DIFFRACTION OF THE PACKET-LIKE AND NON-DIFFRACTING PROPAGATING BEAMS BY AN OBSTACLE IN THE ATMOSPHERE.

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Title of the Thesis**:** Diffraction of the packet-like and non-diffracting propagating beams by an obstacle in atmosphere.

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ABSTRACT

DIFFRACTION OF THE PACKET-LIKE AND NON-DIFFRACTING PROPAGATING BEAMS BY AN OBSTACLE IN ATMOSPHERE.

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This thesis takes two different solutions of homogenous wave equation into consideration. These solutions are named as packet-like solution and non-diffracting beam.First of all the propagation of these waves in the atmosphere is investigated. As a second step, an obstacle (a knife edge) is located on the propagation path of the diffracting beam and the diffraction effects are examined. The results are plotted numerically by using MATLAB.

Keywords: Diffraction,Packet-like Wave,Gaussian Beam.

ÖZ

PAKETİMSİ VE KIRILMAYAN IŞINLARIN
ATMOSFERDE
BİR ENGEL TARAFINDAN KIRILMASI

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Bu tez homojen dalga denkleminin iki çözümünü incelemektedir. Bu çözümler paketimsi çözüm ve kırılmayan ışın olarak adlandırılabilir.İlk olarak bu dalgaların atmosferdeki yayılımı incelenmiştir. İkinci adımda bir engel (bıçak sırtı) ışının yayılım yoluna yerleştirilip kırılma etkileri incelenmiştir. Sonuçlar sayısal olarak MATLAB yardımıyla çizilmiştir.

Anahtar Kelimeler: Kırılma, paketimsi dalga,Gauss ışını.

ACKNOWLEDGEMENT

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CHAPTER 1

INTRODUCTION

A wave packet which can be expressed as a sum of waves with different frequencies is a special feature of wave motion [1]. This type of wave propagation has a wide application in beam and particle optics, electromagnetics and quantum mechanics. Schrödinger equation has solutions which are in the form of wave packets. A wave packet can be expressed as a modulated high-frequency wave with a localized waveform. As mentioned earlier by Bélanger certain packetlike beams are the solutions to homogenous Maxwell equations that were shown by Brittingham who proposed that these solutions are unique electromagnetic pulses satisfying the homogenous Maxwell equations. These solutions are known the beams that remain focused for all time along the propagation path. Because of this peculiarity of these beams Brittingham called them focus wave modes for which he found some mathematical formulations after an extensive study with various differential equations and concluded that focus wave modes are continuous, nonsingular, nondispersive and have a three-dimensional structure with the velocity of light propagating in straight lines [2,3]. Later some authors tried to obtain exact expressions for these modes by using the Maxwell equations or the concept of the complex source [4,5]. Focus wave mode solution of the wave equation, which was first proposed by Brittingham, represents a localized wave in space [6]. It is composed of two waves propagating in opposite directions. He derived an equation, which is in the same form with the paraxial wave equation, by suggesting an ansatz based on the structure of a focus wave mode. He also showed that the solutions of the paraxial wave equation are also valid for his differential equation. All of these attempts in the literature were performed for the electromagnetic or optical waves which propagate with the speed of light. [7,8,9]. The focus wave modes are in the form of Gaussian beams. Their beam parameters depend on space and time. Gaussian beam parameters depend only on space. Gaussian beams are the solution of paraxial wave equation but the focus wavemodes have the solution of wave equation.[10]. Since focus wave modes have undistorted profile with respect to space and time they resemble plane waves with infinite energy which propagate without distortion. Packetlike or focus waves remain focused in some definite time intervals and the envelope of a focus wave is the Gaussian beam.

CHAPTER 2

DIFFERENTIAL EQUATION OF WAVE PACKETS

2.1 Theory

In this section we will derive a differential equation for focus wave modes by using the paraxial approximation and the method of Bélanger [2]. We will propose a wave propagation having equal group and phase velocities.The envelope of this wave is supposed to be a focus wave mode. The homogenous wave equation of the carrier wave can be given by

$$\nabla^2 G(x,y,\eta) - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}(x,y,\eta) = 0. \tag{2.1}$$

Here ∇_T is the Laplacian according to the transverse coordinates and $\eta = z - ct$.To obtain packetlike solutions of this equation we start with the following ansatz[2]:

$$G(x,y,\eta) = U(x,y,\eta)e^{-j\frac{k}{2}(\eta+ct)}, \tag{2.2}$$

which represents a wave packet whose envelope is determined by the function of $U(x,y,\eta)$.

By inserting this ansatz into Equation (1) we obtain an expression which is analogous to the paraxial wave equation given below.

$$\nabla_T^2 U - 2jk\frac{\partial U}{\partial \eta} = 0. \tag{2.3}$$

Here we consider the relation of $\frac{\partial^2 U}{\partial \eta^2} = 0$ as the paraxial approximation.

2.2 Spectral solution of the paraxial wave-packet equation

We can solve the differential equation in Equation (2.3) by means of a Fourier transform and obtain a scattering integral by using the convolution property. The function $U(x, y, \eta)$ can be defined as

$$U(x, y, \eta) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(\alpha, \beta, \eta)\exp\left[-j(\alpha x + \beta y)\right]d\alpha d\beta, \tag{2.4}$$

which is an inverse Fourier integral transform of W(α,β,η). Substituting the Equation (2.4) into the Equation (2.3) we get

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-(\alpha^2 + \beta^2)W - 2jk\frac{\partial W}{\partial \eta}\right]e^{-j(\alpha x + \beta y)}d\alpha d\beta = 0. \tag{2.5}$$

The Equation (2.5) results in

$$\frac{\partial W}{\partial \eta} - j\frac{(\alpha^2 + \beta^2)}{2k}W = 0. \tag{2.6}$$

In order to solve the differential equation in Equation (2.6),let

$$W = C(\alpha, \beta)\exp(r\eta). \tag{2.7}$$

Substituting the Equation (2.7) into (2.6) we get

$$r = \frac{j(\alpha^2 + \beta^2)}{2k}. \tag{2.8}$$

As a result *W* is obtained as

$$W = C(\alpha, \beta)e^{r\eta} = C(\alpha, \beta)e^{j\frac{\alpha^2+\beta^2}{2k}\eta}, \tag{2.9}$$

where $C(\alpha, \beta)$ is a constant according to η. Finally , $U(x, y, \eta)$ is obtained as

$$U(x, y, \eta) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(\alpha, \beta)e^{j\frac{\alpha^2+\beta^2}{2k}\eta}.e^{-j(\alpha x+\beta y)}d\alpha.d\beta. \tag{2.10}$$

The value of $C(\alpha, \beta)$ can be determined by the conditions according to η. If the value of $U(x, y, \eta)$ at $\eta=0$ is known as $U(x,y,0)=U_0(x,y)$ then the relation of

$$U_0(x, y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(\alpha, \beta)e^{-j(\alpha x+\beta y)}d\alpha d\beta, \tag{2.11}$$

is obtained from the Equation (2.10) for $\eta=0$. Equation (2.11) is an inverse Fourier transform from which

$$C(\alpha, \beta) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_0(x, y)e^{-j(\alpha x+\beta y)}dxdy, \tag{2.12}$$

is obtained. Equation (2.12) is known as the Fourier transform of $U_0(x,y)$.

2.3 Diffraction integral

Now we need to derive a diffraction integral for wave-packet solution. A similar approach with Ref.[11] will be used. Equation (2.10) is the inverse Fourier transform of

$$C(\alpha,\beta)e^{j\frac{\alpha^2+\beta^2}{2k}\eta}. \tag{2.13}$$

Therefore $U(x,y,\eta)$ can be written as

$$U(x,y,\eta)=F^{-1}\left\{C(\alpha,\beta)e^{j\frac{\alpha^2+\beta^2}{2k}\eta}\right\}, \tag{2.14}$$

where $F^{-1}\{\}$ represents the inverse Fourier transform.Equation (2.14) leads to the relation of the convolution operation.

$$U(x,y,\eta)=F^{-1}\{C(\alpha,\beta)\}*F^{-1}\left\{e^{j\frac{\alpha^2+\beta^2}{2k}\eta}\right\}, \tag{2.15}$$

The inverse Fourier transform of $C(\alpha,\beta)$ is obtained in Equation (2.11) as $U_0(x,y)$. In order to find the inverse Fourier transform of the exponential term in Equation (2.15), let

$$M(x,y,\eta)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{j\frac{\alpha^2+\beta^2}{2k}\eta}e^{-j(\alpha x+\beta y)}d\alpha d\beta, \tag{2.16}$$

which can be partitioned as

$$M(x, y, \eta) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{j(\frac{\eta}{2k}\alpha^2 - \alpha x)} d\alpha . \int_{-\infty}^{\infty} e^{j(\frac{\eta}{2k}\beta^2 - \beta y)} d\beta . \tag{2.17}$$

The integrals in Equation (2.17) have the same form. In order to arrange the first integral we can perform the following steps:

$$\frac{\eta}{2k}\alpha^2 - \alpha x + A^2 - A^2, \tag{2.18}$$

$$= \left(\sqrt{\frac{\eta}{2k}}\alpha - A \right)^2 - A^2, \tag{2.19}$$

$$= \left(\sqrt{\frac{\eta}{2k}}\alpha \right)^2 - 2\sqrt{\frac{\eta}{2k}}\alpha A + A^2 = \frac{\eta}{2k}\alpha^2 - \alpha x A + A^2, \tag{2.20}$$

$$2\sqrt{\frac{\eta}{2k}}\alpha A = \alpha x, \tag{2.21}$$

and finally

$$A = \frac{x}{2}\sqrt{\frac{2k}{\eta}}, \tag{2.22}$$

is obtained.

Similarly, for the arguments of the second integral we write,

$$\frac{\eta}{2k}\beta^2 - \beta x + B^2 - B^2. \tag{2.23}$$

If we proceed the same steps used for A we obtain B as

$$B = \frac{y}{2}\sqrt{\frac{2k}{\eta}}. \tag{2.24}$$

Now Equation (2.17) can be rewritten as

$$M(x,y,\eta) = \frac{1}{(2\pi)^2}\left[\int_{-\infty}^{\infty} e^{j(\sqrt{\frac{\eta}{2k}}\alpha - A)^2 - A^2} d\alpha . \int_{-\infty}^{\infty} e^{j(\sqrt{\frac{\eta}{2k}}\beta - B)^2 - B^2} d\beta\right] \tag{2.25}$$

and

$$M(x,y,\eta) = \frac{e^{-j(A^2+B^2)}}{(2\pi)^2}\left[\int_{-\infty}^{\infty} e^{j(\sqrt{\frac{\eta}{2k}}\alpha - A)^2} d\alpha . \int_{-\infty}^{\infty} e^{j(\sqrt{\frac{\eta}{2k}}\beta - B)^2} d\beta\right] \tag{2.26}$$

For solving the integrals of the same form we will use a change of variable: Considering the first integral, let

$$j\left(\sqrt{\frac{\eta}{2k}}\alpha - A\right)^2 = -\frac{t^2}{2} \quad , \tag{2.27}$$

then

$$e^{j\frac{\pi}{2}}\left(\sqrt{\frac{\eta}{2k}}\alpha - A\right)^2 = e^{j\pi}\,\frac{t^2}{2} \quad , \tag{2.28}$$

and, as a result we obtain

$$d\alpha = e^{j\frac{\pi}{4}}\sqrt{\frac{k}{\eta}}dt. \tag{2.29}$$

Similarly for the second integral by letting

$$j\left(\sqrt{\frac{\eta}{2k}}\beta - B\right)^2 = -\frac{q^2}{2} \tag{2.30}$$

, we obtain

$$d\beta = e^{j\frac{\pi}{4}}\sqrt{\frac{k}{\eta}}dq. \tag{2.31}$$

Substituting these results into the Equation (2.26) we get

$$M(x,y,\eta) = \frac{e^{-j(A^2+B^2)}}{(2\pi)^2} e^{j\frac{\pi}{2}} \frac{k}{\eta}\left[\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}dt . \int_{-\infty}^{\infty} e^{-\frac{q^2}{2}}dq\right]. \tag{2.32}$$

Since

$$\int_{-\infty}^{\infty} e^{-\frac{p^2}{2}}dp = \sqrt{2\pi}, \tag{2.33}$$

by using this integral result the Equation (2.32) can be rewritten as

$$M(x,y,\eta) = \frac{jk}{2\pi}\frac{e^{-jk\frac{x^2+y^2}{2\eta}}}{\eta}, \tag{2.34}$$

where

$$A^2 + B^2 = \frac{k}{2\eta}(x^2 + y^2), \tag{2.35}$$

is obtained by using the Equations (2.22) and (2.24). Substituting Equation (2.34) in Equation (2.15) we obtain

$$U(x,y,\eta) = U_0(x,y)\,\frac{jk}{2\pi}\,\frac{e^{-jk\frac{x^2+y^2}{2\eta}}}{\eta}, \tag{2.36}$$

which has a Gaussian profile stated by Bélanger [2] as

$$F_{0,0} = \frac{z_0}{z_c + jz_0}\, e^{-j\frac{k}{2}\frac{\rho^2}{z_c + jz_0}}, \tag{2.37}$$

where ρ is the transverse cylidrical coordinate and is equal to $(x^2+y^2)^{1/2}$.

$z_0 = k\,\frac{\omega_0^2}{2}$ where ω_0 is the beam waist at z_c = z-ct = 0. The parameter η and η_0 in U corresponds to z_c and z_0 in $F_{0,0}$ respectively. The following figures are obtained for different parameter values of U and the wavelength is taken as 0.1m.

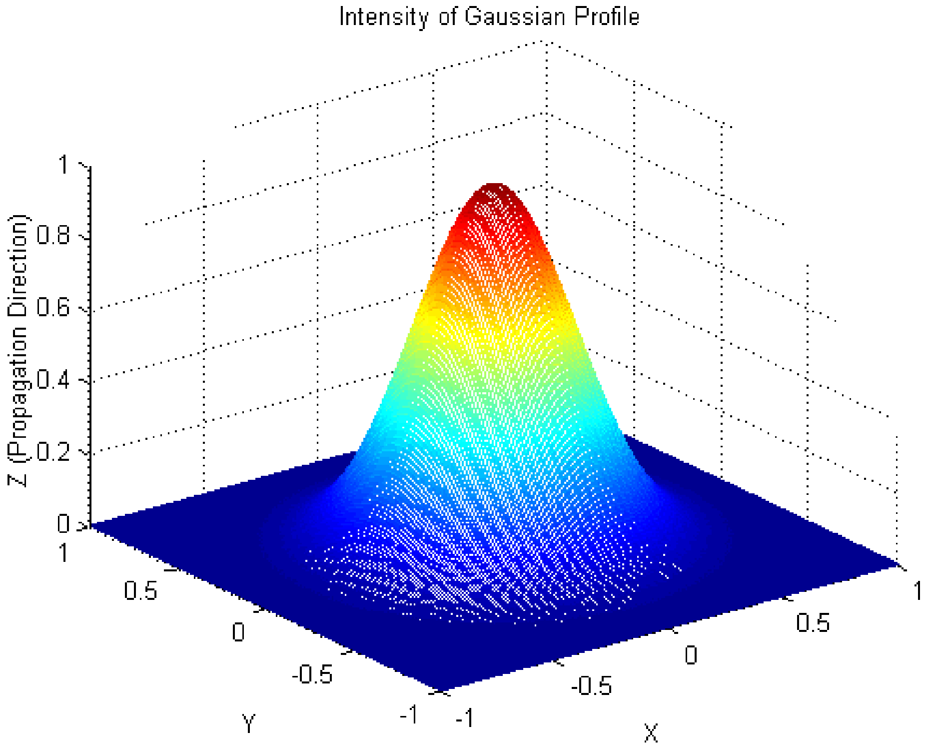

Figure 2.1 Gaussian beam with k, η=2, η_0=1.

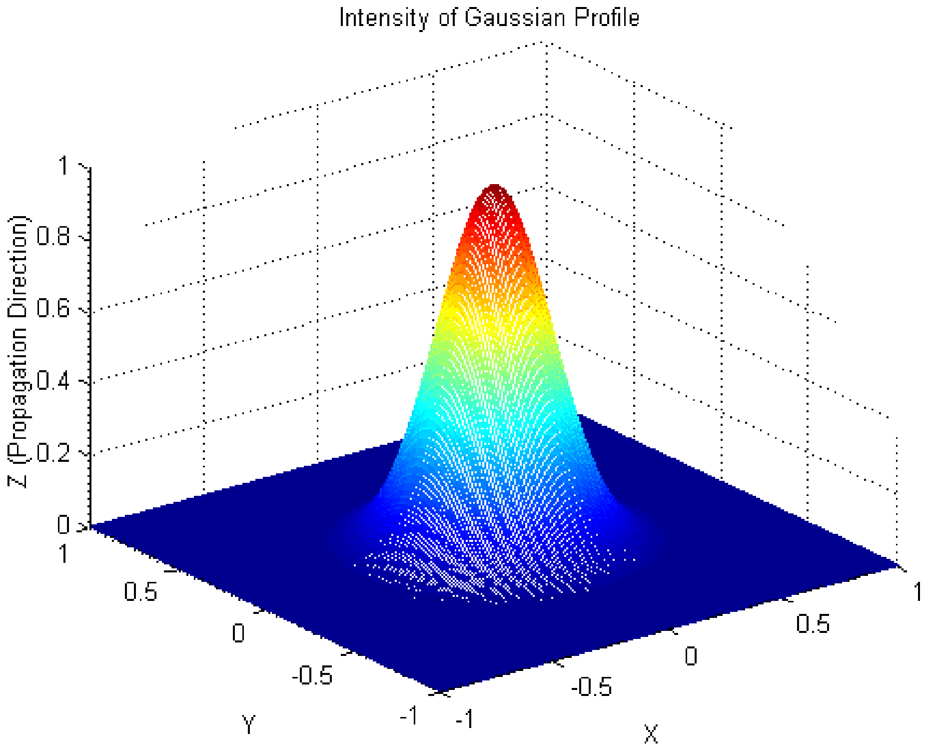

Figure 2.2 Gaussian beam with k_1=2k, η=2, η_0=1.

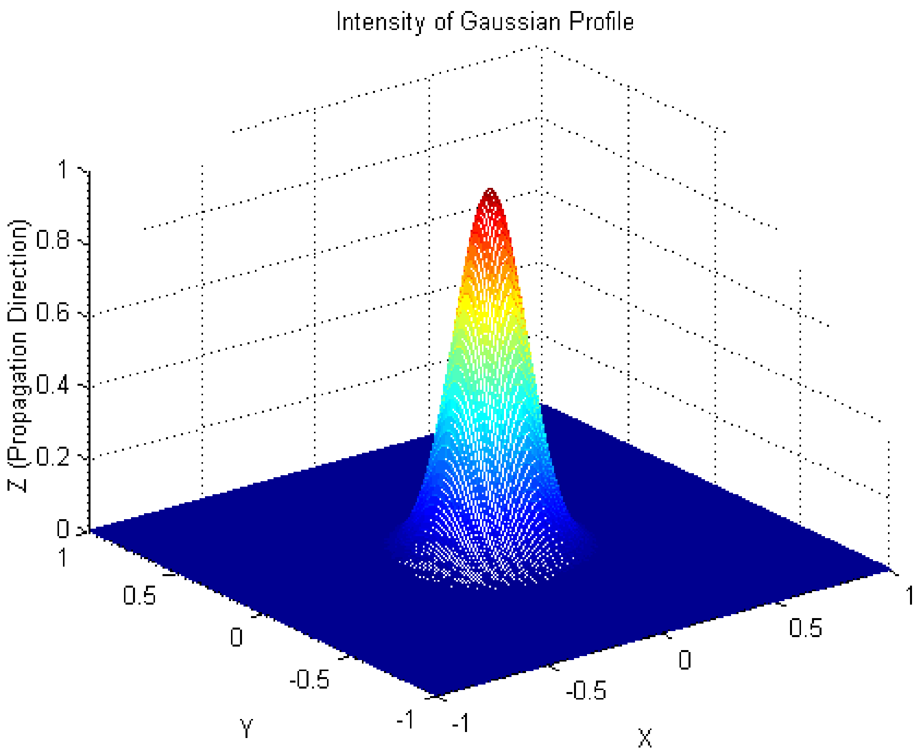

Figure 2.3 Gaussian beam with k_2=5k, η=2, η_0=1.

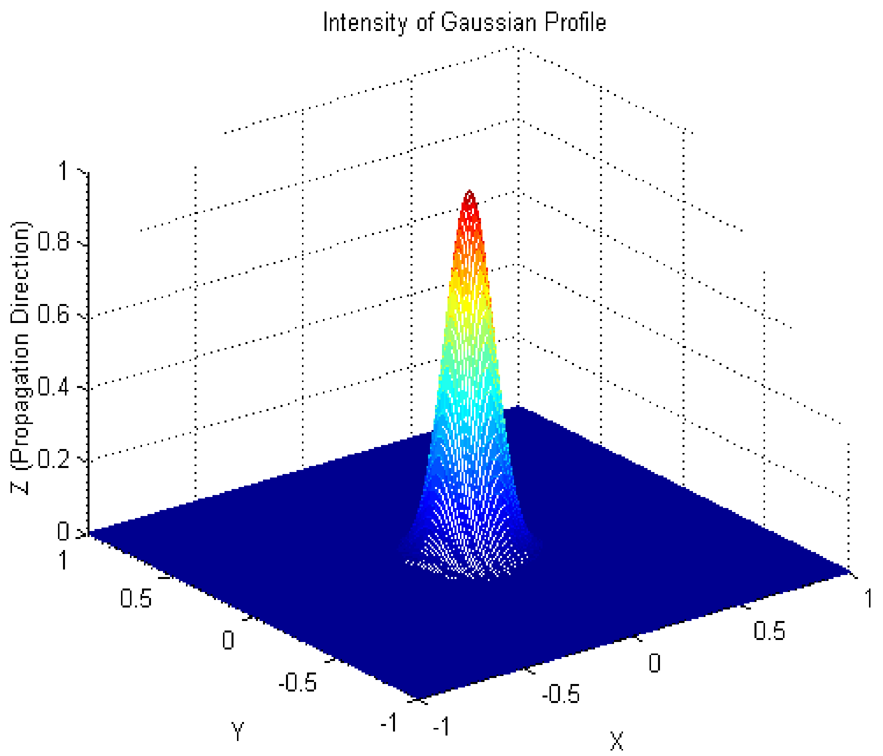

Figure 2.4 Gaussian beam with k_3=10k, η=2, η_0=1.

We see from the above figures that as k increases, the Gaussian profile becomes sharper.

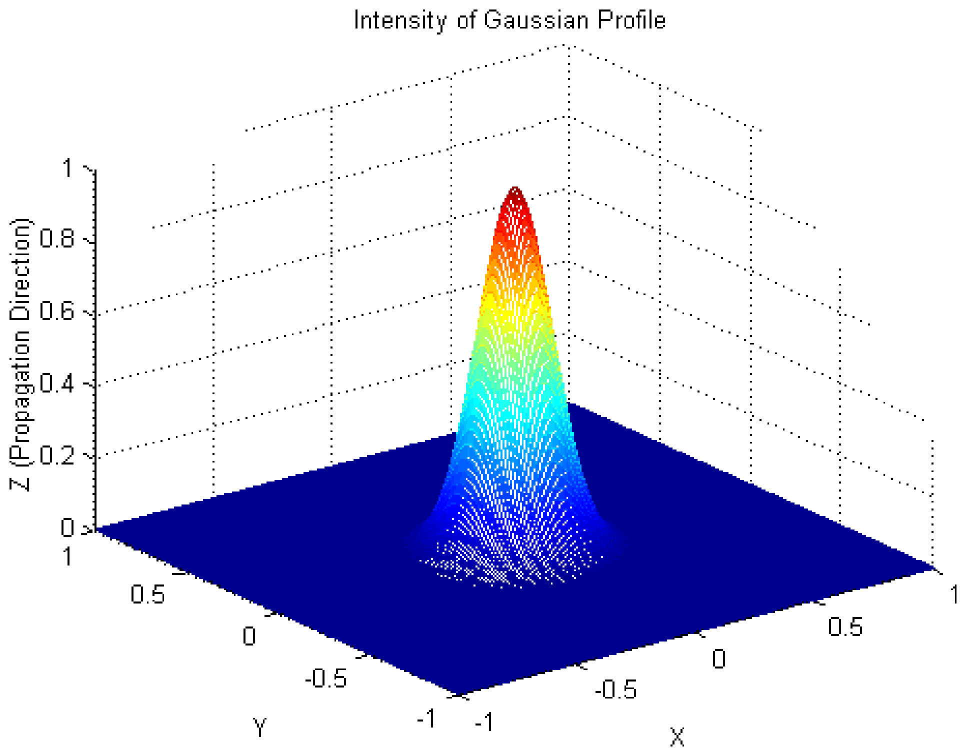

Figure 2.5 Gaussian beam with k_4=10k, η=3, η_0=1.

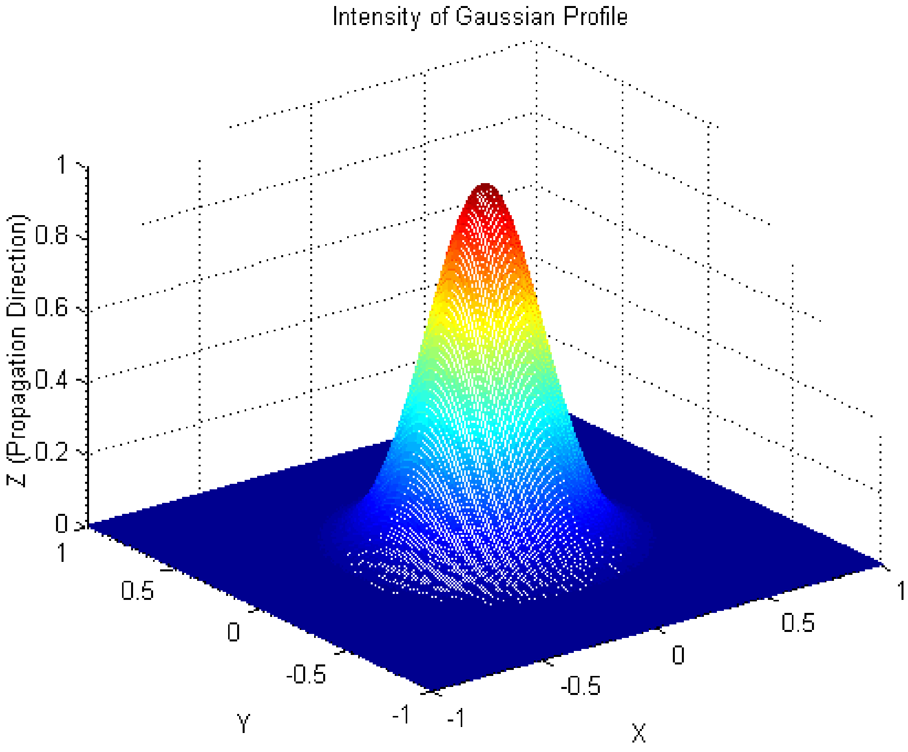

Figure 2.6 Gaussian beam with k_5=10k, η=5, and η_0=1.

We see that as η increases the Gaussian profile becomes broader.

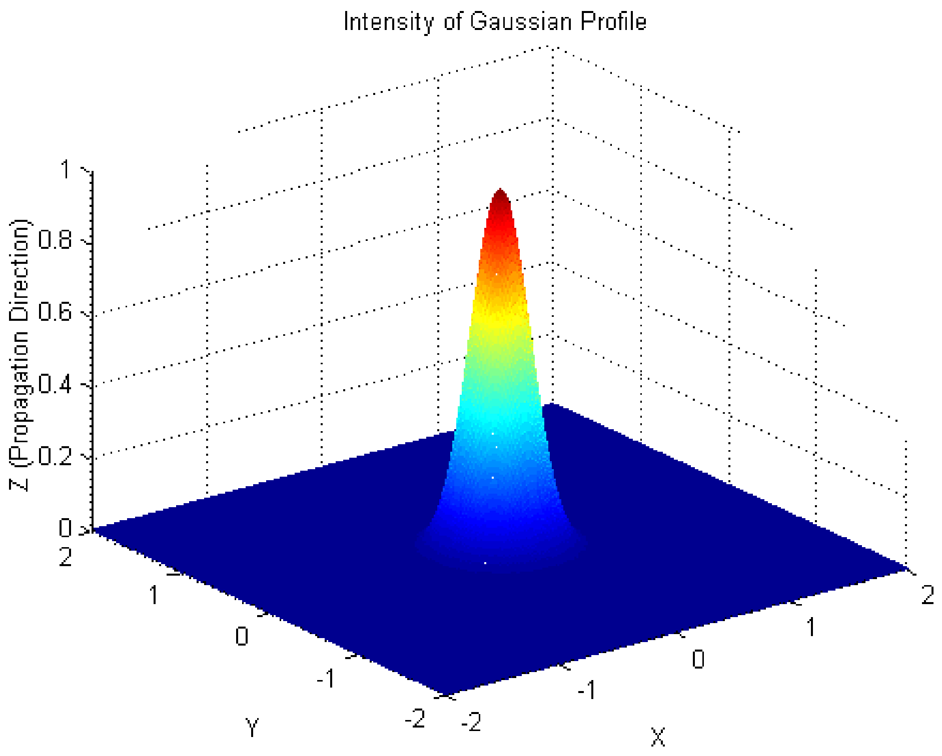

Figure 2.7 Gaussian beam with k_6=10k, η=5, and η_0=1.

Figure 2.7 shows the change in the profile due to the range increase in x and y. As the range in x and y is increased the Gaussian profile becomes sharper. This change is due to the increase in aperture and as a result less dispersion occurs.

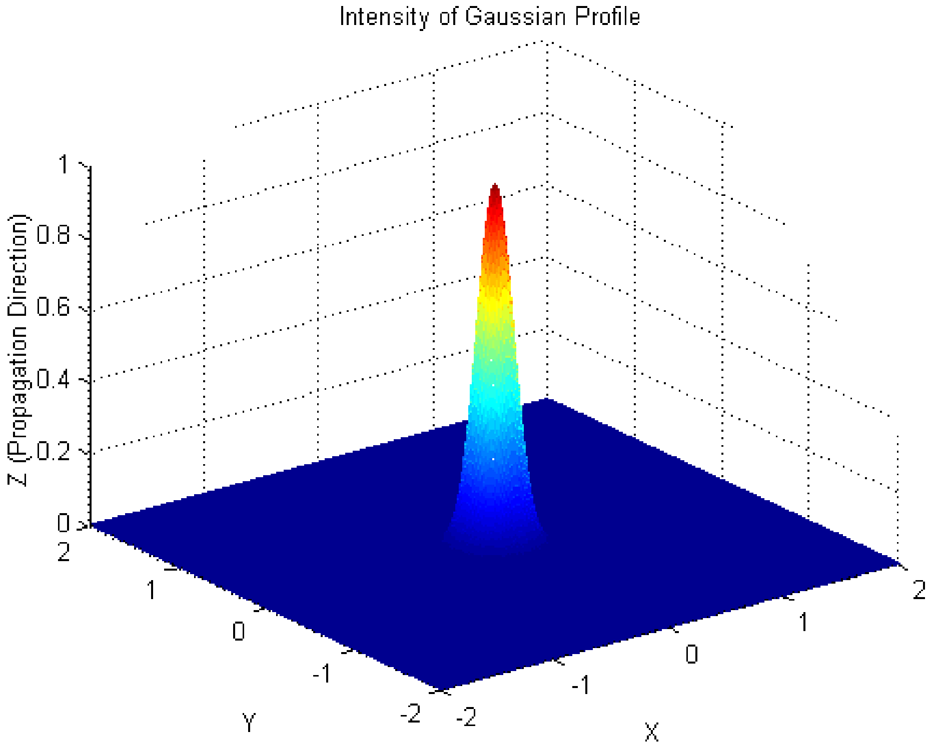

Figure 2.8 Gaussian beam with k_7=10k, η=5, and η_0=5.

Figure 2.8 is obtained from the previous one by just changing the parameter value of η_0 to 5.

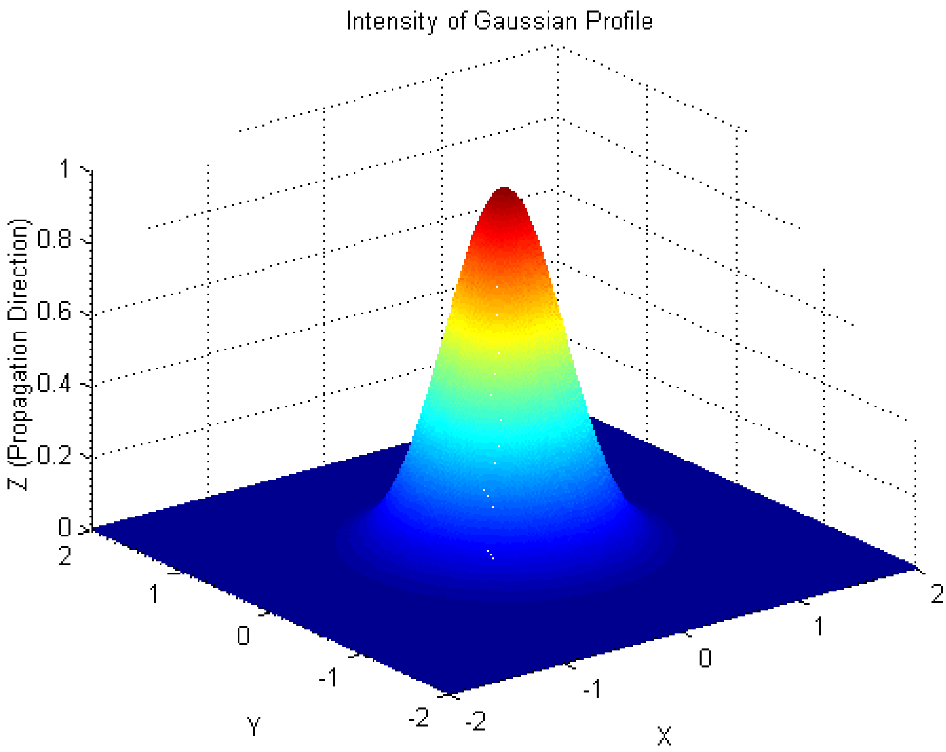

Figure 2.9. Gaussian beam with k_8=10k, η=5, and η_0=100.

But if we increase η further the profile will disperse. If η is replaced by 100 the beam will be as shown in above figure.

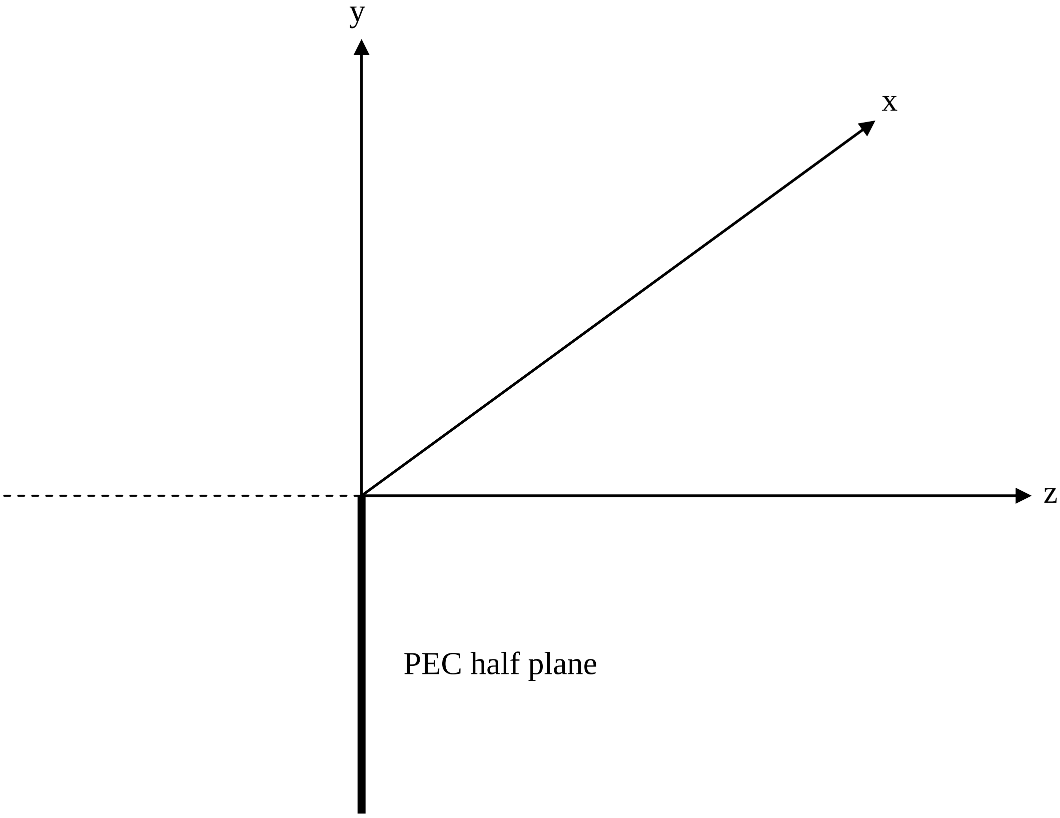

Figure 2.10 A PEC half plane located along y-axis.

The diffraction integral for our problem can be written as

$$U(x, y, \eta) = \frac{jk}{2\pi\eta} \int_{-\infty}^{\infty} \int_{-\infty}^{0} U_0(x', y') e^{-j\frac{k}{2\eta}[(x-x')^2 + (y-y')^2]} dy' dx' \quad , \qquad (2.38)$$

where $U_0(x', y')$ has a unit amplitude. The field is uniformly distributed in the half plane. The paraxial diffraction integral says that the field amplitudes at any z plane are related to the field amplitudes at z=0 plane by a linear,shift-invariant filtering operation with impulse response $M(x, y, \eta)$ which is found in the Equation (2.34).

This means that the impulse response gives the amplitude on a plane at a distance z away from a source which is located at the origin. Turning back to the Equation (2.2) we can rewrite it as

$$G(x,y,\eta)=\frac{jk}{2\pi\eta}e^{-j\frac{k}{2}(z-ct)}\int_{-\infty}^{\infty}\int_{-\infty}^{0}e^{-j\frac{k}{2\eta}[(x-x')^2+(y-y')^2]}dy'\,dx', \tag{2.39}$$

which can be fractioned as

$$G(x,y,\eta)=\frac{jk}{2\pi\eta}e^{-j\frac{k}{2}(z-ct)}\int_{-\infty}^{\infty}e^{-j\frac{k}{2\eta}(x-x')^2}dx'\int_{-\infty}^{0}e^{-j\frac{k}{2\eta}(y-y')^2}dy'. \tag{2.40}$$

In order to solve the first integral in Equation (2.40), by letting

$$-j\frac{k}{2\eta}(x-x')^2=-\frac{t^2}{2}, \tag{2.41}$$

$$dx'=e^{-j\frac{\pi}{4}}\sqrt{\frac{\eta}{k}}dt, \tag{2.42}$$

is obtained. By using the equations (2.41) and (2.42) the first integral is obtained as

$$\int_{-\infty}^{\infty}e^{-j\frac{k}{2\eta}(x-x')^2}dx'=e^{-j\frac{\pi}{4}}\sqrt{\frac{2\pi\eta}{k}}. \tag{2.43}$$

For the second integral in Equation (2.40), by letting

$$\frac{k}{2\eta}(y'-y)^2=t^2, \tag{2.44}$$

$$dy' = \sqrt{\frac{2\eta}{k}}dt, \tag{2.45}$$

is obtained. As a result the second integral becomes as

$$\int_{-\infty}^{0} e^{-j\frac{k}{2\eta}(y-y')^2} dy' = \int_{-\infty}^{-y\sqrt{\frac{k}{2\eta}}} e^{-jt^2} dt, \tag{2.46}$$

which can be written as

$$\int_{y\sqrt{\frac{k}{2\eta}}}^{\infty} e^{-jt^2} dt. \tag{2.47}$$

Since the Fresnel integral is defined as

$$F[x] = \frac{e^{j\frac{\pi}{4}}}{\sqrt{\pi}} \int_{x}^{\infty} e^{-jq^2} dq, \tag{2.48}$$

Equation (2.47) can be written in terms of the Fresnel integral as

$$\sqrt{\pi} e^{-j\frac{\pi}{4}} F\left[y\sqrt{\frac{k}{2\eta}} \right]. \tag{2.49}$$

Substituting the equations (2.49) and (2.43) into Equation (2.40) we obtain $G(x,y,\eta)$ as

$$G(x,y,\eta) = \sqrt{\frac{k}{2\pi\eta}} e^{-j\frac{k}{2}(z-ct)} F\left[y\sqrt{\frac{k}{2\eta}} \right], \tag{2.50}$$

which can be regarded as the final expression of the scattered field. The diffracted field is shown in Figure 2.11 which can be obtained by means of Bessel function [12]. It is important to note that the form of the diffracted field is analogous to the diffraction function introduced by Mohinsky [13].

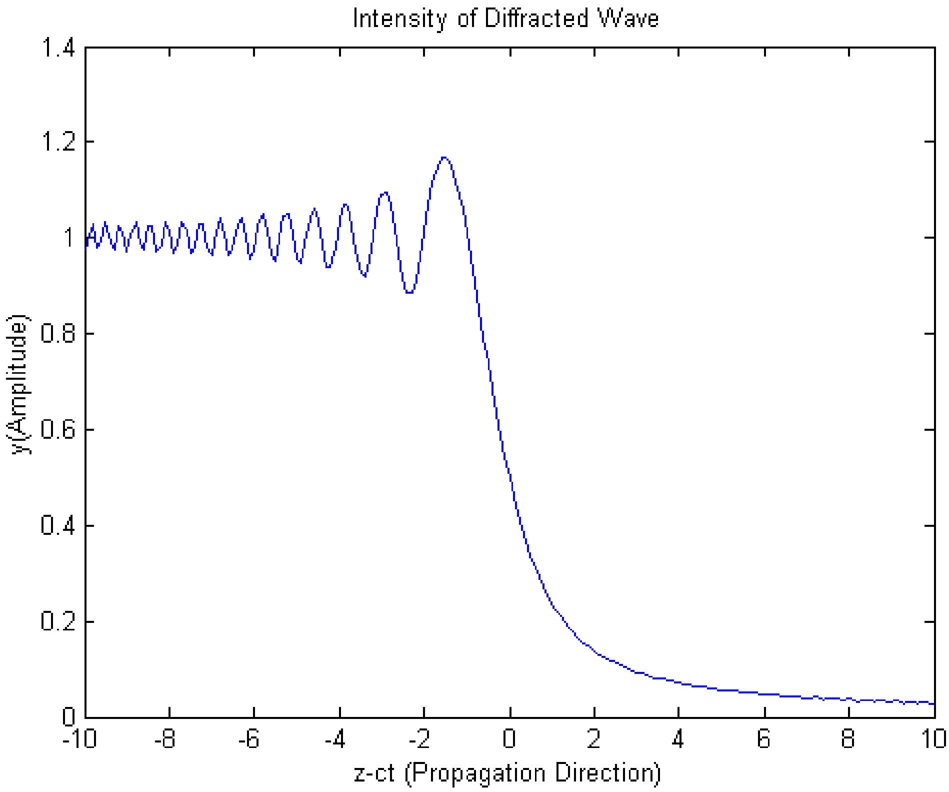

Figure 2.11 Intensity of diffracted wave.

To rewrite the packet wave we substitute Equation (2.36) into Equation (2.2) by taking $U_0(x,y)$ as unity:

$$G(x,y,\eta)=\frac{jk}{2\pi}\frac{e^{-jk\frac{x^2+y^2}{2\eta}}}{\eta}e^{-j\frac{k}{2}(z-ct)}, \tag{2.51}$$

from which we can draw the packet wave as shown in Figure 2.12

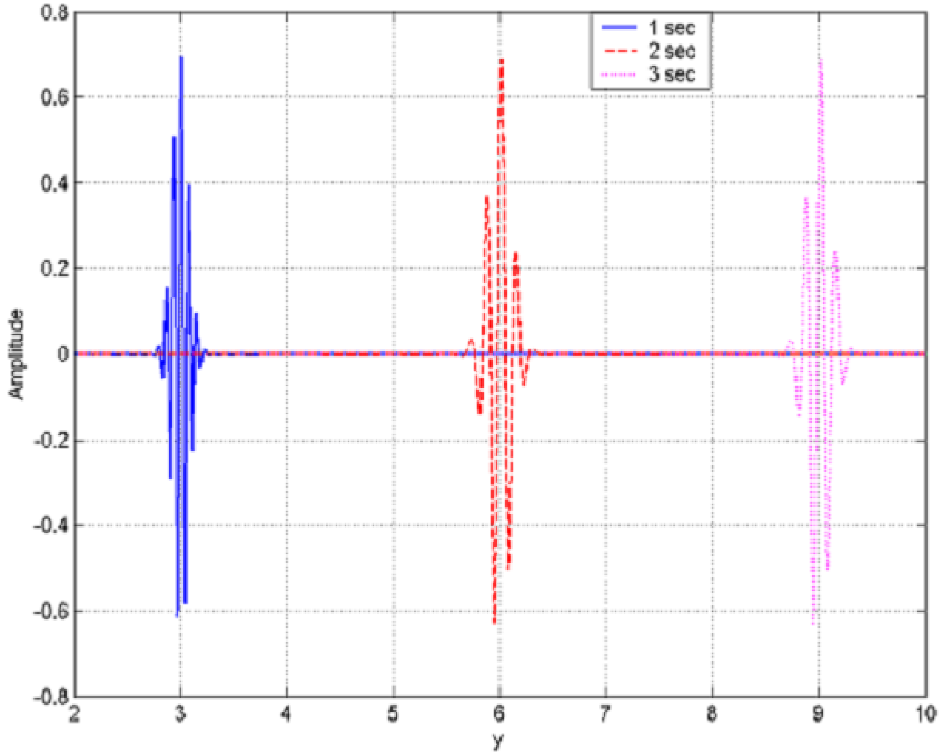

Figure 2.12 Packet wave.

CHAPTER 3

PROPAGATION OF LIGHT IN THE PARAXIAL APPROXIMATION

The classical description of light is as a transverse electromagnetic wave, but to work on many effects we can use a scalar instead of using the full wave equation. In free space, we have

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}. \qquad (3.1)$$

In this equation ψ represents a component of the electric or magnetic field. For monochromatic, coherent light, we can write

$$\psi(x,y,z,t)=\psi(x,y,z,0)e^{-j\omega t}\,. \qquad (3.2)$$

Substituting this into the wave equation, we obtain Helmholtz's equation:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=-k^2\psi \quad , \qquad (3.3)$$

where k=ω/c is the free-space wave number and ω is the radian frequency. Let us assume a propagation parallel to z axis, so that

$$\psi(x, y, z, 0) = f_z(x, y)e^{-jkz} \quad , \tag{3.4}$$

and suppose that $f_z(x,y)$ varies slowly with z.(Similar to a plane wave propagating parallel to the z axis, $f_z(x,y)$ is constant). Substituting Equation (3.4) into Helmholtz's equation,Equation (3.3), yields

$$e^{jkz}\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - 2jk\frac{\partial f}{\partial z} - k^2 f_z\right] + k^2 f_z e^{jkz} = 0 \quad . \tag{3.5}$$

For the paraxial approximation we can neglect $\frac{\partial^2 f}{\partial z^2}$ because of the slowly varying feature of f_z with respect to z. This assumption results in the paraxial wave equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - 2jk\frac{\partial f}{\partial z} = 0. \tag{3.6}$$

CHAPTER 4

GAUSSIAN BEAM OPTICS [14]

Most of the time the laser beams need to be focused, modified, or shaped. By assuming that the laser beam has an ideal Gaussian intensity profile, like TEM_{00} mode, we can approximate the beam propagation. In TEM_{00} mode, the beam is a perfect plane wave at the output of the laser and it has a Gaussian transverse intensity profile. Distortion may be created at some distance by aperture limitations or obstacles along the way. In order to specify the propagation characteristics, we need to determine the beam parameters.

4.1 Gaussian Beams

We will assume a beam propagating in z direction. In order to derive an expression, we will use the Helmholtz's wave equation,

$$\nabla^2 u + k^2 u = 0 \quad , \tag{4.1}$$

where u is the component of the electric or magnetic field. For the beam we can write

$$u = \psi(x, y, z) e^{-jkz} \quad , \tag{4.2}$$

where ψ represents the differences between a laser beam and a planewave.

Following the steps between equations (3.3) and (3.6), we obtain the paraxial wave equation again:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}-2jk\frac{\partial\psi}{\partial z}=0 \quad , \tag{4.3}$$

which has the same form of time dependent Schrödinger equation for ψ and it can be proved that

$$\psi = e^{-j\left(P+\frac{k}{2q}r^2\right)} \quad , \tag{4.4}$$

is a solution of this paraxial equation,where

$$r^2 = x^2 + y^2 \quad . \tag{4.5}$$

Both P and q are the functions of z .P is a complex phase shift that occurs during propagation of the beam and q is the complex beam parameter which expresses the Gaussian variation with respect to r. Substituting Equation (4.4) into Equation (4.3) we get

$$\frac{dq}{dz}=1, \tag{4.6}$$

and

$$\frac{dP}{dz}=-\frac{j}{q} \tag{4.7}$$

then we can write

$$q_2 = q_1 + z \quad , \tag{4.8}$$

which means that the beam parameters in different planes are related in such a way that they are separated by a distance z. Now we will use the relation

$$\frac{1}{q} = \frac{1}{R} - j\frac{\lambda}{\pi\omega^2} \ , \tag{4.9}$$

where ω and R are also the functions of z.R is the radius of curvature which intersects the axis at z and ω is called the beam radius or spot size which is the measure of decrease of the field amplitude. 2ω is called the beam diameter. The Gaussian beam has a minimum diameter $2\omega_0$ at the beam waist where complex beam parameter q can be written as

$$q_0 = j\frac{\pi\omega_0{}^2}{\lambda} \quad , \tag{4.10}$$

and at any distance z from the beam waist q can be written as

$$q = q_0 + z \quad , \tag{4.11}$$

which can be rewritten as

$$q = j\frac{\pi\omega_0{}^2}{\lambda} + z \quad , \tag{4.12}$$

from which we write

$$\frac{1}{q} = \frac{1}{z + \frac{\pi^2\omega_0{}^2}{\lambda}} \ , \tag{4.13}$$

and

$$\frac{1}{q}=\frac{z-j\frac{\pi\omega_0{}^2}{\lambda}}{z^2-\frac{\pi^2\omega_0{}^4}{\lambda^2}} \quad , \tag{4.14}$$

finally we obtain

$$\frac{1}{q}=\frac{z}{z^2+\frac{\pi^2\omega_0{}^4}{\lambda^2}}-j\frac{\frac{\pi\omega_0{}^2}{\lambda}}{z^2+\frac{\pi^2\omega_0{}^4}{\lambda^2}} \quad . \tag{4.15}$$

Now let us equate the real and imaginary parts of Equation (4.15) and Equation (4.9):

For the real terms we write

$$\frac{1}{R}=\frac{\lambda^2 z}{\lambda^2 z^2+\pi^2\omega_0{}^4} \quad , \tag{4.16}$$

and R is obtained as

$$R(z)=z\left[1+\left(\frac{\pi\omega_0{}^2}{\lambda z}\right)^2\right] \quad . \tag{4.17}$$

For the imaginary terms we write

$$\frac{\lambda^2}{\pi\omega^2}=\frac{\frac{\pi\omega_0{}^2}{\lambda}}{z^2+\frac{\pi^2\omega_0{}^4}{\lambda^2}} \quad , \tag{4.18}$$

from which we obtain

$$\omega^2(z)=\omega_0{}^2\left[1+\left(\frac{\lambda z}{\pi\omega_0{}^2}\right)^2\right] \quad . \tag{4.19}$$

By assuming that z is much larger than $\pi\omega_0 / \lambda$ we can say that 1 can be ignored in Equation (4.19). As a result we obtain

$$\omega(z) = \frac{\lambda z}{\pi \omega_0^{\ 2}} \quad . \tag{4.20}$$

Since the beam contour is a hyperbola with asymptotes inclined to the axis at an angle

$$\theta = \frac{\omega(z)}{z} \quad , \tag{4.21}$$

we can write it as

$$\theta = \frac{\dfrac{\lambda z}{\pi \omega_0^{\ 2}}}{z} = \frac{\lambda}{\pi \omega_0} \quad , \tag{4.22}$$

which is the far field diffraction angle. Dividing the Equation (4.17) by Equation (4.19) we write

$$\frac{R}{\omega^2} = \frac{z\left[1 + \left(\dfrac{\pi \omega_0^{\ 2}}{\lambda z}\right)^2\right]}{\omega_0^{\ 2}\left[1 + \left(\dfrac{\lambda z}{\pi \omega_0^{\ 2}}\right)^2\right]} \quad . \tag{4.23}$$

Letting,

$$\left(\frac{\pi \omega_0^{\ 2}}{\lambda z}\right) = P \quad , \tag{4.24}$$

the Equation (4.24) becomes

$$\frac{R}{\omega^2} = \frac{z(1+P^2)}{\omega_0{}^2\left(1+\dfrac{1}{P^2}\right)} = \frac{zP^2}{\omega_0{}^2} \quad . \tag{4.25}$$

Substituting Equation (4.24) into Equation (4.25) we obtain

$$\frac{\lambda z}{\pi\omega_0{}^2} = \frac{\pi\omega^2}{\lambda R} \quad . \tag{4.26}$$

By using this relation we reach the following results:

$$\omega^2 = \omega_0{}^2\left[1+\left(\frac{\pi\omega^2}{\lambda R}\right)^2\right] \quad , \tag{4.27}$$

$$R = z\left[1+\left(\frac{\lambda R}{\pi\omega^2}\right)^2\right] \quad . \tag{4.28}$$

Complex phase shift can be found by integrating dP/dz in Equation (4.7) where q is taken as in Equation (4.12),

$$jP(z) = \ln\sqrt{1+\left(\frac{\lambda z}{\pi\omega_0{}^2}\right)^2} - j\tan^{-1}\left(\frac{\lambda z}{\pi\omega_0{}^2}\right) \quad , \tag{4.29}$$

where the real part is the phase shift Φ between the Gaussian beam and an ideal plane wave, while the imaginary part is the amplitude factor ω/ω_0. Amplitude factor stands for the intensity decrease along the propagation. By using these results we can rewrite the Gaussian beam in Equation (4.1) as

$$u(r,z) = \frac{\omega_0}{\omega} e^{\left[-j(kz-\Phi)-r^2\left(\frac{1}{\omega^2}+\frac{jk}{2R}\right)\right]} \quad , \tag{4.30}$$

where

$$\Phi = \tan^{-1}\left(\frac{\lambda z}{\pi {\omega_0}^2}\right) \quad . \tag{4.31}$$

The Gaussian beam in Equation (4.30) is called fundamental mode.

4.2 Gaussian beam obtained from a complex line source

We use optical components like irises, knife edges, etc. for their blocking and shadowing properties. Gaussian-like optical fields cannot be described by ordinary rays. Therefore when an optical component is illuminated by a beam its shadow is not described by geometrical optics. Optical fields are commonly described in the form of bounded beams. For the Gaussian beams, we consider the rays as emanating from a source which has complex coordinates. These rays are regarded as traveling through complex space and intersecting real space at one point. We already know that when the source coordinates are complex the free-space Green's function yields the fields of a Gaussian beam. If we use the point-source Green's function we obtain a circular cross sectional Gaussian beam. But if we use a line source Green's function we obtain a ribbon having a Gaussian profile. The reason why we use the Green's function to obtain a beam field is that the Green's function we can use the propagation and diffraction expressions that are already derived for the line source excited fields. [15]. For this study we will consider a two-dimensional or ribbon beam propagating parallel to z axis with no variation along x, and the electric field is polarized along x. Let a line source be placed along x-axis as shown in Figure 4.1.

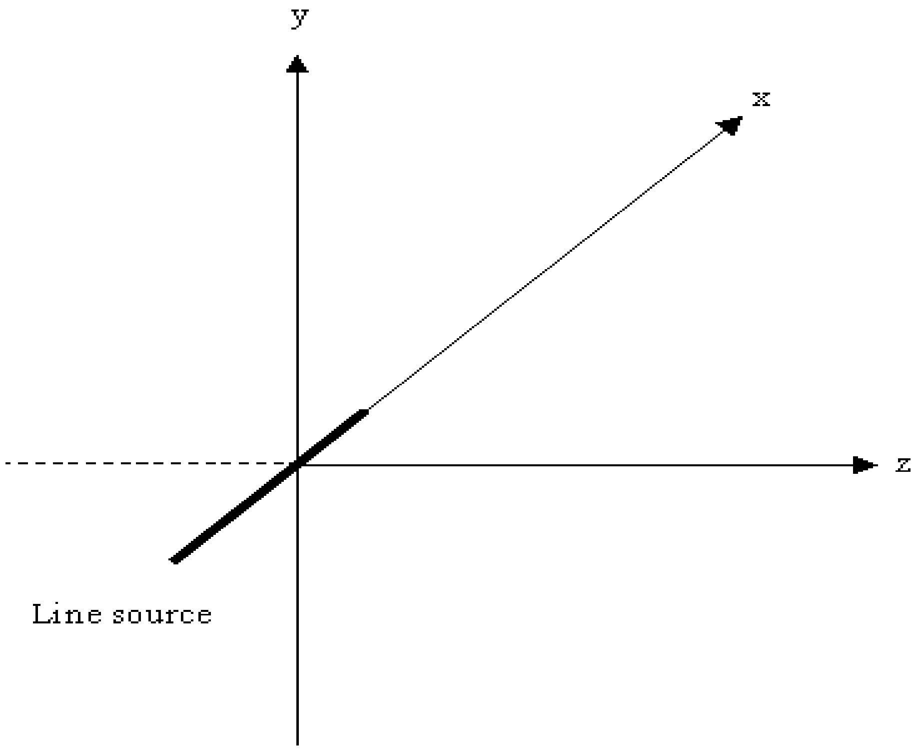

Figure 4.1 Line source located along x-axis.

The vector potential can be written as

$$\vec{A} = \vec{e}_x \frac{w\mu_0 I_0}{4\pi} \iiint_v \frac{e^{-jkR}}{R} dx'\, dy'\, dz' \quad , \qquad (4.32)$$

where

$$\begin{aligned} x' &= (-\infty, \infty) \\ y' &= 0 \\ z' &= 0. \end{aligned} \quad . \qquad (4.33)$$

Then the vector potential becomes as

$$\vec{A} = \overrightarrow{ex}\frac{\omega\mu_0 I_0}{4\pi}\int_{-\infty}^{\infty}\frac{e^{-jkR}}{R}dx' \quad , \tag{4.34}$$

where

$$R = \left[(x-x')^2 + y^2 + z^2\right]^{\frac{1}{2}}. \tag{4.35}$$

Letting,

$$(x - x') = R\sinh\alpha \quad , \tag{4.36}$$

yields

$$dx' = R\cosh\alpha d\alpha \quad , \tag{4.37}$$

$$\vec{A} = \overrightarrow{e_x}\frac{\omega\mu_0 I_0}{4\pi}\int_c e^{-jk\left[R^2 + R^2\sinh^2\alpha\right]^{\frac{1}{2}}}d\alpha \tag{4.38}$$

and

$$\vec{A} = \overrightarrow{e_x}\frac{\omega\mu_0 I_0}{4\pi}\int_c e^{-jkR(1+\sinh^2\alpha)^{\frac{1}{2}}}d\alpha \ . \tag{4.39}$$

Since $(1+\sinh^2\alpha)^{\frac{1}{2}} = \cosh\alpha$,we can write Equation (4.39) as

$$\vec{A} = \overrightarrow{e_x}\frac{\omega\mu_0 I_0}{4\pi}\int_c e^{-jkR\cosh\alpha}d\alpha \quad . \tag{4.40}$$

Since the zero-order Hankel function of the second kind is

$$H_0^{(2)}(kR) = \frac{j}{\pi}\int_c e^{-jkR\cosh\alpha}d\alpha \quad , \tag{4.41}$$

the vector potential can be written in terms of the Hankel function as

$$\vec{A} = \frac{\omega\mu_0 I_0}{4j} H_0^{(2)}(kR) \quad . \tag{4.42}$$

Therefore line source yields the Hankel function which produces a Gaussian profile when complex coordinates are assigned to the source. Figures 4.2 and 4.3 show the Gaussian beams produced by complex line sources:

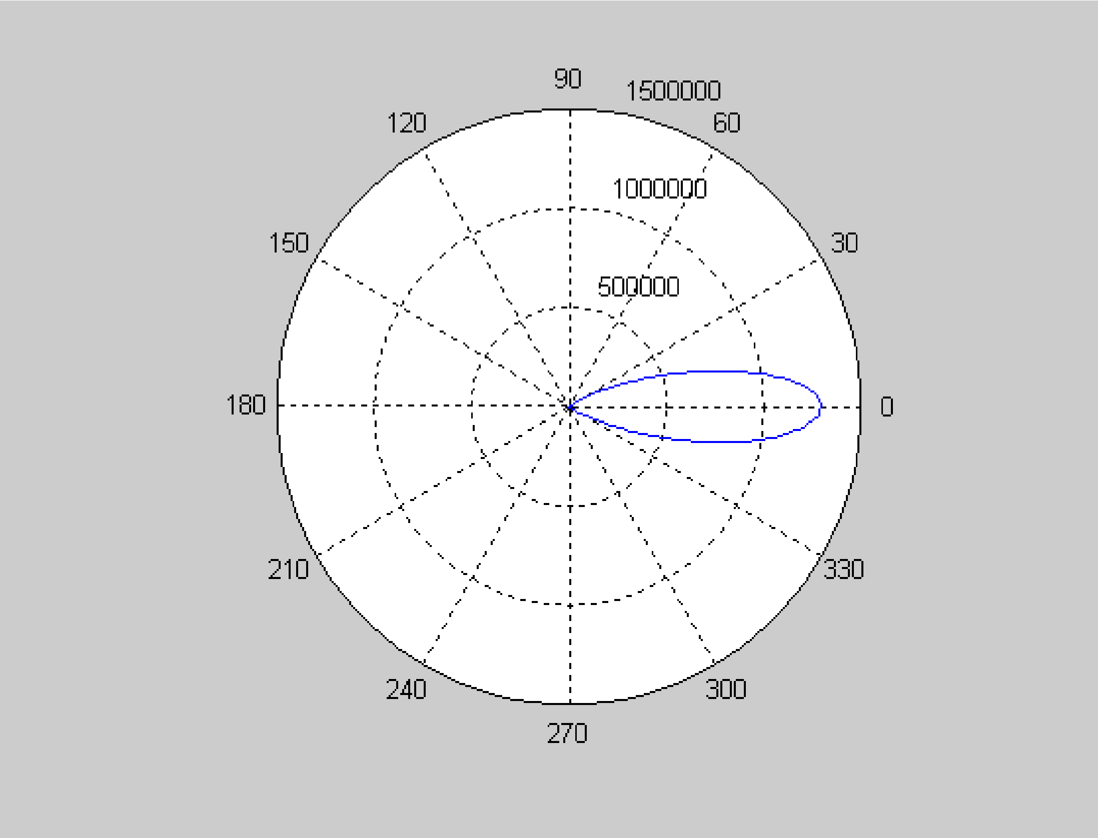

Figure 4.2 Radiation pattern produced by complex line source with ρ being positive.

If ρ, which is the cylindrical coordinate, is chosen positive, radiation pattern depicted in Figure 4.2 in Figure 4.3.

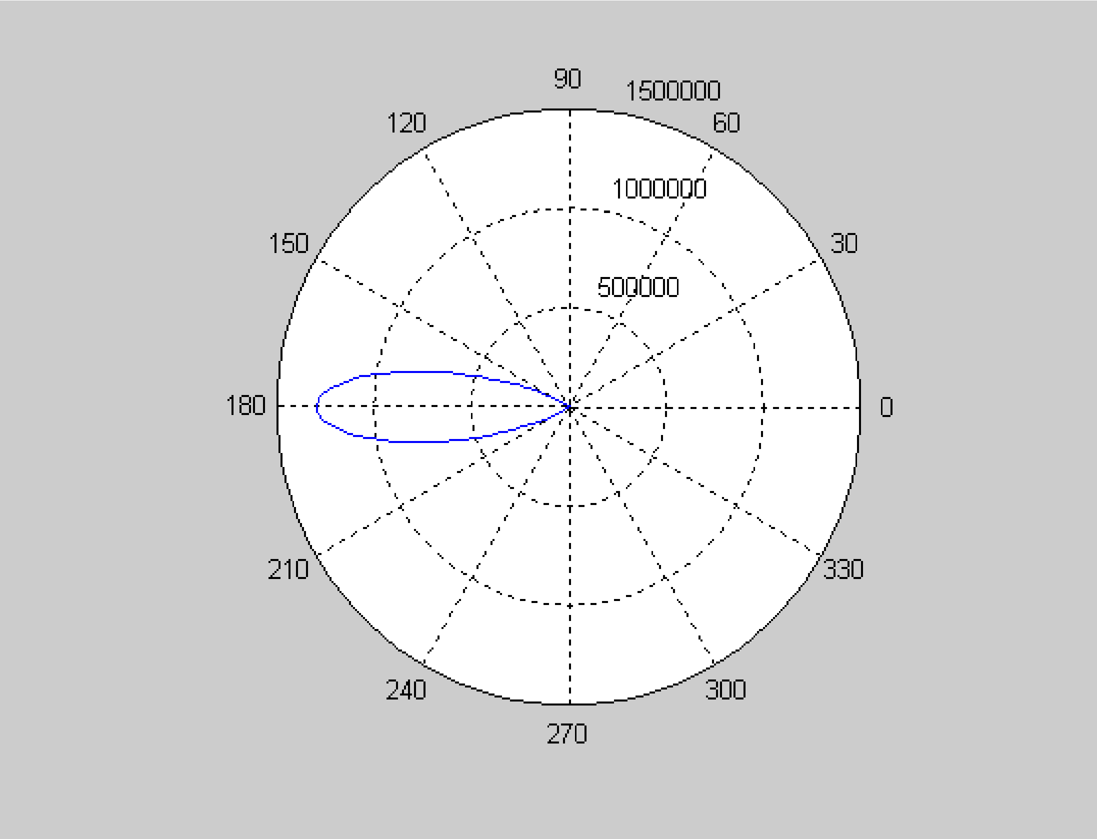

Figure 4.3 Radiation pattern produced by a complex line source with ρ being negative.

Now we will examine this result obtained in Equation (4.42) asymptotically, assuming that kR>>1 which means that either k or R is very large when compared to 1. When k is very large, we say that frequency is very large due to k being $k = \omega\sqrt{\varepsilon\mu}$. When R is very large, we are in the far field region. In order to find an expression by using the asymptotic approach we will start with the stationary phase method which is described in detail in appendix. Let us consider the integral

$$I = \int_a^b e^{jkg(x)} f(x)dx \quad , \tag{4.43}$$

where g(x) is the phase function and f(x) is the amplitude function. At the stationary phase point x_s the first derivative of the phase function will be zero by definition.

$$g'(x) = 0 \Rightarrow x = x_s, \qquad (4.44)$$
$$g'(x_s) = 0.$$

We write the Taylor series expansion of g(x) around the stationary point x_s as

$$g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(x_s)}{n!}(x - x_s)^n. \qquad (4.45)$$

Taking the first three terms and discarding the higher order ones of the Taylor expansion we can write the approximate expression of it as

$$g(x) \cong g(x_s) + g'(x_s)(x - x_s) + \frac{1}{2}g''(x_s)(x - x_s)^2, \qquad (4.46)$$

and

$$g(x) \cong g(x_s) + \frac{1}{2}g''(x_s)(x - x_s)^2, \qquad (4.47)$$

where the term containing the first derivative of g(x) is zero. The value of f(x) at the stationary point can approximately be assumed as $f(x_s)$.

$$f(x) \cong f(x_s). \qquad (4.48)$$

Now let us substitute these results into Equation (4.43),

$$I = \int_{-\infty}^{\infty} f(x_s) e^{-jk\left[g(x_s) + \frac{1}{2}g''(x_s)(x - x_s)^2\right]} dx. \qquad (4.49)$$

Since $f(x_s)$ and $g(x_s)$ are constants,$f(x_s)$ and exp[-jkg(x_s)] can be taken out of the integral yielding,

$$I = f(x_s)e^{-jkg(x_s)} \int_{-\infty}^{\infty} e^{-jk\frac{g''(x_s)}{2}(x-x_s)^2}\, dx. \qquad (4.50)$$

Here the integral has the form of $\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$ which is equal to $\sqrt{2\pi}$.

By using change of variable, let

$$jkg''(x_s)(x-x_s)^2 = y^2, \qquad (4.51)$$

$$(x-x_s)^2 = \frac{e^{-j\frac{\pi}{2}} y^2}{kg''(x_s)}, \qquad (4.52)$$

and

$$dx = \frac{e^{-j\frac{\pi}{4}}}{\sqrt{kg''(x_s)}}\, dy. \qquad (4.53)$$

Substituting these results into Equation (4.49) we get

$$I = f(x_s)e^{-jkg(x_s)} \int_{-\infty}^{\infty} e^{-jk\frac{g''(x_s)}{2}\frac{y^2 e^{-j\frac{\pi}{2}}}{kg''(x_s)}} \frac{e^{-j\frac{\pi}{4}}}{\sqrt{kg''(x_s)}}\, dy, \qquad (4.54)$$

$$I = f(x_s)e^{-jkg(x_s)} \int_{-\infty}^{\infty} e^{-j\frac{y^2 e^{-j\frac{\pi}{2}}}{2}} \frac{e^{-j\frac{\pi}{4}}}{\sqrt{kg''(x_s)}}\, dy, \qquad (4.55)$$

$$I = f(x_s)e^{-jkg(x_s)} \int_{-\infty}^{\infty} e^{-\frac{y^2 e^{-j\frac{\pi}{2}} e^{j\frac{\pi}{2}}}{2}} \frac{e^{-j\frac{\pi}{4}}}{\sqrt{kg''(x_s)}}\, dy, \qquad (4.56)$$

$$I = f(x_s)e^{-jkg(x_s)} \frac{e^{-j\frac{\pi}{4}}}{\sqrt{kg''(x_s)}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy. \qquad (4.57)$$

Finally the

asymptotic evaluation of the integral becomes as

$$I \cong \frac{\sqrt{2\pi}e^{-j\frac{\pi}{4}} f(x_s) e^{-jkg(x_s)}}{\sqrt{kg''(x_s)}}. \tag{4.58}$$

Now let us go back to the Equation (4.39)

$$\vec{A} = \vec{e_x} \frac{kZ_0 I_0}{4\pi} \int_c e^{-jkR\cosh\alpha} d\alpha. \tag{4.59}$$

In this expression the amplitude function *f(x)* is,

$$f(x) = 1, \tag{4.60}$$

and the phase function $g(\alpha)$ is,

$$g(\alpha) = R\cosh\alpha \quad , \tag{4.61}$$

$$g'(\alpha) = R\sinh\alpha = 0, \tag{4.62}$$

is used to find the value of α at the stationary point from which we obtain α_s as

$$R\sinh\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j} = 0 \Rightarrow \alpha = \alpha_s = 0, \tag{4.63}$$

and

$$g''(\alpha) = R\cosh\alpha, \tag{4.64}$$

$$g''(\alpha_s) = R\cosh\alpha_s = \frac{e^{j\alpha_s} + e^{-j\alpha_s}}{2} = R, \tag{4.65}$$

finally,

$$g(\alpha_s) = g(0) = R. \tag{4.66}$$

Substituting these results into Equation (4.58) we get

$$I \cong \frac{\sqrt{2\pi}e^{-j\frac{\pi}{4}}e^{-jkR}}{\sqrt{kR}}. \tag{4.67}$$

As a result the vector potential found in Equation (4.59) can be rewritten as

$$\vec{A} \cong \vec{e_x}\frac{kZ_0 I_0}{4\pi}\frac{\sqrt{2\pi}}{\sqrt{kR}}e^{-jkR}e^{-j\frac{\pi}{4}}, \tag{4.68}$$

and

$$\vec{A} \cong \vec{e_x}\frac{kZ_0 I_0}{4\pi}\frac{e^{-jkR}}{\sqrt{kR}}e^{-j\frac{\pi}{4}}, \tag{4.69}$$

where $\frac{e^{-jkR}}{\sqrt{kR}}$ is called the cylindrical wave factor and the minus sign in front of jkR means that the wave is outgoing. In order to find the electric field we will multiply the vector potential by –jω

$$\vec{E} = -j\omega\vec{A} = -\vec{e_x}\frac{j\omega k Z_0 I_0}{2\sqrt{2\pi}}e^{-j\frac{\pi}{4}}\frac{e^{-jkR}}{\sqrt{kR}}. \tag{4.70}$$

If we let

$$I_0 = \frac{-1}{j\omega k Z_0}, \tag{4.71}$$

Finally we obtain the electric field produced by the line source as

$$\vec{E} \cong \vec{ex}\frac{e^{-j\frac{\pi}{4}}}{2\sqrt{2\pi kR}}e^{-jkR}, \tag{4.72}$$

and

$$E \cong \frac{\exp\left[-jkR - j\frac{\pi}{4}\right]}{2\sqrt{2\pi}\sqrt{kR}}. \tag{4.73}$$

In order to get a Gaussian beam from the line source we need to assign complex values to the line source coordinates as described by Felsen[15]. By adopting the procedure and considering Figure 4.4. [15], we carry out the following steps:

Let,

$$(y, z) = (0, -z_0 - jb), \tag{4.74}$$

then R becomes

$$R = \sqrt{(z + z_0 + jb)^2 + y^2}, \tag{4.75}$$

where

$$(z + z_0 + jb)^2 >> y^2. \tag{4.76}$$

Using Equation (4.76) into Equation (4.75) we write

$$R = \sqrt{(z + z_0 + jb)^2 \left[1 + \frac{y^2}{(z + z_0 + jb)^2} \right]}, \tag{4.77}$$

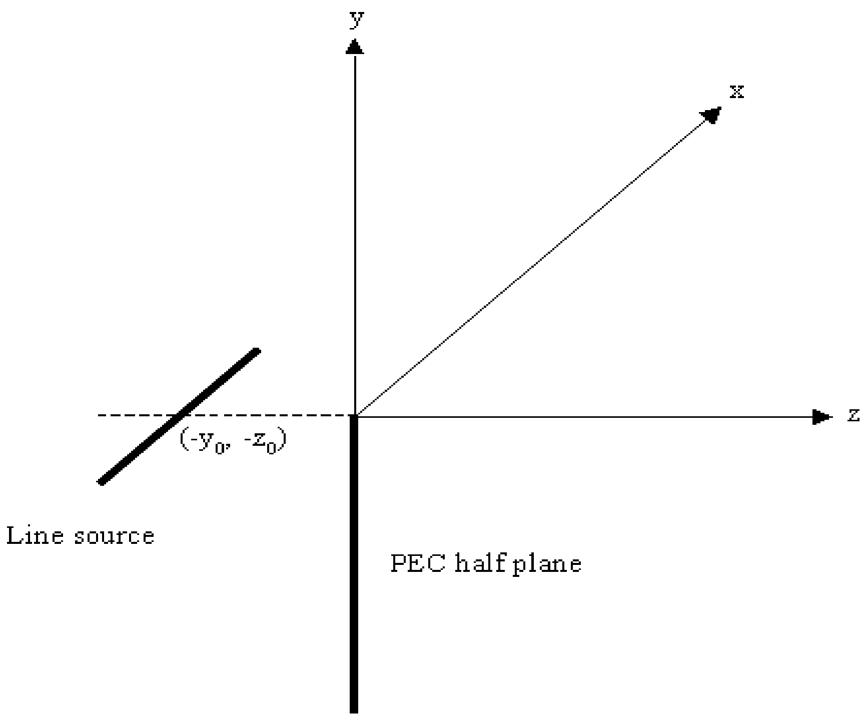

Figure 4.4 Line source located at complex plane to produce a Gaussian beam.

$$R = (z + z_0 + jb)\sqrt{1 + \frac{y^2}{(z + z_0 + jb)^2}}, \tag{4.78}$$

is obtained. Since the binomial expansion is

$$(1 \mp x)^n \cong 1 \mp nx \quad \text{for } x<<1, \tag{4.79}$$

We can use this for the square-root term as

$$\sqrt{1 + \frac{y^2}{(z + z_0 + jb)^2}} = 1 + \frac{1}{2}\frac{y^2}{(z + z_0 + jb)^2} \quad . \tag{4.80}$$

Then R can be written as

$$R = (z + z_0 + jb)\left[1 + \frac{1}{2}\frac{y^2}{(z + z_0 + jb)^2}\right]. \tag{4.81}$$

Substituting Equation (4.81) into Equation (4.73) electric field becomes

$$E \cong \frac{1}{2\sqrt{2\pi}} \frac{e^{-j\frac{\pi}{4}}}{\sqrt{k\left[(z+z_0+jb)^2+y^2\right]^{\frac{1}{2}}}} e^{-jk\left[(z+z_0+jb)\left(1+\frac{y^2}{2(z+z_0+jb)^2}\right)\right]} \quad . \qquad (4.82)$$

Since $(z+z_0+jb)^2 >> y^2$, E can be rewritten as

$$E \cong \frac{1}{2\sqrt{2\pi}} \frac{e^{-j\frac{\pi}{4}}}{\sqrt{k(z+z_0+jb)}} e^{-jk(z+z_0+jb)} e^{-j\frac{k}{2}\frac{y^2}{z+z_0+jb}} \quad . \qquad (4.83)$$

By letting $z_0 = 0$,

$$E \cong \frac{e^{-j\frac{\pi}{4}} e^{-jk(x+jb)}}{2\sqrt{2\pi}\sqrt{k(z+jb)}} e^{-j\frac{k}{2}\frac{y^2}{z+jb}} \quad , \qquad (4.84)$$

is obtained which has the form of the Gaussian beam.

CHAPTER 5

CONCLUSION

In this study, a differential equation is obtained by considering the focus wave mode solution of the wave equation in the sense of paraxial approximation. The equation is solved in the spectral domain by using a Fourier integral transform and a diffraction integral which enables the investigation of the scattering of wave packets by a PEC half plane. It is observed that the results are consistent with the theory.

REFERENCES

1. **JACKSON,J.D**.[1962] *Classical Electrodynamics*,Wiley,New York.

2. **BELANGER,P.A.**[1984] *Packetlike Solutions of the Homogenous-Wave* Equation,Journal of Optical Society of America,A 1,723.

3. **BELANGER,P.A.**[1986] *Lorentz Transformation of Packetlike Solutions of the Homogenous Wave Equation*, Journal of Optical Society of America, A, 3541-542.

4. **SEZGİNER, A**.[1985] *A General Formolation of Focus Wave Modes*, Journal of Applied Physics, 165-650.

5. **ZIOLKOWSKI,R.W.**[1985] *Exact Solutions of The Wave Equation with Complex Source Locations,* J.Math.Phys, 26,861-863.

6. **BRITTINGHAM,J.N**.[1983] *Focus Wave Modes in Homogenous Maxwell's Equations: Transverse Electric Mode*, Journal of Applied Physics, 54,1179.

7. **WU,T.T.** [1984] *Comments on Focus Wave Modes in Homogenous Maxwell Equations: Transverse Electric Mode*, Journal of Applied Physics,56,2587.

8. **HILLION,P**.[1992] *Focus Wave Modes and Diffraction of Plane Waves*, J. Opt. (Paris), 23, 233.

9. **PALMER,M.R.,DONELLEY,R.**[1993] *Focus Wave Modes and the Scalar Wave Equation,* J.Math.Phys.34,4007.

10. **HILLION,P.**[1996] *Diffraction of Focus Wave Modes at a Perfectly Conducting Screen*, Optics Communications, 123,215-224.

11. **SHERMAN,G.C.,BREMERMANN,H.J.**[1969] *Generalization of the Angular Spectrum of Plane Waves and the Diffraction Transform*, Journal of Optical Society of America,59,146.

12. **DURNIN,J.**[1986] *Exact Solutions for Nondiffracting Beams.I.The Scalar Theory*, Journal of Optical Society of America,4,651-654.

13. **MOHINSKY,M**.[1952] *Diffarction in Time*, Physical Review, 88,625-631.

14. **KOGELNIK,H.,LI,T.[1966]** *Laser Beams and Resonators*, Proceedings of The IEEE, 54, 1312.

15. **GREEN,A.C.,BERTONI,H.L.,FELSEN,L.B.[1979]** *Properties of the Shadow Cast by a Half-Screen when Illuminated by a Gaussian Beam*, Journal of Optical Society of America, 503.

16. **STUTZMAN,W.L.,THIELE,G.A.**[1998],*Antenna Theory and Design*, Wiley, New York.

APPENDIX

METHOF OF STATIONARY PHASE [16]

Radiation integrals consist of an amplitude function times a phase function. In many cases an asymptotic evaluation can be made if the amplitude function slowly varies and the phase function rapidly varies. Consider the integral

$$I = \int_a^b f(x) e^{j\beta\gamma(x)} dx \quad , \tag{A1}$$

where f(x) and γ(x) are real functions, and β is a large number. The integral endpoints can be infinite. If f(x) is slowly varying and βγ(x) is a rapidly varying function over the interval of integration due to β being large, the main contribution from the integral comes from the stationary phase point. If there exist more than one stationary points,the contribution of all these stationary points will be summed up. A stationary phase point is defined as a point where the first derivative of the phase function γ is equal to zero:

$\frac{d\gamma}{dx} = 0.$, which means that the function $\gamma(x)$ is maximum at x_0. If $\gamma(x)$ is multiplied by β,the difference between the maximum-valued γ and any other x-valued γ will rapidly increase yielding a rapid exponential growth of the integrand. Therefore the significant contribution to the integral will come from the points that are close to x_0.

If we expand the phase function in a Taylor series around the point of stationary phase,we get the following expression:

$$\gamma(x) \cong \gamma_0 + (x - x_0)\gamma_0^{'} + \frac{1}{2}(x - x_0)^2 \gamma_0^{''} + \ldots \quad , \qquad \text{(A2)}$$

where $\gamma_0^{'}$ and $\gamma_0^{''}$ represent the derivatives of γ with respect to x, evaluated at $x_{0.}$ Now $\gamma_0^{'}$ is assumed as zero by definition in the neighborhood of the point of stationay phase the quantity (x-x_0) is small so that the high-order terms(i.e.,order 3 and higher) in Equation (A2) may be ignored. If there is only one stationay point x_0 in the interval from a to b,and x_0 is not close to either a or b,we can write the Equation (A1) as

$$I = \int_{x-\delta}^{x+\delta} f(x) e^{j\beta\gamma_0} e^{j\beta(x-x_0)^2\gamma_0^{''}/2} dx \quad , \qquad \text{(A3)}$$

if $(\gamma_0^{''} \neq 0)$ where δ represents a small number. By doing this we reduce the range of integration to a small neighborhood about the point of stationary phase. If f(x) is slowly varying,we can approximate it over this small interval. Thus Equation (A3) becomes

$$I_0 = f(x_0) e^{j\beta\gamma_0} \int_{-\infty}^{\infty} e^{j\beta(x-x_0)^2\gamma_0^{''}/2} dx = f(x_0) e^{j\beta\gamma_0} \int_{-\infty}^{\infty} e^{j\beta z^2\gamma_0^{''}/2} dz \quad , \qquad \text{(A4)}$$

where (x-x_0)=z. For convenience, the limits of integration are changed to infinity. This results in a little error which can be neglected.In other regions,the rapid phase variations will cancel each other because of the slowly varying characteristics of f(x). Now let us consider the integral

$$\int_{-\infty}^{\infty} e^{jaz^2} dz \quad , \tag{A5}$$

which can be written by using the Euler's formula as

$$\int_{-\infty}^{\infty} (\cos az^2 + j\sin az^2) dz \quad . \tag{A6}$$

By solving this equation we obtain

$$\sqrt{\frac{\pi}{|a|}} e^{j\frac{\pi}{4}\mathrm{sgn}(a)} \quad , \tag{A7}$$

where

$$\mathrm{sgn(a)} = \begin{cases} 1 & \text{if a>0} \\ -1 & \text{if a<0} \end{cases} \tag{A8}$$

If we use Equation (A5) to evaluate Equation (A4), the stationary phase approximation results in

$$\int_{x-\delta}^{x+\delta} f(x) e^{j\beta\gamma(x)} dx \approx f(x_0) e^{j\beta\gamma(x_0)} \sqrt{\frac{2\pi}{\beta|\gamma_0''|}} e^{j\frac{\pi}{4}\mathrm{sgn}(\gamma_0'')} \quad . \tag{A9}$$

If more than one points of stationary phase exist in the interval of integration and there is no coupling between them,the total value of the integral is obtained by summing the contributions of all these stationary points. Equation (A9) is invalid if the second derivative of the phase function is zero at the point of stationary phase. In this case, we need to use the third-order term in Equation (A2). In addition,Equation (A9) becomes invalid if one of the limits of integration is close to the point of stationary phase x_0. For this case we can express the integral in the form of a Fresnel integral. One more problem occurs if there exist two or more stationay points close together in the range of integration. To obtain the endpoint contribution,it is best to write Equation (A1) as

$$I = \int_{-\infty}^{\infty} f(x) e^{j\beta\gamma(x)} dx - \int_{b}^{\infty} f(x) e^{j\beta\gamma(x)} dx - \int_{-a}^{\infty} f(-x) e^{j\beta\gamma(-x)} dx \quad , \qquad \text{(A10)}$$

or

$$I = I_0 - I_b - I_a \quad . \qquad \text{(A11)}$$

The evaluation of I_0 has been done in Equation (A9).I_b, for instance, can be evaluated by using integration by parts with the wave number to be complex and to have a small amount of loss(i.e.,small α) so that the contribution to the integral by the upper limit at infinity vanishes,and then letting the wave number to be approximated by β,as before. Thus,

$$I_b \cong -\frac{1}{j\beta} \frac{f(b)}{\gamma'(b)} e^{j\beta\gamma(b)} \quad . \qquad \text{(A12)}$$

A similar expression can be found for I_{-a}. Equation (A12) is valid when b is not near or coupled to x_0.

When the stationary point is coupled to the endpoint,we write

$$I_b \cong u(-\mathrm{sgn}(b-x_0))I_0 + \mathrm{sgn}(b-x_0)\, f(b) e^{j\beta\gamma(b) \mp jv^2} \sqrt{\frac{2}{\beta\left|\gamma_0^{"}(b)\right|}} F_{\pm}(v) \qquad \text{(A13)}$$

where

u=unit step function

$F_{\pm}(v)$ is the Fresnel integral

$$\gamma^{"}(b) \neq 0, \qquad \text{(A14)}$$

and

$$v = \sqrt{\frac{\beta}{2\left|\gamma_0^{"}(b)\right|}}\left|\gamma_0^{"}\right|. \qquad \text{(A15)}$$