ON A SHELL FORMULA OF CLOSED CURVES IN RIEMANNIAN MANIFOLDS

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ABSTRACT:

In 1974 Brickell and Hsiung [1, p.184] obtained an extension of the theorem of Fenchel [2], Milnor [3] and Fary [4] on the total absolute curvature of closed curves in Euclidian space. In order to develop the above mentioned theorem Brickell and Hsiung worked on closed curves in a complete simply connected Riemannian n-manifold with nonpositive sectional curvature. Due to the theorem of Hadamard-Cartan such a manifold is diffeomorphic to R^n.

Let O be a point on a closed C^∞ curve C embedded in a Riemannian n - manifold M, and suppose that C lies in a normal neighborhood of O. Construct the shell (Ω, f) on C with the vertex O. Let K be the Gaussian curvature of the induced metric on (Ω, f) and use dA for its area measure. Denote by æ the geodesic curvature of C considered as a curve in (Ω, f), and let s be its arc length. Then the main theorem in Brickell and Hsiung [1] is

$$\int_0^L \text{æ}(s)\, ds = \pi + 1 - \int\!\!\int_\Omega K \; dA$$

This study extends the above mentioned theory to piecewise regular curves C embedded in n - dimensional Riemannian manifolds, and aims to obtain a similar formula for them; moreover, it globalizes their results for two dimensional manifolds and develops a global shell formula depending on certain triangulations of the enclosed area of C in M. Thus, the local theorem will

incorporate outer angles of C at vertices $C(s_i) = Q_i$ for $i=1,.,p$. The shell curve $\bar{C}$ has at the vertices same outer angles as C, if and only if the indicatrix E has a vanishing vertex angle at $E(s_i)$ and E is one to one in a neighborhood of s for $i=1,.,p$.

KAPALI UZAY EGRİLERİNİN KABUK FORMÜLÜ ÜZERİNE

ÖZET:

Brickell ve Hsiung [1, s. 184], 1974 yılında Fenchell [2], Milnor [3] ve Fary [4]' nin Euclid uzaylarındaki kapalı uzay eğrilerinin toplam mutlak eğriliği üzerine olan teorilerini geliştirdiler. Yukarıda adı geçen teoriyi geliştirmek için, Brickell ve Hsiung tam, basit bağlantılı, kesit eğriliği sıfır veya negatif olan n boyutlu Riemann uzaylarında çalıştılar. Hadamard - Cartan teorisine göre bu tür uzaylar R^n uzayına difeomorfiktir.

O noktası, n boyutlu Riemann uzayı M ye gömülmüş, kapalı bir C eğrisinin üzerinde olsun ve C, O noktasının normal komşuluğunda yer alsın. C eğrisinin üzerine O baz noktalı (Ω, f) kabuğu oluşturulsun. K, (Ω, f) üzerine taşınan Riemann uzaklığı için, Gauss eğriliği, dA alan ölçüsü olsun. $\varkappa$, C eğrisinin (Ω, f) kabuk eğrisi olarak düşünüldüğünde, C nin jeodezik eğriliği, s yay uzunluk parametresi olsun.

Brickell ve Hsiung'un ana teoremi aşağıda görülmektedir

$$\int_{0}^{L} \varkappa(s)\, ds = \pi + 1 - \int\int_{\Omega} K\, dA \; .$$

Bu çalışmada, Brickell ve Hsiung' un teorisi, n - boyutlu Riemann uzaylarına gömülmüş, parça parça

regüler uzay eğrileri icin genişletilmekte ve bu eğriler için benzer bir formül elde etmek amaçlanmaktadır; ayrıca iki boyutlu manifoldlar için sonuçlar globalize edilip, C nin çevrelediği alanın üçgen ağı ile kaplanmasına bağımlı olarak global bir kabuk teorisi geliştirilmektedir.

Lokal teori, C uzay eğrisinin $C(s_i) = Q_i$, $i=1,..,p$, köşe noktalarındaki dış açılarını bünyesine almaktadır. Kabuk eğrisi $\bar{C}$ ' nin köşe noktalarındaki dış açıları, uzay eğrisi C nin dış açıları ile aynıdır, ancak bu sadece ve sadece E indikatrisinin $E(s_i)$ noktalarında sıfır dış açısına sahip olması ve s_i noktası çevresinde 1-1 olması ile mümkündür.

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LIST OF SYMBOLS

$B_\epsilon(p)$	Open ball in M around p with radius ϵ
C	Piecewise regular simply closed embedding
C^k	C restricted on $[s_{k-1}, s_k]$, $k = 1,..,p+1$
$\bar{C}$	C considered as a shell curve
$\bar{C}^k$	$\bar{C}$ restricted on $[s_{k-1}, s_k]$, $k = 1,..,p+1$
$\dot{C}(s)$	Tangent vector of C at C(s)
$\dot{\bar{C}}(s)$	Tangent vector of $\bar{C}$ at $\bar{C}(s)$
c^i	$u^i \circ C$, components of C
dA^k	Area element of (Ω^k, f^k)
E	Indicatrix of C
exp	Exponential map of M
$\exp_O$	Exponential map of M at O
F	$F = F(t,a^1,..,a^n) = \exp_O(t \Sigma a^i X_i)$
f	$f = f(y,s) = \exp_O(r(s)E(s))$
f_*	Induced map between tangent bundles
f^*	Induced map between cotangent bundles
f^k	$f\|\Omega^k$

$Gl(R^n,n)$	Group of invertible matrices
g	Riemannian metric
h	$h = \\| w \\|_e$
h^k	$h \mid \Omega^k$
$K:TTM \to TM$	Connection map of the Levi - Civita connection
K, K^k	Gaussian curvature on (Ω,f) and (Ω^k ,f^k)
K^k_M	Sectional curvature of $\text{spann}_{\mathbb{R}} (f^k_* \frac{\partial}{\partial y} , f^k_* \frac{\partial}{\partial s})$
k, k^i	$k^i (s) = h^i (r^i (s),s)$
L	Length of C
l^i	Length of $E \mid [s_{i-1} ,s_i]$
l^i_j	Components of the coframe θ^i
M	n - dimensional manifold
∂M	Boundary of M.
m	Canonical metric function on (M,g)
(M,g)	Riemannian manifold with the metric g
n^k	Normal vector of C^{-k}
∇	Levi-Civita connection
$\\| \quad \\|$	Norm in the tangent space $T_0 M$
$\\| \quad \\|_e$	Norm in R^n

$\otimes$	Tensor product
r, r^k	Radian function and $r\|[s_{k-1}, s_k]$
R	Curvature tensor
R_{ijlm}	Components of the curvature tensor
$\tilde{R}$	Regular region
s	Arc length parameter of C
S^n	n - sphere in R^n
$\langle\ ,\ \rangle_e$	Scalar product in R^n
T_j	Triangle in the triangulation Æ1
TM	Tangent bundle
$T_O M$	Tangent space at O
$T_1 M$	Sphere bundle of M
TTM	Tangent bundle of TM
$u = (u^1, .., u^n)$	Normal coordinates
$X_i, i=1,.,n$	Orthonormal moving frame
α^i	Angle between $\dot{\bar{C}}(s)$ and $\partial/\partial s$
Æ1	Triangulation
δ_{ij}, δ^i_j	Kronecker symbol

Γ^{k}_{M}	Extension of the sectional curvature K^{k}_{M}
Γ^{i}_{jk}	Components of the Levi-Civita connection
$\ae^{k}$	Geodesic curvature of C^{-k}
$\ae^{k}_{M}$	Geodesic curvature of C^{k}
Ω,Ω^{k}	Shell domain,$\{(y,s)\in R^2 \mid 0\le y\le r(s), 0\le s\le L\}$
(Ω,f)	Shell
(Ω^{i},f^{i})	i^{th} - shell pie
Ω^{i}_{ϵ}	i^{th} - shell pie with $y \ge \epsilon$
$\theta^{i}, i=1,..,n$	Moving coframe
θ^{i}_{j}	Components of the Levi-Civita connection form w.r.t. the moving frame
$\wedge$	Wedge product

I. PRELIMINARIES FOR SHELLS

Let (M,g) be a C^∞ Riemannian n-manifold with the metric g. Let C be a piecewise regular simply closed curve embedded in M. We shall denote by Q_i, $i=1,..,p$, the vertices of C. Let O be a point on the curve C different from Q_i, $i=1,..,p$. Assume that C lies in a convex normal neighborhood $U \subset M$ of O. Let s be the arc length parameter and L the total length of C in M. According to our assumption, there is a partition of the interval $[0,L]$ such that

$$0 = s_0 < \ldots\ldots < s_p < s_{p+1} = L \quad , \quad C(s_i) = Q_i$$
$$\text{for } i=1,..,p \quad .$$

We define piecewise C^∞ functions r , E in R and the tangent space $T_O M$ respectively. The curve C is in a normal neighborhood U of the point O. Consequently,

$$C(s) = \exp_O(r(s) E(s)) \ , \ s \in (0,L) .$$

The function r is the radian function and E the indicatrix function of the curve C with respect to the base point O. As the curve C is a topological embedding, both functions are well defined and they are piecewise differentiable on the interval $(0,L)$. We extend by continuity both functions to the closed interval $[0,L]$. Let $\| \quad \|$ denote the norm in the tangent space $T_O M$.

LEMMA 1:

Both functions r and E can be continuously extended at the points 0 and L. The extended functions r and E possess right-hand side and left-hand side derivatives of all orders at $s = 0$ and $s = L$ respectively.

At these points, they have the values

$$r(0) = 0 = r(L) \quad ; \quad \frac{dr}{ds}(0) = 1 = -\frac{dr}{ds}(L) \tag{1}$$

$$E(0) = -E(L) = \frac{dC}{ds}(0) \in T_0 M.$$

PROOF :

Choose a system of normal coordinates determined by an orthonormal frame $X_1, \dots, X_n$ at 0. Let $E_i(s)$, $i = 1, \dots, n$, be the components of E with respect to this frame, and $c^i(s)$ the values of the components of C, i.e.,

$$c^i(s) = u^i(C(s)) \quad \text{for} \quad s \in [0,L] \tag{2}$$

and

$$c^i(s) = r(s)\, E_i(s) \quad \text{for} \quad s \in (0,L)$$

$$c^i(0) = c^i(L) = 0 \quad , \quad i = 1, \dots, n.$$

We can express $c^i(s) = s\, A_i(s)$ by (2), where A_i are C^∞ functions near to 0, and they are different from zero.

$$1 = g(E(s),E(s)) = \sum_{i,j=1}^{n} E_i\, E_j\, \delta_{ij} = \sum_{i=1}^{n} E_i^2(s) \tag{3}$$

$$(4) \quad r(s) = r(s) \left(\sum_{i=1}^{n} E_i^2(s) \right)^{1/2} = \left(\sum_{i=1}^{n} c^{i^2}(s) \right)^{1/2} =$$

$$= s\, \| A(s) \|_e .$$

We denote by $\| \ \|_e$ and $\langle \ , \ \rangle_e$ the standard euclidian norm and metric on R^n respectively.

Using the equation (4) we will calculate $dr/ds|_{s=0}$

$$\frac{dr}{ds} = \| A(s) \|_e + s \sum_{i=1}^{n} A_i(s) \frac{dA_i}{ds} \frac{1}{\|A(s)\|_e}$$

noting that

$$\frac{dc^i}{ds} = A_i(s) + s\,\frac{dA}{ds}{}_i \,,$$

it follows that $\frac{dc^i}{ds}(0) = A_i(0)$ i.e.,

$$\| A(0) \|_e = \left\| \frac{dc^i}{ds}(0) \right\|_e = 1$$

i.e.,

$$\frac{dr}{ds}(0) = \| A(0) \|_e = 1$$

According to our definition,

$$E_i(s) = \frac{A_i(s)}{\|A(s)\|_e} \,, \quad s > 0$$

$$\lim_{s\to 0} E_i(s) = \frac{A_i(0)}{\|A(0)\|_e} = A_i(0) = \frac{dc^i}{ds}(0) \,, \qquad i = 1,\ldots,n \,,$$

or

$$E_i(o) = \frac{dc^i}{ds}(0) \,.$$

In a neighborhood of s = L, for the analysis of the r and E functions we will use similar techniques as above. Knowing that $c^i(L) = 0$ for all $i = 1,..,n$,there are C^∞ functions B_i near to L such that

$$c^i(s) = s\,B_i(s) \,, \quad s \le L.$$

Now, $c^i(L) = L\,B_i(L)$ or $B_i(L) = 0$ for $i = 1,..,n$

Differentiating c^i near L gives

$$\frac{dc^i}{ds}(L) = B_i(L) + L\,\frac{dB_i}{ds}(L) = L\,\frac{dB_i}{ds}(L)$$

and considering that s is an arc length parameter of C, we obtain

$$\left\| \frac{dB}{ds}(L) \right\|_e = \frac{1}{L} \,.$$

The above formula and $c^i(s) = r(s)\, E_i(s)$ yield

$$r(L) = \| c(L) \|_e = 0 \quad .$$

According to the definition of the functions B_i for $s \leq L$ and s sufficiently close to L, $c^i(s) = s\, B_i(s) = r(s)\, E_i(s)$. Therefore, $r(s)\, \|E(s)\|_e = s\, \|B(s)\|_e$. For sufficiently small $h > 0$, we have for the functions B_i we have

$$B_i(L-h) = B_i(L) - h \frac{dB_i}{ds} (L-\theta_i h) = -h \frac{dB_i}{ds} (L-\theta_i h)$$

for $i = 1,.,n$ and $0 < \theta_i < 1$.

We calculate the expression

$$\frac{1}{h}\, r(L-h) = \frac{1}{h}\, (L-h)\ \| B(L-h)\|_e =$$

$$= \quad \frac{1}{h} \ (L-h)\ \Big(\sum_{i=1}^{n} B_i^2\ (L-h)\Big)^{½}$$

$$= \frac{1}{h}\ (L-h)\ \Big(\sum_{i=1}^{n} h^2\ \frac{dB_i}{ds}\ (L - \theta_i h\ )\Big)^{½} \qquad \text{i.e.,}$$

$$\lim_{h \to 0} \frac{r(L - h\ )}{h} = L\ \| \frac{dB}{ds}(L) \|_e$$

i.e., $$\frac{dr}{ds}\ (L) = -1$$

q.e.d.

DEFINITION :

Let Ω denote the set points (y,s) in R^2 such that $0 \leq y \leq r(s)$, $0 \leq s \leq L$ and define the function

$$f : \Omega \longrightarrow M \text{ by}$$

$$f(y,s) = \exp_O(\ y\ E(s)) \quad .$$

$$\Omega^i = \{ (y,s) \in R^2 \mid s_{i-1} \le s \le s_i \}, \quad i=1,..,p+1$$

We call (Ω,f) the shell on C with the base point O and (Ω^i,f^i) the i^{th} shell pie with the base point O.

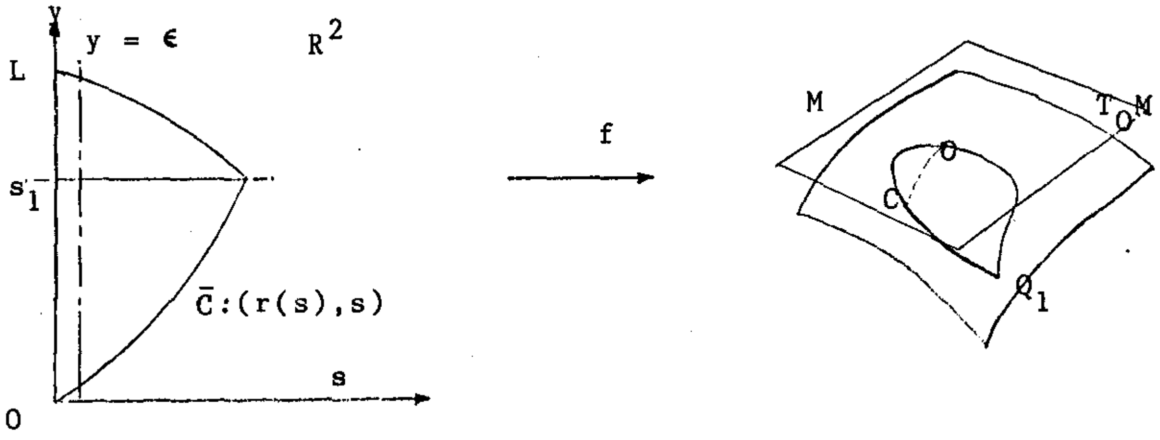
For $\epsilon > 0$, we define

$$\Omega^i_\epsilon = \{ (y,s) \in \Omega \mid y \ge \epsilon \}.$$

For sufficiently small $\epsilon > 0$ the equation $r(s) = \epsilon$ has only two solutions [1, p 178], that is the line $y = \epsilon$ will meet the boundary of Ω^1 and Ω^{p+1} only in two points. We denote by f^i the restriction of f on Ω^i for $i = 1,..,p+1$.

We will induce on Ω^i a Riemannian metric via f^i . However, there are some difficulties because of the singularities of the function f^i . In the following chapter we will see how these difficulties can be handled.

The main theorem 1 provides us with sharper inequalities about the total absolute curvature of closed curves in Euclidean spaces.



II. LOCAL SHELL THEORY FOR N DIMENSIONAL MANIFOLDS

We will make use of the structure equations for a Riemannian n - manifold expressed in polar coordinates. Choose an orthonormal frame $X_1, \dots, X_n$ at O. Extend the frame to a moving frame $X_1, \dots, X_n$ on the normal neighborhood by parallel translation along the geodesic rays through O. We will denote the moving frame again by $X_1, \dots X_n$. We denote by $\theta^1, \dots, \theta^n$ the dual moving coframe, i.e., $\theta^i(X_j)$ is δ^i_j for $i,j = 1,..,n$, and let $\theta^i_j = -\theta^j_i$ be the components of the Levi-Civita connection with respect to these frames.

For the rest of this study the maps are partially defined, unless it is explicitly stated otherwise.

Define the mapping

$$F : R^{n+1} \longrightarrow M \quad \text{by}$$

$$u^i(F(t,a^1,..,a^n)) = t\, a^i \quad , \quad i = 1,..,n.$$

It is shown in [5,p 27] that

$$F^*\theta^i = a^i\, dt + \beta^i \quad , \quad F^*\theta^i_j = \beta^i_j \quad , \; i = 1,\dots,n \; ,$$

where the forms β^i , β^i_j do not involve the form dt. These 1 - forms are zero for $t = 0$. They satisfy the differential equation

$$\frac{\partial \beta^i}{\partial t} = da^i + \sum_{j=1}^{n} a^j \beta^i_j \tag{5}$$

$$\frac{\partial \beta^i_j}{\partial t} = \sum_{k,l=1}^{n} (R^i_{j\,k\,l} \circ F)\, a^k \beta^l \, . \tag{6}$$

R^i_{jkl} are the components of the curvature tensor R with respect to the metric connection ∇ .

LEMMA 2:

We denote by l^i_j the components of the moving coframe θ^i , $i,j = 1,..,n$ with respect to normal coordinates. The functions l^i_j satisfy the equations

$$\beta^i = t \sum_{j=1}^{n} (l^i_j \circ F) \, da^j \quad \text{and} \quad a^i = \sum_{j=1}^{n} a^j (l^i_j \circ F)$$

PROOF:

$$F^* \theta^i = F^* (\sum_{j=1}^{n} l^i_j \, du^j) = \sum_{j=1}^{n} (l^i_j \circ F) \, d(u^j \circ F)$$

$$= \sum_{j=1}^{n} (l^i_j \circ F) \, a^j dt + \sum_{j=1}^{n} t (l^i_j \circ F) \, da^j$$

,i.e.,

$$\beta^i = t \sum_{j=1}^{n} (l^i_j \circ F) \, da^j$$

and

$$F_*(\frac{\partial}{\partial t}) = \sum_{k=1}^{n} a^k \frac{\partial}{\partial u^k} \circ F$$

and

$$F^* \theta^i(\frac{\partial}{\partial t}) = \theta^i (F_*(\frac{\partial}{\partial t})) = \theta^i (\sum_{k=1}^{n} a^k \frac{\partial}{\partial u^k} \circ F)$$

$$= \sum_{k=1}^{n} a^k (\sum_{j=1}^{n} (l^i_j \circ F) \, du^j (\frac{\partial}{\partial u^k} \circ F)$$

$$= \sum_{j,k=1}^{n} a^k (l^i_j \circ F) \, \delta^j_k = \sum_{j=1}^{n} a^j (l^i_k \circ F)$$

q.e.d.

We would like to induce a metric $f^{k\,*}g$ on Ω^k. Therefore, we investigate the singularities of f^k . The mapping f^k is expressed in terms of the normal coordinates $u^1,..,u^n$ by

$u^i\,(f^k(y,s)) = y\,E^k_i(s)$, where E^k_i are the components of the indicatrix of C restricted on $[s_{k-1}, s_k]$ with respect to the frame at O.

We obtain for the tangent vectors

$$f^k_*\left(\frac{\partial}{\partial y}\right)(y,s) = \sum_{i=1}^{n} E^k_i\,(s)\ \frac{\partial}{\partial u^i}\Big|\ f^k(y,s) \qquad (8)$$

$$f^k_*\left(\frac{\partial}{\partial s}\right)(y,s) = y\sum_{i=1}^{n} \frac{dE^k_i}{ds}\ \frac{\partial}{\partial u^i}\ \Big|\ f^k(y,s) \quad .$$

We know that $\| E\,(s) \| = 1$. It follows that the vectors $f^k_*\left(\frac{\partial}{\partial y}\right)$ and $f^k_*\left(\frac{\partial}{\partial s}\right)$ are linearly dependent iff $f^k_*\left(\frac{\partial}{\partial s}\right) = 0$. Therefore, f^k is an immersion except for points on the line $y = 0$ or $s = \alpha$, where α is any number such that $f^k_*\left(\frac{\partial}{\partial s}\right)\Big|_{s=\alpha} = 0$ i.e., the curve C is tangent at the point $f^k(y,\alpha)$ to the geodesic ray τ which is emitted from the base point O.

In order to calculate the induced metric $f^{k\,*}g$ on Ω^k we will make use of the structure equations expressed in polar coordinates .

Define the function $\Phi^k : R^2 \longrightarrow R^{n+1}$ by

$$(y,s) \longmapsto (y,\ E^k_1(s),..,E^k_n(s))$$

for k=1,.,p+1.

Φ^k satisfies $f^k = F \circ \Phi^k$. Now, calculate the 1-forms $\Phi^{k*}\beta^i$ and $\Phi^{k*}\beta^i_j$ on R^2 . Because $\Phi^k_*\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial t}$ and , β^i , β^i_j do not involve dt, we can describe $\Phi^{k*}\beta^i$ and $\Phi^{k*}\beta^i_j$ by functions w^k_i and w^k_{ji} on R^2.

$$\Phi^{k*}\beta^i = \Phi^{k*}\Big(\sum_{j=1}^{n} t\,(l^i_j \circ F\,)\,da^j \qquad (9)$$

$$= \sum_{j=1}^{n} (t \circ \Phi^k)(l^i_j \circ F \circ \Phi^k)\, d(a^j \circ \Phi^k)$$

$$= \sum_{j=1}^{n} y\,(l^i_j \circ f^k)\, \frac{dE^k_j}{ds}\, ds = w^k_i ds$$

$$a^i \circ \Phi^k = E^k_i = \sum_{j=1}^{n} (a^j \circ \Phi^k)(l^i_j \circ f^k) = \sum_{j=1}^{n} E^k_j (l^i_j \circ f^k)$$

$$\Phi^{k*}(\beta^i_j) = w^k_{ji}\, ds \quad , \ i,j = 1,..,n \quad , \ k = 1,...,p+1$$

Calculate the components of $f^m_*(\frac{\partial}{\partial y})$ with respect to the moving frame $X_1,...,X_n$

$$f^m_*(\frac{\partial}{\partial y})(y,s) = \sum_{j=1}^{n} E^m_j(s)(X_j \circ f^m)(y,s) \tag{10}$$

$$\theta^i(f^m_*(\frac{\partial}{\partial s})) = (\sum_{j=1}^{n} (l^i_j \circ f^m)\, du^j)(\sum_{l=1}^{n} y\, \frac{dE^m_l}{ds} \frac{\partial}{\partial u^l} \circ f^m)$$

$$= \sum_{l,j=1}^{n} y\,(l^i_j \circ f^m)\, \frac{dE^m_l}{ds}\, \delta^j_l$$

$$= y \sum_{j=1}^{n} (l^i_j \circ f^m)\, \frac{dE^m_l}{ds} = w^m_i$$

$$f^m_*(\frac{\partial}{\partial s})(y,s) = \sum_{j=1}^{n} w^m_j(y,s)\, X_j \big|_{f^m(y,s)} \quad , m=1,..,p+1. \tag{11}$$

We obtain from (8) and (11) that the functions w^m_{ij} , w^m_i are zero on the line $y = 0$. The impact of the structure equations on the functions w^m_i and w^m_{ij} are

$$\frac{\partial w^m_i}{\partial y} = \frac{dE^m_i}{ds} + \sum_{j=1}^{n} E^m_i\, w^m_{ji} \quad , \quad m = 1,..,p+1 \tag{12}$$

$$\frac{\partial w^m_{ji}}{\partial y} = \sum_{k,l=1}^{n} R_{jikl} \; E^m_k \; w^m_l$$

$$\frac{\partial}{\partial y} \left(\sum_{i=1}^{n} E^m_i \, w^m_i \right) = \sum_{i=1}^{n} E^m_i \frac{\partial w^m_i}{\partial y} = \sum_{i=1}^{n} E^m_i \left(\frac{dE^m_i}{ds} + \sum_{j=1}^{n} E^m_j \, w^m_{ji} \right)$$

$$= \sum_{i=1}^{n} E^m_i \frac{dE^m_i}{ds} + \sum_{i,j=1}^{n} E^m_i E^m_j w^m_{ji} = \sum_{i,j=1}^{n} E^m_i \; E^m_j \, w^m_{ji} \; .$$

Since the indicatrix E^m is normalized ,i.e., $\|E^m(s)\| = 1$,the derivative of $E^m$is perpendicular to E^m. On the other hand since $w^m_{ij} = - w^m_{ji}$ the last equality of the above formula is zero.

Thus, we obtain the equation

$$\langle E^m, w^m \rangle_e = 0, \qquad (13)$$

which will be crucial for the globalization of the shell method in the two dimensional case. The Riemannian metric g on M induces a metric $f^{k*}g$ on the k- th shell pie for $k = 1,..,p+1$

$$f^{k^*} g = f^{k^*} \left(\sum_{i=1}^{n} \theta^i \otimes \theta^i \right) = \sum_{i=1}^{n} f^{k^*} \theta^i \otimes f^{k^*} \theta^i$$

$$f^{k^*} \theta^i = f^{k^*} \left(\sum_{j=1}^{n} l^i_j \, du^j \right) = \sum_{j=1}^{n} (l^i_j \circ f^k) \; d\, (u^j \circ f^k)$$

$$= \sum_{j=1}^{n} (l^i_j \circ f^k) \, E^k_j \, dy + y \sum_{j=1}^{n} \frac{dE^k_j}{ds} (l^i_j \circ f^k) ds$$

$$= E^k_i \, dy + w^k_i \; ds \; .$$

Therefore,

$$\sum_{i=1}^{n} (f^{k*} \theta^i \otimes f^{k*} \theta^i) = \sum_{i=1}^{n} (E^k_i \, dy + w^k_i \, ds) \otimes (E^k_i dy + w^k_i ds)$$

$$= \sum_{i=1}^{n} (E_i^{k^2} dy \otimes dy + E_i^k \, w_i^k \, dy \otimes ds +$$

$$+ \; w_i^k \, E_i^k \; ds \otimes dy \; + \; (w_i^k)^2 ds \otimes ds$$

Since (13), we obtain

$$f^{k*} g = dy \otimes dy + \sum_{i=1}^{n} (w_i^k)^2 \, ds \otimes ds$$

That is to say, unless the vector $w^k = (w_1^k,..,w_n^k)$ is zero, the form $f^{k*}g$ is non singular on the k-th shell pie. Thus, it is a Riemannian metric.

At the vertices $Q_i = C(s_i)$, $i = 1,..,p$ we extend w^i with the right-hand side and left-hand side derivatives of the indicatrix function E^i to the closed interval $[s_{i-1}, s_i]$.

Now, we will compute the Gaussian curvature K^i on the i-th shell pie Ω^i at nonsingular points.

$$\text{Let} \quad h^k = (\sum_{i=1}^{n} w_i^{k^2})^{1/2} = \| w^k \|_e \quad , \; k = 1,..,p+1.$$

If h^k is nonzero, then [6, p.110],[7, p.9] K^k satisfies

$$K^k = - \frac{\partial^2 h^k}{\partial y^2} \, \frac{1}{h^k} \quad . \qquad (14)$$

The area element dA^k on the k-th shell pie Ω^k is

$$dA^k = (\det((\bar{u}_{ij}^k))^{1/2} ; \; i,j=1,..,n) \; dy \wedge ds$$

$$dA^k = h^k \, dy \wedge ds$$

with the metric $d\bar{u} \otimes d\bar{u} = dy \otimes dy + (h^k)^2 \, ds \otimes ds$.

Consequently, we obtain for the expression $K^k dA^k$

$$K^k dA^k = - \frac{\partial^2 h^k}{\partial y^2} \, dy \wedge ds \qquad (15)$$

The objective is to extend this expression to the points where f^k is singular. Let K_M^k denote the sectional curvature of the plane section σ in M, spanned by the vectors $f_*^k(\frac{\partial}{\partial y})$ and $f_*^k(\frac{\partial}{\partial s})$, i.e.,

$$K_M^k(f^k,\sigma) = (1/\det((g_{ij})) \, g(R(f_*^k\frac{\partial}{\partial y}, f_*^k\frac{\partial}{\partial s})f_*^k\frac{\partial}{\partial y}, f_*^k\frac{\partial}{\partial s})$$

$$= (1/h^k)^2 \, g(R(\sum_{j=1}^{n} E_j^k X_j, \sum_{i=1}^{n} w_i^k X_i)\sum_{l=1}^{n} E_l^k X_l, \sum_{s=1}^{n} w_s^k X_s)$$

$$= (1/h^k)^2 \sum_{i,j,s,l=1}^{n} R^i_{jsl} \; E_j^k w_i^k E_l^k w_s^k \quad , (16)$$

and by (12),

$$\frac{\partial^2 w_i^k}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{dE_i^k}{ds} + \sum_{j=1}^{n} E_j^k w_{ji}^k\right) = \sum_{j=1}^{n} E_j^k \frac{\partial w_{ji}^k}{\partial y}$$

therefore

$$K_M^k(f^k,\sigma) = -\sum_{j,s,l=1}^{n} E_j^k \; R_{ijsl} \; E_s^k w_l^k$$

(17)

$$K_M^k(f^k,\sigma) = -(1/h^k)^2 \sum_{i=1}^{n} w_i^k \frac{\partial^2 w_i^k}{\partial y^2} = -(1/h^k)^2 \langle w^k, \frac{\partial^2 w^k}{\partial y^2}\rangle_e$$

LEMMA 3:

The function $\Gamma_M^k : \Omega^k \longrightarrow R$ defined by

$$\Gamma_M^k(y,s) = \begin{cases} -(1/h^k)^2 \langle w^k, \frac{\partial^2 w^k}{\partial y^2}\rangle & \text{if } h^k \text{ is nonzero} \\ 0 & \text{otherwise} \end{cases}$$

is continuous on the k-th shell pie Ω^k.

PROOF:

Obviously, the function Γ_M^k is continuous where h^k is nonzero, and at these points

$$|\Gamma^k_M(y,s)| = \frac{1}{h^k}\,\left|\left\langle w^k, \frac{\partial^2 w^k}{\partial y^2}\right\rangle_e\right| \le$$

$$\le \frac{1}{h^k}\,\|w^k\|_e\,\left\|\frac{\partial^2 w^k}{\partial y^2}\right\|_e = \left\|\frac{\partial^2 w^k}{\partial y^2}\right\|_e$$

But, $\frac{\partial^2 w^k}{\partial y^2}{}_i = \sum_{jml=1}^{n} R_{jiml}\, E^k_j\, E^k_m\, w^k_l$, i.e., the functions $\frac{\partial^2 w^k}{\partial y^2}$ are continuous therefore, Γ^k_M is zero where h^k is zero.

q.e.d.

LEMMA 4:

The function $\frac{\partial h^k}{\partial y}$ is continuous on the k-th shell pie for $k=1,.,p+1$.

It is equal to $\left\|\frac{dE^k}{ds}\right\|_e$ on the line $y = 0$, and is zero at other points where $h^k = 0$.

PROOF:

Let h^k nonzero, then h^k is C^∞ and its partial derivative is

$$\frac{\partial h^k}{\partial y} = \frac{\partial}{\partial y}\langle w^k, w^k\rangle_e^{1/2} = (1/h^k)\left\langle w^k, \frac{\partial w^k}{\partial y}\right\rangle_e . \quad (18)$$

Let h^k be zero. This is the case iff $y = 0$ or $\frac{dE^k}{ds}(\alpha) = 0$.

We will use the equality

$$w^k_i = y\sum_{j=1}^{n}(1^i_j \circ f^k)\,\frac{dE^k_j}{ds} .$$

For $y \geq 0$, we obtain $h^k = y\,\|\mu^k\|_e$ with the functions

$$\mu_i^k = \sum_{j=1}^{n} (l_j^i \circ f^k) \frac{dE^k_j}{ds} \qquad k = 1,..,p+1.$$

Observe that $X_i(0) = \frac{\partial}{\partial u^i}$, $i=1,.,n$ i.e., the transformation matrix $(l_j^i \mid i,j=1,.,n)$ for the covectors θ^i, has the value δ_j^i at the point 0.

The value of μ_i^k on line $y = 0$ is

$$\sum_{j=1}^{n} l_j^i(0) \frac{dE^k_j}{ds} = \frac{dE^k_i}{ds} .$$

Consequently, the derivative of h^k is on line $y = 0$ is

$$\frac{\partial h^k}{\partial y} = \| \mu^k(0,s) \| = \| \frac{dE^k}{ds} \|_e.$$

Other singularities of h^k lie on the lines $s = \alpha$ with $\frac{dE^k_j}{ds}(\alpha) = 0$.

We obtain from the formula $h^k = y \| \mu^k \|_e$ the continuity of $\frac{\partial h^k}{\partial y}$ for points $(0,s)$ where μ^k is nonzero.

Other singularities of h^k are (α,s) such that the derivative $dE^k/ds\ (\alpha)$ of the indicatrix E^k is zero.

Using the inequality (19)

$$\left| \frac{\partial h^k}{\partial y} \right| \leq \frac{1}{h^k} \left| \langle w^k, \frac{\partial w^k}{\partial y} \rangle_e \right| \leq \frac{1}{h^k} \| w^k \|_e \| \frac{\partial w^k}{\partial y} \|_e ,$$

which is valid everywhere on Ω^k, we obtain that $\frac{\partial h^k}{\partial y}$ is continuous at (α,s).

q.e.d.

LEMMA 5:

The function $\frac{\partial^2 h^k}{\partial y^2}$ is continuous on Ω^k_ϵ , $k = 1,..,p+1$ and $\epsilon > 0$.

PROOF:

We obtain from the lemma 4 for h^k nonzero

$$\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}^k\right) = \frac{\partial}{\partial y}\left(\frac{1}{h^k} < w^k , \frac{\partial w}{\partial y}^k >_e\right) = \tag{20}$$

$$= {h^k}^{-3} \left(\left\| \frac{\partial w^k}{\partial y} \right\|_e^2 \left\| w^k \right\|_e^2 - \right.$$

$$\left. - < w^k , \frac{\partial w^k}{\partial y} >_e^2 \right) - \Gamma^k_M$$

,where Γ^k_M is defined as in the lemma 3 and

$$\Gamma^k = \Gamma^k_M - {h^k}^{-3} \left(\| w^k \|_e^2 \left\| \frac{\partial w^k}{\partial y} \right\|_e^2 - < w^k , \frac{\partial w^k}{\partial y} >_e \right|$$

with $$\Gamma^k = - \frac{\partial^2 h^k}{\partial y^2} \quad . \tag{21}$$

Let $h^k = 0$ .i.e., $\frac{dE^k}{ds}(\alpha) = 0$ for $s = \alpha$ then, $\Gamma^k (y,\alpha) = 0$, for $\epsilon \le y \le r(\alpha)$. From (20) and lemma 3, it is obvious that at the points h^k is nonzero the function Γ^k is continuous. For singular points we will show the continuity of $\Gamma^k_M - \Gamma^k$.

By $((\mu^k_{ij})|i,j=1,..,n)$ we define the inverse matrix of

$$((l^i_j \circ f^k)|i,j=1,..,n) \quad \text{with} \quad C^\infty\text{- functions}$$

$$\mu^k_{ij} : R^2 \dashrightarrow R \ .$$

Observe that $y \ge \epsilon > 0$ and

$$\frac{\partial w^k_i}{\partial y} = \sum_{j=1}^{n} (l^i_j \circ f^k) \frac{dE^k_j}{ds} + y \sum_{j=1}^{n} \frac{\partial}{\partial y} (l^i_j \circ f^k) \frac{dE^k_j}{ds}$$

$$= \frac{w^k_i}{y} + y \sum_{j=1}^{n} \frac{\partial}{\partial y} (l^i_j \circ f^k) \frac{dE_j}{ds}$$

$$= \frac{w_i^k}{y} + y \sum_{j,m=1}^{n} \left(\frac{\partial l^i}{\partial u^m} \circ f^k \right) \frac{dE_j^k}{ds} E_m^k$$

Therefore, for $y > 0$ the matrix form of the above formula, with

$w^k = (w_1^k, \ldots, w_n^k)$; $M = ((l_j^r \circ f^k)_{r,j=1,\ldots,n})$;

$\frac{dE^k}{ds} = \left(\frac{dE_1^k}{ds}, \ldots, \frac{dE_n^k}{ds} \right)$,

or

$$(w^k)^t = y M \left(\frac{dE^k}{ds} \right)^t .$$

Thus, we obtain

$$\frac{\partial w_i^k}{\partial y} = \frac{w_i^k}{y} + \sum_{m,j,l=1}^{n} \left(\frac{\partial l_j^i}{\partial u}{}_l \circ f^k \right) \mu_{mj}^k w_l^k E_l^k$$

Define $A_{i,j}(y,s) = \sum_{l,m=1}^{n} \left(\frac{\partial l_m^i}{\partial u^l} \circ f^k \right) E_l^k \mu_{mj}^k$

and $E_{ij} = (1/y) \delta_{ij}$.

We can describe the last equation in operator form

$$\left(\frac{\partial w^k}{\partial y} \right)^t = (E + A)(w^k)^t$$

i.e., there exists $D^k = D^k (\epsilon)$ such that

$$\left\| \frac{\partial w^k}{\partial y} \right\|_e \leq D^k \| w^k \|_e = D^k h^k .$$

Using (20)

$$| \Gamma_M^k - \Gamma^k | = \left| h^{k^{-3}} \left(\| w^k \|^2 \left\|_e \frac{\partial w^k}{\partial y} \right\|_e^2 - \left\langle w^k, \frac{\partial w^k}{\partial y} \right\rangle^2 \right) \right| \leq$$

$$\leq D^{k^2} h^k .$$

Consequently , $\Gamma_M^k - \Gamma^k$ is continuous at points where h^k is zero.

Together with this statement, lemma 3 implies the continuity of Γ^k at zeros of h^k .

q.e.d.

We will calculate the geodesic curvature of the shell curve $\bar{C} : [0,L] \dashrightarrow R^2$ with respect to the induced metric at the nonsingular points. Let $\bar{C}^k$ be the restriction of $\bar{C}$ on $[s_{k-1}, s_k]$. The tangent vector $\dot{\bar{C}}^k$ is

$$\dot{\bar{C}}^k(s) = \left(\frac{dr^k}{ds} \frac{\partial}{\partial y} + \frac{\partial}{\partial s} \right)\Big|_{\bar{C}^k(s)} ,$$

where s is again arc length parameter of C^k.

Therefore, we get

$$d\bar{u}^2 (\dot{\bar{C}}^k(s) , \dot{\bar{C}}^k(s)) = 1 = \left(\frac{dr^k}{ds} \right)^2 + (h^k)^2 (r^k(s),s) \tag{22}$$

We define $k^k(s) = h^k(r^k(s),s)$,i.e.,

$$\left(\frac{dr^k}{ds} \right)^2 + (k^k)^2 (s) = 1 \quad . \tag{23}$$

We will show that the geodesic curvature $\varkappa^k$ of the curve $\bar{C}^k$ is

$$\varkappa^k (s) = \frac{\partial h^k}{\partial y} - \frac{1}{h^k} \frac{d^2 r^k}{ds^2} \quad . \tag{24}$$

The metric components of $f^{k*}g$ satisfy

$$\bar{u}_{11} = 1 \quad , \quad \bar{u}_{12} = 0 = \bar{u}_{21} \quad , \quad \bar{u}_{22} = (h^k)^2$$

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = 0 \; ; \quad \Gamma^1_{22} = - h^k \frac{\partial h^k}{\partial y} \; , \; \Gamma^2_{22} = \frac{1}{h^k} \frac{\partial h^k}{\partial y} \; ;$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{h^k} \frac{\partial h^k}{\partial y} \quad .$$

This is clear from the formula [8, p.84]

$$\sum_{l=1}^{2} \bar{u}_{lk} \Gamma^l_{ij} = (1/2) \left(\frac{\partial \bar{u}_{jk}}{\partial x^i} + \frac{\partial \bar{u}_{ki}}{\partial x^j} - \frac{\partial \bar{u}_{ij}}{\partial x^k} \right)$$

,i,j,k = 1,2,

where $\bar{u}_{ij}$ are the metric components, and Γ^{i}_{jk} are the components of the metric connection. We obtain for

$$\nabla_D \dot{\bar{C}}^k_{|s} =$$

$$= \left(\frac{d^2 r^k}{ds^2} - h^k \frac{\partial h^k}{\partial y} \right) \frac{\partial}{\partial y} \Big|_{\bar{C}^k(s)} + \left(\frac{1}{h^k} \left(2 \frac{dr^k}{ds} \frac{\partial h^k}{\partial y} + \frac{\partial h^k}{\partial s} \right) \right) \frac{\partial}{\partial s} \Big|_{\bar{C}^k(s)}$$

Now, $\quad h^{k\,2}(r^k(s),s) + \left(\frac{dr^k}{ds} \right)^2 = 1$ implies

$$h^k \frac{\partial h^k}{\partial y} \frac{dr^k}{ds} + \frac{dr^k}{ds} \frac{d^2 r^k}{ds^2} + \frac{\partial h^k}{\partial s} h^k = 0$$

Consequently,

$$\frac{\partial h^k}{\partial s} = - \frac{dr^k}{ds} \frac{\partial h^k}{\partial y} - \frac{1}{h^k} \frac{d^2 r^k}{ds^2} \frac{dr^k}{ds}$$

The normal vector of $\bar{C}^k$ at the point $\bar{C}^k(s)$ is [7, p.208]

$$n^k(s) = - h^k \frac{\partial}{\partial y} + \frac{dr}{ds} \frac{1}{h^k} \frac{\partial}{\partial s} \Big|_{\bar{C}^k(s)} .$$

Therefore,

$$æ^k = f^{k\,*} g \left(\nabla_D \dot{\bar{C}}^k, n^k \right) = \left(h^{k\,2} + \left(\frac{dr^k}{ds^2} \right)^2 \right) \frac{\partial h^k}{\partial y} +$$

$$\left(- h^k - \frac{1}{h^k} \left(\frac{dr^k}{ds} \right)^2 \right) \frac{d^2 r^k}{ds^2} = \frac{\partial h^k}{\partial y} - \frac{1}{h^k} \frac{d^2 r^k}{ds^2}$$

Because f^k is an isometry at the points $h^k \neq 0$, it is a a well known fact that [7, p.15]

$$\ae_M^{k\,2} = \ae^{k\,2} + \left[\begin{array}{c}\text{the square of the}\\ \text{length of the second fundamental}\\ \text{form of } (\Omega^k, f^k) \text{ restricted on } C^k\end{array}\right] \quad (25)$$

, where $\ae_M^k$ is the geodesic curvature of C^k .

Therefore , $$| \ae_M^k | \geq | \ae^k |.$$

We will extend the geodesic curvature of $\bar{C}^k$ with respect to the induced metric $dy \otimes dy + h^k(y,s)\ ds \otimes ds$ to a function, defined almost everywhere on $[s_{k-1}, s_k]$.

LEMMA 6:

a) $k^i(s) = h^i(r^i(s),s)$ is absolutely continuous on $[s_{i-1}, s_i]$, $i = 1,\dots,p+1$.

b) k^i is differentiable at the points where it is nonzero. It is differentiable at a zero $s = \alpha$ iff

$$\frac{d\Phi^i}{ds}\Big|_{s=\alpha} = 0$$

with $$\Phi^i(s) = w^i\,(r^i(s),s) \in R^n.$$

PROOF:

a) Because Φ^i is a C^∞ - differentiable on $[s_{i-1}, s_i]$, there exists $B_i > 0$ for $i = 1,\dots,p+1$ such that

$$\left\| \frac{d\Phi^i}{ds} \right\|_e \leq B_i \ .$$

We obtain, using the mean value theorem,

$$| k^i(b) - k^i(a) | = | \|\Phi^i(b)\|_e - \|\Phi^i(a)\|_e | \leq$$

$$\|\Phi^i(b) - \Phi^i(a)\|_e \leq B_i\ |b-a|$$

where $s_{i-1} \leq a\ ,\ b \leq s_i$ for $i = 1,\dots,p+1$.

k^i is Lipschitz bounded, thus it implies that k^i is absolutely continuous.

b) k^i is differentiable at points where it is nonzero.

This is clear because $h^i(r^i(s),s)$ has no singularity there.

If $\Phi^i(\alpha) = 0$ and $\frac{d\Phi^i}{ds}(\alpha)$ is nonzero, then we can factorize the function $\Phi^i(s) = (s - \alpha)\,\tau(s)$ such that τ is C^∞- differentiable and $\tau(\alpha)$ is nonzero. Therefore, $k^i(s) = |s - \alpha|\, \| \tau \|_e$ is nondifferentiable at the point $s = \alpha$. On the other hand, if $d\Phi^i / ds \,|_{s=\alpha} = 0$ then ,

$\Phi^i(s) = (s - \alpha)^2 \beta(s)$ where β is C^∞. Consequently, $k^i(s) = (s - \alpha)^2 \| \beta(s) \|_e$ has at this point a zero derivative.

Define the angular function α^i on $[s_{i-1}, s_i]$ by

$$\sin \alpha^i(s) = \frac{dr^i}{ds} \quad , \; -\pi/2 \le \alpha^i \le \pi/2 \; , \qquad (26)$$

$i = 1,\ldots,p+1$

The formula (23) implies α^i is well defined.

The formula (26) implies

$$\cos \alpha^i(s) = k^i(s) \; . \qquad (27)$$

We compute the angle τ^i between $\dot{C}^i(s)$ and $\partial/\partial s$ in Ω^i equipped with the induced metric $f^{i*}g$

$$\cos \tau^i = \frac{du^{-2} \left(\frac{dr}{ds} \frac{\partial}{\partial y} + \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) \Big|_{(y,s)}}{\| \dot{C}^i(s) \| \, \| \frac{\partial}{\partial s} \|} = \frac{k^{i^2}(s)}{k^i(s)} = \cos \alpha^i$$

Geometrically, the function α^i is the angle between the tangent vector $\dot{C}^i(s)$ and $\frac{\partial}{\partial s}$

LEMMA 7

The function α^i is absolutely continuous on $[s_{i-1}, s_i]$. It is differentiable at a zero point $s = \alpha$ iff k^i is differentiable at α for $i = 1, \dots, p+1$.

PROOF:

The function $\sin^{-1}$ is uniformly continuous on the compact interval $[-1, 1]$. Consequently, there exists a number $\sigma > 0$ such that

$$| \sin^{-1} a - \sin^{-1} b | < \pi/2 \quad \text{where, } | a - b | < \sigma \text{ and } -1 \le a, b \le 1$$

The function r is C^∞ on $[s_{i-1}, s_i]$. There exists a number B_i such that $\left|\frac{d^2 r^i}{ds^2}\right| < B_i$ on $[s_{i-1}, s_i]$. Let s_1, s_2 be with $s_{i-1} \le s_1, s_2 \le s_i$ such that $| s_1 - s_2 | < \sigma / B_i$.

The following equality is obvious from the definition and the setting $\alpha_j = \alpha^i(s_j)$, $\frac{dr}{ds}j = \frac{dr^i}{ds}\Big|_{s = s_j}$

$$\sin(\alpha_2 - \alpha_1) = \frac{dr}{ds}2\, k_1 - \frac{dr}{ds}1\, k_2 = k_1\left(\frac{dr}{ds}2 - \frac{dr}{ds}1\right) - \frac{dr}{ds}1\,(k_2 - k_1)$$

where $k_j = k^i(s_j)$.

Therefore,

$$| \sin(\alpha_2 - \alpha_1) | \le \left| \frac{dr}{ds}2 - \frac{dr}{ds}1 \right| + | k_2 - k_1 | .$$

The mean value theorem implies, that

$$|\sin \alpha_2 - \sin \alpha_1| = \left| \frac{dr}{ds}1 - \frac{dr}{ds}2 \right| \le \left| \frac{d^2 r}{ds^2} \right| \, | s_2 - s_1 | \le \sigma$$

Therefore,

$$| \alpha_2 - \alpha_1 | = | \sin^{-1}(\sin \alpha_2) - \sin^{-1}(\sin \alpha_1) | < \pi/2.$$

Furtheremore,

$$|s| \leq \frac{\pi}{2} \; |\sin s| \quad \text{is on} \quad |s| \leq \frac{\pi}{2} \text{ valid.}$$

We obtain,

$$|\alpha_2 - \alpha_1| \leq \pi/2 \; |\sin(\alpha_2 - \alpha_1)| \leq \pi/2 \quad \left(\left|\frac{dr}{ds}2 - \frac{dr}{ds}1\right| + \right.$$
$$\left. + \; |k_2 - k_1| \right)$$

The function dr^i/ds is C^∞ and according to the lemma 5, k^i is absolutely continuous. This means that the function α^i is absolutely continuous.

For the points s where k^i is nonzero , the equation

$$\sin \alpha^i = dr^i/ds$$

implies

$$\frac{d\alpha^i}{ds} = \frac{d^2 r^i}{ds^2} \; \frac{1}{k^i} \quad .$$

In a neighborhood of a zero of k^i, the function $\sin \alpha^i$ is nonzero. Therefore, the formula $\cos \alpha^i = k^i$ gives

$$\frac{d\alpha^i}{ds} = - \frac{\frac{dk^i}{ds}}{\frac{dr^i}{ds}}$$

iff dk^i/ds exists.

q.e.d.

Consequently, the formula

$$æ^i = \frac{\partial h^i}{\partial y} - \frac{d\alpha^i}{ds} \qquad \text{on } [s_{i-1}, s_i], \; i=1,..,p+1, \tag{29}$$

extends the domain of $æ^i$ to $[s_{i-1}, s_i]$ except for a set of Lebesgue measure zero. Furthermore, if $k^i = 0$, then the extended function on $[s_{i-1}, s_i]$ has a zero at that point. Finally,

$$|\text{æ}_M^k| \geq |\text{æ}^k|$$

holds almost everywhere on $[s_{i-1}, s_i]$.

Now we can formulate the main local theorem.

THEOREM 1 :

Let O be a point on a closed piecewise regular curve C embedded in a n-dimensional Riemannian manifold M. Suppose that C lies in a convex normal neighborhood of the base point O. Let s be the arc length parameter of C such that C(0) = O = C(L), where L is the length of C. Let $Q_m = C(s_m)$ for $m = 1,..,p$ be the vertices of the curve C. Let K^i be the Gaussian curvature and dA^i be the area measure of the induced metric $f^{i*}g$ on the i-th shell pie Ω^i for i=1,..,p+1. æ^i denotes the extended geodesic curvature of the curve $\bar{C}^i$ such that $f(\bar{C}^i(s)) = C^i(s)$ on $[s_{i-1}, s_i]$. We denote by α^i the angle between $\dot{\bar{C}}^i(s)$ and $\frac{\partial}{\partial s}$. Then,

$$\sum_{i=1}^{p+1} \int_{s_{i-1}}^{s_i} \text{æ}^i \, ds = \pi + \sum_{i=1}^{p+1} l^i - \sum_{i=1}^{p+1} \int\!\!\int_{\Omega^i} K^i \, dA^i + \sum_{i=1}^{p} \left(\alpha^{i+1}(s_i) - \alpha^i(s_i) \right)$$

where l^i is the length of the indicatrix function E^i .

PROOF :

First we will integrate $K^i dA^i$ on Ω^i for i =

2,...,p. According to the equation (15), we obtain

$$\int\int_{\Omega^i} K^i \, dA^i = \lim_{\epsilon\to 0} \int\int_{\Omega^i_\epsilon} -\frac{\partial^2 h^i}{\partial y^2} \, dy \, ds =$$

$$= \int_{s_{i-1}}^{s_i} -\frac{\partial h^i}{\partial y} (r^i(s), s) \, ds + \lim_{\epsilon\to 0} \int_{s_{i-1}}^{s_i} \frac{\partial h^i}{\partial y} (\epsilon, s) \, ds \, .$$

Using the extended geodesic curvature $\varkappa^i$, due to (29), we evaluate the first term . Since α is absolutely continuous [9, p.207]

$$\int_{s_{i-1}}^{s_i} -\frac{\partial h^i}{\partial y} (r(s),s) \, ds = - \int_{s_{i-1}}^{s_i} \varkappa^i(s) \, ds + (\alpha^i(s_{i-1}) - \alpha^i(s_i)) \; .$$

Lemma 4 implies that

$$\lim_{\epsilon\to 0} \int_{s_{i-1}}^{s_i} \frac{\partial h^i}{\partial y} (\epsilon,s) \, ds = \int_{s_{i-1}}^{s_i} \left\| \frac{dE^i}{ds} \right\|_e ds = l^i$$

,ie, the second term converges to the euclidean length l^i of the indicatrix on the interval $[s_{i-1}, s_i]$ i.e.,

$$\int\int_{\Omega^i} K^i dA^i = - \int_{s_{i-1}}^{s_i} \varkappa^i(s) \, ds + l^i - (\alpha^i(s_i) - \alpha^i(s_{i-1})).$$

Let $\bar{s}_1$, $L - \bar{s}_p$ be the well defined values of s such that the line $y = \epsilon$ meets the curve $y = r(s)$ once

,i.e., $r(\bar{s}_1) = \epsilon = r(L - \bar{s}_p)$

Therefore,

$$\int\int_{\Omega^1} K^1 \, dA^1 = -\lim_{\epsilon \to 0} \int_{s_0}^{\bar{s}_1} \frac{\partial h^1}{\partial y}(r(s),s)\, ds + \int_{s_0}^{s_1} \frac{\partial h^1}{\partial y}(\epsilon,s)\, ds$$

$$= -\int_{s_0}^{s_1} \text{æ}^1 (s)\, ds - (\alpha^1(s_1) - \alpha^1(s_0)) + l^1$$

and

$$\int\int_{\Omega^{p+1}} K^{p+1} \, dA^{p+1} = \lim_{\epsilon \to 0} \int_{s_p}^{L - \bar{s}} \frac{\partial h^{p+1}}{\partial y}(r^{p+1}(s),s)\, ds -$$

$$- (\alpha^{p+1}(L) - \alpha^{p+1}(s_p)) + l^{p+1}.$$

Consequently ,

$$\sum_{i=1}^{p+1} \int\int_{\Omega} K^i \, dA^i = \sum_{i=1}^{p+1} l^i - \sum_{i=1}^{p+1} \int_{s_{i-1}}^{s_i} \text{æ}^i(s)\, ds - \sum_{i=1}^{p+1} (\alpha^i(s_i) -$$

$$- \alpha^i (s_{i-1})) =$$

$$= \sum_{i=1}^{p+1} l^i - \sum_{i=1}^{p+1} \int_{s_{i-1}}^{s_i} \text{æ}^i(s)\, ds - \alpha^1(s_1) + \alpha^1(s_0) -$$

$$+ \,..\, - \alpha^{p+1}(L) + \alpha^{p+1}(s_p)$$

Lemma 1 shows that $\alpha^{p+1}(L) - \alpha^1(0) = -\pi$.

Therefore, the above expression is

$$= \sum_{i=1}^{p+1} l^i - \sum_{i=1}^{p+1} \int_{s_{i-1}}^{s_i} \text{æ}^i(s)\, ds + \pi + \sum_{i=1}^{p} (\alpha^{i+1}(s_i) - \alpha^i(s_i))$$

q.e.d.

The requirements of the above theorem are satisfied, e.g., for n-manifolds M such that M is a complete simply connected riemannian manifold with nonpositive sectional curvature K_M . It is well known that such a manifold M is a normal neighborhood of each of its points so that the shell (Ω, f) is well defined [10, p.74]. Therefore, theorem 1 could be applied on simply closed regular space curves in n-dimensional Euclidian spaces.

THEOREM 2 :

Let M be a complete simply connected Riemannian manifold with a nonpositive sectional curvature function K_M . Then the geodesic curvature æ_M of any closed piecewise regular C embedded in M satisfies the inequality

$$\sum_{i=1}^{p+1} \int_{s_{i=1}}^{s_i} | \text{æ}_M^i | \, ds \geq 2\pi - \sum_{i=1}^{p+1} \iint_{\Omega^i} K^i_M \, dA + \sum_{i=1}^{p} (\alpha^{i+1}(s_i) - \alpha^i(s_i)),$$

where (Ω,f) is any shell on C.

PROOF :

It is well known that such a manifold is a normal neighborhood of each of its points so that the shell (Ω,f) and the i- th shell pies are defined. According to Lemma 1 the indicatrix of the shell joins a pair of antipodal points on a unit sphere and therefore its length $l \geq \pi$. Consequently as $| \text{æ}^i_M| \geq | \text{æ}^i|$ we obtain, using Theorem 1

$$\sum_{i=1}^{p+1} \int_{s_{i-1}}^{s_i} | \text{æ}^i_M | \, ds \geq \sum_{i=1}^{p+1} \int_{s_{i-1}}^{s_i} | \text{æ}^i | \, ds \geq \sum_{i=1}^{p+1} \int_{s_{i-1}}^{s_i} \text{æ}^i \, ds =$$

$$= \pi + \sum_{i=1}^{p} l^i + \sum_{i=1}^{p} (\alpha^{i+1}(s_i) - \alpha^i (s_i)) - \sum_{i=1}^{p+1} \iint_{\Omega^i} K^i dA^i$$

$$= 2 \pi + \sum_{i=1}^{p} (\alpha^{i+1}(s_i) - \alpha^i(s_i)) - \sum_{i=1}^{p+1} \iint_{\Omega^i} K^i \, dA^i .$$

The proof is completed by $(K^i_M - K^i) \, dA^i \geq 0$ [14, p.250].

q.e.d.

We will compute the outer angle τ^i of the curve C at the vertex $C(s_i)$ and assume that s_i is a nonsingular point of the induced metric $f^{i*}g$.

$$\cos \tau^i = g (\dot{C}^{i+1}(s_i), \dot{C}^i(s_i)) =$$

$$= g \left(\frac{dr^{i+1}}{ds} \frac{\partial f^{i+1}}{\partial y} + \frac{\partial f^{i+1}}{\partial s} , \frac{dr^i}{ds} \frac{\partial f^i}{\partial y} + \frac{\partial f^i}{\partial s} \right)$$

$$= \frac{dr^{i+1}}{ds} \frac{dr^i}{ds} g \left(\frac{\partial f^{i+1}}{\partial y} , \frac{\partial f^i}{\partial y} \right) + \frac{dr^{i+1}}{ds} g \left(\frac{\partial f^{i+1}}{\partial y} , \frac{\partial f^i}{\partial s} \right) +$$

$$+ \frac{dr^i}{ds} g \left(\frac{\partial f^{i+1}}{\partial y} , \frac{\partial f^i}{\partial s} \right) + g \left(\frac{\partial f^{i+1}}{\partial s} , \frac{\partial f^i}{\partial s} \right) =$$

$$= \frac{dr^{i+1}}{ds} \frac{dr^i}{ds} \sum_{k,l=1}^{n} E^{i+1}_k(s_i) E^i_l(s_i) \delta_{kl} + \frac{dr^{i+1}}{ds} \sum_{k,l=1}^{n} E^{i+1}_k(s_i) w^i_l \delta_{kl}$$

$$+ \frac{dr^i}{ds} \sum_{k,l=1}^{n} E_k^i(s_i)\, w_l^{i+1} \delta_{kl} + \sum_{k,l=1}^{n} w_k^{i+1} w_l^i \; \delta_{kl} .$$

As a result of (10), (11),(13) and $E^{i+1}(s\ ) = E^i(s\ )$, this expression is equal to

$$\left.\frac{dr^{i+1}}{ds}\right|_{s=s_i} \left.\frac{dr^i}{ds}\right|_{s=s_i^+} \sum_{k=1}^{n} w_k^{i+1}(r^{i+1}(s_i),s_i)\, w_k^i(r^i(s_i),s_i)$$

We denote by δ^i the angle between $w^{i+1}(s_i)$ and $w^i(s_i)$. Then we obtain

$$\langle w^{i+1} , w^i \rangle_e = \| w^{i+1} \|_e \| w^i \|_e \cos \delta^i \tag{30}$$

$$\cos \tau^i = \tag{31}$$
$$= \sin \alpha^{i+1}(s_i) \sin \alpha^i(s_i) + \cos \delta^i \cos \alpha^{i+1}(s_i) \cos \alpha^i(s_i)$$

We have calculated (31) under the assumption that $k^i(s_i)$ and $k^{i+1}(s_i)$ are nonzero. We will show that we can distort the base point $O \in C([0,L])$ slightly and maintain that both magnitudes $k^{i+1}(s_i)$, $k^i(s_i)$ are different from zero.

Consider the case where $k^i(s_i)$ is zero. This means the tangent vector $\dot{C}(s_i)$ at the vertex point $C(s_i)$ is also tangent to the geodesic ray μ through $C(O)$ and $C(s_i)$. In this case,

either

a) For every $\delta > 0$ there exists s such that

$0 < s < \delta$ and $C(s)$ is nonelement of μ,

or

b) There exists $\delta > 0$ such that for all s

with $0 < s < \delta$, $C(s) \in \mu$.

We will show below that both cases " a) " and " b) " could be avoided by a simple modification of C with the base point O.

Let $p,q \in M$,

$$\Omega_{pq} = \{ \tau:[0,1] \longrightarrow M \mid \tau(0)=p,\ \tau(1)=q,\ \tau \text{ is piecewise differentiable} \}$$

and $L(\tau)$ denote the length of the curve τ. It is well known that the map

$$\begin{aligned} m:\quad & M \times M \dashrightarrow R \\ & (p, q) \dashrightarrow \inf \{ L(\tau) \mid \tau \in \Omega_{p,q} \} \end{aligned}$$

is a metric and (M,m) is a metric space [8, p.156].

We denote by $B_\epsilon(p)$ the open ball around the point $p \in M$ with the radius $\epsilon > 0$.

Let $\bar{R}$ denote the two point compactification of R, and M be a complete manifold equipped with the distance function m. We will define a function s on the unit sphere bundle of M

$$\begin{aligned} & s: T_1 M \dashrightarrow \bar{R} \\ & s(v) = \sup \{ t \in R \mid m(\pi(v), \exp tv) = t \}. \end{aligned}$$

π is the canonical map of the sphere bundle. The function s is continuous [8, p.169]. Moreover, let us define

$$C_p = \{ s(v)v \mid v \in T_p M \cap T_1 M \}$$

and

$$C(P) = \{ \exp_p(w) \mid w \in C_p \} .$$

The set C(P) is the cut locus of M with respect to the point $P \in M$.

First, we introduce a technical lemma.

LEMMA 8 :

Let $A : [a,b] \dashrightarrow M$ be a path and trace A lie in a normal neighborhood of $Q \in M$. Then, there is $\epsilon > 0$ such that trace A lies in a normal neighborhood of $y \in M$ for all $y \in B_\epsilon(Q)$.

PROOF:

Since we deal with a compact set, trace A, we can assume that M is a compact manifold. Therefore, the distance function m is bounded and consequently the function s is bounded. According to the assumption, there is a linear isometry i from R^n onto the tangent space $T_Q M$, and there is a $\sigma > 0$ such that

$$x = (\exp_Q \circ \; i \mid B_\sigma(0))^{-1}$$

is a Riemannian coordinate function on $U = x^{-1}(B_\sigma(0))$.
Since trace $A \subset U$, there is a $\sigma 1$ with $0 < \sigma 1 < \sigma$ with

$$\text{trace } A \subset x^{-1}(B_{\sigma 1}(0)) \subset U \; .$$

We define

$$K = x^{-1} (\{ r \in R^n \mid \| r \|_e = \sigma 1 \}$$

which is diffeomorphic to S^{n-1} and $Q \notin K$.
Therefore, the set $\{ \exp_Q^{-1}(P) \in M \mid P \in K \}$ does not contain the zero vector 0_Q. We denote by $w(P)$ the normalized vector

$$w(P) = \frac{\exp_Q^{-1}(P)}{\| \exp_Q^{-1}(P) \|} \in T_Q M \cap T_1 M \; .$$

Now, we introduce a map

$$\begin{aligned} \Phi : T_1 M &\longrightarrow R \times M \\ w &\longmapsto (s(w), \exp (s(w)\, w)). \end{aligned}$$

Since the components are continuous, Φ is continuous and defined on an open set of the unit sphere bundle.
We choose for $P \in K$, $0 < \epsilon(P) < (1/3) \mid s(w(P)) - m(P,Q) \mid$.

$V = \pi (\Phi^{-1}((s(w(P)) - \epsilon(P), +\infty) \times B_{\epsilon(P)}(P)))$ is an open neighborhood of Q . Note that the projection map π

is open. Let $P1 \in B_{\epsilon(P)}(P) \cap K$ and $Z \in V$, then

$$m(P1,Z) \leq m(P1,P) + m(P,Q) + m(Q,Z) < 2\,\epsilon(P) + m(P,Q)$$
$$< 2\,\epsilon(P) + s(w(P)) - 3\,\epsilon(P) = s(w(P)) - \epsilon(P).$$

Therefore, there is a $v \in T_1 M \cap T_Z M$ and

$$P1 = \exp_Z (m(P1,Z)\, v) .$$

Let $\{ B_{\epsilon(P)}(P) \mid P \in K \}$ be a collection of open balls. Since K is compact, there are finitely many balls $B_{\epsilon(P_i)}(P_i)$

such that $K \subset \bigcup_{i=1}^{k} B_{\epsilon(P_i)}(P_i)$.

We define an open set

$$O = \bigcap_{i=1}^{k} \pi (\Phi^{-1}((s(w(P_i))-\epsilon(P_i),+\infty) \times B_{\epsilon(P_i)}(P_i)) .$$

This open set contains Q. Choose an $\epsilon > 0$ such that

$$Q \in B_{\epsilon}(Q) \subset O .$$

According to the above calculation for all $Z \in B_{\epsilon}(Q)$, the trace A lies on an open normal neighborhood of Z.

Q.E.D.

In the case " a)" we can distort the base point Q on the trace of C, and find a new base point Q such that the geodesic ray τ which emitted from Q to the vertex point $C(s_i)$ is not tangent to the vector $C(s_i)$ at this vertex point.

In the case " b) " we will modify the embedding C itself locally and correlate the geodesic curvature of the modified embedding to the geodesic curvature of the previous one. According to the lemma 8 we choose an $\epsilon > 0$ such that there exists $\delta > 0$ with

$$C([0,\delta] \cup [L-\delta,L]) \subset B_\epsilon(Q) ,$$

and the condition " b) " implies that the curve C restricted on $[0,\delta] \cup [L-\delta]$ is a local geodesic. Therefore, in Riemannian normal coordinates (a,V), we can represent C without restriction of the generality in the form $a(C(s)) = (s, 0, \ldots, 0)$.

Choose a C^∞ function g1 such that

$\operatorname{supp} g1 \subset ([0,\delta) \cup (L-\delta,L])$ and $g1(0) = g1(L)$ are nonzero.For a small $|\beta| \geq 0$ define

$$H_\beta : [0,L] \longrightarrow R^n$$

$$H_\beta(s) = \begin{cases} (s, \beta \cdot g1(s), 0, \ldots, 0) & s \in ([0,\delta) \cup (L-\delta,L]) \\ a(C(s)) & \text{otherwise} \end{cases}$$

and $a^{-1}(H_\beta(0)) \in B_\epsilon(0) \quad M$, $\epsilon > 0$ as in lemma 8.
Observe that $a^{-1}(H_0(s)) = C(s)$ for all s $[0,L]$.
Consider the parameter transformation

$$d^{-1} : [0,L] \longrightarrow [0,\bar{L}]$$

where $\bar{L}$ is the length of the curve H_β , i.e.,

$$\tilde{H}_\beta(\bar{d}^1(s)) = H_\beta(s) \quad .$$

Since $\tilde{H}_\beta(s) = a(C(s))$ on $(\delta, L-\delta)$ and $d^{-1}|(\delta, L-\delta)$ is a translation,

$$\dot{H}_\beta(s) = (H_\beta)_*(D_s) = (\tilde{H}_\beta \circ d^{-1})_*(D_s)$$

$$= \dot{\tilde{H}}_\beta(d^{-1}(s)) \quad \text{for} \quad \delta < s < L - \delta$$

and

$$\nabla_D \dot{H}_\beta \big|_s = K \, (\dot{H}_\beta)_*(D_s) = K \, (\tilde{H}_\beta \circ d^{-1})_*(D_s)$$

$$= K \, (\tilde{H}_\beta)_*(D_{d^{-1}(s)}) = \nabla_D \dot{\tilde{H}}_\beta \big|_{d^{-1}(s)}$$

where K is the connection map of the Levi - Civita connection from TTM into the tangent bundle TM.

Thus, we obtain for the geodesic curvature $\tilde{\text{æ}}_M(d^{-1}(s)) = \text{æ}_M(s)$ on the interval $s \in (\delta , L - \delta)$.

As a result, we calculate the total absolute curvature of the modified curve $\tilde{H}_\beta$

$$\int_0^{\tilde{L}} \tilde{\text{æ}}_M(s) \, ds = \int_0^{d^{-1}(\delta)} \tilde{\text{æ}}_M(s) \, ds + \int_{d^{-1}(\delta)}^{d^{-1}(L-\delta)} \tilde{\text{æ}}_M(s) \, ds + \int_{d^{-1}(L-\delta)}^{\tilde{L}} \tilde{\text{æ}}_M(s) \, ds$$

$$= \int_0^{d^{-1}(\delta)} \tilde{\text{æ}} \, (s) \, ds + \int_{d^{-1}(L-\delta)}^{\tilde{L}} \tilde{\text{æ}} \, (s) \, ds + \int_{d^{-1}(\delta)}^{d^{-1}(L - \delta)} \tilde{\text{æ}} \, (s) \, ds$$

$$= A + \int_\delta^{L-\delta} \text{æ} \, (s) \, ds \, .$$

where A is the sum of the first two integrals.

Since C is a geodesic on $[0, \delta] \cup [L - \delta , L]$ we finally obtain

$$\int_0^{\bar{L}} \tilde{æ}(s)\, ds = A + \int_0^{L} æ(s)\, ds .$$

Thus, we can correlate the total absolute curvature of C to the modified curve up to a translation factor.

Let H_β be the C^∞ deformation as above. The components of the main formula of theorem 1 depends on β continuously [7, p.30]. We will demonstrate this situation in a simple example at the end of this study.

According to the above results, we can assume that for two dimensional cases $\langle w^{i+1}, w^i \rangle_e$ different from zero , i = i,..,p. As a simple consequence of the formula (13) and $E^{i+1}(s_i) = E^i(s_i)$, we know that the vectors $w^{i+1}$and w^i are at the point $(r^i(s_i),s_i)$ colinear. Considering the linear relation of the vectors w^i and the indicatrix E^idue to

$$w^i_m(y,s) = y \sum_{j=1}^{n} l^m_j (f^i(y,s)) \frac{dE^i_j}{ds}\Big|_s = d^i w^{i+1}_m(y,s)$$

$$i = 1,..,p \quad ; \quad m = 1,..,n ,$$

where $Z := ((l^m_j \circ f^i))_{m,j=1} ; \Omega^i \dashrightarrow G\ell(R,n)$ is defined as in lemma 2.

The last equation expressed in operator form is

$$Z\left(\left(d^i \frac{dE}{ds}^{i+1} - \frac{dE}{ds}^{i} \right)^t \right) = 0,$$

where d^i is a proportionality coefficient .

Since the vertex points Q_m are not on the line y=0, the above definition of $Z = Z(y,s)$ is well defined.

We will formulate in this context the behavior of the indicatrix function E at the vertex point Q_m.

LEMMA 9 :

With the above notation for two dimensional cases, $d^k > 0$ if and only if the indicatrix E is 1 -1 near to s_k , $k = 1,..,p$.

PROOF :

Since the above claim is a purely local matter, we assume, for the sake of simplicity, that $s = 0$ and furthermore, there are C^∞ functions a and b such that $a(0) = b\ (0)$ and

$$E \mid [-\epsilon,\epsilon] = \begin{cases} e^{ia(s\ )} & -\epsilon \le s \le 0 \\ e^{i\ b(s)} & 0 \le s \le \epsilon \end{cases} .$$

The right hand side and left hand side derivatives of the functions a,b yield

$$(e^{\ i\ a(o)})' = i\ a'(0)\ e^{i\ a(0)} = (1/d^k)\ i\ b'(0)\ e^{i\ b(0)}$$
$$= (\ e^{ib(0)}\ )'.$$

Therefore, $d^k a'(0) = b'(0)$, and the Taylor expansions of both functions for $0 \le s \le \tilde{\delta}$, $\tilde{\delta}$ is suitable,

$$a(-s) = a(0) - a'(0)\ s + O(s^2)$$
$$b(\ s) = b(0) + b'(0)\ s + O(s^2)\ .$$

Therefore

$$\operatorname{sgn}\ (\ b(s)-b(0)) = \operatorname{sgn}\ b'(0)$$
$$\operatorname{sgn}\ (\ a(-s)-a(0)) = -\operatorname{sgn}\ a'(0) = -\ \operatorname{sgn}\ b'(0)\ .$$

Since $d^k > 0$,

$$\operatorname{sgn}\ (b(s)\ -b(0)) = -\ \operatorname{sgn}\ (a(-s)\ -\ a(0))\ .$$

Define a new function

$$A(s) = \begin{cases} a(s) & \text{for } -\tilde{\delta} \le s \le 0 \\ b(s) & \text{for } 0 \le s \le \tilde{\delta} \end{cases} .$$

We claim there is a small $\delta 1$, $\tilde{\delta} \ge \delta 1 \ge 0$ such that the function A on the interval $[-\delta 1, \delta 1]$ is injective.

Let us assume that the function A is not 1 -1 on $[-\delta 1, \delta 1]$ for each $\delta 1$. Then, there are zero sequences

$$((s_n))_{n \in N}, ((\tilde{s}_n))_{n \in N}$$

such that

$$-\delta \le s_n < 0 < \tilde{s}_n \le \delta \quad \text{and}$$

$$a(s_n) = b(\tilde{s}_n).$$

We have
$$-\operatorname{sgn}(a(-\tilde{s}_n) - a(0)) = \operatorname{sgn}(b(\tilde{s}_n) - b(0))$$
$$= \operatorname{sgn}(a(s_n) - a(0)).$$

Since the function a on $[-\tilde{\delta}, 0]$ is 1-1 and $a(s_n)$ not equal a(0) there is a $\bar{s}_n$ such that either

$$-\tilde{s}_n < \bar{s}_n < s_n \quad \text{or} \quad s_n < \bar{s}_n < -\tilde{s}_n \quad \text{and}$$

$$a(\bar{s}_n) = a(0),$$

which is obvious since a is continuous. $((s_n))_{n \in N}$ is a zero sequence which clearly contradicts to the fact that a is 1 -1 near to zero.

Conversely, let us assume that the indicatrix function E is 1 -1 in a neighborhood of s = 0. We claim

$$d^k = a'(0)/b'(0) > 0$$

or
$$\operatorname{sgn}(a'(0)) = \operatorname{sgn}(b'(0)).$$

Let us assume that $\operatorname{sgn}(a'(0)) = -\operatorname{sgn}(b'(0))$

then, define $d1 := \min\left(\max_{s \in [-\delta, 0]} (a(s)), \max_{s \in [0, \delta]} (b(s))\right)$ such that both functions a,b on $[-\delta, 0]$ and $[0, \delta]$ are respectively monotonic.

For each $0 < d'' < d1$, the line $y = d''$ intersects both functions a,b in a vicinity of zero only once. Choose a zero sequence $((d''_n))_{n \in N}$

such that $0 < d''_n < d1$.

Thus we obtain two zero sequences

$$-\delta < s^*_n < 0 < s^{**}_n < \delta$$

with $a(s^*_n) = d''_n = b(s^{**}_n)$

Therefore, the indicatrix E is not 1-1 in a neighborhood of 0.

q.e.d.

DEFINITION :

The indicatrix E intersects $Q_i, i = 1,..,p$, transversally if and only if E is 1-1 in a neighborhood of s_i .

II. GLOBALIZATION OF THE SHELL THEORY FOR TWO DIMENSIONAL MANIFOLDS

To formulate a global version of the theorem 1, we will introduce some notations from combinatorical topology. Let M be a Riemannian manifold with boundary ∂M and J a simplicial complex, and $t : J \longrightarrow \partial M$ be a C^r triangulation of the boundary.
An extension of t is a C^r triangulation $G: L \longrightarrow M$ of M such that $G^{-1} o\, t$ is a linear isomorphism of J with a subcomplex of L. It is well known that [11,p.101], when M is a manifold having a boundary, any C^r triangulation of the boundary may be extended to a C^r triangulation of M.

Let S be a two dimensional manifold, i.e.,a surface. A region $R \subset S$ is said to be regular if R is compact and its boundary ∂R is the finite union of simple closed piecewise regular curves which do not intersect. Let S be an oriented surface, and $\{ x_\alpha \mid \alpha \in A \}$ be a family of parametrizations compatible with the orientation of S. Let $R \subset S$ be a regular region of S. Then,there is a triangulation $\mathfrak{E}$ of R such that every triangle $T_j \in \mathfrak{E}$ is contained in some coordinate neighborhood of the family $\{ x_\alpha \mid \alpha \in A \}$. Furthermore, if the boundary of every triangle ∂T_j of $\mathfrak{E}$ is positively oriented, adjacent triangles determine opposite orientations in the common edge [12, Chp.1], [13, p.127].

Let Q_k, $k = 1,\dots,p$ be the vertices of the boundary ∂R. We denote by τ_k the outer angles of ∂R. Let $\mathfrak{E} = \{ T_j \mid j = 1,\dots,F \}$ be an extension of a triangulation $\mathfrak{E}$ of the boundary ∂R. Moreover, let each triangle T_j lie in a coordinate neighborhood of the family $\{ x_\alpha \mid \alpha \in A \}$ such that each ∂T_j be positively oriented.

To clarify the relationship between the outer angles τ_i, $-\pi \le \tau_i \le \pi$, of the space curve C at the vertices Q_i and the shell angles α^i, α^{i+1}, $-\pi/2 \le \alpha^i, \alpha^{i+1} \le \pi/2$, we investigate the orientation of the function

$$f^k(y,s) = \exp_0 (y\, E^k(s))$$

at a vertex point Q_i. Let M be an oriented two dimensional manifold. Let C be parametrized such that the normal vector of C shows inside of C. We choose $X_1 = \dot{C}(0)$ and X_2 = the normal vector of C at the point C(0). Using the normal coordinates, with the help of the formulas (8), we can identify the tangent map $f^k_{*}\big|_{(r^k(s_i),s_i)}$ with the matrix

$$\left.\begin{pmatrix} E_1^k(s) & E_2^k(s) \\ y\,\dfrac{dE_1^k}{ds}(s) & y\,\dfrac{dE_2^k}{ds}(s) \end{pmatrix}\right|_{(y,s) = (r^k(s_i),s_i)} \qquad (32)$$

$i = 1,\dots,p$ and $k = 1,\dots,p+1$.

Since $r^k(s_i) > 0$,

$$\operatorname{sgn} (\det f^k_{*|(r^k(s_i),s_i)}) = \operatorname{sgn} (\det (E^k(s_i), \frac{dE^k}{ds}(s_i)))$$

We define f^k is at Q_i orientation preserving iff

$$\operatorname{sgn} (\det (f^k_{*|(r^k(s_i),s_i)}) > 0 , \quad k = i,i+1.$$

Let us assume that E intersects the embedding C at the point Q_i nontransversally [p.37] in the sense of previous definition. This means

$$\operatorname{sgn} (\det (f^i_{*|(r^i(s_i),s_i)}) = - \operatorname{sgn} (\det (f^{i+1}_{*|(r^i(s_i),s_i)}).$$

For the transversal case, with f^i, f^{i+1}, both orientation preserving we obtain by (31)

$$\tau_i + (\alpha^{i+1} - \alpha^i) = 0 \quad . \qquad (33)$$

If both functions are orientation reversing then

$$\tau_i - (\alpha^{i+1} - \alpha^i) = 0 \quad .$$

THEOREM 3 :

Let $\tilde{R} \subset S$ be a regular region of an oriented surface and let $C_1 , \ldots , C_q$ be simple closed piecewise regular curves which form the boundary $\partial \tilde{R}$ such that Q_k , $k=1,..,p$ be the vertices of $C_l, l=1,..,q$. Let $\mathcal{T} = \{ T_j \mid j =1,..,F \}$ be a triangulation of the region R such that every triangle T_j is contained in a normal neighborhood of B_j, which is a nonvertex boundary point of T_j and let the boundaries be positively oriented. Construct (Ω_j , f_j) for every triangle $j = 1,..,F$ with the base points B_j. We shall denote by Ω^a_j the a-th shell pie of Ω_j with the vertices Q^a_j .

Let K_j^a be the Gaussian curvature and dA_j^a be the area measure of the shell pie (Ω^a, f^a_j) , $a = 1,..,4$. We denote by l_j the length of the indicatrix function E_j of (Ω_j, f_j) where every E_j intersects Q^a_j transversally. Let f^{a+1}_j, f^a_j be orientation preserving for each vertex point, then

$$\sum_{l=1}^{q} \int_0^{L_l} æ_l(s)\, ds + \sum_{j=1}^{F} \sum_{a=1}^{4} \iint_{\Omega^a_j} K^a_j \, dA^a_j + \pi F + \sum_{k=1}^{p} \tau_k$$

$$= 2\pi X(\widetilde{R}) + \sum_{j=1}^{F} l_j \quad .$$

$X(\widetilde{R})$ denotes the Euler-Characteristic of the enclosed region $\widetilde{R}$, and $æ_l$ is the extended geodesic curvature of C_l and τ_k, $k = 1,..,p$, are the external angles of the curves C_l.

PROOF :

We will apply the local shell theory to every triangle T_j and add up the results. Let $T_j \in$ Æ1 be a triangle with $B_j \in T_j$, a nonvertex base point. Since T_j lies in a normal neighborhood of B_j, we can apply local shell theory on the boundary of T_j . We choose a realization of ∂T_j and again denote it by ∂T_j, i.e., $\partial T_j : [0, \bar{L}_j]$ $\dashrightarrow T_j$, $\partial T_j = \partial T_j(s)$ where s is the arc length parameter. Thus,

$$\sum_{a=1}^{4} \int_{s^{a-1}_j}^{s^a_j} æ^a_j(s)\, ds + \sum_{a=1}^{4} \int\int_{\Omega^a_j} K^a_j \, dA^a_j = \pi + \sum_{a=1}^{4} l^a_j -$$

$$-(\alpha^1_j + \alpha^2_j + \alpha^3_j) \quad .$$

α_j^d , d =1,.,3 denote the external angles of T_j .

We shall now introduce the interior angles of T_j , given by

$$\alpha_j^d = \pi - \beta_j^d \quad , d = 1,.,3$$

Thus,

$$\sum_{j=1}^{F} \sum_{d=1}^{3} \alpha_j^d = \sum_{j=1}^{F} \sum_{d=1}^{3} \pi - \sum_{j=1}^{F} \sum_{d=1}^{3} \beta_j^d = 3\pi F - $$

$$- \sum_{j=1}^{F} \sum_{d=1}^{3} \beta_j^d \quad .$$

Let E_e = the number of external edges of Æ1

E_i = the number of internal edges of Æ1

V_e = the number of external vertices of Æ1

V_i = the number of internal vertices of Æ1

$E = E_e + E_i \; ; \; V = V_e + V_i$

Since the curves C_k are closed $E_e = V_e$. We obtain by induction

$$3 F = 2 E_i + E_e .$$

Thus,

$$\sum_{j=1}^{F} \sum_{d=1}^{3} \alpha_j^d = 2 \pi E_i + \pi E_e - \sum_{j=1}^{F} \sum_{d=1}^{3} \beta_j^d$$

We observe that we can collect the numbers of external vertices Æ1 in two groups, vertices of some curve C_k and vertices introduced by the triangulation, i.e.,

$$V_e = V_{ec} + V_{et},$$

where V_{ec} is the number of vertices of the curves C and V_{et} the number of external vertices of Æ1, which are not vertices of some curves C_k. Notice that the sum of angles around each internal vertex is 2π, thus we get

$$\sum_{j=1}^{F}\sum_{d=1}^{3}\alpha_j^d = 2\pi E_i + \pi E_e - 2\pi V_i - \pi V_{et} - \sum_{k=1}^{p}(\pi - \tau_k).$$

Since $E_e = V_e$, we conclude that

$$\sum_{j=1}^{F}\sum_{k=1}^{3}\alpha_j^k = 2\pi E_i + 2\pi E_e - 2\pi V_i - \pi V_e - \pi V_{et} - \pi V_{ec} + \sum_{k=1}^{p}\tau_k = 2\pi E - 2\pi V + \sum_{k=1}^{p}\tau_k .$$

This implies, with the theorem 1,

$$\sum_{a=1}^{4}\int_{s_j^{a-1}}^{s_j^a} \text{æ}_j^a(s)\,ds + \sum_{a=1}^{F}\int\int_{\Omega_j^a} K_j^a\, dA_j^a + (\alpha_j^1 + \alpha_j^2 + \alpha_j^3)$$

$$= \pi + \sum_{a=1}^{4} l_j^a ,$$

with $0 = s_j^0 < s_j^1 < .. < s_j^4 = \bar{L}_j$

$$\sum_{j=1}^{F}\sum_{a=1}^{4}\int_{s_j^{a-1}}^{s_j^a} \text{æ}_j^a(s)\,ds + \sum_{j=1}^{F}\sum_{a=1}^{4}\int\int_{\Omega_j^a} K_j^a\, dA_j^a - \sum_{j=1}^{F}\sum_{a=1}^{4}\alpha_j^a =$$

IV. DEMONSTRATION

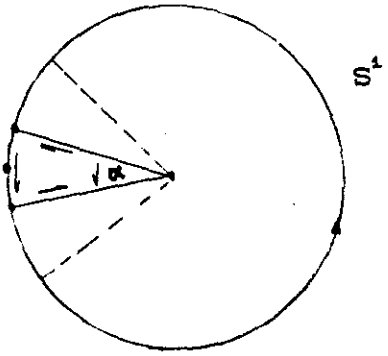
a)

Consider the standard embedding of S^1 into R^2 plane

$$C : [0,2\pi] \dashrightarrow R^2$$

$$s \longmapsto e^{is} .$$

We choose a triangulation as below



Let α be an angle such that $m\alpha = 2\pi$.

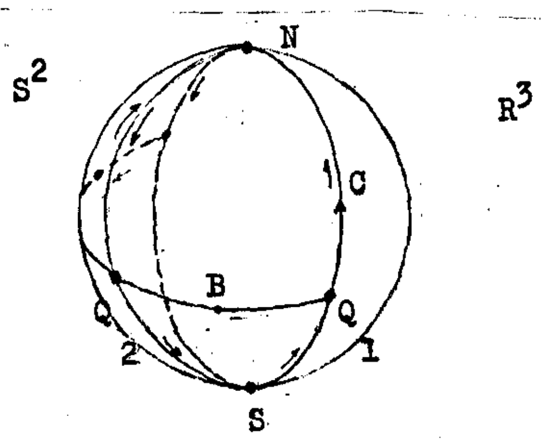
Then,

$$\int_0^{2\pi} \ae \; ds \; + \; m \; \pi \; = \; 2\pi + (\, l_1 + \ldots . + l_m \,) \, .$$

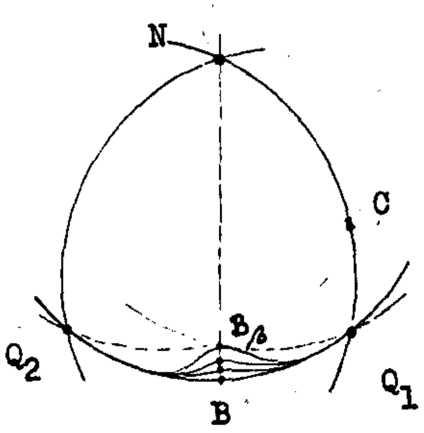
Because of the convexity of the almost triangle shaped shells we obtain

$$\int_0^{2\pi} \ae \; ds \; = \; 2 \, \pi \, , \quad \text{as expected.}$$

b) In the second example we take as manifold the S^2 sphere and as C, a great circle through north- and south-pole. In contrast to the first example the embedding C has no point such that C lies in a normal neighborhood of it. We choose a triangulation of the left hemisphere of S^2 by two intersecting great semi circles as below .



The triangles of this triangulation are made of minimal geodesics. In order to apply the theorem 2 on this triangulation, choose a nonvertex base point B on a geodesic. Since B is on a geodesic, we make, according chapter two, a small deformation inwards of the triangle. We can reach every vertex point of the triangle from the "top" of this deformation. Since the bump is inwards, the intersection of the geodesics, which are emitted from the top of the deformation, to the vertices, are transversal. Therefore, we can apply the theorem 2 on the shells with the base point B which is the top of the hill. If we let the deformation parameter β converge to zero, the outer angles of the triangle are not affected by this limiting process, and $\iint_{\Omega^\beta} K_\beta \, dA_\beta$, l_β, $\int_0^{L_\beta} \varkappa_\beta \, ds$ converge to $\iint_\Omega K \, dA$, l, $\int_0^L \varkappa \, ds$ [7, p.30]



$$\sum_{i=1}^{4} \int_{s_{i-1}}^{s^{i}} \text{æ}^{i}\, ds + \pi/2 + 3\cdot(\pi/2) = \pi + \pi$$

or for the global formula

$$\int_{0}^{L_C} \text{æ}\, ds + \sum_{i=1}^{4} \int\int_{\Omega^{i}} dA^{i} + 4\pi = 2\pi + (l_1 + \cdots + l_4),$$

i.e.,

$$\int_{0}^{L_C} \text{æ}\, ds = - 2\pi - 4\pi + 2\pi + 4\pi = 0$$ as expected.

EPILOG

This study shows that, applying essentially Gauss - Bonnet theorem, we can find a global shell formula for simply closed curves embedded in two dimensional manifolds. The global formula of theorem 2 relates purely differential geometrical magnitudes of curve C with a pure topological invariant which is the Euler characteristic of the area enclosed by C.

As usual,in the applications of the Gauss-Bonnet theorem, we can play topology and geometry one against the other to gain more information about curve C.
As we have shown, the formula in theorem 2 depends on certain triangulations. For an arbitrary triangulation, the relationship between the shell angle and curve angle is more complicated than it is in Formula 33. Although it is easy to find a general formula, it is impractical and difficult in use. However in view of the above mentioned duality, one could probably use this formula to prove the existence of convex triangulations of the area enclosed by C.

If the manifold M is n - dimensional and the curve C lies in a two dimensional submanifold S, we can again use the globalization theorem. Taking the second fundamental form of S into consideration, we obtain more information about the total absolute curvature of C in M, especially when S is a totally geodesic manifold.

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