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SPARE PARTS INVENTORY MANAGEMENT WITH DEMAND PRIORITIES

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Title of Thesis : **Spare Parts Inventory Management with Demand Priorities**

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ABSTRACT

SPARE PARTS INVENTORY MANAGEMENT WITH DEMAND PRIORITIES

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In this thesis, we analyze a single-echelon inventory management problem of a manufacturing company which is a subsidiary company of an engineering company. The company manufactures machines and also supplies supplementary equipment and spare parts to its customers. Based on the priorities in meeting the demands, the customers of the manufacturing company are classified as high priority and low priority customers. The engineering company which is the most important customer is treated as the high priority customer whereas all other customers having the same priority are assumed to be the low priority customers. In this study, considering these two types of customers, we focus on an (R, Q) inventory model with a critical level policy where backorders and lost sales are allowed. Below the critical level the demands of low priority customers are not met to retain stock for high priority customer and therefore the unmet demands of low priority customers are lost. The demands of high priority customer are satisfied until inventory level reaches zero and under stock out any unsatisfied demand of high priority customer is backordered. Considering the possible levels of reorder point (R) and critical point (c), three different cases are defined as $c=R$, $c>R$, and $c<R$. Each case is formulated as a

continuous time Markov chain. The objective is to determine reorder level and ordering quantity for each case that will minimize the system-wide total inventory cost including ordering, holding, backordering and lost sale costs.

Keywords: Inventory, single-echelon, rationing, backorder, lost sales, Markov chain.

ÖZ

TALEP ÖNCELİKLİ YEDEK PARÇA STOK YÖNETİMİ

ÖZEN, Duygu

Yüksek Lisans, Endüstri Mühendisliği Ana Bilim Dalı
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Bu tez kapsamında, tek kademeli stok yönetimi problemi bir mühendislik firmasının bağlı firması durumundaki bir üretici firma için analiz edilmiştir. Firma müşterileri için makineler üretmekte ve ayrıca müşterilerine ilave donanım ve yedek parça tedarik etmektedir. Talepleri karşılamadaki öncelikler temel alınarak, üretici firmanın müşterileri yüksek öncelikli ve düşük öncelikli müşteriler olarak sınıflandırılmıştır. En önemli müşteri olan Mühendislik şirketi yüksek öncelikli müşteri olarak, diğer tüm müşteriler ise hepsi aynı önceliğe sahip olmak üzere düşük öncelikli müşteriler kabul edilmiştir. Bu çalışmada, bu iki tür müşteri göz önünde bulundurularak sonradan karşılama ve kayıp satışların yer aldığı bir (R, Q) stok modelinde kritik seviye politikası üzerine odaklanılmıştır. Kritik seviyenin altında, yüksek öncelikli müşteri için stok tutmak amacıyla düşük öncelikli müşterilerin talepleri karşılanmamakta ve bu nedenle düşük öncelikli müşterilerin karşılanmayan talebi kaybedilmektedir. Yüksek öncelikli müşterinin talepleri stok düzeyi sıfıra ulaşıncaya kadar karşılanmakta ve stok kalmaması durumunda bu müşterinin talebi sonradan karşılanmaktadır. Yeniden sipariş noktası (R) ve kritik noktanın (c) olası seviyeleri göz önüne alındığında $c=R$, $c>R$, and $c<R$ olmak üzere üç farklı durum tanımlanmıştır. Her durum sürekli zamanlı Markov zinciri olarak formüle edilmiştir. Her durum için sipariş verme, stokta tutma, sonradan karşılama ve satış kaybı maliyetlerini içeren toplam stok maliyetini en küçükleyecek şekilde yeniden sipariş noktasını ve kritik seviyeyi belirlemek amaçlanmaktadır.

Anahtar Kelimeler: Stok, tek kademeli stok, farklı öncelikli iki müşteri, sonradan karşılama, satış kaybı, Markov zinciri.

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LIST OF ABBREVIATIONS

c	Critical Level
EOQ	Economic Order Quantity
FCFS	First Come First Serve
R	Reorder Level
Q	Order Quantity

INTRODUCTION

Nowadays, cost of holding inventory is considered as a significant part of the overall costs, and the companies want to reduce the inventory related costs by investigating inventory management policies. Therefore, inventory systems have been an important study area for many researchers. There are various types of costs associated with inventories such as holding cost, production or purchasing costs, ordering cost, and backordering and/or lost sale costs. Hence, it is important to make optimal decisions on the amount of inventory to be held and when to place an order since such decisions affect the total cost. Other significant inputs for the inventory problems are the replenishment lead time and the demand rate that cause fluctuations in inventory levels. Also, the review policy (periodic or continuous), performance measures, and the prioritization of customers (rationing or no rationing) are the other considerations for inventory systems. Based on all such characteristics inventory problems can be analyzed in different ways.

In this thesis, we study on an inventory management problem for a particular manufacturing company. The company manufactures machinery and supplies equipment and spare parts for the machines to its customers. This company is also a subsidiary group of an engineering company and therefore it supplies spare parts mainly to the engineering company and also to the other customers. Therefore the engineering company is treated as the first priority customer and we assume that its demands are more important than those of the others. With the aim of meeting the demands of its customers on time the company considers stocking a lot of spare parts and equipment in their inventory, which nearly includes 800 items. However, when they want to stock, for example, 10 units of each item, the annual cost of holding inventory is nearly 600,000 USD. The company has limited manufacturing capacity, and hence it encounters problems in supplying demands on time. If the company cannot keep enough stock, it backorders the demands of the engineering company and loses sales to other customers. To avoid backorders and lost sales, the company has to stock enough but this time it may cause high inventory costs. However, not meeting the demands of the engineering company causes more significant penalties than the cost of unmet demands of the other customers.

Based on this environment, we consider a critical level based stock rationing problem of a make-to-order manufacturing company, which is a supplier and a subsidiary of a larger parent company. The company serves two different customer types; the parent company which is the most important customer and the other customers all having equal importance. Therefore, the parent company is considered as the high priority customer while the rest are considered as the second -or low- priority customers. As long as some inventory exists, the demand of the high priority customer is always satisfied and it is allowed to be backordered when inventory level drops to zero. Nevertheless, backordering becomes an expensive option for the company, most probably more expensive than the cost of losing low priority customers. The low priority customers are also important, however, if the inventory level drops to a critical level as a company policy it is preferred to ignore the sales to these customers in order to supply required items to the high priority customer.

Under this setting, the objective of this thesis is to propose an inventory policy for the manufacturing company that will minimize the system-wide total inventory cost including ordering, holding, backordering, and lost sale costs. To differentiate the customer types, we propose a rationing policy.

To formulate the problem, we consider a single-echelon inventory system. We assume that the demands occur according to a Poisson distribution. The stocks are replenished within an exponentially distributed lead time every time with an order size of Q units.

To analyze the inventory system, considering reorder level (R) and critical level (c), we defined three cases for the stock rationing problem; (i) the critical level is equal to the reorder level in CASE I (c=R), (ii) the critical level is greater than the reorder level in CASE II ($c>R$), and (iii) the critical level is less than the reorder level in CASE III ($c<R$). In all cases, the company places an order when the inventory level drops to R. When the inventory level drops to c, the demands of low priority customers are not satisfied any more while the demand of high priority customer is still met from stocks so long as there is an available part in stock. The unsatisfied demand for high priority customer is backordered when inventory level becomes zero while the unsatisfied demand for low priority customers is lost below critical level c. Our objective is to determine the inventory policy parameters R (reorder point), Q (order quantity) and c (critical level) that will minimize the total inventory holding, ordering, backordering and lost sale cost. The representation of three cases is shown in Figure 1.

CASE	CASE I: $c=R$	CASE II: $c>R$	CASE III: $c<R$
Maximum Inventory Level	$R+Q$	$R+Q$	$R+Q$
Reorder Level / Critical Level	c, R	c	R
Reorder Level / Critical Level		R	c
Backorder	0	0	0

Figure 1 Representation of Three Cases

We formulate each case as a continuous time Markov chain and study the cost minimization problem for each case. To develop the average cost function the steady state probabilities are computed using MATLAB. We implement and test our model by generating some test data. After collecting and analyzing the results, we compare c and R values and try to determine the best policy that will minimize the cost.

The organization of the thesis is as follows. Chapter 1 includes a literature review in which we summarize the work related to our study. In Chapter 2, the problem is defined and the formulation of the three cases is presented. In Chapter 3, we define the settings for the experimentation and then illustrate the formulations of the cases by running them in MATLAB and presenting the numerical analyses. Finally, conclusion and discussion is presented.

CHAPTER I

LITERATURE REVIEW

In this chapter, we present a summary of the literature related to our field. This mainly includes studies on inventory control system with rationing. Therefore, our review does not include other studies on spare parts inventory control systems that do not consider rationing since we think that it is out of the main scope of this thesis.

We classify the studies in the literature based on (i) the number of echelons (single vs. multi echelon), (ii) number of items (single vs. multiple), (iii) type of inventory policies considered, and (iv) how shortages are handled (lost sales and backordering) as usually done in other reviews on inventory control literature. Although our main concern is the studies on rationing, we also review some studies that do not consider the rationing problem since they are interesting for the analytical point of view. Therefore, we also classify the studies based on (v) whether rationing policy is considered or not, (vi) what type of a supply channel is considered (production, inventory and supply chain). The taxonomy of these studies is given in Table 1.

Our main focus is the studies on continuous review policies since a continuous review policy is considered in this study. However, our review also considers some studies on periodic review systems. Mainly three policies dominate the relevant literature on continuous review system. These are (S-1, S), (s, S) and (R, Q) inventory policies. Under an (S-1, S) inventory policy, which is also called continuous review base stock policy in the literature, the system orders one unit every time when the inventory position reduces by one unit to bring the inventory position to level S again. Under (s, S) system, the objective is to place an order to bring the inventory up to level S, only when the inventory level drops to s, the order point. Under (R, Q) – also called (Q, R) – policy, an order of Q units is placed when the inventory level drops to *R*. In this policy, the maximum inventory level could be *R*+*Q*. Since the company of concern in this study orders with the same order quantity every time, (*R*, *Q*) policy is considered to be more appropriate in our case and therefore we focus on it. Apart from these inventory policies, there are also extensions of these policies for systems that face different demand types and want to differentiate them. These types of policies are known as critical level policies. For instance, under a critical level base stock policy, system still replenishes orders according to a (*S*-1, *S*) policy but only with a single modification: below a critical level, *c*, low priority customers are not satisfied to retain stock for high priority customer. In a similar way, the critical level polices are obtained as extensions of (*S*-1, *S*), (*s*, *S*) and (*R*, *Q*) type inventory policies. In this review only (*S*-1, *S*), (*s*, *S*), (*R*, *Q*) and rationing policies are considered.

1.1 STUDIES ON INVENTORY RATIONING PROBLEMS WITH BACKORDERING

There are a number of studies assuming a backordering for their systems. Among these Kaplan (1969) studies stock rationing policy with backordering. He assumes periodic review and models the rationing level as a function of time to replenishment. When the high priority demand is received, it is supplied if the stock on hand is issuable. If it is a low priority demand, it is supplied from stock. If demand is not satisfied, it is backordered. He solves his model by an algorithm that computes reserve levels and analyzes this model by unsophisticated policies.

Nahmias and Demmy (1981) studies on different inventory systems which are a single period model, a multi-period model with zero lead-time, and an approximate continuous review model. They modeled the problem for high and low priority demand classes. In the single period model, y is the level of starting stock and c is the support level where $0<c<y$. When the inventory level is c ($c>0$), all low priority customers' demand are backordered. The (*s*, *S*) policy is used in the multi-period model with zero lead-time where $0<c<s<S$. In third model, the policy is determined as (*R*, *Q*) with $R>c>0$, the demand arrivals form a Poisson process from each class.

Ha (1997) considers a stock rationing problem for a make-to-stock production control with two priority classes of customers and backordering. The customers have to join the backorder queues of their classes. Backorders of low priority classes are supplied when there is not any backordering for high priority classes of customers. Moreover, when there is not backorder, the production is continued up to safety stock level. He conjectures optimal policy for the backorder queue lengths of all priority classes.

Carr and Duenyas (2000) study a production system producing two classes of products. The first class of products is make-to-stock and the firm meets the demand. The other class of products is make-to-order and the firm has the option to accept or reject a particular order. They model a simple two-class M/M/1 queue for making accept/reject decisions, determining the type of product to produce next, and deciding what quantity of orders to be supplied and when to sign a contract to produce the make-to-stock products.

Wang et al. (2001) study on rationing for two service classes on the basis of delivery lead time. They analyze single location model and its extension to two-echelon systems with numerical solutions.

Wang et al. (2002) study on rationing for two classes of service. They analyze the evaluation of service levels and the determination of required base stock levels for the target service level. This study is done for a single location and extended for two-echelon systems.

Deshpande, Cohen, and Donohue (2003) study on inventory rationing considering different customer classes with the policy of (R, Q). They consider a static threshold-based rationing policy being characterized by different arrival rates and shortage (stock out and delay) costs. The threshold inventory level is c and demands arrive according to a Poisson process. After this level, only the demand of high priority customers is supplied and the demand of low priority customers is backordering.

Ghalebsaz-Jeddi et al. (2004) study on a single echelon inventory system with multi items. They model a system with backordering when the estimation of marginal backorder cost is available and the payment is due upon order arrival. They solve the problem with a Lagrange multiplier technique.

Kocağa and Şen (2004) study (*S*-1, *S*) spare parts service system with backordering. The system has rationing for their customers; if the down orders, the orders are supplied immediately for the equipment failures of the customers and when the lead time orders, the orders are supplied a future date for the scheduled maintenance activities. They develop an approximation model and a simulation model to analyze and optimize the critical levels.

Bulut (2005) analyzes the stock rationing policies for continuous review systems. He analyzes (*R*, *Q*) inventory systems with backordering under rationing policy. He provides a recursive procedure to generate the transition probabilities of the embedded Markov chain to obtain the steady-state probabilities. In his study, he conducts a simulation study to evaluate the performance of the proposed policy.

Cesaro and Pacciarelli (2011) study on Spare Part Management System (SPMS) of a supply chain with three actors, which are the airport authority, a logistics company and the equipment supplier. The company has a two echelon (*R*, *Q*) policy with backordering and rationing, but the company managers are interested in evaluating the potential benefits derived from the adoption of a single echelon policy in their SPMS (Cesaro and Pacciarelli (2011), page 3). They model a Markov chain to obtain the steady state probabilities, but they realize that computing them in difficult data set gets large. Therefore, they adapt three approximation methods to compute the overall service level, memory requirement, and error with respect to the exact value, when available.

Mayorga and Ahn (2011) evaluate advantages of coordinating capacity and inventory decisions in a make-to-stock production environment. The company meets multi-class demand and has additional capacity options, which are temporary and randomly available. They formulate a model a Markov Decision Process (MDP). They find to meet to the optimal joint control problem exists according to solving their model. They also present several simpler heuristics and evaluate their performances.

Okonkwo and Obaseki (2011) study on (S, S-1) continuous review inventory system for different classes of customers. These customers are classified according to their priorities; when the high priority customers have zero demand lead time, low priority customers have a positive demand lead time. The system backorders when there is no stock on hand. They analyze the system using a stochastic computer solution with a user interface.

In this thesis, as opposed to these studies that assumes backordering for shortages, we consider backordering for the high priority customers and lost sales for the low priority customers.

Apart from studies on single echelon models, there are a number of works studying multi-echelon inventory rationing problems under the backordering assumption. Axsäter et al. (1998) study (S-1, S) inventory policies including a warehouse and N-retailer inventory system. All demands arrive the system with Poisson distribution. The critical levels are defined for each retailer. Below the critical level, retailer demand is backordered. They develop a heuristic method and solve using numerical experiments to optimize policy parameters.

Wong et al. (2006) study two-echelon inventory system for multi items. The system operates according to First Come First Serve (FCFS) when the system does not have any product and the customer is backordering. To solve this system, they develop a heuristic that performs as a greedy approach. Wong et al. (2007) study the same system but they develop four different heuristics that include greedy approach.

There are also studies that consider dynamic rationing policies. One can refer to Teunter and Haneveld (2008) for a review with this type of problems. Teunter and Haneveld (2008) study inventory systems with critical and non-critical of two demand classes. The demands arrive to the system Poisson. The aim of the study is to analyze dynamic rationing strategies including a numerical example for obtaining optimal rationing levels while backordering.

Satır (2010) studies on rationing for multi-echelon inventory system and the decentralized spare parts network model for after sales service. He analyzes the system using of Continuous Time Markov Decision Process. The optimal policy is computed for information sharing under decentralized and centralized systems, service pooling and inventory benefits for spare parts management system.

1.2 STUDIES ON INVENTORY RATIONING PROBLEMS WITH LOST SALES

Apart from studies on single echelon models, there are a number of works studying single-echelon inventory rationing problems under the lost sales assumption. Cohen et al. (1988) study (R, Q) inventory system with two priority customer classes with lost sales for single-echelon inventory system. They embed their study for single product and single echelon from another study of Cohen et al. (1986), which is for multi echelon inventory system. When there is excess demand, lost sales occur. They model a Markov chain which is derived as approximate, renewal-based model. They analyze the performance of two models.

Dekker et al. (1997) study (S-1, S) lost sales inventory model with several demand classes. In their study, they consider a critical stock level for each demand class, where the level is determined by stock policy. When the inventory level is below of the critical stock level, order of low priority demand classes is not satisfied from stock on hand and it is lost. They develop efficient solution methods for optimal policies and present numerical data.

Frank et al. (1999) study a periodic review system for deterministic and stochastic demands. The system has to supply deterministic demand, but for stochastic demand there is lost sales when the demand is not met. They formulate a dynamic programming model to characterize optimal policies.

Melchiors (1999) analyzes an (*R*, *Q*) inventory model with unit Poisson demand, several demand classes, and lost sales. The demand classes are controlled by critical levels with is $c \leq s+1$ for single item. He develops a simple policy and optimal policy and after solving two policies, decides on the simple policy that is much easier to implement than the optimal policy.

Melchiors et al. (2000) analyze a continuous review (*R*, *Q*) model with lost sales and two demand classes having priorities. They consider different critical levels c where $c<s$ and $c \geq s$. Using Poisson demand and deterministic lead times, they present an exact formulation for the average inventory cost with numerical examples.

Frank et al. (2003) study a periodic review inventory system with two priority demand classes, one deterministic and the other stochastic. While deterministic demand is supplied, the stochastic demand is not satisfied during the period; therefore, it is defined as lost sales in their study. They characterize an optimal policy and a simple heuristic policy. They have numerical results for these policies and find out that the simple heuristic policy works extremely well and is very easy to compute.

Kranenburg and Van Houtum (2006) study single item and continuous review model for multiple demand classes for the (*S*-1, *S*) lost sales inventory model. The system has been modeled with several classes of critical levels, and therefore different penalty cost parameters are defined when there are occurring lost sales. Three accurate and efficient heuristic algorithms are defined to optimize critical levels while minimizing inventory holding and penalty costs.

Isotupa (2006) works on (R, Q) inventory policy for two types of customers, namely, ordinary and priority customers. When the inventory level gets to R, the demands of ordinary customers are lost. The arrivals of two types of customers form independent Poisson processes. He develops a computationally efficient algorithm to determine optimal values for the reorder level and reorder quantity to minimize cost rate.

As opposed to the studies on rationing problems, our study is the combination of these two subclasses of polices: unsatisfied demand for high priority customers is backordered while the one for low priority customers is lost. In this sense, our problem is a mix of backordering and lost sales case, which makes our study different from the literature. This policy has received little attention in the literature.

Cattani and Souza (2002) study for Poisson demand and a single product type, exponential and replenishment server for two customer classes. They compare several policies, which are rationing policies (R, Q) and First Come First Serve (FCFS). Their aim is to minimize customer response delay: lost sales, backorder and a combination of lost sales and backorder.

Enders et al. (2008) study (S, c) inventory policy, on multi class customers, backorders for primarily distinguished customers and lost sales for the others. They develop for each item an efficient algorithm to determine the optimal critical level policy. Although Van Houtum and Zijm (2000) do not consider a rationing problem, they study backordering with lost sales for the multi echelon inventory system. To analyze relations between service and cost; they define four different models (1) periodic review and backordering, (2) periodic review and lost sales, (3) continuous review and backordering, and (4) continuous review and lost sales.

1.3 OTHER RELATED STUDIES

The studies on the analysis of (Q, R) policy is also of our interest since a (Q, R) policy is considered in this thesis. The reader may refer to Axsäter (2006) and the references for an overview of studies on single item single echelon continuous review (R, Q) inventory policies. Nevertheless, among studies on (Q, R) policy, the paper that is most relevant to our study is Nordmann and Altiok (1998) who study on a Markovian approach for the evaluation of inventory systems with (R, Q) policy and backordering. They develop three models based on the number of outstanding orders; at most one outstanding order, at most two outstanding orders, and up-to m outstanding orders. They also develop closed–form solutions for these models. In this thesis, we use the results of Nordmann and Altiok (1998) to obtain closed-form steady-state probabilities for the developed Markov chain model.

Table 1 Taxonomy of Inventory Policies

Authors	Policy Optimization	Number of Items	Number of Echelons	Demand Distribution	Inventory Policy	Cost vs. Service Constrained Model	Production	Inventory	Supply Chain	Backorder	Lost Sales	Rationing	Critical Level
Axsäter et al. (1998)	Heuristics	Multi	Multi	Poisson	(*S*-1, *S*)	Cost		X		X		X	$c \geq S$
Bulut (2005)	Heuristics		Single	Poisson	(*S*, *S*-1)			X		X		X	
Carr and Duenyas (2000)	Both	Two	Single	Poisson	M/M/1 queue (OEM)	Profit	X			X		X	
Cattani and Souza (2002)	Heuristics	Single	Single	Poisson	(*S*-1, *S*), FCFS	Cost	X			X	X	X	
Cesaro and Pacciarelli (2011)	Heuristics	Single	Single	Poisson	(*S*-1, *S*)	Cost		X		X		X	
Cohen et al. (1988)	Both	Single	Single		(s, S)	Cost	X	X			X	X	
Dekker et al. (1997)	Heuristics	Single	Single	Poisson	(*S*-1, *S*)	Service level optimization and cost optimization		X			X	X	
Deshpande et al. (2003)	Heuristics		Single	Poisson	(*Q*, *r*), (*Q*, *r*, *K*)	Cost		X		X		X	$K<r$
Enders et al. (2008)	Heuristics	Multi	Single	Poisson	(*S*, *c*)	Cost		X		X	X	X	
Frank et al. (1999)	Heuristics	Multi	Single	Deterministic and stochastic demand	(*s*, *S*)	Expected total discounted cost					X	X	
Frank et al. (2003)	Heuristics		Single	Deterministic and stochastic demand	(*s*, *k*, *S*)	Cost	X	X			X	X	
Ghalebsaz-Jeddi et al. (2004)	Optimum	Multi	Single			Cost		X		X		X	
Ha (1997)	Heuristics	Single	Single	Poisson	M/M/1 queue	Cost	X			X		X	
Isotupa (2006)	Heuristics	Single	Single	Poisson	(*s*, *Q*)	Cost		X			X	X	

Table 1 (continued)

Authors	Policy Optimization	Number of Items	Number of Echelons	Demand Distribution	Inventory Policy	Cost vs. Service Constrained Model	Production	Inventory	Supply Chain	Backorder	Lost Sales	Rationing	Critical Level
Kaplan (1969)	Optimum	Multi	Single	Poisson		Cost		X		X		X	
Kocağa and Şen (2004)	Both	Single	Single	Poisson	(*S*-1, *S*)	Service level constrained		X		X		X	
Kranenburg and Van Houtum (2006)	Heuristics	Single	Single	Poisson	(*S*-1, *S*)	Inventory holding and penalty cost		X			X	X	
Mayorga and Ahn (2011)	Both		Single	Poisson	Markov decision process: static production/rationing policy	Cost		X		X		X	
Melchiors (1999)	Heuristics	Single	Single	Poisson	(*s*, *Q*)	Cost		X			X	X	$c \leq s+1$
Melchiors et al. (2000)	Heuristics	Multi	Single	Poisson	(*s*, *Q*)	Cost		X			X	X	$c<s$, $c \geq s$
Nahmias and Demmy (1981)	Develop methods for comparing fill rates		Single	Erlang-2 & Poisson	$0<K<y$, (*s*, *S*) $0<K<s<S$ & (*Q*, *R*) $R>K>0$	Cost	X	X		X		X	
Nordmann and Altiok (1998)		Single		Poisson	(*r*, *Q*)			X	X	X			
Okonkwo and Obaseki (2011)	Both	Multi	Single	Stochastic	(*S*, *S*-1)	Service level constrained		X		X		X	
Satır (2010)	Optimum		Multi				X	X		X		X	
Song and Lau (2003)	Heuristics		Single		EOQ	Cost		X		X			
Teunter and Haneveld (2008)	Heuristics		Single	Poisson	(*r*, *Q*)	Cost		X		X		X	
Van Houtum and Zijm (2000)	Heuristics		Multi			Cost		X		X	X		
Wang et al. (2002)	Heuristics		Multi-Single	Poisson	(*S*-1, *S*)	Cost		X		X		X	
Wang et al. (2007)		Multi	Single – extended two	Poisson	(*S*-1, *S*)	Service level constrained		X				X	
Wong et al. (2006)	Heuristics	Multi	Multi	Poisson		Cost		X		X		FCFS	
Wong et al. (2007)	Heuristics	Multi	Multi	Poisson		Cost		X		X		FCFS	

CHAPTER II

PROBLEM DEFINITION AND MATHEMATICAL MODEL

2.1 PROBLEM DEFINITION

We consider a continuous review inventory system that faces the demand from two different types of customers. The demand occurs according to a Poisson process with rate λ_T for high priority customer and λ_O for other customers. The current system operating under an (R, Q) policy is considered. That is, when inventory level drops to R, an order of Q units is placed. The lead time is assumed to be exponentially distributed with parameter μ for each order. For the sake of simplicity, we also assume that there is at most one outstanding order at a time. In this way, we consider a finite capacity system.

In this thesis, we want to test the performance of using a rationing policy in place of the currently used (R, Q) policy. For the rationing policy, we assume a critical level (R, Q) policy. Under this policy, the orders are given in the same manner as (R, Q) system. However, unlike (R, Q), at or below the critical level c, low priority customers are not satisfied to retain stock for high priority customer and unmet demand for low priority customers is lost. That is, although the low priority customers are also important, if the inventory level drops to critical level the company ignores the demand of these customers in order to supply the required items to the high priority customers. When the inventory level becomes zero, unsatisfied demand of high priority customer is backordered. Our model assumes a single item setting though we also make experiments to evaluate the performance of critical level policy for multi-items.

Due to the nature of critical level (R, Q) policy, we define below three cases for the purpose of analyzing the critical level for each and then comparing them to determine the best policy.

2.1.1 CASE I: $c=R$

For $c=R$, the company serves both customer types until the inventory level becomes R. At this level the company places an order of size Q and continues meeting the demands of high priority customer, whereas the demands of the other customers at or below level R are lost. When there is no stock available in the system, the demand for high priority customers is backordered. Rationing, lost sales and backordering are shown in Figure 2 whereas the state transition diagram is shown in Figure 3.

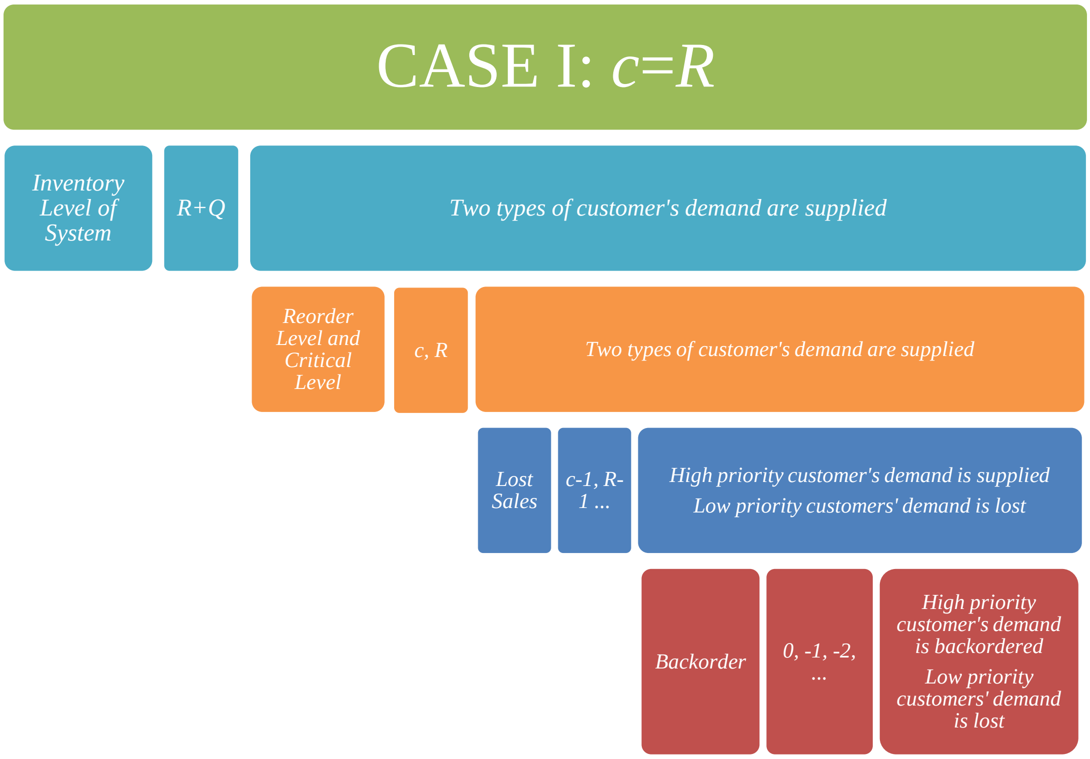

Figure 2 Rationing, Lost Sales and Backordering in Case I ($c=R$)

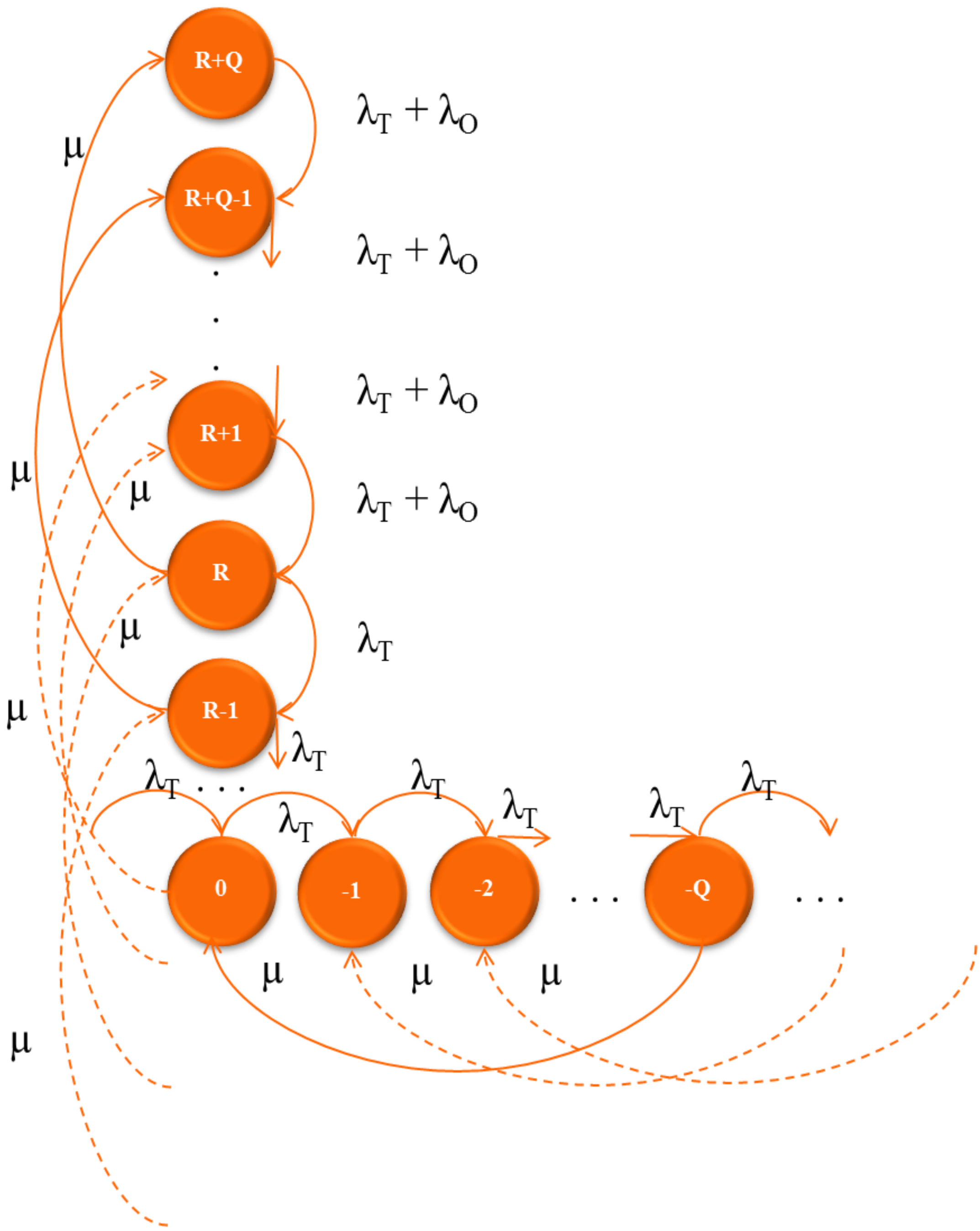

Figure 3 State Transition Diagram of Case I (*c*=*R*)

2.1.2 CASE II: *c>R*

For *c>R*, the company serves both types of customers until the inventory level decreases to the critical level *c*. Below level *c*, the demand of low priority customers are lost and the demands of the high priority customer will continue to be met. The company places an order of size *Q* when the level becomes *R*. When there is no stock available in the system, the demand for high priority customers is backordered. Rationing, lost sales and backordering are shown in Figure 4 whereas the state transition diagram is shown in Figure 5.

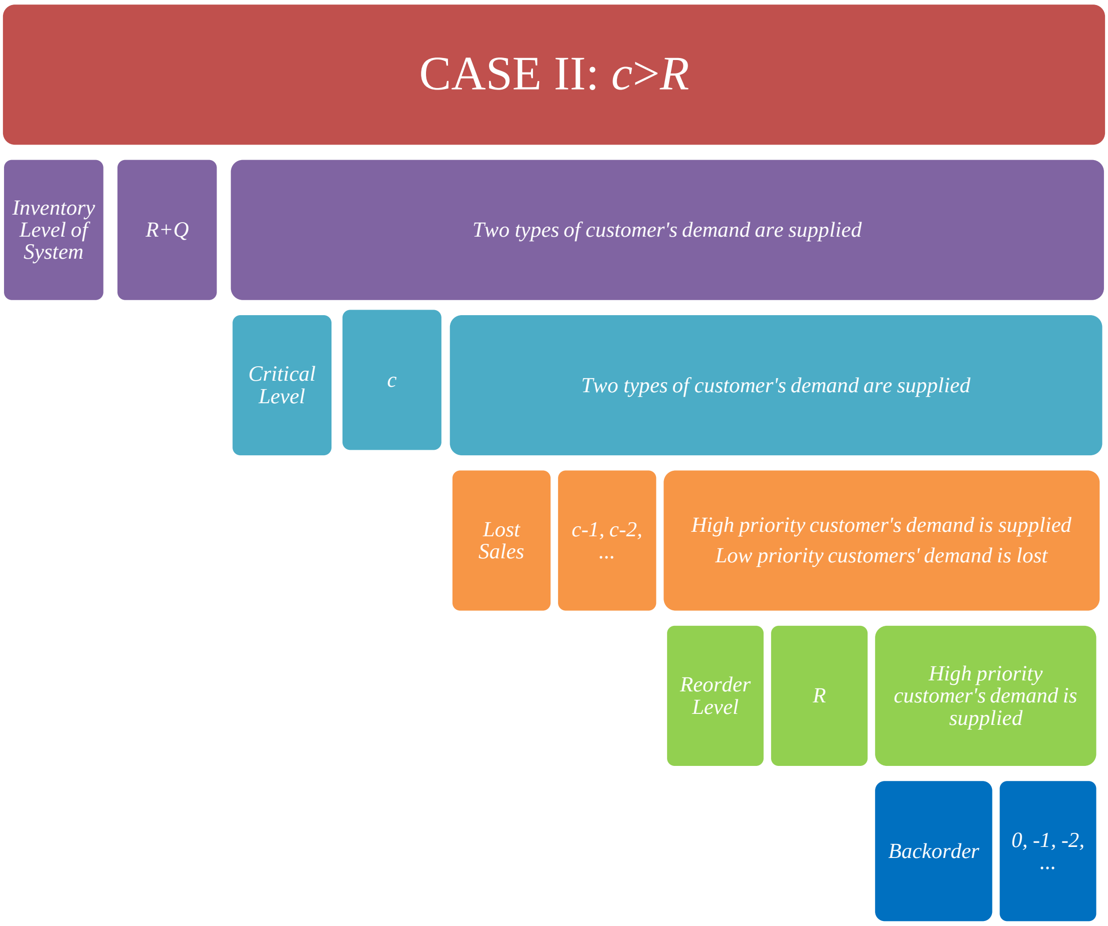

Figure 4 Rationing, Lost Sales and Backordering in Case II (*c*>*R*)

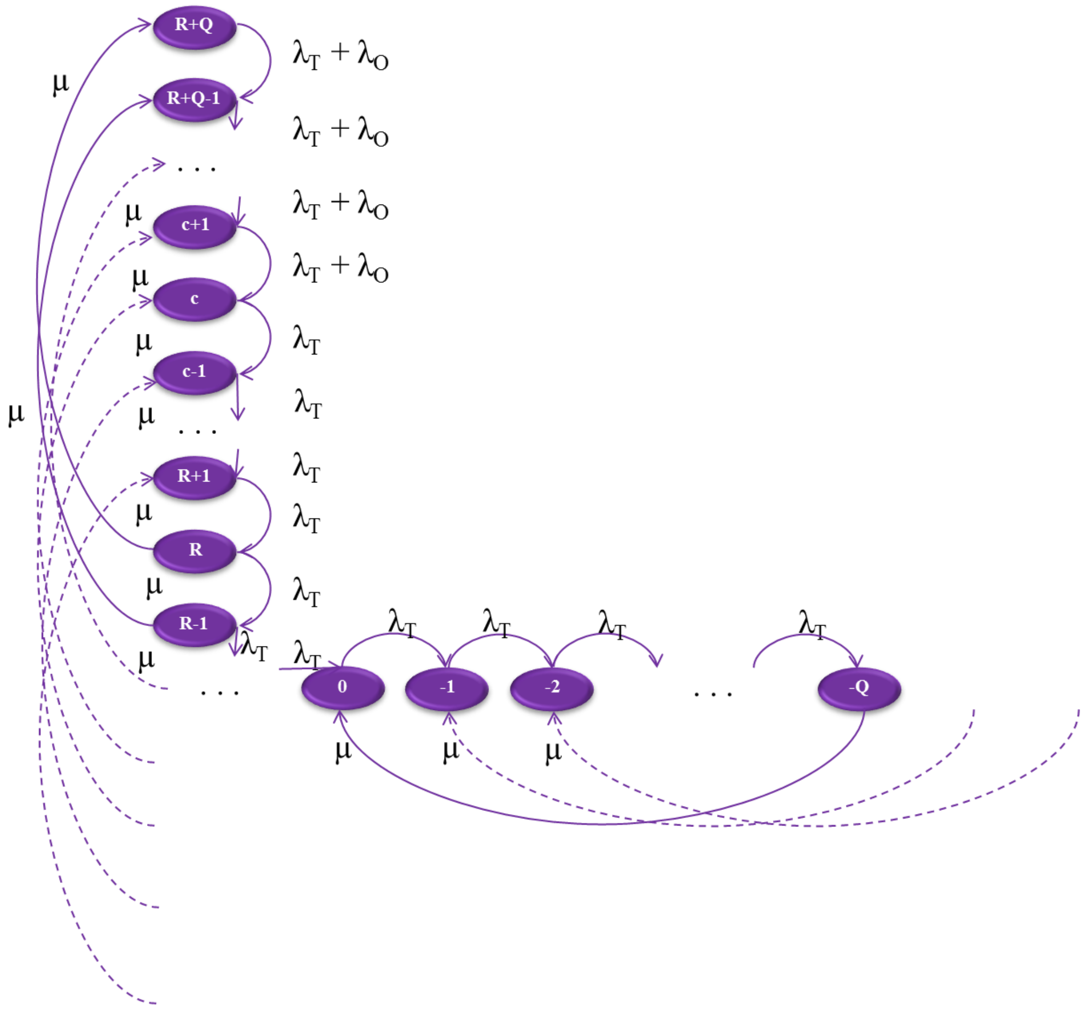

Figure 5 State Transition Diagram of Case II (*c*>*R*)

2.1.3 CASE III: *c*<*R*

In this case the company places an order of size Q when inventory level becomes *R* and goes on meeting the demands of both types of customers until the inventory level drops to the critical level *c*. Below level c, the demand of low priority customers are lost. Only after when the inventory level becomes zero, the company starts backordering for the high priority customer. Rationing, lost sales and backordering are shown in Figure 6 whereas the state transition diagram is shown in Figure 7.

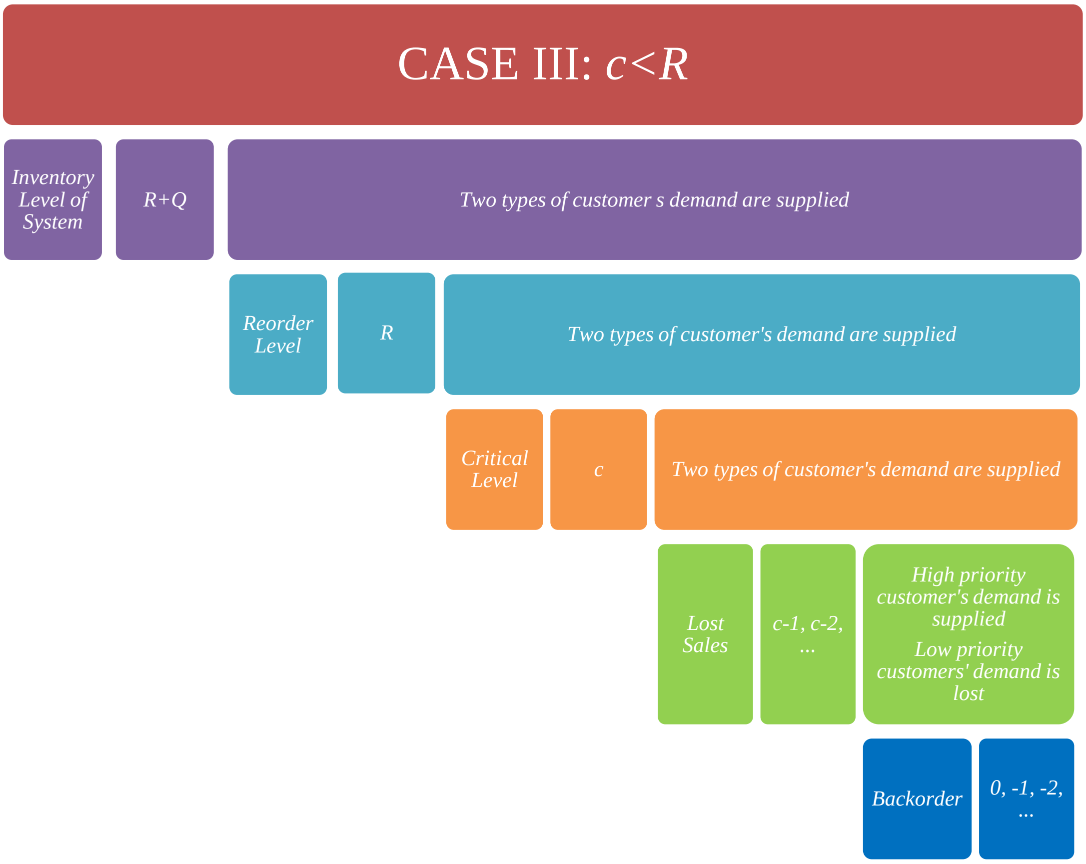

Figure 6 Rationing, Lost Sales and Backordering in Case III ($c<R$)

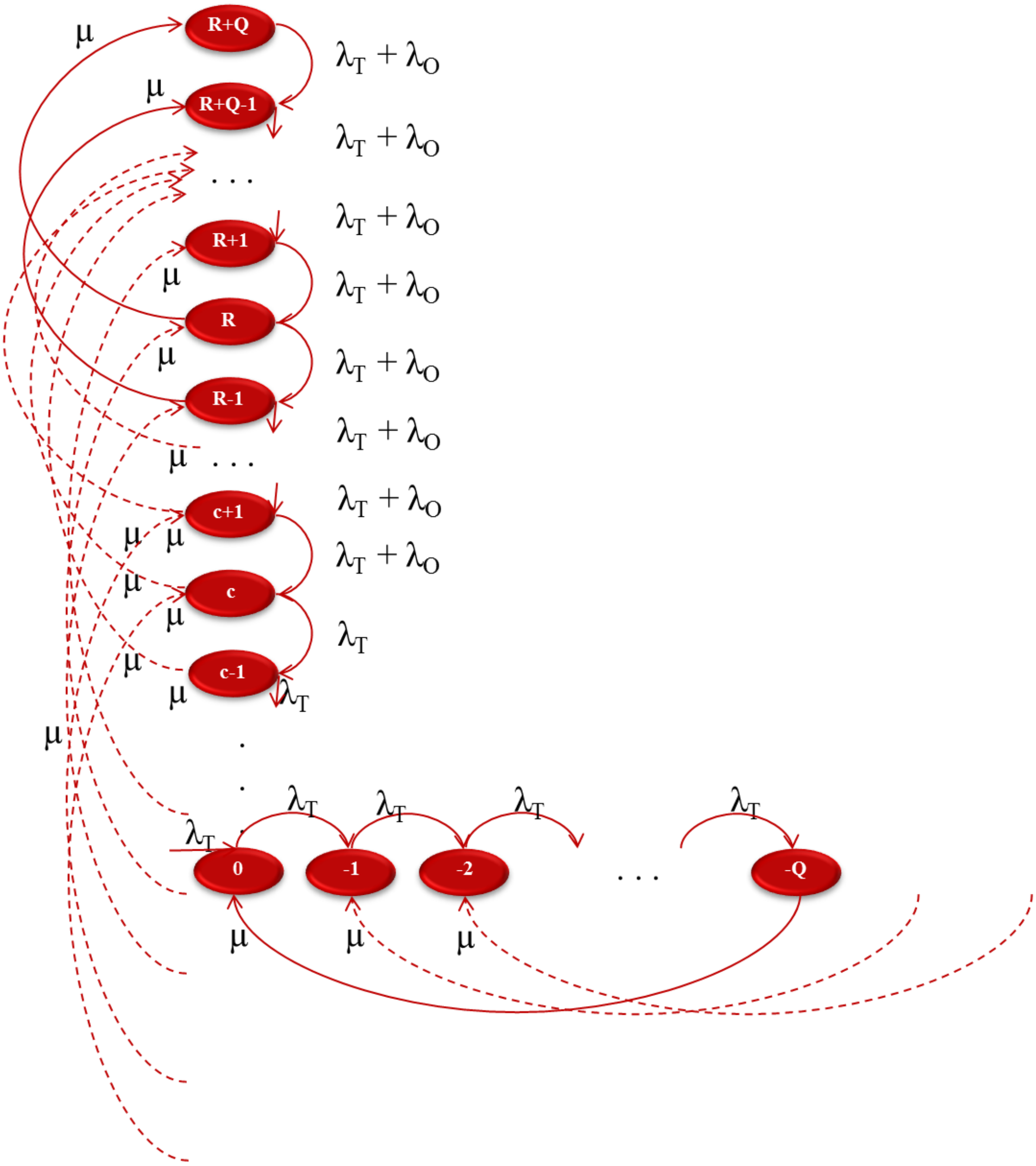

Figure 7 State Transition Diagram of Case III ($c<R$)

Considering the relations between c and R, since balance equations for the states differ in each case, each of these three cases is formulated separately. The decision variables and the parameters used in the formulation followed by the mathematical formulation are given in the following sections.

2.2 MATHEMATICAL FORMULATION

In this section, we introduce the mathematical model formulation for the three cases of the problem introduced.

Notation

The notation that we use in the development of the mathematical formulation is as follows.

i: inventory level ($-\infty, \ldots, R+Q$)

j: demand classes (T, O)

λ_T: the demand rate for high priority customer, $\lambda_T>0$

λ_O: the demand rate for low priority customers, $\lambda_O>0$

μ: replenishment service rate (interarrival rate) ($\mu>0$)

h: holding cost/unit/year

K: fixed ordering cost

γ_b: backorder cost/unit/year of demand from class T

γ_l: lost sales cost/unit of demand from class O

π_i: steady state probability that inventory level is i

π_R: steady state probability of reordering

π_B: steady state probability of backordering

π_L: steady state probability of lost sale

c: critical level

R: reorder level

Q: ordering quantity

$I(c, R, Q)$: annual average inventory level

$B(c, R, Q)$: annual average backorder level

$L(c, R, Q)$: annual number of lost sales

$C(c, R, Q)$: annual total cost

Our objective for the problem is defined as to minimize the total annual average inventory holding cost, fixed ordering cost, backordering cost and lost sales costs, which are expressed by the following objective function.

Minimize $C(c, R, Q) = h\, I + K\, (\lambda_T + \lambda_O) / Q + \gamma_b\, B + \gamma_l\, L$

Note that the transition rate diagrams in Section 3.1 are different for each case. This leads to different inflow and outflow equations (balance equations) as well as different steady state probabilities for each case. In Section 3.2.1-3.2.3, we provide the balance equations, which are used to obtain the steady state probabilities for cases I-III, respectively.

2.2.1 Formulation for CASE I: *c*=*R*

This is the case where we assume that critical point and the reorder point are the same. Considering certain ranges of states within which the same flow balance relation is valid we can write the balance equations for Case I as follows.

Balance Equation	Range	
$\pi_{R+Q}(\lambda_T + \lambda_O) = \pi_R\, \mu$	$R + Q$	**(1)**
$\pi_i(\lambda_T + \lambda_O) = \pi_{i+1}(\lambda_T + \lambda_O) + \pi_{i-Q}\, \mu$	$R + 1 \leq i \leq R + Q - 1$	**(2)**
$\pi_R(\lambda_T + \mu) = \pi_{R+1}(\lambda_T + \lambda_O) + \pi_{R-Q}\, \mu$	R	**(3)**
$\pi_i(\lambda_T + \mu) = \pi_{i+1}\lambda_T + \pi_{i-Q}\, \mu$	$\infty \leq i \leq R - 1$	**(4)**

Equations (1)-(4) immediately follow from the transition diagram in Figure 3. Note that Equation (4) describes the tail of the steady state probabilities. This is exactly the same structure with that in Nordmann and Altiok (1998), who study the steady state probabilities for an (R, Q) inventory policy under backordering. Therefore, it follows from Nordmann and Altiok (1998) that the steady state probabilities having a general structure described by (4) leads to

$$\pi_{R-i} = q^i \pi_R \qquad 0 \leq i \qquad \textbf{(5)}$$

$$\frac{\lambda_T}{\mu} = \frac{q - q^{1-Q}}{1-q} \qquad \textbf{(6)}$$

where $q \geq 0$. From Nordmann and Altiok (1998), we also know that there always exists a $q \geq 0$ satisfying Equation (6) and the stability condition for this system is expressed by $\lambda_T/\mu \leq Q$.

Note that we can show the validity of equation (6) by using the following the argument. First, by rewriting Equation (4) for $i = R - 1$, we establish

$$\pi_R = \frac{\pi_{R-1}(\lambda_T+\mu)-\pi_{R-1-Q}\,\mu}{\lambda_T}.$$

Then, by using this result and Equation (5), we obtain.

$$\pi_{R-i} = q^i \frac{\pi_{R-1}(\lambda_T+\mu)-\pi_{R-1-Q}\,\mu}{\lambda_T},$$

which can also be written as

$$\lambda_T(\pi_{R-i} - q^i\pi_{R-1}) = q^i\mu(\pi_{R-1} - \pi_{R-1-Q}).$$

Finally, by using Equation (5), we obtain

$$\lambda_T(q^i\pi_R - q^i q^1\pi_R) = q^i\mu(q^1\pi_R - q^{1-Q}\pi_R),$$

which reduces to Equation (6).

Now we use these results to establish a closed form statement for the steady state probabilities. First, from Equation (1) we have

$$\pi_{R+Q} = \frac{\pi_R\,\mu}{\lambda_T+\lambda_O}. \quad \textbf{(7)}$$

Similarly, from Equation (2) we have

$$\pi_{R+Q-1}(\lambda_T + \lambda_O) = \pi_{R+Q}(\lambda_T + \lambda_O) + \pi_{R-1}\,\mu \quad \textbf{(8)}$$

Using (7) and (8) gives

$$\pi_{R+Q-1}(\lambda_T + \lambda_O) = \frac{\pi_R\,\mu}{\lambda_T+\lambda_O}(\lambda_T + \lambda_O) + \pi_{R-1}\,\mu,$$

which simplifies to

$$\pi_{R+Q-1} = \frac{\mu}{\lambda_T+\lambda_O}(\pi_R + \pi_{R-1}).$$

Using the same idea, we obtain

$$\pi_{R+i} = \frac{\mu}{\lambda_T+\lambda_O}\left(\sum_{n=0}^{i}\pi_{R-n}\right) \text{ for } 1\leq i \leq Q-1$$

which can be then generalized to

$$\pi_{R+i} = \frac{\mu}{\lambda_T+\lambda_O}\left(\sum_{n=0}^{i}\pi_{R-n}\right) \text{ for } 0 \leq i \leq Q. \qquad \textbf{(9)}$$

Now we use equation (5) to obtain

$$\pi_{R+i} = \frac{\mu}{\lambda_T+\lambda_O}\left(\sum_{n=0}^{i}q^n\pi_R\right) \text{ for } 0 \leq i \leq Q$$

which is further simplified by

$$\pi_{R+i} = \frac{\mu}{\lambda_T+\lambda_O}\left(\sum_{n=0}^{i}q^n\right)\pi_R$$

$$= \frac{\mu}{\lambda_T+\lambda_O}\left(\frac{1-q^{i+1}}{1-q}\right)\pi_R \text{ for } 0 \leq i \leq Q \qquad \textbf{(10)}$$

To ensure the sum of all probabilities be equal to 1, we use

$$\sum_{i=-\infty}^{R+Q}\pi_i = 1 \qquad \textbf{(11)}$$

Then, we use Equation (5) and (11) to obtain an expression in terms of π_R and this yield

$$\sum_{i=-\infty}^{R}\pi_i + \sum_{i=R+1}^{R+Q}\pi_i = \pi_R(1+q+q^2+\cdots) + \pi_R\frac{\mu}{\lambda_T+\lambda_O}\frac{1}{1-q}\sum_{i=1}^{Q}1-q^{i+1}$$

which can further be simplified as

$$\sum_{i=-\infty}^{R}\pi_i + \sum_{i=R+1}^{R+Q}\pi_i = \pi_R\frac{1}{1-q} + \pi_R\frac{\mu}{\lambda_T+\lambda_O}\frac{1}{1-q}\left(Q - q^Q\sum_{i=1}^{Q-1}q^{-i}\right)$$

$$= \pi_R\frac{1}{1-q}(1 + \pi_R\frac{\mu}{\lambda_T+\lambda_O}\left(Q - q^Q\sum_{i=1}^{Q-1}\left(\frac{1}{q}\right)^i\right)$$

$$= \pi_R\frac{1}{1-q}(1 + \frac{\mu}{\lambda_T+\lambda_O}[Q - q\frac{1-q^Q}{1-q}])$$

By using the result that this summation is equal to 1, we obtain an expression for π_R.

$$\pi_R = \frac{1-q}{1+\frac{\mu}{\lambda_T+\lambda_O}[Q-q\frac{1-q^Q}{1-q}]} \qquad \textbf{(12)}$$

Now, Equation (13) can be used to obtain a closed form expression for the steady state performance measures. Accordingly, the probability of losing a customer is given by

$$\pi_L = \sum_{i=R}^{-\infty} \pi_i$$
$$= \sum_{i=0}^{\infty} q^i \pi_R$$
$$= \frac{1}{1+\frac{\mu}{\lambda_T+\lambda_O}[Q-q\frac{1-q^Q}{1-q}]} \quad \textbf{(13)}$$

Similarly, the average inventory level and the average backorder level are obtained by

$$I(c, R, Q) = \sum_{i=0}^{R+Q} i \cdot \pi_i \quad \textbf{(14)}$$
$$B(c, R, Q) = -\sum_{i=-\infty}^{0} i \cdot \pi_i \quad \textbf{(15)}$$

Finally, by using the fact that the arrival process follows a Poisson process and due to the PASTA property, the average lost sales per unit period can be described by

$$L(c, R, Q) = \lambda_O \cdot \pi_L = \sum_{i=-\infty}^{R} \lambda_O \pi_i. \quad \textbf{(16)}$$

Then, for given values of Q, R and c, we obtain annual total cost $C(c, R, Q)$ by using Equations (14)-(16).

2.2.2 Formulation for CASE II: *c>R*

This is the case where we assume that critical point is greater than the reorder point. Considering certain ranges of states within which the same flow balance relation is valid we can write the balance equations as follows.

Balance Equation	**Range**	
$\pi_{R+Q}(\lambda_T + \lambda_O) = \pi_R \mu$	$R + Q$	**(17)**
$\pi_i(\lambda_T + \lambda_O) = \pi_{i+1}(\lambda_T + \lambda_O) + \pi_{i-Q}\mu$	$c + 1 \leq i \leq R + Q - 1$	**(18)**
$\pi_c \lambda_T = \pi_{c+1}(\lambda_T + \lambda_O) + \pi_{c-Q}\mu$	c	**(19)**
$\pi_i \lambda_T = \pi_{i+1}\lambda_T + \pi_{i-Q}\mu$	$R + 1 \leq i \leq c - 1$	**(20)**
$\pi_i(\lambda_T + \mu) = \pi_{i+1}\lambda_T + \pi_{i-Q}\mu$	$-\infty \leq i \leq R$	**(21)**

Equations (17) - (21) are constructed based on the problem definition (see Figure 5). Equation (22) is defined in the light of the study of Nordmann, and Altiok (1998) where *q≥0*. This is same equation constructed in their study for at most one outstanding order for the (*R*, *Q*) inventory policy with backordering.

$$\pi_{R+1-i} = q^i \pi_{R+1} \qquad 0 \leq i \qquad \textbf{(22)}$$

$$\frac{\lambda_T}{\mu} = q \frac{1-q^Q}{1-q} \qquad \textbf{(23)}$$

Note that we can show the validity of Equation (23) by using the following the argument. First, by rewriting Equation (20) for $i = R + 1$, we establish

$$\pi_{R+1} = \frac{\pi_{R+2}(\lambda_T+\mu) - \pi_{R+1-Q}\,\mu}{\lambda_T}.$$

Then, by using this result and Equation (22), we obtain.

$$\pi_{R+1-i} = q^i \frac{\pi_{R+2}(\lambda_T+\mu) - \pi_{R+1-Q}\,\mu}{\lambda_T},$$

which can also be written as

$$\lambda_T\left(\pi_{R+1-i} - q^i \pi_{R+2}\right) = q^i \mu\left(\pi_{R+2} - \pi_{R+1-Q}\right).$$

Finally, by using Equation (22), we obtain

$$\lambda_T\left(q^i \pi_{R+1} - q^i q^1 \pi_{R+1}\right) = q^i \mu (q^1 \pi_{R+1} - q^{1-Q} \pi_{R+1}),$$

which reduces to Equation (23).

Now we use these results to establish a closed form statement for the steady state probabilities. First, from Equation (17) we have

$$\pi_{R+Q} = \frac{\pi_R\,\mu}{\lambda_T+\lambda_O} \quad \textbf{(24)}$$

Similarly, from Equation (18) we have

$$\pi_{R+Q-1}(\lambda_T+\lambda_O) = \pi_{R+Q}(\lambda_T+\lambda_O) + \pi_{R-1}\,\mu \quad \textbf{(25)}$$

Using (24) and (25) gives

$\pi_{R+Q-1}(\lambda_T+\lambda_O) = \frac{\pi_R\,\mu}{\lambda_T+\lambda_O}(\lambda_T+\lambda_O) + \pi_{R-1}\,\mu,$

which simplifies to

$\pi_{R+Q-1} = \frac{\mu}{\lambda_T+\lambda_O}(\pi_R+\pi_{R-1}).$

Using the same idea, we obtain

$\pi_{R+i} = \frac{\mu}{\lambda_T+\lambda_O}\left(\sum_{n=0}^{i}\pi_{R-n}\right)$ for $1 \le i \le Q-1$

which can be then generalized to

$$\pi_{R+i} = \frac{\mu}{\lambda_T+\lambda_O}\left(\sum_{n=0}^{i}\pi_{R-n}\right) \text{ for } 0 \le i \le Q. \quad \textbf{(26)}$$

Now we use equation (22) to obtain

$\pi_{R+i} = \frac{\mu}{\lambda_T+\lambda_O}\left(\sum_{n=0}^{i}q^n\pi_R\right)$ for $0 \le i \le Q$

which is further simplified by

$$\begin{aligned}\pi_i &= \frac{\mu}{\lambda_T+\lambda_O}\left(\sum_{n=i-Q+1}^{R}\pi_n + \pi_{R+Q}\right)\\ &= \frac{\mu}{\lambda_T+\lambda_O}\left[\left(\frac{1-q^{R-R+Q+1-i}}{1-q}q^{R+1}\right)\pi_R\frac{R(R+1)}{2} + \pi_{R+Q}\right]\\ &= \frac{\mu}{\lambda_T+\lambda_O}q\frac{1-q^{Q+1-i}}{1-q}\pi_{R+1};\, c+1 \le i \le R+Q \quad \textbf{(27)}\end{aligned}$$

We use these results to establish a closed form statement for the steady state probabilities from Equation (19) we have

$$\pi_c = \frac{\mu}{\lambda_T}\sum_{n=i-Q}^{R}\pi_n \quad \textbf{(28)}$$

Similarly, from Equation (20) we have

$$\pi_i = \frac{\mu}{\lambda_T} \sum_{n=i-Q}^{R} \pi_n \tag{29}$$

And from Equation (21) we have

$$\pi_i = \frac{\mu}{\lambda_T} \sum_{n=i-Q}^{i-1} \pi_n \; ; \; n \leq R \tag{30}$$

which is further simplified by

$$\begin{aligned} \pi_i &= \frac{\mu}{\lambda_T} \left(\sum_{n=i-Q+1}^{R} \pi_n + \pi_{R+Q} \right); 1 \leq i \leq Q \\ &= \frac{\mu}{\lambda_T} \left[\left(\frac{1-q^{R-R+Q+1-i}}{1-q} q^{R+1} \right) \pi_R \frac{R(R+1)}{2} + \pi_{R+Q} \right]; 1 \leq i \leq Q \\ &= \frac{\mu}{\lambda_T} q \frac{1-q^{Q+1-i}}{1-q} \pi_{R+1}; R+1 \leq i \leq c \end{aligned} \tag{31}$$

To ensure the sum of all probabilities be equal to 1, we have

$$\sum_{i=-\infty}^{R+Q} \pi_i = 1 \tag{32}$$

After obtaining explicit equations for the steady state probabilities, we can easily derive specific performance measures. The average level of inventory and backordering are computed in Equations (33) and (34) respectively.

$$I(c, R, Q) = \sum_{i=0}^{R+Q} i * \pi_i \tag{33}$$

$$B(c, R, Q) = -\sum_{i=-\infty}^{0} i * \pi_i \tag{34}$$

Finally, by using the fact that the arrival process follows a Poisson process and due to the PASTA property, the average lost sales per unit period can be described by

$$L(c, R, Q) = \lambda_O \cdot \pi_L = \sum_{i=-\infty}^{c} \lambda_O \pi_i. \tag{35}$$

Then the annual total cost $C(c, R, Q)$ can be evaluated using Equations (33)-(35).

2.2.3 Formulation for CASE III: *c<R*

This is the case where we assume that critical point is greater than the reorder point. Considering certain ranges of states within which the same flow balance relation is valid we can write the balance equations as follows.

Balance Equation	**Range**	
$\pi_{R+Q}(\lambda_T + \lambda_O) = \pi_R \mu$	$R + Q$	**(36)**
$\pi_i(\lambda_T + \lambda_O) = \pi_{i+1}(\lambda_T + \lambda_O) + \pi_{i-Q}\mu$	$R + 1 \leq i \leq R + Q - 1$	**(37)**
$\pi_i(\lambda_T + \lambda_O + \mu) = \pi_{i+1}(\lambda_T + \lambda_O) + \pi_{i-Q}\mu$	$c + 1 \leq i \leq R$	**(38)**
$\pi_c(\lambda_T + \mu) = \pi_{c+1}(\lambda_T + \lambda_O) + \pi_{c-Q}\mu$	c	**(39)**
$\pi_i(\lambda_T + \mu) = \pi_{i+1}\lambda_T + \pi_{i-Q}\mu$	$-\infty \leq i \leq c - 1$	**(40)**

Equations (36) – (40) are constructed based on the problem definition (see Figure 3). Equation (41) is defined in the light of the study of Nordmann, and Altiok (1998) where $q \geq 0$. This is same equation constructed in their study for at most one outstanding order for the (R, Q) inventory policy with backordering.

$$\pi_{c-i} = q^i \pi_c \qquad 0 \leq i \qquad \textbf{(41)}$$

Equation (42) can be obtained from Equation (38) and Equation (41) or Equation (40) and Equation (41), and the ratio λ_T/μ. There always exists a non-negative q satisfying Equation (42) in Nordmann, and Altiok (1998). The stability condition for this system can be expressed as $\lambda_T/\mu \leq Q$.

$$\frac{\lambda_T}{\mu} = q\,\frac{1-q^Q}{1-q} \qquad \textbf{(42)}$$

Note that we use to establish a closed form statement for the steady state probabilities. First, from Equation (39) we have

$$\pi_c = \mu \sum_{n=c+3}^{c-Q} \pi_n \qquad \textbf{(43)}$$

Similarly, from Equation (40) we have

$$\pi_i = \frac{\mu\, \lambda_T \sum_{n=i+2}^{i-Q} \pi_n + \pi_{i+1-Q\mu^2}}{(\lambda_T+\mu)^2} \quad \textbf{(44)}$$

Using (42), (43) and (44) gives

$$\pi_i = \frac{\mu}{\lambda_T+\lambda_O}\left(\sum_{n=i-Q+1}^{R} \pi_n + \pi_{R+Q}\right); c+1 \le i \le R+Q$$

$$= \frac{\mu}{\lambda_T+\lambda_O}\left[\left(\frac{1-q^{R-R+Q+1-i}}{1-q} q^{R+1}\right)\pi_R \frac{R(R+1)}{2} + \pi_{R+Q}\right]; c+1 \le i \le R+Q$$

which simplifies to

$$\pi_i = \frac{\mu}{\lambda_T+\lambda_O} q \frac{1-q^{c+Q-i}}{1-q} \pi_{R+1}, c+1 \le i \le R+Q \quad \textbf{(45)}$$

Substituting Equation (43) and Equation (44) into Equation (45) we obtain also

$$\pi_c = \frac{\lambda_T}{\mu} q \frac{1-q^Q}{1-q} \quad \textbf{(46)}$$

Equation (47) is constructed to ensure the sum of all probabilities be equal to 1.

$$\sum_{i=-\infty}^{R+Q} \pi_i = 1 \quad \textbf{(47)}$$

After obtaining explicit equations for the steady state probabilities, we can easily derive specific performance measures. Average inventory level and average backorder level are obtained using Equations (48) and (49) respectively.

$$I(c,R,Q) = \sum_{i=0}^{R+Q} i * \pi_i \quad \textbf{(48)}$$

$$B(c,R,Q) = -\sum_{i=-\infty}^{0} i * \pi_i \quad \textbf{(49)}$$

Finally, by using the fact that the arrival process follows a Poisson process and due to the PASTA property, the average lost sales per unit period can be described by

$$L(c,R,Q) = \lambda_O * \pi_L = \sum_{i=-\infty}^{c} \lambda_O * \pi_i \quad \textbf{(50)}$$

Similar to the previous cases, total cost $C(c, R, Q)$ can be computed using Equations (48)-(50). Calculations for a selected data set using the formulation prevented here and the numerical results are given in the following chapter.

CHAPTER III

SOLUTION AND NUMERICAL RESULTS

When we search through the literature, we observe that the inventory problems with backordering and/or lost sales are generally solved by developing an optimal solution algorithm. In our study, we first formulate the problem as a continuous time Markov chain for three cases which are $c=R$, $c>R$ and $c<R$. Then, we implement the model to obtain steady state probabilities and average cost based on company's selected problem instances for 10 items as well as artificially generated data. In this study, we do not consider the exact solution of the optimization problem. Instead, we formulate and run an algorithm in MATLAB. In this algorithm, order quantity Q is computed as an upper bound for the economic order quantity (EOQ value) for each solution as it is commonly observed in practice. Therefore, we will enumerate reorder level (R) and critical level (c) to find the optimal R and c values. EOQ value is computed as given by Equation (51) below. The demand quantity in EOQ formula is computed by summing the demand rate for high priority customer and the demand rate for low priority customers.

$$EOQ = \sqrt{\frac{2*K*(\lambda_T+\lambda_O)}{h}} \quad \textbf{(51)}$$

3.1 EXPERIMENTAL DESIGN

In our experimentation we first generate some sample problems by setting some values for the parameters K, h, γ_b, γ_l, λ_T, λ_O and μ to compute the best values of R and c. These are base samples that are used to check whether the formulations and the algorithm are correct and to observe the behavior of the system. The number of these test problems, considering λ_T and λ_O as a pair, is $3^6 = 729$. We solved these 729 test problems and 10 real problems using MATLAB on a personal computer with Intel (R) Core (TM) i7-260QM CPU @ 2.20 GHz 2.20 GHz and 4 GB RAM under Windows XP 64 bit operating system. It was observed that the performance of the algorithm is good in terms of the solution time.

We consider a solution algorithm to search for the optimal solution for the best value of annual total cost (C_{best}), optimal value of reorder level (R^{opt}), optimal value of ordering quantity (Q^{opt}) and optimal value of critical level (c^{opt}) by enumerating over a range of values of the decision variables c, R, and Q. The search space is truncated at maximum ordering quantity (Q_{max}) and maximum reorder level (R_{max}). The outline of the algorithm can be summarized as follows.

Initialize the values for K, h, γ_b, γ_l, λ_T, λ_O and μ

$For\ Q = EOQ : Q_{max}$

    $For\ R = 0 : R_{max}$

        $For\ c = 0 : R_{max}$

            If $c<R$

                Calculate $C(c, R, Q)$ by using the results for Case III.

                If $C(c, R, Q) < C_{best}$

                    Update $C_{best} = C(c, R, Q)$, $R^{opt} = R$, $Q^{opt} = Q$ and c^{opt}=c,

            Otherwise do nothing and continue to search

            Else if $c=R$

                Calculate $C(c, R, Q)$ by using the results for Case I.

                If $C(c, R, Q) < C_{best}$

Update $C_{best} = C(c, R, Q)$, $R^{opt} = R$, $Q^{opt} = Q$ and c^{opt}=c,

Otherwise do nothing and continue to search

Else if c>R

Calculate $C(c, R, Q)$ by using the results for Case II.

If $C(c, R, Q) < C_{best}$

Update $C_{best} = C(c, R, Q)$, $R^{opt} = R$, $Q^{opt} = Q$ and c^{opt}=c,

Otherwise do nothing and continue to search

End

End

End

End

3.2 SOLUTIONS AND NUMERICAL RESULTS

Here we first obtained approximate data for 10 problem instances that the company supplies, and defined 10 problems using this data. Then, using this real or approximate data related to some of company's products, 10 problems were defined and solved. The input data for these problems is shown in Table 2 in decreasing λ_T values. In these real problems values of K, γ_b, and γ_l are assumed to be fixed in accordance with the real situation. Fixed ordering cost of each problem instance is the same because these problem instances are supplied from the same company and there is not any variety for each of them. The holding cost of each problem instance is changeable because of their volume in the warehouse. Based on these data the rationing policy and annual total cost to the company are determined.

Table 2 Input Data Set

	PROBLEM INSTANCE									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
K	2	2	2	2	2	2	2	2	2	2
h	0.3	0.7	1.2	0.2	2.5	0.5	1	2.3	2.1	2.3
γ_b	1	1	10	1	10	1	1	1	1	1
γ_l	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
μ	10	5	5	5	4	3	3	4	3	3
λ_T	50	25	25	20	20	10	10	7	5	5
λ_O	10	5	5	10	3	5	5	2	2	2

Using this input data set for problems, the steady state probabilities (π_i) are computed first. Then using these probabilities, the probabilities of reordering, backordering, and lost sales (π_R, π_B, π_L), average backorders $B(c, R, Q)$ and average inventory level $I(c, R, Q)$ are obtained. The results are summarized in Table 3. When the replenishment service rate (μ) is increased, the lost sales probability is decreased, and vice versa, as expected.

Table 3 Numerical Results

	PROBLEM INSTANCE									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
π_R	$2.55*10^{-13}$	$7.63*10^{-07}$	0.0001	$1.40*10^{-13}$	0.0034	$2.20*10^{-07}$	$2.00*10^{-05}$	0.0002	0.0001	0.0001
π_B	$6.25*10^{-13}$	$3.07*10^{-06}$	0.0004	$1.4*10^{-13}$	0.0313	$3.90*10^{-07}$	0.0001	0.0002	0.0001	0.0001
π_L	0.2382	0.4665	0.6193	0.2725	0.8030	0.4874	0.6197	0.6856	0.6898	0.6898
***B* (*c*, *R*, *Q*)**	$4.50*10^{-12}$	$2.00*10^{-05}$	0.0036	0.0000	0.0313	$2.20*10^{-06}$	0.0003	0.0011	0.0006	0.0006
$\bar{I}$ **(*c*, *R*, *Q*)**	153.42	70.45	47.42	131.90	27.21	55.62	38.77	18.99	19.09	19.09

In the light of Table 3, when the holding cost is increased while the other values are fixed, average backorder level is increased and average inventory level and steady state probabilities are decreased, as observed for problem instances 8 and 9. When holding cost is increased from 0.70 (problem instance 2) to 1.20 (problem instance 3), average backorder increases from 0.00000307 (problem instance 2) to 0.0004 (problem instance 3) and average backorder decreases from 70.45 (problem instance 2) to 47.42.

Another observation in Table 3 is that the lost sales probability is always higher than the backordering probability as can be seen in Figure 8. This results from the selected values of demand rates for both customer types and replenishment rate. Also, considering the input data set of the test problems (Table 2), the effect of the changes in the replenishment rate (μ) can be seen. When μ value is increased, the lost sale and backorder probabilities are decreased as expected.

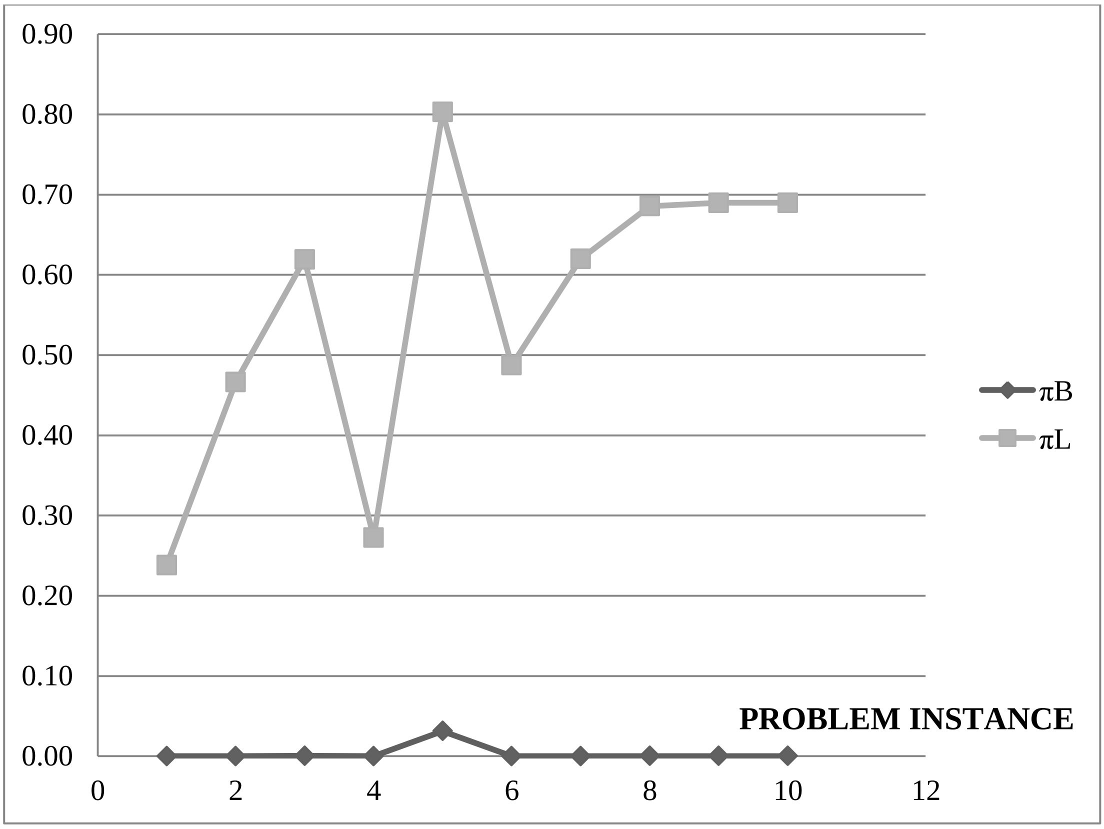

Figure 8 Backordering and Lost Sale Probabilities of Problems

Using the data of the test problems, *EOQ* are computed. Ordering quantities (Q) are obtained rounding the *EOQ* values and Q_{max} is computed as 29. R and c values are also computed for all problems and the results for the problems are given in Table 4. As seen in this table some rationing situations are observed in the case $c<R$ for problem instances 1, 4, 5, and 6, whereas for the other problem instances critical level and the reorder point are the same ($c=R$). However, critical level is obtained as $c=1$ for all problem instances. The critical level helps avoiding backordering for high priority customer, and also yields supply of all customers' demands until inventory level reaches "1". As seen in Table 4 R_{max} is computed as 16.

Table 4 Solution Results (Order Quantity, Reorder and Critical Levels)

	PROBLEM INSTANCE									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
EOQ	28.28	13.09	10.00	24.49	6.07	10.95	7.75	3.96	3.65	3.49
Q	29	14	10	25	7	11	8	4	4	4
R	3	1	1	5	16	2	1	1	1	1
c	1	1	1	1	1	1	1	1	1	1

During the computations, minimum annual total costs for problems are in Figure 9. In these problems it is seen that when the holding cost increases, keeping the other costs fixed, the annual total cost also increases. For example for problem instances 6 and 7, holding costs are 0.5 and 1.0, where all the other data is the same, and the minimum annual total costs are obtained as 6.78 and 10.37 respectively. Similar situation can be observed for problem instances 2 and 3, for which holding costs are 0.70 and 1.20, and annual total costs are computed as 8.29 and 9.38.

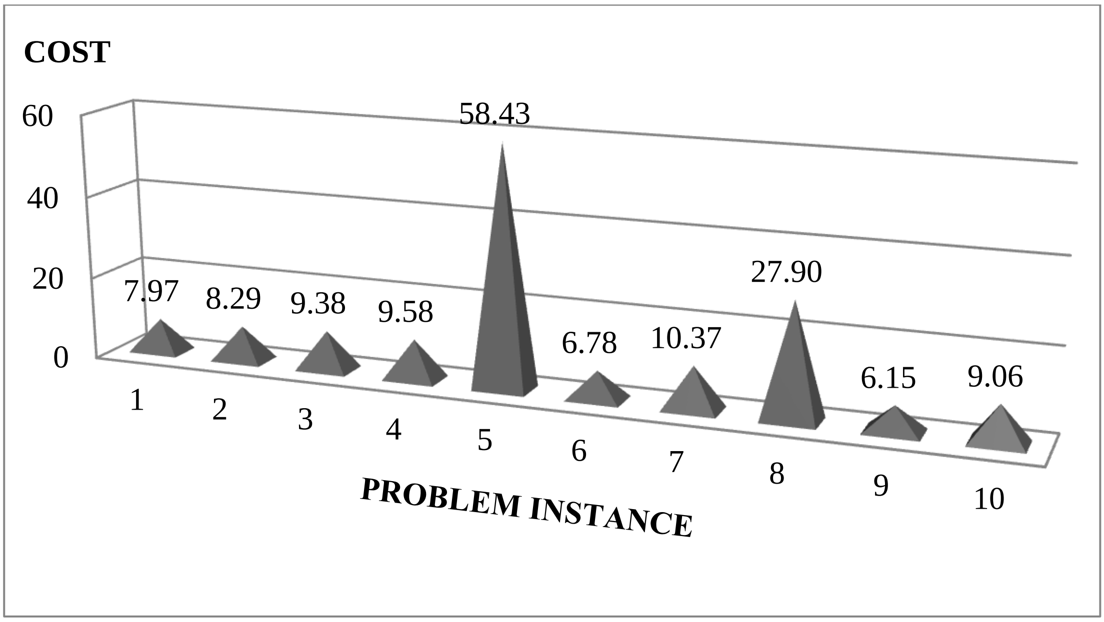

Figure 9 Minimum Annual Total Costs of Problem Instance

While solving these problem instances in Matlab, we searched its advantage instead of the other software packages. We found the solutions are computed in very small time. Mostly problem instances are solved in 100 seconds which is shown in Figure 10.

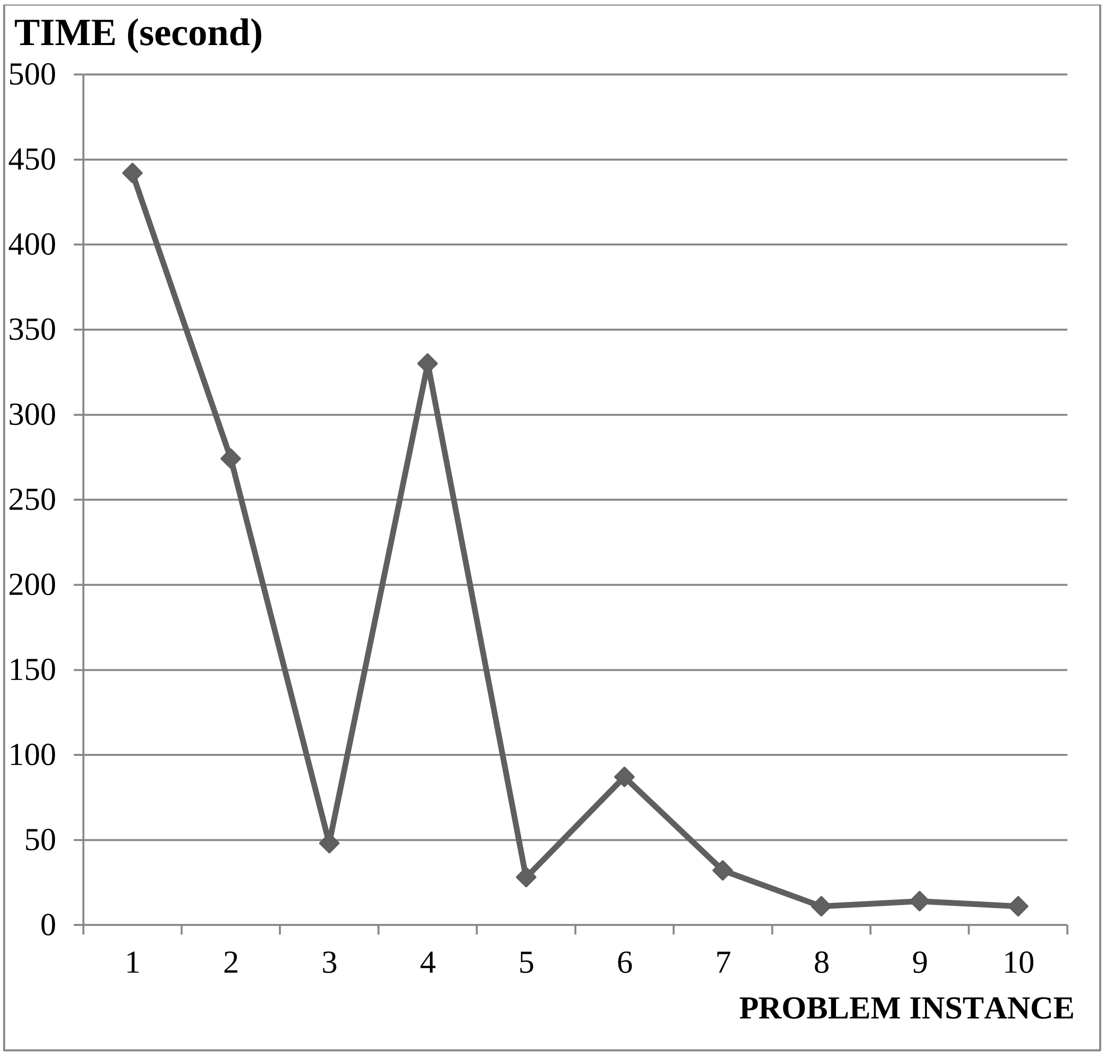

Figure 10 Computational Time in Matlab

CONCLUSION

In this thesis, we present an analysis of a continuous review inventory system. In particular we consider a manufacturing company that supplies machines, equipment and spare parts to its customers. We consider two types of customers having different priorities. High priority customer is another company the demands of whom should be satisfied as long as there is available inventory. All other customers are treated as low priority customers and their demands may be rejected at certain inventory levels.

In this system both sales and backordering are allowed for high priority customer and low priority customers, respectively. It is assumed that the demands of high priority customer are backordered when there is no inventory available, whereas the demands of low priority customers are lost below a critical level. The system is modeled as a continuous time Markov Chain with Poisson demand rates for the arrival of two types of customers and exponential replenishment times for supplying machines, spare parts and equipment.

First, a comprehensive literature survey is done and the articles related to our study are reviewed. Then formulation of the problem is formulated based on three different cases that defined according to the relations between critical level (c) and reorder level (R), which are defined as $c<R$, $c=R$, and $c>R$. While this literature survey, we found a best place. Lost sales and backordering studied in literature separately; however, we studied both of them for each case and solved three cases together for computing the best level of each problem. Finally, test problems are generated and also some problem instance is used to verify the model and to analyze the critical level policy under the lost sales for the low priority customers and the backordering for the high priority customers, with the objective of minimizing the annual total cost and finding the best values of the reorder point (R) and the order quantity (Q).

To the best of our knowledge, this study is the first in inventory management that involves lost sales and backorders together and considers three possible cases depending on the relations between critical level and reorder level.

In most problems we observe that a critical level policy may not be so crucial. However, depending on the input data some problems yields critical levels. Therefore, as a future work an extensive experimentation may be done by generating new problems and increasing the number of problems allowing a number of different values for each input parameters.

As an extension, one may try to obtain closed form solutions analytically; allowing backorders and/or lost sales, for each case and may develop an improved solution method instead of the search algorithm presented here. Also increasing the number of customer classes, with changing penalties for lost sales and backorders might be another consideration.

We believe that analysis of such systems are useful in developing the inventory management policies, and making decisions on when to order and how much to order based on the product and customer related characteristics such as demand rate and replenishment rate as well as ordering, holding, backordering, lost sale costs. These solutions are more important for each company; therefore, the manufacturing company can give the order with computed ordering quantity to avoid lost sales and backorder. This type of problems can be solved with developing different models and methods in the correct time.

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Appendix A: Curriculum Vitae

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Özen, Duygu

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Degree	Institution	Year of Graduation
BS	Çankaya University Industrial Engineering	2009
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WORK EXPERIENCE

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2011–Present	SFS Teksomak Ltd. Şti.	International Sales and Marketing Manager
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2007 – 2008	Çankaya University	Assistant Student
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FOREIGN LANGUAGES

Advanced English, Elementary Spanish