ACCRETION DISCS AROUND COMPACT OBJECTS

by

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Abstract

Accretion onto a compact object is the process that is thought to power unexpectedly intense radiation of some astrophysical objects like Active Galactic Nuclei and X-Ray Binaries. A similar process is known to exist in Cataclysmic Variables and Young Stellar Objects. When the accreting matter has enough angular momentum it forms a disc around the compact object (a black hole, neutron star or white dwarf) and this disc is called an accretion disc.

In this theses accretion discs around compact objects are studied both analytically and numericaly. Simplifying hydrodynamic equations in their full generality by the assumptions of the standart accretion disc theory both time-independent and time-dependent equations were solved analytically. Time-dependent solutions are supported by numerical results. In all solutions a viscosity prescription which is a little more generalized than the standart prescription is used. The solution of the time-dependent equations with the generalized viscosity belong to the same family with the earlier ones.

Özet

Bir tıkız nesnenin üzerine madde yığılmasının Aktif Galaktik Çekirdek ve X-ışını çiftleri gibi astrofiziksel nesnelerin beklenmedik yeğinlikteki ışımasını güdümleyen süreç olduğu düşünülmektedir. Benzer bir sürecin Coşkun Değişenler ve Genç Yıldızsı Nesneler'de de yer aldığı bilinmektedir. Yığılan madde yeterli açısal momentuma sahipse tıkız nesnenin (karadelik, nötron yıldızı veya beyaz cüce) etrafında bir disk oluşur ve bu diske birikim diski adı verilir.

Bu çalışmada tıkız nesnelerin etrafindaki birikim diskleri hem analitik hem de sayısal olarak incelenmiştir. Hidrodinamik denklemler en genel biçimlerinden Standart Birikim Diski Kuramı'nın varsayımları ile sadeleştirilerek, hem zamandan bağımsız hem de zamana bağlı durum için, analitik olarak çözülmüştür. Zamana bağlı çözümler sayısal sonuçlarla desteklenmiştir. Tüm çözümlerde standart teorininkinden biraz daha genelleştirilmiş bir viskozite reçetesi kullanılmıştır. Zamana bağlı denklemlerin genelleştirilmiş viskozite ile yapılan analitik çözümü daha öncekilerle aynı ailedendir.

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a	binary seperation
b	thickness of the boundary layer
$\mathbf{B}$	magnetic induction vector field
c	speed of light
c_s	speed of sound
D	dissipation per unit surface area
e	total energy per unit mass
E	radiation energy density
$\mathbf{E}$	electric vector field
$\mathbf{f}$, $\mathbf{f_{mag}}$	force density and magnetic force density
$\mathbf{F}$	radiative flux vector
G	gravitational constant
H	thickness of the disc
J	specific angular momentum
$\mathbf{J}$	current density vector field
k	Boltzmann's constant
L, $L_\odot$	luminosity, solar luminosity
L_1	inner Lagrangian point
m	time parameter in the numerical scheme
$mMax$	maximum number of time steps
m_p	proton rest mass
M_1	mass of the primary (accreting) star
M_2	mass of the secondary (mass donor) star
M_{disc}	mass of the disc
$M_\odot$	solar mass
$\dot{M}$	accretion rate
P, P_{mag}	pressure and magnetic pressure
P_{gas}, P_{rad}	gas and radiation pressure
pc	parsec

$\mathbf{q}$	conductive flux of heat
r	radial distance from the center of the accreting object
r_1	distance of the primary from the CM of the binary system.
r_{circ}	circularization radius
R_1	radius of the accreting object
R_{in}, R_{out}	inner and outer radius of the accretion disc
S_ρ	mass source term
$t_{dyn},\ t_{th},\ t_{visc}$	dynamical, thermal and viscous time scales
T	temperature
$\mathbf{v}$	velocity vector
$v_{Alfv\grave{e}n}$	Alfven Velocity
v_r, v_φ, v_z	radial, circularization and vertical velocity
α	viscosity parameter
ϵ	internal energy per unit mass
η	dynamical viscosity
$\bar{\kappa}$	opacity
σ	conductivity, Stephan's constant
Σ	surface mass density
μ	mean molecular weight
μ_0	magnetic permeabilty of vacuum
$\nu,\ \nu_{turb}$	kinematic viscosity and turbulent viscosity
ρ	volume mass density
τ	optical thickness
$\tilde{\tau}$	viscous stress tensor
τ_{ij}	cartesian components of the viscous stress tensor
Φ	dissipation function
χ	bulk viscosity
Ψ_R	Roche potential
ω_{orb}	orbital angular velocity
Ω	angular velocity
$\Re,\ \Re_{mag}$	Reynold's number and Magnetic Reynold's number

1. INTRODUCTION

Accretion is a process in which an astrophysical object aggregates matter from its surrounding by its gravitational attraction. The source of the accreting matter could be the interstellar medium or a star accompanying the accreting star in a binary system.

The importance of accretion was first recognized in the context of the formation of planets emerging from the early solar nebula. Jeffreys (1924) correctly described the evolution of a differentially rotating viscous disc. In the beginning of 1940's Peek and von Weizsäcker concluded that the early solar nebula would seperate into two parts –a central core containing most of the mass and a disc containing most of the angular momentum–by the action of viscous torques which are increased by turbulence (Pringle, 1981). The equations of motion, which are derived in section (4.2), were first derived by von Weizsäcker in 1948 who also put forward the argument that turbulent viscosity must be the dominant dissipation process and hypothesized a mixing length prescription in which the mixing length varies as a given power of radius. A somewhat more general solution was given by Lüst in 1952.

Accretion is spherical when the accreting matter has no angular momentum with respect to the accreting object. Hoyle and Lyttleton (1939) examined the possible change in luminosity of a main sequence star due to its passage through an interstellar (cold) gas cloud and derived the accretion rate. Bondi (1952) calculated the accretion rate and gave a full analytic solution of the flow of a polytropic gas falling onto a static gravitating body. These equations can also describe the case of outflow, i.e. stellar winds.

Accreting matter forms a disc if its specific angular momentum J is too large for it to hit the accreting object directly. That is, if the circularization radius

$$r_{circ} = \frac{J^2}{GM_1} \tag{1.1}$$

is greater than the effective size of the accreting object an accretion disc will form. This condition is amply satisfied in some binary systems where one of the stars fills its Roche lobe and overflows matter through the inner Lagrange point, described in page(6), onto the other star a compact object (a white dwarf, neutron star or a black hole). An accretion disc is more likely to form if the accreting star is compact because then the scale length associated with the mass donor star is much larger than the object onto which accretion occurs and so the circularization radius is more likely to exceed the effective size of the accreting object. While accreting onto a compact object the gas has to reduce its specific angular momentum by up to ~ 3 orders of magnitude. As is described below this presents a severe problem to accretion theory because the ordinary (molecular) viscous torques and other trivial torques on the disc are not enough to account this much of transport of angular momentum.

In 1960's were discovered the quasars (Schmidt, 1962) which had extremely large luminosities ($L \sim 10^{12} L_{\odot}$) produced in comparatively small regions ($r \sim 1\ pc$) and galactic X-ray sources (Giacconi et al., 1962) emitting large powers at very high temperatures ($T \sim 10^7 K$). Salpeter (1964) and Zel'dovich were the first to recognize the possible importance of accretion as an astrophysical energy source in quasars. Shlovsky (1967) was the first to propose that a galactic X-ray source (Scorpius X-1) could be a binary system in which accretion onto a neutron star supplies the observed luminosity. The role of an accretion disc in XRBs and the similarity with the cataclysmic variables were pointed out by Prendergast and Burbidge (1968). The importance of disc accretion around a massive black hole was recognized by Lynden-Bell (1969) and Pringle and Rees (1972) and a detailed discussion of accretion discs with a computation of emission spectrum was given by Shakura and Sunyaev (1973).

In an accretion disc the matter follows almost keplerian orbits. That is, the angular velocity Ω of matter at radius r is given by

$$\Omega = \sqrt{\frac{GM_1}{r^3}} \tag{1.2}$$

This expression for angular velocity manifests that keplerian rotation is a differential rotation $d\Omega/dr \neq 0$ and so there will be viscous stresses acting between the adjacent layers of disc. The action of the viscous stresses will have two results: (a)The inner sides that has greater angular velocity will drag the outer parts to have greater velocity and the act of the outer parts will delay the motion of the inner parts. So the inner parts will loose kinetic energy and sink down to keplerian orbits with smaller radius. However, in order to conserve the angular momentum, the outer parts which gained energy will recede from the centre, (b) Because of the differential rotation between adjacent layers the viscous stresses will dissipate energy and make the disc radiate at a power proportional to the compactness of the accreting object.

Accretion discs, today, appear in many diverse contexts in astrophysics such as Cataclysmic Variables (CVs), X-ray binaries (XRBs), Young Stellar Objects (YSOs) and Active Galactic Nuclei (AGN). The direct evidence for accretion discs is irrefutable in CVs, convincing in YSOs and remarkable in AGN (Papaloizou and Lin, 1995). Releasing of gravitational energy through accretion is the only possible known way of producing the luminosity output of XRBs.

The study of accretion discs in Cataclysmic Variables is of special importance. The discs in these systems are the most easy to observe and the best understood. As the theory of accretion discs suffer from many problems, it is hoped that improvements in the understanding of the discs will be achieved by understanding the discs in these "easier" systems first. A subtype of cataclysmic variables called dwarf novae exhibit interesting time dependent phenomena understanding of which is important in revealing the fundamental secret of the accretion discs, the viscosity.

2. HYDRODYNAMICAL EQUATIONS FOR ACCRETION DISCS

In this section the equations of hydrodynamics describing an accretion disk in a binary system are given. The accreting (primary) star is at the origin and the system is rotating around its center of mass. The equations are written in a frame of reference rotating with the binary system (corotating frame) so that the stars are fixed.

It is assumed that the mass of the material in the disc is much less than the mass of the central star, $M_{disc} \ll M_1$ so that the self-gravity of the disc can be neglected. It is also assumed that the relativistic effects can be neglected:

$$\frac{v}{c} \ll 1 \qquad \frac{GM_1}{rc^2} \ll 1 \tag{2.1}$$

Here r is the distance from the center of the accreting object. This assumption is valid everywhere for discs around neutron stars and white dwarfs and everywhere except at $r \lesssim 3r_{in}$ for discs around black holes (Lightman, 1974).

The state of a moving fluid is determined by five quantities: The three components of the velocity $\mathbf{v}$ and any two of the three thermodynamical quantities (pressure P and density ρ, for example)(Landau and Lifshitz, 1959). Accordingly a complete system of equations of fluid dynamics should be five in number. These are conservation of mass, conservation of momentum (three equations for three components) and conservation of energy. The third thermodynamic quantity (T in this case) can be found by means of an equation of state after these five equations are solved.

The intense radiation fields of accretion discs can influence heavily the momentum and energy gains and loses of the material, and hence its motions. Thus, one must consider the fluid of accretion discs consisting of both material particles and photons, and calculate the contributions of both types of particles to the equations of motion and of energy conservation. In this way one obtains the equations of radiation hydrodynamics, which describe the coupled flow of the gas and the radiation.

2.1. Mass and Momentum Equations

We write the equations in a conservative form where the LHS has the time derivative of a quantity plus the divergence of the flux of that quantity. Then anything on the RHS would be a source (if positive) or a sink (if negative) term. If the RHS is zero then the quantity is conserved for the system.

The equation of continuity which ensures the conservation of mass is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.2}$$

Here ρ is the mass per unit volume and $\mathbf{v}$ is the velocity field. There may be added a source term S_ρ at the RHS of (2.2) which could either represent the effect of the mass transfer stream ($S_\rho > 0$) or the effects of mass loss through a wind ($S_\rho < 0$).

The equation for the conservation of momentum is

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = -\nabla P + \nabla \cdot \tilde{\tau} - \rho \nabla \Psi_R - 2\rho\omega_{orb} \times \mathbf{v} \tag{2.3}$$

Here P, τ and Ψ_R represents the pressure, the viscous stress tensor and the Roche potential, respectively. The last term on the RHS is the Coriolis force where ω_{orb} is the angular velocity vector of the binary system

$$\omega_{orb} = \sqrt{\frac{G(M_1 + M_2)}{a^3}}\hat{\mathbf{z}} \tag{2.4}$$

a being the binary seperation.

The Roche potential in cylindrical coordinates is

$$\begin{aligned} \Psi_R(r, \varphi, z) &= -\frac{GM_1}{\sqrt{r^2 + z^2}} - \frac{GM_2}{\sqrt{r^2 + a^2 - 2ar\cos\varphi + z^2}} \\ &\quad -\frac{1}{2}\omega_{orb}^2(r^2 + r_1^2 - 2rr_1\cos\varphi) \end{aligned} \tag{2.5}$$

where the first term is the gravitational potential of the primary (the accreting star) and the second term is the gravitational potential of the secondary (mass donor star). Here the mass of the accretor (primary) and the donor star (secondary) are denoted by M_1 and M_2 respectively. The third term will give the centrifugal force due to the rotation about the center of mass when the gradient is acted upon. Also note that

$$r_1 = \frac{M_2}{M_1 + M_2}a \tag{2.6}$$

is the distance of accreting star from the center of mass of the system. The innermost equipotential surface pass through the *inner Lagrange point* L_1 which is a saddle point of Φ_R on the line connecting the center of the two stars. This special equipotential surface is called the *Roche lobe* and has a dumbell shape in three dimensions.

The cartesian components of the viscous stress tensor (described in appendices and in the references therein) are

$$\tau_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \delta_{ij}(\chi - \frac{2}{3}\eta) \frac{\partial v_k}{\partial x_k} \tag{2.7}$$

where η is the dynamical viscosity and χ is the bulk viscosity. Bulk viscosity is negligible except in the study of the structure of shock waves and in the absorption and attenuation of acoustic waves (Anderson et al. 1984). It is zero for monatomic gases (Hughes and Brighton, 1991). The first part of the viscous stress tensor represents the contribution from the shearing between different layers of the fluid whereas the second part is the contribution due to the compressibility of the fluid. This second part is usually neglected because the shear contribution (especially $\tau_{r\varphi}$ is overwhelmingly large in discs. However there may be locations in the disc (the central parts near the star) where the bulk viscosity (so the second term) is important (Kley et al., 1993)

2.2. The Energy Equation

The equation for the conservation of energy is

$$\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) = -\nabla \cdot (P\mathbf{v}) + \nabla \cdot (\tilde{\tau}\mathbf{v}) - \nabla \cdot \mathbf{q} - \nabla \cdot \mathbf{F} \tag{2.8}$$

where $e = \frac{1}{2}v^2 + \epsilon + \Psi_R + E/\rho$ is the total energy per unit mass. In (2.8) the first term on the right represents the pressure work per unit volume per unit time. The second term on the right is the work done by the viscous stresses per unit volume per unit time. In the third term $\mathbf{q}$ is the conductive flux of heat and measures the rate at which random motions in the gas transport thermal energy, smoothing out the temperature differences. ϵ is the internal (thermal) energy per unit mass. E is radiation energy density and $\mathbf{F}$ is the radiative flux vector. The term $-\nabla \cdot \mathbf{F}$ gives the rate at which radiant energy is being lost by emission, or gained by absorption per unit volume of the gas.

The second term on the RHS of (2.8) can be written as $\nabla \cdot (\tilde{\tau}\mathbf{v}) = \Phi + \mathbf{v} \cdot (\nabla \cdot \tilde{\tau})$ where

$$\Phi = (\tilde{\tau} \cdot \nabla)\mathbf{v} \tag{2.9}$$

is called the viscous dissipation function which is the rate at which deviatoric streses do irreversible work on the fluid. The viscous dissipation function in cylindrical coordinates is given in the appendix C.

The energy equation (2.8) is useful but has the momentum equation (2.3) hidden in it. When the momentum equation (ignoring the Coriolis force) is multiplied by $\mathbf{v}$ and subtracted from the energy equation one obtains

$$\frac{\partial}{\partial t}(\rho\epsilon + E) + \nabla \cdot [(\rho\epsilon + E)\mathbf{v}] = -P\nabla \cdot \mathbf{v} + \Phi - \nabla \cdot \mathbf{q} - \nabla \cdot \mathbf{F} \tag{2.10}$$

Surely many other simplifications can be made in these equations in order to make them tractable and indeed that is what is done in chapter 4.

3. MAGNETOHYDRODYNAMIC EQUATIONS

The gas flowing in an accretion disc around a compact object is ionized (and thus is conducting) and it is immersed in the magnetic field of the accreting compact object. This means that for a full discussion of the accretion disc problem one must consider the effects of the magnetic field. In this section we will write the MHD equations for this situation.

3.1. MHD Equations

The complete set of MHD equations for an accretion disc in a binary system are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{3.1}$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = -\nabla P + \nabla \cdot \tilde{\tau} + \mathbf{J} \times \mathbf{B} - \rho \nabla \Psi_R - 2\rho \omega_{orb} \times \mathbf{v} \tag{3.2}$$

$$\frac{\partial (\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) = -\nabla \cdot (P\mathbf{v}) + \nabla \cdot (\tilde{\tau} \mathbf{v}) + \mathbf{E} \cdot \mathbf{J} - \nabla \cdot \mathbf{q} - \nabla \cdot \mathbf{F} \tag{3.3}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{3.4}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{3.5}$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{3.6}$$

where the first three equations stand for the conservation of mass, momentum and energy ($e = \frac{1}{2}v^2 + \epsilon + \Psi_R + E/\rho$), respectively. The third term on the RHS of (3.2) is the magnetic force. The equations (3.4) and (3.5) are Faraday's and Ampére's law, respectively. Equation (3.6) is the Ohm's law (Jackson, 1975). These equations are to be supplemented by an equation of state to find the temperature.

The displacement term $\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$ is neglected from the Ampére's law (3.5), on the grounds that we are not dealing with relativistic velocities. This can be justified as follows: (3.4) implies that $\mathbf{E}$ is of the order of BL/τ where L is a characteristic length and τ is a characteristic time of the system. Equation (3.1) indicates that $L/\tau \sim v$ and thus $E \sim vB$. Now, the term $\nabla \times \mathbf{B}$ in (3.5) which is of the order of B/L is much greater than the order of the displacement current term $E/(c^2\tau) = (v/c)B/(c\tau) = (v/c)B/L$ if one is not dealing with relativistic velocities.

3.2. Magnetic Diffusion and Freezing of the Field Lines

An equation for the evolution of the magnetic field will be obtained. Eliminating J between (3.6) and (3.5)

$$\mathbf{E} = \frac{1}{\mu_0 \sigma} \nabla \times \mathbf{B} - \mathbf{v} \times \mathbf{B} \tag{3.7}$$

and using this in (3.4) one finds

$$\frac{1}{\mu_0\sigma}\nabla\times(\nabla\times\mathbf{B})-\nabla\times(\mathbf{v}\times\mathbf{B})+\frac{\partial\mathbf{B}}{\partial t}=0 \tag{3.8}$$

This can be simplified, by using the vector identity $\nabla\times(\nabla\times\mathbf{B}) = -\nabla^2\mathbf{B}-\nabla(\nabla\cdot\mathbf{B})$ and the Maxwell equation $\nabla\cdot\mathbf{B}=0$, as follows

$$\frac{\partial\mathbf{B}}{\partial t}=\frac{1}{\mu_0\sigma}\nabla^2\mathbf{B}+\nabla\times(\mathbf{v}\times\mathbf{B}) \tag{3.9}$$

The two limit cases that can give insight for the evolution of magnetic field in a plasma are examined below:

(A)The fluid at rest: For $\mathbf{v}=0$ (3.9) reduces to

$$\frac{\partial\mathbf{B}}{\partial t}=\frac{1}{\mu_0\sigma}\nabla^2\mathbf{B} \tag{3.10}$$

which is a diffusion equation. This means that an initial configuration of magnetic field will decay away in a diffusion time

$$\tau=\mu_0\sigma L^2 \tag{3.11}$$

where L is a length characteristic of the spatial variation of $\mathbf{B}$.

(B)The fluid with infinite conductivity: For $\sigma\to\infty$ equation (3.9) reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \tag{3.12}$$

which means that the magnetic flux through any loop moving with the local fluid velocity is constant in time. This effect is also described by saying that the lines of force are frozen into the fluid and are carried along with it. A plasma with $\sigma \to \infty$ is called an ideal plasma. Note that for an ideal plasma $\mathbf{E} + \mathbf{v} \times \mathbf{B}$ in (3.6) must vanish so that $\mathbf{J}$ remains finite. Thus

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} \tag{3.13}$$

3.3. Magnetic Pressure

Using $\mathbf{J}$ from equation (3.5) in its place in equation (3.2) one obtains for the magnetic force density

$$\mathbf{f}_{mag} = \mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \tag{3.14}$$

which, by the vector identity $\mathbf{B} \times (\nabla \times \mathbf{B}) = \frac{1}{2}\nabla(\mathbf{B} \cdot \mathbf{B}) - (\mathbf{B} \cdot \nabla)\mathbf{B}$, becomes

$$\mathbf{f}_{mag} = -\frac{1}{2\mu_0}(\nabla B^2) + \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B} \tag{3.15}$$

The first term on the RHS of this equation shows that the magnetic force is equivalent to the gradient of a magnetic pressure

$$P_{mag} = \frac{B^2}{2\mu_0} \tag{3.16}$$

plus a term which can be thought of as additional term along the lines of force (Jackson, 1975).

Just as the sound waves travel with velocity $c_s = \sqrt{P/\rho}$ there is another type of wave called the magnetohydrodynamic wave (Alfvén wave) which travels with a speed

$$v_{Alfvén} = \sqrt{\frac{P_{mag}}{\rho}} = \frac{B}{\sqrt{2\mu_0\rho}} \tag{3.17}$$

Alfvén waves are associated with the transverse motion of lines of magnetic induction. The tension in the lines of force tends to restore them to straight-line form, thereby causing a transverse oscillation.

3.4. Magnetic Reynolds Number

A unitless number called magnetic Reynolds number is defined in order to distinguish between situations in which diffusion of the field lines relative to the fluid occurs and those in which lines of force are frozen in. If V is a typical velocity of the problem and L is a typical length, then the magnetic Reynolds number is defined as

$$\Re_{mag} = \frac{V\tau}{L} \tag{3.18}$$

where τ is the diffusion time given in (3.11). Thus

$$\Re_{mag} = \mu_0 \sigma V L \tag{3.19}$$

If $\Re_{mag} \gg 1$ then the diffusive term in (3.9) can be ignored and we arrive at (3.12) that is, the transport of the lines of force with the fluid dominates over diffusion. In astrophysical cases this is indeed the case (Jackson, 1975) and the magnetic field lines flow with the fluid to a very good approximation.

4. THE ANGULAR MOMENTUM TRANSPORT PROBLEM

4.1. The Problem

The specific angular momentum of a blob of gas at a Keplerian orbit is $J = L/m = rv_\varphi = (GMr)^{1/2}$. As the angular momentum increases with r, in order to accrete, this gas has to loose angular momentum as well as energy. However angular momentum is a conserved quantity and if some matter is to accrete because of kinetic energy loss, angular momentum is to be redistributed among the gas particles such that while some part of the gas moves inward to a Keplerian orbit with less angular momentum, some other part should move outwards to take over the rest of the angular momentum. As keplerian rotation is a differential rotation there is shearing between adjacent layers and the viscous stresses acting between these layers will cause a redistrubition of the angular momentum. So particles with more angular momentum will gain more angular momentum and move outwards and particles with less angular momentum will loose angular momentum and move inwards.

"Viscosity governs the local structure, and the time scale of the evolution of the disc" (Pringle, 1981). However observations imply that ordinary molecular viscosity is too small to give reasonable time scales for the evolution. So one must propose different mechanisms for the redistribution of the angular momentum in the disc. The unknown nature of the viscosity (or rather, the process which redistribute the angular momentum) is the main problem of accretion disc theory. The uncertainty in the functional form and the magnitude of the viscosity limits predictive power of the theory.

Consider the equation (3.2) with (3.15) and (3.16):

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\nabla\cdot(\rho\mathbf{v}\mathbf{v})=-\nabla(P+P_{mag})+\nabla\cdot\tilde{\tau}+\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}-\rho\nabla\Psi_R-2\rho\omega_{orb}\times\mathbf{v} \quad (4.1)$$

Any change in the angular momentum of the fluid in the disc is due to a torque $\mathbf{N}=\mathbf{r}\times\mathbf{f}$ where $\mathbf{f}$ is the force density and represents the terms on the RHS of (4.1). Thus one can conclude that, if there is no exotic physics associated with discs, torques can only be due to pressure gradients, magnetic forces, an anomalous viscosity and nonaxisymmetric gravitational field. All the sources of angular momentum transport remain marginal when compared to the anomalous viscosity that could be generated by turbulence.

4.2. Turbulence as a Source of Anomalous Viscosity

A steady flow of an incompressible fluid is specified by three parameters ν, u and l where ν is the kinematic viscosity ($\nu=\eta/\rho$), v is the velocity of the main stream and l is one linear dimension. These quantities have the following dimensions:

$$[\nu]=L^2T^{-1},\quad [l]=L,\quad [u]=LT^{-1} \quad (4.2)$$

Only one dimensionless quantity can be formed from the above three. This combination

$$\Re=\frac{lu}{\nu} \quad (4.3)$$

is called the Reynold's number and any other dimensionless parameter can be written as a function of $\Re$. A steady flow can not be realized for all values of $\Re$ and flows tend to become turbulent for $\Re$ greater than a critical value $\Re_{cr}$.

The low molecular viscosity implies high Reynold's number associated with astrophysical accretion discs. Laboratory shear flows tend to become turbulent for Reynolds numbers greater than around 10^3 (Townsend, 1976). The high Reynold's number associated with accretion discs have lead authors to conlude that the flow must be unstable to turbulent motions.

The possibility of the existence of turbulence implied by high Reynold's numbers suggests a mechanism called turbulent viscosity which could act as an effective viscosity for the redstribution of angular momentum. The idea is this: turbulence consists of a hierarchy of eddies with different lengthscales. In any turbulent flow the largest eddies play an important part. The size of the largest eddies are of the order of the dimensions of the region in which the flow takes place. We denote this order of magnitude for any turbulent flow by l. The velocity in the largest eddies is comparable with the variation of the mean velocity over the distance l; let Δu denote the order of magnitude of this variation. Since ν_{turb} characterises the properties of the turbulent flow, its order of magnitude must be determined by ρ, l and Δu. The only quantity that can be formed from these and that has the dimensions of kinematic viscosity is $l\Delta u$, and therefore

$$\nu_{turb} \sim l\Delta u \tag{4.4}$$

For the turbulence in accretion discs, the scale of the largest eddies is less than the disc thickness H (certainly true) and the turbulence is subsonic (probably true, otherwise the turbulent motions would be thermalised by shocks). Then we can write

$$\nu_{turb} \lesssim c_s H \tag{4.5}$$

where c_s is the sound speed. We can convert this inequality into an equation

$$\nu = \alpha c_s H \tag{4.6}$$

where α is a dimensionless parameter less than unity. This is the famous prescription of Shakura and Sunyaev (1973) for the anomalous viscosity. Note that this is only an ad hoc prescription and apart from expecting $\alpha \lesssim 1$ one gains nothing as α is not necessarily a constant.

The time scale for variability during outburst of dwarf novae enables estimates to be made for the magnitude of the viscosity (Pringle, 1981). Such works (Webbing, 1976) imply an effective Reynolds number of $\Re \sim 10^3$. For dwarf novae outbursts, values of $\Re \sim 10^2 - 10^3$ (or values of α in the range $0.1 - 1$) provide reasonable fits to the data (Lynden-Bell and Pringle, 1974), Bath and Pringle, 1982).

The origin of the turbulence in accretion discs poses another problem because discs are linearly stable to hydrodynamic perturbations by the Rayleigh criterion. The Rayleigh criterion for the stability of a fluid disc to axisymmetric modes is that the specific angular momentum not decrease outward:

$$\frac{d}{dr}(r^2\Omega) > 0 \tag{4.7}$$

This criterion is not violated in accretion discs and this puts the existence of turbulence in accretion discs into doubt. In order to save their turbulent viscosity Shakura and Sunyaev (1973) suggested that the discs might be nonlinearly unstable. However non-linear instabilities of keplerian discs are not well demonstrated (Hawley and Balbus, 1995). There could be other ad hoc suggestions (like global instability) but 'turbulence should be universal property of an accretion disk, not a consequence of one of a number of special conditions each unique to a particular type of disc' (Hawley and Balbus, 1995).

In 1990's a hydromagnetic instability in shearing flow (which was considered by Velhikov in 1959 and Chandrasekhar (1961) before) was applied to accretion disc flows (Hawley and Balbus, 1995). This instability is favored by many authors because it is *local* (its cause is a characteristic of the underlying flow field) and *linear*, and its existence is independent of the magnetic field and its orientation but for sufficiently weak fields, depends only on the angular velocity profile in the disc. Hawley-Balbus instability also offers the possibility of starting from a simple nonturbulent disc flow with a simple magnetic field configuration and calculating the angular momentum transport resulting from the fully developed turbulence produced by the instability (Papaloizou and Lin, 1995).

Hawley-Balbus instability can be viewed as arising because of a modification of Rayleigh criterion in the presence of a (even weak) poloidal magnetic field because of the tendency of the field to enforce corotation. The requirement of this modified stability criterion is that the angular velocity not decrease outward:

$$\frac{d\Omega}{dr} > 0 \tag{4.8}$$

This is obviously violated in a Keplerian disc.

4.3. Magnetic Stresses

Magnetic viscosity is the transfer of angular momentum by magnetic stresses. Since the early days of accretion discs the possible importance of magnetic stresses in producing significant angular momentum transport has been considered by many authors (Eardley and Lightman, 1975), Galeev et al. 1979). The magnetic fields must be maintained by some local dynamo action otherwise they would decay with some timescale.

The magnetic (Maxwell) stress is always larger than the Reynolds (velocity) stress by an average factor of four. The Shakura and Sunyaev (1973) model assumed that the net stress $\tau_{r\varphi}$ is proportional to the total disc pressure P. The simulations of Hawley-Balbus instability imply that $\tau_{r\varphi}$ is proportional to the magnetic pressure

$$\tau_{r\varphi} = \alpha_{mag} P_{mag} \tag{4.9}$$

with $\alpha_{mag} \sim 0.5 - 0.6$ (Hawley and Balbus, 1995).

A radical idea is the transfer of angular momentum by relativistic magnetized winds, as suggested by Blandford (1976) and by Lovelace (1976), (also Blandford and Znajek, 1977) They show that if the disc has embedded in it an ordered magnetic field with a sufficiently large perpendicular component, then the disc can act as a kind of two dimensional pulsar. The energy and the angular momentum can be carried away directly in the form of hydromagnetic winds which are called jets. These winds are highly collimated and the evidence for the existence of them goes back to the early radio observations of twin lobes in extended radio galaxies.

5. STANDART ACCRETION DISC THEORY

5.1. Assumptions and Approximations

In this section the hydrodynamical equations of accretion discs are simplified by many assumptions and approximations. Cylindrical coordinates where $r = 0$ corresponds to the center of the disc and $z = 0$ corresponds to the disc's midplane are used. The approximations referred are listed below:

1. *The Coriolis and centripetal forces are neglected.*

2. *The gravitational attraction of the secondary star is neglected.* This assumption means that the analysis of the outermost regions of the disc where this assumption is not valid is left out. The analysis is confined to the inner portions ($r \lesssim 0.1R_{out}$) where the influence of the gravitation of the secondary and the streaming matter are negligible. With these two assumptions the momentum equation (2.3) becomes

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla \cdot (\rho\mathbf{v}\mathbf{v}) = -\nabla P + \nabla \cdot \tilde{\tau} - \rho\nabla\Psi \tag{5.1}$$

where

$$\Psi(r, z) = -\frac{GM_1}{\sqrt{r^2 + z^2}} \tag{5.2}$$

3. *The disc is axially symmetric.* This means that the derivative of any disc property with respect to φ vanishes.

4. *No motion of matter in the vertical direction:* $v_z = 0$. With these two asumptions mass conservation equation (2.2) and the three components of the momentum equation (5.1) become:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) = 0 \tag{5.3}$$

$$\frac{\partial(\rho v_r)}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r^2) - \frac{\rho v_\varphi^2}{r} = -\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) - \frac{\tau_{\varphi\varphi}}{r} - \frac{GM_1\rho r}{(r^2+z^2)^{3/2}} \tag{5.4}$$

$$\frac{\partial(\rho v_\varphi)}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r v_\varphi) + \frac{\rho v_r v_\varphi}{r} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{r\varphi}) + \frac{\tau_{r\varphi}}{r} \tag{5.5}$$

$$\frac{\partial P}{\partial z} = -\frac{GM_1\rho z}{(r^2+z^2)^{3/2}} \tag{5.6}$$

where

$$\tau_{rr} = 2\eta\frac{\partial v_r}{\partial r} + (\chi - \frac{2}{3}\eta)\frac{1}{r}\frac{\partial(rv_r)}{\partial r} \tag{5.7}$$

$$\tau_{\varphi\varphi} = 2\eta\frac{v_r}{r} + (\chi - \frac{2}{3}\eta)\frac{1}{r}\frac{\partial(rv_r)}{\partial r} \tag{5.8}$$

$$\tau_{r\varphi} = \eta r\frac{\partial}{\partial r}(\frac{v_\varphi}{r}) \tag{5.9}$$

5. *The disc is thin.* If H is the thickness of the disc at r

$$\frac{H}{r} \ll 1 \tag{5.10}$$

This assumption has many important implicatons. They are listed below:

- The vertical hydrostatic equilibrium equation simplifies as

$$\frac{\partial P}{\partial z} = -\frac{GM_1\rho z}{r^3} \tag{5.11}$$

- One can write, to order of magnitude, $\partial P/\partial z \approx P/H$. Therefore (5.11) can be written as

$$\frac{P}{H} = \frac{GM_1 \rho H}{r^3}. \tag{5.12}$$

Using $c_s^2 = P/\rho$ and refferring the (5.10) this becomes

$$c_s^2 \ll \frac{GM_1}{r} \tag{5.13}$$

This shows that the order of the last term in (5.4) is much greater than $c_2^2 \rho/r = P/r$ and thus *radial pressure gradient can be neglected compared to the gravitational term.*

- An order of magnitude analysis to (5.3) and (5.5) using α-prescription $\nu \lesssim c_s H$ implies that

$$\frac{v_r}{r} \sim \frac{\nu}{r^2} \lesssim \frac{H}{r}\frac{c_s}{r} \ll \frac{c_s}{r}$$

and so

$$v_r \ll c_s \tag{5.14}$$

that is, *the radial velocity is highly subsonic.*

- Equations (5.7) and (5.8) imply that

$$\frac{\tau_{rr}}{r} \sim \frac{\tau_{\varphi\varphi}}{r} \sim \frac{\eta}{r}\frac{v_r}{r} \ll \frac{\nu\rho}{r}\frac{c_s}{r} < c_s \frac{H}{r}\rho\frac{c_s}{r} \ll \frac{\rho c_s^2}{r} \ll \frac{GM_1\rho}{r^2} \tag{5.15}$$

so *the second and the third terms on the RHS of (5.4) can be neglected.*

- Using (5.14) in calculating the order of magnitude of the radial acceleration terms imply

$$\frac{\rho v_r^2}{r} \ll \frac{\rho c_s^2}{r} \tag{5.16}$$

so that *the radial accelaration terms can also be ignored.*

- The four items above imply that (5.4) simplify as

$$v_\varphi = \left(\frac{GM_1}{r}\right)^{1/2} \tag{5.17}$$

that is v_φ is Keplerian. Note that (5.13) means that the *Keplerian velocity is highly supersonic.* Using $\Omega = v_\varphi / r$ the radial and angular momentum equations (5.4) and (5.5) become

$$\Omega = \left(\frac{GM_1}{r^3}\right)^{1/2} \tag{5.18}$$

$$\frac{\partial}{\partial t}(r^2\rho\Omega) + \frac{1}{r}\frac{\partial}{\partial r}(r^3\rho\Omega v_r) = \frac{1}{r}\frac{\partial}{\partial r}(r^2\tau_{r\varphi}) \tag{5.19}$$

and note that (5.9) can be written as

$$\tau_{r\varphi} = \eta r\frac{\partial\Omega}{\partial r} = \nu\rho r\frac{\partial\Omega}{\partial r} \tag{5.20}$$

- The result that the gas of the disc moves in Keplerian orbits is equivalent to the fact that gravitational force of the central object is much greater than the internal stress and pressure gradents in the disc gas. From this follows that the *gravitational energy of the disc gas is much greater than its internal energy.*
- *The energy flux is only in the vertical direction.* So only $\partial F/\partial z$ contributes to the divergence in the energy equation (2.8)

6. *The timescale for gas to drift radially inward is long compared with the timescales for energy (heat) and sound waves to travel vertically through the disc.* The implication of this assumption is that we may equate the energy generation rate to the divergence of the energy flux.

$$\frac{\partial F}{\partial z} = \Phi \tag{5.21}$$

7. *The principle sources of opacity are free-free and Compton scattering.*

$$\bar{\kappa} = \bar{\kappa}_{ff} + \bar{\kappa}_{es} \tag{5.22}$$

where free-free opacity is

$$\bar{\kappa}_{ff} = 0.64 \times 10^{22} \rho T^{-7/2} \tag{5.23}$$

and electron scattering is

$$\bar{\kappa}_{es} = 0.04 \tag{5.24}$$

Novikov and Thorne (1973) at p.378 gives a standart derivation of these opacities.

8. *The optical depth is large everywhere in the disc* ($\tau \gg 1$), so that the radiation field is locally very close to the blackbody value.

9. *The gas is a fully ionized plasma.* This assumption is true for the typical disc temperatures of 10^6K in hydrogen gas.

10. *Radiation emitted from either the disc or the compact object does not reimpinge on the disc.*

11. *The energy generated in the disc is transported to the surface by radiation rather than by convection.* Thermal conduction as a mechanism for transporting energy to the surface is at least eight orders of magnitude smaller than radiative transport for models considered here (Lightman, 1974). Thus we can drop the conductive term $-\nabla \cdot \mathbf{q}$ from the energy equations (2.8) and (2.10).

12. The pressure is the sum of gas and radiation pressures

$$P = P_{gas} + P_{rad} \tag{5.25}$$

where

$$P_{gas} = \frac{\rho k T}{\mu m_p} \tag{5.26}$$

$$P_{rad} = \frac{4\sigma}{3c} T^4 \tag{5.27}$$

here k, σ and c are Boltzmann constant, Stefan-Boltzmann constant and the speed of light, respectively. μ is the mean molecular weight and m_p is the proton mass.

5.2. Vertically Averaged Equations

In this section all equations are averaged vertically by integrating them over the thickness of the disc. Following new variables are defined

$$\Sigma(r,t) = \int_0^H \rho dz \tag{5.28}$$

$$D(r,t) = \int_0^H \Phi dz \tag{5.29}$$

Here Σ is the surface mass density and D is the dissipation per unit surface area. The same symbols for the other variables are pursued to use although they are also vertically averaged and are now functions of only r and t. Because of uncertainty in vertically averaging the vertically averaged equations will be approximate up to factors of order unity.

The vertically averaged equations are listed below:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma v_r) = 0 \tag{5.30}$$

$$\Omega = \sqrt{\frac{GM_1}{r^3}} \tag{5.31}$$

$$\frac{\partial}{\partial t}(r^2\Sigma\Omega) + \frac{1}{r}\frac{\partial}{\partial r}(r^3\Sigma\Omega v_r) = \frac{1}{r}\frac{\partial}{\partial r}(r^2\tau_{r\varphi}) \tag{5.32}$$

$$c_s^2 = \frac{GM_1H^2}{r^3} \tag{5.33}$$

$$\frac{4\sigma}{3\tau}T^4 = D(r) \tag{5.34}$$

$$P = \frac{\rho kT}{\mu m_p} + \frac{4\sigma}{3c}T^4 \tag{5.35}$$

$$\bar{\kappa} = 0.04 + 0.64 \times 10^{22}\rho T^{-7/2} \tag{5.36}$$

$$\rho = \frac{\Sigma}{H} \tag{5.37}$$

$$c_s^2 = \frac{P}{\rho} \tag{5.38}$$

$$\tau = \Sigma\kappa_R \tag{5.39}$$

$$\tau_{r\varphi} = \nu\Sigma r\frac{\partial\Omega}{\partial r} \tag{5.40}$$

$$D(r) = \frac{1}{2}\nu\Sigma(r\frac{\partial\Omega}{\partial r})^2 \tag{5.41}$$

In the last equation the 1/2 factor is because of the fact that the disc has two faces. By referring (5.31) and (5.39) to use in (5.34) one obtains

$$\frac{4\sigma}{3\Sigma\bar{\kappa}}T^4 = \frac{9}{8}\upsilon\Sigma\frac{GM_1}{r^3} \tag{5.42}$$

and using (5.40) in (5.32)

$$\frac{\partial}{\partial t}(r^2\Sigma\Omega) + \frac{1}{r}\frac{\partial}{\partial r}(r^3\Sigma\Omega v_r) = \frac{1}{r}\frac{\partial}{\partial r}(r^3\nu\Sigma\frac{\partial\Omega}{\partial r}) \tag{5.43}$$

From this equation and (5.30) one can derive

$$v_r = \frac{1}{r\Sigma(r^2\Omega)'}\frac{\partial}{\partial r}(r^3\nu\Sigma\frac{\partial\Omega}{\partial r}) \tag{5.44}$$

which by (5.31), become

$$v_r = -\frac{3}{r^{1/2}\Sigma}\frac{\partial}{\partial r}(\nu\Sigma r^{1/2}) \tag{5.45}$$

Using this in (5.30) one obtains

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(\nu\Sigma r^{1/2})\right] \tag{5.46}$$

This is a non-linear (for a general form of ν) diffusion equation governing the time evolution of surface density in a keplerian disc. If ν is not a function of Σ then (5.46) is linear.

To complete the set we only need a viscosity prescription. Arguments about the magnetic nature of the viscosity imply that the functional dependency of α can be carried in $\alpha = \alpha_0 (H/r)^n$ (Vishniac et al., 1990). In order to make our solutions more general we will use this modified α-prescription:

$$\nu = \alpha_0 (\frac{H}{r})^n c_s H \tag{5.47}$$

Note that for $n = 0$ and $\alpha_0 = \alpha$ one obtains the familiar α prescription of Shakura and Sunyaev (1973). This slighthly generalized prescription was first proposed by Meyer and Meyer-Hoffmeister (1983) and used to model soft X-ray transients by Mineshige and Wheeler (1989).

$\dot{M}(r,t)$, the accretion (mass transfer) rate is

$$\dot{M} = 2\pi r \Sigma (-v_r) \tag{5.48}$$

Finally we obtained a full set of equations for the disc structure. Once the set of equations(5.33), (5.35), (5.36), (5.37), (5.38), (5.39), (5.42), (5.46) and (5.47) are solved simultaneously v_r can be found by (5.45) and $\dot{M}$ can be found from (5.48) afterwards.

5.3. Regions on the Disc

The disc is assumed to be of three regions. In the outer region the gas pressure is dominant over the radiation pressure and free-free opacity is dominant over the electron scattering opacity. In the middle region the gas pressure is still dominant, but the main source of opacity is electron scattering now. In the inner region radiation pressure is more important and the main source of opacity is again electron scattering.

5.4. Timescales on the Disc

There are three fundamental timescales asociated with standart acretion discs; dynamical, thermal and viscous. A forth timescale is the one associated with the propogation of transition fronts, but this is not as fundamental as the other timescales.

Dynamical time is the period of a keplerian revolution. It is also the sound crossing time of the disc thickness t_z or response time to a perturbation of vertical hydrostatics by (5.33):

$$t_{dyn} \sim \Omega^{-1} \sim \frac{r}{v_\varphi} \sim \frac{H}{c_s} = t_z \tag{5.49}$$

The thermal timescale is the ratio of the thermal content to the local dissipation rate.

$$t_{th} \sim \frac{\Sigma c_s^2}{D(r)} \sim \frac{\Sigma c_s^2}{\nu \Sigma \Omega^2} \sim \frac{1}{\alpha} t_{dyn} \tag{5.50}$$

The viscous timescale is the time it takes material to viscously drift inwards. This is the timescale on which changes occur in local surface density:

$$t_{visc} \sim \frac{r^2}{\nu} \sim \frac{r^2}{\alpha c_s H} \sim \frac{1}{\alpha}\left(\frac{r}{H}\right)^2 t_{dyn} \tag{5.51}$$

Since $\alpha \lesssim 1$ and in standart thin discs $H/r \ll 1$,

$$t_{dyn} < t_{th} < t_{visc} \tag{5.52}$$

5.5. Steady Thin Discs

The changes in radial structure in a thin disc occur on timescales $\sim t_{visc} = r^2/\nu$. In this section it is assumed that the external conditions change on timescales longer than t_{visc} such that the disc settled to a steady state. We can examine this situation by setting $\partial/\partial t = 0$ in the equations of section above. The conservation of mas equation (5.30) in the steady state becomes

$$\frac{\partial}{\partial r}(r\Sigma v_r) = 0 \tag{5.53}$$

which from (5.48) gives

$$\dot{M} = 2\pi r\Sigma(-v_r) = Const. \tag{5.54}$$

that is the inward flux i.e. the accretion rate is constant. The angular momentum equation (5.32) with (5.40), in the steady state simplify as

$$\frac{\partial}{\partial r}(v_r \Sigma r^3 \Omega) = \frac{\partial}{\partial r}(\nu \Sigma r^3 \frac{\partial \Omega}{\partial r}) \tag{5.55}$$

and this when integrated, will give $v_r \Sigma r^r \Omega = \nu \Sigma r^3 \frac{\partial \Omega}{\partial r} + Const.$ which may be written as

$$-\nu \Sigma \frac{\partial \Omega}{\partial r} = \Sigma(-v_r)\Omega + \frac{Const}{2\pi r^3} \tag{5.56}$$

The constant in this equation will be determined by the conditions at the boundary layer. In a realistic case the star will be rotating with a smaller angular momentum than the keplerian angular velocity at its surface. That means that the matter is to be slowed down in order to accrete onto the surface of the star. This will take place at the so called boundary layer of radial extend b which is much smaller than R_1 and so the angular velocity will have a maximum at $R_1 + b$. This means that $\Omega'(R_1 + b) = 0$. Thus $Const. = 2\pi(R_1 + b)^3 \Sigma v_{R_1+b} \Omega(R_1 + b)$. At $R_1 + b$ the angular velocity has not yet departed from its keplerian value $\Omega = \sqrt{\frac{GM_1}{(R_1+b)^3}}$. Thus $Const = 2\pi(R_1 + b)^{3/2} \Sigma v_{R_1+b} \sqrt{GM_1}$. As the accretion rate is a constant as implied by (5.54):

$$Const. = -\dot{M}(GM_1 R_1)^{1/2} \tag{5.57}$$

Substituting this into (5.56)

$$\nu \Sigma \frac{\partial \Omega}{\partial r} = \Sigma v_r \Omega + \dot{M}\sqrt{GM_1 R_1} \tag{5.58}$$

and by using the keplerian angular velocity (26) in this equation one obtains

$$\nu\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{R_1}{r}\right)^{1/2}\right] \tag{5.59}$$

By (5.59) and (5.54)

$$v_r = -\frac{3\nu}{2r}\left[1 - \left(\frac{R_1}{r}\right)^{1/2}\right]^{-1} \tag{5.60}$$

Now, the eight equations (5.33), (5.35), (5.36), 5.37), (5.38), (5.42), (5.47), (5.59) constitute a full set and can be solved for eight unknowns ρ, Σ, H, c_s, P, T, κ and ν as functions of $\dot{M}$, M, r and parameter α. Once these equations are solved v_r can be found by (5.60). The solutions for the outer region are listed below:

$$T = C_1 \alpha_0^{-\frac{2}{10+n}} \dot{M}^{\frac{3}{10+n}} M^{\frac{2n+5}{2(10+n)}} r^{-\frac{2n+15}{2(10+n)}} f^{\frac{3}{10+n}} \tag{5.61}$$

$$\Sigma = C_2 \alpha_0^{-\frac{8}{10+n}} \dot{M}^{\frac{14-n}{2(10+n)}} M^{\frac{13n+10}{4(10+n)}} r^{-\frac{7n+30}{4(10+n)}} f^{\frac{14-n}{2(10+n)}} \tag{5.62}$$

$$\nu = C_3 \alpha_0^{\frac{8}{10+n}} \dot{M}^{\frac{3(n+2)}{2(10+n)}} M^{-\frac{13n+10}{4(10+n)}} r^{\frac{7n+30}{4(10+n)}} f^{\frac{3(n+2)}{2(10+n)}} \tag{5.63}$$

$$H = C_4 \alpha_0^{-\frac{1}{10+n}} \dot{M}^{\frac{3}{2(10+n)}} M^{-\frac{15}{4(10+n)}} r^{\frac{4n+45}{4(10+n)}} f^{\frac{3}{2(10+n)}} \tag{5.64}$$

$$\tau = C_5 \alpha_0^{-\frac{8}{10+n}} \dot{M}^{\frac{-8}{10+n}} M^{\frac{6n}{2(10+n)}} r^{\frac{-n}{10+n}} f^{\frac{2-n}{10+n}} \tag{5.65}$$

$$P = C_6 \alpha_0^{-\frac{9}{10+n}} \dot{M}^{\frac{17-n}{2(10+n)}} M^{\frac{17n+35}{4(10+n)}} r^{\frac{-15n+105}{4(10+n)}} f^{\frac{34-2n}{10+n}} \tag{5.66}$$

$$v_r = -\alpha_0^{\frac{8}{10+n}} \dot{M}^{\frac{3(n+2)}{2(10+n)}} M^{-\frac{13n+10}{4(10+n)}} r^{\frac{3n-10}{4(10+n)}} f^{\frac{2n-28}{10+n}} \tag{5.67}$$

where we defined

$$f = \left[1 - \left(\frac{R_1}{r}\right)^{1/2}\right] \tag{5.68}$$

and the coefficients are

$$C_1 = (39.75 \times 10^3)^{\frac{1}{10+n}} \tag{5.69}$$

$$C_2 = 9.15 \times 10^{-4} (0.142 \times 10^8)^{-(n+1)} (39.75 \times 10^3)^{-\frac{n+2}{2(10+n)}} \tag{5.70}$$

$$C_3 = 1.16 \times 10^2 (0.142 \times 10^8)^{n+1} (39.75 \times 10^3)^{\frac{n+2}{2(10+n)}} \tag{5.71}$$

$$C_4 = 0.142 \times 10^8 (39.75 \times 10^3)^{\frac{1}{2(10+n)}} \tag{5.72}$$

$$C_5 = 53.58 \times 10^{16} \times (0.142 \times 10^8)^{-2n-3} (39.75 \times 10^3)^{-\frac{n+6}{n+10}} \tag{5.73}$$

$$C_6 = 87 \times 10^{-8} \times (0.142 \times 10^8)^{-(n+1)}(39.75 \times 10^3)^{-\frac{n+1}{2(10+n)}} \tag{5.74}$$

$$C_7 = 174(0.142 \times 10^8)^{n+1}(39.75 \times 10^3)^{\frac{n+2}{2(n+10)}} \tag{5.75}$$

The middle and inner regions can be solved in a similar way. The important thing to note is that the solution of the time-independent equations is of the form $r^l(1 - (R_1/r)^{1/2})^m$. This may give insight for the solution of the time-dependent equations.

5.6. Time Dependent Discs

We postulate that α_0 and n are independent of any disc variable we want to solve, but may depend on other variables like magnetic fields etc. Of course magnetic fields depend on the disc variables we want to solve and this requires that we have to assume the coupling to be weak. The vertically averaged equations to be solved are (Frank et al., 1992)

$$\rho = \frac{\Sigma}{H} \tag{5.76}$$

$$H = c_s \left(\frac{r^3}{GM}\right)^{1/2} \tag{5.77}$$

$$c_s^2 = \frac{P}{\rho} \tag{5.78}$$

$$P = P_{gas} + P_{rad} = \frac{\rho k T}{\mu m_p} + \frac{4\sigma}{3c} T^4 \tag{5.79}$$

$$\frac{4\sigma}{3\tau} T^4 = \frac{9}{8} \nu \Sigma \frac{GM}{r^3} \tag{5.80}$$

$$\tau = \Sigma \bar{\kappa} \tag{5.81}$$

$$\nu = \alpha_0 \left(\frac{H}{r}\right)^n c_s H \tag{5.82}$$

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R} \frac{\partial}{\partial r} \left\{ R^{1/2} \frac{\partial}{\partial r} (\nu \Sigma r^{1/2}) \right\} \tag{5.83}$$

We have eight equations for the eight unknowns ρ, Σ, H, c_s, P, T, τ and ν which are functions of r and t. Once this set of equations is solved, $\dot{M}$ may be found from

$$2\pi R \frac{\partial \Sigma}{\partial t} = \frac{\partial \dot{M}}{\partial r} \tag{5.84}$$

and boundary conditions. Then v_R can be found by

$$\dot{M} = 2\pi r \Sigma (-v_r) \tag{5.85}$$

Following Bath and Pringle (1981) the boundary conditions will be imposed as

$$\Sigma(R_{in}, t) = 0, \qquad \frac{\partial \Sigma}{\partial r} \mid_{r=R_{out}} = 0 \tag{5.86}$$

The first of these boundary conditions allow any material there to be accreted on the central star. This also ensures that the viscous torque acting on the disc due to the central star is zero (Pringle, 1977). At the outer edge of the disc in a close binary system, tidal effects due to the secondary star remove angular momentum efficiently from the disc material and limit the disc radius to about 0.8-0.9 of the size of the primary's Roche lobe (Lin and Pringle, 1976); Papaloizou and Pringle, 1977). Thus an appropriate boundary condition is to fix the outer radius $R = R_{out}$ and to take $\partial\Sigma/\partial R = 0$ at that point. This ensures $v_r = 0$ at $R = R_{out}$ and removes angular momentum from matter at that radius at the required rate.

The same opacity law, as before, is used:

$$\bar{\kappa} = \bar{\kappa}_{ff} + \bar{\kappa}_{es} = 6.4 \times 10^{21} \rho T^{-7/2} + 4.0 \times 10^{-2} \tag{5.87}$$

5.6.1. The solution of the algebraic equations

The equations can be seen to consist of two parts. The first seven equations are algebraic and the last one is a differential equation for the surface density. The first seven equations can be solved as a function of M, Σ and r (temporary solution) and the solution for the viscosity can then be placed in (5.83) which then can be solved for Σ. This solution for Σ as a function of r and t then can be placed in the temporary solutions to get ρ, H, c_s, P, T, τ and ν as functions of r and t.

For the outer region where $\bar{\kappa}_{ff} \gg \bar{\kappa}_{es}$ and $P_{gas} \gg P_{rad}$ the temporary solution of the equations in terms of M, r and Σ is as shown below:

$$T = C_{11}\alpha_0^{\frac{2}{14-n}} M^{\frac{2-n}{14-n}} r^{\frac{n-6}{14-n}} \Sigma^{\frac{6}{14-n}} \tag{5.88}$$

$$\nu = C_{12}\alpha_0^{\frac{16}{14-n}} M^{-\frac{13n+10}{2(14-n)}} r^{\frac{7n+30}{2(14-n)}} \Sigma^{\frac{3(n+2)}{14-n}} \tag{5.89}$$

$$H = C_{13}\alpha_0^{\frac{1}{14-n}} M^{-\frac{6}{14-n}} r^{\frac{18-n}{14-n}} \Sigma^{\frac{3}{14-n}} \tag{5.90}$$

$$\rho = C_{14}\alpha_0^{-\frac{1}{14-n}} M^{\frac{6}{14-n}} r^{\frac{n-18}{14-n}} \Sigma^{\frac{11-n}{14-n}} \tag{5.91}$$

$$P = C_{15}\alpha_0^{\frac{1}{14-n}} M^{\frac{8-n}{14-n}} r^{\frac{2(n-12)}{(14-n)}} \Sigma^{\frac{17-n}{14-n}} \tag{5.92}$$

And the radiated flux $F = (4\sigma/3\tau)T^4$ is

$$F = C_{16}\alpha_0^{\frac{16}{14-n}} M^{\frac{18-15n}{2(14-n)}} r^{\frac{13n-54}{2(14-n)}} \Sigma^{\frac{2(n+10)}{14-n}} \tag{5.93}$$

In these equations

$$C_{11} = \left[\frac{27}{32\sigma} 6.4 \times 10^{21} \left(\frac{k}{\mu m_p}\right)^{n+1} G^{\frac{2-n}{2}}\right]^{\frac{2}{14-n}} \tag{5.94}$$

$$C_{12} = \left[\frac{k}{\mu m_p} \left(\frac{27}{32\sigma} 6.4 \times 10^{21} (\frac{k}{\mu m_p})^{n+1}\right)^{\frac{1}{14-n}}\right]^{n+2} G^{-\frac{13n-10}{2(14-n)}} \tag{5.95}$$

$$C_{13} = \frac{k}{\mu m_p} \left(\frac{27}{32\sigma} 6.4 \times 10^{21} (\frac{k}{\mu m_p})^{n+1} \right)^{\frac{1}{14-n}} G^{\frac{-6}{14-n}} \tag{5.96}$$

$$C_{14} = (\frac{k}{\mu m_p})^{-1} \left(\frac{27}{32\sigma} 6.4 \times 10^{21} (\frac{k}{\mu m_p})^{n+1} \right)^{-\frac{1}{14-n}} G^{\frac{n-2}{2(14-n)}} \tag{5.97}$$

$$C_{15} = \frac{k}{\mu m_p} \left(\frac{27}{32\sigma} 6.4 \times 10^{21} (\frac{k}{\mu m_p})^{n+1} \right)^{\frac{1}{14-n}} G^{\frac{8-n}{14-n}} \tag{5.98}$$

$$C_{16} = \frac{9}{8} \frac{k}{\mu m_p} \left(\frac{27}{32\sigma} 6.4 \times 10^{21} (\frac{k}{\mu m_p})^{n+1} \right)^{\frac{1}{14-n}} G^{\frac{38-15n}{2(14-n)}} \tag{5.99}$$

5.6.2. Solution of the diffusion equation

When either of the temporary solutions for ν is placed in (5.83) it takes the form

$$\frac{\partial \Sigma}{\partial t} = \frac{A_0}{r} \frac{\partial}{\partial r} \left\{ r^{1/2} \frac{\partial}{\partial r} \left[r^a \Sigma^b \right] \right\} \tag{5.100}$$

where

$$A_0 = 3C_{12} \alpha_0^{\frac{16}{14-n}} M^{-\frac{13n+10}{2(14-n)}} \tag{5.101}$$

$$a = \frac{7n+30}{2(14-n)} + \frac{1}{2} \tag{5.102}$$

$$b = \frac{3(n+2)}{14-n} + 1 \tag{5.103}$$

The temporary solutions for the middle and the inner regions would result in different values for A_0, a and b. These solutions could then easily be incorporated into the solution of (5.100).

Now (5.100) will be solved: The form of the solution of the time-independent equations were of the form $r^l(1-(R_1/r)^{1/2})^m$ and one can find the form of the temporal part by applying a seperation of variables to (5.100). When this is done one can propose a trial solution

$$\Sigma(r,t) = K(t+\delta)^X r^Y \left(1 - (\frac{\xi^{1/2}}{r^{1/2}})\right)^Z \tag{5.104}$$

where K, δ, X, Y, Z and ξ are constants to be determined. When this ansatz equation is placed in (5.100) one obtains:

$$X = \tfrac{1}{1-b}, \qquad Y = \tfrac{2a-5}{2(1-b)}$$
$$K = \left\{A_0\left[\tfrac{2a-5b}{2(1-b)} - \tfrac{2a-5b}{4}\right]\right\}^{\frac{1}{1-b}} \tag{5.105}$$

The integration constants, ξ and Z are to be determined from the boundary conditions. The boundary conditions (5.86) respectively imply that

$$\xi = R_{in} \tag{5.106}$$

$$Z = \frac{2a-5}{1-b}\left[1 - (\frac{R_{out}}{R_{in}})^{1/2}\right] \tag{5.107}$$

Thus, the solution can be written as

$$\Sigma(r,t) = \left\{A_0\left[\frac{2a-5b}{2(1-b)} - \frac{2a-5b}{4}\right]\right\}^{\frac{1}{1-b}} (t+\delta)^{\frac{1}{1-b}} r^{\frac{2a-5}{2(1-b)}} f^{\mu\frac{2a-5}{1-b}} \tag{5.108}$$

where

$$\mu = 1 - (\frac{R_{out}}{R_{in}})^{1/2} \tag{5.109}$$

Thus, by equations (5.101), (5.102) and (5.103), the solution for the outer region is:

$$\Sigma(R,t) = C_{out}^{\frac{n-14}{3n+6}} \alpha_0^{\frac{-16}{3n+6}} M^{\frac{13n+10}{6(n+2)}} (t+\delta)^{\frac{n-14}{3n+6}} r^{\frac{26-11n}{6(n+2)}} f^{\mu\frac{26-11n}{3n+6}} \tag{5.110}$$

$$C_{out} = \left\{\frac{B_1(n+14)(3n+2)}{(n+2)(14-n)}\right\} \tag{5.111}$$

Now remains only to determine δ from the initial conditions. This is the point where the problem arises. In all our calculations we assumed that δ was a constant. Using a general form of initial condition in (5.108) would give δ as a function of r and this would put us into inconsistancy with our initial assumption that δ is constant. However two choices of $\Sigma(r,0)$ would not lead to any inconsistenty: (1) $\Sigma(r,0) = 0$ giving $\delta = 0$ and (2)

$$\Sigma(r,0) = Br^{\frac{2a-5}{2(1-b)}} f^{\mu \frac{2a-5}{1-b}} \tag{5.112}$$

giving

$$\delta = B^{1-b} \left\{ A_0 \left[\frac{2a-5b}{2(1-b)} - \frac{2a-5b}{4} \right] \right\}^{-1} \tag{5.113}$$

That is, one should have an initial condition of the same form of the solution itself. The only parameter one is free to determine initially is the constant B. Before going into the reason for our inability in incorporating a general initial condition to the ansatz equation, it should be emphasized that whatever the solution of the radial and (non-linear) part of (5.100) the same problem would arise, beacuse the integration of the the dependent part is trivial and the problem arises because of the form of it. The way out of this dilemma is that there is a "similarity" (Boltzmann) transformation which, by amalgamating two independent variables into one ($\varsigma = r^{\alpha} t^{\beta}$), turns (5.100) into a second-order ordinary differential equation which requires only two boundary conditions. This could only be achieved by a consolidation of the initial condition with one of the boundary conditions. Such solutions are called similarity (self-similar) solutions. Although a similarity variable is not used in the solution above, the existence of the similarity variable manifests itself by forcing an initial condition of the same form of the solution itself. Such solutions of the equation (5.100) are described in Pringle (1974) and the solution found here is from the same family with the one found therein. A more recent reference for self similar solutions is by Mineshige (1981).

5.6.3. The complete solutions

In this final step, solution for (5.100) is implemented into its place in the temporary solutions.

$$T = C_{11} C_{out}^{\frac{-2}{n+2}} \alpha_0^{\frac{2}{14-n}} M^{\frac{2-n}{14-n}} (t+\delta)^{\frac{-2}{n+2}} r^{\frac{n-6}{14-n}} f^{\mu \frac{6}{14-n}} \tag{5.114}$$

$$\nu = C_{12} C_{out}^{\frac{-2}{n+2}} \alpha_0^{\frac{16}{14-n}} M^{-\frac{13n+10}{2(14-n)}} (t+\delta)^{\frac{-2}{n+2}} r^{\frac{7n+30}{2(14-n)}} f^{\mu \frac{3(n+2)}{14-n}} \tag{5.115}$$

$$H = C_{13} C_{out}^{\frac{-2}{n+2}} \alpha_0^{\frac{1}{14-n}} M^{-\frac{6}{14-n}} (t+\delta)^{\frac{-2}{n+2}} r^{\frac{18-n}{14-n}} f^{\mu \frac{3}{14-n}} \tag{5.116}$$

$$\rho = C_{14} C_{out}^{\frac{-2}{n+2}} \alpha_0^{-\frac{1}{14-n}} M^{\frac{6}{14-n}} (t+\delta)^{\frac{-2}{n+2}} r^{\frac{n-18}{14-n}} f^{\mu \frac{11-n}{14-n}} \tag{5.117}$$

$$P = C_{15} C_{out}^{\frac{-2}{n+2}} \alpha_0^{\frac{1}{14-n}} M^{\frac{8-n}{14-n}} (t+\delta)^{\frac{-2}{n+2}} r^{\frac{2(n-12)}{(14-n)}} f^{\mu \frac{17-n}{14-n}} \tag{5.118}$$

$$F = C_{16} C_{out}^{\frac{-2}{n+2}} \alpha_0^{\frac{16}{14-n}} M^{\frac{18-15n}{2(14-n)}} (t+\delta)^{\frac{-2}{n+2}} r^{\frac{13n-54}{2(14-n)}} f^{\mu \frac{2(n+10)}{14-n}} \tag{5.119}$$

where δ is given by (5.113).

6. NUMERICAL APPROACH

In this part of the thesis a simple computer program to solve the time-dependent equations of the standart accretion disc theory is presented. We list the equations below again

$$\rho = \frac{\Sigma}{H} \tag{6.1}$$

$$H = c_s \left(\frac{r^3}{GM}\right)^{1/2} \tag{6.2}$$

$$c_s^2 = \frac{P}{\rho} \tag{6.3}$$

$$\frac{4\sigma}{3\tau}T^4 = \frac{9}{8}\nu\Sigma\frac{GM}{r^3} \tag{6.4}$$

$$\tau = \Sigma\bar{\kappa} \tag{6.5}$$

$$\nu = \alpha_0 \left(\frac{H}{r}\right)^n c_s H \tag{6.6}$$

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left\{r^{1/2}\frac{\partial}{\partial r}(\nu\Sigma r^{1/2})\right\} \tag{6.7}$$

The opacity is generalized as

$$\bar{\kappa} = a_1 \rho^{a_2} T^{a_3} \tag{6.8}$$

where a choice of $a_1 = 6.4 \times 10^{21}$, $a_2 = 1$ and $a_3 = -7/2$ would represent free-free, and a choice of $a_1 = 0.040$, $a_2 = 0$ and $a_3 = 0$ would represent electron scattering opacity. One can choose some other forms which exist in the literature (Faulkner et al., 1983), as well. In a similar way the pressure is generalized as

$$P = b_1 \rho^{b_2} T^{b_3} \tag{6.9}$$

so that the combination $b_1 = \frac{k}{\mu m_p}$, $b_2 = 1$ and $b_3 = 1$ corresponds to the gas pressure whereas the combination $b_1 = \frac{4\sigma}{3c}$, $b_2 = 0$ and $b_3 = 4$ corresponds to the radiation pressure.

The only differential equation (6.7) in the list, by a change of variables $X = 2r^{1/2}$ and $S = X\Sigma$ becomes

$$\frac{\partial S}{\partial t} = \frac{12}{X^2} \frac{\partial^2}{\partial X^2}(S\nu) \tag{6.10}$$

This form of the equation is obviously more convenient to integrate by a finite difference method. Also, as $\Delta X = \Delta r / r^{1/2}$, constant grid sizes in $X - domain$ correspond to variable grid sizes, which becomes denser in the inner region of the disc, in $r - domain$. Short spatial steps near the center is something that we highly prefer. The integration of (6.10) using the standart first-order explicit method gives

$$S_i^{m+1} = S_i^m + \frac{12\Delta t}{X_i^2 \Delta X^2}(S_{i-1}^m \nu_{i-1}^m + S_{i+1}^m \nu_{i+1}^m - 2S_i^m \nu_i^m) \tag{6.11}$$

which is the new value of S at the ith zone after m time steps. For this numerical scheme to be stable the time step Δt is required to satisfy

$$\Delta t < \frac{1}{2}\frac{X_i^2 \Delta X^2}{12\nu_i} \tag{6.12}$$

Just like its done in the analytic solution, the algebraic equations are to be solved in terms of r and Σ. Once such a solution of ν is obtained it can easily be transformed into $X - domain$ and inserted into (6.11). Every variable is then solved by placing $\Sigma = S/X$ into their place in the temporary solutions. As is usually prefered in the literature, a Gaussian density distribution is used as the initial condition:

$$\Sigma(r,0) = 10^{-5} e^{-(\frac{r-r_0}{r_{in}})^2} \tag{6.13}$$

which of course was represented in the $X - domain$ as

$$S(X,0) = 10^{-5} X e^{(\frac{X^2 - X_0^2}{X_{in}^2})^2} \tag{6.14}$$

The solutions were written into data files which were then plotted in order to visualise. Some of the solutions with parameters $iMax = 130$, $mMax = 355000$, $dt = 1$, $\alpha_0 = 0.1$, $n = 0$, $\bar{\kappa} = \bar{\kappa}_{ff}$, $P = P_{gas}$, $R_{in} = 10km$, $M_1 = 3M_\odot$ are presented below:

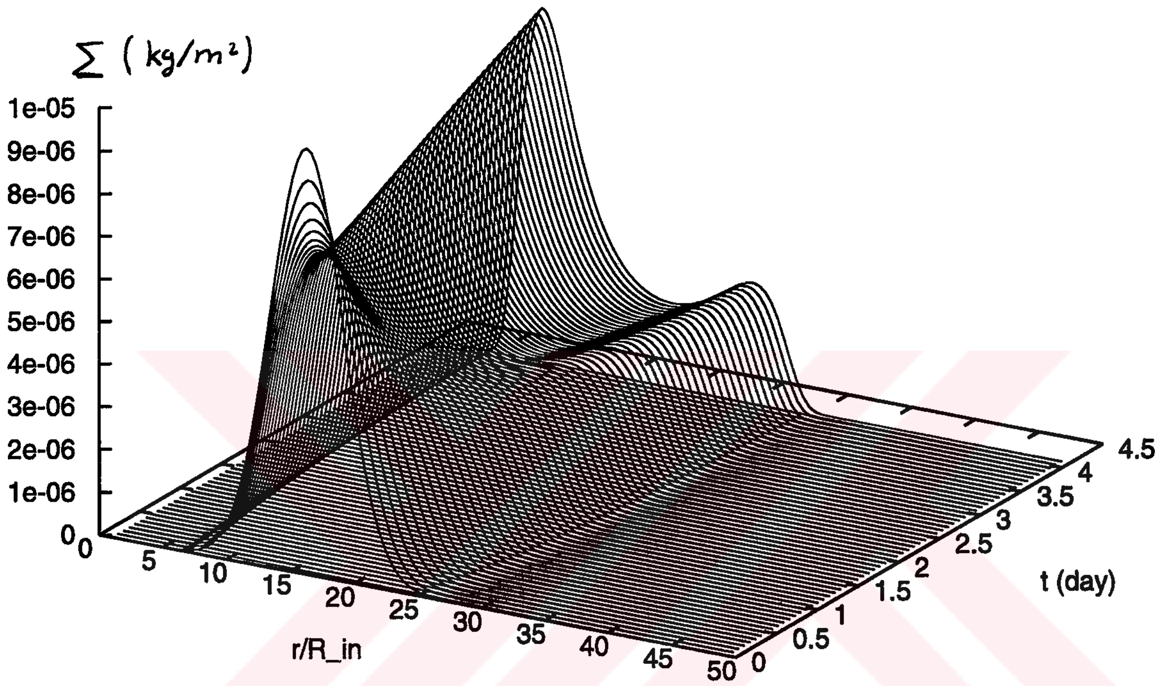

FIGURE 6.1. Evolution of the surface density

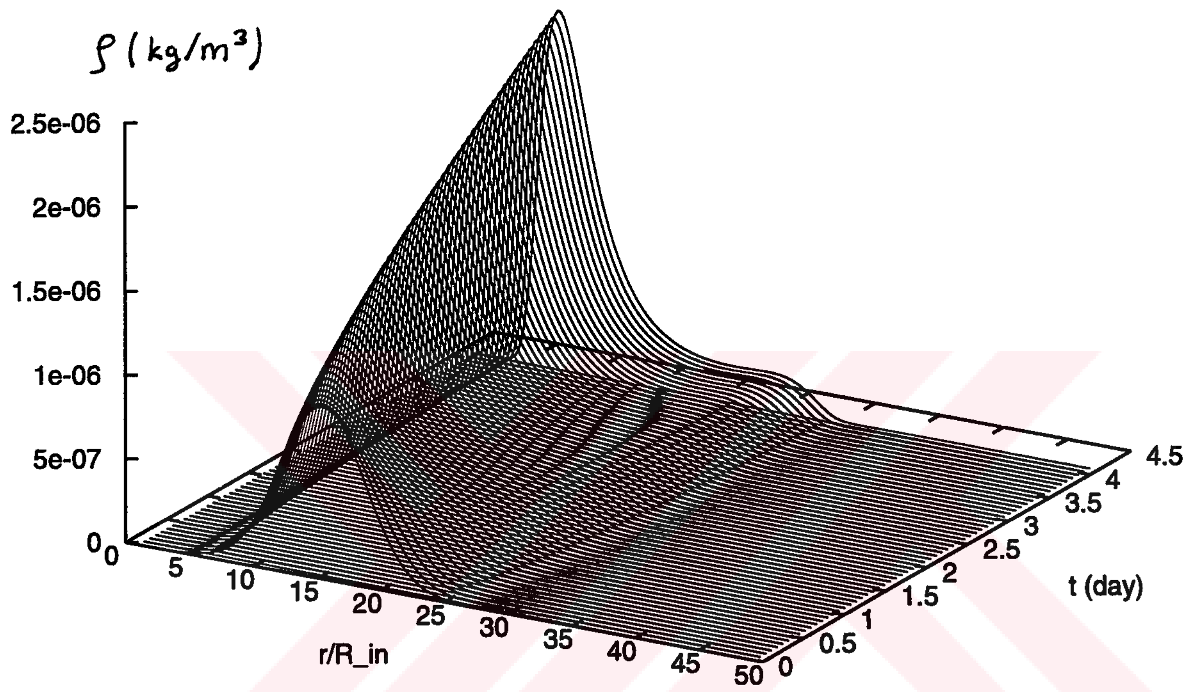

FIGURE 6.2. Evolution of the density

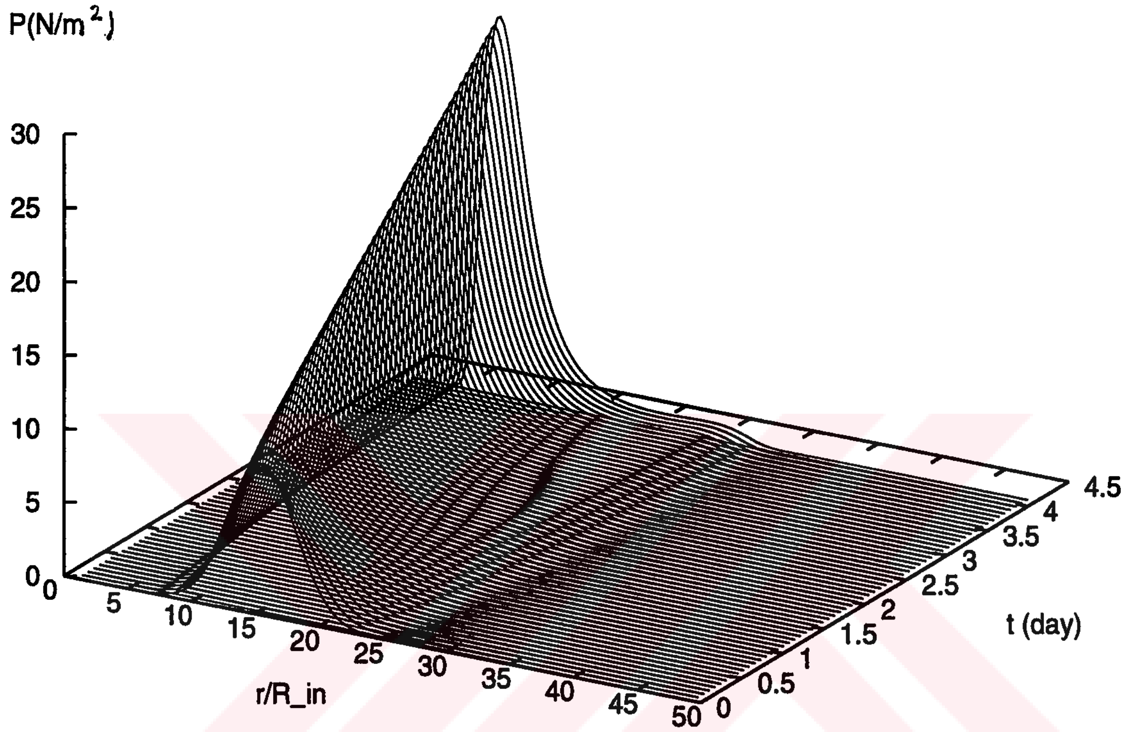

FIGURE 6.3. Evolution of the pressure

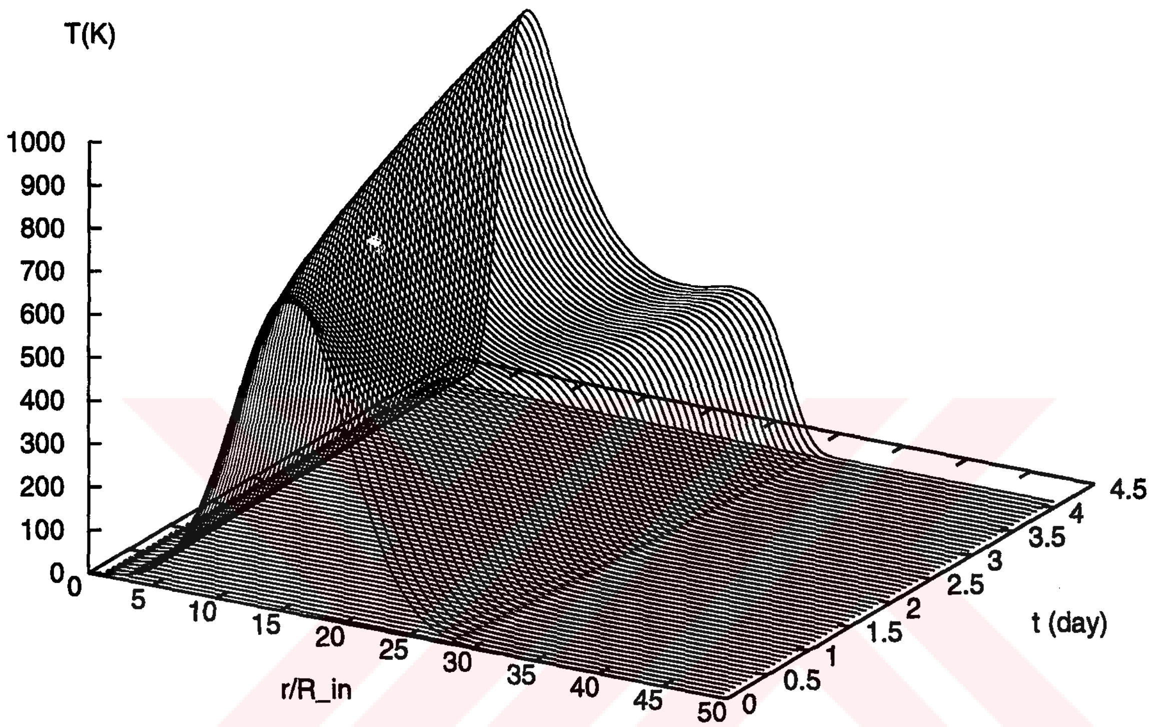

FIGURE 6.4. Evolution of the temperature

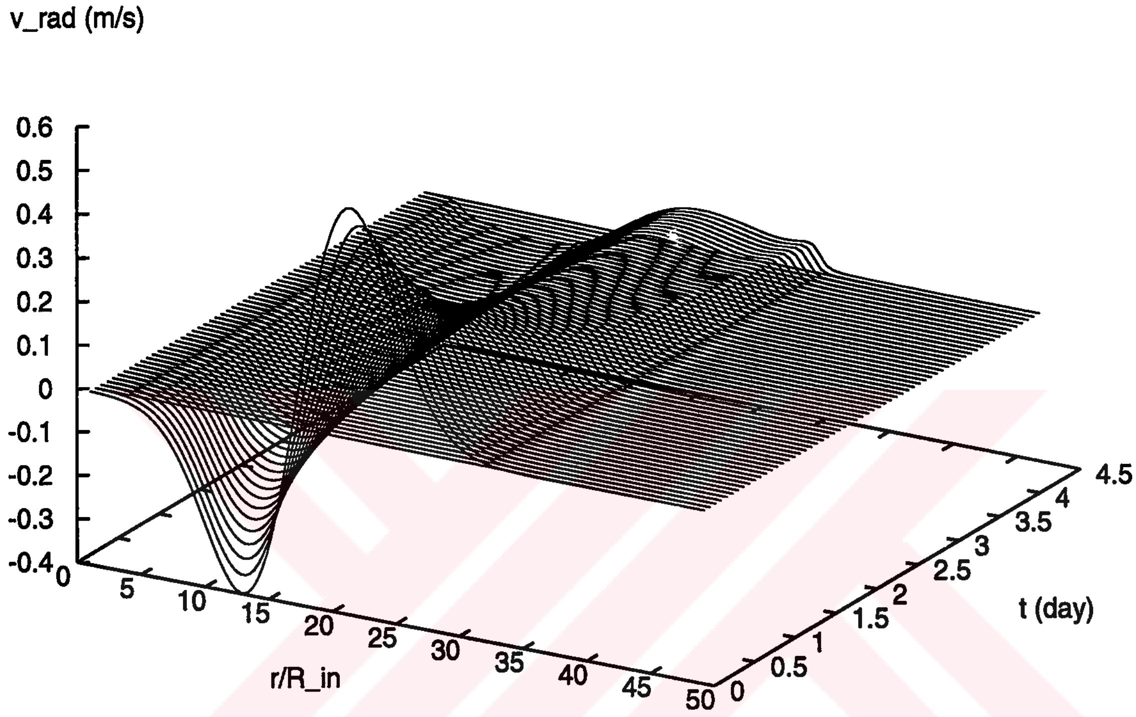

FIGURE 6.5. Evolution of radial velocity

It should be confessed that, although one is free to choose any prescription for pressure and opacity by playing with the constants in (6.8) and (6.9), the program is not capable of handling the different regions of the disc (see section(4.3)) by applying appropriate pressure and opacity prescriptions in each region and extrapolating between. This weakness of the program can be overcomed by finding two solutions with both the radiation and gas pressures in every step and then making a weighted average of every solution. A similar procedure can be applied for opacity as well. That is, greater the value of electron scattering compared to free-free scattering greater weight is given to values found by it.

7. CONCLUSION

It requires a lot more study to master the physics of accretion discs. With this thesis only an introductory small step is taken. Standart accretion disc theory is only one of the four branches of solutions which today exist in the literature. Other solution branches are more complex to get involved in.

The analytic and numerical solutions found in the thesis are both consistent with the literature. The numerical solution can be improved and a way for it is explained. The next step after this improvement could be to incorporate magnetic fields into the numerical solution. In fact the possibility that the turbulence in the disc is of hydromagnetic origin implies that any theory of accretion discs without magnetic fields is naive. However it requires three dimensions for turbulence to emerge naturally in the simulations and global three dimensional simulations of discs are beyond the capability of todays's computers.

The steady and time-dependent analytic solutions are rather general and the free parameters n and α_0 can be chosen freely to fit the observations. This could bring constraints on the functional form of the viscosity which is the main unknown of the accretion disc theory.

APPENDIX A:

Vector Operators in Curvilinear Coordinates

In generalized curvilinear coordinates the divergence of any vector **V** is:

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 V_1) + \frac{\partial}{\partial x_2}(h_1 h_3 V_2) + \frac{\partial}{\partial x_3}(h_1 h_2 V_3)\right] \tag{1}$$

The components of the divergence of any tensor $\tilde{T}$ in generalized coordinates is (Anderson et al., 1984)

$$\begin{aligned}(\nabla \cdot \tilde{T})_{x_1} &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 T_{x_1 x_1}) + \frac{\partial}{\partial x_2}(h_1 h_3 T_{x_1 x_2}) + \frac{\partial}{\partial x_3}(h_1 h_2 T_{x_1 x_3})\right] \\ &\quad + \frac{T_{x_1 x_2}}{h_1 h_2}\frac{\partial h_1}{\partial x_2} + \frac{T_{x_1 x_3}}{h_1 h_3}\frac{\partial h_1}{\partial x_3} - \frac{T_{x_2 x_2}}{h_1 h_2}\frac{\partial h_2}{\partial x_1} - \frac{T_{x_3 x_3}}{h_1 h_3}\frac{\partial h_3}{\partial x_1}\end{aligned} \tag{2}$$

$$\begin{aligned}(\nabla \cdot \tilde{T})_{x_2} &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 T_{x_2 x_1}) + \frac{\partial}{\partial x_2}(h_1 h_3 T_{x_2 x_2}) + \frac{\partial}{\partial x_3}(h_1 h_2 T_{x_2 x_3})\right] \\ &\quad + \frac{T_{x_2 x_2}}{h_2 h_3}\frac{\partial h_2}{\partial x_3} + \frac{T_{x_1 x_2}}{h_1 h_2}\frac{\partial h_2}{\partial x_1} - \frac{T_{x_3 x_3}}{h_2 h_3}\frac{\partial h_3}{\partial x_1} - \frac{T_{x_1 x_1}}{h_1 h_2}\frac{\partial h_1}{\partial x_1}\end{aligned} \tag{3}$$

$$(\nabla \cdot \tilde{T})_{x_3} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 T_{x_3 x_1}) + \frac{\partial}{\partial x_2}(h_1 h_3 T_{x_3 x_2}) + \frac{\partial}{\partial x_3}(h_1 h_2 T_{x_3 x_3})\right]$$

$$+\frac{T_{x_1 x_2}}{h_1 h_2}\frac{\partial h_1}{\partial x_2} + \frac{T_{x_1 x_3}}{h_1 h_3}\frac{\partial h_1}{\partial x_3} - \frac{T_{x_2 x_2}}{h_1 h_2}\frac{\partial h_2}{\partial x_1} - \frac{T_{x_3 x_3}}{h_1 h_3}\frac{\partial h_3}{\partial x_1} \tag{4}$$

The gradient of any scalar function Θ in any generalized coordinates is

$$\nabla \Theta = \frac{1}{h_1}\frac{\partial \Theta}{\partial x_1}\mathbf{i} + \frac{1}{h_2}\frac{\partial \Theta}{\partial x_2}\mathbf{j} + \frac{1}{h_3}\frac{\partial \Theta}{\partial x_3}\mathbf{k} \tag{5}$$

In cylidrical coordinates where

$$\begin{array}{lll} x_1 = r & h_1 = 1 & V_1 = V_r \\ x_2 = \varphi & h_2 = r & V_2 = V_\varphi \\ x_3 = z & h_3 = 1 & V_3 = V_z \end{array} \tag{6}$$

the divergence of any vector field $\mathbf{V}$ becomes

$$\nabla \cdot \mathbf{V} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_r) + \frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial}{\partial z}(rV_z)\right] \tag{7}$$

The components of the divergence of any tensor becomes

$$(\nabla \cdot \tilde{T})_r = \frac{1}{r}\left[\frac{\partial}{\partial r}(rT_{rr}) + \frac{\partial T_{r\varphi}}{\partial \varphi} + \frac{\partial}{\partial z}(rT_{rz})\right] - \frac{T_{\varphi\varphi}}{r} \tag{8}$$

$$(\nabla \cdot \tilde{T})_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}(rT_{\varphi r}) + \frac{\partial T_{\varphi\varphi}}{\partial \varphi} + \frac{\partial}{\partial z}(rT_{\varphi z})\right] + \frac{T_{\varphi r}}{r} \tag{9}$$

$$(\nabla \cdot \tilde{T})_z = \frac{1}{r}\left[\frac{\partial}{\partial r}(rT_{zr}) + \frac{\partial T_{z\varphi}}{\partial \varphi} + \frac{\partial}{\partial z}(rT_{zz})\right] \tag{10}$$

The gradient of any scalar function in cylindrical coordinates becomes

$$\nabla \Theta = \frac{\partial \Theta}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial \Theta}{\partial \varphi}\varphi + \frac{\partial \Theta}{\partial z}\mathbf{z} \tag{11}$$

APPENDIX B:

The Components of the Strain Tensor

The components of the strain tensor in generalized coordinates are as follows (Hughes and Brighton, 1991):

$$e_{11} = \frac{1}{h_1}\frac{\partial v_1}{\partial x_1} + \frac{v_2}{h_1 h_2}\frac{\partial h_1}{\partial x_2} + \frac{v_3}{h_1 h_3}\frac{\partial h_1}{\partial x_3} \tag{1}$$

$$e_{22} = \frac{1}{h_2}\frac{\partial v_2}{\partial x_2} + \frac{v_3}{h_2 h_3}\frac{\partial h_2}{\partial x_3} + \frac{v_1}{h_1 h_2}\frac{\partial h_2}{\partial x_1} \tag{2}$$

$$e_{33} = \frac{1}{h_3}\frac{\partial v_3}{\partial x_3} + \frac{v_1}{h_1 h_3}\frac{\partial h_3}{\partial x_1} + \frac{v_2}{h_2 h_3}\frac{\partial h_3}{\partial x_2} \tag{3}$$

$$e_{12} = \frac{h_1}{h_2}\frac{\partial}{\partial x_1}(\frac{v_2}{h_2}) + \frac{h_1}{h_2}\frac{\partial}{\partial x_2}(\frac{v_1}{h_1}) \tag{4}$$

$$e_{13} = \frac{h_3}{h1}\frac{\partial}{\partial x_1}(\frac{v_3}{h_3}) + \frac{h_1}{h_3}\frac{\partial}{\partial x_3}(\frac{v_1}{h_1}) \tag{5}$$

$$e_{23} = \frac{h_3}{h_2}\frac{\partial}{\partial x_2}(\frac{v_3}{h_3}) + \frac{h_2}{h_3}\frac{\partial}{\partial x_3}(\frac{v_2}{h_2}) \tag{6}$$

These, in cylindrical coordinates become:

$$e_{rr} = \frac{\partial v_r}{\partial r} \tag{7}$$

$$e_{\varphi\varphi} = \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} \tag{8}$$

$$e_{zz} = \frac{\partial v_z}{\partial z} \tag{9}$$

$$e_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}(\frac{v_\varphi}{r}) + \frac{1}{r}\frac{\partial v_r}{\partial \varphi}\right] \tag{10}$$

$$e_{rz} = \frac{1}{2}\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right] \tag{11}$$

$$e_{\varphi z} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial v_z}{\partial \varphi} + r\frac{\partial}{\partial z}(\frac{v_\varphi}{r})\right] \tag{12}$$

APPENDIX C:

The Viscous Stress Tensor and the Dissipation Function

The viscous stress tensor in cartesian tensor notation is

$$\tau_{ij} = 2\eta e_{ij} + \delta_{ij}(\chi - \frac{2}{3}\eta)\nabla \cdot \mathbf{v} \tag{1}$$

The components of the viscous stress tensor in cylindrical coordinates thus becomes

$$\tau_{rr} = 2\eta e_{rr} + (\chi - \frac{2}{3}\eta)\nabla \cdot \mathbf{v} \tag{2}$$

$$\tau_{\varphi\varphi} = 2\eta e_{\varphi\varphi} + (\chi - \frac{2}{3}\eta)\nabla \cdot \mathbf{v} \tag{3}$$

$$\tau_{zz} = 2\eta e_{zz} + (\chi - \frac{2}{3}\eta)\nabla \cdot \mathbf{v} \tag{4}$$

$$\tau_{r\varphi} = 2\eta e_{r\varphi} \tag{5}$$

$$\tau_{rz} = 2\eta e_{rz} \tag{6}$$

$$\tau_{\varphi z} = 2\eta e_{\varphi z} \tag{7}$$

The viscous dissipation function Φ is defined in generalized orthogonal coordinates as

$$\begin{aligned} \Phi &= \eta\left[2\left(e_{11}^2 + e_{22}^2 + e_{33}^2\right) + (2e_{23})^2 + (2e_{31})^2 + (2e_{12})^2\right] \\ &+(\chi - \frac{2}{3}\eta)\left(e_{11} + e_{22} + e_{33}\right)^2 \end{aligned} \tag{8}$$

This, in cylindrical coordinates become

$$\begin{aligned} \Phi &= \eta\left[2\left(e_{rr}^2 + e_{\varphi\varphi}^2 + e_{zz}^2\right) + (2e_{\varphi z})^2 + (2e_{zr})^2 + (2e_{r\varphi})^2\right] \\ &+(\chi - \frac{2}{3}\eta)\left(e_{rr} + e_{\varphi\varphi} + e_{zz}\right)^2 \end{aligned} \tag{9}$$

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