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MASTER THESIS

THE DISCRETE LAPLACE TRANSFORM

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ABSTRACT

The Discrete Laplace Transform

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Integral transforms have not only been effective methods to solve ordinary and partial differential equations, but helped in the development to the theory of these equations as well. The Laplace transform method is considered one of the strongest among them. The question arises here whether there is a discrete analogue of the Laplace transform which can be used to solve difference equations.

In this thesis, using the definition of the Laplace Transform on an arbitrary time scale, the Discrete Laplace Transform is introduced. The main theorems related to this transform are mentioned. The Discrete Laplace Transform of elementary discrete functions are developed and some applications are given.

Keywords: Time Scale, Discrete Laplace Transform, Convolution, Exponential Order.

ÖZ

Ayrık Laplace Dönüşümü

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İntegral dönüşümleri, adi ve kısmi diferansiyel denklemleri çözmek için etkili bir yöntem olmakla kalmayıp, aynı zamanda bu denklemler teorisinin gelişimine de yardımcı olmuştur. Laplace dönüşümü yöntemi aralarında en güçlülerinden biri olarak kabul edilir. Bu da fark denklemlerini çözmek için Laplace dönüşümünün ayrık bir analogu olup olmadığı sorusunu ortaya çıkarır.

Bu tezde, keyfi bir zaman skalası uzerindeki Laplace Dönüşümünun tanımını kullanarak, ayrık Laplace dönüşümü tanıtılmıstır. Bu dönüşüm ile ilgili temel teoremler belirtilmiştir. Temel ayrik fonksiyonların Laplace dönüşümleri geliştirilmiş ve bazı uygulamalar verilmiştir.

Anahtar kelimeler: Zaman skalasi, Ayrık Laplace dönusumü, Konvolüsyon, Üstel Basamaklı.

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CHAPTER 1

INTRODUCTION

Integral transforms play an important role in the development of the theory of initial and boundary value problems for linear differential and integral equations that arise in applied mathematics, mathematical physics and engineering, because they provide strong methods for solving such equations [1].

The Laplace transform, one of the strongest integral transforms, has been regarded as a powerful tool for solving differential equation, both ordinary and partial.

The Laplace transform was, firstly, used in the work of Euler who utilized the inverse Laplace transform for solving a second order linear differential equation in 1763. Spitzer attached the transform:

$$y = \int_a^b e^{sx} \emptyset(s) ds,$$

which was used by Euler, the name of Laplace in 1878.

Poincare and Pincherle extended the Laplace transform to its complex form by the end of the nineteenth century. [2]

The modern Laplace transform was firstly used by Bateman in 1910 and Bemotein in 1920. In 1920's Doctch applied the Laplace transform to differential, integral and integro-differential equation giving this transform a more modern approach [3].

Analogous to the Laplace transforms applied to continuous linear systems of differential equations, the Z-transform is applied to solve linear systems of difference equations. This transform was known to Laplace, but developed by Huriwicz in 1947 [4].

In [5], Bohner, M., Guseinov used definition of the Laplace transform, on an arbitrary time scale, in order to specify the h-Laplace and consequently the discrete Laplace transform.

We have to mention that Stefan Hilger initiated the theory of the time scales which is a tool to unify continuous and discrete models [6,7].

In [8], the fractional discrete Laplace transform in order to be applied to difference equations with fractional order. But the applicability of this transform to difference equations was not discussed properly.

To our knowledge, on body has attempted to go further and develop the theory of discrete Laplace transform in order to make it applicable to linear systems of difference equations.

We think that this transform will be an alternative to the Z-transform.

In chapter 2, of this thesis, basic concepts of time scales are introduced.

In chapter 3, the discrete Laplace transform is defined and its basic properties are introduced.

In chapter 4, the discrete Laplace transform of basic functions are discussed.

In chapter 5, some applications are given.

Chapter 6 is devoted to the conclusion.

CHAPTER 2

THE TIME SCALES

2.1 Preliminaries on Time Scales

Definition 2.1.1. [6] A time scale set $\mathbb{T}$ is a non empty closed subset of $\mathbb{R}$.

Definition 2.1.2. [6] The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T}: s > t\}, \forall\, t \in \mathbb{T}, \tag{2.1.1}$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t)(t) = \sup\{s \in \mathbb{T}: s < t\}, \forall\, t \in \mathbb{T}, \tag{2.1.2}$$

where

$$\inf \emptyset = \sup \mathbb{T} \text{ and } \sup \emptyset = \inf \mathbb{T}.$$

Definition 2.1.3. [6] A point $t \in \mathbb{T}$ is called

a) right-dense, if $\sigma(t) = t$,
b) left-dense, if $\rho(t) = t$,
c) right-scattered, if $\sigma(t) > t$,
d) left-scattered, if $\rho(t) < t$.

Definition 2.1.4. [6] The graininess function $\mu: \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t. \tag{2.1.3}$$

Definition 2.1.5. [6] The set $\mathbb{T}^k$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise it is $\mathbb{T}$ without this left-scattered maximum.

Definition 2.1.6. [6] We say that a function $f: \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^k$ if there is a number $f^{\Delta}(t)$ with the property that $\forall\, \varepsilon > 0$ there is a neighborhood

$$U = (t - \delta, t + \delta) \cap \mathbb{T} \text{ of } t \text{ for some } \delta > 0$$

such that

$$\left|\left[f\big(\sigma(t)\big) - f(s)\right] - f^{\Delta}(t)[\sigma(t) - s]\right| \le \varepsilon|\sigma(t) - s|, \forall\, s \in U \tag{2.1.4}$$

The number $f^{\Delta}(t)$ is called the delta derivative of f on $\mathbb{T}^k$. We have the following theorem:

Theorem 2.1.7. [6] Assume that $f\colon \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$.

Then we have:

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f\big(\sigma(t)\big) - f(t)}{\mu(t)} \tag{2.1.5}$$

(iii)In this case If t is right-dense, then f is differentiable at t iff the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s} \tag{2.1.6}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{2.1.7}$$

(iv)If f is differentiable at t, then

$$f\big(\sigma(t)\big) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.1.8}$$

Property 2.1.8. [6] If f is a delta differentiable function then :

a) $f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t),$ (2.1.9)

b) $(fg)^{\Delta}(t) = f^{\Delta}(t) g^{\sigma}(t) + f(t) g^{\Delta}(t),$ (2.1.10)

where $f^{\sigma}(t) = f\big(\sigma(t)\big)$.

Definition 2.1.9. [6] A function $f\colon \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right-dense and its left-sided limit exists at left-dense points $t \in \mathbb{T}$. The set of all rd-continuous functions is denoted by C_{rd} and the set of all differentiable functions with rd-continuous derivative is denoted by C_{rd}^{1} .

Theorem 2.1.10. [6] If $f \in C_{rd}$, then f possesses an antiderivative. That is there exists a function F with $F^{\Delta} = f$, and in this case an integral is defined by

$$\int_s^t f(\tau)\,\Delta\tau = F(t) - F(s). \tag{2.1.11}$$

2.2 The Exponential Functions on Time Scales

Let $\mathbb{T}$ be a time scale with σ as the forward jump operator and Δ as the delta differentiation operator.

Definition 2.2.1. [6] A function $p: \mathbb{T} \to \mathbb{C}$ is called regressive if $1 + \mu(t)p(t) \neq 0$, $\forall\, t \in \mathbb{T}$.

The set R of all regressive and rd-continuous function forms an abelian group under the addition $\oplus$ defined by

$$(p + q)(t) = p(t) + q(t) + \mu(t)p(t)q(t), \forall\, t \in \mathbb{T}. \tag{2.2.1}$$

The additive inverse of $p \in R$, denoted by $\ominus\, p$ is

$$(\ominus\, p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)} \tag{2.2.2}$$

Theorem 2.2.2. [6] Let $p \in R$ and $t_0 \in \mathbb{T}$. Then the I.V.P

$$y^{\Delta} = p(t)y, \qquad y(t_0) = 1, \tag{2.2.3}$$

has a unique solution on $\mathbb{T}$.

The solution of the I.V.P. (2.2.3) is called the exponential function and is denoted by

$$e_p(.,t_0). \tag{2.2.4}$$

Below are some properties of the exponential functions on time scales.

Theorem 2.2.3. [6] If p and q are regressive, then

a) $e_0(t,s) \equiv 1 \; and \; e_p(t,t) \equiv 1$;
b) $e_p(\sigma(t),s) = \big(1 + \mu(t)p(t)\big)e_p(t,s)$;
c) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

d) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

e) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

f) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

g) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;

h) $\left(\frac{1}{e_p(.,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(.,s)}$.

CHAPTER 3

The Laplace Transform

In [5], the Laplace transform on a general time scale was defined by

$$\mathcal{L}\{y(t)\}(s) = \int_{t_0}^{\infty} y(t)e_{\ominus s}(\sigma(t), t_0)\Delta t, \qquad for\ s \in \mathcal{D}\{y\}, \tag{3.1}$$

for any function $y : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$.

Now to find the Discrete Laplace Transform we have first to define the discrete exponential function.

If we consider the time scale

$$\mathbb{T} = \mathbb{N} \cup \{0\} \text{ and } t_0 = 0,$$

Then we have

$$\sigma(t) = t + 1 \text{ and } \mu(t) = 1.$$

For a function $f : \mathbb{T} \to \mathbb{C}$ we have

$$f^{\Delta}(t) = f(t+1) - f(t) \quad \text{for all } t \in \mathbb{T}.$$

Therefore for any complex number s, the initial value problem

$$y^{\Delta} = sy, \qquad t \in \mathbb{T}, \qquad y(0) = 1 \tag{3.2}$$

takes the form

$$y(t+1) = (1+s)y(t), \qquad t \in \mathbb{T}, \qquad y(0) = 1. \tag{3.3}$$

Hence $e_s(t, 0)$ has (for $s \neq -1$) reads

$$e_s(t, t_0) = (1+s)^t \quad \text{for all } t \in \mathbb{T}.$$

Next, we have

$$\begin{aligned} \ominus s &= \frac{-s}{1 + \mu(t)s} \\ &= \frac{-s}{1+s} \end{aligned} \tag{3.4}$$

so that the initial value problem

$$y^{\Delta} = \ominus s\, y, \qquad t \in \mathbb{T}, \qquad y(0) = 1 \tag{3.5}$$

takes the form

$$y\,(t\,+1)\;=\frac{1}{1+s}y(t),\qquad t\in\mathbb{T},\qquad y(t_0)\;=\;1. \tag{3.6}$$

Hence $e_{\ominus s}(t,0)$ has the form

$$e_{\ominus s}(t,0)=\;(1\;+s)^{-t}\;\text{ for all } t\in\mathbb{T}.$$

Consequently, for any function $y:\mathbb{N}_0\to\mathbb{C}$, its Laplace transform $Y(s)$ has, according to (3.1) the form

$$\begin{aligned} Y(s)=\;\mathcal{L}\{y\}(s)&=\sum_{t\in\mathbb{N}_0}y(t)(1+s)^{-(t+1)}\\ &=\;\sum_{k=0}^{\infty}\frac{y(k)}{(1+s)^{k+1}}, \end{aligned} \tag{3.7}$$

Thus the Discrete Laplace Transform can be defined as the following

Definition 3.1. The "$\mathcal{L}_d$-transform" of a sequence $\{y_k\}_{k=0}^{\infty}$ is a function $Y(s)$ of a complex variable defined by

$$Y(s)=\mathcal{L}_d\{y_k\}=\sum_{k=0}^{\infty}\frac{y_k}{(s+1)^{k+1}}, \tag{3.8}$$

for all values of s where the series converges.

and we say that the $\mathcal{L}_d$-transform "exists" provided there is a number $N>0$ such that :

$$\sum_{k=0}^{\infty}\frac{y_k}{(s+1)^{k+1}}\text{ converges for } |s+1|>N. \tag{3.9}$$

If $R=\limsup|y_k|^{\frac{1}{k}}$, then one of the following cases holds:

(i) If $0<R<\infty$, the series (3.8) converges for $|s+1|>R$ and diverges otherwise;

(ii) If $R=0$, the series (3.8) converges for all values of s except possibly for $s=-1$;

(iii) If $R=0$, the series (3.8) diverges everywhere.

Definition 3.2. The sequence $\{y_k\}_{k=0}^{\infty}$ is said to be "exponentially bounded" if there is an $M>0$ and a $c>1$ such that

$$|y_k|\le Mc^k\quad\text{for } k\ge 0.$$

Theorem 3.3. If the sequence $\{y_k\}$ is exponentially bounded then the $\mathcal{L}_d$-transform of $\{y_k\}$ exists.

Proof. Assume that the sequence $\{y_k\}$ is exponentially bounded. Then there is an $M > 0$ and a $c > 1$ such that

$$|\mathrm{y_k}| \leq \mathrm{Mc^k} \quad \text{for } \mathrm{k} \geq 0.$$

We have :

$$\sum_{k=0}^{\infty}\left|\frac{y_k}{(s+1)^{k+1}}\right| \leq \sum_{k=0}^{\infty}\frac{|y_k|}{|s+1|^{k+1}} \leq \frac{M}{s+1}\sum_{k=0}^{\infty}\left|\frac{c}{(s+1)}\right|^k$$

and the last series is a geometric series that converges when $|s+1| > c$.

It follows the $\mathcal{L}_d$-transform of the sequence $\{y_k\}$ exists.

Lemma 3.4. If $r > R,$ the series (3.8) is uniformly convergent for values of s where $|s+1| \geq r.$

Proof. Because $r > R,$ there exists $\varepsilon > 0$ such that

$r > R + \varepsilon$ and since $=$ lim sup$|y_k|^{\frac{1}{k}}$, for the same ε, there exists a natural number and such that

$$|y_k| \leq (R+\varepsilon)^k, \text{ for all } k \geq n,$$

Now,

$$\left|\sum_{k=n}^{\infty}\frac{y_k}{(s+1)^{k+1}}\right| \leq \sum_{k=n}^{\infty}|y_k|\left|\frac{1}{s+1}\right|^{k+1} \leq \sum_{k=n}^{\infty}\left(\frac{R+\varepsilon}{r}\right)^{k+1}.$$

Therefore,

$$\left|\sum_{k=n}^{\infty}\frac{y_k}{(s+1)^{k+1}}\right| \leq \frac{r}{r-R-\varepsilon}\left(\frac{R+\varepsilon}{r}\right)^{n+1} \to 0 \; as \; n \to \infty.$$

Below we mention the basic theorems related to the Discrete Laplace Transform

Theorem 3.5. (Linearity Property)

If a and b are constant, then

$$\mathcal{L}_d\{af_k + bg_k\} = a\mathcal{L}_d\{f_k\} + b\mathcal{L}_d\{g_k\}. \tag{3.10}$$

Proof.

$$\mathcal{L}_d\{af_k + bg_k\} = \sum_{k=0}^{\infty} \frac{af_k + bg_k}{(s+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{af_k}{(s+1)^{k+1}} + \sum_{k=0}^{\infty} \frac{bg_k}{(s+1)^{k+1}}$$

$$= a\sum_{k=0}^{\infty} \frac{f_k}{(s+1)^{k+1}} + b\sum_{k=0}^{\infty} \frac{g_k}{(s+1)^{k+1}}$$

Therefore, we have

$$\mathcal{L}_d\{af_k + bg_k\} = a\mathcal{L}_d\{f_k\} + b\mathcal{L}_d\{g_k\}$$

Theorem 3.6. (Discrete Laplace Transform of a shifted sequence)

Let $\{y_k\}_{k=0}^{\infty}$ be a sequence such that its Discrete Laplace Transform exists for $|s+1| > R$ and n a positive integer, then for $|s+1| > R$, the following holds:

$$\mathcal{L}_d\{y_{k+n}\} = (s+1)^n\mathcal{L}_d\{y_k\} - \sum_{m=0}^{n-1} y_m(s+1)^{n-m-1} \tag{3.11}$$

Proof. First observe that :

$$\mathcal{L}_d\{y_{k+n}\} = \sum_{k=0}^{\infty} \frac{y_{k+n}}{(s+1)^{k+1}}$$

$$= \frac{1}{s+1}\sum_{k=0}^{\infty} (s+1)^{-k} y_{k+n}$$

$$= \frac{1}{s+1}\sum_{k=n}^{\infty} (s+1)^{-k+n} y_k$$

$$= \frac{(s+1)^n}{s+1}\left[\sum_{k=0}^{\infty}(s+1)^{-k}y_k - \sum_{m=0}^{n-1}(s+1)^{-m}y_m\right]$$

$$= (s+1)^n\left[\sum_{k=0}^{\infty}\frac{y_k}{(s+1)^{k+1}} - \sum_{m=0}^{n-1}\frac{y_m}{(s+1)^{m+1}}\right]$$

Therefore, we have

$$\mathcal{L}_d\{y_{k+n}\} = (s+1)^n\mathcal{L}_d\{y_k\} - \sum_{m=0}^{n-1} y_m(s+1)^{n-m-1}.$$

Theorem 3.7. (Initial Value and Final Value Theorem)

a) If $Y(s) = \mathcal{L}_d\{y_k\}$ exists for $|s+1| > R$, then

$$\lim_{s\to\infty}(s+1)\,Y(s) = y_0\,, \lim_{s\to\infty} Y(s) = 0 \tag{3.12}$$

b) If $Y(s)$ exists for $|s+1| > 1$ and $sY(s)$ is analytic at $s = 1$, then

$$\lim_{n\to\infty} y_n = \lim_{s\to 0} s\mathcal{L}_d\{y_k\} \tag{3.13}$$

proof.

part (a): Since

$$\mathcal{L}_d\{y_k\} = \sum_{k=0}^{\infty}\frac{y_k}{(s+1)^{k+1}},$$

we have

$$(s+1)\mathcal{L}_d\{y_k\} = \sum_{k=0}^{\infty}\frac{y_k}{(s+1)^k}$$

$$= y_0 + \frac{y_1}{s+1} + \frac{y_2}{(s+1)^2} + \cdots$$

Thus,

$$\lim_{s\to\infty}(s+1)L_d\{y_k\} = y_0$$

$$\lim_{s\to\infty}(s+1)\,Y(s) = y_0.$$

Consequently,

$$\lim_{s\to\infty} Y(s) = \lim_{s\to\infty} \frac{y_0}{s+1} = 0.$$

Part (b): Since,

$$\mathcal{L}_d\{y_{k+1} - y_k\} = \sum_{k=0}^{\infty} \frac{y_{k+1}}{(s+1)^{k+1}} - \sum_{k=0}^{\infty} \frac{y_k}{(s+1)^{k+1}}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{y_{k+1}}{(s+1)^{k+1}} - \frac{y_k}{(s+1)^{k+1}} \right)$$

$$= \lim_{n\to\infty} \left[-\frac{y_0}{s+1} + \left(\frac{1}{s+1} - \frac{1}{(s+1)^2} \right) y_1 + \right.$$

$$\left. \left(\frac{1}{(s+1)^2} - \frac{1}{(s+1)^3} \right) y_2 + \cdots + \left(\frac{1}{(s+1)^n} - \frac{1}{(s+1)^{n+1}} \right) y_n + \frac{y_{n+1}}{(s+1)^{n+1}} \right]$$

we have,

$$\lim_{s\to 0} \mathcal{L}_d\{y_{k+1} - y_k\} = \lim_{s\to 0} \lim_{n\to\infty} \left[-\frac{y_0}{s+1} + \left(\frac{1}{s+1} - \frac{1}{(s+1)^2} \right) y_1 + \right.$$

$$\left. \left(\frac{1}{(s+1)^2} - \frac{1}{(s+1)^3} \right) y_2 + \cdots + \left(\frac{1}{(s+1)^n} - \frac{1}{(s+1)^{n+1}} \right) y_n + \frac{y_{n+1}}{(s+1)^{n+1}} \right].$$

Thus,

$$\lim_{s\to 0} \mathcal{L}_d\{y_{k+1} - y_k\} = \lim_{n\to\infty} \lim_{s\to 0} [-y_0 + y_{n+1}]$$

$$\lim_{s\to 0} \mathcal{L}_d\{y_{k+1} - y_k\} = \lim_{n\to\infty} [-y_0 + y_{n+1}]$$

$$\lim_{s\to 0} \mathcal{L}_d\{y_{k+1} - y_k\} = \left(\lim_{n\to\infty} y_{n+1} \right) - y_0.$$

Hence,

$$\lim_{s\to 0} [\mathcal{L}_d\{y_{k+1}\} - \mathcal{L}_d\{y_k\}] = \left(\lim_{n\to\infty} y_{n+1} \right) - y_0.$$

On the other side, using Theorem 3.6., we have

$$\lim_{s\to 0}[(s+1)\mathcal{L}_d\{y_k\} - y_0 - \mathcal{L}_d\{y_k\}] = \left(\lim_{n\to\infty} y_{n+1}\right) - y_0$$

$$\lim_{s\to 0} s\mathcal{L}_d\{y_k\} - y_0 = \left(\lim_{n\to\infty} y_{n+1}\right) - y_0.$$

Hence,

$$\lim_{n\to\infty} y_{n+1} = \lim_{s\to 0} s\mathcal{L}_d\{y_k\}$$

Definition 3.8. [9] Let $\{f_k\}_{k=0}^{\infty}$ and $\{g_k\}_{k=0}^{\infty}$ be two sequences. Then the convolution of f_k and g_k is defined by

$$f_k * g_k = \sum_{m=0}^{k} f_{k-m} g_m. \tag{3.14}$$

The following theorem presents the Discrete Laplace Transform of the convolution of two sequences.

Theorem 3.9. (Convolution Theorem)

If $F(s)$ exists for $|s+1| > a$ and $G(s)$ exists for $|s+1| > b$, then:

$$\mathcal{L}_d\{f_k * g_k\} = (s+1)\mathcal{L}_d\{f_k\}\mathcal{L}_d\{g_k\}, \tag{3.15}$$

for $|s+1| > max\{a, b\}$.

Proof.

$$\mathcal{L}_d\{f_k\}.\mathcal{L}_d\{g_k\} = \sum_{k=0}^{\infty} \frac{f_k}{(s+1)^{k+1}} \sum_{k=0}^{\infty} \frac{g_k}{(s+1)^{k+1}}$$

$$= \frac{1}{(s+1)^2} \sum_{k=0}^{\infty} \frac{f_k}{(s+1)^k} \sum_{k=0}^{\infty} \frac{g_k}{(s+1)^k}$$

$$= \frac{1}{(s+1)^2} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{f_{k-m} g_m}{(s+1)^k}$$

$$= \frac{1}{s+1} \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} f_{k-m} g_m \right) \frac{1}{(s+1)^{k+1}}$$

$$= \frac{1}{s+1} \sum_{k=0}^{\infty} \frac{f_k * g_k}{(s+1)^{k+1}}$$

$$= \frac{1}{s+1} \mathcal{L}_d\{f_k * g_k\}.$$

Therefore, we have

$$\mathcal{L}_d\{f_k * g_k\} = (s+1)\mathcal{L}_d\{f_k\}.\mathcal{L}_d\{g_k\}.$$

Or

$$\mathcal{L}_d\{f_k * g_k\} = (s+1)F(s).G(s).$$

Example 3.10. In this example we find the Discrete Laplace Transform of $\{f_k = 1\}$.

$$F(z) = \mathcal{L}_d\{1\} = \sum_{k=0}^{\infty} \frac{1}{(s+1)^{k+1}}$$

$$= \frac{1}{(s+1)} \sum_{k=0}^{\infty} \frac{1}{(s+1)^k}$$

$$= \frac{1}{(s+1)} \frac{1}{1 - \frac{1}{s+1}}$$

$$= \frac{1}{s},$$

for $|s+1| > 1$.

Corollary 3.11. If $\mathcal{L}_d\{f_k\}$ exists for $|s+1| > r$, then

$$\mathcal{L}_d\left\{\sum_{m=0}^{k} y_m\right\} = \mathcal{L}_d\{1 * y_k\} = \frac{s+1}{s}\mathcal{L}_d\{y_k\}, \qquad \text{for } |s+1| > max\{1, r\}. \tag{3.16}$$

Proof. By Definition 3.8. we have

$$\sum_{m=0}^{k} y_m = 1 * y_k.$$

Then by the convolution Theorem 3.9. we have

$$\mathcal{L}_d\left\{\sum_{m=0}^{k} y_m\right\} = \mathcal{L}_d\{1 * y_k\}$$

$$= (s+1)\mathcal{L}_d\{1\}\mathcal{L}_d\{y_k\}.$$

Using Example 3.10. we get

$$\mathcal{L}_d\left\{\sum_{m=0}^{k} y_m\right\} = (s+1)\frac{1}{s}\mathcal{L}_d\{y_k\}.$$

Hence,

$$\mathcal{L}_d\left\{\sum_{m=0}^{k} y_m\right\} = \frac{s+1}{s}\mathcal{L}_d\{y_k\}.$$

Theorem 3.12.

$$\mathcal{L}_d\{y_k\} = 0 \text{ for } |s+1| > R \iff y_k = 0, \forall\, k = 0,1,2,3, \ldots$$

Proof.

If $y_k = 0$, then $\mathcal{L}_d\{y_k\} = 0, \forall\, k = 0,1,2,3, \ldots$ by definition.

Now suppose that $\mathcal{L}_d\{y_k\} = 0$, then

$$\sum_{k=0}^{\infty} \frac{y_k}{(s+1)^{k+1}} = 0$$

$$\frac{y_0}{s+1} + \frac{y_1}{(s+1)^2} + \cdots + \frac{y_n}{(s+1)^{n+1}} + \cdots = 0 \qquad (3.17)$$

Multiply 3.17 by $(s+1)$ and then taking the limit as $s \to \infty$, we get

$$y_0 = 0.$$

Multiply 3.17 by $(s+1)^2$ and then taking the limit as $s \to \infty$, we get

$$y_1 = 0.$$

Similarly one can prove that,

$$y_2 = y_3 = y_4 = \cdots = 0.$$

Thus,

$$y_k = 0 \quad \forall\, k = 0,1,2,3, \dots\ .$$

Corollary 3.13. If $\mathcal{L}_d\{x_k\} = \mathcal{L}_d\{y_k\}$ for $|s+1| > R$, then

$$x_k = y_k \quad \forall\, k = 0,1,2,3, \dots\ .$$

Proof.

Since $\mathcal{L}_d\{x_k\} = \mathcal{L}_d\{y_k\} \implies \mathcal{L}_d\{x_k - y_k\} = 0.$

Using Theorem 3.12. we have,

$$x_k - y_k = 0 \quad \forall\, k = 0,1,2,3, \dots\ .$$

That is

$$x_k = y_k \quad \forall\, k = 0,1,2,3, \dots\ .$$

CHAPTER 4

Discrete Laplace Transform of Elementary Functions

In this chapter, we discuss the Discrete Laplace Transforms of some elementary discrete functions.

In the following theorem, the Discrete Laplace Transforms of the discrete exponential function is given.

Theorem 4.1. Let a be a real number. Then

$$\mathcal{L}_d\{a^k\} = \frac{1}{s+1-a}, for \ |s+1| > |a| \tag{4.1}$$

Proof.

$$\mathcal{L}_d\{a^k\} = \sum_{k=0}^{\infty} \frac{a^k}{(s+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(s+1)} \frac{a^k}{(s+1)^k}$$

$$= \frac{1}{s+1} \sum_{k=0}^{\infty} \left(\frac{a}{(s+1)}\right)^k$$

$$= \frac{1}{s+1} \frac{1}{1-\frac{a}{s+1}}$$

$$= \frac{1}{s+1} \frac{(s+1)}{(s+1-a)}$$

$$= \frac{1}{s+1-a}, \qquad |s+1| > |a|.$$

Using Theorem 4.1 one can find the Discrete Laplace Transforms of some trigonometric functions.

Corollary 4.2.

a) For a real number a and $|s+1|>1$,

$$\mathcal{L}_d\{\cos(ak)\} = \frac{s+1-\cos a}{s^2-2s\cos a+2} \tag{4.2}$$

$$\mathcal{L}_d\{\,sin(ak)\} = \frac{\sin a}{s^2-2s\cos a+2} \tag{4.3}$$

b) For a real number a and $|s+1| > \max\{e^a, e^{-a}\}$,

$$\mathcal{L}_d\{\,\cosh(ak)\} = \frac{s+1-\cosh(a)}{s^2+2(s+1)(1-\cosh(a))} \tag{4.4}$$

$$\mathcal{L}_d\{\,\sinh(ak)\} = \frac{\sinh(a)}{s^2+2(s+1)(1-\cosh(a))} \tag{4.5}$$

Proof. a) By Theorem 4.1 we have :

$$\mathcal{L}_d\left\{\left(e^{ia}\right)^k\right\} = \frac{1}{s+1-e^{ia}} \qquad for \quad |s+1|>1,$$

$$\mathcal{L}_d\left\{e^{iak}\right\} = \frac{1}{s+1-e^{ia}}.$$

Since $e^{iak} = \cos(ak) + i\,sin(ak)$, by Theorem 3.5 we have :

$$\mathcal{L}_d\{\cos(ak) + i\,sin(ak)\} = \frac{1}{s+1-\cos a - i\sin a},$$

$$= \frac{1}{s+1-\cos a - i\sin a}\frac{s+1-\cos a + i\sin a}{s+1-\cos a - i\sin a}$$

$$= \frac{s+1-\cos a + i\sin a}{(s+1-\cos a)^2 + sin^2 a}$$

$$= \frac{s+1-\cos a + i\sin a}{s^2+1+cos^2 a - 2s\cos a + sin^2 a}$$

$$= \frac{s+1-\cos a + i\sin a}{s^2-2s\cos a+2}$$

$$= \frac{s+1-\cos a}{s^2-2s\cos a+2} + i\,\frac{\sin a}{s^2-2s\cos a+2}$$

Hence, we conclude:

$$\mathcal{L}_d\{\cos(ak)\} = \frac{s+1-\cos a}{s^2-2s\cos a+2},$$

$$\mathcal{L}_d\{\ sin(ak)\} = \frac{\sin a}{s^2-2s\cos a+2}.$$

b)

$$\mathcal{L}_d\{\cosh(ak)\} = \mathcal{L}_d\left\{\frac{e^{ak}+e^{-ak}}{2}\right\}$$

$$= \mathcal{L}_d\left\{\frac{1}{2}\ e^{ak} + \frac{1}{2}\ e^{-ak}\right\}$$

$$= \frac{1}{2}\ \mathcal{L}_d\{e^{ak}\} + \frac{1}{2}\mathcal{L}_d\{\ e^{-ak}\}$$

$$= \frac{1}{2}\left[\frac{1}{s+1-e^{a}} + \frac{1}{s+1-e^{-a}}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+1-\sinh(a)-\cosh(a)} + \frac{1}{s+1-\sinh(-a)-\cosh(-a)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+1-\cosh(a)-\sinh(a)} + \frac{1}{s+1-\cosh(a)-\sinh(a)}\right]$$

$$= \frac{1}{2}\left[\frac{s+1-\cosh(a)+\sinh(a)+s+1-\cosh(a)-\sinh(a)}{(s+1-\cosh(a)-\sinh(a))(s+1-\cosh(a)+\sinh(a))}\right]$$

$$= \frac{1}{2}\left[\frac{2s+2-2\cosh(a)}{(s+1-\cosh(a)-\sinh(a))(s+1-\cosh(a)+\sinh(a))}\right]$$

$$= \frac{s+1-\cosh(a)}{(s+1-\cosh(a)-\sinh(a))(s+1-\cosh(a)+\sinh(a))}$$

$$= \frac{s+1-\cosh(a)}{(s+1-\cosh(a))^2-\sinh^2(a)}$$

$$= \frac{s+1-\cosh(a)}{s^2+1+2s-2(s+1)\cosh(a)+\cosh^2(a)-\sinh^2(a)}$$

$$= \frac{s+1-\cosh(a)}{s^2+2s(1-\cosh(a))+2(1-\cosh(a))}.$$

Therefore, we have

$$\mathcal{L}_d\{\cosh(ak)\} = \frac{s+1-\cosh(a)}{s^2+2(s+1)(1-\cosh(a))}.$$

$$\mathcal{L}_d\{\sinh(ak)\} = \mathcal{L}_d\left\{\frac{e^{ak}-e^{-ak}}{2}\right\}$$

$$= \mathcal{L}_d\left\{\frac{1}{2}\,e^{ak} - \frac{1}{2}\,e^{-ak}\right\}$$

$$= \frac{1}{2}\,\mathcal{L}_d\{e^{ak}\} - \frac{1}{2}\mathcal{L}_d\{\,e^{-ak}\}$$

$$= \frac{1}{2}\left[\frac{1}{s+1-e^{a}} - \frac{1}{s+1-e^{-a}}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+1-\sinh(a)-\cosh(a)} - \frac{1}{s+1-\sinh(-a)-\cosh(-a)}\right]$$

$$= \frac{1}{2}\left[\frac{s+1-\cosh(a)+\sinh(a)-s-1+\cosh(a)+\sinh(a)}{(s+1-\cosh(a)-\sinh(a))(s+1-\cosh(a)+\sinh(a))}\right]$$

$$= \frac{1}{2}\left[\frac{2\sinh(a)}{(s+1-\cosh(a)-\sinh(a))(s+1-\cosh(a)+\sinh(a))}\right]$$

$$= \frac{\sinh(a)}{(s+1-\cosh(a))^2-\sinh^2(a)}$$

$$= \frac{\sinh(a)}{s^2+1+2s-2(s+1)\cosh(a)+\cosh^2(a)-\sinh^2(a)}$$

$$= \frac{\sinh(a)}{s^2+2s(1-\cosh(a))+2(1-\cosh(a))}.$$

Therefore, we have

$$\mathcal{L}_d\{\sinh(ak)\} = \frac{\sinh(a)}{s^2+2(s+1)(1-\cosh(a))}.$$

Theorem 4.3. If $\mathcal{L}_d\{y_k\} = Y(s)$ exists for $|s+1| > r$, then for any constant $a \neq 0$,

$$\mathcal{L}_d\{a^k y_k\} = \frac{1}{a} Y\left(\frac{s+1-a}{a}\right). \tag{4.6}$$

Proof.

$$Y(s) = \mathcal{L}_d\{a^k y_k\}$$

$$= \sum_{k=0}^{\infty} \frac{a^k y_k}{(s+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{a} \frac{a^{k+1} y_k}{(s+1)^{k+1}}$$

$$= \frac{1}{a} \sum_{k=0}^{\infty} \frac{y_k}{\left(\frac{s+1}{a}\right)^{k+1}}$$

$$= \frac{1}{a} \sum_{k=0}^{\infty} \frac{y_k}{\left(\frac{s}{a} + \frac{1}{a}\right)^{k+1}}$$

$$= \frac{1}{a} Y\left(\frac{s+1}{a} - 1\right).$$

Therefore, we have

$$\mathcal{L}_d\{a^k y_k\} = \frac{1}{a} Y\left(\frac{s+1-a}{a}\right).$$

Theorem 4.4. If $Y(s) = \mathcal{L}_d\{y_k\}\ for\ |s+1| > r$ then :

$$\mathcal{L}_d\{(k+n)^{\underline{n}} y_k\} = (-1)^n (s+1)^n \frac{d^n}{ds^n} Y(s). \tag{4.7}$$

Proof.

$$Y(s) = \sum_{k=0}^{\infty} \frac{y_k}{(s+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty} (s+1)^{-k-1} y_k$$

$$Y'(s) = -\sum_{k=0}^{\infty}(k+1)(s+1)^{-k-2}y_k$$

$$Y''(s) = \sum_{k=0}^{\infty}(k+1)(k+2)(s+1)^{-k-3}y_k$$

$$Y'''(s) = -\sum_{k=0}^{\infty}(k+1)(k+2)(k+3)(s+1)^{-k-4}y_k.$$

Taking the derivative continuously with respect to s, we reach

$$Y^{(n)}(s) = (-1)^n\sum_{k=0}^{\infty}(k+1)(k+2)\dots(k+n)(s+1)^{-k-n-1}y_k.$$

Therefore,

$$(-1)^n Y^{(n)}(s)(s+1)^n = \sum_{k=0}^{\infty}\frac{(k+1)(k+2)\dots(k+n)}{(s+1)^{k+1}}y_k.$$

Hence,

$$\mathcal{L}_d\{(k+n)^{\underline{n}}y_k\} = (-1)^n(s+1)^n\frac{d^n}{ds^n}Y(s).$$

Corollary 4.5. For any integer $n > 0$,

$$\mathcal{L}_d\{(k+n)^{\underline{n}}\} = \left(1+\frac{1}{s}\right)^n\frac{n!}{s} \tag{4.8}$$

Proof. From Theorem 4.4., we know,

$$\mathcal{L}_d\{(k+n)^{\underline{n}}y_k\} = (-1)^n(s+1)^n\frac{d^n}{ds^n}Y(s).$$

Letting $y_k = 1$, we have

$$\mathcal{L}_d\{(k+n)^{\underline{n}}1\} = (-1)^n(s+1)^n\frac{d^n}{ds^n}Y(s)$$

$$= (-1)^n(s+1)^n\frac{d^n}{ds^n}\left(\frac{1}{s}\right)$$

$$= (-1)^n (s+1)^n (-1)^n \frac{n!}{s^{n+1}}$$

$$= \left(\frac{s+1}{s}\right)^n \frac{n!}{s}.$$

Therefore, we have

$$\mathcal{L}_d\{(k+n)^{\underline{n}}\} = \left(1+\frac{1}{s}\right)^n \frac{n!}{s}.$$

Theorem 4.6. For $n = 1,2,3, \ldots,$

$$\mathcal{L}_d\{k^{\underline{n}}\} = \frac{n!}{s^{n+1}} \quad for \quad |s| > 0. \tag{4.9}$$

Proof. We will use mathematical induction.

For n=1,

$$\mathcal{L}_d\{k\} = \sum_{k=0}^{\infty} \frac{k}{(s+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(s+1)} \cdot \frac{k}{(s+1)^k}$$

$$= \frac{1}{(s+1)} \sum_{k=0}^{\infty} \frac{k}{(s+1)^k}$$

$$= \frac{1}{(s+1)} \sum_{k=1}^{\infty} \frac{k}{(s+1)^k}$$

$$= \frac{1}{(s+1)} \sum_{k=0}^{\infty} \frac{k+1}{(s+1)^{k+1}}$$

$$= \frac{1}{(s+1)} \left[\sum_{k=0}^{\infty} \frac{k}{(s+1)^{k+1}} + \sum_{k=0}^{\infty} \frac{1}{(s+1)^{k+1}}\right]$$

$$\mathcal{L}_d\{k\} = \frac{1}{(s+1)} [\mathcal{L}_d\{k\} + \mathcal{L}_d\{1\}]$$

$$(s+1)\mathcal{L}_d\{k\} = \mathcal{L}_d\{k\} + \mathcal{L}_d\{1\}$$

$$(s+1)\mathcal{L}_d\{k\} - \mathcal{L}_d\{k\} = \mathcal{L}_d\{1\}$$

$$\mathcal{L}_d\{k\}(s+1-1) = \mathcal{L}_d\{1\}$$

$$s\mathcal{L}_d\{k\} = \mathcal{L}_d\{1\}$$

$$s\mathcal{L}_d\{k\} = \frac{1}{s}.$$

Thus,

$$\mathcal{L}_d\{k\} = \frac{1}{s^2}.$$

Now suppose it is true for $n \leq m$.

$$\mathcal{L}_d\{k^{\underline{m+1}}\} = \sum_{k=0}^{\infty} \frac{k^{\underline{m+1}}}{(s+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty} k^{\underline{m+1}} \left(\frac{1}{s+1}\right)^{k+1}$$

$$= \left[-k^{\underline{m+1}} \frac{1}{s(s+1)^k} \bigg|_0^{\infty} - \sum_{k=0}^{\infty} (m+1)k^{\underline{m}} \left(\frac{-1}{s}\right) \frac{1}{(s+1)^{k+1}} \right]$$

$$= \frac{1}{s} \sum_{k=0}^{\infty} (m+1)k^{\underline{m}} \frac{1}{(s+1)^{k+1}}$$

$$= \frac{m+1}{s} \sum_{k=0}^{\infty} k^{\underline{m}} \frac{1}{(s+1)^{k+1}}$$

$$= \frac{m+1}{s} L_d\{k^{\underline{m}}\}$$

$$= \frac{m+1}{s} \frac{m!}{s^{m+1}}.$$

Therefore, we have

$$\mathcal{L}_d\{k^{\underline{m+1}}\} = \frac{(m+1)!}{s^{m+2}}.$$

Theorem 4.7. For a positive integer n,

$$\mathcal{L}_d\left\{\binom{k}{n}\right\} = \frac{1}{s^{n+1}}. \tag{4.10}$$

Proof.

$$\mathcal{L}_d\left\{\binom{k}{n}\right\} = \mathcal{L}_d\left\{\frac{k^{\underline{n}}}{n!}\right\}$$

$$= \frac{1}{n!}\mathcal{L}_d\{k^{\underline{n}}\}$$

$$= \frac{1}{n!}\frac{n!}{s^{n+1}}$$

$$= \frac{1}{s^{n+1}}.$$

Therefore, we have

$$\mathcal{L}_d\left\{\binom{k}{n}\right\} = \frac{1}{s^{n+1}}.$$

Theorem 4.8. For any positive real number r,

$$\mathcal{L}_d\left\{\binom{r}{k}\right\} = \left(\frac{s+2}{s+1}\right)^r \frac{1}{s+1}, \qquad |s+1| > 1\,. \tag{4.11}$$

Proof. We know that for any positive real number r and $|x| < 1$, we have the formula

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n.$$

Replacing x by $\frac{1}{s+1}$, we get,

$$\left(1 + \frac{1}{s+1}\right)^r = \sum_{k=0}^{\infty} \binom{r}{k} \left(\frac{1}{s+1}\right)^k$$

$$= \sum_{k=0}^{\infty} \binom{r}{k} \frac{1}{(s+1)^k}$$

$$\left(\frac{s+2}{s+1}\right)^r = \sum_{k=0}^{\infty} \binom{r}{k} \frac{1}{(s+1)^k}$$

$$= (s+1)\mathcal{L}_d \left\{\binom{r}{k}\right\}.$$

Therefore, we have

$$\mathcal{L}_d \left\{\binom{r}{k}\right\} = \left(\frac{s+2}{s+1}\right)^r \frac{1}{s+1}.$$

The unit impulse sequence $\delta(n), n \geq 1,$ is given in the following definition.

Definition 4.9. [9]

$$\delta_k(n) = \begin{cases} 1 & if\ k = n \\ 0 & if\ k \neq n \end{cases} \tag{4.12}$$

In the following theorem the Discrete Laplace Transform of the unit impulse sequence is given.

Theorem 4.10.

$$\mathcal{L}_d\{\ \delta_k(n)\} = \left(\frac{1}{s+1}\right)^{n+1}. \tag{4.13}$$

Proof.

$$\mathcal{L}_d\{\ \delta_k(n)\} = \sum_{k=0}^{\infty} \frac{\delta_k(n)}{(s+1)^{k+1}}$$

$$= \frac{1}{(s+1)^{n+1}}.$$

Below after we define the unit step sequence we percent its Laplace Transform.

Definition 4.11. [9]

$$u_k(n) = \begin{cases} 0 & if\ 0 \leq k \leq n-1 \\ 1 & if \qquad k \geq n \end{cases}. \tag{4.14}$$

Theorem 4.12. For any positive integer n and $|s+1| > 1$,

$$\mathcal{L}_d\{\, u_k(n)\} = \frac{1}{s(s+1)^n}. \tag{4.15}$$

Proof.

$$\mathcal{L}_d\{\, u_k(n)\} = \sum_{k=0}^{\infty} \frac{u_k(n)}{(s+1)^{k+1}}$$

$$= \sum_{k=n}^{\infty} \frac{1}{(s+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(s+1)^{n+k+1}}$$

$$= \frac{1}{(s+1)^n} \sum_{k=0}^{\infty} \frac{1}{(s+1)^{k+1}}$$

$$= \frac{1}{(s+1)^n} \mathcal{L}_d\{\, 1\}$$

$$= \frac{1}{(s+1)^n} \frac{1}{s}.$$

Theorem 4.12. can be generalized as follows.

Theorem 4.13. For any positive integer n,

$$\mathcal{L}_d\{y_{k-n} u_k(n)\} = (s+1)^{-n} \mathcal{L}_d\{y_k\}. \tag{4.16}$$

Proof.

$$\mathcal{L}_d\{y_{k-n} u_k(n)\} = \sum_{k=0}^{\infty} \frac{y_{k-n} u_k(n)}{(s+1)^{k+1}}$$

$$= \sum_{k=n}^{\infty} \frac{y_{k-n}}{(s+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{y_k}{(s+1)^{k+n+1}}$$

$$= \frac{1}{(s+1)^n} \sum_{k=0}^{\infty} \frac{y_k}{(s+1)^{k+1}}.$$

Therefore, we have

$$\mathcal{L}_d\{y_{k-n} u_k(n)\} = \frac{1}{(s+1)^n} \mathcal{L}_d\{y_k\}.$$

CHAPTER 5

APPLICATIONS

In this chapter we are going to discuss some applications of the Discrete Laplace Transform.

5.1 Second Order Difference Equations with Constant Coefficients

Let's consider the general second order difference equation

$$a_2 y_{k+2} + a_1 y_{k+1} + a_0 y_k = f_k \,, \qquad k = 0,1,2,\dots \tag{5.1.1}$$

with the following initial conditions

$$y_0 = b_0 \;\;,\;\; y_1 = b_1 \tag{5.1.2}$$

Applying the Discrete Laplace Transform to both sides of equation 5.1.1 and using 5.1.2 we get

$$\mathcal{L}_d\{a_2 y_{k+2} + a_1 y_{k+1} + a_0 y_k\} = \mathcal{L}_d\{f_k\},$$

Or

$$a_2[(s+1)^2 Y(s) - (s+1)y_0 - y_1] + a_1[(s+1)Y(s) - y_0] + a_0 Y(s) = F(s),$$

where

$$Y(s) = \mathcal{L}_d\{y_k\},$$

$$F(s) = \mathcal{L}_d\{f_k\}.$$

Hence,

$$Y(s) = \frac{[(s+1)a_2 + a_1]y_0 + a_2 y_1}{a_2(s+1)^2 + a_1(s+1) + a_0} + \frac{F(s)}{a_2(s+1)^2 + a_1(s+1) + a_0}.$$

Thus,

$$y(t) = \mathcal{L}_d{}^{-1}\left\{\frac{[(s+1)a_2 + a_1]y_0 + a_2y_1}{a_2(s+1)^2 + a_1(s+1) + a_0} + \frac{F(s)}{a_2(s+1)^2 + a_1(s+1) + a_0}\right\}.$$

We can also apply the Discrete Laplace Transform to solve system of linear equations. For example, consider the system

$$\begin{aligned} u_{k+1} - v_k &= -1 \\ -u_k + v_{k+1} &= 3 \end{aligned} \tag{5.1.3}$$

where,

$$u_0 = 0 \ \ , \ \ v_0 = 2. \tag{5.1.4}$$

Now, if we apply the Discrete Laplace Transform to the equations in the system 5.1.3 and using the conditions 5.1.4, we get the algebraic system,

$$\begin{aligned} (s+1)U(s) - V(s) &= -\frac{1}{s} \\ -U(s) + (s+1)V(s) &= \frac{3+2s}{s} \end{aligned} \tag{5.1.5}$$

where,

$$\begin{aligned} U(s) &= \mathcal{L}_d\{u_k\} \\ V(s) &= \mathcal{L}_d\{v_k\}. \end{aligned}$$

Solving 5.1.5 for $U(s)$ and $V(s)$, we get

$$U(s) = \frac{1}{s^2}$$

$$V(s) = \frac{1}{s^2} + \frac{2}{s}.$$

Hence,

$$u_k = \mathcal{L}_d{}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= k$$

and

$$v_k = \mathcal{L}_d{}^{-1}\left\{\frac{1}{s^2} + \frac{2}{s}\right\} = k + 2.$$

5.2 Volterra Summation Equation of Convolution Type

we consider the following equation

$$y_k = f_k + \sum_{m=0}^{k-1} u_{k-m-1} y_m \qquad k \geq 0 \tag{5.2.1}$$

where f_k and u_{k-m-1} are given.

obviously $y_0 = f_0$,

and

$$y_{k+1} = f_{k+1} + \sum_{m=0}^{k} u_{k-m} y_m \,. \tag{5.2.2}$$

Applying the Discrete Laplace Transform to both side of equation 5.2.2, we get

$$\mathcal{L}_d\{y_{k+1}\} = \mathcal{L}_d\{f_{k+1}\} + \mathcal{L}_d\left\{\sum_{m=0}^{k} u_{k-m} y_m\right\},$$

or

$$(s+1)Y(s) - y_0 = (s+1)F(s) - f_0 + (s+1)U(s)Y(s),$$

where

$$Y(s) = \mathcal{L}_d\{y_k\},$$

$$F(s) = \mathcal{L}_d\{f_k\},$$

$$U(s) = \mathcal{L}_d\{u_k\}.$$

We find $Y(s)$ as

$$Y(s) = \frac{F(s)}{1-U(s)}$$

$$= \frac{1}{1-U(s)} F(s)$$

$$= (s+1)\frac{1}{(s+1)\big(1-U(s)\big)}F(s).$$

Thus,

$$y(t) = \sum_{m=0}^{k} v_{k-m} f_m$$

where,

$$v_k = {\mathcal{L}_d}^{-1}\left\{\frac{1}{(s+1)\big(1-U(s)\big)}\right\}.$$

Table 1. List of Discrete Laplace Transform

Sequence	d- Laplace Transform
1	$\frac{1}{s}$
a^k	$\frac{1}{s+1-a}$
k	$\frac{1}{s^2}$
k^2	$\frac{1}{s^2}+\frac{2}{s^3}$
$k^{\underline{n}}$	$\frac{n!}{s^{n+1}}$
$sin(ak)$	$\frac{\sin a}{s^2-2s\cos a+2}$
$\cos(ak)$	$\frac{s+1-\cos a}{s^2-2s\cos a+2}$
$\sinh(ak)$	$\frac{\sinh a}{s^2+2(s+1)(1-\cosh a)}$
$\cosh(ak)$	$\frac{s+1-\cosh a}{s^2+2(s+1)(1-\cosh a)}$

Table 2. List of Discrete Laplace Transform

Sequence	d- Laplace Transform
$\delta_k(n)$	$\left(\frac{1}{s+1}\right)^{n+1}$
$u_k(n)$	$\frac{1}{s(s+1)^n}$
ky_k	$-(s+1)Y'(s) - Y(s)$
$f_k * g_k$	$(s+1)F(s).G(s)$
$\sum_{m=0}^{k} y_m$	$\frac{s+1}{s}Y(s)$
$a^k y_k$	$\frac{1}{a}Y\left(\frac{s+1-a}{a}\right)$
y_{k+n}	$(s+1)^n \mathcal{L}_d\{y_k\} - \sum_{m=0}^{n-1} y_m (s+1)^{n-m-1}$
$y_{k-n} f_k(n)$	$\frac{1}{(s+1)^n}Y(s)$

CHAPTER 6

CONCLUSION

The Laplace transform is considered as the one the most important among various integral transforms that contributed in the development of the theory of differential equations. The question whether these transforms have discrete analogue, still needs an answer.

In this thesis, the discrete analogue of the Laplace transform is defined using the definition of the Laplace transform on a general time scale. The main theorems related to this transform are stated and the Laplace transforms of the most important functions are derived.

Unlike differential equations which can be solved by several transform methods, there is only one summation transform called Z-transform used to solve difference equations. However solving difference equations using Z-transform needs tedious and intricate calculations. Even though there is a strong relation between the Z-transform and Discrete Laplace Transform, the Discrete Laplace Transform can be considered to be a rival to the Z-transform that reduces these calculations to minimum.

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APPENDIX A

CURRICULUM VITAE

PERSONAL INFORMATION

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EDUCATION

Degree	Institution	Year of Graduation
MS Çankaya Univ.	Mathematics And Computer Science	
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WORK EXPERIENCE

Year	Place	Enrollment
1992-1998	high schools	Math teacher
1998-2003	high schools	Computer teacher
2003-2004	The Directorate of Education in Kirkuk Governorate	Official Department of Computer & Programmer
2004- 2011	The Directorate of Education in Kirkuk Governorate	Educational Specialist Supervisor On Computer Teachers

FOREIGN LANGUAGES

Arabic, English, Turkish, Kurdish