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FIXED POINT THEOREMS FOR CYCLIC CONTRACTION MAPPINGS

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ABSTRACT

FIXED POINT THEOREMS FOR CYCLIC CONTRACTION MAPPINGS

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M.Sc., Department of Mathematics and Computer Science

Supervisor: Prof. Dr. Kenan Taş

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In this thesis we do a survey about cyclic contractions on some special metric spaces and also we examine fixed point theorems for the cyclic contraction mappings.

Keywords: Cyclic Mappings, Fixed Point, Contraction Mappings, Proximity Point.

ÖZ

DEVÍRLÍ BÜZÜLME DÖNÜŞÜMLERÍ ÍÇÍN SABÍT NOKTA TEOREMLERÍ

BALPETEK, Güzin

M.Sc., Matematik–Bilgisayar Bölümü

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Bu tezde bazi özel metrik uzaylardaki devirli büzülme dönüsümleri ile ilgili olarak bugüne kadar yapılmıs olan çalısmaları derledik ve devirli büzülme dönüsümleri için sabit nokta teoremlerini inceledik.

Anahtar Kelimeler: Devirli dönüsümler, Sabit Nokta, Büzülme Dönüsümleri, Yakınlık Noktası.

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TABLE OF CONTENTS

INTRODUCTION

Let's start with the fixed point problem $Tx = x$ .We are looking for the solutions of this equation. If a solution exists then it is called a "fixed point" of T. One of the most important results in fixed point theory is the Banach Contraction Mapping Principle.

Definition 0.1. *Let (X, d) be a metric space. A function $T : X \to X$ is called* **contraction** *if there exists a number $k \in \mathbb{R}$ with $0 \leq k < 1$ such that $\forall x, y \in X$ the following inequality*

$$d(T(x), T(y)) \leq kd(x, y).$$

holds. The smallest value of k is called **Lipschitz constant** *of T. If the above condition is instead satisfied for $k \leq 1$, then the mapping is said to be a* **non-expansive map**.

Theorem 0.2 (Banach Contraction mapping principle). *Let (X, d) be complete and $T : X \to X$ be a contraction mapping . Then $\exists$ a unique point x in X such that $T(x) = x$.*

In this thesis we do a survey about some types of cyclic contractions on some special metric spaces and also we examine fixed point theorems for these various types of cyclic contraction mappings.

Some good references are Sh. Rezapour, M. Derafshpour and N. Shahzad [11, 14], Ralph De Marr [19], S. Karpagam and Sushama Agrawal [1], W.A. Kirk and P.S. Srinavasan and P. Veeramani [2], A.A. Eldered and P. Veeramani [3, 4], G.Petruşhel [5], M.A.Al-Thafai and N.Shahzad [6].

The organization of this thesis is as follows.

In chapter I, we will give some results on fixed point theory of cyclic contraction mappings which are generalization of contraction mappings.

In chapter II, we give some results on cyclic contractions mappings and fixed point theorems on Reflexive Banach spaces.

In chapter III, we will give some important results on existence of the best proximity point of cyclic ϕ-contractions in ordered metric spaces and we will introduce partially ordered metric spaces.

In chapter IV, we will give some important recently developments on best proximity theorems about KT- type cyclic orbital contraction mappings.

In chapter V, we give existence and convergence results for best proximity points considering different contractive type conditions for cyclical contractive operators.

We should note that the existence and convergence results for best proximity points of different cyclic contractive mappings for probabilistic and partial metric spaces are still open and they can be studied.

CHAPTER 1

CYCLIC CONTRACTION MAPPINGS AND FIXED POINT THEOREMS.

In this chapter we will give several new results on fixed point theory of cyclic contraction mappings which are generalization of contraction mappings. The fundamental reference is [2]. Throughout this article, $\mathbb{R}^+$ denotes the set of all non-negative numbers, and $\mathbb{N}$ is the set of all natural numbers. Let us define the following sets that we need in the following.

Definition 1.1. *Let (X, d) be a metric space and let $A, B \subset X$. Then we define the following notations.*

$$dist(A, B) = \inf\{d(x, y) : x \in A, y \in B\},$$

Definition 1.2. *Let X be a metric space and $A \subseteq X$ and $B \subseteq X$. A map $T : A \cup B \to A \cup B$ is called* **cyclic contraction map** *if it satisfies the following conditions*

(1) $T(A) \subseteq B$ and $T(B) \subseteq A$

(2) For some $k \in (0, 1)$, we have $d(Tx, Ty) \leq kd(x, y) + (1 - k)dist(A, B)$, $\forall x \in A, y \in B$.

If $A \cap B \neq \emptyset$, then $dist(A, B) = 0$ and T is a contraction on $A \cap B$ and hence from the Banach contraction principle we can obtain the fixed point. So it is interesting to study when $dist(A, B) > 0$. In this case, the cyclical contraction map defined above does not induce $A \cap B \neq \emptyset$.

Definition 1.3. *Let (X, d) be a metric space, and A and B be nonempty subsets of X, and $\phi \in \Phi$. Assume that $T : A \cup B \to A \cup B$ is a cyclic map such that, for some $x \in A$, and for each $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\begin{aligned} &\phi(R(T^{2n-1}x, y)) < \phi(d(A, B)) + \varepsilon + \delta \\ \textit{implies} \quad &\phi(d(T^{2n}x, Ty)) < \phi(d(A, B)) + \varepsilon, n \in \mathbb{N}, y \in A \end{aligned} \tag{1.1}$$

where $R(T^{2n-1}x, y) = \frac{1}{3}[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$. *Then* T *is said to be a* ϕ*-type orbital Meir-Keeler cyclic contraction.*

Proposition 1. *Let* A *and* B *be nonempty and closed subsets of a metric space* X *and* $T : A \cup B \to A \cup B$ *is a* ϕ*-type orbital Meir-Keeler cyclic contraction. If* $x \in A$ *satisfies condition (1.1) then* $d(T^{n+1}x, T^n x) \to d(A, B)$*, as* $n \to \infty$*.*

Proof. Suppose T is a ϕ-type orbital Meir-Keeler cyclic contraction. Take $x \in A$ for which (1.1) is satisfied. Since either n or $n+1$ is even, then for each $x \in A$, we have $\frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] \geq d(A, B)$.

Consider the case

$$R(T^n x, T^{n-1}x) = \frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] = d(A, B).$$

So, $\phi(R(T^n x, T^{n-1}x)) = \phi(d(A, B))$. Then due to (1.1) we have

$$\phi(d(T^{n+1}x, T^n x)) = \phi(\frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)]). \quad (1.2)$$

Set $d_n = d(T^n x, T^{n-1}x)$ for each $n \in \mathbb{N}$. Then, (1.2) turns into

$$\phi(d_{n+1}) = \phi(\frac{1}{3}[2d_n + d_{n+1}])$$

Since ϕ is a strictly increasing function, $d_n = d_{n+1}$. Hence, we have

$$\alpha_{n+1} = \phi(d_{n+1}) = \phi(d_n) = \alpha_n$$

Now, consider the other case:

$$R(T^n x, T^{n-1}x) = \frac{1}{3}[2d_n + d_{n+1}] > d(A, B).$$

Since ϕ is a strictly increasing function, we have

$$\phi(R(T^n x, T^{n-1}x)) = \phi(\frac{1}{3}[2d_n + d_{n+1}]) > \phi(d(A, B)).$$

Set $\varepsilon_1 = \phi(\frac{1}{3}[2d_n + d_{n+1}]) - \phi(d(A, B)) > 0$. Du to (1.1), for this ε_1, there exists a δ such that

$$\phi(d_{n+1}) = \phi(d(T^{n+1}x, T^n x)) < \phi(d(A, B)) + \varepsilon_1 = \phi(\frac{1}{3}[2d_n + d_{n+1}]).$$

Hence, $\phi(d_{n+1}) < \phi(\frac{1}{3}[2d_n + d_{n+1}])$ for all $n \in \mathbb{N}$. Regarding ϕ is a strictly increasing function, we have $d_{n+1} < d_n$.

Hence $\{d_n\}$ is a non-increasing sequence which is bounded below by $d(A, B)$. Therefore $\{d_n\}$ converges to some d with $d \geq d(A, B)$.

We assert that $d = d(A, B)$. Suppose not, that is, $d > d(A, B)$ and hence $\phi(d) > \phi(d(A, B))$. Set $\varepsilon = \phi(d) - \phi(d(A, B)) > 0$. Thus, there exists a $\delta > 0$ which satisfies (1.1). Regarding $\{d(T^{n+1}x, T^n x)\} \to d$, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\phi(d) &\leq \phi(\tfrac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+2}x, T^{n+1}x) + d(T^{n+1}x, T^n x)]) \\ &< \phi(d) + \delta = \varepsilon + \phi(d(A, B)) + \delta, \quad \forall n \geq n_0.\end{aligned}$$

Thus,

$$\phi(d(T^{n+2}x, T^{n+1}x)) < \phi(d(A, B)) + \varepsilon = \phi(d), \ \forall n \geq n_0$$

which is a contradiction. Hence $d = d(A, B)$ □

Proposition 1.4. *Let A and B be nonempty and closed subsets of a metric space X and $T : A \cup B \to A \cup B$ is a ϕ-type orbital Meir-Keeler cyclic contraction. Suppose $d(A, B) = 0$. Then, for each $\varepsilon > 0$, there exist $n_1 \in \mathbb{N}$ and a $\delta > 0$ such that*

$$\phi(d(T^p x, T^q x)) < \varepsilon + \delta \ \textit{ implies that } \ \phi(d(T^{p+1}x, T^{q+1}x)) < \varepsilon \tag{1.3}$$

where p and q are opposite parity, with $p, q \geq n_1$.

Proof. Take $x \in X$ for which (1.1) is satisfied. Since T is a ϕ-type orbital Meir-Keeler cyclic contraction, for a given $\varepsilon > 0$, there exists a $\delta > 0$ satisfies (1.1). That is,

$$\begin{aligned}&\phi(\tfrac{1}{3}[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]) < \varepsilon + \delta \\ &\qquad \text{implies } \ \phi(d(T^{2n}x, Ty)) < \varepsilon, n \in \mathbb{N}, y \in A\end{aligned} \tag{1.4}$$

Without loss of generality we can choose

$$\delta < \varepsilon. \tag{1.5}$$

Regarding that $d(A, B) = 0$, ϕ is strictly increasing and Proposition 1, one can choose $n_1 \in \mathbb{N}$ in a way that

$$\phi(d(T^n x, T^{n+1}x)) < \frac{\delta}{2}, \ \text{ for each } \ n \geq n_1. \tag{1.6}$$

We claim that $\phi(d(T^p x, T^q x)) < \varepsilon + \delta$ implies that $\phi(d(T^{p+1}x, T^{q+1}x)) < \varepsilon$.

Fix $n \geq n_1$. Take $p, q \in \mathbb{N}$ which are opposite parity with $p, q \geq n_1$. Suppose that $\phi(d(T^p x, T^q x)) < \varepsilon + \delta$. Without loss of generality we may assume $T^p x \in A$ and $T^q x \in B$ with $p = 2n$ and $q = 2m - 1$. Otherwise, revise the indices respectively.

Thus we have $\phi(d(T^p x, T^q x)) = \phi(d(T^{2n} x, T^{2m-1} x)) < \varepsilon + \delta$, for $m \geq n$. Then, regarding (1.6) we get

$$\begin{aligned} \phi(d(T^{2n}x, T^{2m-1}x)) &\leq \phi(\tfrac{1}{3}[d(T^{2m-1}x, T^{2n}x) + d(T^{2m}x, T^{2m-1}x) + d(T^{2n+1}x, T^{2n}x)]) \\ &\leq \tfrac{1}{3}[\varepsilon + \delta + \tfrac{\delta}{2} + \tfrac{\delta}{2}] \\ &< \varepsilon + \delta. \end{aligned} \tag{1.7}$$

Consider (1.4) under the assumption $y = T^{2n} x$, the inequality (1.7) yields that

$$\phi(d(T^{2n+1}x, T^{2m}x)) = \phi(d(T^{p+1}x, T^{q+1}x)) < \varepsilon$$

Thus, we observe that for a given $\varepsilon > 0$, there exist $n_1 \in \mathbb{N}$ and a $\delta > 0$ such that

$$\phi(d(T^p x, T^q x)) < \varepsilon + \delta \text{ implies that } \phi(d(T^{p+1}x, T^{q+1}x)) < \varepsilon \tag{1.8}$$

where p and q are opposite parity, with $p, q \geq n_1$.

□

Lemma 1.5. *Let X be a complete metric space, A and B non-empty, closed subsets of X such that $d(A, B) = 0$. Suppose $T : A \cup B \to A \cup B$ be a ϕ Type orbital Meir-Keeler cyclic contraction and $d(A, B) = 0$. Then*

$$\phi(d(T^{2n}x, Ty)) < \phi(R(T^{2n-1}x, y)) \quad \textit{if} \quad T^{2n-1}x \neq y. \tag{1.9}$$

Proof. To get (1.9), it is sufficient to show that (1.1) is equivalent to the following condition: For each $\varepsilon > 0$ there exists δ such that

$$\begin{aligned} \varepsilon \leq \phi(R(T^{2n-1}x, y)) < \varepsilon + \delta \\ \text{implies } \phi(d(T^{2n}x, Ty)) < \varepsilon, n \in \mathbb{N}, y \in A \end{aligned} \tag{1.10}$$

where $R(T^{2n-1}x, y) = \frac{1}{3}[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$ and recall that $d(A, B) = 0$.

It is clear that (1.1) implies (1.10). Now, suppose (1.10) holds. Fix $T^{2n-1}x, y \in A \cup B$ and $\varepsilon > 0$. If $\phi(R(T^{2n-1}x, y)) < \varepsilon$, since (1.10) we have $\phi(d(T^{2n}x, Ty)) \leq \phi(R(T^{2n-1}x, y))$ and consequently $\phi(d(T^{2n}x, Ty)) < \varepsilon$. If $\phi(R(T^{2n-1}x, y)) \geq \varepsilon$, then immediately (1.1) holds. Thus, (1.10) and (1.1) are equivalent under the condition $d(A, B) = 0$.

We show now if (1.10) holds then we have $\phi(d(T^{2n}x, Ty)) \leq \phi(R(T^{2n-1}x, y))$. If $\phi(R(T^{2n-1}x, y)) = 0$ then $T^{2n-1}x = y$. Thus $\phi(d(T^{2n}x, Ty)) \leq \phi(R(T^{2n-1}x, y))$. Suppose $\phi(R(T^{2n-1}x, y)) \neq 0$ and fix $\varepsilon \leq \phi(R(T^{2n-1}x, y))$. Choose a $\delta > 0$ such that (1.10) holds. Notice that if $\phi(R(T^{2n-1}x, y)) \leq \phi(d(T^{2n}x, Ty))$, we get a contradiction with (1.10). □

Theorem 1.6. *Let X be a complete metric space, A and B non-empty, closed subsets of X such that $d(A, B) = 0$. Suppose $T : A \cup B \to A \cup B$ be a ϕ Type orbital Meir-Keeler cyclic contraction. Then, there exists a fixed point, say $z \in A \cap B$, such that for each $x \in A$ satisfying (1.1), the sequence $\{T^{2n}x\}$ converges to z.*

Proof. Take $x\mathbb{N}A$. We show that $\{T^n x\}$ is a Cauchy sequence. Suppose not. Then there exists an $\varepsilon > 0$ and a subsequence in $\{T^{n(i)}\}$ of $\{T^n x\}$ with

$$\phi(d(T^{n(i)}x, T^{n(i+1)}x)) > 2\varepsilon. \tag{1.11}$$

For this ε, there exists $\delta > 0$ such that

$$\phi(R(T^{2n-1}x, y)) < \varepsilon + \delta \text{ implies that } \phi(d(T^{2n}x, Ty)) < \varepsilon \tag{1.12}$$

where $\phi(R(T^{2n-1}x, y)) = \phi(\frac{1}{3}[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)])$. Set $r = \min\{\varepsilon, \delta\}$ and $d_m = d(T^m x, T^{m+1}x)$. Due to Proposition 1, one can choose $n_0 \in \mathbb{N}$ such that

$$\phi(d^m) = \phi(d(T^m x, T^{m+1}x)) < \frac{r}{4}, \quad \text{for } m \geq n_0. \tag{1.13}$$

Let $n(i) \geq N$. Suppose $\phi(d(T^{n(i)}x, T^{n(i+1)-1}x)) \leq \varepsilon + \frac{r}{2}$. Then triangle inequality implies that

$$\begin{aligned} \phi(d(T^{n(i)}x, T^{n(i+1)}x)) &\leq \phi(d(T^{n(i)}x, T^{n(i)-1}x) + d(T^{n(i+1)-1}x, T^{n(i+1)}x)) \\ &\varepsilon + \tfrac{r}{2} + d_{n(i+1)-1} < 2\varepsilon \end{aligned} \tag{1.14}$$

which contradict the assumption (1.11). Thus, there are values of k with $n(i) \leq k \leq n(i+1)$ such that $\phi(d(T^{n(i)}, T^k x)) > \varepsilon + \frac{r}{2}$.opposite parity. Assume that $\phi(d(T^{n(i)}x, T^{n(i)+1}x)) \geq \varepsilon + \frac{r}{2}$. Then

$$\phi(d_{n(i)}) = \phi(d(T^{n(i)}x, T^{n(i)+1}x)) \geq \varepsilon + \frac{r}{2} > r + \frac{r}{2} > \frac{r}{4}$$

which is a contradiction with (1.13). Hence, there are values of k with $n(i) \leq k \leq n(i+1)$ such that $\phi(d(T^{n(i)}, T^k x)) < \varepsilon + \frac{r}{2}$ where k and $n(i)$ are opposite parity. Choose smallest integer k with $k \geq n(i)$ such that $\phi(d(T^{n(i)}x, T^k x)) \geq \varepsilon + \frac{r}{2}$. Therefore,

$$\phi(d(T^{n(i)}x, T^{k-1}x)) < \varepsilon + \frac{r}{2}. \tag{1.15}$$

Thus,

$$\phi(d(T^{n(i)}x, T^{k-1}x)) \leq \phi(d(T^{n(i)}x, T^{k-1}x) + d(T^{k-1}x, T^k x)) < \varepsilon + \frac{r}{2} + \frac{r}{4} = \varepsilon + \frac{3r}{4}. \tag{1.16}$$

Then there exists an integer k satisfying $n(i) \leq k \leq n(i+1)$ such that

$$\varepsilon + \frac{r}{2} \leq \phi(d(T^{n(i)}x, T^k x)) < \varepsilon + \frac{3r}{4}. \tag{1.17}$$

Due to the facts

$$\phi(d(T^{n(i)}x, T^k x)) < \varepsilon + \frac{3r}{4} < \varepsilon + r$$
$$\phi(d(T^{n(i)}x, T^{n(i)+1}x)) = \phi(d_{n(i)}) < \frac{r}{4} < \varepsilon + r$$
$$\phi(d(T^k, T^{k+1}x)) = \phi(d_k) < \frac{r}{4} < \varepsilon + r$$

we have

$$\begin{aligned}\phi(R(T^{n(i)}x, T^k x)) &= \phi(\tfrac{1}{3}[d(T^{n(i)}x, T^k x) + d(T^{n(i)}x, T^{n(i)+1}x) + d(T^{k+1}x, T^k x)]) \\ &\leq \tfrac{1}{3}[\varepsilon + r + \varepsilon + r + \varepsilon + r] = \varepsilon + r\end{aligned} \tag{1.18}$$

which implies $\phi(d(T^{n(i)+1}, T^{k+1}x)\phi(< \varepsilon$. But,

$$\begin{aligned}\phi(d(T^{n(i)+1}x, T^{k+1}x)) &\geq \phi(d(T^{n(i)}x, T^k x) - d(T^{n(i)}x, T^{n(i)+1}x) - d(T^k x, T^{k+1}x)) \\ &> \varepsilon + \tfrac{r}{2} - \tfrac{r}{4} - \tfrac{r}{4} = \varepsilon\end{aligned}$$

which contradicts the preceding inequality.

Hence $\{T^n x\}$ is a Cauchy sequence. Thus $\{T^n x\}$ to some $z \in A$. Consider now

$$0 \leq \phi(d(T^{2n-1}x, z)) \leq < (d(T^{2n-1}x, T^{2n}x) + d(T^{2n}x, z)) \tag{1.19}$$

which tends to zero as well. Thus

$$\lim_{n\to\infty} \phi(d(T^{2n-1}x, z)) = 0. \tag{1.20}$$

Since $\{T^{2n-1}x\}$ is a sequence in B, it converges to $z \in B$. Taking into account both A and B are closed, we get $z \in A \cap B$.

Let us show $Tz = z$.

Taking account into Lemma 1.5

$$\begin{aligned}\phi(d(Tz,z)) &= \lim_{n\to\infty} \phi(d(T^{2n}x, Tz)) < \phi(R(T^{2n-1}x, z)). \\ & \lim_{n\to\infty} \phi(\tfrac{1}{3}[d(T^{2n-1}x, z) + d(T^{2n}x, T^{2n-1}x) + d(Tz, z)])\end{aligned}$$

which implies that

$$\phi(d(Tz,z)) < \frac{1}{3}\phi(d(Tz,z)).$$

This is a contradiction and hence $Tz = z$.

Lastly, we show z is a unique fixed point of T. Suppose not, so there exists a point $w \in A \cap B$ such that $z \neq w$ and $Tw = w$. Due to Lemma 1.5

$$\begin{aligned}\phi(d(w,z)) &= \phi(d(Tw,z)) = \lim_{n\to\infty} \phi(d(T^{2n}x, Tw)) < \lim_{n\to\infty} \phi(R(T^{2n-1}x, w)) \\ &\leq \lim_{n\to\infty} \phi(\tfrac{1}{3}[d(T^{2n-1}x, w) + d(T^{2n}x, T^{2n-1}x) + d(Tw, w)]) \\ &\leq \phi(\tfrac{1}{3}[d(z,w) + d(z,z) + d(Tw,w)]) = \tfrac{1}{3}\phi(d(z,w))\end{aligned}$$

which is a contradiction. Hence, $z = w$. □

Example 1.7. *Let $X = [-2, 2]$ and $A = B = [0, 1]$. Suppose* $\phi(t) = \begin{cases} t^2 & \text{if } x \in [0,1) \\ \ln t & \text{if } x \in [1,\infty) \end{cases}$ *and* $Tx = \begin{cases} 0 & \text{if } x \in [0,\frac{1}{2}) \\ \frac{1}{8} & \text{if } x \in [\frac{1}{2},1] \end{cases}$

Since (1.1) is equivalent to (1.10), it is enough to show if T satisfies (1.10) we have the result.

Fix $x = 0 \in A$. Then $Tx = T0 = 0$ and $T^n 0 = 0$ for all $n \in \mathbb{N}$. Thus, $|T^{2n}0 - T^{2n-1}0| = 0$

Moreover, $|T^{2n-1}0 - y| = \begin{cases} 0 & \text{if } y \in [0,\frac{1}{2}) \\ \frac{1}{8} & \text{if } y \in [\frac{1}{2},1], \end{cases}$ *and* $|Ty - y| = \begin{cases} |y| & \text{if } y \in [0,\frac{1}{2}) \\ |y - \frac{1}{8}| & \text{if } y \in [\frac{1}{2},1]. \end{cases}$

Notice also that $|T^{2n}0 - Ty| = \begin{cases} 0 & \text{if } y \in [0,\frac{1}{2}) \\ \frac{1}{8} & \text{if } y \in [\frac{1}{2},1] \end{cases}$

Thus, $R(T^{2n-1}x, y) = \begin{cases} \frac{y}{3} & \text{if } y \in [0,\frac{1}{2}) \\ \frac{y}{3}[\frac{1}{8} + |y - \frac{1}{8}|] = y3 & \text{if } y \in [\frac{1}{2},1]. \end{cases}$

$$d(T^{2n}x, Ty) = \begin{cases} 0 & \text{if } y \in [0, \frac{1}{2}) \\ \frac{1}{8} & \text{if } y \in [\frac{1}{2}, 1]. \end{cases}$$

Case (i) $y \in [0, \frac{1}{2})$. *Take an arbitrary* $\varepsilon > 0$. *Then for any* $\delta > 0$, $\phi(R(T^{2n-1}x, y)) = \phi(\frac{y}{3}) = \frac{y^2}{9} < \varepsilon + \delta$ *implies that* $d(T^{2n}x, Ty) = 0 < \varepsilon$.

Case (ii) $y \in [\frac{1}{2}, 1]$. *Take an arbitrary* $\varepsilon > 0$ *which satisfies the condition (1.10). Choose* $\delta = \min\{\frac{5\varepsilon}{9}, 1 - \frac{5\varepsilon}{9}\}$

$$\varepsilon < \phi(R(T^{2n-1}x, y)) = \phi(\frac{y}{3}) = \frac{y^2}{9} < \varepsilon + \delta$$

Since $\frac{1}{2} \leq y \leq 1$ *implies that* $\frac{1}{36} \leq \frac{y^2}{9} \leq \frac{1}{9}$. *Thus one of the following case holds:* $\frac{1}{56} < \varepsilon$ *or* $\frac{1}{52} < \varepsilon$. *In any case*

$$\phi(d(T^{2n}x, Ty)) = \phi(\frac{1}{8}) = \frac{1}{64} \leq \frac{y^2}{9} - \delta < \varepsilon.$$

Example 1.8. *Let* $X = \mathbb{N}$ *and* $A = B = \{1, 2\} \cup \{5, 6, 7, \cdots\}$. *Suppose* $\phi(t) = \sqrt{3t}$ *and* $Tx = \begin{cases} 1 & \text{if } x \in \{1, 2\} \\ 2 & \text{if } x \geq 5 \end{cases}$

Since (1.1) is equivalent to (1.10), it is enough to show if T *satisfies (1.10) we have the result.*

Fix $x = 1$. *Then* $Tx = T1 = 1$ *and* $T^n 1 = 1$ *for all* $n \in \mathbb{N}$. *Thus,* $|T^{2n}1 - T^{2n-1}1| = 0$. *Moreover,*

$$|T^{2n-1}1 - y| = \begin{cases} |1 - y| & \text{if } y \in \{1, 2\} \\ |1 - y| & \text{if } y \geq 5, \end{cases} \quad \text{and } |Ty - y| = \begin{cases} |1 - y| & \text{if } y \in \{1, 2\} \\ |2 - y| & \text{if } x \geq 5 \end{cases}$$

Notice also that $|T^{2n}0 - Ty| = \begin{cases} 0 & \text{if } y \in \{1, 2\} \\ 1 & \text{if } x \geq 5 \end{cases}$

Thus, $R(T^{2n-1}x, y) = \begin{cases} \frac{2(y-1)}{3} & \text{if } y \in \{1, 2\} \\ \frac{2y-3}{3} & x \geq 5 \end{cases}$

$$d(T^{2n}x, Ty) = \begin{cases} 0 & \text{if } y \in \{1, 2\} \\ 1 & \text{if } x \geq 5 \end{cases}$$

Case (i) $y \in \{1, 2\}$. *Take an arbitrary* $\varepsilon > 0$. *Then for any* $\delta > 0$, $\phi(R(T^{2n-1}x, y)) = \phi(\frac{2(y-1)}{3}) = \sqrt{2y - 2} < \varepsilon + \delta$ *implies that* $\phi(d(T^{2n}x, Ty)) = \phi(0) = \sqrt{0} = 0 < \varepsilon$.

Case (ii) $y \geq 5$. *Take an arbitrary* $\varepsilon > 0$ *which satisfies the condition (1.10).*

Choose $\delta = \varepsilon, \frac{\varepsilon}{2}$

$$\phi(R(T^{2n-1}x, y)) = \phi(\frac{2y-3}{3}) = \sqrt{2y-3} < \varepsilon + \delta$$

Since $y \geq 5$ *then* $7 \leq 2y - 3 < \infty$ *and thus* $\varepsilon \geq 7$.

$$\phi(d(T^{2n}x, Ty)) = \phi(1) = \sqrt{1} = 1 < \sqrt{2y-3} - \delta < \varepsilon.$$

Indeed, for any $\varepsilon > 0$ *which satisfies the condition (1.10), Theorem is satisfied for any* $\delta > 0$.

Proposition 1.9. *Let* X *be a metric space and* $A \subseteq X$ *and* $B \subseteq X$. *Choose* $T : A \cup B \to A \cup B$ *as a cyclic contraction map. Then starting with any* $x_0 \in A \cup B$ *we have* $d(x_n, Tx_n) \to dist(A, B)$ *where* $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$.

Proof. Now we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq kd(x_{n-1}, x_n) + (1-k)dist(A, B) \\ &\leq k(d(x_{n-1}, x_{n-2}) + (1-k)dist(A, B)) + (1-k)dist(A, B) \\ &= k^2 d(x_{n-1}, x_{n-2}) + (1-k^2)dist(A, B). \end{aligned}$$

By induction on n, we have

$$d(x_n, x_{n+1}) \leq k^n d(x_1, x_0) + (1-k^n)dist(A, B).$$

Thus $d(x_n, x_{n+1}) \to dist(A, B)$. □

Now we are ready to give the existence theorem for a best proximity point.

Proposition 1.10. *Let* X *be a metric space and* $A \subseteq X$ *and* $B \subseteq X$ *and* $T : A \cup B \to A \cup B$ *is a cyclic contraction map, let* $x_0 \in A$ *and define* $x_{n+1} = Tx_n$. *Suppose* x_{2n} *has a convergent subsequence in* A. *Then* $\exists$ *an element* x *in* A *s.t.* $d(x, Tx) = dist(A, B)$.

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ convergent to x in A. By definition of distance between two sets and triangle inequality, we have

$$dist(A, B) \leq d(x, x_{2n_k-1}) \leq d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}).$$

Thus $d(x, x_{2n_k-1})$ converges to $dist(A, B)$. Since

$$dist(A, B) \leq d(x_{2n_k}, Tx) \leq d(x_{2n_k-1}, x),$$

$d(x, Tx) = dist(A, B)$.

□

Theorem 1.11. *Let X be a metric space and $A \subseteq X$ and $B \subseteq X$ be closed. Suppose $T : A \cup B \to A \cup B$ be a cyclic contraction map. If one of the A or B is boundedly compact then $\exists$ an element $x \in A \cup B$ with $d(x, Tx) = dist(A, B)$.*

Corollary 1.12. *Let A and B be closed subsets of a normed space X. Suppose $T : A \cup B \to A \cup B$ be a cyclic contraction map. If the space generated by either A or B is a finite dimensional subspace of X, then there exists an element $x \in A \cup B$ with $d(x, Tx) = dist(A, B)$.*

Example 1.13. *Given k in $(0, 1)$, let $A, B \subseteq \ell^p$ with $1 \leq p \leq \infty$, defined by*

$$A = \{((1 + k^{2n})e_{2n}) : n \in \mathbb{N}\} \quad \textit{and} \quad B = \{((1 + k^{2m-1})e_{2m-1})m \in \mathbb{N}\}.$$

Suppose

$$T((1 + k^{2n})e_{2n}) = (1 + k^{2n+1})e_{2n+1} \quad \textit{and} \quad T((1 + k^{2m-1})e_{2m-1}) = (1 + k^{2m})e_{2m}.$$

Then T is a cyclic contraction on $A \cup B$.

Note that the sets A is no best proximity point.
Now we can give the existence,uniqueness and convergence result of this section.

Lemma 1.14. *Given $A \subseteq X$ a closed and convex subset and $B \subseteq X$ a closed subset and X be a uniformly convex Banach space . Let $\{x_n\} \in A$, $\{z_n\} \in A$ and $\{y_n\} \in B$ satisfies the conditions.*

(1) $\|z_n - y_n\| \to dist(A, B)$

(2) $\forall \epsilon > 0$ $\exists$ *an integer* $N_0 > 0$ *s.t.* $\forall m > n \geq N_0$, $\|x_m - y_n\| \leq dist(A, B) + \epsilon$.

Then, $\forall \epsilon > 0$ $\exists$ a positive integer N_1 such that for all $m > n \geq N_1$, $\|x_m - z_n\| \leq \epsilon$.

Similarly, we can give the following lemma.

Lemma 1.15. *Given $A \subseteq X$ a closed and convex subset , $B \subseteq X$ a closed subset and X be a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and y_n be a sequence in B satisfying the following conditions.*

(1) $\|x_n - y_n\| \to dist(A, B)$.

(2) $\|z_n - y_n\| \to dist(A, B)$.

Then $\|x_n - z_n\| \to 0$.

Corollary 1.16. *Given* $A \subseteq X$ *a closed and convex subset ,* $B \subseteq X$ *a closed subset and* X *be a uniformly convex Banach space. Let* $\{x_n\}$ *be a sequence in* A *and* $y_0 \in B$ *such that* $\|x_n - y_0\| \to dist(A, B)$*. Then* x_n *converges to* $P_A(y_0)$ *where* P_A *is the closest point of* A *to* y_0*.*

Proof. Since $dist(A, B) \leq \|y_0 - P_A(y_0)\| \leq \|y_0 - x_n\|$, we have $\|y_0 - P_A(y_0)\| = dist(A, B)$. Now put $y_n = y_0$ and $z_n = P_A(y_0)$ in lemma 1.15. □

Theorem 1.17. *Given* $A \subseteq X$ *a closed and convex subset ,* $B \subseteq X$ *a closed subset and* X *be a uniformly convex Banach space.Suppose* $T : A \cup B \to A \cup B$ *is a cyclic contraction map, then* $\exists$ *a unique best proximity point* x *in* A*.*

If we delete the convexity condition from the assumption of Theorem 1.17, then both the convergence and uniqueness are not guaranteed even in a finite dimensional space.

CHAPTER 2

CYCLIC CONTRACTION MAPPINGS AND FIXED POINT THEOREMS ON REFLEXIVE BANACH SPACES.

In this chapter we give some results on Cyclic Contractions Mappings and fixed point theorems on Reflexive Banach spaces. The fundamental reference is [11].

Let (X, d) be a complete metric space. If a function $T : X \to X$ is continuous, satisfying inequality

$$d(T(x), T^2(x)) \leq kd(x, T(x)) \quad \forall x \in X, \quad \text{where}\, k \in (0, 1),$$

then T has a fixed point in X. From the condition on T we may conclude that the sequence $\{T^n(x)\}$ is Cauchy $\forall x \in X$.

Theorem 2.1. *Let (X, d) complete metric space, $A \subseteq X$ and $B \subseteq X$ be two closed subsets. Suppose $T : X \to X$ satisfies* (1) *and* (2) *above. Then T has a unique fixed point in $A \cap B$.*

Definition 2.2. *Let (X, d) complete metric space, $A \subseteq X$ and $B \subseteq X$ be two closed subsets. Let $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. Then $T : A \cup B \to A \cup B$ is called a* **cyclic ϕ-contraction map** *if it satisfies the following conditions.*

(1) $T(A) \subseteq B$ and $T(B) \subseteq A$.

(2) $d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) + \phi(d(A, B))$,

for all $x \in A, y \in B$.

Also $x \in A \cup B$ is called a **best proximity point** *if $d(x, Tx) = d(A, B)$. As a special case, when $\phi(t) = (1 - \alpha)t$,in which $\alpha \in (0, 1)$ is a constant, T is called cyclic contraction.*

As an extension of Proposition **??** for cyclic ϕ-contraction map we can give the following result.

Theorem 2.3. *Given $\phi : [0, \infty) \to [0, \infty)$ as a strictly increasing map. Let $A, B \subseteq X$ subsets of a metric space (X, d), $T : A \cup B \to A \cup B$ a cyclic ϕ-contraction map, $x_0 \mathbb{N} A \cup B$, and $x_{n+1} = Tx_n$, for all $n \geq 0$. Then, $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.*

Now by using this important result, we provide the main result which give positive answer to the question. Their proofs are basically due to Al-Thagafi and Shahzad in [6].

Now we need the following definitions about weak topology.

2.1 WEAK TOPOLOGY

The fundamental reference is [16] for this section.

One may call subsets of a topological vector space **weakly closed** if they are closed with respect to the weak topology.

Let $\mathbb{K}$ be either the field of complex numbers or the field of real numbers with the usual topologies. Let X be a topological vector space over $\mathbb{K}$. We may define a possibly different topology on X using the continuous dual space $X^\star$.

Definition 2.4. *Let V be a vector space over $\mathbb{K}$.*

A subset C in V is said to be **convex** *if $\forall$ x and $y \in C, tx + (1-t)y \in C$ for every t in the unit interval.*

A subset C in V is said to be **circled** *if $\forall x \in C, \lambda x$ is in C if $|\lambda| = 1$.*

A subset C in V is called **cone** *if $\forall x \in C$ and $0 \leq \lambda \leq 1$, λx is in C.*

A subset C in V is called **balanced** *if for all $x \in C, \lambda x$ is in C if $|\lambda| \leq 1$.*

From definition, a balanced set is a circled cone.

A subset C in V is called **absolutely convex** *if it is both balanced and convex.*

We need the following concept which was known as the Hahn-Banach theorem.

Definition 2.5. *Suppose that X is a normed space and X^{**} is the second dual space of X. The canonical map $x \to \hat{x}$ defined by $\hat{x}(f) = f(x), \forall x \in X, f \in X^*$ gives an isometric linear isomorphism from $X \to X^{**}$. The space X is called* **reflexive** *if this map is surjective.*

We know that, all finite-dimensional normed spaces and Hilbert spaces are reflexive.

Theorem 2.6. *Suppose X be a reflexive Banach space, $\phi : [0,\infty) \to [0,\infty)$ is strictly increasing and $A \subseteq X$, $B \subseteq X$ be weakly closed subsets of X and $T : A \cup B \to A \cup B$ a cyclic ϕ-contraction map.Then $\exists$ a pair $(x, y) \in A \times B$ s.t. $\|x - y\| = d(A, B)$.*

Definition 2.7. *Let $A, B \subseteq X$ be on a normed space X, $T : A \cup B \to A \cup B$, $T(A) \subseteq B$, and $T(B) \subseteq A$. We say that T satisfies the* **proximal property** *if*

$$x_n \overset{w}{\to} x \in A \cup B, \quad \|x_n - Tx_n\| \to d(A, B) \Longrightarrow \|x - Tx\| = d(A, B).$$

Theorem 2.8. *Let $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing map. Let $A, B \subseteq X$ be subsets of a reflexive Banach space X s.t. A is weakly closed and $T : A \cup B \to A \cup B$ a cyclic ϕ-contraction map. Then,$\exists$ $x \in A$ s.t. $x \in A$ such that $\|x - Tx\| = d(A, B)$ provided that one of the following conditions is satisfied*

(1) T is weakly continuous on A.

(2) T satisfies the proximal property.

Proof. If $d(A, B) = 0$, the result follows from Theorem 1 in [6]. So we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for all $n \geq 1$. By Theorem 2.3, the sequence $\{x_{2n}\}$ is bounded. Since X is reflexive and A is weakly closed, the sequence $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ s.t. $x_{2n_k} \to x \in A$ as $k \to \infty$.

(1) Since T is weakly continuous on A and $T(A) \subseteq B$, we have $x_{2n_k+1} \to Tx \in B$ as $k \to \infty$. Thus $x_{2n_k} - x_{2n_k+1} \to x - Tx \neq 0$ as $k \to \infty$. The rest of the proof is similar to the proof of the last theorem.

(2) By Theorem 3 in [6], we have

$$\|x_{2n_k} - Tx_{2n_k}\| = \|x_{2n_k} - x_{2n_k+1}\| \to d(A, B)$$

as $k \to \infty$.

Since T satisfies the proximal property, we have $\|x - Tx\| = d(A, B)$. $\square$

For the next theorem we need definitions of strictly convex set and strictly convex spaces.

Definition 2.9. *A set S in a vector space V is called* **strictly convex** *if $\alpha x + (1-\alpha)y \in Interior(S)$ for every $x, y \in S$ with $x \neq y$ and $\alpha \in (0,1)$.*

A **strictly convex space** *is a normed topological vector space $(V, \|\|)$ whose the unit ball is a strictly convex set.*

Theorem 2.10. *Let X be reflexive and strictly convex Banach space and $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing map. Suppose $A, B \subseteq X$ be closed and convex on X. and $T : A \cup B \to A \cup B$ a cyclic $\phi-$ contraction map. If $(A - A) \cap (B - B) = \{0\}$, then $\exists$ a unique $x \in A$ s.t. $T^2 x = x$ and $\|x - Tx\| = d(A, B)$.*

Theorem 2.11. *Let X be a reflexive and strictly convex Banach space , and $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing map. Suppose $A, B \subseteq X$ are closed and convex on X. If $T : A \cup B \to A \cup B$ a cyclic $\phi-$contraction map. Then $\exists$ a unique $x \in A$ s.t. $T^2 x = x$ and $\|x - Tx\| = d(A, B)$ provided that one of the following conditions is satisfied*

(1) T is weakly continuous on A.

(2) T satisfies the proximal property.

Proof. If $d(A, B) = 0$, the result follows from Theorem 1 in [6]. So we assume that $d(A, B) > 0$. Since A is closed and convex, it is weakly closed. By Theorem 2.8, there exists an element $x \in A$ with $\|x - Tx\| = d(A, B)$. Thus, $T^2 x = x$. Indeed, if we assume that $T^2 x - Tx \neq x - Tx$, then from the convexity of A and the strict convexity of X, we have

$$\|\frac{T^2 x + x}{2} - Tx\| = \|\frac{T^2 x - Tx}{2} + \frac{x - Tx}{2}\| < d(A, B),$$

which is a contradiction. The uniqueness of x follows as in the proof of Theorem 8 in [6]. $\square$

CHAPTER 3

ON EXISTENCE OF BEST PROXIMITY POINTS OF CYCLIC CONTRACTIONS IN ORDERED METRIC SPACES

In this chapter we will give some important results on existence of the best proximity point of cyclic ϕ-contractions in ordered metric spaces. The fundamental reference is [14].

3.1 PARTIALLY ORDERED METRIC SPACES

In this section we will introduce partially ordered metric spaces. The fundamental reference for this section is [19].

In a partially ordered space it is possible to introduce a notion of convergence which is similar to that defined for real numbers. This convergence is called **order convergence**. On the other hand every metric space can be embedded in a partially ordered space so that convergence in the metric space can be obtained from order convergence. Also a standard fixed point theorem in complete metric spaces can be obtained as a special case of a fixed point theorem in partially ordered spaces.

Definition 3.1. *A partially ordered space is an abstract set X with a binary relation $\leqq$, called a* **partial order***, which satisfies three conditions:*
1)$x \leqq x$ *for all* $x \in X$;
2)$x \leqq y$ *and* $y \leqq z \implies x \leqq z \ \forall \ x, y, z \in X$;
3)$x \leqq y$ *and* $y \leqq x \implies x = y \ \forall \ x, y \in X$;

If M is a nonempty subset of X , then $\inf M$ denotes the infumum or greatest lower bound of M and $\sup M$ denotes the supremum or least upper bound of M. Thus, $\inf M$ is defined to be that element $v \in X$ such that $v \leqq x$ for all $x \in M$ and if $w \leqq x$ for all $x \in M$, then $w \leqq v$. Similarly, $\sup M$ is defined to be that element $u \in X$ such that $x \leqq u$ for all $x \in M$ and if $x \leqq w$ for all $x \in M$, then $u \leqq w$. We know that $\sup M$ and $\inf M$ are unique if they exist.

A sequence $\{x_n\}$ of elements from X is said to be convergence to $x \in X$ if there exist two sequences $\{y_n\}$ and $\{z_n\}$ such that $y_1 \leqq y_2 \leqq ... \leqq x \leqq ... \leqq z_2 \leqq z_1$, $y_n \leqq x_n \leqq z_n$, and $\sup\{y_n\} = x = \inf\{z_n\}$. Convergence defined in this way is called **order convergence** and it is denoted by o-convergence. If the sequence $\{x_n\}$ o-converges to x, we will write $o - \lim x_n = x$. For example if we take X as the set of real numbers with the usual order $\leqq$, then $o - convergence$ is the same as the usual convergence for real numbers.

Theorem 3.2. *Let $\{p_n\}$ be a sequence in S and let $\{x_n\}$, where $x_n = (p_n, 0)$, be the corresponding sequence in X. Then $\lim p_n = p \in S$ iff $o - \lim x_n = x = (p, 0)$.*

We will now show that a fixed point theorem in complete metric spaces can be obtained as a special case of a fixed point theorem in partially ordered spaces, but first we need some preliminaries. If X is a partially ordered space, then a chain L is a nonempty subset of X such that if $x, y \in L$, then $x \leqq y$ or $y \leqq x$.

Lemma 3.3. *Let S be a complete metric space and let X be the partially ordered space constructed from S. If L is a chain in X and is bounded above, then $\sup L$ exists.*

Definition 3.4. *If X is a partially ordered space, then a mapping $F : X \to X$ is said to be* **isotone** *if $F(x) \leqq F(y)$ whenever $x \leqq y$.*

Now let S be any metric space and let X be the partially ordered space constructed from S. If $f : S \to S$ is a map such that $d(f(p), f(q)) \leqq \alpha d(p, q)$ for all $p, q \in S$, then f determines an isotone mapping $F : X \to X$ as follows: if $x = (p, \lambda) \in X$, then $F(x) = (f(p), \alpha\lambda)$. We note that if the mapping F has a fixed point, then so does f. In case S is a complete metric space and $\alpha < 1$, then it is well known that f has a fixed point.

Theorem 3.5. *Let S be a complete metric space and let X be the partially ordered space constructed from S. Let $F : X \to X$ be isotone. If there exist $x_0, x_1 \in X$ such that $x_0 \leqq F(x_0) \leqq F(x_1) \leqq x_1$, then F has a fixed point.*

3.2 EXISTENCE OF BEST PROXIMITY POINTS OF CYCLIC CONTRACTIONS.

In this section we will give some results about best proximity points of cyclic ϕ-contractions in ordered metric spaces. Main reference of this section is [15].

Let X be a nonempty set and T be a self map on X. We denote the set of all nonempty subsets of X by 2^X and the set of all invariant subsets of X by $I(T)$, that is $I(T) = \{Y \in 2^X : T(Y) \subseteq Y\}$. For each pair of sets X and Y and self maps $T : X \to X$ and $S : Y \to Y$, we define the self map $T \times S : X \times Y \to X \times Y$ by $T \times S(x, y) = (Tx, Sy) = (Tx, Sy)$. If $(X, \leq)$ is a partially ordered set, then we define

$$X_{\leq} = \{(x, y) \in X \times X : x \leq y \quad \text{or} \quad y \leq x\}.$$

Let $(X, d, \leq)$ be an ordered metric space and $T : X \to X$ a self map on X. For each nonempty subset C of X and $x* \in X$, we define

$$E_T, C(x^\star) = \{x \in C : \lim_{n\to\infty} T^{2n}x = x^\star\}.$$

We say that X has a property (C) whenever for each monotone sequence $\{x_n\}$ in X with $x_n \to x$ for some $x \in X$, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every element of a self map $T : X \to X$ is called **"orbitally continuous"** whenever for each $x \in \{n(i)\}_{i\geq 1}$ with $T^{n(i)}x \to a$ for some $a \in X$,we have $T^{n(i)+1} \to Ta$. Here, $T^{m+1} = T(T^m)$.

Theorem 3.6. *Let $(X, d, \leq)$ be an ordered metric space, $A, B \in 2^X$ and T a decreasing self map on $A \cup B$ on $A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. Suppose that $\exists$ $x_0 \in A$ such that $x_0 \leq T^2x_0 \leq Tx_0$ and*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) + \phi(d(A, B))$$

for all $x \in A$ and $y \in B$ with $x \leq y$, where $\phi : [0, \infty) \to [0, \infty)$ is strictly increasing. If $x_{n+1} = Tx_n$ and $d_n = d(x_{n+1}, x_n)$ for all $n \geq 0$, then $d_n \to d(A, B)$.

Proof. First note that we have

$$x_0 \leq x_2 \leq \ldots \leq x_{2n} \leq x_{2n+1} \leq \ldots \leq x_3 \leq x_1$$

for all $n \geq 1$. Thus we get

$$0 \leq d_{n+1} \leq d_n - \phi(d_n) + \phi(d(A, B))$$

for all $n \geq 1$.So, the sequence $\{d_n\}$ is decreasing and bounded from below. If $d_{n_0} = 0$ for some n_0, then $d_n \to d(A, B) = 0$. Suppose that $d_n > 0$ $\forall n \geq 1$ and $d_n \to t_0$ for some $t_0 \geq d(A, B)$. Since

$$\phi(d(A,B)) \leq \phi(d_n - d_{n+1} + \phi(d(A,B))$$

, we have $\phi(d_n) \to \phi(d(A, B))$. This implies that $\phi(t_0) = \phi(d(A, B))$. So $t_0 = d(A, B)$ because ϕ is strictly increasing. □

Theorem 3.7. *Let $(X, d, \leq)$ be an ordered metric space, $B \in 2^X, A$ a closed subset of X and T be a decreasing self map on $A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. Suppose $\exists$ $x_0 \in A$ such that $x_0 \leq T^2 x_0 \leq T x_0$ and*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) + \phi(d(A, B))$$

$\forall$ $x \in A$ and $y \in A \cup B$ with $x \leq y$.If T is orbitally continuous or X has a property (C), then $\exists$ an element $x \in A$ such that $d(x, Tx) = d(A, B)$.

The following is another example for a cyclic ϕ-contraction. Note that we should improve Example 3 in [6] because T is not a cyclic ϕ-contraction in this example. For seeing this, it is sufficient that we put $x = \frac{-1}{2}$ and $y = \frac{1}{2}$. Then

$$\frac{2}{3} = d(Tx, Ty) > d(x, y) - \phi(d(x, y)) + \phi(d(A, B)) = \frac{1}{2}.$$

Now for improving, it is sufficient to replace the function ϕ by $\phi(t) = \frac{t^2}{2(1+t)}$.

Example 3.8. *Consider the Euclidian ordered metric space $X = \mathbb{R}$ with the usual norm. Suppose that $A = [-1, 0]$, $B = [0, 1]$ and $T : A \cup B \to A \cup B$ is defined by $Tx = \frac{-x}{3}$ $\forall x \in A \cup B$. If $\phi : [0, \infty) \to [0, \infty)$ is defined by $\phi(t) = \frac{t}{2}$, then ϕ is strictly increasing and T is a cyclic ϕ-contraction map.*

Theorem 3.9. *Let $(X, d, \leq)$ be an ordered metric space, $A, B \in 2^X$ and $T : A \cup B \to A \cup B$ be a map such that $T(A) \subseteq B$, $T(B) \subseteq A$ and $((A \times B) \cup (B \times A)) \cap X_{\leq} \in I(T \times T)$. Suppose that $\exists x_0 \in A$ such that $x_0, Tx_0 \in X_{\leq}$ and*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) + \phi(d(A, B))$$

for all $x \in A$ and $y \in B$ with $(x, y) \in \mathbb{N}X_{\leq}$, where $\phi : [0, \mathbb{N}fty) \to [0, \mathbb{N}fty)$ is a strictly increasing map. If $x_{n+1} = Tx_n$ and $d_n = d(x_{n+1}, x_n)$ for all $n \geq 0$, then

$$d_n \to d(A, B).$$

Proof. First note the inequality

$$d(T^{2n+1}x_0, T^{2n}x_0) \leq d(T^{2n-1}x_0) - \phi(d(T^{2n}x_0, T^{2n-1}x_0)) + \phi(d(A, B))$$

for all $n \geq 1$. Thus, we obtain

$$0 \leq d_{n+1} \leq d_n - \phi(d_n) + \phi(d(A, B))$$

for all $n \geq 1$. Hence, the sequence $\{d_n\}$ is decreasing and bounded from below. If $d_{n_0} = 0$ for some n_0, then $d_n \to d(A, B) = 0$. Suppose that $d_n > 0$ for all $n \geq 1$ and $d_n \to t_0$ for some $t_0 \geq d(A, B)$. Since

$$\phi(d(A, B)) \leq \phi(d_n) \leq d_n - d_{n+1} + \phi(d(A, B)),$$

we have $\phi(d_n) \to \phi(d(A, B))$. From this we obtain $\phi(t_0) = \phi(d(A, B))$. So $t_0 = d(A, B)$ because ϕ is strictly increasing. □

Theorem 3.10. *Let $(X, d, \leq)$ be an ordered metric space, $A, B\mathbb{N}2^X$ and T a self map on $A \cup B$ such that $T(A) = B, T(B) \subseteq A$ and $((A \times B) \cup (B \times A)) \cap X_{\leq}\mathbb{N}I(T \times T)$. Assume that for each $x, y\mathbb{N}A$ $\exists$ $z\mathbb{N}A$ such that $(x, z), (y, z)\mathbb{N}X_{\leq}$. Suppose that there exist $x_0, x^{\star}\mathbb{N}A$ s.t. $x_0\mathbb{N}E_{T,A}(x_0, Tx_0)\mathbb{N}X_{\leq}$ and*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) + \phi(d(A, B))$$

for all $x\mathbb{N}A$ and $y\mathbb{N}B$ with $(x, y)\mathbb{N}X_{\leq}$, where $\phi : [0, \mathbb{N}fty) \to [0, \mathbb{N}fty)$ is a strictly increasing map. Suppose that $y\mathbb{N}A, (x, y)\mathbb{N}X_{\leq}$ and $x\mathbb{N}E_{T,A}(x^{\star})$ imply that $y\mathbb{N}E_{T,A}(x^{\star})$. Then, $E_{T,A}(x^{\star}) = A$ and the following statement holds.

$$E_{T,B}(Tx^{\star}) = B \quad and d(x^{\star}, Tx^{\star}) = d(A, B) \Leftrightarrow T$$

is orbitally continuous.

The following example shows that the assumption

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) + \phi(d(A, B))$$

for all $x\mathbb{N}A$ and $y\mathbb{N}B$ with $(x, y)\mathbb{N}X_{\leq}$, does not imply the following assumption:

$$y\mathbb{N}A, (x, y)\mathbb{N}X_{\leq}, x\mathbb{N}E_{T,A}(x^{\star}) \Rightarrow y\mathbb{N}E_{T,A}(x^{\star}).$$

Example 3.11. *Consider the subsets* $A = \{x_1 = (6,3), x_2 = (1,3)\}$ *and* $B = \{y_1 = (2,0), y_2 = (0,4)\}$ *of* $\mathbb{R}^2$ *via the following order:*

$$(a,b) \leq (c,d) \Leftrightarrow a \leq c \quad \textit{and} \quad b \leq d.$$

Define $T : A \cup B \to A \cup B$ *by* $Tx_1 = y_2, Tx_2 = y_1, Ty_1 = x_2, Ty_2 = x_1$. *Note that* $x_2 \leq x_1$ *and* $y_1 \leq x_1$, *and other elements are not comparable. We have* $d(Tx_1, Tx_2) = d(x_2, y_2) = d(A,B) = \sqrt{2}$ *and* $d(x_1, y_1) = \sqrt{25}$. *Consider the map* $\phi : [0, \mathbb{N}fty) \to [0, \mathbb{N}fty)$ *by* $\phi(x) = \frac{x}{2}$. *Then we have*

$$d(Tx_1, Ty_1) \leq d(x_1, y_1) - \phi(d(x_1, y_1)) + \phi(d(A,B)),$$

while $T^{2n}x_1 \to x_1$ *and* $T^{2n}x_2 \to x_2$.

CHAPTER 4

RECENT DEVELOPMENTS ON BEST PROXIMITY THEOREMS; KT- TYPE CYCLIC ORBITAL CONTRACTION MAPPINGS

If there is no exact solution to the fixed point equation $Tx = x$ for a non-self mapping $T : A \to B$, then it is desirable to find an approximate solution x such that $d(x, Tx)$ is minimum.

Let A and B be non-empty subsets of a metric space X. A map $T : A \cup B \to A \cup B$ is called **cyclic map** if $T(A) \subseteq B$ and $T(B) \subseteq A$. From [2] we have the following result.

Theorem 4.1. *Let X be a complete metric space. Let $A \subseteq X$ and $B \subseteq X$ be closed and T be a cyclic map. Suppose that $\exists\ 0 < \alpha < 1$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y), x \in A, y \in B.$$

Then T has a unique fixed point in $A \cap B$.

In [3] Eldred et al. modified this condition for the case $A \cap B = \emptyset$ as follows:

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) dist(A, B), \forall x \in A, y \in B.$$

A self mapping T on $A \cup B$ is said to be a **cyclic contraction** if
a) T is cyclic and
b) it satisfies the following condition

$$\exists 0 < \alpha < 1; d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) dist(A, B), x \in A, y \in B.$$

Note that this condition does not entail that $A \cap B \neq \emptyset$. Therefore it makes no sense to ask for a fixed point of T. However we may ask a **best proximity point**, that is, a point $x \in A \cup B$ such that $d(x, Tx) = dist(A, B)$. The notion of cyclic mapping and best proximity points are studied by many authors, such as, *S. Karpagam and Sushama Agrawal [1], W.A. Kirk and P.S. Srinavasan and P.*

Veeramani [2], A.A. Eldered and P. Veeramani [3, 4], G.Petruşhe [5], M.A.Al-Thafai and N.Shahzad [6].

In 2011, M. Gabeleh and A. Abkar [10], give a generalization of the cyclic contraction, which is called **semi-cyclic contraction pair**.

Definition 4.2. *Let (X,d) be a complete metric space, $A \subseteq X$ and $B \subseteq X$ be closed . Suppose that $S, T : A \cup B \to A \cup B$ be maps satisfying the conditions;*
a) $S(A) \subseteq B, T(B) \subseteq A$,
b) $\exists \alpha \in (0,1)$ s.t.

$$d(Sx, Ty) \leq \alpha d(x,y) + (1-\alpha) dist(A,B), x \in A, y \in B$$

Then (S,T) is called a semi-cyclic contraction pair.

Note that in the case $S = T$,a semi-cyclic contraction pair reduces to a cyclic contraction.

They give the following theorem under semi-cyclic contraction condition:

Theorem 4.3. *Let X be a uniformly convex Banach space and $A \subseteq X$ and $B \subseteq X$ be closed and convex. Suppose that (S,T) is a semi-cyclic contraction pair.*
a) If $dist(A,B) = 0$, then S, T have a unique common fixed point in $A \cap B$
b) If $dist(A,B) > 0$, then each mapping has a unique best proximity point.
Moreover either of fixed point or best proximity points can be approximated by some iterative sequences.

Note that if the space X is not uniformly convex, then the uniqueness of best proximity point may fail.

Example 4.4. *Let $X = \mathbb{R}$ and for all $(x,y) \in \mathbb{R}^2$ define $||(x,y)|| = \max\{|x|, |y|\}$.*
Let $A = \{(x,y) \in \mathbb{R}^2 : 1/2 \leq x \leq 1, y = 0\}$
$B = \{(x,y) \in \mathbb{R}^2 : x = 0, 1 \leq y \leq 2\}$.
Clearly A and B are closed and $dist(A,B) = 1$. Define $S, T : A \cup B \to A \cup B$ by

$$S(x,y) = \begin{cases} (0,1), & \text{if } (x,y) \in A \\ (x,y), & \text{if } (x,y) \in B \end{cases}$$

$$T(x,y) = \begin{cases} (\frac{y}{2},0), & \text{if } (x,y) \in B \\ (x,y), & \text{if } (x,y) \in A \end{cases}$$

Obviously $S(A) \subseteq B, T(B) \subseteq A$. Note also that neither S nor T is cyclic. On the *other hand, if* $b = (0, y) \in B$ *and* $a = (x, 0) \in A$ *then* $||Tb - Sa|| = ||T(0, y) - S(x, 0)|| = ||(\frac{y}{2}, 1)|| = 1$. *Similarly* $||a - b|| = \max\{x, y\} = y$. *Therefore* $||T(b) - S(a)|| = 1 \leq (\frac{1}{2})|y| + \frac{1}{2} = (\frac{1}{2})||b - a|| + (\frac{1}{2})dist(A, B)$.

Recently, Al-Tagafi and Shahzad in [6] introduced the notion of **cyclic ϕ -contractions** and S.N. Mishra et al in extend this to the (ϕ, ψ)**- weakly contractions** and obtained some existence results for this new class of mappings.

Let ϕ be the family of functions $\phi : [0, \infty) \to [0, \infty)$ such that :
a) ϕ is continuous and nondecreasing,
b) $\phi(t) = 0$ iff $t = 0$
This function is called an **altering distance function**.

Definition 4.5. *Let* X *be a metric space,* $A, B \subseteq X$ *and* T *a cyclic mapping.* T *is called a cyclic* (ϕ, ψ)*- weakly contraction if*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) + \phi(d(A, B))$$

for all $x \in A, y \in B$.

Remark 2. *a) A cyclic contraction is cyclic* (ϕ, ψ)*- weakly contraction with* $\psi(t) = t, \phi(t) = (1 - \lambda)t$ *for* $t \geq 0, \lambda \in (0, 1)$.
b) A cyclic ϕ*- contraction is a cyclic* (ϕ, ψ)*- weakly contraction with* $\psi(t) = t$ *for* $t \geq 0$.

Theorem 4.6. *Let* X *be a uniformly convex Banach space and* $A \subseteq X$ *and* $B \subseteq X$ *with* A *is closed. Let* T *be a cyclic* (ϕ, ψ)*- weakly contraction mapping. For* $x_0 \in A$*, define* $x_{n+1} = Tx_n$*. If* $d(A, B) = 0$*, then* T *has a unique fixed point* $z \in A \cap B$.

Here if we let $\psi(t) = t$ we get theorem 6 of M. A. Al-Thagafi and N. Shahzad (2009) in [6].

Very recently Karpagam and Agrawal [1] introduced the definition.

Definition 4.7. *Let* (X, d) *be a metric space,* $A \subseteq X$ *and* $B \subseteq X$ *. And* $T : A \cup B \to A \cup B$ *be cyclic. If for some* $x \in A$ *there exists a* $k_x \in (0, 1)$ *such that*

$$d(T^{2n}x, Ty) \leq k_x d(T^{2n-1}x, y), \quad n \in \mathbb{N}, \quad y \in A \tag{4.1}$$

then T *is called a* ***cyclic orbital contraction****.*

In 2012 , E.Karapinar, G. Petrushel and K.Tas generalized the above definition in the following ways.

Definition 4.8. *, Let (X, d) be a metric space, $A \subseteq X$ and $B \subseteq X$. If for some $x \in A$ there exists a $k_x \in (0, \frac{1}{2})$ such that*

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, x) + d(Ty, y)]; n \in \mathbb{N}; y \in A \tag{4.2}$$

then the cyclic map $T : A \cup B \to A \cup B$ is called a ***Kannan type cyclic orbital contraction***

Now let us look at the theorem given by Karpagam and Agrawal in [1] for cyclic orbital contractions.

Theorem 4.9. *Let (X, d) be a metric space, $A \subseteq X$ and $B \subseteq X$. Suppose that $T : A \cup B \to A \cup B$ is a cyclic orbital contraction. Then $A \cap B$ is non empty and T has a unique fixed point.*

E.Karapinar, G. Petrushel and K.Tas give a generalization of this Theorem and Kirk's theorem for Kannan type cyclic orbital contractions.

Theorem 4.10. *Let (X, d) be a metric space, $A \subseteq X$ and $B \subseteq X$ and $T : A \cup B \to A \cup B$ be a Kannan type cyclic orbital contraction. Then $A \cap B$ is non-empty and T has a unique fixed point.*

Corollary 4.11. *Let T be a self map on a complete metric space (X, d). If for some $x \in X$, there exists a $k_x \in (0, \frac{1}{2})$ such that*

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, x) + d(Ty, y)], n \in \mathbb{N}; y \in A \tag{4.3}$$

then, T has a unique fixed point.

Example 4.12. *Let $A = [-1, 0]$ and $B = [0, 1]$ on $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$. Define*

$$T(x) = \begin{cases} -x & \text{if } x \in A \\ -\frac{x}{2} & \text{if } x \in B \end{cases}$$

Then $T(A) \subseteq B$ and $T(B) \subseteq A$. On the other hand, $T^{2n}x = \frac{x}{2^n}$ and $T^{2n-1}x = -\frac{x}{2^n}$, for every $x \in A$. Therefore, for every $y \in [-1, 0], Ty = -y$. Thus, $d(T^{2n}x, Ty) = |\frac{x}{2^n} + y|$ and $d(T^{2n-1}x, y) = | - \frac{x}{2^n} - y| = d(T^{2n}x, Ty)$. There *is no a* $k_x \in (0,1)$. *Thus, Theorem 2.2 in [?] is not applicable. But, we have* $d(T^{2n-1}x, x) = |\frac{x}{2^n} + x|$, $d(Ty, y) = 2|y|$, *therefore the Kannan type cyclic orbital contraction condition* $d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, x) + d(Ty, y)]; \forall n \in \mathbb{N}; \forall y \in [0,1]$ *satisfies for* $k_x = \frac{1}{2}$. *So, T has a unique fixed point, that is* $x = 0$.

Remark 3. *Note that the statement of Equation (1) in the Definition of Cyclic Orbital Contraction could not be generalized to the following condition:*

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n}x, y) + d(Ty, x)]; n \in \mathbb{N}; y \in A. \tag{4.4}$$

since both $T^{2n}x$ *and* y *lies in* A, *the above statement fails to be cyclic. To avoid such cases, we define and use the notion "opposite parity": We say that* $p, q, n \in \mathbb{N}$ *are opposite parity if either* $T^p x \in A$, $T^q x \in B$ *or* $T^p x \in B, T^q x \in A$ *holds.*

Definition 4.13. *Let A and B be non-empty subsets of a metric space* (X, d). *A cyclic map* $T : A \cup B \to A \cup B$ *is said to be a* ***Chatterjee type cyclic orbital contraction*** *if for some* $x \in A$ *there exists a* $k_x \in (0, \frac{1}{2})$ *such that*

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, y) + d(x, Ty)]; n \in \mathbb{N}; y \in A. \tag{4.5}$$

We prove a similar theorem for Chatterjee type cyclic orbital contractions.

Theorem 4.14. *Let* (X, d) *be a metric space,* $A \subseteq X$ *and* $B \subseteq X$ *and* $T : A \cup B \to A \cup B$ *be a* ***Chatterjee type cyclic orbital contraction***. *Then* $A \cap B$ *is non-empty and T has a unique fixed point.*

Corollary 4.15. *Let* (X, d) *be a complete metric space and* $T : X \to X$. *If for some* $x \in A$ $\exists k_x \in (0, \frac{1}{2})$ *such that*

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, y) + d(x, Ty)]; n \in \mathbb{N}; y \in A \tag{4.6}$$

then, T has a unique fixed point.

Similarly, we can give the following definition.

Definition 4.16. *Let* (X, d) *be a metric space,* $A \subseteq X$ *and* $B \subseteq X$. *A cyclic map* $T : A \cup B \to A \cup B$ *is called a* ***Reich type cyclic orbital contraction*** *if for some* $x \in A$ $\exists$ *a* $k_x \in (0, \frac{1}{3})$ *such that* $\forall n \in \mathbb{N}, y \in A$

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)], \tag{4.7}$$

We obtain a similar theorem for Reich type cyclic orbital contractions.

Theorem 4.17. *Let (X, d) be a metric space, $A \subseteq X$ and $B \subseteq X$ and $T : A \cup B \to A \cup B$ be a **Reich type cyclic orbital contraction**. Then $A \cap B$ is non empty and T has a unique fixed point.*

Corollary 4.18. *Let (X, d) be a complete metric space , $T : X \to X$ be given. If for some $x \in A$ $\exists$ $k_x \in (0, \frac{1}{3})$ s.t. $\forall n \in \mathbb{N}, y \in A$*

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)] \tag{4.8}$$

then, T has a unique fixed point.

On the other hand, Meir and Keeler proved their well known fixed point theorem.

Theorem 4.19. *Let (X, d) be a complete metric space and $T : X \to X$ any map. Suppose that the following condition is satisfied. $\forall \epsilon > 0$, $\exists$ $\delta > 0$ s.t. $\forall x, y \in X$,*

$$\epsilon \leq d(x, y) < \epsilon + \delta \implies d(Tx, Ty) < \epsilon$$

. Then T has a unique fixed point $z \in X$ and $\forall x \in X, T^n x \to z$.

Later on, Jachymski presented several modifications of this condition and get interesting variants of the theorem.

Very recently, E. Karapinar, S. Romeguera and K.Tas (2013) introduced a very general notion which is called *cyclic orbital generalized contraction*.

Definition 4.20. *Let (X, d) be a metric space, $A \subseteq X$ and $B \subseteq X$ and $T : A \cup B \to A \cup B$ a cyclic map. If $\exists$ $x_0 \in A$ and a function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) < t$ $\forall t > 0$, and*

$$d(T^{2n}x_0, Ty) \leq \phi(M(T^{2n-1}x_0, y))$$

*for all $y \in A$ and $n \in \mathbb{N}$, then T is called a **COG (cyclic orbital generalized contraction)** for x_0 and ϕ, where*

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(Tx, y)]\}.$$

E. Karapinar, S. Romeguera and K.Tas (2013) give a fixed point theorem which generalizes both the Boyd and Wong's fixed point theorem and Matkowski's fixed point theorem.

Theorem 4.21. *Let (X, d) be a metric space, $A \subseteq X$ and $B \subseteq X$ and $T : A \cup B \to A \cup B$ be a COG-contraction for an $x_0 \in A$ and $\phi : [0, \infty) \to [0, \infty)$. If ϕ satisfies the condition:*
$\forall \epsilon > 0 \; \exists \delta > 0$

$$\epsilon < t < \epsilon + \delta \implies \phi(t) \leq \epsilon.$$

Then T has a fixed point $z \in A \cap B$ such that $T^n x_0 \to z$.

CHAPTER 5

BEST PROXIMITY POINTS FOR CYCLICAL CONTRACTIVE OPERATORS

In this section our aim is to obtain existence and convergence results for best proximity points considering different contractive type conditions. Kirk, Srinivasan and Veeramani in [2], introduced the notion of contractions under cyclic conditions. Actually in this case the problem is solved under the hypothesis that the intersection of the sets involved in the cyclic contraction is nonempty, that is $A \cap B \neq \emptyset$. Moreover, the fixed points are situated in the intersection set. In the case $A \cap B = \emptyset$, it is natural to try to find an approximative solution for the fixed point problem, that is a best proximity point.

The best proximity point theorems are obtained in the framework of a uniformly convex Banach space, and the contractive condition imposed to the operator is weakened. Basically, we are speaking about weak cyclic Kannan contractions, weak cyclic Chatterjea contractions and weak cyclic Reich-Rus contractions. Let us mention that an operator is said to be **nonexpansive** if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$.

Now we will introduce a new class of contraction, called **weak cyclic Kannan contraction**,and we will give convergence and existence results for best proximity points. For the details see [17].

Definition 5.1. *Let (X, d) be a metric space, $A \subseteq X$ and $B \subseteq X$. Then a cyclic mapping $T : A \cup B \to A \cup B$ is a* ***weak cyclic Kannan contraction*** *if it satisfies the condition*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] + (1 - 2a)d(A, B)$$

for all $x \in A, y \in B, a \in [0, \frac{1}{2})$.

Example 5.2. *Let $X = \ell^p, 1 \leq p < \infty$ and $k \in (0, 1)$. Define $A = \{(1 + k^{2n})e_{2n} : n \in \mathbb{N}\}$ and $B = \{(1 + k^{2m-1})e_{2m-1} : m \in \mathbb{N}\}$. Define $T : A \cup B \to A \cup B$ by*

$$T(1 + k^{2n})e_{2n} = (1 + k^{2n+1})e_{2n+1}$$

and

$$T(1+k^{2m-1})e_{2m-1} = (1+k^{2m})e_{2m}.$$

Then $d(A,B) = 2^{1/p}$ *and* T *is a cyclic contraction. Also* T *is a weak cyclic Kannan contraction.*

Example 5.3. *Let* $A = B = \mathbb{R}$ *be two subsets of* $X = \mathbb{R}$ *with the usual norm, and* $T : \mathbb{R} \to \mathbb{R}$ *defined by*

$$T(x) = \begin{cases} 0 & \text{if } x \leq 2 \\ -\frac{1}{2} & \text{if } x > 2 \end{cases}$$

Then $d(A,B) = 0$ *and hence* T *is not a cyclic contraction but* T *is a weak cyclic Kannan contraction.*

The following result gives a necessary condition for the existence of a best proximity point, for weak cyclic Kannan contractions.

Theorem 5.4. *Let* (X,d) *be a uniformly convex Banach space and* $A \subseteq X$ *and* $B \subseteq X$ *Suppose* $T : A \cup B \to A \cup B$ *is a weak cyclic Kannan contraction. Then*
a) T *has a best proximity point* z *in* A *and this point is unique.*
b) The sequence $\{T^{2n}x\}$ *converges to* z *for any starting point* $x \in A$
c) z *is the unique fixed point of* T^2.
d) Tz *is a best proximity point of* T *in* B.

From [18] we have the following definitions.

Definition 5.5. *Given non-empty subsets* A, B *and* C *of a metric space* (X,d), *an element* $x^* \in A$ *is called a* ***common best proximity point of the non-self mappings*** *if* $S : A \to B$ *and* $T : A \to B$ *if it satisfies the condition*

$$d(x^*, Sx^*) = d(x^*, Tx^*) = d(A,B).$$

A common best proximity point is an element at which both real valued functions $x \to d(x,Sx)$ and $x \to d(x,Tx)$ attain global minimum, since $d(x,Sx) \geq d(A,B)$ and $d(x,Tx) \geq d(A,B)$ for all x.

Definition 5.6. *Let* $S : A \to B$ *and* $T : A \to B$ *be given maps.The pair* (S,T) *is called a contractive pair iff*

$$d(Sx_1, Tx_2) < d(x_1, x_2)$$

whenever x_1 *and* x_2 *are in* A.

If T is a contractive map, then (T, T) is a contractive pair. On the other hand, even if S and T are contractive mappings, the pair (S, T) is not necessarily a contractive pair.

Definition 5.7. *Given mappings $S : A \to B$ and $F : B \to A$, the pair (S, F) is called a* ***cyclic contractive pair*** *iff*

$$d(Sx, Fy) < d(x, y)$$

whenever $x \in A$ and $x \in B$ satisfy the condition that $d(x, y) > d(A, B)$.

By this definition of S. Basha et al, if the self-maps are defined on the same set, then the notion of cyclic contractive pair reduces to that of contractive pair.

Theorem 5.8. *Let (X, d) be a metric space and $A \subseteq X$ and $B \subseteq X$. Suppose $S : A \to B$, $T : A \to B$ and $F : B \to A$, $G : B \to A$ are contractive mappings satisfying the conditions:*
a) (FS, GT) and (SF, TG) are contractive pairs.
b) (S, F) and (T, G) are cyclic contractive pairs.

Then, S and T have a common best proximity point x^* and F and G have a common best proximity point y^* such that $d(x^*, y^*) = d(A, B)$.

If S and T are identical, and F and G are identical, then the preceding theorem gives rise to the following result as a corollary.

Example 5.9. *Consider $\mathbb{R}$. Let $A = [1, 2]$ and $B = [-2, -1]$. Define the mappings $S : A \to B$, $T : A \to B$ and $F : B \to A$, $G : B \to A$ be defined as follows.*

$$Sx = \frac{1}{2}(1 - x) - 1$$
$$Tx = \frac{1}{4}(1 - x) - 1$$
$$Fy = \frac{1}{6}(-y - 1) + 1$$
$$Gy = \frac{1}{3}(-y - 1) + 1$$

It can be easily shown that all conditions of the Theorem are satisfied. And $x = 1$ is a common best proximity point of S and T, and $y = -1$ is a common best proximity point of F and G.

We should note that the condition (a) of the Theorem 5.8 is essential. Consider the following example about this idea.

Example 5.10. *Consider the space* $\mathbb{R}^2$*. Let* $A = \{(0, y) : 0 \leq y \leq 1\}$ *and* $B = \{(1, y) : 0 \leq y \leq 1\}$*. Suppose the mappings* $S : A \to B$*,* $T : A \to B$ *and* $F : B \to A$ *,* $G : B \to A$ *are defined as follows.*
$S(0, y) = (1, \frac{1-y^2}{2})$, $T(0, y) = (1, \frac{y^2}{2})$
$F(1, y) = (0, \frac{1-y^2}{2})$, $G(1, y) = (0, \frac{y^2}{2})$.
Then S, T, F *and* G *are contractive and* (S, F) *and* (T, G) *are cyclic contractive pairs. But the condition* (a) *of Theorem 5.8 does not hold. Further, it can be seen that* S *and* T *have no common best proximity point; nor do the mappings* F *and* G *have a common best proximity point.*

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