_o^n$$

$$b_n = C_1^n$$

and E(.) is the expectation operator.

Let us now discuss two applications

3.4.1. TONE DETECTION

The P.D.F. expressions for both hypotheses can be used in tone detection problem. Thus we concentrate our efforts on evaluating the performance of the likelihood ratio test (L_{m_n}) and calculating probability the detection (P_D) and probability of false alarm (P_F). Evaluation of the detection performance of the MESA moment detector for the general (involving M AC lags) case is a difficult task. We demonstrate here the detection performance for M=2, the result presented can be viewed as a lower bound on the MESA moment detection performance.

The sufficient statistic obtained from the likelihood ratio test is [73].

$$L_{Mn} = Mn\left[0.5\left(\frac{1}{\sigma_{o,n}^2} - \frac{1}{\sigma_{1,n}^2}\right)Mn\left(\frac{\lambda_{o,n}}{\sigma_{o,n}^2} - \frac{\lambda_{1,n}}{\sigma_{1,n}}\right)\right] \underset{H_o}{\overset{H_1}{\gtrless}} M_t \qquad 3.20$$

with

$$M_T = \frac{\lambda_{1,n}^2}{2\,\sigma_{1,n}^2} - \frac{\lambda_{o,n}^2}{2\,\sigma_{o,n}^2} + \ln\tau - \ln\left(\frac{\sigma_{1,n}}{\sigma_{o,n}}\right)$$

being the threshold of the test. The approach is to compute the n´th moment by the MESA method as data is obtained and then test its value against the following hypotheses.

$$m_n \leq m_T \quad : H_o$$
$$m_n \geq m_T \quad : H_1$$

Using (3.19) and (3.20), P_F may be evaluated as

$$P_F = \int_{M_T}^{\infty} P(M_n|H_o)\, dM_n \qquad 3.21$$

which yields

$$P_F = Q\left(\frac{M_T - \lambda_{o,n}}{\sigma_{o,n}}\right) \qquad 3.22$$

where Q(.) denotes the error function. Similarly the P_D computed from (3.19) and (3.20) may be expressed as

$$P_D = Q\left(\frac{M_T - \lambda_{1,n}}{\sigma_{1,n}}\right) \qquad 3.23$$

These expressions enable to use MESA moment technique as a signal detector.

3.4.2. MEAN FREQUENCY ESTIMATION

The first moment can be used to estimate the mean frequency as follows:

$$\hat{f} \quad \frac{0.25}{T} - \frac{0.2}{T} \sum_{k\ 1}^{M/2} \frac{r_{2k-1}}{(2k-1)^2\, r_o} \qquad 3.24$$

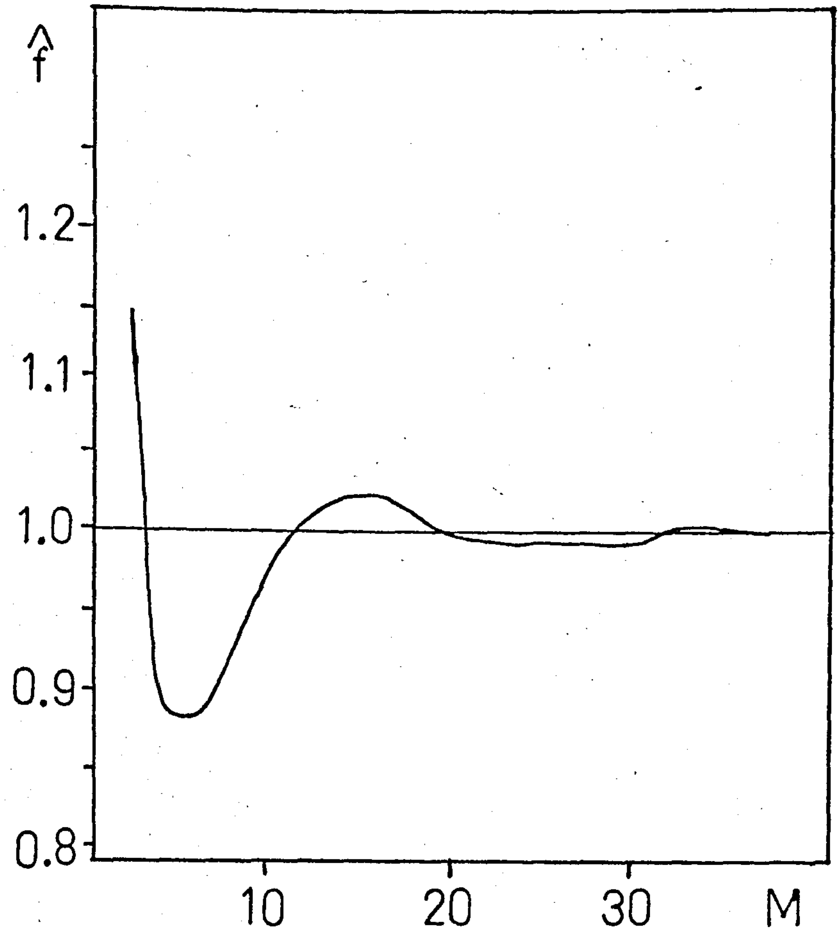


Fig.III.2 Estimation tone frequency plotted parametrically versus number of AC lags.

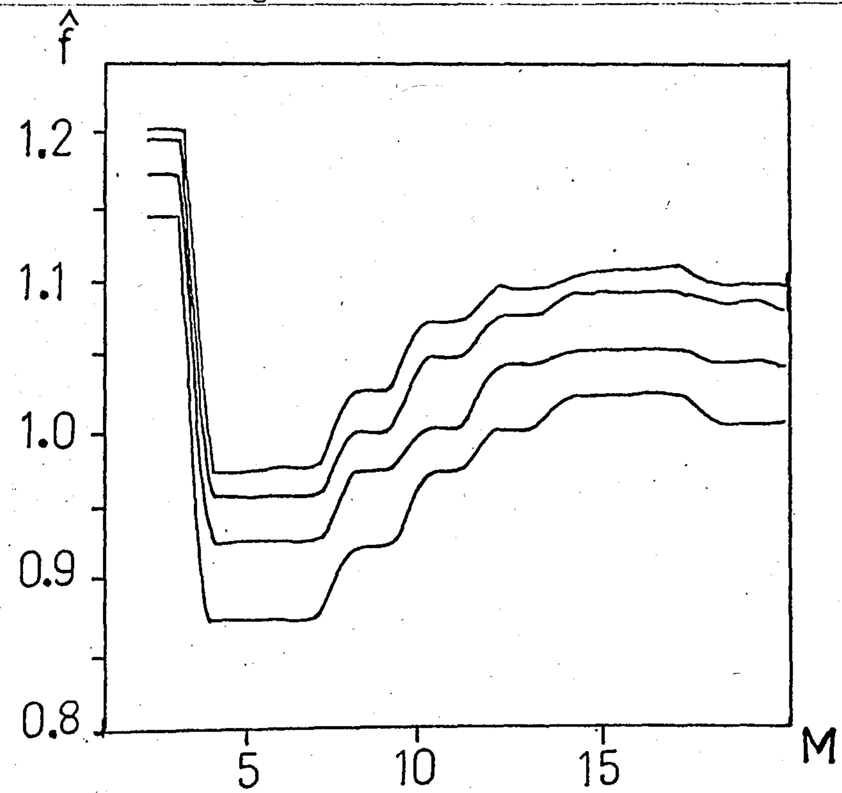


Fig.III.3. Variation of the mean frequency estimation versus number of AC terms.

In Fig.III.2, the estimated mean frequency is plotted versus the number of known AC lags. (Extrapolation goes from 3 to 40 starting with the two known AC lags.) The simulated tone frequency is 1 Hz and sampling rate 20 Hz. One observes an oscillatory behaviour for $\hat{f}$ up to 30, however higher order AC terms can be obtained by extrapolation through Eq.(3.3). In Fig.III.3 for the estimation of tone frequency both the known and estimated AC lags are used. Typical situation which needs mentioning again is stated as: The frequency error is not inversely proportional to the number of AC lags which are estimated or known in a certain limiting values of M. For example the minimum frequency error is obtained by estimated AC with the initial phase (ϕ) is zero for M is between 4 and 7, and known AC for M is between 12 and 20.The phase dependency of MESA mean frequency estimator is shown in Fig.III.4. It can be observed that the frequency error depends on the initial phase of the tone. For example minimum frequency error is attained for $6 \leq M < 8$ at about $\phi=180$ while for M=18, good performance is attained, at about $\phi=90^{\circ}$ and $210.^{\circ}$ Also the values of the estimated frequency using the known AC are straight lines as shown in Fig.III.4.

3.4.3. VARIANCE OF THE MEAN FREQUENCY

A measure of the accuracy of the mean frequency estimate of the MESA method is given by the variance analysis. In the calculation of the variance of f use has been made of the variance expressions for the AC terms as below:

The variance of the unbiased estimate of the i'th AC lag is [84]

$$\mathrm{Var}\ r_i = \frac{1}{(N-i)^2} \sum_{k=-(N-i)}^{n-i} (N-i-|k|)\,(r_k^2 + r_{k+i}\, r_{k-i}) - E^2(r_i) \qquad 3.25$$

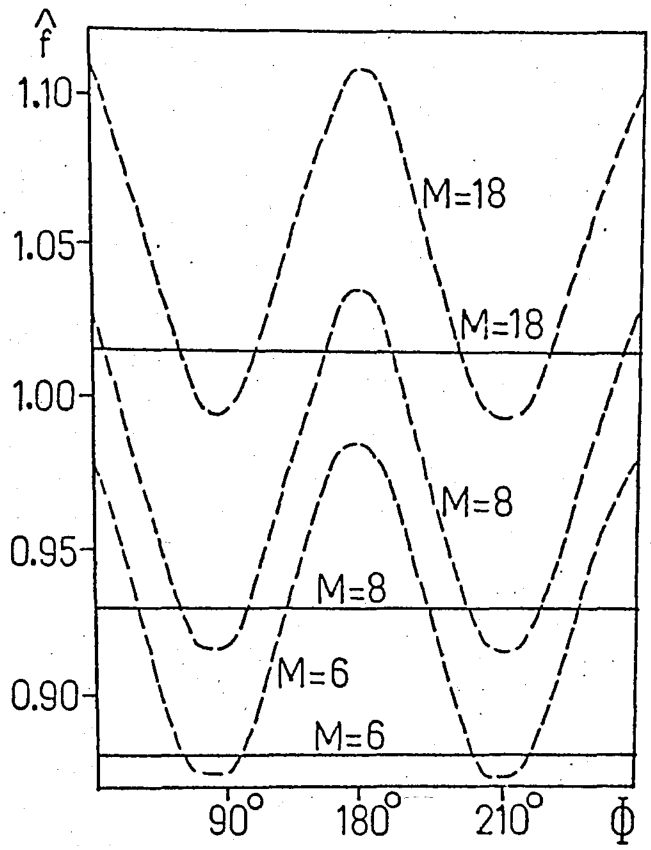


Fig.III.4. Variation of the mean frequency estimation with the initial phase (N=100).

Using again second order approximation technique as in [74] and assuming that the error in estimating r_{2k-1} is uncorrelated with r_o, we have finally

$$\mathrm{Var}\ \hat{f} \triangleq \sigma_{\hat{f}}^2 = 0.04T^{-2} \sum_{k=1}^{M/2} \frac{\mathrm{Var}\ (r2k-1)}{(2k-1)^2} \qquad 3.26$$

where

$$\mathrm{Var}\,(\Gamma_{2k-1}) = \left(\frac{\partial \Gamma_{2k-1}}{\partial r_{2k-1}}\right)^2 \sigma^2_{r_{2k-1}} + \left(\frac{\partial \Gamma_{2k-1}}{\partial r_o}\right)^2 \sigma^2_{r_o}$$

$$\Gamma_{2k-1} = \frac{2k-1}{r_o}$$

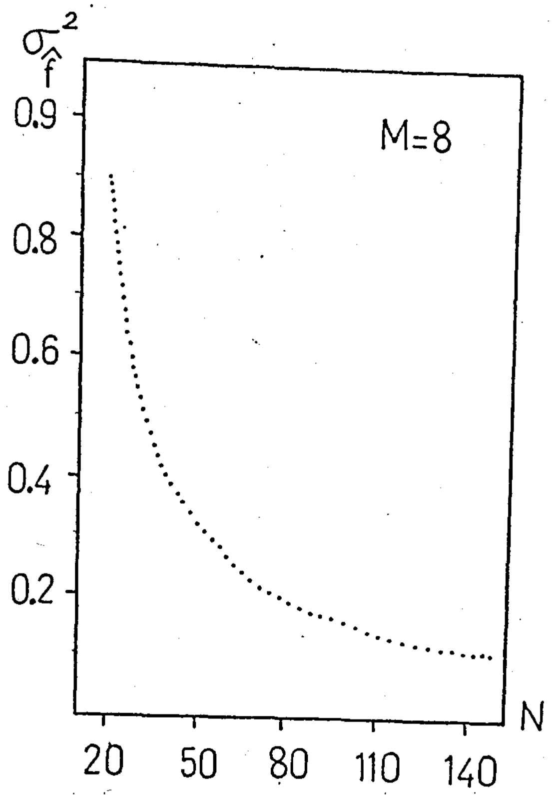


Fig.III.9. Variation of the mean frequency estimation variance vs. N

Let us again specialize to the case of a sinusoidal signal in white noise and consider two situations separately for analytical simplicity. The variance of the mean frequency estimate is then expressed in terms of the number of data samples (N), sampling period and mean frequency.

Noise Free ($\sigma_n^2 = 0$)

For the known AC lag Eq.(3.25) can be written as

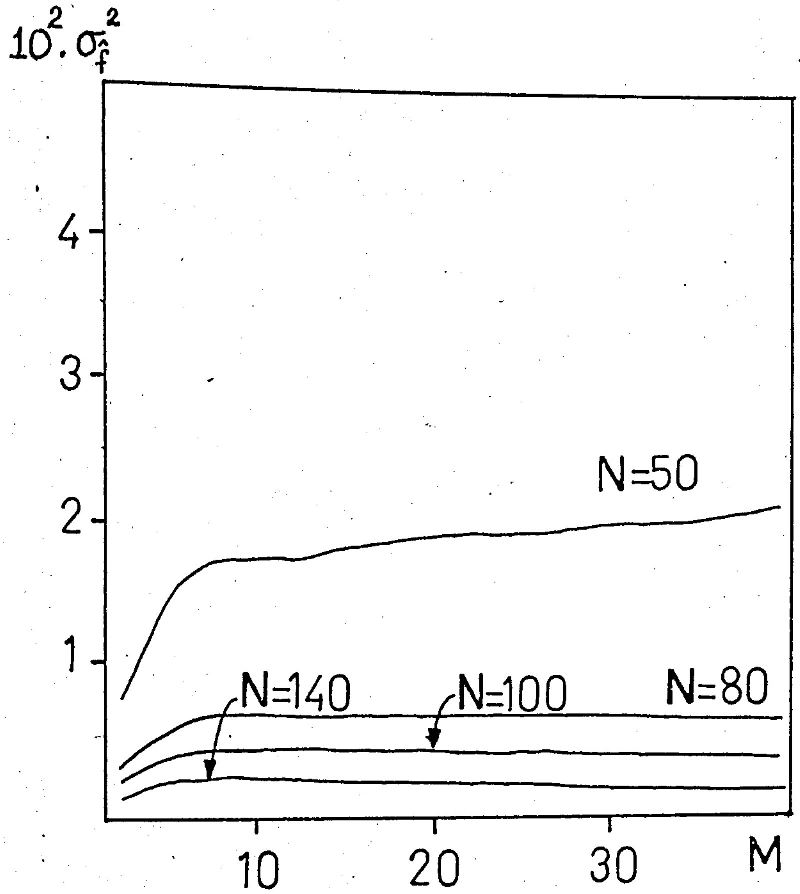


Fig.III.5. Variation of the mean frequency estimation variance versus number of AC terms (noise free case).

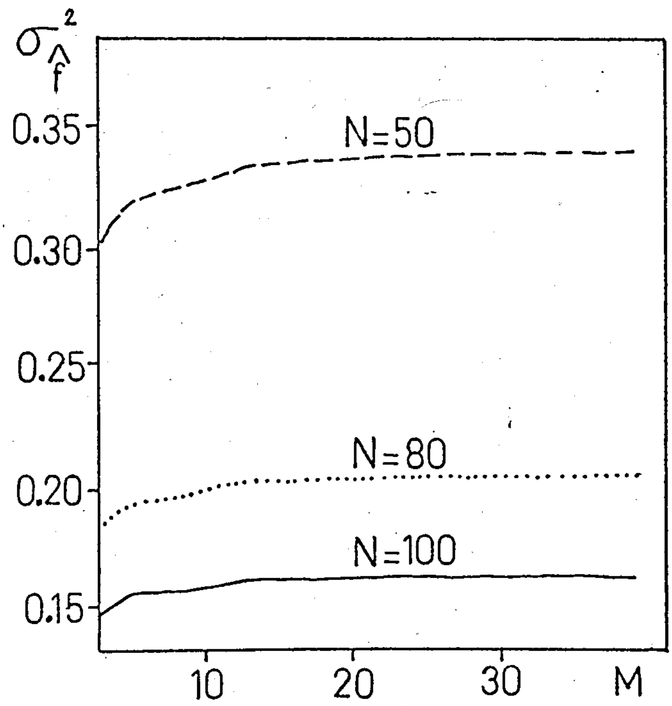


Fig.III.6. $\sigma^2_{\hat{f}}$ versus number of AC terms (μ =5).

$$\text{Var } r_i = \frac{1}{(N-i)^2} \sum_{k=-(N-i)}^{(N-i)_2} A^2 \quad N-i- k \quad (\text{Cos}^2 k\omega_1 + \text{Cos } (k+i)\omega_1 \text{Cos}(k-i)\omega_1) -$$

$$A^2 \text{Cos}^2 i \omega_1 \qquad 3.27$$

Since

$$\sum_{k=1}^{N-i} (N-i)(1+\text{Cos}\, i\omega_1) = (N-i)^2 \quad 1+\text{Cos}2i\omega_1 \qquad 3.28.a$$

$$\sum_{k=1}^{N-i} k(1+\text{Cos}2i\omega_1) \quad \frac{(N+i)(N-i+1)}{2} \quad (1+\text{Cos}2i\omega_1) \qquad 3.28.b$$

one obtains as

$$\text{Var } r_i = \frac{A^2}{(N-i)} \quad 1+ \frac{2}{(N-i)} \sum_{k=1}^{N-i} (N-i-k)\ \text{Cos}2k\omega_1) \qquad 3.29$$

and finally

$$\sigma_f^2 = 0.04 \qquad \beta_k + 2h_{jk} + \frac{f(k)}{N} \ (1 \quad \frac{2g(j)}{N}\ ) \qquad 3.30$$

where we have used the definitions :

$$\beta_k = \sum_{k=1}^{M/2} \frac{1}{(N-2k+1)(2k-1)^2}$$

$$h_{jk} = \sum_{k=1}^{M/2} \sum_{j=1}^{M/2} \frac{(N-2k+1-j)}{(N-2k+1)^2} \quad \frac{\text{Cos}2j\omega_1}{(2k-1)^2}$$

Noisy case ($\sigma n^2 \neq 0$)

From previous section, we can write similarly

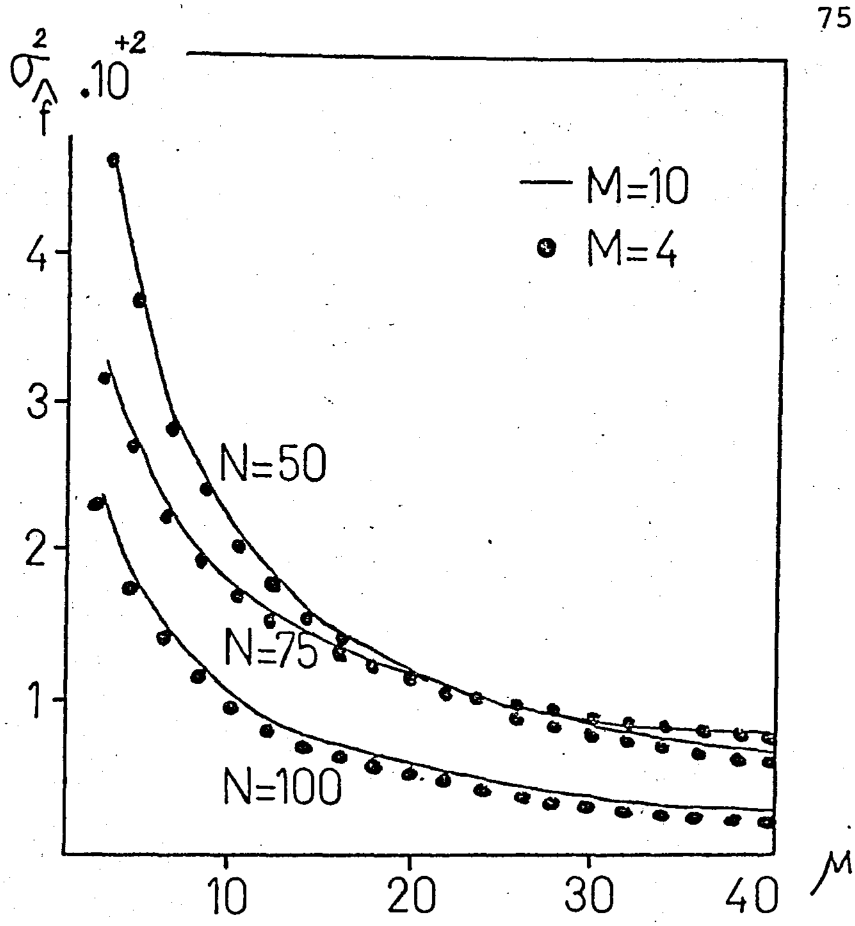


Fig.III.7. Variation of the mean frequency estimation variance with respect to $\mathcal{M} = A_1^2/\sigma_n^2$

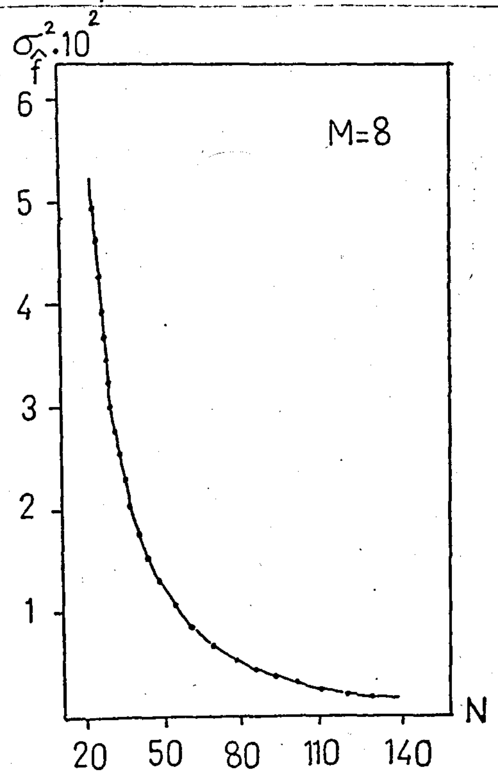


Fig.III.8. $\sigma^2_{\hat{f}}$ versus number of data samples (noisy situation, $\mathcal{M}=5$)

$$\mathrm{ar}(r_i)=\mathrm{Var}(r_i)\big|_{\sigma_n^2=0}+\frac{1}{(N-i)^2}\left[(N-i)\sigma_n^4+2(N-i)A\sigma_n^2+2(N-2i)A\sigma_n^2\mathrm{Cos}2i\omega_1\right] \quad 3.31$$

and

$$\mathrm{ar}(r_o)=\mathrm{Var}(r_o)\big|_{\sigma_n^2=o}+\frac{1}{N^2}\left[N\sigma_n^4+2N\sigma_n^2A+2NA\sigma_n^2\right] \quad 3.32$$

Then

$$\frac{0.04T^{-2}\,\mu^2 f(k)}{N(1+\mu)^4}\quad \mu^2+\frac{2\mu^2}{N}\,g(j)+(1+4\mu)+\frac{0.04T^{-2}}{(1+\mu)^2}\quad \beta_k\,\mu^2+$$
$$2\mu^2 h_{jk}+\beta_k+2\,\mu(\beta_k+\xi_k) \quad 3.33$$

where

$$\xi_k=\sum_{k=1}^{M/2}\cdot\frac{(N-4k+2)\mathrm{Cos}\,(4-2)\omega_1}{(N-2k+1)^2\,(2k-1)^2}$$

and h_{jk}, $f(k)$, β_k, $g(j)$ are defined previously.

The variation of the mean frequency estimation variance with respect to the number of AC terms and data samples as well as with respect to the (signal to noise ratio) is shown in Figs.IV.5, IV.6 and IV.7-8 respectively It can be observed from Figs.IV.5-6 that the variance of the mean frequency estimate does not decrease with increasing M, since with more AC terms, the more estimation error is introduced. Also for fixed N, the AC estimation errors at higher order lags account for this behaviour. The saturation in Fig.IV.5 and Fig.IV.6 is due to the k^{-2} term in Eq.(3.33).

3.4.4. PDF OF THE MEAN FREQUENCY

In here we give the expressions for P.D.F. of the mean frequency when only two AC terms are used, i.e., $f=0.25T^{-1}-0.2T^{-1}r_1/r_o$. derivation of the P.D.F.'s of the mean frequency follows the determination of the distribution of the ratio r_1/r_o first and then finding $P(f|H_o)$ and $P(f|H_1)$ respectively. Actually, since derivation steps used in here with obtaining both the P.D.F. of MESA moment and the P.D.F. of mean frequency for argument method are similar, therefore we will omit the derivation.

For H_o

For noise only case, P.D.F. is given by

$$P_{\hat{f}}(f|H_0)=\frac{1}{\left|0.2T^{-1}\right|g(f)}\left[\frac{\sqrt{N}}{\sqrt{\Pi}}\frac{1}{2\left[g(f)\right]^{1/2}}+\frac{1}{\sqrt{2\Pi^2}}\exp\left[-\frac{N}{2}\left(1-\frac{1}{g(f)}\right)\right]\right] \quad 3.34$$

where

$$g(f)=2\left[\frac{f-0.25T^{-1}}{-0.2T^{-1}}\right]+1$$

In Fig.IV.10, the P.D.F. for H_o is plotted versus frequency with various values of number of data samples. As in the previous section, the simulated sampling frequency is 20 Hz. The density of mean frequency displays arithmetic symmetry around $0.25T^{-1}=5$ Hz as expected.

For H_1

For signal plus white noise, the P.D.F. of mean frequency is

$$P_{\hat{f}}(f|H_1)\ \frac{A}{2A_3A_2^{1/2}}\left[\frac{1}{A_2^{1/2}}-\frac{\Pi\ A_4}{2A_3^{1/2}}\right]\exp\left[-A_5A_2+\frac{A_2A_4^2}{4A_3}\right] \quad 3.35$$

where

$$A_1=\frac{1}{2\Pi\sigma x\ \sigma_y\sqrt{1-r^2}}$$

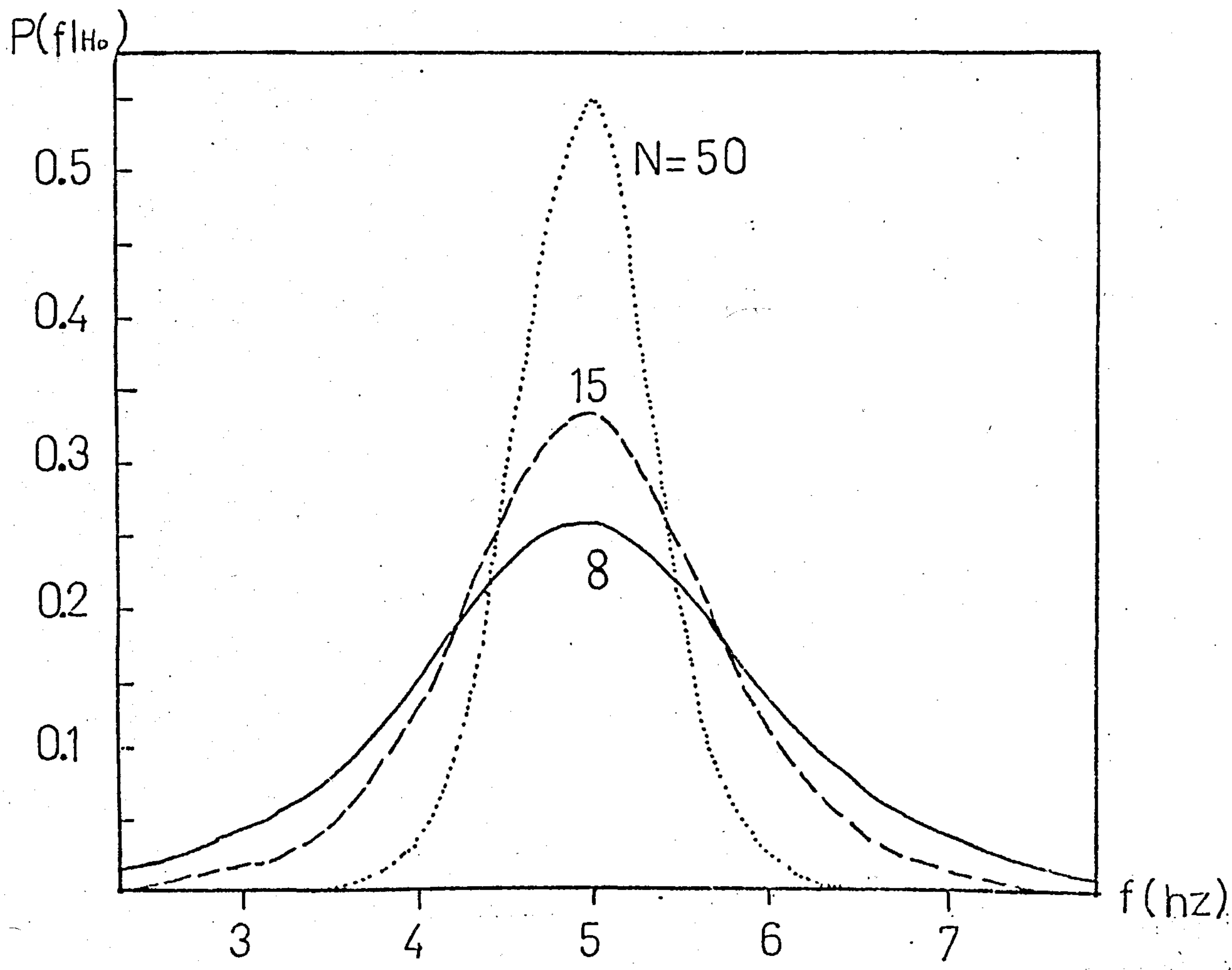
$$A_2=\frac{1}{2(1-r^2)}$$

$$A_3 = \frac{z^2}{\sigma_x{}^2} + \frac{1}{\sigma_y{}^2} - \frac{2rz}{\sigma_x \sigma_y}$$

$$A_4 = \frac{2rz\bar{y}}{\sigma_x \sigma_y} + \frac{2r\bar{x}}{\sigma_x \sigma_y} - \frac{2\bar{x}z}{\sigma_x} - \frac{2\bar{y}}{\sigma_y{}^2}$$

$$A_5 = \frac{\bar{x}^2}{\sigma_x{}^2} - \frac{2r\bar{x}\bar{y}}{\sigma_x \sigma_y} + \frac{\bar{y}^2}{\sigma_y{}^2}$$

all A_i, $i=1,\ldots,5$ are evaluated at $z=(f-0.25T^{-1})/(0.2T^{-1})$ and σ_x, $\bar{x}$, σ_y, $\bar{y}$ are statistics for the real part of r_1 and for r_o defined in Appendix (A), respectively.



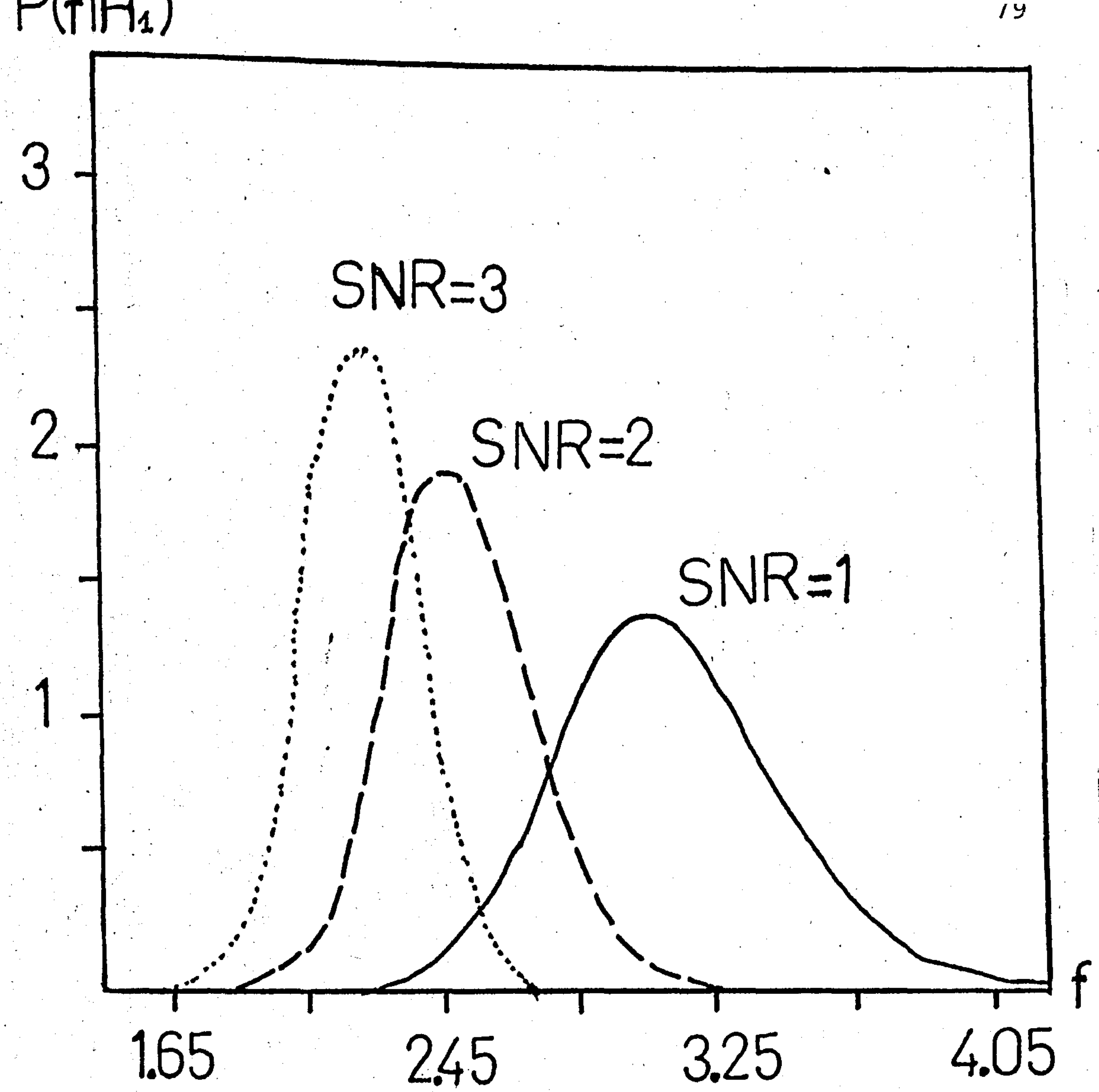


Fig.IV.11. P.D.F. of mean frequency for H_1. The simulated tone frequency is 1 Hz. N and T^{-1} are 50 and 20 Hz, respectively.

Also the P.D.F for H_1 is shown in Fig.IV.11 with different values of SNR. If we consider the P.D.F. of mean frequency error as $P(f-f|H_1)$ one can see that the mean frequency estimate at low SNR using two AC terms does not yield satisfactory results. This undesired behaviour can be reduced by using more AC terms with extrapolation technique. By examining Fig.IV.1 and IV.11, for high SNR condition it can be seen that the frequency error takes the reasonable value. Another important point in Fig.IV.11, the frequency of tone and sampling rate. Now if we keep the sampling rate constant and take the frequency of sinusoid, let's say 4 Hz, the frequency error will decrease and the P.D.F. of frequency error will shift to the left with respect to the before one for constant SNR condition.

CHAPTER IV: PROPERTIES OF AR METHOD

4.1. EXACT SOLUTION OF AR PARAMETERS

It is well known that for a known autocorrelation $\left\{ r_k, \ k=0,1,..p-1 \right\}$ for a wide sense stationary sequence, the AR parameters $\left\{ a_i, \ i=1,2,\ldots .p \right\}$ are a solution to the normal equations known as Yule-Walker [26]

$$\underline{\underline{R}}_x \underline{a} = \underline{r} \quad \text{or} \quad \underline{a} = \underline{\underline{R}}_x^{-1} \underline{r} \tag{4.1}$$

where $\underline{\underline{R}}_x$ is the Toeplitz matrix of correlation with elements $\left\{ r(i,j) = r_{i-j} \ i, j = 0,1,\ldots .p \right\}$ and vectors (This type of matrix is defined as the one having all the elements of each diagonal equal [85])

$$\underline{a} = \left[a_1, a_2, a_3 \ldots\ldots a_p \right]^T$$

$$\underline{r} = \left[r_1, r_2, r_3, \ldots\ldots, r_p \right]^T$$

Employing the matrix inversion expressions , it is possible to obtain exact solution of the AR parameters expressed by Eq. (4.1), i.e., the analysis of the exact solution of AR parameters will accordingly be based on the analytical study of inversion of autocorrelation matrix , even though in implementation one has to use the Levinson-Durbin algorithm or another suggested popular methods which are mentioned in the first chapter.

4.1.1 COMPLEX TOEPLITZ

Considering the case of a complex sinusoidal signal in white noise, the autocorrelation sequence for such a data may be expressed as

$$r_k = \sigma_n^2 \; (k) + A\, e^{jw_1 k} \qquad 4.2$$

where $\delta(k)$ is Kronecker delta , σ_n^2,A and ω_1 are noise power ,signal power and frequency of sine wave. Exact solution of AR parameters are given by

$$\underline{a} = \left[\underline{\underline{R}}_x^c\right]^{-1} \underline{r} \qquad 4.3$$

or it can be augmented to include σ_n^2 as follows

Using Eq. (C-5) into Eq. (4.3) , we have finally

$$\begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ \cdot \\ \cdot \\ a_p \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\gamma}{1-\gamma} \exp(-j\omega_1) \\ \exp(-j2\omega_1) \\ \vdots \\ \vdots \\ \frac{\gamma}{1-\gamma} \exp(-jp\omega_1) \end{bmatrix} \qquad 4.4$$

or

$$a_i = \frac{\gamma}{1-\gamma} \exp(-ji\omega_1) \qquad i = 1 , 2 ,\ldots.p \qquad 4.5$$

and $a_o = 1$

where

$$\gamma = \frac{SNR}{1 \div (p+1)SNR} \quad \text{and } SNR = \frac{A}{\sigma_n^2}$$

4.1.2 REAL SYMMETRIC MATRIX

For the case of a real sinusoidal signal in white noise , the autocorrelation sequence and exact solution of AR parameters are given respectively

$$r_k = \sigma_n^2 \delta(k) + A \cos (\omega_1 k) \qquad 4.6$$

and

$$\underline{a} = \left[\underline{\underline{R}}_x^r\right]^{-1} \underline{r} \qquad 4.7$$

Employing Eq. (C-11) into (4.7)

$$a_i = \frac{2\mu \left[\mu \dfrac{\mathrm{Sin}w_1(P+1)}{\mathrm{Sin}w_1} \mathrm{Cos}w_1(p-i) - \mathrm{Cos}\,iw_1 \right]}{(1-\mu)^2 + \mu^2 \dfrac{\mathrm{Sin}\,(2p+1)w_1}{\mathrm{Sin}w_1} - \mu \left[\dfrac{\mathrm{Sin}w_1(P+1)}{\mathrm{Sin}w_1}\right]^2} \qquad 4.8$$

for i =1 , 2 ,......p where

$$\mu = \frac{SNR}{2 + (p+1)\,SNR} \qquad SNR = \frac{A}{\sigma_n^2}$$

The above expression yields the exact value of the AR parameters for certain classes of matrices as a function of SNR , frequency of sinusoid, ω_1 and the AR model order, p for a sinusoidal (complex or real) signal in white noise if the true autocorrelation function is known. Note that for the noise alone case all a_i will be zero.

4.2 DEVIATION OF AR PARAMETERS

Our primary interest in obtaining the coefficient deviation of AR polynomial is to investigate the performance of AR method. In particular once the deviation of the estimated AR parameters has been computed, then the determination of the roots displacement is immediately available.

Assume that x_n is an AR process to which white noise w_n with variance has been added. Then

$$y_n = x_n + w_n$$

is the observed process. The true and estimated AR parameters for this process are given

$$\underline{a} = \underline{\underline{R}}_x^{-1}\, \underline{r} \qquad 4.9$$

and

$$\hat{\underline{a}} = \underline{\underline{R}}_y^{-1}\, \underline{r} \qquad 4.10$$

respectively, where

$$R_y = R_x + \sigma_n^2 I$$

It was shown that [28]

$$\hat{a} = P\ B\ P^T a \qquad 4.11$$

or

$$\hat{a} = P\left[\Lambda + \sigma_n^2 I\right]\Lambda P^T a \qquad 4.12$$

where B is a diagonal elementary matrix with ith diagonal element is

($\lambda_i/\lambda_i + \sigma_n^2$), λ_i is the ithh eigenvalue of R_x and P is the modal matrix i.e., ith column of P is the ith normalized eigenvector correponding to eigenvalue λ_i , and Λ=diag ($\lambda_1, \lambda_2, \ldots\ldots\lambda_p$).

It is of interest to find the deviation of the estimated AR para- meters a i.e., $\Delta a = \hat{a} - a$ which is

$$\Delta a = P\ K\ P^T a \qquad 4.13$$

where

$$K = \text{diag}\left(\frac{-1}{1+\frac{\lambda_1}{\sigma_n^2}}, \ldots\ldots\ldots \frac{-1}{1+\frac{\lambda_p}{\sigma_n^2}}\right)$$

Therefore , additive white noise causes the estimated AR parameters de - viate from the true parameters.

4.3 ROOT DISPLACEMENT DUE TO INEXACT COEFFICIENT

When the coefficient of AR polynomial, $a_1 . a_2, \ldots a_p$ are inexact due to the white noise, corresponding roots of AR polynomial are in errors. If z is obtained as a root of the AR polynomial with es- timated a , then

$$\hat{A}(z) = z_k^p + \hat{a}_1 z_k^{p-1} + \ldots\ldots \hat{a}_p = 0 \qquad 4.14$$

If we denote the exact coefficients by $a = \hat{a} + \Delta a$, then the true root $(z_k + \Delta z_k)$ must satisfy the equation

$$\hat{A}_1(z) = (z_k + \Delta z_k)^p + (\hat{a}_1 + \Delta a_1)(z_k + \Delta z_k)^{p-1} \qquad 4.15$$

$$\ldots\ldots\ldots\ldots + (\hat{a}_p + \Delta a_p) = 0$$

If the Eq. (4.14) is subtracted from Eq. (4.15) and if it is assumed that the relative errors are sufficiently small to permit neglect of higher-order terms, i.e.,

$$(z_k + \Delta z_k) = z_k^i + i\,(\Delta z_k)\, z_k^{i-1} \quad i=1, 2, \ldots p \qquad 4.16$$

it follows to a first order approximation, we have

$$\left[p z_k^{p-1} + (p-1)\, z_k^{\bar{p}-2}\, \hat{a}_1 + \ldots . a_{p-1}\right] \Delta z_k + z_k^{p-1} \Delta a_1 + z_k^{p-2}\, a_2 + \ldots \Delta a_p = 0 \qquad 4.17$$

and hence [86]

$$\Delta z_k = - \frac{z_k^{p-1} \Delta a_1 + \ldots\ldots\ldots \Delta a_{p-1}}{p\, z_k^{p-1} + (p-1)\, z_k^{p-1} \hat{a}_1 + \ldots\ldots\ldots\ \hat{a}_{p-1}} \qquad 4.18$$

or

$$\Delta z_k = - \frac{z_k^{p-1} \Delta a_1 + \ldots\ldots\ldots \Delta a_{p-1}}{\left|\frac{dA}{dz}\right|\ z = z_k} \qquad 4.19$$

In particular if each coefficient is known to be in error by no more than ε,

$$|\Delta a_i| \leqslant \varepsilon \qquad i = 1, 2, \ldots p \qquad 4.20$$

there follows, within the same degree of approximation, the maximum root displacement is

$$|\Delta z_k| \cong \frac{(1 + |z_k| + |z_k|^2 + \ldots\ldots\ldots\ldots + |z_k|^{p-1})\,\varepsilon}{|A'(z_k)|} \qquad 4.21$$

or

$$|z_k|_{max} \cong \frac{1}{|A'(z_k)|}\;\frac{z_k^{p} - 1}{(|z_k| - 1)} \qquad 4.22$$

When $|z_k| \cong 1$, this approximate bound becomes as

$$|z_k|_{max} \cong \frac{p\varepsilon}{|A'(z_k)|} \qquad 4.23$$

In terms of relative errors , Eq. (4.19) yields the approximation

$$\frac{\Delta Z_k}{Z_k} \approx - \sum_{\ell=1}^{p} C_{k\ell} \frac{\Delta a_\ell}{\hat{a}_\ell} \qquad 4.24$$

where

$$C_{k\ell} = \frac{Z_k^{p-\ell-1}\hat{a}_\ell}{A(Z_k)}$$

Hence , in particular if the magnitude of the relative errors in which coefficient does not exceed i.e.,

$$\frac{\Delta a_i}{a_i} < \eta \qquad i=1,2,\ldots.p \qquad 4.25$$

then

$$\left|\frac{Z_k}{Z_k}\right|_{max} \approx \eta \sum_{=1}^{p} |C_{k\ell}| \qquad 4.26$$

Eq. (4.19) and (4.24) give the root displacement and relative root displacement due to the inexact coefficients of AR polynomial which are going to be used in the later section.

4.3.1 CIRCULANT MATRIX CASE

Before using circular matrix , a few key properties of these matrices will be presented [85]. A circulant matrix of order M can be represented by its first row and has the form illustrated in Eq. (4.27)

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{M-1} \\ c_{M-1} & c_0 & c_1 & \cdots & c_{M-2} \\ \vdots & & & & \vdots \\ c_1 & c_2 & \cdots & & c_0 \end{bmatrix} \quad 4.27$$

It will be also useful to use Z-transform of the first row,

$$C(z^{-1}) = c_0 + c_1 z^{-1} + \ldots \quad c_{M-1} z^{-M+1} \quad 4.28$$

Let F^* be the discrete Fourier transform matrix of dimension M ,

$$F = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & & w^{M-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & w^{M-1} & w^{M-2} & \cdots & w \end{bmatrix} \frac{1}{\sqrt{M}} \quad 4.29$$

where

$$w = \exp(2\pi j \; / \; M) \quad , \quad j = \sqrt{-1}$$

An important theorem is that if **C** is circulant it is diagonalized by **F** , that is $\mathbf{C} = \mathbf{F}^{*}\mathbf{\Lambda}\mathbf{F}$ where

$$\Lambda = \text{diag}\left\{ C(1),\ C(w),\ldots C(w^{M-1}) \right\}$$

Equivalently , the eigenvalues of a circulant matrix are simply the scaled DFT of the first row of the matrix, i.e.,

$$\lambda_k = \sum_{\ell=0}^{M-1} C_\ell \exp\left[-2\, j\ell k/m\right] \quad k=0,1,\ldots(M-1) \tag{4.30}$$

and the corresponding eigenvectors are given by

$$V_k = \left[1\ ,\ \exp(-j2\pi k/M),\ldots\exp(-2\pi k(M-1)/M\right]\frac{1}{\sqrt{M}} \tag{4.31}$$

for k =0 , 1 , 2.........(M-1)

It can be also shown that any matrix expressible in the form of as $\mathbf{F}^{*}\mathbf{\Lambda}\mathbf{F}$ is a circulant matrix. Details are found in [85].

Considering the properties of circulant matrices as we mentioned so far, one can immediately obtain the deviation of AR coefficients using Eq. (4.13) as

$$\Delta a = \begin{bmatrix} \sum_{\ell=1}^{M} a_\ell \sum_{k=0}^{M-1} \mu_k W^{k(\ell-1)} \\ \sum_{=1}^{M} a_\ell \sum_{k=0}^{M-1} \mu_k W^{k(\ell-2)} \\ \vdots \\ \sum_{\ell=1}^{M} a_\ell \sum_{k=0}^{M-1} \mu_k W^{k(\ell-M)} \end{bmatrix} \tag{4.32}$$

where

$$\mu_i = \frac{-\sigma_n^2}{\lambda_i + \sigma_n^2} \qquad i = 0,1,\ \ldots\ (M-1)$$

or for ith coefficient it can be expressed as

$$\Delta a_i = \sum_{\ell=1}^{M} a_\ell \sum_{k=0}^{M-1} \mu_k \exp\left[\frac{j2\pi k}{M}(\ell - i)\right] \quad i=1,2,\ldots M \qquad 4.33$$

After having derived the deviation of AR coefficients , the n´th root displacement due to these coefficients can be given by using Eq. (4.19)

$$z_n = \frac{\sum_{k=1}^{M} \sum_{\ell=0}^{M-1} \sum_{m=0}^{M-1} a_k \mu_\ell z_n^{-m} W^{\ell(k-m-1)}}{\sum_{m=0}^{M-1} (M-m)\hat{a}_m z_n^{-m}} \qquad 4.34$$

In this representation a , z represnt estimated coefficient and root (pole or zero) respectively which are found for M´th order AR procees. Also , the root displacement depends on the eigenvalue seperation of autocorrelation matrix, signal to noise ratio as expected and order of AR process.

4.3.2 TRANSFORMED CIRCULANT MATRIX CASE

There exists a case the circulant matrix becomes a diagonal matrix. This can be achieved by using properties of discrete Fourier transform matrix i.e., **F F = I** . Equivalently this is in other words, the case where the discrete Fourier transform matrix yields thé unitary transform.

The solution of true AR parameters for the transform domain is obtained as follows

$$\mathbf{a} = \left[\mathbf{F}\ \mathbf{R}_x \mathbf{F}^*\right]^{-1} \mathbf{F}^* \mathbf{r} \qquad 4.35$$

where $\mathbf{R}_x$ is circular matrix. Similarly we have an expression for the estimated AR parameters

$$\hat{\mathbf{a}} = \left[\mathbf{F}\ \mathbf{R}_y \mathbf{F}^*\right]^{-1} \mathbf{F}\ \mathbf{r} \qquad 4.36$$

Now since

$$F R_x F = \Lambda \tag{4.37}$$

one can obtain easily

$$F R_y F^* = \Lambda + \sigma_n^2 I \tag{4.38}$$

and

$$\Lambda \mathbf{a} = \mathbf{F}^* \mathbf{r}$$

Using Eq. (4.35) into Eq. (4.36) and considering the fact that $\Delta \mathbf{a} = \hat{\mathbf{a}} - \mathbf{a}$, we have

$$\Delta \mathbf{a} = \left[(\Lambda + \sigma_n^2 \mathbf{I}) \Lambda - \mathbf{I} \right] \mathbf{a} \tag{4.39}$$

or

$$a_i = \frac{\sigma_n^2}{\lambda_i \ \sigma_n} \cdot a_i \qquad \text{for } i=1,2,... \tag{4.40}$$

It can be concluded that from Eq.(4.40), if the autocorrelation matrix is of the same form Eq.(4.37) in other words, if the transform domain technique is used for autocorrelation matrix, the deviation of ith AR coefficient is independent of other coefficients in contrary to case of circulant matrix.

The n´th root displacement for this case can be given as

$$\Delta Z_n = \frac{\sum_{i=0}^{M-1} Z_n^{M-i-1} \, a_{i+1} \, \mu_i}{\sum_{i=0}^{M-1} (M-i) \, -\hat{a}_i \, Z_n^{M-i-1}} \tag{4.41}$$

The difference in root displacement of circular and transformed circular matrices can perhaps best be demonsrated by examining Eq.(4.41) and Eq. (4.34).

4.3.3 COMPLEX TOEPLITZ MATRIX CASE

For the case of complex Toeplitz matrix, R is given

$$R_y = \sigma_n^2 I + a\, V^* V^T \qquad 4.42$$

Since R_y is Hermitian there exists a unitary transformation **P** which diagonalizes R_y as

$$P\, R_y P = \text{diag}\, [\, \lambda_1\ \lambda_2 \cdots \lambda_p\ ] \qquad 4.43$$

where

$$P = \frac{1}{\sqrt{p}} [\, V^*,\ V\ ,\ \ldots\ldots\ V_{p-1}]$$

i.e., V_1 , V_2 , .. V_{p-1} are (p-1) mutually orthonormal vectors also orthogonal to V . Also the associated eigenvalues are

$$\lambda_1 = \sigma_n + p\, a \qquad 4.44.a$$

$$\lambda_1 = \lambda_3 \cdots\cdots = \sigma_n^2 \qquad 4.44.b$$

The coefficient deviation can be expressed using (4.43) and (4.13). Another formulation can be made by using (4.9) and (4.10) so that we have

$$\Delta a = -\sigma_n^2 R_y^{-1} a \qquad 4.45$$

The n′th root displacement is given

$$z = -\frac{\sigma_n^2 Z^T R_y^{-1} a}{\sum_{i=0}^{M-1} (\, M-i)\, \hat{a}_i\, z_n^{M-i-1}} \qquad 4.46$$

where

$$Z = [\, z_k^{p-1},\ z_k^{p-2},\ \ldots\ldots\ 1\ ]^T$$

4.3.4 REAL SYMMETRIC MATRIX CASE

It is well known that when a matrix is real and symmetric, it has real eigenvalues and orthogonal eigenvectors. However it is not in general positive definite and therefore may have positive, negative or zero eigenvalues. But for the real symmetric autocorrelation matrix case all eigenvalues are positive. Since the analytic expression for the eigenvalues is not available let us formulate the problem using (4.9) and (4.10). By following the steps in complex Toeplitz case, we have

$$\Delta a = \left\{ I - \frac{2a}{\beta_1} Re[VV^{*T}] - \frac{aRe}{\sigma_n^2 \beta_1} Re[c\; a(\mu(M+1)-1)VV^T] \right\} a \qquad 4.47$$

where

$$\beta_I = 2\sigma_n + a\left[(M+1) - \mu c_1 c_2\right]$$

For the case real symmetric matrix, the closed form expression of the root displacement is straightforward but it needs some tedious work.

4.3.5 VARIANCE OF THE AR PARAMATERS

It was shown that all the common methods of estimating the AR parameters produces same estimate (i.e., in numerical value) for N M . Hence we will restrict the discussion to the Yule- Walker method or autocorrelation method [26]. It has been shown that asymptotically , the Yule-Walker method produces an MLE of the Ar parameters [87]. The asymptotic probability density function for the a parameters is Gaussian and is given by

$$P(a) = (2\pi)\, I\, C\, I^{1/2} \exp\left[-\; (\hat{a} - a)\, C_{\hat{a}}^{-1} (\hat{a} - a) \right] \qquad 4.48$$

where

$$C_a = \frac{\sigma_n^2}{N} R_x^{-1}$$

for the real symmetric Toeplitz matrix, i.e., $[\ r\ ] = r$. Therefore the variance of the AR parameters using the closed form expression of **R** found in Appendix C, can be obtained as

$$\text{Var}\ a_{j+1} = \frac{1}{\beta_1 N}\left[\beta_1 - 2a + 2a\,\mu \frac{\text{Sin}w_1 (M+1)}{\text{Sin}w_1} \text{Cos}\left(M-2j\right) w_1\right] \text{ for } j=0,1,\ldots\frac{M}{2}$$

and

4.49

$$\text{Var}\ \hat{a}_{2j} + \frac{M}{2} = \frac{1}{\beta_1 N}\left[\beta_1 - 2a + 2a\mu \frac{\text{Sin}w_1 (M+1)}{\text{Sin}w_1} \text{Cos}\ (2jw_1)\right] \text{ for } j=1,2,\ldots\frac{M}{2} \qquad 4.50$$

respectively. One can see easily that there is a symmetrical structure in variance expression by examinig the above equations ; for example ;

$$\text{Var}\ (\ a_{M/2}) = \text{Var}\ (\ a_{M/2\ +2})$$

Eqs. (4.49) and (4.50) are needed in the frequency error analysis section.

4.4. FREQUENCY ERROR FOR AR PROCESS

If one is interested in only estimating the frequency of sinusoid with modern spectrum techniques there will be a problem which is the bias in the position of spectral peaks with respect to the true frequency location of those peaks. Frequency bias in processing sinusoidal signal with Burg algorithm has been experimentally investigated by Chen and Stegen [20] for noisy case and theoretically analyzed by Swingler [18] for noiseless case by considering the second order case.

In this section, as frequency measurement (estimation) accuracies, the statistical fluctuation of a peak frequency is investigated by using Newton's and Sakai's methods. A major task in determinig the performance of frequency estimate obtainig by modern spectrum techniques is to investigate the frequency error when the model order or data size or signal to noise ratio is changed.

4.4.1 SAKAI'S METHOD

Let us define

$$S(w) = \frac{\sigma_n^2}{A(w)\,A(-w)} \quad , \quad \hat{S}(w) = \frac{\sigma_n^2}{\hat{A}(w)\,\hat{A}(-w)} \qquad 4.51$$

true and estimated power spectrum respectively. Let us consider the statistical fluctuation of the estimator for the frequency at which S (w) has a peak value. Since S (w) and S (w) take the maximum values at w and w respectively it follows that

$$\frac{d}{dw} A(w)\,A(-w)\Big|_{w=w} = \frac{d}{dw}\hat{A}(w)\,\hat{A}(-w)\Big|_{w=\hat{w}} \qquad 4.52$$

or

$$A'(w_1)\,A(-w_1) + A(w_1)\,A'(-w_1) = 0 \qquad 4.53.a$$

$$\hat{A}'(\hat{w}_1)\,\hat{A}(-\hat{w}_1) + \hat{A}(\hat{w}_1)\,\hat{A}'(-\hat{w}_1) = 0 \qquad 4.53.b$$

For large N, the frequency error can be assumed to be small so

$$\hat{A}'(\hat{w}_1) = \hat{A}'(w) + \hat{A}''(w_1)\,\Delta w \qquad 4.54.a$$

$$\hat{A}(\hat{w}_1) = \hat{A}(-w_1) + \hat{A}'(-w_1)\,\Delta w \qquad 4.54.b$$

i.e., $\hat{A}'(\hat{w}_1)$ and $\hat{A}(-\hat{w}_1)$ can be approximated near w_1 by means of Taylor series expansion and also by defining $\hat{\mathbf{a}} \triangleq \mathbf{a} - \Delta\mathbf{a}$ as the AR coefficient vector deviation, we can write

$$\hat{A}(w_1) = A(w_1) - \Delta a\ F(w_1) \qquad 4.55.a$$

$$\hat{A}'(w) = A'(w) - \Delta a^T F'(w_1) \qquad 4.55.b$$

where

$$F(w) = \left[e^{-jw} e^{-jw2} \ldots\ldots e^{-jwM} \right]$$

Eq.(4.55 a-b) shows the estimated spectrum at $w=w$ contains two parts: one part which gives the exact spectral peak value, the second part reflects the bias term due to the coefficient deviation in AR fliter. (It should be noted that is interpreted as the difference between the main peak frequency of A (w) and $\hat{A}$ (w) in this investigation.)

By substituting these into (4.53.a) and (4.53.b) , we have

$$\left[\hat{A}'(w_1) + \hat{A}''(w_1)\Delta w\right]\left[\hat{A}(-w_1) + \hat{A}'(-w_1)\Delta w\right] + \left[\hat{A}(w_1) + \hat{A}'(w_1)\Delta w\right]\left[\hat{A}'(-w_1) + \hat{A}''(-w_1)\ \Delta w\right] = 0 \qquad 4.56$$

and assuming $\Delta a^T \Delta w = 0$, $\Delta w^2 = 0$, $\Delta a^T \Delta a = 0$, we write the frequency error expression as follows :

$$\Delta w = \Delta a^T \left[\frac{A'(w_1)\ F(-w_1) + F'(w_1)\ A(-w_1) + A(w_1)\ F'(-w_1) + F(w_1) A'(-w_1)}{2\ A'(w_1) A'(-w_1) + A''(w_1)\ A(-w_1) + A(w_1)\ A''(-w_1)} \right] \qquad 4.57$$

Since Eq. (4.57) establishes the relationship between Δw and Δa , one can determine the frequency error for AR process using the expressions of coefficient deviation presented in the previous section. Using the result of the second order AR process when the input data is sinusoidal signal in white noise, we can show an interesting fact namely that A (w_1) and A ($-w_1$) become zero. Actually for this case, the frequency error expression can be rewritten as

$$\Delta W = \Delta a^T \left[\frac{A'(w_1)\ F(-w_1) + A'(-w_1)\, F(w_1)}{2A'(w_1) \cdot A'(-w_1)} \right] \tag{4.58}$$

Also since w is directly related to Δ a, we have to calculate Δ w depending upon the situation in which one can have a specific form the autocorrelation matrix.

4.4.2 NEW METHOD

Another particularly simple method which is actually associated with the Newton method between w and w is investigated in this section.

In order to examine the behaviour of the frequency error w, A(w) is approximated near w by its Taylor series expansion as

$$|\hat{A}(w)|^2 = |\hat{A}(w_1)|^2 + \left.\frac{d|\hat{A}(w)|^2}{dw}\right|_{w=w_1} (w-w_1) + \frac{1}{2} \left.\frac{d^2\ \hat{A}(w)^2}{dw^2}\right|_{w=w_1} (w-w_1)^2 + \ldots \tag{4.59}$$

By neglecting higher order terms and the above expression is differentiated and setting the result equal to zero at $w = \hat{w}_1$ in order to find the frequency error in the equivalent form

$$\left.\frac{d|\hat{A}(w)|^2}{dw}\right|_{w=\hat{w}_1} = \left.\frac{d|\hat{A}(w)|^2}{dw}\right|_{w=w_1} + \left.\frac{d^2\ \hat{A}(w)^2}{dw}\right|_{w=w_1} (\hat{w}_1 - w_1) = 0 \tag{4.60}$$

Since $|A(w_1)|^2$ is a constant. Finally we have

$$w_1 - \hat{w}_1 = \frac{\left.\frac{d|\hat{A}(w)|^2}{dw}\right|_{w=w_1}}{\left.\frac{d^2|\hat{A}(w)|^2}{dw}\right|_{w=w_1}} \tag{4.61}$$

Now let us try to find the closed form expression for $|\hat{A}(w)|^2$ and it's derivatives with respect to w . Since

$$\hat{A}(w) = \sum_{k=0}^{M} \hat{a}_k e^{-jwk}$$

we obtain

$$|\hat{A}(w)|^2 = \sum_{k=0}^{M} \hat{a}_k^2 + 2 \sum_{k=0}^{M-1} \hat{a}_k \hat{a}_{k+1} \operatorname{Cos}w + \ldots . 2\hat{a}_o \hat{a}_M \operatorname{Cos}wM \tag{4.63}$$

or

$$|\hat{A}(w)^2| = 2 \sum_{i=0}^{M} \operatorname{Cos}w_i \sum_{k=0}^{M-i} \hat{a}_k \hat{a}_{k+i} \tag{4.64}$$

From Eq. (4.63), it follows that

$$\frac{d}{dw}|\hat{A}(w)|^2 = -2 \sum_{i=0}^{M} i \operatorname{Sin}wi \sum_{k=0}^{M-i} \hat{a}_k \hat{a}_{k+i}$$

and finally (Note that the first summation index begins from one.)

$$\frac{d^2}{dw^2} |\hat{A}(w)|^2 = -2 \sum_{i=1}^{M} i^2 \operatorname{Cos}wi \sum_{k=0}^{M-i} \hat{a}_k \hat{a}_{k+1}. \tag{4.65}$$

Using (4.65)and (4.64) into Eq.(4.61), the estimation error $w = w - w$ is expressed as

$$w_1 - \hat{w}_1 = \frac{\left[\sum_{i=1}^{M} i \operatorname{Sin}wi \sum_{k=0}^{M-i} \hat{a}_k \hat{a}_{k+i} \right]\Bigg|_{w=w_1}}{\left[\sum_{i=1}^{M} i^2 \operatorname{Cos}wi \sum_{k=0}^{M-i} \hat{a}_k \hat{a}_{k+i} \right]\Bigg|_{w=w_1}} \tag{4.66}$$

Although we derived detailed expression for frequency error here so far using the mentioned method, exact evaluation of Eq.(4.66) requires the solution of AR parameters which is available in the previous section for the users. However, another interesting part is the determining the second order statistics of AR coefficients in order to obtain the expected value of the estimation error. Also the specific structure of autocorrelation matrix plays a role in frequency error analysis as one can see easily. Finally, the asymptotic statistics of AR parameter estimates permits us to evaluate the asymptotic behaviour of the expected value of the difference between true and estimated peak frequency location. Using the exact solution of AR parameters for complex Toeplitz matrix case, Eq.(4.66) becomes; after some further work

$$w_1-\hat{w}_1 = \text{Re}\left[\frac{\sum_{i=1}^{M} i(M-i)\left\{\frac{1}{2}\text{Sin}2iw_1 - j\ \text{Sin}^2 iw_1\right\}}{\sum_{i=1}^{M} i^2\ (M-i)\left\{\text{Cos}^2 iw_1 - \frac{j}{2}\ \text{Sin}\ i2w_1\right\}}\right] \qquad 4.67$$

Other matrices can be found by making use of the exact solution of AR parameters in a similar fashion.

4.5. ZEROES OF THE AR POLYNOMIAL

When the input consists of a complex sine wave in additive complex white noise, the (M-1) ′th order (odd) AR polynomial has one of its (M-1) zeroes at exp ± j (+) ¢. The remaining (M-2) zeroes represent the presence of complex white noise. Whereas for real sinusoidal signal case; it has two of its M zeros located at exp ± + j (+) ¢ and (M-2) extranous noise zeroes. In AR polynomial, the noise zeroes tend to distribute themselves with approximately uniform angular separation and constant radius inside the unit circle so as to account for the uniform spectrum of the additive white noise. For M′th order (real signal case) AR polynomial (M-2) zeros due to noise, 2 zeroes due to signal can be written as follows by assuming the location of zeroes shown in Fig. (IV.1), and frequency error;

$$z_{1,2} = r_s\ \exp\left[\mp j\ (W_1+\Delta W)\right] \qquad 4.68$$

and

$$z_i = r_n\ \exp\left[j\ (i\ \theta_n + W_1 + \Delta w)\right] \qquad 4.69$$

It is clear that also one can see

$$(M-1)\,\theta_n + 2(w_1+\Delta w) = 2\pi \qquad 4.70$$

Now, from the basic polynomial properties we have

$$a_{k,k} = \prod_{i=1}^{k} Z_i = r_s^2 \; r_n^{k-2} \exp\left[j \sum_{\ell=}^{k-2} \theta_n + w_1 + \Delta w \right] \quad k=2,3..M \qquad 4.71$$

or

$$a_{k,k} = r_s^2 \; r_n^{M-2} \quad \exp\left[jM\pi \right] \qquad 4.72$$

Let us define

$$\beta = \frac{r_n}{r_s} \qquad 4.73$$

then Eq.(4.72) is rewritten as

$$a_{k,k} = r_s^k \, \beta^{k-2} \quad \exp\left(jk\pi \right) \qquad 4.74$$

Depending upon whether k is even or odd, the sign of $a_{k,k}$ will change, so

$$a_{k,k} = \begin{cases} \beta^{k-2} \quad r_s^k & \text{for } k \text{ even} \\ \beta^{k-2} \quad r_s^k & \text{for } k \text{ odd} \end{cases} \qquad 4.75$$

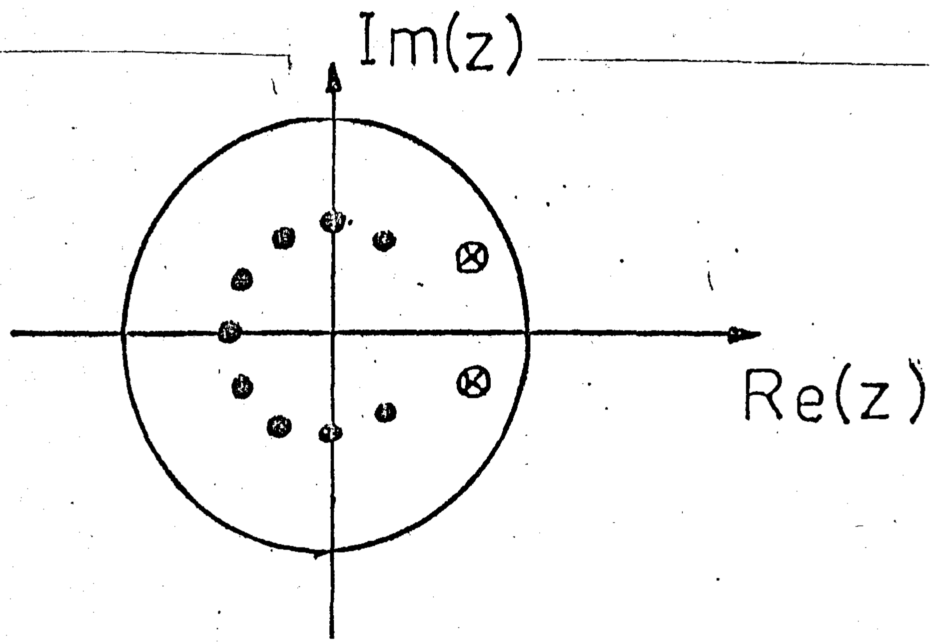


Fig.IV.1. Zeros of the AR polynomial (:Signal zeros, :Noise zeros)

Therefore $a_{k,k}$ is always real as expected. The above equation gives the relationship between the reflection coefficient and radius of the signal zero. Since β is the ratio of the radius of the noise zero to the radius of the signal zero, it is a function of both SNR and model order. To determine the effect of the zeros on the reflection coefficient or directly on the AR spectral estimate it is necessary to examine location of these zeros as a function of both SNR and model order. It was shown that in [28], for high SNR the signal zeros are near to the origin and noise zeros are located far enough within the unit circle so as not affect the spectrum and $\beta \ll 1$.

As SNR decreases the radius of the signal zero decreases until it is equal to the radii of the noise zeros which produce equiripple approximation to the flat noise spectrum. Furthermore as the model order is increased, the radius of the signal zero increases for a constant SNR resulting in a higher resolution spectrum estimate and $\beta \ll 1$. In a practical sense however, model order can not be increased independently since (r_k, k=1, . . . ,M) must be estimated from the finite data samples to obtain the coefficients of AR polynomial. Ulyrch and Bishop [7] report that $N/3 < M < N/2$. As the model order is increased beyond these limits, the radii of the noise zeros approach that of the signal zeros for constant SNR, then poor spectral estimate is obtained. The fact that signal zeros approach the unit circle faster than the noise zeros accounts for the better spectral estimate. However, when the autocorrelation function must be estimated, estimation errors will give rise to zero perturbations. Thus a small perturbation of a noise zero may result in a spurious peak if the radius of the noise zero is nearly equal to that of the signal zeros. Lacoume [88] has formulated this variation of zero locations with model order analytically as follows:

i) Signal zero is at a distance

$$d = \frac{2}{M(M+1)\ SNR} \qquad 4.76$$

from the unit circle and its angular position is when SNR

ii) The (M-1) other zeros are regularly distributed inside the unit circle in the ring

$$(2M)^{-1/M} < |Z| < 1 - \frac{1}{M} \qquad 4.77$$

It is noted that this limit allows us to use the radius of zero in carrier detection problem with a threshold [88]

$$|Z| > 1 - \frac{1}{M} \qquad 4.78$$

Also recently Kay [77] has investigated the use of radius of zero of AR polynomial for the detection of sinusoid in white noise (In fact he determined the lower bound of AR detector).

4.5.1. STATISTICAL ANALYSIS OF ZEROS

Now let us establish the relationship between reflection coefficient and radius of zero for the noise only case. Since AR spectrum for H_o produces approximately flat spectrum, then

$$a_{k,k} = r_n^k \exp\left[j \sum_{\ell=1}^{k} \ell \quad \theta_n \right] \qquad 4.79$$

or

$$a_{k,k} = \begin{cases} r_n^k & \text{for } k \text{ even} \\ -r_n^k & \text{for } k \text{ odd} \end{cases} \qquad 4.80$$

One can easily see that this formulation contradicts the result of the exact solution of AR parameters. The problem arises from the fact that finite observation i.e., error made during the estimation of autocorrelation function. Secondly the characteristics of the distribution of the zeros satisfies with normalized Butterworth filter zeros.

The PDF for the estimated values of the reflection coefficient is given [89] for H_o

$$P_a^{Ho}(a) = \frac{\Gamma(\ell-1/2)}{\sqrt{\pi}\,\Gamma(\ell-2/2)}(1-a^2)^{\ell-4/2} \qquad \ell > 2 \qquad 4.81$$

where

l=N-M+2

$a=a_{M,M}$= Reflection coefficient estimate for any M´th order model and $\Gamma(\,.\,)$ denotes the Gamma Function.

For H_1, the PDF of the reflection coefficient estimate $a_{M,M}$ for any M´th model order is given if the true value of reflection coefficient is b

$$P_{\hat{a}}^{H_1}(a) = \frac{(1-b^2)^{\frac{\ell-1}{2}}}{\pi(\ell-3)}(1-a^2)^{\frac{\ell-4}{2}} \frac{d^{\ell-2}}{d(ba)^{\ell-2}}\left[\frac{\cos^{-1}(-ba)}{\sqrt{1-b^2a^2}}\right] \qquad 4.82$$

where l is as defined above and it reflects the effect of data samples and model order. By making use of Eqs.(4.81) and (4.82), the PDF of the radius of signal or noise zeros can be obtained as

$$P_{r_s}^{H_1}(r) = \left[\frac{dr_s}{da\ M.M}\right]^{-1} P_a^{H_1}(a)\Big|_{a=\beta^{M-2}\, r_s^{M}} \qquad 4.83$$

for H_1

$$P_{r_n}^{Ho}(r) = \left[\frac{d\, r_n}{a_{M.M}} \right]^{-1} P_a^{Ho}(a) \Bigg|_{a\ =\ r_n^M} \qquad 4.84$$

for H_o respectively. Eqs.(4.83) and (4.84) give the final expressions for the PDF of radius of root of AR polynomial. Although these expressions appear in complicated form, they can be used for carrier detection problem.

4.5.2. FURTHER ASSUMPTIONS ABOUT THE STATISTICS OF ZEROS

We know that the roots of AR polynomial are complex conjugate order. Suppose that the real and imaginary parts of zeros for H_o are Gaussian distributed both zero mean and the same variance . Let us define

$$Re[z_i] \triangleq x_i$$
$$Im[z_i] \triangleq y_i \qquad \text{for } i = 1, 2, \ldots, M \qquad 4.85$$

Then their joint density function is

$$P_{xy}(x_i, y_i) = \frac{1}{2\pi\sigma_n^2} \exp\left[-(x_i^2 + y_i^2)/2\sigma_n^2 \right] \qquad 4.86$$

One can obtain from [74]

$$P_{r,w}(r_i, w_i) = \frac{r_i}{2\pi\sigma_n^2} \exp\left[-r_i^2/2\sigma_n^2 \right] \qquad 4.87$$

where

$$r_i^2 = x_i^2 + y_i^2$$

$$w_i = \arctan\left(\frac{y_i}{x_i} \right)$$

Finally we have

$$P_r(r_i) = \frac{r_i}{\sigma_n^{2}} \exp\left[-r_i^2 / 2\sigma_n^2\right] \quad \text{for } r_i > 0 \qquad 4.88$$

which is the well known Rayleigh density distribution.

For H_1 we assume that for signal zeros $E[x\ ] = \mu_1$, $E[y\ ] = \mu_2$, and $\sigma_{x_s}^2 = \sigma_{y_s}^2 = \sigma_n^2$ where x and y denote the real and imaginary parts of the signal zero. Similarly [74]

$$P_{r_s}(r_s) = \begin{cases} \dfrac{r_s}{\sigma_n^2} \, I_o \, \dfrac{r_s(\mu_1^2+\mu_2^2)^{1/2}}{\sigma_n^2} \exp - \dfrac{r_s^2+\mu_1^2+\mu_2^2}{2\sigma_n^2} & r_s \quad 0 \\ 0 & r_s \quad 0 \end{cases} \qquad 4.89$$

where I_o is the Modified Bessel Function of order zero which is written as

$$I_o(x) \triangleq \frac{1}{2\pi} \int_0^{2\pi} e^{x\cos\theta} d\theta = \sum_{n=0}^{\infty} \frac{x^{2n}}{z^{2n}(n!)^2}$$

Let us determine the probability P that r_s is the largest root. The decision rule is given

$$R = \max \{r_1, r_2, \ldots\ldots\, r_m\} \qquad 4.90$$

But we must note that the zeros of AR polynomial are complex conjugate therefore by the defined decision rule there are two largest roots due to real sinusoidal signal. In order to solve this problem, we are interested in upper side of unit circle i.e., $\omega_i = 0, \pi$ or $0 \leq \omega_i \leq \pi$, then

$$P_{r_i}(r_i) = \int_0^{\pi} P_{r,w}(r_i, w_i)\, dw_i = \frac{r_i}{2\sigma_n^2} \exp\left[-r_i^2/2\sigma_n^2\right] \qquad 4.91$$

and $P_{r_s}(r_s)$ does not change. Now

$$P = \left\{ P_r \quad \text{all } r_i < r_s \right\} \qquad 4.92$$

or

$$P = Pr\{\text{ all } r_i < r_s \;\; r_s = r_T\} \; P_r\{r_s = r_T\} \tag{4.93}$$

and

$$Pr\{\text{all } r_i < r_s \mid r_s = r_T\} = P_r\{r_i < r_s \mid r_s = r_T\}^{\frac{M}{2}-1} \tag{4.94}$$

where r_T is the threshold . Using Eqs.(4.89) and (4.91)

$$P = \int_0^1 P_{r_s}(R_T)\left[\int_0^{r_T} (r_1)\, dr_1\right]^{\frac{M}{2}-1} d\, r_T \tag{4.95}$$

Since

$$\int_0^{r_T} P_{r_1}(r_1)\, dr_1 = \frac{1}{2}\left[1 - \exp\left(-\frac{r_T^2}{2\sigma_n^2}\right)\right] \tag{4.96}$$

Finally we obtain

$$P = \frac{1}{2^{\frac{M}{2}-1}} \int_0^1 1 - \exp\left(-\frac{r_T^2}{2\sigma_n^2}\right)^{\frac{M}{2}-1} \frac{r_T}{\sigma_n^2}\, I_0\, \frac{r_T(\mu_1^2+\mu_2^2)^{1/2}}{\sigma_n^2} \exp \frac{-(r_T^2+\mu_1^2+\mu_2^2)}{2\sigma_n^2}\, dr_T \tag{4.97}$$

When this method is used, the probability of error is equal to (1-P). In this formulation, simulation studies are needed to gain idea about the performance. Although the integral involving the Bessel Function cannot be evaluated in closed form, there exist some cases where $I_0(x)$ can be approximated and the closed form solution can be obtained accordingly.

4.5.3. AR SPECTRUM IN TERMS OF THE RADII OF ZEROS

Let us continue to analyze the role of radii of zeros in AR spectrum. It is of interest to formulate for the case of L sinusoidal signals with noise free known autocorrelation function. Since the zeros of AR polynomial are located in z-plane as

$$z_{1,2} = r_i \exp(+jw_i) \qquad \text{for } i = 1,2,\ldots,L$$

depending upon the power of each sinusoid; then one can find immediately the AR spectrum in terms of the radii of signal zeros.

$$|A(e^{jw})|^2 = \prod_{j=1}^{L} \left[1-2r_i \cos w_i Z^{-1} + r_i^2 Z^{-2}\right]\Big|_{Z=e^{-jw}} \left[1-2r_i \cos w_i Z + r_i^2 Z^2\right]\Big|_{Z=e^{jw}} \qquad 4.99$$

or

$$|A(E^{jw})|^2 = \prod_{i=1}^{L} \left\{1-4r_i \left[(1+r_i^2)\cos w + r_i \cos w_i\right] \cos w_i + r_i^4 + 2r_i^2 \cos 2w\right\} \qquad 4.100$$

It is clear that for any i, i.e., $r_i=1$ and $w=w_i$

$$|A(e^{jw})|^2\Big|_{w=w_i, r_i=1} = 0$$

To illustrate the result of Eq.(4.100) we let L=1, w=0.2 , the resulting AR power spectrum has been numerically calculated and is shown in Fig.IV.2 for different values of radius. Fig.IV.2 also describes the effect of value of radius on the notch width.

The presence of noise in observation destroys this formulation. Let us consider again L sinusoids in white noise with N-point discrete time observations. In this case, AR spectrum whose roots satisfy Fig.IV.1 can be given

$$A(Z) = \prod_{i=1}^{M/2} \left[1 - 2r_i \cos w_i Z^{-1} + r_i^2 Z^{-2}\right]$$

and

$$|A(e^{jw})|^2 = \prod_{i=1}^{M/2} \left[1-4 r_i (k+r_i^2) \cos w_i \cos w + 2r_i^2 \cos 2w + 4r_i^2 \cos^2 w_i + r_i^4\right] \qquad 4.103$$

or

$$|A(e^{jw})|^2 = \sum_{i=L+1}^{M/2} \left[1 - \frac{4r_{s,i}}{\beta_i}\left(1+\frac{r_{s,i}^2}{\beta_i}\right) \cos w_i \cos w + \frac{2r_{s,i}^2}{\beta_i^2} \cos 2w\right.$$

$$+ \frac{4r^2_{s,i}}{\beta_i^2} \ Cos^2 w_i + \frac{r^4_{s,i}}{\beta_i^4} \Bigg] \cdot \Bigg\{ \prod_{j=1}^{L} \ (1-4r_{s,j}(1+r^2_{s,j}) Cosw_j Cosw + 2r^2_{s,j} Cos2w$$

$$+4r^2_{s,j} \ Cos^2 \ w_j + r^4_{s,j}) \Bigg\}$$

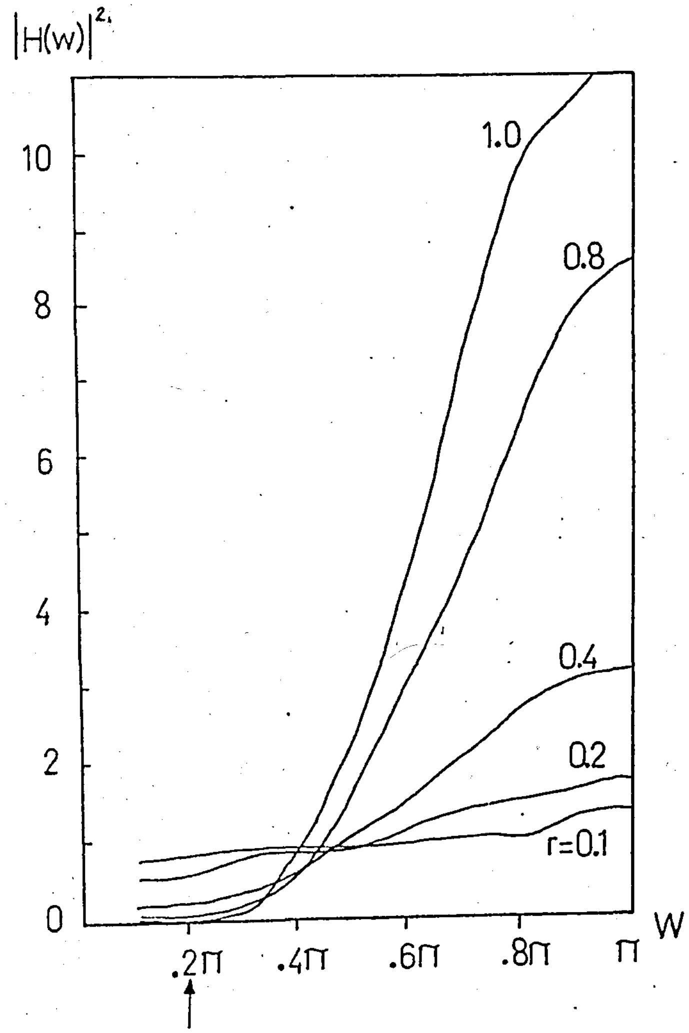


Fig.IV.2. AR spectrum for different values of radius (L=1, noise free, second order model)

Now let us return to the case where L=1. Eq.(4.104) is written as

$$|A(e^{jw})|^2 = \left\{ \left[1 - \frac{4r_s}{\beta}\left(1+\frac{r_s^2}{\beta^2}\right) \mathrm{Cos}w\mathrm{Cos}\left[(i-1)w_N+w_1\right] + \frac{2r_s^2}{\beta^2} \right.\right.$$

$$\left.\left. \mathrm{Cos}2w + \frac{4r_s^2}{\beta^2}\mathrm{Cos}^2\ (i-1)\ w_N + w_1 + \frac{r_s^4}{\beta^4} \right] \right\} \left[1-4r_s(1+4r_s^2) \right.$$

$$\left. .\mathrm{Cos}w_1\mathrm{Cos}w + 2r_s^2\ \mathrm{Cos}2w + 4r_s^2\ \mathrm{Cos}^2\ w_1 + r_s^4 \right] \qquad 4.105$$

where

$$\beta_i = \beta \quad \text{and} \quad r_{s,1} = r_s$$

Now let us consider noise only case. Similarly the AR spectrum whose roots satisfy Fig.IV.1 is

$$A(Z) = \prod_{i=1}^{M/2} \left[(1-2r_n(\mathrm{Cos}jw)Z^{-1} + r_n^2\ Z^{-2} \right] \qquad 4.106$$

Since $w_i = i2\pi/M$ and

$$\prod_{k=1}^{M-1} (X^2-2X\ \mathrm{Cos}\ k\pi/M+1) = \frac{X^{2M}-1}{X^2-1} \qquad 4.107$$

we have after some tedious work

$$|A(e^{xw})|^2 = \frac{(1+r_n^2+2r_n\mathrm{Cos}w)\ (1+r_n^{2M}+2r_n^M\ \mathrm{Cos}wM)}{(1+r_N^2-2r_N\ \mathrm{Cos}w)} \qquad 4.108$$

4.5.4. THE RELATIONSHIP BETWEEN ΔZ AND ΔW

Much of the useful information contained for frequency of a signal is obtained by locating spectral peaks or solving the roots of AR polynomial, i.e., selecting a root with maximum magnitude. Now let us investigate the effect of deviation of signal zeros on the frequency estimation problem by using pole or zero method.

Considering the worst case; the true and estimated signal zeros can be given respectively for the case of a single tone (real or complex)

$$Z_1 = e^{\mp j w_1} \qquad 4.109.a$$

$$\hat{Z}_1 = r_1\, e^{\mp j \hat{w}_1} \qquad 4.109.b$$

where

$$\hat{w}_1 = w_1 + \Delta w \quad \text{and} \quad r_1 = 1-\Delta r_1 \;, \quad r_1 < 1$$

The deviation of signal zero is

$$\Delta Z = Z_1 - \hat{Z}_1 \qquad 4.110$$

or

$$\Delta Z = Z_r + jZ_I \qquad 4.111$$

where

$$Z_r = \text{Cos}w_1 - r_1 \text{ Cos } (w_1 + \Delta w)$$

$$Z_I = \text{Sin}w_1 - r_1 \text{ Sin } (w_1 + \Delta w)$$

Assuming that Δw is very small and $(\Delta r\ \Delta w) \approx 0$, we have

$$\Delta Z = \Delta \bar{r}_1 \text{ Cos}w_1 + r_i \Delta w \text{ Sin}w_1 \; + j\; \Delta r_1 \text{Sin}w_1 - r_1\ \nabla w \text{ Cos}w_1 \qquad 4.112$$

and

$$\measuredangle \Delta Z = \tan^{-1} \frac{r_1 \text{Sin}w_1 - r_1 \Delta w \text{ Cos}w_1}{r_1 \text{ Cos}w_1 + r_1\ \Delta w \text{ Sin}w_1} \qquad 4.113$$

Now using the limit for radius given by Eq.(4.76)

$$\measuredangle \Delta Z = \tan^{-1} \frac{2\text{Sin}w_1 - (M\ M+1\ \text{SNR}-2)\Delta w\ \text{Cos}w_1}{2\text{Cos}w_1 - (M\ M+1\ \text{SNR}-2\ \Delta w\ \text{Sin}w_1} \qquad 4.114$$

Eq.(4.114) gives the deviation of signal zero in terms of frequency of signal, frequency error, SNR and order of the AR process.

Special case: $r_1 = 1$ (Fig.IV.3.d)

$$\Delta Z = \left[\text{Cos}w_1 - \text{Cos } (w_1 + \Delta w)\right] + j\left[\text{Sin}w_1 - \text{Sin } (w_1 + \Delta w)\right] \qquad 4.115.a$$

or again assuming w is small

$$\Delta Z = w \sin w_1 - j \cos w_1 \qquad 4.115.b$$

$$\measuredangle \Delta Z = \tan^{-1}\left[\frac{\cos w_1}{\sin w_1}\right] \qquad 4.115.c$$

Δz is real: (Fig.IV.3.a)

$$\Delta Z = \cos w_1 = r_1 \sin (w_1 + \Delta w) \qquad 4.116.a$$

$$\sin w_1 = r_1 \sin (w_1 + \Delta w) \qquad 4.116.b$$

$$\measuredangle \Delta Z = 0^{o} \qquad 4.116.c$$

$$r_1 = \frac{1}{\cos \Delta w + \dfrac{\cos w_1}{\sin w_1} \sin \Delta w} \qquad 4.116.d$$

For small Δw

$$\Delta Z = (1-r_1) \cos w_1 + r_1 \Delta w \sin w_1 \qquad 4.117.a$$

$$r_1 = \frac{1}{1 + \Delta w \dfrac{\cos w_1}{\sin w_1}} \qquad 4.117.b$$

$$\Delta Z = 0^{o} \qquad 4.117.c$$

Δz is imaginary : (Fig.IV.3.b)

$$\Delta Z = j \left[\sin w_1 - r_1 \sin (w_1 + \Delta w)\right] \qquad 4.118.a$$

$$\cos w_1 = r_1 \cos w_1 + \Delta w \qquad 4.118.b$$

$$\measuredangle \Delta Z = \pi/2 \qquad 4.118.c$$

$$r_1 = \frac{1}{\cos \Delta w - \tan w_1 \sin \Delta w} \qquad 4.118.d$$

For small Δw

$$r_1 = \frac{1}{1 - \Delta w \tan w_1} \qquad 4.119.a$$

$$\Delta Z = j (1-r_1) \sin w_1 - r_1 \Delta w \cos w_1 \qquad 4.119.b$$

$$\Delta Z = \pi/2 \qquad 4.120.c$$

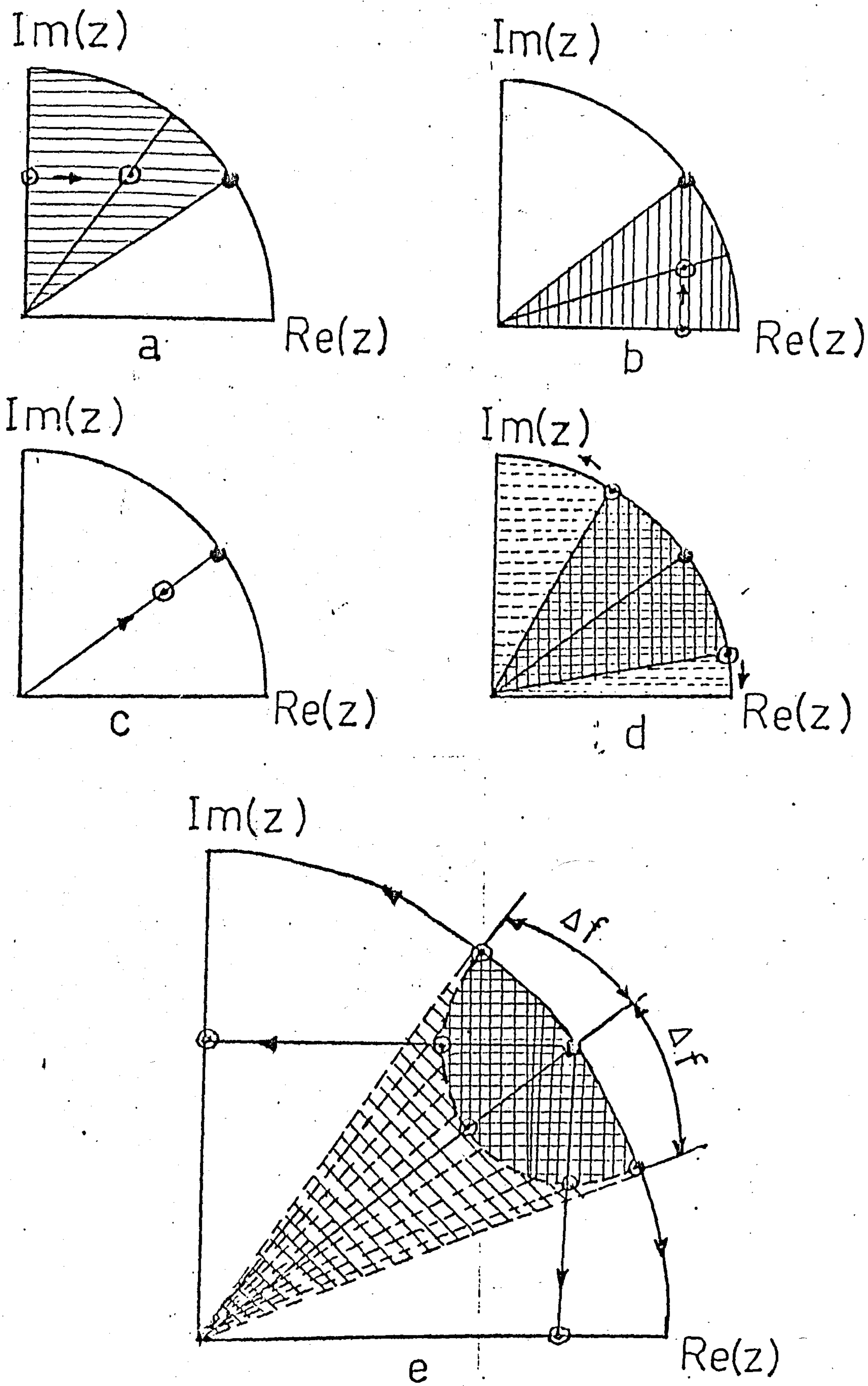


Fig.IV.6. Signal zero and frequency error.

Δz is complex : (Fig.IV.3.c), but no frequency error

$$\Delta Z = (\text{Cos}w_1 - r_1\ \text{Cos}w_1) + j\ (\text{Sin}w_1 - r_1\text{Sin}w_1) \qquad 4.120$$

$$\measuredangle\ \Delta Z = w_1$$

The equations given in this section establish the relationship between Δz and the frequency error.

4.5.5. THE RESULTS FOR SECOND ORDER CASE

As stated in Section 4.2, the structure of the autocorrelation matrix plays an important role in frequency determination. In this section we limit ourselves to second order case i.e., M=2 in order to give an insight about the lower bound of the mentioned AR method. As indicated in detail in several papers [32, 33, 34, 46], usually the accuracy of the spectrum estimator needs to minimize the frequency errors to achieve good performance. It can be seen that the structure of the known 2x2 AC matrix of a sinusoid is accepted Toeplitz or circulant matrix. This implies that for the second order case the performances of both circulant and real symmetric (Toeplitz) matrices are identical.

Sakai's Method

For the second order AR case we have

$$A(w) = 1 - 2\ \text{Cos}w_1 e^{-jw} + e^{-2jw}$$

$$A'(w_1) = -2e^{-jw_1}\ \text{Sin}w_1 \qquad 4.121$$

$$A''(w_1) = 2e^{-jw_1}\ \text{Cos}w_1 - 2\ e^{-jw_1}$$

Now let us determine the denominator and numerator of Eq.(4.66). Since

$$F^T(w_1) = \left[e^{-jw_1}, e^{-2jw_1}\right] \qquad 2.122$$

and using the fact that A(w) and A(-w) are equal to zero, we have

$$A'(w_1)\ F(-w_1) + F(w_1)\ A'(-w_1) = \begin{bmatrix} 1 \\ \cos w_1 \end{bmatrix} (-4\sin w_1) \qquad 4.123$$

and

$$2\ A'(w_1)\ A'(-w_1) = 4\ \sin^2 w_1 \qquad 4.124$$

Finally

$$\Delta W = \frac{\Delta a_1 + \Delta a_2 \cos w_1}{-\sin w_1} \qquad 4.125$$

From Eq.(4.9) one can easily obtain

$$\Delta a_1 = \frac{2 + 3\mu}{1 + 2\mu + \mu^2 \sin^2 w_1} \qquad 4.126.a$$

$$\Delta a_2 = \frac{-(1+2\mu + \mu \cos 2w_1)}{(1+2\mu + \mu^2 \sin^2 w_1} \qquad 4.126.b$$

Substituting (4.126) into (4.125), we have

$$\Delta W = \frac{-(1+\mu/2)\ \cos w_1 + \mu/2\ \cos 3w_1}{(1+2\mu + \mu^2 \sin^2 w_1)\ \sin w_1} \qquad 4.127$$

One can also obtain $\Delta w = 0$ as $\mu \to \infty$

To illustrate the result of (4.126) and (4.127), we let several values of w and SNR, the resulting parameter and frequency deviations have been numerically calculated and are plotted in Figs.IV.4, IV.5 and IV.6

New Method

The closed form expression of I A(w) I and its derivatives can be given for the second order AR process as

$$|A(w)|^2 = 1 + 2\hat{a}_1 \cos w + 2\hat{a}_2 \cos 2w + 2\hat{a}_1\hat{a}_2 \cos w + \hat{a}_2^2$$

$$\left.\frac{d\,|A(w)|^2}{dw}\right|_{w=w_1} = -2\ \ \hat{a}_1 \sin w_1 + 2a_2\ \sin w_1 2 + \hat{a},\hat{a}_2\ \sin w_1$$

$$\frac{d^2\ A(w)^2}{dw^2} = -2\ \ \hat{a}_1 \cos w_1 + 4\hat{a}_2 \cos 2w_1 + \hat{a},\hat{a}_2 \cos w_1 \qquad 4.128$$

Substituting (4.128) into (4.66) the frequency error expression is obtained in terms of estimated AR parameter and frequency of sinusoid as

$$\Delta W = \frac{\left[\hat{a}_1 + \hat{a}_1\hat{a}_2 + 4a_2 \operatorname{Cos} w_1\right] \operatorname{Sin} w_1}{\left[\hat{a}_1 + \hat{a}_1\hat{a}_2\right] \operatorname{Cos} w_1 + 4a_2 \operatorname{Cos} 2w_1} \qquad 4.129$$

From Section 4.1 the exact solution of AR parameter for the second order case can be found

$$\hat{a}_1 = \frac{-\mu^2 \operatorname{Cos} w_1}{\Gamma} \cdot \left[2 \operatorname{Sin}^2 w_1 + \frac{1}{\mu}\right] \qquad 4.130$$

and

$$\hat{a}_2 = \frac{\mu^2 (\operatorname{Sin}^2 w_1 - \frac{1}{\mu} \operatorname{Cos} w_1)}{\Gamma} \qquad 4.131$$

where

$$\Gamma = 1 + 2\mu + \mu^2 \operatorname{Sin}^2 w_1$$

By making use of these exact solutions the final form of the frequency error expression is

$$\Delta W = \frac{g(\mu, w_1) \operatorname{Sin} w_1}{g(\mu_1 w_1) \operatorname{Cos} w_1 - h(\mu_1 w_1)} \qquad 4.133$$

where

$$g(\mu_1 w_1) = \frac{-\mu \operatorname{Cos} w_1}{\Gamma^2} \left[(1+\mu) + 4(1+2\mu)\operatorname{Cos}^2 w_1 - 4\Gamma \operatorname{Sin}^2 w_1\right] \qquad 4.134$$

and

$$h(\mu_1 w_1) = \frac{4}{\Gamma} \left[(\mu^2 + 2\mu) \operatorname{Sin}^4 w_1 - \mu \operatorname{Sin}^2 w_1\right]$$

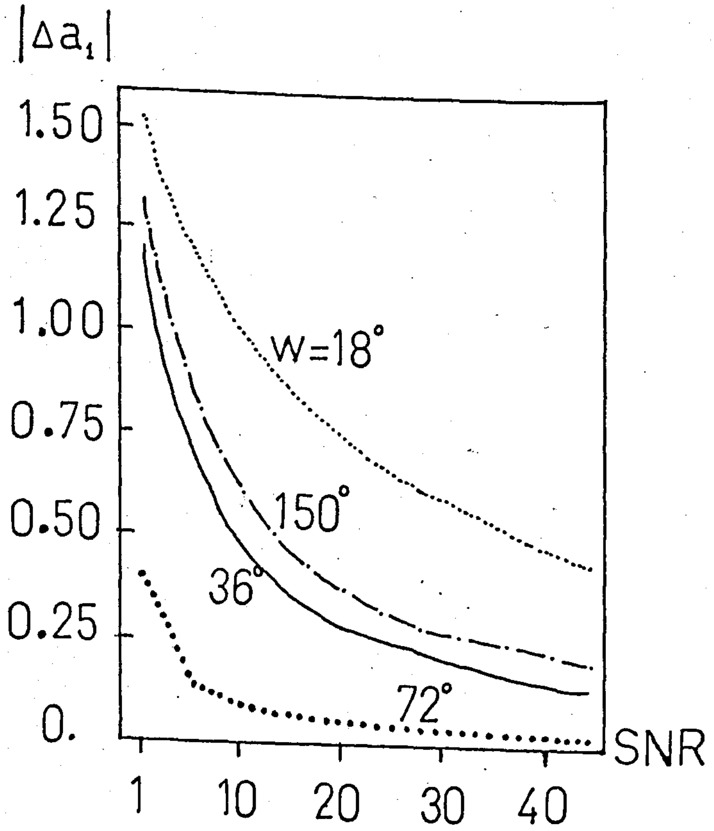
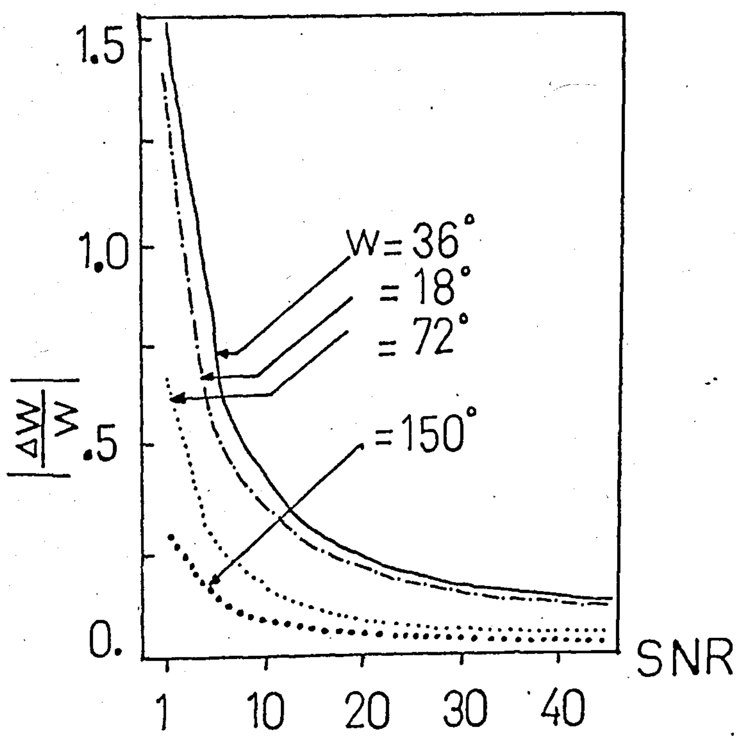
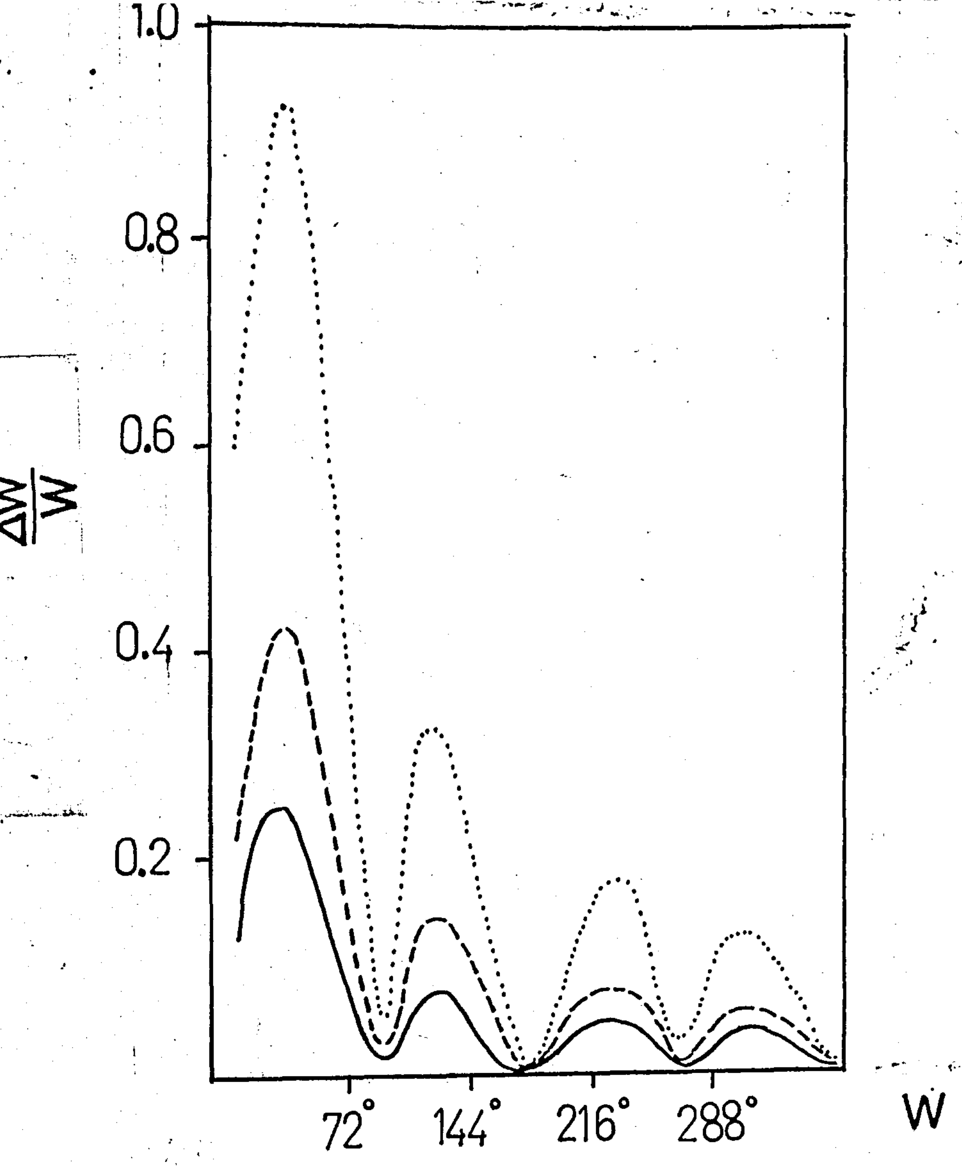
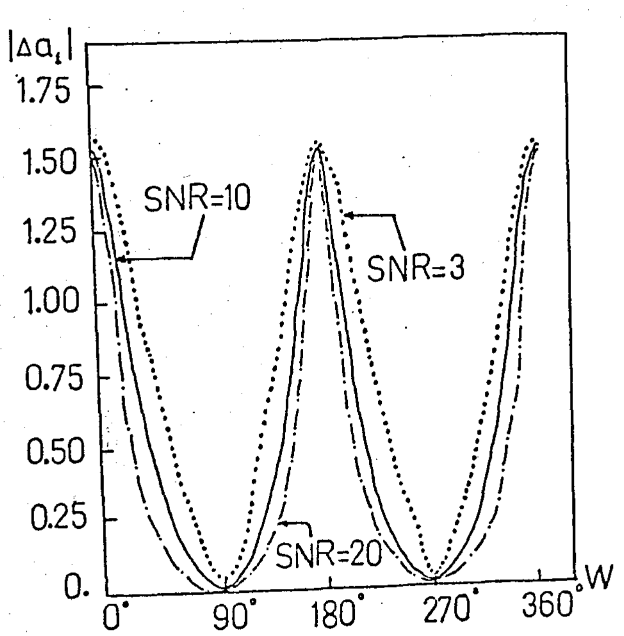
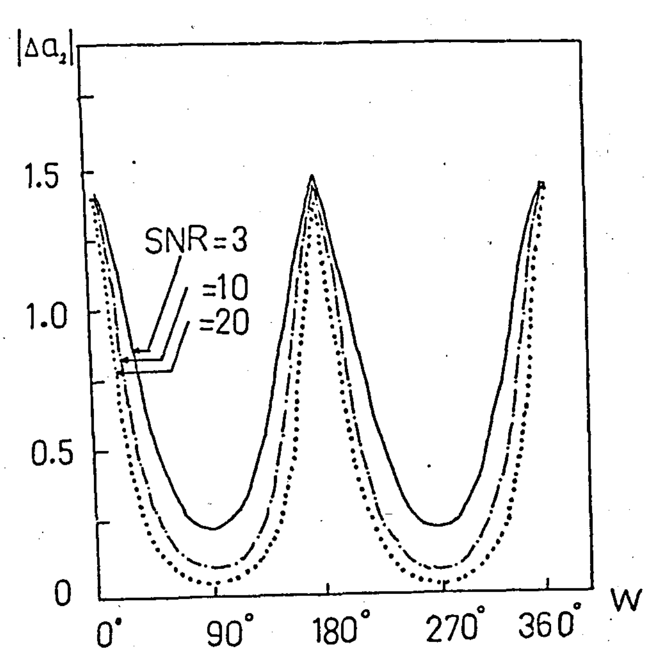


Fig.IV.3. Coefficient deviation versus SNR.




1.0
0.8
0.6
0.4
0.2
ΔW/W
72°
144°
216°
288°
W


|Δa₁|
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.
SNR=10
SNR=3
SNR=20
0°
90°
180°
270°
360°
W


|Δa₂|
1.5
1.0
0.5
0
SNR=3
=10
=20
0°
90°
180°
270°
360°
W

CHAPTER V : CONCLUSIONS AND RECOMMENDATIONS

5.1 CONCLUSIONS

In this dissertion , various modern spectrum approaches to both frequency and spectral moments estimation and their ability (performance) have been investigated and formulated in detail.

Most of the previously appeared popular estimation techniques used in frequency estimation , their performance and drawbacks are summarized in Chapter I.

In Chapter II , we suggest a reasonably accurate and computationally simple technique for estimating the frequency of a sinusoidal signal. Two points are emphasized. The first one deals with the great simplicity of the frequency determination by the proposed method and second one with the accuracy of the obtained results. This technique does not require the computation of the entire spectrum or autocorrelation function which implies that from the point of view of data processing complexity , the Argument method is obviously much simpler. Additionally it is shown that it is an unbiased estimator.

A statistical analysis of the argument method is made for both a a single tone and two tone case and examined the effect of autocorrelation lag on the estimation performance to obtain optimal result. Then the variance of estimated frequency is derived analytically as functions SNR - number of data and autocorrelation lag. This expression gives a hint that the optimal performance can be obtained with proper choice of the autocorrelation lag. Ithas been observeo that the estimation performance of the suboptimum frequency estimator is comparable to that of the corresponding optimum estimator known as ML when autocorrelation lag is 1/3 the data length. In other words it can be concluded that the parformance of the proposed method can be greatly improved by using optimal value of autocorrelation lag which is an integer number close to N/3 bringing the performance close to that of ML frequency estimator. Another important conclusion is that the thres- occurs at about 10 decibel. Therefore only /3'th autocorrelation fun- is needed to determine the frequency of a single tone optimally. This situation is also valid for another popular methods such as Modified Covariance, Covariance, MEM etc, but they do need certainly some additional complex processing. Also this technique yields estimate which is independent of initi phase of sinusoid as demonstreted in[34].

Final study of the Chapter II is the derivation of PDF of estimated frequency for both H_0 and H_1 cases. In a short, this technique gives the expected result for noise only case (H_0) ,like that uniformly distributed probability density function. The PDF for H_1 is expressed in terms of the frequency of sinusoid, number of data samp autocorrelation lag and SNR. Also it displays arithmetic symmetry around the true frequency which implies again ; the estimator is unbiased.

Remarkable points which are still open for further research are to determine the receiver operation charecteristics of this tecnnique using the PDF expressions obtained in Chapter II and the optimal performance for bandwidth estimation problem.

The chapter III gives the results of a study to determine the asymptotic behaviour and statistical properties of MESA spectral moments. The elementary properties of MESA moments, such as filtering, windowing and shifting are formulated. A good estimate of the moments can be obtained by using only a few autocorrelation lags i.e., $5 < M < 10$. The first and second moments can be reliably estimated using two lags.

The analysis of statistical properties reveals that probability of detection and probability of false alarm can be written as functions of the expected value and variance of n'th moment for the tone detection problem. It will be interest to compute the hypothesis tests on the n'th moment (n= 1, 2 .....) based on their asymptotic relative efficiency.

Simulation studies have shown that after a certain value of M (typically 10 to 20) there is a little improvement in the tone frequency esti mate. In other words it is observed that from the derived analytical exp ression, the variance of the mean frequency estimate does not decrease with increasing the number of autocorrelation terms. For fixed number of data samples, the autocorrelation estimation errors at higher order lags and k term in the formulation account for this behaviour.

Finally; the derived expression for PDF using only two autocorrelation terms with signal plus noise case does not yield satisfactory re - sult at low SNR condition. However for noise only case, it displays arithmetic symmetry around 0.25 T as expected.

Chapter IV deals with the effectiveness of AR method in frequency estimation problem. The fidst step of our analysis is to obtain the exact solution of AR parameters by employing standard matrix inversion lemma for different structured matrices such as complex Toeplitz and real symmetric matrices. In the second section the coefficient deviation of AR polynomial is formulated in order to investigate the performance of AR method. It is shown that the presence of white noise causes the estimated AR parameters deviate from the true parameters. Actually the main thrust of this formulation is to determine analytically the root displacement of AR polynomial whose coefficients similarly obtained by various matrices such as ; circular, complex Toeplitz, real symmetric and transformed circular. For circular matrix case Δa and Δz are expressed in terms of SNR, eigenvalue seperation of autocorrelation matrix and order of AR polynomial. It is shown that if the circular matrix is transformed via DFT, the deviation of ith AR coefficient is independent of other coefficients in contrary to case of circular matrix.

In Section 4 , as frequency estimation accuracy the statistical fluctuation of a spectral peak is investigated by using Sakai's and Taylor series approximation methods. It is demonstrated that one can determine the frequency error for AR process using the expressions of coefficients deviation mentioned above in Sakai's method. In Taylor series approximation method the exact evaluation of frequency error expression requires the solution of AR parameters which is available. Another interesting point is the determinig the second order statistics of AR parameters in order to obtain the expected value of estimation error. Also the spesific structure of autocorrelation matrix plays a role in frequency analysis.

Finally the statistical properties of roots of AR polynomial are investigated. The PDF of roots are derived for the cases H_0 and H_1 respectively. Although this expressions appear in comlicated form, they can be used for carrier detection problem. It is demonstrated that the probability of error for (pole -zero) method can be obtained with reasonable assumption and making use of structure of roots. Another important result established in the remainig part of this Section is to illustrate the several possible positions of signal zero in Z-plane and give some analytic expressions for those cases related to the frequency estimation problem. Also second order case makes use of the derived expressions in Section 4.

5.2 RECOMMENDATIONS FOR FURTHER RESEARCHES

This dissertation suggests topics of further research can be outlined as follows

i. The performance of bandwidth estimation can be optimized by examining autocorrelation lag or selecting the optimal value of autocorrelation lag in obtained variance expression of bandwidth estimate.

ii. Computational complexity-variance of frequency estimate product can be taken as a basis to illustrate the attractivity of the analytic signal model and modern spectrum estimation methods.

iii. Signal detection statistics of

- Frequency estimation via analytic signal method
- Bandwidth estimation via analytic signal method
- MESA spectral moments method
- AR method by making use of the statistics of roots

can be found by using the expressions in the dissertation and compared to each others.

iv. The comparative performance of maximum root and spectral decision rule can be obtained by using the result of the statistics of roots and spectral coefficients ior frequency estimation problem.

v. Simulation studies are needed in order to understand and analyze the results of Section 3, i.e., the upper and lower limits of the roots and coefficients displacement varying with the matrix structure. This investigation also gives an oppurtunity to evaluate performance of frequency estimation by making use of the expressions derived in last section of chapter IV.

vi. It will be of interest to compute the efficacies of the MESA spectral moment test and to compare the hypothesis tests on the n'th moment (n=1,2,....) based on their asymptotic relative efficiency (ARE) figures.

APPENDIX A

The purpose of this appendix is to obtain the statistics of the real and imaginary parts of the autocorrelation function of a complex sinusoidal signal in white noise.

Let us assume that the sample vector is

$$Z = \begin{cases} \mathbf{x}+\mathbf{y} & \text{for } H_1 \\ \mathbf{y} & \text{for } H_0 \end{cases} \qquad \text{(A.1)}$$

where

$$\mathbf{x} = [\, x_0, x_1, \ldots\ldots\ldots\ldots x_{H-1}]$$
$$\mathbf{y} = [\, y_0, y_1, \ldots\ldots\ldots\ldots y_{N-1}]$$

H_0 and H_1 mean the null, alternative hypotheses, i.e., the noise only case and signal plus noise $x_n = s_n + j\tilde{s}_n$ and $y_n = w_n + j\tilde{w}_n$ and '$\sim$' denotes the Hilbert Transform. Also **y** is a complex Gaussian vector and $E[\tilde{w}_n]=0$, $Var[\tilde{w}_n]=\sigma^2/2$, $E[\tilde{w}w]=0$.

Define the circular autocorrelation function estimate as in [77].

$$\hat{r}^c_k = \frac{1}{N} \sum_{i=0}^{N-1} z_i \, z_{i+k} \qquad \text{(A.2)}$$

(Some of the steps of derivation follow that of [77]) where the unobserved data points $z_N, z_{N+1}, \ldots z_{N+k-1}$ are defined as $z_N = z_0, z_{N+1} = z_1, \ldots$ Thus z_i can be viewed as a periodic sequence with period N.

For $N>>p$ where $k=1, \ldots, p$

$$\hat{r}_k^c \simeq \hat{r}_k = \frac{1}{N} \sum_{i=0}^{N-k-1} z_i^* \; z_{i+k} \qquad (A.3)$$

since the added terms in the sum are small compared to the rest of the sum.

Now define the permutation matrix J with NxN

$$J = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ \cdot & & & & & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \qquad (A.4)$$

Then

$$\hat{r}_1 = Z^H J\, Z \qquad (A.5)$$

where H denotes conjugate transpose and $Z = | z_0, z_1, \ldots z_{N-1} |^T$

Let

$$Z' = M^H Z \qquad (A.6)$$

where M is the modal matrix of J which is a circular matrix, then

$$[M]_{ik} = \frac{1}{N} \exp \left[j \frac{2\pi ik}{N} \right] \qquad k,i = 0,1,\ldots (N-1) \quad (A.7)$$

and the eigenvalues of J are given

$$\lambda_\ell = \exp \left| j \frac{2\pi \ell}{N} \right| \qquad \ell = 0,1,\ldots (N-1) \quad (A.8)$$

Rewriting Eq.(A.5) we have

$$\hat{r}_1 = \left[(M Z')^H J \,(M Z') \right] / N \qquad (A.9)$$

Since M is a unitary transformation so that $M^{-1} = M^H$ and

$$M^H J\, M = \operatorname{diag} \left\{ \lambda_0 \lambda_1 \ldots \lambda_{N-1} \right\} \triangleq \Lambda \qquad (A.10)$$

$$\hat{r}_1 = Z'^H \Lambda Z' = \sum_{i=0}^{N-1} \frac{\lambda_i \, |z_i'|^2}{N} \qquad (A.11)$$

For any lag it can be shown that |77|

$$\hat{r}_k \cong \frac{1}{N}\sum_{i=0}^{N-1} |z_i'|^2 \exp\left| j\frac{2\pi ik}{N} \right| \qquad (A.12)$$

For H_o

We proceed to consider the noise only case. Since $Z' = M^H Z$ and $E[Z]=0$, $E[ZZ^H]=\sigma_n^2 I$ then it is clear that

$$E[Z']=0 \qquad (A.13)$$

and

$$E[Z'Z']=\sigma_n^2 I \qquad (A.14)$$

which has the same statistical properties as Z.

From [74, pp.194] we have $E|Z_i|^2 = \sigma_n^2$ and $Var\,|Z_i|^2 = \sigma_n^4$ $Cov\,|Z_i|^2, |Z_j|^2 = {}_{ij}\,\sigma_n^4$ for all i. The statistics for can be summarized as follows:

$$E\{R_e\,\hat{r}_k\} = 0 \quad , \quad Var\;R_e(\hat{r}_k) = \frac{\sigma_n^4}{2N}$$

$$E\;I_m\,\hat{r}_k = 0 \quad , \quad Var\;I_m(\hat{r}_k) = \frac{\sigma_n^4}{2N}$$

$$E\;\hat{r}_o = \sigma_n^2 \quad , \quad Var\;\hat{r}_o = \frac{\sigma_n^4}{N} \qquad (A.15)$$

$$C_{ov}\;R_e(\hat{r}_k), I_m(\hat{r}_k) = 0 \; , \; C_{ov}\;I_m(\hat{r}_k), \hat{r}_o = 0$$

$$C_{ov}\;I_m(\hat{r}_k), \hat{r}_o = 0 \qquad (A.16)$$

We can say that $\hat{r}_o$, $Re\,\hat{r}_k$, $I_m\,\hat{r}_k$ are independent random variables. Although probabilty density function for is exponential for large N, the densities become Gaussian by means of the central limit theorem. Hence

$$Re\;\hat{r}_k \sim N\;(o, \frac{\sigma_n^4}{2N})$$

$$I_m\;\hat{r}_k \sim N\;(0, \frac{\sigma_n^4}{2N}$$

$$\hat{r}_o \sim N\;({}_n^2, \frac{\sigma_n^4}{N} \qquad (A.17)$$

where N denotes normal random variable

For H_1

For signal plus noise case, the input sample is

$$Z_i = X_i + xy_i \tag{A.18}$$

and

$$|Z_i|^2 = \tilde{S}_i^2 + S_i^2 + w_i^2 + \tilde{w}_i^2$$

$$E\ |Z_i|^2 = (S_i)^2 + (\tilde{S}_i)^2 + \sigma_n^2 \tag{A.19}$$

Now let us determine

$$E\ R_e(\hat{r}_k) = \sum_{i=0}^{N-1} \frac{E\ |Z_i|^2 \cos(\frac{2\pi k i}{N})}{N} = \frac{1}{N} \sum_{i=0}^{N-1} (S_i)^2 + (\tilde{S}_i)^2 \cos(\frac{2\pi k i}{N})$$

$$E\ I_m(\hat{r}_k) = \frac{1}{N} \sum_{i=0}^{N-1} E\ |Z_i|^2 \sin(\frac{2\pi i k}{N}) = \frac{1}{N} \sum_{i=0}^{N-1} (S_i)^2 + (\tilde{S}_i)^2 \sin(\frac{2\pi i k}{N})$$

$$E\ \hat{r}_o = \frac{1}{N} \sum_{i=0}^{N-1} E\ |Z_i|^2 = \frac{1}{N} \sum_{i=0}^{N-1} (S_i)^2 + (\tilde{S}_i)^2 + \sigma_n^2$$

If we define the circular signal autocorrelation function as

$$\hat{r}^c_{sk} = \frac{1}{N} \sum_{i=0}^{N-1} S_i^* S_{i+k} \tag{A.20}$$

where etc. Since

$$S_i + j\tilde{S}_i = \sum_{\ell=0}^{N-1} M^H_{il}\ S + j\tilde{S}_\ell \triangleq \frac{1}{\sqrt{N}} C_i \tag{A.21}$$

where is the DFT coefficient of and thus

$$\hat{r}^c_{sk} = \frac{1}{N} \sum_{\ell=0}^{N-1} (S_\ell)^2 + (\tilde{S}_\ell)^2 \exp \frac{j2\pi k\ell}{N}$$

Rearranging the expected value of the previous term as

$$E\ Re(\hat{r}_k) = Re\ \hat{r}^c_{sk}$$

$$E\ I_m(\hat{r}_k) = I_m\ \hat{r}^c_{sk}$$

$$E\ \hat{r}_o = \sigma_n^2 + \hat{r}^c_{so} \qquad (A.22)$$

Now let us find the variance of

$$Var(|Z_i|^2) = E\ |Z_i|^2 - |\ _i|^2\ \sigma_n^2\ ^2 \qquad (A.23)$$

where

$$|\ _i|^2 = |S_i + j\tilde{S}_i|^2$$

After some tedious work, we have

$$Var\ (|Z_i|^2) = E\ (\beta_i + 2\alpha_i - \sigma_n^2)^2 \qquad (A.24)$$

where

$$\beta_i = (w_i)^2 + (\tilde{w}_i)^2$$

$$\alpha_i = \tilde{w}_i\ \tilde{S}_i + w_i\ S_i$$

By using the moment feature of normal random variables stated in [74]

$$E\ X^n = \begin{cases} 1,2,\ldots.\ (n-1)\ \sigma^n & n - \text{even} \\ 0 & n\ \text{odd} \end{cases} \qquad (A.25)$$

where $E\ X = 0 \quad E\ X^2 = \sigma^2$

Then

$$Var\ (\ |Z_i|^2) = 2\sigma_n^2\ |\ _i|^2 + \sigma_n^4 \qquad (A.26)$$

and

$$Cov\ |Z_i|^2,\ |Z_j|^2 = Var\ |Z_i|^2\ \cdot\ _{ij} \quad i \neq j$$

$$Var\ Re(\hat{r}_k) = \frac{1}{N^2}\sum_{i=0}^{N-1} Var\ (|Z_i|^2)\ Cos\frac{2\pi ik}{N} = \frac{\sigma_n^4}{2N} + \frac{\sigma_n^2}{N}\ \hat{r}^o_{so} + R_e(\hat{r}^c_{s2k}$$

$$\text{Var } I_m(\hat{r}_k) = \frac{1}{N^2} \sum_{i=0}^{N-1} \text{Var } |Z_i|^2 \text{ Sin } \frac{2\pi ik}{N} = \frac{\sigma_n^4}{2N} + \frac{\sigma_n^2}{N} \hat{r}_{so}^o + R_e(\hat{r}_{s2k}^c)$$

$$\text{Var } \hat{r}_o = \frac{1}{N^2} \sum_{i=0}^{N-1} \text{Var } (|Z_i|^2) = \frac{\sigma_n^4}{N} + 2\sigma_n^2 \frac{r_{so}^c}{N}$$

$$\text{Cov } R_e(\hat{r}_k), I_m(\hat{r}_k) = \frac{\sigma_n^2}{N} I_m \hat{r}_{s2k}^c$$

$$\text{Cov } Re(\hat{r}_k), \hat{r}_o = \frac{2\sigma_n^2}{N} Re \hat{r}_{sk}^c$$

$$\text{Cov } I_m(\hat{r}_k), \hat{r}_o = \frac{2\sigma_n^2}{N} I_m \hat{r}_{sk}^c \qquad (A.27)$$

In a summary

$$R_e \hat{r}_k \sim N\left(Re \hat{r}_{sk}^c, \frac{\sigma_n^4}{2N} + \frac{\sigma_n^2}{N} \hat{r}_{so}^c + Re(\hat{r}_{s2k}^c)\right)$$

$$I_m \hat{r}_k \sim N\left(I_m \hat{r}_{sk}^c, \frac{\sigma_n^4}{2N} + \frac{\sigma_n^2}{N} \hat{r}_{so}^c - Re(\hat{r}_{s2k}^c)\right)$$

$$r_o \sim N\left(\hat{r}_{so}^c + \sigma_n^2, \frac{2\sigma_n^2 \hat{r}_{so}^c}{N} + \frac{\sigma_n^4}{N}\right) \qquad (A.28)$$

For known signal autocorrelation function case i.e $\hat{r}_{sk}^c = \frac{A^2(N-k)}{N}$ $\text{Cos}w_1k$, we have

$$R_e \hat{r}_k \sim N\left(\frac{A^2(N-k)}{N} \text{Cos}w_1k, \frac{\sigma_n^4}{N} \frac{1}{2} + \mu \left(1 + \frac{(N-2k)}{N} \text{Cos}2w_1k\right)\right)$$

$$I_m \hat{r}_k \sim N \left(\frac{A^2 (N-k)}{N} \right) \text{Sin} w_1 k \, , \frac{n^4}{N} \; \frac{1}{2} + \mu \left(1 - \frac{(N-2k)}{N} \text{Cos} 2 w_1 k \right)$$

$$r_o \sim N \left(A^2 + \sigma_n^2 \, , \frac{2A^2 \sigma_n^2}{N} + \frac{\sigma_n^4}{N} \right) \qquad \text{(A.29)}$$

where A^2, μ are signal power, SNR respectively

APPENDIX - B

In this Appendix, we now study certain properties of two jointly random variables (x,y) and the probability density function of their ratio . The joint density of x and y is given [74].

$$P_{x,y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left[\left(\frac{-1}{2(1-r^2)}\right)\left\{\frac{(x-\bar{x})^2}{\sigma_x^2} \frac{2r(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y} + \frac{(y-\bar{y})}{\sigma_y^2}\right\}\right] \quad \text{B.1}$$

for $x>0$, $y>0$

Let us find the distribution of the ratio $z = x/y$. From [74, pp196] we write

$$P_Z(Z) = \int_0^\infty y\, P_X(yz,y)\, dy \quad \text{B.2}$$

Define

$$T_1 = \frac{(yz-\bar{x})^2}{\sigma_x^2} = \frac{y^2z^2 - 2\bar{x}\,zy + \bar{x}^2}{\sigma_x^2}$$

$$T_2 = -\frac{2r\ (x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y} = -\frac{2r}{\sigma_x\sigma_y} \quad Zy^2 - Zy\bar{y} - \bar{x}y + \bar{x}\bar{y}$$

$$T_3 = \frac{(y-\bar{y})}{\sigma_y^2} = \frac{(y^2-2y\bar{y} + \bar{y}^2}{\sigma_y^2} \quad \text{B.3}$$

Then

$$f_Z(Z) = A_1 \int_0^\infty y \exp\ -A_2\ (T_1+T_2+T_3)\ \ dy \quad \text{(B.4)}$$

where

$$A_1 = \frac{1}{2\pi\ \sigma_x\ \sigma_y\sqrt{1-r^2}} \quad \text{and} \quad A_2 = \frac{1}{2(1-r^2)}$$

Now

$$T_1 + T_2 + T_3 \stackrel{\Delta}{=} A_3\, y^2 + A_4 Y + A_5 \quad \text{B.5}$$

where

$$A_3 = \frac{z^2}{\sigma_x^2} + \frac{1}{\sigma_y^2} - \frac{2rZ}{\sigma_x\sigma_y}$$

$$A_4 = \frac{2rZ\bar{y}}{\sigma_x \sigma_y} + \frac{2r\bar{x}}{\sigma_x \sigma_y} - \frac{2\bar{x}Z}{\sigma_x^2} - \frac{2\bar{y}}{\sigma_y^2}$$

$$A_5 = \frac{\bar{x}^2}{\sigma_x^2} - \frac{2r\bar{x}\bar{y}}{\sigma_x \sigma_y} + \frac{\bar{y}^2}{\sigma_y^2}$$

By collecting these terms, Eq. (B.4) becomes as

$$f_Z(Z) = \left[e^{-A_5A_2 + \frac{A_2A_4^2}{4A_3}}\right] \cdot A_1 \int_0^\infty y \exp\left(-A_3A_2\left[y + \frac{A_4}{2A_3}\right]^2\right) dy \qquad \text{B.6}$$

Define

$$§ = \left(y + \frac{A_4}{2A_3}\right)\sqrt{A_3A_2} \qquad \text{B.7}$$

then

$$f_Z(Z) = A_1 \exp\left[-A_5A_2 + \frac{A_2A_4^2}{4A_3}\right]\left\{\int_0^\infty \frac{1}{A_3A_2} § e^{-§^2} d_§ - \int_0^\infty \frac{A_4}{2A_3\ A_3A_2} e^{-§^2} d_§\right\} \text{B.8}$$

Since

$$\int_0^\infty § \ e^{-§^2} d_§ = \frac{1}{2} \qquad \text{and}$$

$$\int_0^\infty e^{-§^2} d_§ = \frac{\sqrt{\pi}}{2} \qquad \text{B.9}$$

we have finally

$$f_Z\ (Z) = \frac{A_3}{2A_3\ A_2^{1/2}}\left[\frac{1}{A_2^{1/2}} - \frac{\sqrt{\pi}\ A_4}{2A_3^{1/2}}\right] \exp\left[-A_5A_2 + \frac{A_2A_4^2}{4A_3}\right] \qquad \text{B.10}$$

APPENDIX C

i. COMPLEX TOEPLITZ

For the signal which is a sinusoidal (complex) with power " a " and frequency ω_1 , the autocorrelation matrix can be written as follows :

$$R = \sigma_n^2 I + a\, V^* V^T$$

where σ_n^2 is the power of the white noise and

$$V = \left[1 \;,\; \exp(j\,\omega_1) \;, \ldots\ldots \exp(j\,\omega_1 M) \right]^T$$

and ' * ' , ' T ' denote complex conjugate transpose respectively.

By using the Matrix Inversion Lemma (MIL)

$$\left[A + a\, V^* V^T \right]^{-1} = A^{-1} - \frac{a A^{-1} V^* V^T A^{-1}}{I + a\, V^T A^{-1} V^*}$$

we have

$$\left[R_x^c \right]^{-1} = \frac{I}{\sigma_n^2} \left[I - \frac{a\, V^* V^T}{\sigma_n^2 + a\, V^T V^*} \right]$$

or more clearly

$$= \frac{1}{\sigma_n^2} \left[I - \gamma V^* V^T \right]$$

where

$$\gamma = \frac{SNR}{I + (M+I)\,SNR} \qquad \text{and} \qquad SNR = \frac{a}{\sigma_n^2}$$

Finally

$$\left[R_x^c \right]^{-1} = \begin{bmatrix} I-\gamma & \gamma \exp(j\omega_1) & \cdots & \gamma \exp(j\omega_1 M) \\ \gamma \exp(-j\,\omega_1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \gamma \exp(-jM\,\omega_1) & \cdots & \exp(-j\,\omega_1) & I-\gamma \end{bmatrix}$$

ii. REAL SYMMETRIC

Real symmetric matrix can be written as

$$R_x^r = \sigma_n^2 I + \frac{a}{2}\left[V V^{*T} + V^* V^T\right]$$

Let us define

$$A = \sigma_n^2 I + \frac{a}{2} V V^{*T}$$

$$R_x^r = A + \frac{a}{2} V^* V^T$$

and

$$\left[R_x^r\right]^{-1} = A + \frac{a}{2} V^* V^T = A^{-1} - \frac{A^{-1} V^* V^T A^{-1}}{I + \frac{a}{2} V^T A^{-1} V^*}$$

where

$$A^{-1} = \frac{1}{\sigma_n^2}\left[I - \mu V V^{*T}\right] \text{ and } \mu = \frac{SNR}{2 + (M+I)\, SNR}$$

then

$$\left[R_x^r\right]^{-1} = \frac{1}{\sigma_n^2}\left(I - \mu V V^{*T}\right) -$$

$$\frac{\frac{a}{2\sigma_n^2}\left(I - V V^{*T}\right)\left(V^* V^T\right)\frac{1}{\sigma_n^2}\left(I - V V^{*T}\right)}{1 + \frac{a}{2\sigma_n^2} V^T \left(I - \mu V V^{*T}\right) V^*}$$

Observe that

$$c_1 \triangleq V^T V = \sum_{i=0}^{M} \exp(j2i\,\omega_1) = \frac{\sin((M+1)\,\omega_1)}{\sin \omega_1}$$

$$V^{*T} V^* = c_2 = c_1^*$$

$$V^T V^* = M+1$$

After some tedious work,

$$\left[R_x^r\right]^{-1} = \frac{\beta_1 I - 2a\,\mathrm{Re}\left[V V^{*T}\right] + 2\mu a\,\mathrm{Re}\, V V^T c_2}{2(1+\xi)\,\sigma_n^4}$$

$$\xi = \frac{a}{2\sigma_n^{\,2}} \left[(M+1) - \mu C_1 C_2 \right]$$

$$\beta_1 = 2\sigma n^2 + a \left[(M+1) - \mu C_1 C_2 \right]$$

$$VV^{*T} = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-M} \\ z & \ddots & \ddots & \vdots \\ z^2 & \ddots & \ddots & z^{-1} \\ \vdots & & & \vdots \\ z^M & z^{M-1} & \cdots z & 1 \end{bmatrix}$$

$$VV^T = \begin{bmatrix} 1 & z & \cdots & z^M \\ z & z^2 & & z^{M+1} \\ z^2 & & z^4 & \vdots \\ \vdots & & \ddots & \vdots \\ z^M & z^{M+1} & \cdots & z^{2M} \end{bmatrix}$$

$$\mathrm{Re}\left[c_2 VV^T\right] = \mathrm{Re} \begin{bmatrix} C_2 & C_2 Z & \cdots & C_2 Z^M \\ C_2 Z & & & C_2 Z^{M+1} \\ \vdots & & & \vdots \\ C_2 Z^M & C_2 Z^{M+1} & & Z^{2M} C_2 \end{bmatrix}$$

Or, another representation is

$$\left[\mathbf{R}_x^r\right]^{-1} = \frac{1}{\sigma_n^{\,2}} \left[I - \frac{2a \ \mathrm{Re}\left[VV^{*T}\right]}{\beta_1} + \frac{2a\mu \,\mathrm{Re}\left[VV^T C_2\right]}{\beta_1} \right]$$

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