EVALUATION OF INSPECTION POLICIES AND PROCESSES FOR DETERIORATING SYSTEMS SUBJECT TO CATASTROPHIC FAILURE

by

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ABSTRACT

The purpose of this thesis is to study deteriorating systems subject to catastrophic failure in order to evaluate deterministic inspection policies and stochastic inspection processes. The underlying deterioration process is assumed to be an increasing Markov renewal process so that the system deteriorates over time. An important feature of the model is that the true state of the system cannot be known by simple observation; instead, some tests have to be carried out in order to detect if the system has positive deterioration or not. However the results of the tests are not perfect so that the probability of true and false detections depend on the unobserved state of the system.

The system can be inspected in two ways. Inspections are done either deterministically at some prespecified points in time which constitute deterministic inspection policies. On the other hand, inspections at random times are also possible and they constitute stochastic inspection processes. These inspection policies and processes are evaluated in various ways; explicit expressions to compute the expected

number of tests are presented and some practical applications of our results are illustrated with some interesting examples.

KATASTROFİK ŞEKİLDE BOZULAN SİSTEMLERDE MUAYENE POLİTİKA VE SÜREÇLERİNİN DEĞERLENDİRİLMESİ

Ö Z E T

Bu çalışmanın amacı katastrofik şekilde aşınan sistemlerin deterministik muayene politikalarını ve stokastik muayene süreçlerini değerlendirmeye yönelik olarak incelemektir. Sistemin aşınma süreci artan bir Markof yenileme süreci olarak alınmıştır ve dolayısıyla sistem zaman içinde bozulmaktadır. Modelin önemli bir özelliği sistemin asıl durumunun bilinmemesi ve sadece bazı testlerle sistemde aşınma olup olmadığının belirlenebilmesidir. Ancak, gözönüne alınması gereken önemli bir nokta test neticelerinin kesin doğru olmayışı ve gözlenemeyen gerçek sistem durumuna göre bazı olasılıklarla doğru veya yanlış tesbitler yapılabilmesidir.

Sistemin iki şekilde muayene edilmesi mümkündür. Deterministik muayene politikalarıyla sistem önceden belirlenmiş deterministik zamanlarda muayene edilebilir. Veya kesinlikle bilinmeyen rassal zamanlarda yapılan muayenelerle bir stokastik muayene süreci söz konusu olabilir. Bu tezde deterministik muayene politikaları ve stokastik muayene süreçleri değişik metotlarla değerlendirilmekte, beklenen test sayıları için açık ifadeler sunulmakta ve böylece elde edilen çeşitli uygulama sonuçları birçok ilgi çekici örneklerle açıklanmaktadır.

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I. INTRODUCTION

The purpose of this thesis is to evaluate deterministic inspection policies and stochastic inspection processes for deteriorating systems which are subject to catastrophic failure. This section includes an overview of the research together with a detailed literature survey of related maintenance models.

1.1. OVERVIEW OF THE RESEARCH

It has become a common experience in reliability theory to represent the evolution of the state of a reliability system by a deterioration process. A mechanical part developing cracks or an individual developing cancer can be given as examples of deteriorating systems sharing two important features. First, the true state of the system cannot be known by simple observations. However, some tests can be carried out in order to detect if the system has positive deterioration or not, and then corrective action can be taken to reduce the likelihood of failure. Moreover the results of the tests themselves are not perfect so that the probability of true and false detections depends on the unobserved state of the system. Secondly, when failure occurs, it is catastrophic; that is the failed system cannot be repaired or replaced. The cost of failure is so great that it cannot be compared with the cost of inspections and corrective actions.

The objective of this thesis is to propose and evaluate inspection or testing policies or processes for deteriorating systems subject to catastrophic failure. Our model will have two significant features. First the underlying deterioration process will be taken to be a Markov renewal process. This is a reasonable choice since these processes are well-known for their applicability in many fields of pure and applied sciences. Moreover, the stochastic structure of Markov renewal processes is of sufficient complexity and generality to be an actual model of deterioration. For example, these processes can be used to model the configuration of cracks in a mechanical part or the size of a tumor. We shall see in the next section that most inspection/decision models in the literature involve discrete-time and discrete-state Markov chains which are Markov renewal processes with a rather restrictive special form. Thus, the Markov renewal process representation of deterioration treated in this thesis will provide sufficient generalization to the literature.

The second feature of our model is that it allows for imperfect observations during inspections. This is important for many applications, because inspections usually give inaccurate estimates of the underlying state. For example, a mammogram in breast cancer screening may or may not reveal the presence of the tumor, and the detection is further complicated by the possibility of false positive and false negative results. The literature survey presented in the next section indicates that the amount of research carried out so far which involves imperfect information about the underlying state of the system is quite limited.

As a result of these important features, our model is anticipated to have considerable utility for a variety of applications. Mechanical systems involving parts which are subject to catastrophic failure where the cost of failure cannot quantitatively be compared with the inspection and repair costs, or cancer screening problems in medicine where failure is equivalent to the death of a person can be given as examples.

In the next section a review of the literature is presented. The deterioration process and the inspection problem is formulated in Chapter II. A brief review of Markov renewal processes will be presented together with formal definitons of deterministic and stochastic inspections. Chapter III and Chapter IV are devoted to the analysis of deterministic inspection policies and stochastic inspection processes respectively. The main emphasis in these chapters is placed on the computation of the measures of effectiveness like the expected number of tests performed, the expected number of tests with positive, negative results, etc. Chapter V treats some examples where the results of Chapters III and IV will be used to evaluate certain policies. In particular, a management problem to determine the optimal inspection schedule to maximize the probability of detection before failure will be analyzed. Finally, Chapter VI concludes this thesis by summarizing the main points emphasized and by indicating problems and areas for further research.

1.2. LITERATURE SURVEY

There is an extensive literature on the optimal control of deteriorating systems, especially on the optimal maintenance of systems subject to failure. In 1965, Mc Call(1) presented a survey with 88 references in scheduling policies for stochastically failing equipment. A survey, reviewing the area of maintenance models and including 259 references was done by Pierskalla and Voelker(2) in 1976. Sherif and Smith(3) updated this survey in 1981 by providing 524 references. A more recent survey which listed 243 references was provided by Bosch and Jensen(4,5) in 1983.

Much of this literature deals with classical maintenance models on the inspection of systems subject to stochastic deterioration. Each time, the deteriorating system is inspected in order to decide whether to replace it, repair it or let it continue as it is. Hence the main issue is the decision on the time of repair or replacement, not the time of inspection. The true state of the system being known at each decision epoch, these optimal replacement/repair models are generally solved using Markov decision theory as in Derman[6] or Özekici[7], or renewal theory as in Barlow and Proschan[8]. Our interest lies in the models where the issue is not only to determine the time when to take corrective action for a deteriorating system, but also to decide on when to inspect it. The problem is further complicated by the fact that the true state of the system cannot be known with certainty even after the inspections which are costly; so there is a trade-off between the inspection cost and the accuracy of the information about the true state of the system. For this reason, conventional Markov decision theory cannot be used to analyse such models. Instead it is usually necessary to use what is called a partially observable Markov decision process (POMDP) model. Monahan[9] provides a survey of POMDP models where he presents properties of these models as summarized below.

One of the main characteristics of the POMDP is the transformation of the information vector from period to period via Bayes' rule. There is considerable literature on Bayesian control of sequential decision processes which is only indirectly related to POMDP models. In this literature, elements of the decision process are unknown. For example, the decision maker may not know the transition probability matrix governing the movement of the process, as treated in Satia and Lave(10). In a POMDP, however all the elements of the decision process are assumed to be known. Only, information regarding the current state of the nonobservable underlying process is obtained.

There is also literature dealing with the acquisition of information for various continuous time partially observable stochastic processes. For example, Anderson and Friedman(11,12) made a comprehensive study of a continuous time model where the underlying state is taken to be a Brownian motion process.

The quality control models in the literature can be classified on the basis of the source and degree of partial information. Many studies are carried on the two-state model where the system or the process is either in good or bad states. In general the underlying process represents the condition of a machine which is deteriorating over time. The true condition of the machine is not known with certainty, but information can be obtained either by observing the machine's output or by directly inspecting the machine. Girshick and Rubin(13) were first to consider the two-state model under the assumption that perfect information is available after inspection and no information can be obtained without inspection. They showed that the optimal decision should be based on the probability that the system is in the bad state, and so the optimal policy can be specified by the regions of the unit interval that correspond to the three possible actions, namely inaction, inspection and replacement. Furthermore, they conjectured that the region corresponding to an individual action is a single interval, and so the unit interval is divided into at most three subintervals, one for each possible action. But, this three-subinterval conjecture was shown to be false by Taylor(14) who provided a four-subinterval counter-example, where the "inaction" region consisted of two subintervals separated by the inspection region and the replace subinterval included probability one, which is indeed very logical since, if the system is in the bad state with probability one, the decision maker has only the choice of replacing the system.

Swallwood and Sondik(15), Eckles(16), Ross(17), and Albright(18) examined similar models. Ross(17) proved that, the most general optimal policy is the four-subinterval structure of Taylor(14). Eckles(16) investigated a closely related model, he organized however the structure of the optimal policy in a different way without focusing on the probability distribution of the underlying state. Rosenfield(19) considered another variation of the model and obtained similar results for the optimal maintenance policy.

In practically all of the optimal inspection models in the literature an inspection, when it occurs, reveals the true state of the system with perfect information. An important exception is the model by White[20,21], who allows the decision maker to receive imperfect information about the true state of the underlying deterioration process. As a special case, he considered a model where perfect information was obtained by inspection and imperfect information was obtained without inspection, thus generalizing Ross'[17] model. He came out with the same results by proving that the optimal policy has the four-subinterval structure. However, in the discussion of the general partially observable model, where only imperfect information can be obtained with or without inspection, the characterization of the optimal policy remained an open question.

Virtually all of the models in the literature are presented in the context of a maintenance problem, usually one where the system can be restored to be as good as new. Eddy[22,23] studied a significantly different optimal inspection situation: a problem of preventive medicine that seeks for the optimal schedule to screen for a disease. In his problem, the system is identified by the person on whom screening tests are to be applied and failure is the sickness and consequently the death which can occur at most once for a system and is usually more catastrophic than in maintenance context. An important objective may be to identify a diseased state as early as possible thereby maximizing the probability of a cure. In 1982, Eddy and Shwartz[24] presented screening problem in cancer and explained the application of mathematical models in their paper where they especially emphasized the difference between so-called deep and surface models.

The surface models consider only clinical events that can be observed directly and are on the surface. The basic function of these models is to tabulate observations and estimate the consequences of existing screening programs. They do not attempt to describe the underlying disease pathophysiology or screening dynamics that caused the observed events, and therefore cannot be used to estimate the consequences of screening programs that have not yet been conducted. Bailar[25] gave a good example for a surface model that used data from the Health Insurance Plan of Greater New York to estimate the number of new cases of breast cancer generated by the x-rays delivered in performing mammograms versus the number of cancer deaths prevented by the addition of mammography to yearly screening programs. He estimated that the five years of screening with mammography prevented not more than 12 to 14 breast cancer deaths, while the radiation was expected to induce about 16 new cases of breast cancer.

Deep models, on the other hand, explicitely consider the pathophysiology of the underlying disease and how the course of the disease is affected by screening. The importance of this difference is that deep models can be used to estimate the value of screening programs that have never been studied in clinical trials. Models proposed by Eddy[22,23], Kirch and Klein[26], and Schwartz[27] can be given as examples for deep models. Most of the important questions asked of deep models concern the time factor: What is the effect of screening a population for a certain number of years? What is the optimal frequency of screening? At what ages should screening be started and stopped? To answer these and similar questions one must be able to describe first, how a disease progresses or develops over time, how detecting a disease at a particular time in its development affects important outcome measures such as mortality, and how the detection capabilities of screening tests vary as the disease progresses. In the models developed by Eddy[22,23], Kirch and Klein[26], and Schwartz[27], the disease progression is taken to be continuous over time. Eddy's[22,23] model tracks the change in mortality as the disease develops, Kirsh and Klein's[26] and Schwartz's[27] models are concerned with the growth of cancer in size and the probability of spread to axillary lymph nodes. In most cases, discrete disease states are defined. For example, Schwartz[27] defined 21 disease states consisting of seven tumor-size categories defined through the tumor diameter and for each size category three lymph node involvement levels. Moreover, he formulated quantitatively some hypothesis concerning the rate of disease progression, the tendency of the disease to be detected without benefit of scheduled screening examinations. He estimated parameters by fitting his model statistically to published data on breast cancer. On the basis of the model, he calculated the benefits of screening under alternative assumptions about the women screened, the number of screens given, and the ages at which the screens are given.

The detection capability of screening tests, which is one of the main components of deep models, is defined in terms of true positive and false positive rates. The probability that a test will detect an existing cancer, or the true positive rate, obviously varies with the state of development of the cancer. For a cancer that is in its first week of life and consists of only a few cells, that probability is almost zero. On the other hand, for a cancer which is decades old with considerable tumor size, the true positive probability is virtually one. Since the state of development varies with time, and one of the main purposes of a deep model is to analyse time-related problems, it is desirable to model the detection capability of screening tests as a function of the state of development of the cancer. However, only in a few models, such as those of Eddy[22,23] and Shwartz[27], the true positive probabilities vary as the disease progresses.

Many of the models differ also in the outcome measures that can be estimated. Some models such as those of Shwartz[27], Kirch and Klein[26] estimate the probability of detection before a terminal state such as axillary lymph node involvement is reached. Shwartz's[27] model can estimate the probabilty that a woman will have a recurrence of her disease whereas Eddy's[22,23] model estimates the probability that she will die of a recurrence.

In many of the models, there are also differences in the screening programs that are analyzed. Kirch and Klein[26] were concerned with whether a non-periodic schedule, involving the same expected number of examinations per patient as a periodic schedule, could reduce the average time to detect a given disease; or whether a non-periodic schedule involving fewer expected examinations per patient could, on the average, lead to disease detection as early as a given periodic schedule. Kirch and Klein[26] show that an optimal examination schedule which minimizes the expected detection delay would be non-periodic, and that the frequency of examinations would either be approximately or exactly proportional to the square root of the age-specific incidence probability of the disease. They also derived optimal schedules for breast cancer examinations; they found that optimal non-periodic schedules result in a savings of 2 % to 3 % in the expected number of examinations when periodic and non-periodic schedules have the same detection delays.

Lincoln and Weiss[28] considered the efficiency of different policies for scheduling medical examinations; the formal problem is to evaluate the effects of random delay between examinations on the diagnosis and outcome of the disease. They treat both periodic and random examinations and allow for imperfect diagnosis depending on how long the disease has been present. The examination times $\{\tau_i\}$ form a renewal process. Their problem, as usual, is the determination of an examination schedule that is optimal in some sense or if not optimal satisfies certain reasonable constraints. They use two different criteria to measure the effectiveness of the policy. The first, and perhaps simplest criterion that one can think of involves the setting of a level $\varepsilon<1$, such that no more than a fraction ε of those people who eventually have a tumor will have an undetected tumor for more than a specified time. The second criterion is to require that the mean undetected time of tumor growth exceed a given time. Lincoln and Weiss(28) also derived formulas concerning periodic and non-periodic policies and applied the results they have found to data on cancer of the cervix.

Another deep model on screening for cancer is given by Eddy(22,23). Rather than being based on a detailed mathematical description of pathophysiologic characteristics of the disease such as tumor size, growth rate, and lymph node involvement, this model is concerned only with the detectability of the cancer as a function of the age of the cancer at the time it is detected and treated. In this model there are three states: healthy, diseased or sick. The untreated person proceeds from healthy to diseased and then to sick where he remains forever. Superimposed on this underlying process is a schedule of costly tests. The outcome of a test during the healthy state can be positive, in which case a false positive cost is incurred. In the diseased state, the person is not aware of the disease, and the outcome of a test while in this state can be negative. But if a test here is positive then treatment commences and due to earliness of detection, cure is enhanced. If the process reached the sick state then the disease becomes apparent to the person and there is no need for additional tests.

As a function of a specified inspection schedule Eddy[22,23] derived formulas for some quantities of interest such as the expected number of false positives, the probability that the disease will be detected before entering the sick state and the probabilty of getting sick. One main assumption, called the progression assumption is used in deriving these formulas: Once a cancer has grown to the point that it is detectable by a test, it is always detectable by that test.

In many of the models in literature, there exist also interesting cost analyses by which the effectiveness of the screening program can be evaluated. Some of the important costs considered are the costs of delivering the tests, the costs of checking by a more definitive work-up, whether a positive result obtained is true or not.

It is clear that the most important feature of deteriorating models involve the stochastic structure of the underlying process. Almost all deterioration models in the literature satisfy a Markovian structure in one form or other. An excellent account of deterioration models with continuous Markov, continuous semi-Markov, right-continuous Markov, Markov additive, and general semi-Markov structures is given by Çınlar[29].

This review on the studies of deteriorating systems reveals the novelty of this thesis. In addition to the three-state model examined by Eddy[22,23], a model with a countable number of states will be treated by describing the deterioration process through a Markov renewal process which generalizes most of the papers surveyed above. Deterministic inspection policies and stochastic inspection processes will be examined offering more generality to literature, since stochastic and deterministic inspections cover respectively random inspections and periodic examination schedules. Many formulas will be derived using Markov renewal theory and evaluations of the results obtained will be made.

II. FORMULATION OF THE INSPECTION PROBLEM

In this chapter the Markov renewal structure of the deteriorating system will be explained together with a brief review of the theory of Markow renewal processes. Then the definition of deterministic inspection policies will be given and the structure of stochastic inspection processes will be formulated. Finally, some important measures of effectiveness in the evaluation of inspection policies and processes will be explained.

2.1. STOCHASTIC STRUCTURE OF DETERIORATION PROCESS

We are interested in a system which deteriorates over time, the deterioration level increasing as time goes on to reach finally a failing or terminal state. Since the true state of the system is not directly observable, the aim of the inspector is to detect the disorder by carrying out some tests, as early as possible, so that he could somehow find a remedy for it.

We define X_t as the deterioration level at time t and call $X = \{X_t : t \geq 0\}$ the deterioration process.

X is increasing, has state space $E_\Delta = E \cup \{\Delta\}$ where $E=\{0,1,2,\ldots\}$ and Δ is an absorbing state called the terminal state. In medical applications such as screening for cancer, state 0 corresponds to the case where tumor is not present, while states 1,2,3,... correspond to increasing sizes of the tumor and state Δ is a terminal state where it is no longer possible to cure the disease. Define

$$T_o = 0, \; T_{n+1} = \inf\{t \geq T_n : X_t \neq X_{T_n}\}$$

$$Z_o = X_o, \; Z_n = X_{T_n}$$

so that Tn is the time of the n'th jump and Z_n is the n'th stage of deterioration.

The relation between the processes X and Z is also given by

$$X_t = \begin{cases} Z_n & \text{if} \quad T_n \leq t < T_{n+1} \\ \Delta & \text{if} \quad t \geq \sup_n T_n \end{cases} \tag{1.1}$$

We now state the main assumption on the stochastic structure of the deterioration process.

<u>ASSUMPTION (1.1)</u> (Z,T) is a Markov renewal process where X is the minimal semi-Markov process associated with it.

We now include a brief survey of Markov renewal processes as presented in Çınlar[30].

DEFINITION (1.1) The stochastic process $(Z, T) = \{(Z_n, T_n) : n \in \mathbb{N}\}$ is said to be a Markov renewal process with state space E_Δ provided that

$$P\{Z_{n+1} = j,\ T_{n+1} - T_n \leq t \mid Z_o, \ldots, Z_n;\ T_o, \ldots, T_n\} = P\{Z_{n+1} = j,\ T_{n+1} - T_n \leq t \mid Z_n\}$$

for all $n \in \mathbb{N}$, $j \in E_\Delta$, and $t \in \mathbb{R}_+$, where $\mathbb{N} = \{0,1,2,\ldots\}$ and $\mathbb{R}_+ = [0,\infty)$.

The process (Z,T) is assumed to be time-homogeneous: that is

$$P\{Z_{n+1} = j,\ T_{n+1} - T_n \leq t \mid Z_n = i\} = Q(i,j,t)$$

for any $i,j \in E_\Delta$, $t \in \mathbb{R}_+$, independent of n. The family of probabilities $Q = \{Q(i,j,t) : i,j \in E,\ t \in \mathbb{R}_+\}$ is called a semi-Markov kernel over E. We assume $Q(i,j,0) = 0$ for all $i,j \in E$ and defining

$$P(i,j) = \lim_{t\to\infty} Q(i,j,t) \quad i,j \in E_\Delta$$

it follows that P is a Markov matrix. Furthermore Z is a Markov chain with state space E_Δ and transition probability matrix P. Since X is increasing, the matrix P is upper triangular, that is

$$P(i,j) = P\{Z_{n+1} = j \mid Z_n = i\} = 0, \; j \leq i, \; n \in \mathbb{N}$$

If $P(i,j) = 0$ for some pair (i,j) then $Q(i,j,t) = 0$ for all t. We then define $Q(i,j,t)/P(i,j) = 1$. With this convention we define $G(i,j,t) = Q(i,j,t)/P(i,j)$, $i,j \in E$, $t \in \mathbb{R}_+$. Then, for each pair (i,j) the function $t \to G(i,j,t)$ is a distribution function of the sojourn in state i given that the next state is j, since

$$G(i,j,t) = P\{T_{n+1} - T_n \leq t \mid Z_n = i, \; Z_{n+1} = j\} \quad i \in E, \; j \in E_\Delta \quad (1.2)$$

This explains the stochastic structure of the (Z,T) process. The increments $\{T_{n+1} - T_n\}$ are conditionally independent given Z_o, $Z_1, \ldots$ with respective distributions $\{G(Z_n, Z_{n+1}, \cdot)\}$ and the Markov chain evolves according to the transition probabilities specified by P. Hence, we can see that the evolution of the deterioration level X of the reliability system is such that the sequence of states visited form a Markov chain with transition matrix P where the sojourn in a given state has a distribution which depends on the state being visited and the next state to be visited. Thus the sojourn in some state i has the distribution $G(i,j,\cdot)$ if j is the next level of deterioration. This explains the applicability of this model to many real life situations. For example, it is reasonable to assume that the stages of a disease, tumor, etc. evolve as a Markov chain with sojourns satisfying(1.2).

For simplicity of notation we will let $P_i\{\cdot\}$ denote the conditional probability $P\{\cdot \mid Z_o = i\}$ and E_i be the corresponding expectation. If we define

$$Q^n(i,j,t) = P_i\{Z_n = j,\ T_n \leq t\} \quad i,j \in E_\Delta,\ t \in \mathbb{R}_+,\ n \in \mathbb{N},$$

Then

$$Q^o(i,j,t) = I(i,j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

and

$$Q^1 = Q \text{ by definition.}$$

Furthermore, a renewal argument shows that

$$Q^{n+1}(i,k,t) = \sum_{j \in E} \int_o^t Q(i,j,du)\, Q^n(j,k,t-u)$$

and the expected number of visits to state j, by the process Z until time t, given that the initial state is i, can be computed by

$$R(i,j,t) = E_i \left[\sum_{n=0}^{\infty} I_{\{Z_n = j,\ T_n \leq t\}} \right] = \sum_{n=0}^{\infty} Q^n(i,j,t)$$

where R is called the Markov renewal kernel corresponding to Q. Since, in our model, each state is visited at most once by the process X, the number of visits to any state j, from any initial state i, is either zero or one. Hence, the expected number of visits from an initial state i, to any state j, during the time interval $[0,t]$, is nothing but the cumulative probability distribution of the entrance time to state j, starting from state i. If we let

$$T^j = \inf\{t \geq 0 : X_t = j\}$$

denote the time of entrance to state j, we can write

$$R(i,j,t) = P\{T^j \leq t \mid Z_o = i\} = P_i\{T^j \leq t\}$$

The state of the deteriorating system at any time cannot be determined by simple observation. However, information about the state of the system can be obtained by carrying out some tests, and the results of the tests are only probabilistically related to the true state of the system. The results of a test is either positive (+) or negative (-), where + shows that the deterioration level is different from zero, that is, the system is in $E_\Delta \setminus \{0\}$.

Associated with any test time t, we define

$$Y_t = \begin{cases} 1 \text{ if test at time t is positive} \\ 0 \text{ if test at time t is negative} \end{cases}$$

We assume that the probabilistic relation between the true state of the system and the test results can be determined from statistical data, so that

$$P\{Y_t = 1 \mid X_t = i\} = p_i \quad i \in E, \ t \geq 0$$

are known beforehand. Note that p_o is the probability of a false positive, and p_i, $i \in E \setminus \{0\}$ is the probability of detecting the deterioration when in state $i \geq 1$.

Note that the detection probabilities may be taken to increase as the deterioration level increases which implies

$$0 \leq p_o \leq p_1 \leq \ldots \leq p_\Delta \leq 1$$

This is a reasonable assumption since one cannot always expect a test to detect a disorder which has just started its development, while a disorder which has progressed for a long time is more likely to be detected by the same test. This is also consistent with Eddy's[22,23] and Shwartz's[27] models where the true positive probabilities vary as the disease progresses.

It is also assumed that a false positive outcome does not interrupt the sequence of tests. Each time a positive test result is obtained, a more elaborate test is carried so as to detect false positive results. Hence, inspection and testing is stopped whenever all inspection times scheduled are exhausted or when a true positive detection is made, whichever occurs first.

2.2. DETERMINISTIC INSPECTION POLICIES

The deteriorating system as described by the deterioration process X is such that the underlying state is not directly observable; so the system has to be inspected in order to obtain information about its true state.

The inspection policies to be analyzed in this thesis will have considerable generality. They will permit one to realistically model many new and interesting real-life problems. They will be appropriate for health screening problems as well as maintenance applications, and for systems degrading in time as well as for ones that abruptly fail. However, most of the examples will be related to cancer screening problems in order to provide interesting and appropriate motivations.

An inspection policy is an increasing sequence of inspection times where the inspector has to apply some tests. These inspection times and the maximum number of inspections that have to be performed are predetermined by the inspector, after considering historical data. For example, the decision of a physician upon a particular inspection policy depends on some specific conditions like age, sex, heredity, environment, on the results of some preliminary tests and on statistical data.

To give a formal and mathematical definition of a deterministic inspection policy as such, let

$$\tau_n = \{(t_1,\ldots,t_n) : 0 \leqq t_1 \leqq \ldots \leqq t_n,\ t_i \in \mathbb{R}^+,\ i=1,\ldots,n\}\ n \geqq 1$$

be the set of all n-tuples with positive valued increasing coordinates and let $\tau = \bigcup_{n=1}^{\infty} \tau_n$ be their union. So any element $\mathbf{t} = (t_1,\ldots,t_n)$ of τ is an inspection policy or a schedule since it belongs to one and only one of the τ_n's where n is the maximum number of inspections that can be performed and the coordinates of the n-tuple represent the corresponding inspection times. In other words t_i is the time of the i'th inspection. For notational convenience we will represent an inspection policy or a schedule $\mathbf{t} = (t_1 \ldots, t_n) \in \tau$ by a right-continuous step function s(.) which increases by jumps of size one only defined on $\mathbb{R}_+$ by

$$\begin{aligned} &s(0) = 0 \\ &s(t) = \sup \{n: t_n \leqq t\},\ t \geqq 0 \end{aligned} \tag{2.1}$$

Note that s(t) is the total number of tests scheduled until time t. We also define $\bar{s} = \sup_t s(t) = \lim_{t\to\infty} s(t)$ as the total number of tests scheduled. It can easily be established that any element $\mathbf{t} = (t_1,\ldots,t_n) \in \tau$ can represent a step function s(.) as described in (2.1). Similarly, if $\mathbf{S}$ the set of all right-continuous positive valued step functions s(.) on $\mathbb{R}_+$ and increasing by jumps of size one only with s(0) = 0, then for any $s \in \mathbf{S}$, we can find the corresponding element $\mathbf{t} = (t_1,\ldots,t_n) \in \tau$ by defining recursively

$$\begin{aligned} &t_o = 0 \\ &t_{k+1} = \inf\{t>t_k: s(t) \neq s(t-)\},\ k < s\ (\infty) \end{aligned}$$

So, in the following chapters a deterministic inspection policy will be represented by a step function s as described above and the necessary calculations will be carried out accordingly.

2.3. STOCHASTIC INSPECTION PROCESSES

In general, the examination or inspection schedule for the deteriorating system is prepared deterministically by the inspector as explained in the previous section. However, in some cases the inspection times may be random so that the patient decides on the time of the inspections in some probabilistic way possibly depending on many social, psychological and environmental factors.

Let V_1, V_2, V_3,... be random variables representing the times of arrivals in an arrival process $S = \{S_t: t \geq 0\}$ where S_t is the number of arrivals until time t. The stochastic process S is called an inspection process where V_1, V_2, V_3, ... correspond to random inspection times. The stochastic structure of S may depend on the state of deterioration and on time so that the conditional distribution is given by

$$P\{S_{t+u}-S_t = k \mid X_t = X_{t+u} = j\} = h(j,t,u,k) \qquad (3.1)$$

where h is a distribution function in k for fixed j, t and u. As it can be seen from (3.1) the distribution of the process S depends on the state of deterioration j, and on the time t where the dependence on t shows the non-stationarity of the process.

If we assume the function h(j,t,u,k) takes on the following form

$$h(j,t,u,k) = \frac{e^{-b(j,t,u)}(b(j,t,u))^k}{k!}$$

then the random inspection times constitute a possibly non-stationary Poisson process with an expectation function depending on the state of deterioration.

The following special cases where

$$b(j,t,u) = \lambda u \tag{3.2}$$

$$b(j,t,u) = b(t,u) \tag{3.3}$$

$$b(j,t,u) = \lambda(j)u \tag{3.4}$$

can be considered. Note that (3.2) and (3.3) correspond to stationary and non-stationary Poisson processes respectively. Non-stationary inspections are more realistic than stationary inspections, especially when we think in terms of medical applications. The random visits of an individual to a physician may depend on so many different factors that the rate of these visits cannot be assumed constant. For example, a perfectly healthy looking woman, 40 to 45 years old, may want to have a breast cancer examination (mammogram for example), and starting from that time may continue to have screening examinations at randomly chosen times depending on many factors such as her psychological state or her environment.

The case where the rate of inspections depends only on the state of the system as in (3.4) is also realistic since it is logical to assume that a person who does not feel well will visit the physician more frequently than a person who is in perfect health.

2.4. MEASURES OF EFFECTIVENESS TO EVALUATE INSPECTION POLICIES AND PROCESSES

Having specified the model, the next step is to compute various measures of interest so that deterministic inspection policies and stochastic inspection processes could be evaluated.

We consider T^{Δ}, the time of entrance to the failing state, so that

$$T^{\Delta} = \inf \{t \geq 0: X_t = \Delta\}$$

and we let D be the minimum of T^{Δ} and the time of the first true positive test result, that is

$$D = \inf \{t_i \in \mathbf{t}: X_{t_i} > 0, Y_{t_i} = 1\} \wedge T^{\Delta}$$

where $\mathbf{t} = (t_1, \ldots, t_n)$ is an inspection policy.

An important evaluation criterion for any inspection policy or process is $P_i\{D<T^{\Delta}\}$ since this is the probability that the testing strategy will detect deterioration before catastrophic failure given that the initial state is i. More-

over, $P_i\{X_D = j\}$, $j \geq i$ is also an important measure since it yields the distribution of the state of the system at the time of detection given that the initial state is i.

The probability distribution of the earliness of the detection of the deterioration, given that the initial state is i, $P_i\{T^\Delta - D \leq t\}$, or the expected value of the earliness, given that the initial state is i, $E_i[T^\Delta - D]$, are also important evaluation criteria for inspection policies or processes.

Finally, the last measures of effectiveness to be mentioned are on the economics of a strategy. They concern the expected number of tests performed and are denoted by $E_i[N]$, $E_i[N_+]$, $E_i[N_-]$, $E_i[N_{F_+}]$, $E_i[N_{F_-}]$, $E_i[N_{T_+}]$, $E_i[N_{T_-}]$ where N, N_+, N_-, N_{F_+}, N_{F_-}, N_{T_+}, N_{T_-}, denote the total number of tests performed, the number of tests with positive, negative, false positive, false negative, true positive, true negative results respectively. We also note that the expected number of tests with true positive results is nothing but the probability that a true positive detection is made before catastrophic failure so that

$$E_i[N_{T_+}] = P_i\{D < T^\Delta\}, \quad i \in E.$$

In the following chapters, some of the measures of effectiveness mentioned above will be computed for deterministic inspection policies and stochastic inspection processes.

III. ANALYSIS OF DETERMINISTIC INSPECTION POLICIES

In this chapter, deterministic inspection policies will be considered and computational formulas on the total number of tests performed, the number of tests with positive, negative, true positive, true negative, false positive and false negative results will be derived.

3.1. EXPECTED NUMBER OF TESTS

Let s be a deterministic inspection policy so that the maximum number of tests $\bar{s}$ that can be carried out and the times when they have to be performed, are known with certainty. Since testing stops when a true positive detection is made, the total number of tests performed will be less than or equal to $\bar{s}$; in fact if the initial state is zero, and all the tests are scheduled before T_1, time of the first jump, then exactly $\bar{s}$ tests will have to be performed.

We shall first compute f(j,m), the expected number of tests during a sojourn in state j, given that the total number of tests scheduled during that sojourn is m. Note that f(j,0)=0 for all j, and f(Δ,m)=0 for all m since no tests are performed once state Δ is entered. For $m \geq 1$

$$f(j,m) = E\left[N_{T_{n+1}} - N_{T_n} \mid Z_n = j,\ s(T_{n+1}) - s(T_n) = m\right]$$

where N_t denotes the total number of tests performed until time t. We can easily see that, for a non-zero state $j>0$

$$P\{N_{T_{n+1}} - N_{T_n} = k \mid Z_n = j,\ s(T_{n+1}) - s(T_n) = m\} = p_j(1-p_j)^{k-1}$$

for $k = 1,\ldots,m-1$, and

$$P\{N_{T_{n+1}} - N_{T_n} = m \mid Z_n = j,\ s(T_{n+1}) - s(T_n) = m\} = (1-p_j)^{m-1}$$

Then

$$E\left[N_{T_{n+1}} - N_{T_n} \mid Z_n = j,\ s(T_{n+1}) - s(T_n) = m\right] = \sum_{k=1}^{m-1} k p_j (1-p_j)^{k-1} + m(1-p_j)^{m-1}$$

Letting $1-p_j = q_j$, we obtain

$$E\left[N_{T_{n+1}} - N_{T_n} \mid Z_n = j,\ s(T_{n+1}) - s(T_n) = m\right] = p_j \sum_{k=1}^{m-1} q_j^{k-1} + m q_j^{m-1} = (1-q_j^m)/p_j$$

so that $f(j,m) = (1-q_j^m)/p_j$ for $j > 0$.

If the initial state is zero, then all the tests scheduled for the sojourn in that state have to be performed because of the fact that a false positive result does not stop the testing procedure; so that

$$f(0,m) = m.$$

Hence

$$f(j,m) = \begin{cases} (1-q_j^m)/p_j & \text{if} \quad j > 0 \\ m & \text{if} \quad j = 0 \end{cases} \tag{1.1}$$

Let N be the total number of tests performed, then N can be written as

$$N = \sum_{n=0}^{\infty} (N_{T_{n+1}} - N_{T_n})$$

so that, for all $i \in E$

$$E_i[N] = \sum_{n=0}^{\infty} E_i[N_{T_{n+1}} - N_{T_n}] \qquad (1.2)$$

where

$$E_i[N_{T_{n+1}} - N_{T_n}] = E_i[E_i[N_{T_{n+1}} - N_{T_n} | T_n, T_{n+1}, Z_n]]$$

$$= E_i [f(Z_n, s(T_{n+1}) - s(T_n))]$$

$$= \sum_{j \in E} \int_o^{\infty} Q^n(i,j,dz) \int_o^{\infty} F_j(du) f(j, s(z+u) - s(z))$$

where

$$F_j(u) = P_j \{T_1 \leq u\} = \sum_{k \in E_\Delta} Q(j,k,u) \quad \text{for} \quad j \in E.$$

Now, from (1.2)

$$E_i[N] = \sum_{n=0}^{\infty} \sum_{j \in E} \int_o^{\infty} Q^n(i,j,dz) \int_o^{\infty} F_j(du) f(j, s(z+u) - s(z))$$

But since $\sum_{n=0}^{\infty} Q^n(i,j,z) = R(i,j,z)$ we finally obtain

$$E_i[N] = \sum_{j \in E} \int_o^{\infty} R(i,j,dz) \int_o^{\infty} F_j(du) f(j, s(z+u) - s(z)) \qquad (1.3)$$

for any $i \in E$.

3.2. EXPECTED NUMBER OF TESTS WITH POSITIVE RESULTS

We shall first compute $f_+(j,m)$ defined to be the expected number of tests with positive results, during a sojourn in state j, given that the total number of tests scheduled during that sojourn is m. Note that $f_+(j,0) = 0$ for all j, and $f_+(\Delta,m) = 0$ for all m. For $m \geq 1$

$$f_+(j,m) = E\left[N^+_{T_{n+1}} - N^+_{T_n} \mid Z_n = j,\ s(T_{n+1}) - s(T_n) = m\right]$$

where N^+_t denotes the total number of tests with positive results, performed until time t. For a non-zero state $j > 0$, we can obtain either one positive result or none, so that

$$P\{N^+_{T_{n+1}} - N^+_{T_n} = 0 \mid Z_n = j,\ s(T_{n+1}) - s(T_n) = m\} = (1-p_j)^m = q_j^m$$

and

$$P\{N^+_{T_{n+1}} - N^+_{T_n} = 1 \mid Z_n = j,\ s(T_{n+1}) - s(T_n) = m\} = 1 - q_j^m$$

Then

$$E\left[N^+_{T_{n+1}} - N^+_{T_n} \mid Z_n = j,\ s(T_{n+1}) - s(T_n) = m\right] = 1-q_j^m$$

so that $f_+(j,m) = 1-q_j^m = p_j f(j,m)$ for $j>0$.

If the initial state is zero, then all the tests scheduled for the sojourn in that state have to be performed and a positive result is obtained with probability p_o. Hence, this becomes a Bernouilli process with success probability p_o so that

$$f_+(0,m) = p_o m = p_o f(0,m)$$

Hence

$$f_+(j,m) = p_j f(j,m) = \begin{cases} 1-q_j^m & \text{if} \quad j>0 \\ p_o m & \text{if} \quad j=0 \end{cases} \tag{2.1}$$

If N_+ is the total number of tests performed with positive results then

$$N_+ = \sum_{n=0}^{\infty} (N^+_{T_{n+1}} - N^+_{T_n})$$

and carrying out the same calculations as in the previous section we obtain

$$E_i[N_+] = \sum_{j\in E} \int_o^{\infty} R(i,j,dz) \int_o^{\infty} F_j(du) f_+(j,s(z+u)-s(z)) \tag{2.2}$$

for any $i\in E$.

Note that if the initial state i is non-zero, then the expected number of tests with positive results, $E_i[N_+]$, and the expected number of tests with true positive results, $E_i[N_{T+}]$ are the same, since a positive result obtained in a non-zero state is in fact a true positive result. Hence $f_{T+}(j,m)$, the expected number of tests with true positive results, during a sojourn in a state j, given that the total number of tests scheduled during that sojourn is m, is equal to $f_+(j,m)$ for $j > 0$, and $f_{T+}(0,m) = 0$. So

$$f_{T_+}(j,m) = \begin{cases} 1-q_j^m & \text{if} \quad j>0 \\ 0 & \text{if} \quad j=0 \end{cases} \tag{2.3}$$

As a result, $E_i[N_{T_+}] = E_i[N_+]$ for $i > 0$ and using (2.3) we can write

$$E_i[N_{T_+}] = \sum_{j\in E} \int_o^\infty R(i,j,dz) \int_o^\infty F_j(du) f_{T_+}(j,s(z+u) - s(z)) \tag{2.4}$$

for any $i \in E$.

Note that false positive results are only obtained if the initial state is zero. So, letting $f_{F_+}(j,m)$ be the expected number of tests with false positive results, during a sojourn in state j, given that the total number of tests scheduled for that state is m, we can easily deduce that

$$f_{F_+}(j,m) = \begin{cases} 0 & \text{if} \quad j>0 \\ f^+(j,m) & \text{if} \quad j=0 \end{cases}$$

so that

$$f_{F_+}(j,m) = \begin{cases} 0 & \text{if} \quad j>0 \\ P_o m & \text{if} \quad j=0 \end{cases} \tag{2.5}$$

Clearly $E_i[N_{F_+}] = 0$ if $i > 0$ and using $f_{F_+}(j,m)$, we can write

$$E_i[N_{F_+}] = \sum_{j\in E} \int_o^\infty R(i,j,dz) \int_o^\infty F_j(du) f_{F_+}(j,s(z+u)-s(z)) \tag{2.6}$$

for any $i \in E$.

3.3. EXPECTED NUMBER OF TESTS WITH NEGATIVE RESULTS

If the initial state i is non-zero then the expected number of tests with negative results, $E_i[N_-]$, and the expected number of tests with false negative results, $E_i[N_{F-}]$ are the same, since a negative result obtained in a non-zero state is in fact a false negative result. Moreover, since the only true negative results are obtained in the interval $[0, T_1]$ when the initial state is zero, then

$$E_i[N_{T-}] = 0 \quad \text{for} \quad i \neq 0$$

We shall first compute as usual, $f_-(j,m)$, $f_{F-}(j,m)$, $f_{T-}(j,m)$ the expected number of tests with negative, false negative, true negative results respectively, during a sojourn in state j, given that the total number of tests scheduled during that sojourn is m. We can easily see that, for a non-zero state j>0

$$P\{N^-_{T_{n+1}} - N^-_{T_n} = k | Z_n = j,\ s(T_{n+1}) - s(T_n) = m\} = p_j q_j^k$$

for k=1,...,m-1, and

$$P\{N^-_{T_{n+1}} - N^-_{T_n} = m | Z_n = j,\ s(T_{n+1}) - s(T_n) = m\} = (1-p_j)^m = q_j^m$$

where N^-_t denotes the total number of tests with negative results performed until time t.

Then

$$E\left[N^-_{T_{n+1}} - N^-_{T_n} \middle| Z_n=j,\ s(T_{n+1}) - s(T_n) = m\right] = p_j \sum_{k=1}^{m-1} kq_j^k + mq_j^m$$

$$= (q_j/p_j)(1-q_j^m)$$

so that $f_-(j,m) = (q_j/p_j)(1-q_j^m) = q_j f(j,m)$ for $j>0$, $j \in E$.

Clearly, $f_{F-}(j,m) = f_-(j,m)$ if $j>0$,

and

$$f_{F_-}(j,m) = 0 \qquad \text{if} \quad j=0 .$$

If the initial state is zero, all of the tests scheduled for the sojourn in that state have to be performed and a negative result is obtained with probability q_o. Hence

$$f_-(0,m) = q_o m = q_o f(0,m)$$

Clearly $f_{T_-}(0,m) = f_-(0,m)$

and

$$f_{T_-}(j,m) = 0 \quad \text{for} \quad j>0 .$$

Hence, we have shown that

$$f_-(j,m) = q_j f(j,m) = \begin{cases} (q_j/p_j)(1-q_j^m) & \text{if} \quad j>0 \\ q_o m & \text{if} \quad j=0 \end{cases} \tag{3.1}$$

$$f_{F_-}(j,m) = \begin{cases} (q_j/p_j)(1-q_j^m) & \text{if} \quad j>0 \\ 0 & \text{if} \quad j=0 \end{cases} \tag{3.2}$$

$$f_{T_-}(j,m) = \begin{cases} 0 & \text{if} \quad j>0 \\ q_o{}^m & \text{if} \quad j=0 \end{cases} \tag{3.3}$$

Then by similar calculations as in the previous sections we obtain

$$E_i[N_-] = \sum_{j \in E} \int_o^\infty R(i,j,dz) \int_o^\infty F_j(du) f_-(j,s(z+u)-s(z)) \tag{3.4}$$

$$E_i[N_{F_-}] = \sum_{j \in E} \int_o^\infty R(i,j,dz) \int_o^\infty F_j(du) f_{F_-}(j,s(z+u)-s(z)) \tag{3.5}$$

$$E_i[N_{T_-}] = \sum_{j \in E} \int_o^\infty R(i,j,dz) \int_o^\infty F_j(du) f_{T_-}(j,s(z+u)-s(z)) \tag{3.6}$$

for all $i \in E$.

Note that, for notational simplicity, the results in (1.3), (2.2), (2.4), (2.6), (3.4), (3.5), (3.6) can be combined in a single identity as

$$E_i[N_k] = \sum_{j \in E} \int_o^\infty R(i,j,dz) \int_o^\infty F_j(du) f_k(j,s(z+u) - s(z)) \tag{3.7}$$

where $k \in \{\infty, +, T_+, F_+, -, T_-, F_-\}$ with

$N_\infty = N$ and $f_\infty(j,m) = f(j,m)$

So, as it can be seen in (3.7) all the previous results can be summarized in a compact form. Note that the results obtained in this chapter have considerable importance from an economical point of view, since each test performed has a particular cost.

IV. ANALYSIS OF STOCHASTIC INSPECTION PROCESSES

In this chapter, we will consider stochastic inspection processes in general and extend the results obtained for deterministic inspection policies to stochastic case. Since the steps in the derivation of some of the formulas are similar to the ones of Chapter III, only the important points will be emphasized. Special cases of stochastic inspection processes which are stationary and non-stationary Poisson processes will be considered.

4.1. EXPECTED NUMBER OF TESTS

Let $S = \{S_u : u \geq 0\}$ be a stochastic inspection process for which

$P\{S_{t+u} - S_t = k | X_t = X_{t+u} = j\} = h(j,t,u,k)$ where $h: E \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N} \rightarrow [0,1]$ is a distribution function in k for fixed j, t and u. Here S_t is the total number of tests performed until time t, and in medical applications for example, it represents the stochastic behavior of the visits of an individual to a physician.

We will first compute $f(j,t,u)$ or the expected number of tests performed during the time interval $[t,t+u)$ given that the deterioration level is j during the same time domain. Hence

$$f(j,t,u) = E\left[N_{t+u} - N_t \mid X_t = X_{t+u} = j\right]$$

From the previous chapter we know that

$$E\left[N_{t+u} - N_t \mid X_t = X_{t+u} = j,\ S\right] = \frac{1-q_j^{S_{t+u}-S_t}}{p_j}$$

for a non-zero state j, and hence

$$f(j,t,u) = E\left[N_{t+u} - N_t \mid X_t = X_{t+u} = j\right] \sum_{k=0}^{\infty} \frac{1-q_j^k}{p_j}\, h(j,t,u,k)$$

If $j=0$, then it is clear that

$$f(0,t,u) = \sum_{k=0}^{\infty} k\, h(0,t,u,k)$$

As a result,

$$f(j,t,u) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{1-q_j^k}{p_j}\, h(j,t,u,k) & \text{if} \quad j>0 \\ \displaystyle\sum_{k=0}^{\infty} k\, h(0,t,u,k) & \text{if} \quad j=0 \end{cases}$$

So, the expected number of tests performed, $E_i[N]$, for any initial state $i \in E$, can be computed as follows.

$$E_i[N] = \sum_{n=0}^{\infty} E_i[N_{T_{n+1}} - N_{T_n}]$$

$$= \sum_{n=0}^{\infty} E_i[E_i[N_{T_{n+1}} - N_{T_n} | Z_n]]$$

$$= \sum_{n=0}^{\infty} E_i[f(Z_n, T_n, T_{n+1} - T_n)]$$

$$= \sum_{n=0}^{\infty} \sum_{j\in E} \int_o^{\infty} Q^n(i,j,dt) \int_o^{\infty} F_j(du) f(j,t,u)$$

Hence

$$E_i[N] = \sum_{j\in E} \int_o^{\infty} R(i,j,dt) \int_o^{\infty} F_j(du) \; f(j,t,u) \qquad (1.1)$$

for any $i\in E$.

4.2. EXPECTED NUMBER OF TESTS WITH POSITIVE RESULTS

The analysis made in the preceeding section can be duplicated to obtain the expected number of tests with positive, true positive and false positive results. We shall not present proofs to avoid repetition.

Defining $f_+(j,t,u)$, $f_{T+}(j,t,u)$, and $f_{F+}(j,t,u)$ as the expected number of tests during $[t,t+u)$ with positive, true positive and false positive results respectively given that $X_t = X_{t+u} = j$ one can obtain

$$f_+(j,t,u) = p_j f(j,t,u)$$

$$f_{T_+}(j,t,u) = \begin{cases} f_+(j,t,u) & \text{if} \quad j>0 \\ 0 & \text{if} \quad j=0 \end{cases}$$

$$f_{F_+}(j,t,u) = \begin{cases} 0 & \text{if} \quad j>0 \\ f_+(0,t,u) & \text{if} \quad j=0 \end{cases}$$

and that

$$E_i[N_k] = \sum_{j\in E} \int_o^\infty R(i,j,dt) \int_o^\infty F_j(du) f_k(j,t,u) \qquad (2.1)$$

for $k\in\{+, T_+, F_+\}$

4.3. EXPECTED NUMBER OF TESTS WITH NEGATIVE RESULTS

To find the expected number of tests with negative, true negative and false negative results one can simply define

$$f_-(j,t,u) = q_j f(j,t,u)$$

$$f_{T_-}(j,t,u) = \begin{cases} 0 & \text{if} \quad j>0 \\ f_-(0,t,u) & \text{if} \quad j=0 \end{cases}$$

$$f_{F_-}(j,t,u) = \begin{cases} f_-(j,t,u) & \text{if} \quad j>0 \\ 0 & \text{if} \quad j=0 \end{cases}$$

and the end result for the case of stochastic inspection processes can be summarized as

$$E_i[N_k] = \sum_{j \in E} \int_0^\infty R(i,j,dt) \int_0^\infty F_j(du)\ f_k(j,t,u) \tag{3.1}$$

for all $k \in \{\infty, +, -, T_+, T_-, F_+, F_-\}$

where $N_\infty = N$ and $f_\infty(j,t,u) = f(j,t,u)$

4.4. SOME INSPECTION PROCESSES

An important and interesting inspection process can be obtained by assuming that S is a conditionally and possibly non-stationary Poisson process given the deterioration process X. This implies that h must be of the form

$$h(j,t,u,k) = \frac{e^{-b(j,t,u)}(b(j,t,u))^k}{k!}$$

In this special case the functions $f_k(j,t,u)$ for any $k \in \{\infty, +, -, T_+, T_-, F_+, F_-\}$ take on simple forms. For example,

$$f_\infty(j,t,u) = \begin{cases} \sum_{k=0}^{\infty} \left(\frac{1-q_j^k}{p_j}\right)\left(\frac{e^{-b(j,t,u)}(b(j,t,u))^k}{k!}\right) & \text{if } j>0 \\ \sum_{k=0}^{\infty} \frac{k\ e^{-b(0,t,u)}(b(0,t,u))^k}{k!} & \text{if } j=0 \end{cases}$$

So,

$$f_\infty(j,t,u) = \begin{cases} (1/p_j)(1-e^{-b(j,t,u)p_j}) & \text{if } j>0 \\ b(0,t,u) & \text{if } j=0 \end{cases} \tag{3.2}$$

Similar expressions can be obtained for the functions $f_k(j,t,u)$, $k \in \{+, -, T_+, T_-, F_+, F_-\}$.

Note that if $b(j,t,u) = b(t,u)$ independent of the state of deterioration j, then S is a non-stationary Poisson process. Similarly, if $b(j,t,u) = \lambda u$ independent of j and t, then the inspection process S is an ordinary Poisson process with rate λ. As an illustration, we present for this case the expected number of tests with true positive results.

So if $b(j,t,u) = \lambda u$ then

$$f_{T_+}(j,t,u) = \begin{cases} 1-e^{-\lambda p_j u} & \text{if} \quad j>0 \\ 0 & \text{if} \quad j=0 \end{cases}$$

Hence

$$E_i\left[N_{T_+}\right] = \sum_{j\in E} \int_0^\infty R(i,j,dt) \int_0^\infty F_j(du)(1-e^{-\lambda p_j u}) \quad \text{if} \quad i>0$$

and

$$E_o\left[N_{T_+}\right] = \sum_{j\in E} \int_0^\infty R(0,j,dt) \int_0^\infty F_j(du)(1-e^{-\lambda p_j u}) - \int_0^\infty F_o(du)(1-e^{-\lambda p_o u})$$

This concludes our analysis of inspection processes. Now, in Chapter V we will use some of the results obtained in this and in the previous chapter to evaluate certain policies or processes.

V. EVALUATION OF DETERMINISTIC INSPECTION POLICIES AND STOCHASTIC INSPECTION PROCESSES

In the preceeding chapters, various formulas were derived to compute expected number of tests to be performed for deterministic inspection policies and stochastic inspection processes. Of course, it is important to know how and where these results can be used. In this chapter, applications of the results obtained will be illustrated with many examples, and measures of effectiveness other than those on the number of tests will be considered. Numerical examples will be solved for some interesting cases. It should be mentioned that our aim in this thesis is to evaluate deterministic inspection policies and stochastic inspection processes and not to deal with the optimal inspection problem. Therefore we will only mention how our results can be used rather than work on a general optimization problem.

5.1. SOME IMPORTANT MEASURES OF EFFECTIVENESS

Perhaps the most important measure of effectiveness for inspection policies is the probability that a true positive detection is made before catastrophic failure. We have seen that this probability is nothing else but the expected number of tests with true positive results, or

$$P_i\{D<T^{\Delta}\} = E_i\left[N_{T_+}\right]$$

for any initial state $i \in E$. So, given the total number of tests that can be performed one may be interested to find the optimal schedule which maximizes $P_i\{D<T^\Delta\}$.

An economic model can be constructed where c_p is the cost of each test performed, c_f is the cost of each false positive result obtained and c_ν the cost of catastrophic failure. The objective here may be to minimize the total expected cost C, where

$$C = c_p E_i[N] + c_f E_i[N_{F_+}] + c_\nu (1 - E_i[N_{T_+}])$$

by choosing an inspection policy.

The probabilty distribution of the state of the system at the time of detection, the distribution function of the earliness of detection and the expected value of the earliness of detection denoted by

$$P_i\{X_D=j\}, \quad P_i\{T^\Delta - D \leq t\}, \quad E_i[T^\Delta - D]$$

respectively are other measures of effectiveness that have considerable importance since they concern the time of the true positive detection and the state of the system at that time.

5.2. MODELS WITH SINGLE INSPECTION

This section will be devoted to the analysis of the case where the inspector is limited with one inspection. In other words, he can perform only one test to obtain information about the underlying state of the system.

5.2.1. A General Model

If initially there is no deterioration or $X_o=0$, then scheduling the test too early will not be desirable since the chance to detect a possible deterioration will be lost. Moreover, even if the initial state is non-zero, scheduling the test at time zero, cannot guarantee that the disorder will certainly be detected since there is always a probability of obtaining false negative results.

To analyze the problem with single inspection at a time t, let s(u) be defined as:

$$s(u) = \begin{cases} 0 & u<t \\ 1 & u\geq t \end{cases} \qquad (2.1)$$

Hence, only one inspection is available at time t which determines our inspection policy.

We will first simplify the expression for the expected number of tests with true positive results when s(.) has the form given in (2.1). In Chapter III, $E_i[N_{T_+}]$ was found to have the following form:

$$E_i[N_{T_+}] = \sum_{j\in E} \int_o^\infty R(i,j,dz) \int_o^\infty F_j(du)\ f_{T_+}(j,s(z+u)-s(z))$$

where

$$f_{T_+}(j,m) = \begin{cases} 1-q_j^m & j>0 \\ 0 & j=0 \end{cases}$$

Note that $s(z+u)-s(z)=1$ if and only if $z\in(0,t)$ and $u\in(t-z,\infty)$. So,

$$h(i,t) = E_i\left[N_{T_+}\right] = \sum_{j\in E} \int_o^t R(i,j,dz) \int_{t-z}^{\infty} f_{T_+}(j,1)F_j(du)$$

$$= \sum_{j\in E} \int_o^t R(i,j,dz)(1-q_j)(1-F_j(t-z))$$

$$= \sum_{j\in E} \int_o^t R(i,j,dz)p_j\bar{F}_j(t-z) \qquad (2.2)$$

where $\bar{F}_j(u) = 1-F_j(u)$.

The expression for $E_o\left[N_{T_+}\right]$ will be different since $f_{T_+}(0,n)=0$. One can easily see that

$$h(0,t)=E_o\left[N_{T_+}\right] = \sum_{j\in E_o} \int_o^t R(0,j,dz)p_j\bar{F}_j(t-z) \qquad (2.3)$$

where $E_o=E\setminus\{0\}$.

Now letting $R_{ij}(u) = R(i,j,u)$

and assuming R_{ij} and F_j are differentiable with respective derivatives r_{ij} and f_j we can find a necessary condition which has to be satisfied by t to maximize $P_i\{D<T^{\Delta}\}$ by differentiating h(i,t) with respect to t.

$$dh(i,t)/dt = \sum_{j\in E} p_j r_{ij}(t) - \sum_{j\in E} \int_o^t p_j\ R(i,j,dz)f_j(t-z)$$

for any initial state i>0.

Therefore setting this expression equal to zero we obtain

$$\sum_{j\in E} p_j r_{ij}(t^*) = \sum_{j\in E} \int_o^{t^*} p_j\ R(i,j,dz) f_j(t^*-z)$$ where t^* denotes the value of t which maximizes $P_i\{D<T^\Delta\}$

Since $r_{ij}(t) = 0$ for $j\leq i$, we let $p_j^! = p_j / \sum_{k>i} p_k, j>i$

Clearly

$\sum_{j>i} p_j^! = 1$, and we get

$$\sum_{j>i} p_j^! r_{ij}(t^*) = \sum_{j>i} \int_o^{t^*} p_j^!\ R(i,j,dz) f_j(t^*-z) \qquad (2.4)$$

Now, $r_{ij}(t)$ can be interpreted as the probability that the time of entrance to state j is t if the initial state is i, or

$$r_{ij}(t) \overset{\Delta}{=} P_i\{T^j=t\}$$

and $\int_o^t R(i,j,dz)\ f_j(t-z)$ can be interpreted as the probability that the time of exit from state j is t given that the initial state is i. Let U^j be the time of exit from state j, then

$$\int_o^t R(i,j,dz) f_j(t-z) \overset{\Delta}{=} P_i\{U^j=t\}$$

and (2.4) can be written as

$$\sum_{j\in E_i} p_j^!\ P_i\{T^j=t^*\} = \sum_{j\in E_i} p_j^!\ P_i\{U^j=t^*\} \qquad (2.5)$$

for any non-zero initial state i, where $E_i=E\setminus\{0,1,...,i\}$.

Clearly (2.5) is a mixture of probability density functions. So, the necessary conditions to be satisfied by t* can be written as

$$P_i\{T^K=t^*\} = P_i\{U^K=t^*\} \tag{2.6}$$

where K is a random variable satisfying

$$P\{K=j\} = p_j' \qquad j \in E_i .$$

Note that the expression obtained in (2.6) will also be the same if the initial state is zero.

Roughly speaking (2.6) implies that the inspector has to choose a random state K and schedule the test at a time where the system is in that state. Hence, a necessary condition for a time t* to be optimal, is that it has to be a time where the system should be in that random state K.

This optimization problem requires further research since one needs assumptions on the structure of $F_j(.)$ and $R_{ij}(.)$ to make characterizations on t*. We will study a special case in section 5.2.2 where we will investigate the conditions necessary to make (2.6) a sufficient condition of optimality.

In the general model with single inspection the expressions for $E_i[N]$, $E_i[N_+]$, $E_i[N_-]$, $E_i[N_{F_+}]$, $E_i[N_{F_-}]$, $E_i[N_{T_-}]$ can be simplified as

$$E_i[N] = \sum_{j\in E} \int_0^t R(i,j,dz)\ \bar{F}_j(t-z)$$

$$E_i[N_+] = \sum_{j\in E} \int_0^t p_j\ R(i,j,dz)\ \bar{F}_j(t-z)$$

$$E_i[N_-] = \sum_{j\in E} \int_0^t q_j\ R(i,j,dz)\ \bar{F}_j(t-z)$$

$$E_i[N_{F_+}] = \begin{cases} 0 & i>0 \\ p_o\ \bar{F}_o(t) & i=0 \end{cases}$$

$$E_i[N_{F_-}] = \begin{cases} \sum_{j\in E} \int_0^t q_j\ R(i,j,dz)\ \bar{F}_j(t-z) & i>0 \\ \sum_{j\in E_o} \int_0^t q_j\ R(0,j,dz)\ \bar{F}_j(t-z) & i=0 \end{cases}$$

$$E_i[N_{T_-}] = \begin{cases} 0 & i>0 \\ q_o \bar{F}_o(t) & i=0 \end{cases}$$

An optimization problem can be modeled to minimize the total expected cost. Recall that c_p is the cost of performing a test, c_f the cost of obtaining a false positive and c_ν the cost of catastrophic failure, with $c_p \leqq c_f \leqq c_\nu$. Then the total expected cost c(i,t) where i is the initial state and t the time of the single inspection, has the following form.

$$c(i,t) = \begin{cases} c_p E_i[N] + c_\nu(1-E_i[N_{T_+}]) & , \quad i>0 \\ c_p E_o[N] + c_f E_o[N_{F_+}] + c_\nu(1-E_o[N_{T_+}]), & \quad i=0 \end{cases} \tag{2.7}$$

By differentiating c(i,t) with respect to t and setting the derivative equal to zero we obtain the necessary condition to be satisfied by optimal t*.

$$dc(i,t)/dt = \sum_{j \in E} (c_p - c_\nu p_j) r_{ij}(t) - \sum_{j \in E} (c_p - c_\nu p_j) \int_0^t R(i,j,dz) f_j(t-z)$$

so that dc(i,t)/dt = 0 implies

$$\sum_{j \in E_i} (c_p - c_\nu p_j) P_i\{T^j = t^*\} = \sum_{j \in E_i} (c_p - c_\nu p_j) P_i\{U^j = t^*\} \quad (2.8)$$

for $i \in E_o$.

For i=0, a similar analysis yields the following necessary condition to be satisfied by t* which minimizes the total expected cost.

$$\sum_{j \in E_o} (c_p - c_\nu p_j) P_o\{T^j = t^*\} = \sum_{j \in E_o} (c_p - c_\nu p_j) P_o\{U^j = t^*\} \quad (2.9)$$

The similarity between (2.5) and (2.8) should be pointed out. Note that if $c_p = 0$ then (2.5) and (2.8) are the same. Otherwise, assuming $c_p < c_\nu p_j$, for all $j \in E$ we can take

$$p_j' = (c_\nu p_j - c_p) / \sum_{j \in E_i} (c_\nu p_j - c_p)$$

so that (2.8) reduces to (2.5). The assumption $c_p < c_\nu p_j$ for all $j \in E_o$ is a logical assumption since it implies that the cost of catastrophic failure c_ν is so great that it remains bigger than c_p, the cost of performing a test, even when it is multiplied by p_j the probability of getting a true positive result.

Note also that, c_f, the cost of a false positive result appears only in (2.9) which gives the condition to be satisfied by t*, if the initial state is zero, since one cannot talk of a false positive outcome for a non-zero initial state.

5.2.2. A Simplified Model

In this subsection we consider a simplified model where $E_\Delta = \{0,1,\Delta\}$ with

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

A typical sample path of the process X is given in Figure 1 where

$$T_1 = \inf\{t \geq 0 : X_t = 1\} \ , \quad T_\Delta = \inf\{t \geq 0 : X_t = \Delta\}$$

are the first passage time to states 1 and Δ respectively. It follows that

$T_1 = T^1$ and $T_\Delta = U^1$

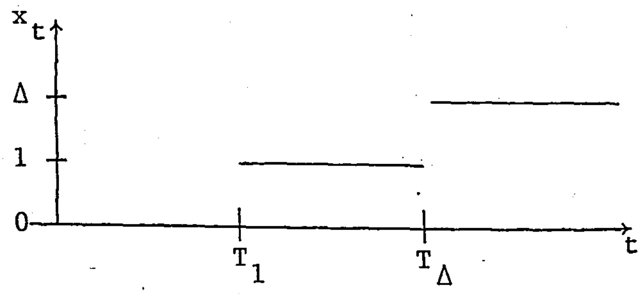

FIGURE 2.1. A Typical Sample Path of X.

This model is similar to the one examined by Eddy[22,23] with state 0 being the healthy state, state 1 the defective state and state Δ the state where the sickness becomes evident. From the transition matrix P of the process, we can see that only state 1 can be reached from state 0, and state Δ is to be reached from state 1.

We let

$$F(t) = R(0,1,t) = P_o\{T^1 \leqq t\}$$

$$G(t) = R(1,\Delta,t) = P_1\{T^\Delta \leqq t\}$$

be the distribution function of T^1 and T^Δ respectively. If we let t be the time of inspection, then (2.3) reduces to

$$E_o[N_{T_+}] = \int_o^t p_1 \; R(0,1,dz)(1-G(t-z)) = p_1 F*(1-G)(t) \qquad (2.10)$$

Now, assuming that F is differentiable with derivative f, (2.10) implies that the necessary condition of optimality is

$$f(t^*) = \int_o^{t^*} f(u)g(t^*-u)du \tag{2.11}$$

which can be interpreted as

$$P\{T_1=t^*\}=P\{T_\Delta=t^*\}$$

and this is a special case of (2.6).

We next present a theorem which gives the properties to be satisfied by f and G so that (2.11) becomes a sufficient condition of optimality.

THEOREM (2.1). If f is a unimodal differentiable function and G has increasing failure rate (IFR), then F*(1-G) is unimodal.

Proof. By the differentiability of f we can write

$$d(F^*(1-G)(t)/dt) = \int_o^t f'(u)(1-G(t-u))du$$

$$= \int_o^t f'(u)K(t,u)du$$

where $K(t,u) = \begin{cases} 1-G(t-u) & u \leqq t \\ 0 & u>t \end{cases}$

K is totally positive of order 2(TP_2) since G is IFR by our assumption. Moreover, unimodality of f implies that f' changes sign from plus to minus so that by the variation diminishing property (F*(1-G))' also changes sign from plus to minus. This clearly implies that F*(1-G) is unimodal and that F*(1-G)'=0 has a maximum value.

Therefore, under the assumptions of the theorem, (2.11) is both necessary and sufficient for optimality.

As a numerical applictaion if F and G both have the exponential distribution

$$F(t) = 1-e^{-at} \quad , \qquad t \geq 0$$

$$G(t) = 1-e^{-bt} \quad , \qquad t \geq 0$$

with rates a>0 and b>0 respectively, then optimal t* can be obtained by solving (2.11) as

$$t^* = (1/a-b)\ \ell n(a/b) \tag{2.12}$$

for $a \neq b$, and if $a = b$ then,

$$\lim_{b \to a} (1/a-b)\ \ell n(a/b) = 1/a$$

so that $t^* = 1/a$ which is the expected time of the first jump or the expected time at which the disorder starts to develop.

The computation of the probability distribution and mean of the earliness of detection can easily be made for this simplified model. We first compute $P_o\{T^{\Delta}-D \leq z,\ D=t\}$ where t is the time of the scheduled inspection.

$$P_o\{T^\Delta - D \leqq z,\ D = t\} = P_o\{T^\Delta - D \leqq z \mid D=t\}P_o\{D=t\}$$

$$= P_o\{T^\Delta \leqq z+t \mid D=t\}P_o\{D=t\}$$

$$= P_o\{T^\Delta \leqq z+t \mid X_t=1,\ Y_t=1\}P_o\{X_t=1, Y_t=1\}$$

$$= p_1\ P_o\{T^\Delta \leqq z+t,\ T_1 \leqq t < T^\Delta\}$$

$$= p_1\ P_o\{T_1 \leqq t,\ t < T^\Delta \leqq z+t\}$$

$$= p_1 \int_o^t F(du)(G(z+t-u)-G(t-u))$$

It is clear that

$$P_o\{T^\Delta - D \leqq z\} = P_o\{T^\Delta - D \leqq z, D=t\} + P_o\{T^\Delta - D \leqq z, D=T^\Delta\}$$

and since

$$P_o\{T^\Delta - D \leqq z, D=T^\Delta\} = P_o\{T^\Delta - D \leqq z \mid D=T^\Delta\}P_o\{D=T^\Delta\}$$

$$= P_o\{D=T^\Delta\} = 1-P_o\{D=t\} = 1-p_1\ F*(1-G)(t)$$

we obtain

$$P_o\{T^\Delta - D \leqq z\} = p_1 \int_o^t F(du)G(z+t-u) - p_1\ F*G(t) + 1 - p_1\ F*(1-G)(t)$$

$$= 1-p_1\ F(t) + p_1 \int_o^t F(du)G(z+t-u)$$

$$= 1-p_1 \int_o^t F(du)(1-G(z+t-u))$$

Now, the expected earliness can easily be computed as

$$E_o\left[T^\Delta - D\right] = \int_o^\infty P_o\{T^\Delta - D > z\}dz$$

$$= p_1 \int_o^t F(du) \int_o^\infty (1-G(t+z-u))dz \qquad (2.13)$$

Another objective in the optimal inspection problem may be to find an optimal t which maximizes $E_o\left[T^\Delta - D\right]$. Differentiating (2.13) with respect to t, and setting the derivative equal to zero we obtain

$$f(t^*) = (F^*(1-G)(t^*))/m \qquad (2.14)$$

where $m = \int_o^\infty (1-G(t))dt = E\left[T_2 - T_1\right]$, so (2.14) is the necessary condition of optimality to be satisfied by t* which maximizes $E_o\left[T^\Delta - D\right]$.

Note that although (2.11) and (2.14) are different, we obtain the same t* given by (2.12) when F and G are both exponential functions with rates a and b respectively.

5.2.3. An Extended Model

In this subsection we take $E_\Delta = \{0,1,2,\ldots,M-1, M\}$ where state $M \equiv \Delta$ is the terminal state, and $P(i,i+1) = 1$ so that the stages of deterioration increase deterministically to the adjacent state. Letting $T_1, T_2, \ldots, T_M$ represent the times of the jumps to states 1, 2, ..., M we assume that the interarrival durations $T_1, T_2-T_1, T_3-T_2, \ldots, T_M-T_{M-1}$ are independent with distributions $F_o, \ldots, F_{M-1}$ respectively. Moreover, the distribution of T_k is for any k = 1, 2, ..., M $G_k = F_o*\ldots*F_{k-1}$

We can easily show that

$$P_o\{D<T_M\} = E_o\left[N_{T_+}\right] = \sum_{k=1}^{M-1} P_k(G_k(t)-G_{k+1}(t)) \qquad (2.15)$$

where t is the time of inspection. Then a necessary condition to be satisfied by optimal t* maximizing (2.10) is given by

$$\sum_{k=1}^{M-1} p_k g_k(t^*) = \sum_{k=1}^{M-1} p_k g_{k+1}(t^*) \qquad (2.16)$$

where $g_k(t) = dG_k(t)/dt \qquad k = 1, \ldots, M-1$

Then, (2.16) can be rewritten as

$$P_o\{T_K = t^*\} = P_o\{T_{K+1} = t^*\} \qquad (2.17)$$

where K is a random variable satisfying

$$P\{K = j\} = p'_j \qquad j = 1, \ldots, M-1$$

$$p'_j = p_j / \sum_{k=1}^{M-1} p_k$$

Note that (2.17) is a special case of (2.6).

5.3. MODELS WITH MULTIPLE INSPECTIONS

In these models, there are n inspections available at times $t_1, \ldots, t_n$, so that the deterministic policy s(.) can be written as

$$s(u) = \begin{cases} 0 & u < t_1 \\ 1 & t_1 \leq u < t_2 \\ \vdots & \\ n & u \geq t_n \end{cases}$$

It follows from III.(3.7) that $E_i[N_k]$ for $k \in \{\infty, +, T_+, F_+, -, F_-, T_-\}$ can be written as

$$E_i[N_k] = \sum_{j \in E} \int_0^\infty R(i,j,du) \int_o^\infty F_j(dz)\, f_k(j, s(z+u) - s(z))$$

$$= \sum_{j \in E} \sum_{m=0}^{n-1} \sum_{\ell=m+1}^{n} \int_{t_m}^{t_{m+1}} R(i,j,du) f_k(j,\ell-m) \, (F_j(t_{\ell+1}-u) - F_j(t_\ell - u))$$

(3.1)

where $t_o = 0$ and $t_{n+1} = \infty$.

We now consider the model of (5.2.2) under the assumption that only two inspections are available to the inspector. Hence

$$s(u) = \begin{cases} 0 & u < t_1 \\ 1 & t_1 \leqq u < t_2 \\ 2 & u \geqq t_2 \end{cases}$$

and the inspector has to choose two inspection times t_1 and t_2 so as to maximize the probability of detecting the disorder before it becomes evident. Using (3.1) we can se that

$$P_o\{D<T^{\Delta}\} = E_o[N_{T_+}] = \sum_{m=0}^{1} \sum_{\ell=m+1}^{2} \int_{t_m}^{t_{m+1}} F(du) f_{T_+}(1,\ell-m)\ (G(t_{\ell+1}-u)-G(t_{\ell}-u))$$

$$= \int_o^{t_1} F(du)p_1(G(t_2-u)-G(t_1-u)) + \int_o^{t_1} F(du)(1-q_1^2)(1-G(t_2-u)) + \int_{t_1}^{t_2} F(du)p_1(1-G(t_2-u)$$

$$= p_1(F(t_1)-F*G(t_1)+F*(1-G)(t_2)-p_1\int_o^{t_1} F(du)(1-G(t_2-u)))$$

So, letting $h(t_1,t_2) = P_o\{D<T^{\Delta}\}$ we obtain

$$h(t_1,t_2) = P_o\{D<T^{\Delta}\} = p_1(F(t_1)-F*G(t_1)+F*(1-G)(t_2)-p_1\int_o^{t_1} F(du)(1-G(t_2-u))) \quad (3.2)$$

The necessary conditions to be satisfied by optimal t_1^* and t_2^* maximizing $P_o\{D<T^{\Delta}\}$ are found by differentiating $h(t_1,t_2)$ with respect to t_1 and t_2 and setting the derivatives equal to zero. This implies that the conditions

$$\begin{aligned} (1-p_1)f(t_1^*)+p_1 f(t_1^*)G(t_2^*-t_1^*) &= F*g(t_1^*) \\ f(t_2^*)+p_1 \int_o^{t_1^*} F(du)g(t_2^*-u) &= F*g(t_2^*) \end{aligned} \quad (3.3)$$

must be satisfied by optimal t_1^* and t_2^*.

As a numerical example we study the case when F and G are both exponential with respective rates a and b, (a,b>0) such that

$$F(t) = 1 - e^{-at} \qquad t \geq 0$$

$$G(t) = 1 - e^{-bt} \qquad t \geq 0$$

in the special case where $p_1 = 1$ so that the testing procedure gives no false negative results. Then (3.3) reduces to

$$-ae^{-at_1^*} - (b-a)p_1 e^{-at_1^*} e^{-b(t_2^*-t_1^*)} = -be^{-bt_1^*}$$

$$-ae^{-at_2^*} + be^{-at_1^*} e^{-b(t_2^*-t_1^*)} = 0 \qquad (3.4)$$

and this can be solved to yield

$$t_1^* = (1/b-a)\ \ell n\ (b/(a+(b-a)\ (b/a)^{b/a-b}))$$

$$t_2^* = (1/b-a)\ \ell n\ (b^2/(a^2+a(b-a)(b/a)^{b/a-b}))$$

as the optimal solution for b≠a. When b=a, it can be shown that

$$t_1^* = (1/a)\ (e-1/e)$$

$$t_2^* = (1/a)\ (2e-1/e)$$

is the optimal solution.

Note that when T_1 and T_Δ are exponentially distributed with the same rate a, and when no false negative result can be obtained, then the optimal schedule is such that

$$t_1^* < E[T_1] < t_2^* < E[T_\Delta]$$

which is reasonable since testing detects the disorder for sure.

Now, by similar computations as in (5.2.3) the probability distribution of earliness can be computed to be

$$P_o\{T^\Delta - D \leq Z\} = 1 - p_1 F(t_1) - P_1 F(t_2) + p_1^2 F(t_1) - p_1^2 \int_o^{t_1} F(du)\, G(t_2+z-u)$$

$$+ p_1 \int_o^{t_2} F(du) G(t_2+z-u) + p_1 \int_o^{t_1} F(du)\, G(z+t_1-u)$$

and

$$E_o[T^\Delta - D] = p_1 \int_o^{t_2} F(du) \int_o^\infty (1-G(t_2+z-u)dz + p_1 \int_o^{t_1} F(du) \int_o^\infty (1-G(t_1+z-u))\, dz$$

$$- p_1^2 \int_o^{t_1} F(du) \int_o^\infty (1-G(t_2+z-u))dz$$

Differentiating this expression with respect to t_1 and t_2 we obtain the following two conditions of optimality.

$$\begin{aligned} mf(t_1^*) &= F*(1-G)(t_1^*) + p_1 f(t_1^*)\, L(t_2^*-t_1^*) \\ mf(t_2^*) &= F*(1-G)(t_2^*) + p_1 H(t_1^*, t_2^*) \end{aligned} \tag{3.4}$$

where $m = E[T_\Delta - T_1] = \int_o^\infty (1-G(t'))dt'$

$$L(t_2^*-t_1^*) = \int_{t_2^*-t_1^*}^\infty (1-G(x))dx$$

$$H(t_1^*,t_2^*) = \int_0^{t_1^*} F(du)\ G(t_2^*-u) - F(t_1^*)$$

Note that if t_2 tends to infinity then we obtain a model with single inspection.

Since

$$\lim_{t_2^* \to \infty} L(t_2^*-t_1^*) = 0$$

$$\lim_{t_2^* \to \infty} H(t_1^*,t_2^*) = 0$$

then the conditions of (3.4) reduce to

$$mf(t_1^*) = F*(1-G)\ (t_1^*)$$

$$mf(\infty) = 0$$

so that we obtain the result of (2.14).

5.4. MODELS WITH STOCHASTIC INSPECTIONS

In case of stochastic inspections, an interesting optimization problem may be to find the optimal rate of inspections to minimize the total expected cost.

Consider a stationary Poisson inspection process with rate λ. If c_p, c_f and c_ν are the costs of performing a test, the cost of obtaining a false positive result and the cost of catastrophic failure respectively, the expected cost $c(i,\lambda)$ for a deterioration process with initial state i can be written as

$$c(i,\lambda) = \begin{cases} c_p E_i[N] + c_\nu (1-E_i [N_{T_+}]) & i > 0 \\ c_p E_o[N] + c_f E_o[N_{F_+}] + c_\nu(1-E_o[N_{T_+}]) & i = 0 \end{cases}$$

Using IV (3.1), IV (3.2) we obtain

$$c(i,\lambda)=c_p \sum_{j\in E} \int_o^\infty R(i,j,dz) \int_o^\infty F_j(du)(1-e^{-\lambda p_j u})(1/p_j)$$

$$+c_\nu(1- \sum_{j\in E} \int_o^\infty R(i,j,dz) \int_o^\infty F_j(du)(1-e^{-\lambda u p_j})$$

for i>0. The optimal rate λ^* can be found by differentiating $c(i,\lambda)$ with respect to λ and setting the derivative equal to zero. Hence we obtain

$$\sum_{j\in E} \int_o^\infty R(i,j,dz) \int_o^\infty e^{-\lambda^* u p_j} \, u \, (c_p - c_\nu p_j) \, F_j(du) = 0 \qquad (4.1)$$

Similarly, if the initial state is zero then the necessary condition of optimality for the rate λ is given by

$$\sum_{j\in E} \int_o^\infty R(0,j,dz) \int_o^\infty e^{-\lambda^* u p_j} \, u \, (c_p - c_\nu p_j) \, F_j(du) = -c_f \int_o^\infty p_o u F_o \, (du) \qquad (4.2)$$

Now, consider the extended model of (5.2.3) with M = 3. So $E_\Delta = \{0,1,2,3\}$, $F_o(t) = P_o\{T^1 \leqq t\}$, $F_1(t) = P_1\{T^2 \leqq t\}$ and $F_2(t) = P_2\{T^3 \leqq t\}$. Then, assuming that initially the system is in state 1, (4.1) becomes

$$\int_o^\infty e^{-\lambda^* u p_1} \, u \, (c_p - c_\nu p_1) F_1(du) + \int_o^\infty F_1(dz) \int_o^\infty e^{-\lambda^* u p_2} \, u \, (c_p - c_\nu p_2) \, F_2(du) = 0$$

and by taking F_1 and F_2 to be exponential distributions with rates a and b respectively, we obtain the condition to be satisfied by the optimal λ^* as

$$\frac{(c_p - c_\nu p_1)\, a}{(\lambda^* p_1 + a)^2} + \frac{(c_p - c_\nu p_2)\, b}{(\lambda^* p_2 + b)^2} = 0 \qquad (4.3)$$

The solution is given by

$$\lambda^* = (a\sqrt{D} - b) \;/\; (p_2 - p_1\sqrt{D})$$

where $D = (c_\nu p_2 - c_p)b \;/\; (c_p - c_\nu p_1)a$ under the assumption that $c_\nu p_2 > c_p > c_\nu p_1$.

In case where our assumption does not hold true, for example when $c_p = 0$, then the optimal rate λ^* will be infinite. In fact if performing a test has no cost then testing at an infinite rate is of course optimal.

VI. SUMMARY OF RESULTS AND FUTURE RESEARCH TOPICS

In this thesis the inspection problem of deteriorating systems subject to catastrophic failure is analyzed. Deterministic inspection policies and stochastic inspection processes are evaluated under the assumption that the deterioration process possesses a Markov renewal structure.

Chapter I presented an introduction to the problem with a review of past and current research on similar inspection models. The deterioration process and the inspection problem are formulated in detail in Chapter II. Chapter III and Chapter IV are devoted to the analysis and evaluation of deterministic inspection policies and stochastic inspection processes where various formulas are derived to compute the expected number of inspections. Some applications of the results are presented in the context of decision models in Chapter V. Interesting examples on finding the optimal inspection schedule which maximizes the probability of early detection are considered. The main purpose in our research has been to evaluate inspection policies and processes with respect to some measures of effectiveness, and not to present a general theory on the optimal inspection or control problem. Naturally, some possible extensions should be emphasized to point out future research topics. Of course, the optimal inspection problem mentioned throught the thesis needs analysis. Since the deterioration process is partially observable, the theory on partially observable Markov decision processes may be used or one may enjoy Markov renewal theory.

Note that in our model the result of a test was either positive or negative so that the inspector could only identify the deterioration as being zero or non-zero. A natural extension could be to consider a model where testing gives information on the present unobserved state of deterioration in which case Y_t would be a process with state space E such that

$$p_{ij} = P\{Y_t = j \mid X_t = i\}$$

for any $i,j \in E$, gives the probabilistic relationship between the underlying process X and the information process Y. Recall that in our model, the cost of a false positive result c_f was needed only when the initial state was zero. However, this will not necessarily be true in this extension since many types of false results may exist even when the initial state is non-zero.

Another extension of our model may be to consider deterministic and stochastic inspections simultaneously. This also would be more realistic since in medical applications a patient who is under the control of a physician may also require some extra tests done at some random points in time.

Note also that our model considers tests only as a means of obtaining some kind of information about the underlying state of the system. However, in some cases these tests may alter the system state to a higher deterioration level as in mammograms performed to screen for breast cancer where radiation may induce more deterioration. Hence the possibility of an immediate increase in the deterioration after each test has to be included in the model.

Finally, there is a single deterioration process in our model which presents the state of the system. A natural extension of this model will be the case where the state of the system is represented by multiple dependent deterioration processes which are inspected through several different tests. The determination of the inspection and testing times for the various deterioration processes and test combinations will indeed be very interesting because of the economies of scale involved in performing multiple tests at the same time.

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