ON DISCRETE FRACTIONAL CALCULUS WITH APPLICATIONS

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ABSTRACT

ON DISCRETE FRACTIONAL CALCULUS WITH APPLICATIONS

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In this thesis, we present the discrete fractional calculus in the frame of the Delta difference operator and discuss the most important properties and theorems. In order to solve delta fractional difference initial value problems, we discuss the Laplace transform related to this calculus and give the main formulas that are needed to solve such problems. The discrete fractional calculus in the frame of the Nabla difference operators and the related Laplace transforms are discussed and some Nabla fractional difference initial value problems are solved as well.

Keywords: Gamma Function, Delta Operator, Nabla Operator, Fractional Sum, Fractional Difference, Laplace Transform, Exponential Order.

ÖZ

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Bu tezde, Delta fark operatörü çerçevesinde ayrık kesirli analiz sunulmuş ve bu operatörün en önemli özellikleri ve ilgili teoremler tartışılmıştır. Kesirli mertebeden başlangıç değer Delta fark denklemleri çözmek için Laplace dönüşümü ele alınmış ve bunun gibi problemleri çözmek için gerekli olan temel formüller verilmiştir. Nabla fark operatörlü Kesirli analiz ve ilgili Laplace dönüşümü de ele alınmış ve kesirli mertebeden Nabla fark denklemleri için bazı başlangıç değer problemleri de çözülmüstür.

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AnahtarKelimeler: Gamma fonksiyonu, Delta operatörü, Nabla operatörü, Kesirli toplam, Kesirli fark, Laplace dönüşümü, Ustel mertebe.

DEDICATION

Dedicated to My Parents and My Family

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Also, I want to express my deep love to my sons Mustafa and Hussain and my daughters Zainab and Zahraa who suffer of the expatriation during my study period.

My God bestows health and happiness to all of them.

TABLE OF CONTENTS

LIST OF SYMBOLS

Γ	Gamma
$\int$	Integral
$\prod$	Product
Δ	Delta (Forward Operator)
∇	Nabla (Backward Operator)
$\mathcal{L}$	Laplace Transform
σ	Sigma
$\sum$	Summation

CHAPTER 1

INTRODUCTION

The fractional calculus deals with integrals and derivatives of any real or complex order. This type of calculus is as old as the usual calculus. Many great scientists like Euler, Bernoulli, Liouville and others who were the pioneers in developing the usual calculus contributed also in the birth of the fractional calculus. Since that time many people have been working on this field to help the development and applicability to various areas of mathematics, physics, engineering and other sciences. And it was founded that the fractional calculus can be extensively used in a big number of physical phenomena as a strong and effective tool for mathematical modeling [1, 2, 3].

Now the question is whether the discrete version of this calculus can gain this fame and importance arises. To the extent of our knowledge, the first article on the discrete fractional calculus appeared in the middle of the twentieth century by Diaz and Osler [4]. Then, an article by Gray and Zhang appeared in 1988 [5] followed by an article by Miller and Ross [6].

The discovery of the theory of the time scales [7,8], a theory which is used to combine the continuous and the discrete calculus, started a new era because it made it possible for scientists to make more development in the theory of the discrete fractional calculus as many authors used the tools in this theory of the time scales to report new results in the discrete fractional calculus [9-26].

Similar to the continuous fractional calculus, it turned out the discrete fractional calculus is a remarkable tool in mathematics to describe some physical and real world phenomena and applications [27, 28, 29, 30, 31, 32-59]. Since the physical phenomena are described by equations but methods of solving such equations are needed. Due to that, very significant and well known transform called the Laplace transform was developed not only to solve such equations, but also to help go further in the theory of the discrete fractional equations.

The main goal of this thesis is to collect all basic scattered pieces of information on the discrete fractional calculus and gather them in one source that help people working on this subject to reach all facts and results of this calculus for using one reference only.

This thesis is organized as follows:

In the second chapter, we discuss the main features and theorems related to the fractional calculus in the frame of the Delta difference operator.

In the third chapter, we discuss the main features and theorems related to the fractional calculus in the frame of the Nabla difference operator.

In the fourth chapter, we present the Laplace transform in the calculus of Delta fractional sums and fractional differences and give some examples to show how to solve initial value problems using this transform.

In the fifth chapter, the Laplace transform, used to solve initial value problems of Nabla fractional sums and differences is discussed and interesting results will obtained.

The sixth chapter is devoted to the conclusions.

CHAPTER 2

THE DELTA DISCRETE FRACTIONAL CALCULUS

Before we discuss the discrete fractional calculus, we have to introduce two important concepts that will be continuously used in this thesis.

2.1. Gamma Function and Falling Factorial

In this subsection, we introduce the Gamma function, the falling functions and discuss some of their properties which are used in our work.

2.1.1. Gamma function and its properties

The Gamma function was first introduced by Euler in order to generalize the factorial function.

Definition 1 [35]**:** The Gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt. \tag{2.1}$$

It follows that the Gamma function $\Gamma(z)$ is well defined for $z > 0$ and

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1. \tag{2.2}$$

We can use (2.1) to obtain

$$\Gamma(2) = 1.\Gamma(1) = 1 = 1!,$$

$$\Gamma(3) = 2.\Gamma(2) = 21! = 2!.$$

Some fundamental properties of the Gamma function are as below [2]:

(i.) $\Gamma(z) > 0, for\ z > 0,$ (2.3)

(ii.) $\Gamma(z+1) = z\Gamma(z),$ (2.4)

(iii.) $\Gamma(n+1) = n!, for\ n \in \mathbb{N}_0,$ (2.5)

(iv.) $\frac{\Gamma(z+k)}{\Gamma(z)} = (z+k+1) - (z-1)z, for\ z \in C(-\mathbb{N}_0)\ and\ k \in \mathbb{N},$ (2.6)

where $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0$ is the set of the non-negative integers.

2.1.2. The Falling Factorial Function

The falling factorial power $t^{\underline{v}}$(read t to the v falling) is defined as follows:

$$t^{\underline{v}} = t(t-1)(t-2)\ldots(t-(v-1)) = \prod_{k=0}^{v-1}(t-k) = \frac{\Gamma(t+1)}{\Gamma(t+1-v)}, \tag{2.7}$$

where $v \geq 0$ and Γ denotes the Gamma function.

Below are some properties of the falling factorial function.

Theorem [35]:

(i.) $\Delta t^{\underline{v}} = v\, t^{\underline{v-1}}$, (2.8)

where Δ is the difference operator, (ii.) $(t-v)t^{\underline{v}} = t^{\underline{v+1}}, where\ v \in \mathbb{C}(-\mathbb{N}_0)$,

(iii.) $v^{\underline{v}} = \Gamma(v+1)$,

(iv.) $t^{\underline{v+u}} = (t-u)^{\underline{v}} t^{\underline{u}}$.

Proof [35]: Depending on the definition of the falling factorial, its properties and proofs can be shown directly as follows:

(i.) $\Delta t^{v} = \Delta \frac{\Gamma(t+1)}{\Gamma(t-v+1)}$

$$= \frac{\Gamma((t+1)+1)}{\Gamma\big((t+1)-v+1\big)} - \frac{\Gamma(t+1)}{\Gamma(t-v+1)} = \frac{\Gamma(t+2)}{\Gamma(t-v+2)} - \frac{\Gamma(t+1)}{\Gamma(t-v+1)}$$

$$= \frac{(t+1)\Gamma(t+1)}{(t-v+1)\Gamma(t-v+1)} - \frac{\Gamma(t+1)}{\Gamma(t-v+1)}$$

$$= \frac{\Gamma(t+1}{\Gamma(t-v+1)}\left(\frac{t+1}{t+1-v} - 1\right) = \frac{\Gamma(t+1)-v}{\Gamma(t-v+1)(t-v+1)}$$

$$= v\ \frac{\Gamma(t+1)}{\Gamma(t-v+2)} = v\ t^{\underline{v}}.$$

(ii.) $(t-v)t^{\underline{v}} = (t-v)\frac{\Gamma(t+1)}{\Gamma(t+1-v)} = (t-v)\frac{\Gamma(t+1)}{(t-v)\Gamma(t-v)} = t^{\underline{v+1}}$.

(iii.) $v^{(v)} = \Gamma(v+1)$ *it comes directly from definition of falling factorial.*

(iv.) $t^{\underline{v+u}} = \frac{\Gamma(t+1)}{\Gamma(t+1-v+u)} = \frac{\Gamma(t-u+1)}{\Gamma(t+1-v-u)}\frac{\Gamma(t+1)}{\Gamma(t-u+1)} = (t-u)^{v} t^{\underline{u}}$.

2.2. The Differences Operators (Δ)

Definition 2: [35] Let $f(x)$ be a function of a real or complex variables. The differences operator Δ is defined by:

$$\Delta f(x) = f(x+1) - f(x). \tag{2.9}$$

We introduce some properties of this operator namely;

Theorem [39] Let $n, m > 0$ be integers (i.) $\Delta^m(\Delta^n y(x)) = \Delta^{m+n} y(x),$ (2.10)

(ii.) $\Delta(y(x) + z(x)) = \Delta y(x) + \Delta z(x)$ (2.11)

(iii.) $\Delta(cy(x)) = c(\Delta y(x))$, If c is constant, (2.12)

(iv.) $\Delta[y(x).z(x)] = y(x).\Delta z(x) + z(\sigma(x))\,\Delta y(x) = y(\sigma(x)).\Delta z(x) + z(x)\,\Delta y(x)$, (2.13)

(v.) $\Delta\left(\frac{y(x)}{z(x)}\right) = \frac{z(x)\Delta y(x) - y(x)\Delta z(x)}{z(x)z(\sigma(x))}$, (2.14)

where $\sigma(x) = x + 1$.

Proof [35]: The proofs of (i.), (ii.) and (iii.) are trivial. We will prove iv while v is proved similarly.

$\Delta(y(x)z(x)) = y(x+1)z(x+1) - y(x)z(x)$

$= y(x+1)z(x+1) - y(x+1)z(x) + (y(x+1) - y(x))z(x)$

$= y(x+1)\big(z(x+1) - z(x)\big) + z(x)\big(y(x+1)\big) - y(x)$

$= y\big(\sigma(x)\big)\Delta z(x) + z(x)\Delta y(x).$

For the second equality we have:

$\Delta\left(y(x)\, z(x)\right) = y(x+1)z(x+1) - y(x)z(x)$

$= y(x+1)z(x+1) - z(x+1)y(x) + z(x+1)y(x) - y(x)z(x)$

$= z(x+1) + y(x+1) - y(x) + y(x)\big(z(x+1) - z(x)\big)$

$= z\big(\sigma(x)\big)\,\Delta y(x) + y(x)\,\Delta z(x).$

Remark [35]: From the equation (2.13),we have

$g(x)\,\Delta f(x) + \Delta\,(f(x)g(x)) - f(\sigma(x)\Delta\, g(x).$

Applying

$\sum_{x=1}^{b-1}$ − operator to both sides give

$$\sum_{x=1}^{b-1} g(x)\Delta f(x) = \sum_{x=1}^{b-1} \Delta\big(f(x)g(x)\big) - \sum_{x=1}^{b-1} f\big(\sigma(x)\big)\Delta g(x)$$

$$\sum_{x=1}^{b-1} g(x)\Delta f(x) = f(x)g(x)|_1^b - \sum_{x=1}^{b-1} f\big(\sigma(x)\big)\Delta g(x), \qquad (2.15)$$

where $b > 1$ is an integer. This is known as the summation by parts formula in discrete calculus [35].

2.3. The Delta Fractional Sums and Differences

Definition 3[35]:

If $f\colon \mathbb{N}_a \to \mathbb{R}$ *and* $v > 0$ *is given, then the* v^{th} *order fractional sum of* f *is defined by*:

$$(\Delta_a^{-v} f)(t) := \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-\sigma(s))^{\underline{v-1}} f(s), \qquad for\ t \in \mathbb{N}_{a+v},$$

where $(t-\sigma(s))^{\underline{v-1}}$ *is the generalized falling function and*

$\mathbb{N}_a = \{a, a+1, a+2, \dots\}$.

The fractional sum of order 0 *is defined by* $\Delta_a^{-0} f(t) := f(t)$, *for* $t \in \mathbb{N}_a$.

Definition 4 [11]:

Let

$f: \mathbb{N}_a \to \mathbb{R}$ *and* $v \geq 0$ *be given, and let* $N \in \mathbb{N}$ *be chosen such that* $N - 1 < v \leq N$. *Then the* v^{th} order fractional difference of f *is given by*

$$(\Delta_a^v f)(t) = \Delta_a^v f(t) := \Delta^N \Delta_a^{-(N-v)}, for\ t \in \mathbb{N}_{a+N-v}.$$

Definition 5 [11]:

Let $f: \mathbb{N}_a \to \mathbb{R}$ *and* $v > 0$ *be given. Then*

(i.) *the* v^{th} – fractional sum of f is given by

$$\Delta_a^{-v} f(t) := \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t - \sigma(s))^{\underline{v-1}} f(s), t \in \mathbb{N}_{a+v},$$

(ii.) *the* v^{th} – fractional difference of f is written as

$$\Delta_a^v f(t) := \begin{cases} \frac{1}{\Gamma(-v)} \sum_{s=a}^{t+v} (t - \sigma(s))^{\underline{-v-1}} f(s), v \notin \mathbb{N}, t \in \mathbb{N}_{a+N-v}, \\ \Delta^N f(t), v = N \in \mathbb{N}. \end{cases} \quad (2.16)$$

2.4. Composing Delta Fractional Sums and Differences

Here we present the rules of composing a fractional sum with a fractional sum, a fractional difference with a fractional sum, fractional sum with fractional difference and a fractional difference with fractional difference.

Theorem [11]:(Composing a Sum with a Sum)

Let $f: \mathbb{N}_a \to \mathbb{R}$ be a given and suppose $v, \mu > 0$. Then

$$\Delta_{a+\mu}^{-v} \Delta_a^{-\mu} f(t) = \Delta_a^{-v-\mu} f(t), for\ t \in \mathbb{N}_{a+\mu+v}. \quad (2.17)$$

Proof [11]: Suppose $f: N_a \to R$ *and* $v, \mu > 0$. Then for $t \in \mathbb{N}_{a+\mu+v}$.

$$\Delta_{a+\mu}^{-v} \Delta_a^{-\mu} f(t) = \frac{1}{\Gamma(v)} \sum_{s=a+\mu}^{t-v} (t - \sigma(s))^{\underline{v-1}} \left(\frac{1}{\Gamma(\mu)} \sum_{r=a}^{s-\mu} (s - \sigma(r))^{\underline{\mu-1}} f(r)\right)$$

$$= \frac{1}{\Gamma(v) r(\mu)} \sum_{s=a+\mu}^{t-v} \sum_{r=a}^{s-\mu} (t - \sigma(s))^{\underline{v-1}} (s - \sigma(r))^{\underline{\mu-1}} f(r)$$

$$= \frac{1}{\Gamma(v)\Gamma(\mu)} \sum_{r=a}^{t-(v+\mu)} \sum_{s=r+\mu}^{t-v} (t - \sigma(s))^{\underline{v-1}} (s - \sigma(r))^{\underline{\mu-1}} f(r)$$

Let $x = s - \sigma(r)$ and continue with

$$= \frac{1}{\Gamma(v)\,\Gamma(\mu)} \sum_{r=\alpha}^{t-(v+\mu)} \left[\sum_{x=\mu-1}^{t-v-r-1} (t-x-r-2)^{\underline{v-1}} x^{\underline{\mu-1}} \right] f(r)$$

$$= \frac{1}{\Gamma(v)\,\Gamma(\mu)} \sum_{r=\alpha}^{t-(v+\mu)} \left[\sum_{x=\mu-1}^{t-v-r-1} (t-r-1)-\sigma(x))^{\underline{v-1}}\, x^{\underline{\mu-1}} \right] f(r)$$

$$= \frac{1}{\Gamma(v)\,\Gamma(\mu)} \sum_{r=\alpha}^{t-(v+\mu)} \left([\Delta_{\mu-1}^{-v}(t^{\underline{t-1}}]|_{t-r-1}\, f(r) \right)$$

$$= \frac{1}{\Gamma(\mu)} \sum_{r=\alpha}^{t-(v+\mu)} \frac{\Gamma(\mu)}{\Gamma(\mu+v)} (t-r-1)^{\underline{m-1+v}} f(r)$$

$$= \frac{1}{\Gamma(v+\mu)} \sum_{r=\alpha}^{t-(v+\mu)} (t-\sigma(r))^{\underline{(v+\mu)-1}} f(r)$$

$$= \Delta_{\alpha}^{-(v+\mu)} f(t) = \Delta_{\alpha}^{-v-\mu} f(t).$$

Since v and μ are arbitrary, we conclude more generally that

$\Delta_{\alpha+\mu}^{-v} \Delta_{\alpha}^{-\mu} f(t) = \Delta_{\alpha}^{-v-\mu} f(t) = \Delta_{\alpha+v}^{-\mu} \Delta_{\alpha}^{-v} f(t), for\ t \in N_{\alpha+v+\mu}.$

Before considering the general composition $\Delta_{\alpha+\mu}^{v}$ *and* $\Delta_{\alpha}^{-\mu}$, we first restrict v to be a natural number.

Lemma [9]: Let $f: \mathbb{N}_{\alpha} \to R$ *be a given for only* $k \in \mathbb{N}_0$ *and* $\mu > 0$ *with* $M-1 < \mu \leq M$, *we have*

$\Delta^{k} \Delta_{\alpha}^{-\mu} f(t) = \Delta_{\alpha}^{k-\mu} f(t), for\ t \in \mathbb{N}_{\alpha+\mu},$

$$\Delta^{k} \Delta_{\alpha}^{\mu} f(t) = \Delta_{\alpha}^{k+\mu} f(t), for\ t \in \mathbb{N}_{\alpha+M-\mu}.$$

Theorem [9]:(Composing a Difference with Sum)

Let $f: \mathbb{N}_{\alpha} \to R$ *be given, and suppose* $v, \mu > 0$ *with* $N-1 < v \leq N$.

Then we have,

$$\Delta_{\alpha+\mu}^{v} \Delta_{\alpha}^{-\mu} f(t) = \Delta_{\alpha}^{v-\mu} f(t), for\ t \in \mathbb{N}_{\alpha+\mu+N-v}.$$

Proof [9]: Let f, v, N *and* μ be given as n the statement of the theorem and let

$t \in \mathbb{N}_{\alpha+\mu+N-v}$. Then, we obtain:

$$\Delta_{a+\mu}^{v}\Delta_a^{-\mu} f(t) = \Delta^N\Delta_{a+\mu}^{-(N-v)}\Delta_a^{-\mu} f(t) = \Delta^N\Delta_a^{-(N-v+\mu)} f(t),$$

$$= \Delta_a^{-(N-v+\mu)} f(t) = \Delta_a^{v-\mu} f(t).$$

Theorem [11]: (Composing a Sum with a Difference)

Let $f: \mathbb{N}_a \to \mathbb{R}$ *be given and suppose* $k \in N_0$ *and* $v > 0$. *Then for* $t \in \mathbb{N}_{a+v}$

we have

$$\Delta_a^{-v}\Delta^k f(t) = \Delta_a^{k-v} f(t) - \sum_{j=0}^{k-1} \frac{\Delta^j f(a)}{\Gamma(v-k+j+1)} (t-a)^{\underline{v-k+j}}.$$

Moreover, if $\mu > 0$ *with* $M-1 < \mu \leq M$, *then for* $t \in N_{a+M-\mu+v}$,

we have

$$\Delta_{a+M-\mu}^{-v}\Delta_a^{\mu} f(t) = \Delta_a^{\mu-v} f(t) - \sum_{j=0}^{M-1} \frac{\Delta_a^{j-(M-\mu)} f(a+M-\mu)}{\Gamma(v-M+j+1)} (t-a-M+\mu)^{\underline{v-M+j}}.$$

Proof: It can be found in [15]**.**

Theorem [9]: (Composing a Difference with a Difference)

Let $f: \mathbb{N}_a \to \mathbb{R}$ *be given and suppose* $v, \mu > 0$ *with* $N-1 < v \leq N$ *and* $M-1 \leq M$.

Then for $t \in N_{a+M-\mu+N-v}$,

$$\Delta_{a+M-\mu}^{v}\Delta_a^{\mu} f(t) = \Delta_a^{v+\mu} f(t) - \sum_{j=0}^{M-1} \frac{\Delta_a^{j-M+\mu} f(a+M-\mu)}{\Gamma(-v-M+j+1)} (t-a-M+\mu)^{\underline{-v-M+j}}.$$

Proof: The proof of this theorem can be seen in [9].

2.5. Taylor Monomials and Power Rule

The Taylor monomials are very important when we want to find the Laplace transform of the discrete fractional and sums. These monomials are defined and developed in the time scale theory in [7].

The Taylor Monomials related to delta difference theory are defined recursively as

$$\begin{cases} h_0(t,a) := 1, \\ h_{n+1}(t,a) := \int_a^t h_n(s,a)\Delta s, \text{ for } n \in \mathbb{N}_0. \end{cases} \tag{2.18}$$

For the specific domain $\mathbb{N}_a$, the Taylor Monomials can be written explicitly as

$$h_n(t,a) = \frac{(t-a)^{\underline{n}}}{n!}, for\ n \in \mathbb{N}_0, t \in \mathbb{N}_a. \tag{2.19}$$

In general we can write

$$h_v(t,a) = \frac{(t-a)^{\underline{v}}}{\Gamma(v+1)}, for\ v \geq 0, t \in \mathbb{N}_a, \tag{2.20}$$

where the above generalized falling functions is given by:

$$t^{\underline{\mu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}, for\ t, \mu \in \mathbb{R}.$$

Here, we take the convention that $t^{\underline{\mu}} = 0$ whenever $t + 1 - \mu \in -\mathbb{N}_0$. This generalized falling function allows us to extend (2.18) to define a general Taylor Monomial that will serve us well in the discrete fractional calculus setting.

CHAPTER 3

THE NABLA DISCRETE FRACTIONAL CALCULUS

In this chapter we discuss the fractional discrete calculus in the frame of the Nabla difference. Before start the discussion let us introduce the Nabla difference operator**.**

3.1 Nabla Difference Operators

Definition 1[14]: For any function $f: \mathbb{N}_a \to \mathbb{R}$ we define the backward operator, ∇ by

$$\nabla f(t) = f(t) - f(t-1) \; for \; t \; \in \mathbb{N}_{a+1}. \tag{3.1}$$

The higher order difference is defined by

$$\nabla^n f(t) = \left(\Delta^{n-1} f(t)\right) for \; t \in \mathbb{N}_{a+n}, n \in \mathbb{N}.$$

Moreover, we consider ∇^0 the identity operator.

Theorem [15]:(Fundamental Theorem of Nabla Calculus)

Let $f: \mathbb{N}_a \to \mathbb{R}$ *and let* F *be an* $-$ *Nabla differences of* f *on* $\mathbb{N}_a$*, then for any*

$c, d \in \mathbb{N}_a$ *we have:*

$$\int_c^d f(t)\nabla t = \sum_{t=c+1}^{d} f(t) = F(d) - F(c). \tag{3.2}$$

Proof [15]:

The Nabla product rules for two functions $u, v: \mathbb{N}_a \to \mathbb{R}$ and $t \in \mathbb{N}_{a+1}$ is given by

$\nabla(u(t)v(t)) = u(t)v(t) + v(\rho(t))\nabla u(t)$, where $(t) = t - 1$.

This immediately leads to the summation by parts formula for Nabla calculus:

$$\sum_{s=b+1}^{c} u(t)\nabla\, v(t) = u(t)\, v(t)|_b^c - \sum_{s=b+1}^{c} v(\rho(t))\nabla\, u(t).$$

Below we define the rising function which plays an important role in the theory of the Nabla fractional calculus. For $n, k \in \mathbb{N}$, the rising factorial function is defined by

$$k^{\overline{n}} := k(k+1)\dots(k+n-1) = \frac{(k+n-1)!}{(k-1)!}.$$

This definition can be generalized as follows by using the gamma function as follows:

Definition 2 [15]: *The rising function is defined by*

$$k^{\overline{v}} := \frac{\Gamma(k+v)}{\Gamma(k)}, \tag{3.3}$$

*for those values*of$k, v \in \mathbb{R}$ where the right side is well defined. Moreover, if

$k \in \{0, -1, -2, \dots\}$ *and* $k + v \notin \{0, -1, -2, \dots\}$, *then* $k^{\overline{v}} = 0$.

We observe the following regarding the rising factorial function:

Theorem [15]:

(i.) $\nabla(t^{\overline{\alpha}}) = \alpha\, t^{\overline{\alpha-1}}$, (3.4)

(ii.) $t^{\overline{\alpha}} = (t + \alpha - 1)^{(\alpha)}$, (3.5) (iii.) $\Delta_t(s-p(t))^{\overline{\alpha}} = -\alpha(s-p(t))^{\overline{\alpha-1}}$, (3.6)

Proof: It can proved directly using the definition of Rising Factorial.

3.2 Nabla Fractional Sums and Differences

Here we define the fractional discrete fractional sums and differences within the Nabla operator.

Definition 3 [15]:

Let $f: \mathbb{N}_a \to R$ and $v > 0$, then the v^{th} order fractional sum is given by

$$\nabla_a^{-v} f(t) := \frac{1}{\Gamma(v)} \int_a^t (t-\rho(s))^{\overline{v-1}} f(s) \nabla s. \quad (3.7)$$

Next we define the fractional difference in terms of a fractional sum.

Definition 4[15]:

Let $f: \mathbb{N}_a \to R, v > 0$, and choose $N \in \mathbb{N}$ such that $N-1 < v < N$. Then, the v^{th} order fractional difference is given by

$$\nabla_a^{v} f(t) := \nabla^N \nabla_a^{-(N-v)} f(t), t \in \mathbb{N}_{a+N}. \quad (3.8)$$

3.3 Fractional Taylor Monomials

Here, we define the fractional Taylor monomials, which will help us solving initial value problems for Nabla fractional equations.

Definition 5 [15]: *The (Nabla) Taylor monomials, $h_n(t,a)$, $n \in \mathbb{N}_0$*

are defined recursively by

$$\begin{cases} h_0(t,a) := 1, \\ h_n(t,a) := \int_a^t h_{n-1}(\tau,a)\nabla\tau, n \in \mathbb{N}. \end{cases} \tag{3.9}$$

It is well known that this implies that

$$h_n(t,a) = \frac{(t-a)^{\overline{n}}}{n!}, \quad n \in \mathbb{N}_0.$$

The generalization is given below.

Definition 6 [15]: (Fractional Order Taylor Monomials)

For $v \in R\backslash\{-1,-2,\ldots\}$, the $v-th$ Taylor Monomials is given by

$$h_v(t,a) = \frac{(t-a)^{\overline{v}}}{\Gamma(v+1)}, t \in \mathbb{N}_a. \tag{3.10}$$

Now we can apply the fractional power rule in the following theorem.

Theorem [16]: *For $\mu, v \in R$ such that v and $v+\mu$ are not negative integers*:

$$\nabla_a^{\mu} h_v(t,a) = h_{v-\mu}(t,a) \tag{3.11}$$

for

$$\begin{cases} t \in \mathbb{N}_a, \quad \mu < 0, \\ t \in \mathbb{N}_{a+N}, \mu < 0, \end{cases}$$

where $N-1 < v \leq N$.

Proof [15]:

$$\nabla_a^{\mu} h_v(t,a) = \frac{1}{\Gamma(v+1)} \nabla_a^{\mu} (t-a)^{\overline{v}} = \frac{1}{\Gamma(v+1)} \frac{\Gamma(v+1)}{\Gamma(v-\mu+1)} (t-a)^{\overline{v-\mu}}$$

$$= h_{v-\mu}(t,a).$$

The next Lemma relates two Taylor monomials based at values that differ by one.

Lemma [15]: (One Step Taylor Monomial Shifting)

$For\ v \in \mathbb{R}\backslash\{-1,-2,\dots\}\ and\ N \in \mathbb{N},$

$$h_{v-n}(t,a+1) = h_{v-N(t,a)-h_{v-N-1}(t,a)}. \tag{3.12}$$

Proof [15]: Let us consider:

$$h_{v-N}(t,a) - h_{v-N-1}(t,a) = \frac{(t-a)^{\overline{v-N}}}{\Gamma(v-N+1)} - \frac{(t-a)^{\overline{v-N-1}}}{\Gamma(v-N)}$$

$$= \frac{\Gamma(t-a+v-N)}{\Gamma(v-N+1)\Gamma(t-a)} - \frac{\Gamma(t-a+v-N-1)}{\Gamma(v-N)\Gamma(t-a)}$$

$$= \frac{\Gamma(t-a+v-N-1)}{\Gamma(t-a)\Gamma(v-N+1)}\left(t-a+v-N-1(v-N)\right)$$

$$= \frac{\Gamma(t-a+v-N-1)}{\Gamma(t-a)\Gamma(v-N+1)} = h_{v-N}(t,a+1)$$

The previous theorem can be extended to the following general formula.

Theorem [15]: (General Taylor Monomial Shifting)

$For\ v \in \mathbb{R}\backslash\{-1,-2,\dots\}\ and\ N,m\ \in \mathbb{N},$

$$h_{v-N}(t,a+m) = \sum_{k=0}^{m}\binom{m}{k}(-1)^k h_{v-N-k}(t,a).$$

Proof: The proof is by induction on m. The base case, $m = 1$, follows from the previous Lemma. Assume that:

$$h_{h-N}(t,a+m) = \sum_{k=0}^{m}\binom{m}{k}(-1)^k h_{v-N-k}(t,a)\ for\ m \geq 1.$$

From the previous Lemma we obtain

$$h_{v-N}(t,a+m+1) = h_{v-N}(t,a+m) - h_{v-N}(t,a+m).$$

Applying the induction hypothesis to both terms on the right side of this equation gives

$$h_{v-N}(t, a+m+1)$$

$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^k h_{v-N}(t, a+m) - \sum_{k=0}^{m} \binom{m}{k} (-1)^k h_{v-N-k}(t, a)$$

$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^k h_{v-N-k}(t, a) - \sum_{k=0}^{m} \binom{m}{k-1} (-1)^k h_{v-N-k}(t, a)$$

$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^k h_{v-N-k}(t, a) - \binom{m}{m+1} (-1)^{m+1} h_{v-N-m-1}(t, a)$$

$$= \sum_{k=0}^{m+1} \binom{m}{k-1} (-1)^k h_{v-N-k}(t, a) + \sum_{k=0}^{m} \binom{m}{-1} (-1)^{-1} h_{v-N}(t, a)$$

$$= \sum_{k=0}^{m+1} \binom{m}{k} + \binom{m}{k-1} (-1)^k h_{v-N-k}(t, a)$$

$$= \sum_{k=0}^{m+1} \binom{m+1}{k} (-1)^k h_{v-N-k}(t, a).$$

3.4 Composition Rules

Here we present the rules of composing a fractional sum with a fractional sum, a fractional difference with a fractional sum, fractional sum with fractional difference and a fractional difference with fractional difference.

Theorem [13]: (Nabla Sum Composed with a Nabla Sum)

Let $v, \mu > 0$ *be given. Then*:

$$\nabla_a^{-\nu}\nabla_a^{-\mu} f(t) = \nabla_a^{-\nu-\mu} f(t). \tag{3.13}$$

Proof [13]: We will use the definition of the Laplace transformation which will be discussed later:

$$\mathcal{L}\{\nabla_a^{-\nu}\nabla_a^{-\mu} f\}(s) = \frac{1}{s^{\nu}}\mathcal{L}\{\nabla_a^{-\mu} f\}(s) = \frac{1}{s^{\nu+\mu}}\mathcal{L}\{f\}(s) = \mathcal{L}\{\nabla_a^{-\nu-\mu} f\}(s).$$

Because the Laplace transform is unique we have:

$$\nabla_a^{-\nu}\nabla_a^{-\mu} f(t) = \nabla_a^{-\nu-\mu} f(t).$$

Now we will first consider the case of whole order differences composed with fractional sums and fractional differences.

Lemma [13]: (Whole Order Differences Composed with Fractional Sums and Differences)

Let $k \in \mathbb{N}_0, \mu > 0,$ *and choose* $M \in \mathbb{N}$ *such that* $M - 1 < \mu \leq M.$ *Then*

$$\nabla^k \nabla_a^{-\mu} f(t) = \nabla_a^{k-\mu} f(t), \tag{3.14}$$

and

$$\nabla^k \nabla_a^{\mu} f(t) = \nabla_a^{k+\mu} f(t). \tag{3.15}$$

Proof [13]:

Let $k \in \mathbb{N}_0, \mu > 0,$ and choose $M \in \mathbb{N}$ such that $M - 1 < \mu \leq M$ be given.

Case 1: $\mu = M$.

First note the following:

$$\nabla\nabla^{-1} f(t) = \nabla \frac{1}{\Gamma(1)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{1-1}} d(s) = \nabla \sum_{s=a+1}^{t} f(s)$$

$$= \sum_{s=a+1}^{t} f(s) - \sum_{s=a+1}^{t-1} f(s) = f(t).$$

So, then, for the case of $\mu = M$ we have

$$\nabla^k \nabla^{-M} f(t) = \nabla^{k-1}(\nabla \nabla^{-1}\left(\nabla^{-(M-1)} f(t)\right))$$

$$= \nabla^{k-1} \nabla^{-(M-1)} f(t)$$

$$= \nabla^{k-2} \nabla^{-(M-2)} f(t)$$

. . .

$$= \nabla^{k-M} f(t).$$

This shows for this particular case.

Case 2: Let $M - 1 < \mu < M$.

First we will show that $\nabla \nabla_a^{\mu} f(t) = \nabla_a^{1+\mu} f(t)$.

We have

$$\nabla \nabla_a^{\mu} f(t) = \nabla\left(\frac{1}{\Gamma(-\mu)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{-\mu-1}}\right)$$

$$= \frac{1}{\Gamma(-\mu)} \sum_{s=a+1}^{t} (-\mu - 1)(t - \rho(s))^{\overline{-\mu-2}} f(s)$$

$$= \frac{1}{\Gamma(-\mu - 1)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{-\mu-2}}$$

$$= \nabla_a^{-(-\mu-1)} f(t) = \nabla_a^{1+\mu} f(t).$$

So for any $k \in \mathbb{N}_0$,

$$\nabla^k \nabla_a^{\mu} f(t) = \nabla^{k-1}(\nabla \nabla_a^{\mu} f(t))$$

$$= \nabla^{k-1} \nabla_a^{1+\mu} f(t)$$

$$= \nabla^{k-2} \nabla_a^{2+\mu} f(t)$$

…

$$= \nabla_a^{1+\mu} f(t).$$

Theorem [15]: (Nabla Differences Composed with a Nabla Sum)

For $v, \mu > 0$, we have $\nabla_a^{v} \nabla_a^{-\mu} f(t) = \nabla_a^{v-\mu} f(t)$. (3.16)

Proof [15]: $Let\ v, \mu > 0$ be given, and $N \in \mathbb{N}$ such that $N - 1 < v \leq N$.

Then, we have:

$$\nabla_a^{v} \nabla_a^{-\mu} f(t) = \nabla^N \nabla_a^{-(N-v)} \nabla_a^{-\mu} f(t)$$

$$= \nabla^N \nabla_a^{-(N-v)-\mu} f(t) = \nabla_a^{v-\mu} f(t).$$

Theorem [14]: (Nabla Sum Composed with a Whole Order Nabla Difference)

$Let\ \alpha > 0\ and\ k \in \mathbb{N}_0\ be\ given. Then$

$$\nabla_{a+k}^{-\alpha} \nabla^k f(t) = \nabla_a^{k-\alpha} f(t) - \sum_{j=0}^{k-1} \frac{\nabla j f(a+j+1)}{\Gamma(\alpha-k+j+1)} (t-a-j)^{\overline{\alpha-k+j}}\ for\ t \in \mathbb{N}_{a+k}. \quad (3.17)$$

Proof [14]:

Let $k \in \mathbb{N}_0$ be given and suppose $\notin \{1, 2, \dots, k-1\}$. We first state the following identity which follows from the product rule for the Nabla operator:

$$\nabla_s\left[(t-\rho(s))^{\overline{\alpha-1}} f(s)\right] = t-\rho(s))^{\overline{\alpha-1}} \nabla_s f(s) - (\alpha-1)\left(t+1-\rho(s)\right)^{\overline{\alpha-2}} f(\rho(s)).$$

Then we have

$$\nabla_{a+k}^{-\alpha}\nabla^k f(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=a+k+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\nabla^k f(s)$$

$$= \frac{(t-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)}\nabla^{k-1}f(s)|_{a+k}^{t} + \frac{(\alpha-1)}{\Gamma(\alpha)}\sum_{s=a+k+1}^{t}(t+1-\rho(s))^{\overline{\alpha-2}}\nabla^{k-1}f(\rho(s))$$

$$= \nabla^{k-1} f(t) - \frac{(t-a-k+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\nabla^{k-1}f(a+k) + \frac{1}{\Gamma(\alpha-1)}\sum_{s=a+k}^{t-1}(t-\rho(s))^{\overline{\alpha-2}}\nabla^{k-1}f(s)$$

$$= -\frac{(t-a-k+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\nabla^{k-1}f(a+k) + \frac{1}{\Gamma(\alpha-1)}\sum_{s=a+k}^{t}(t-\rho(s))^{\overline{\alpha-2}}\nabla^{k-1}f(s)$$

$$= \nabla_{a+k-1}^{-(\alpha-1)}\nabla^{k-1} f(t) - \frac{(t-a-k+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\nabla^{k-1} f(a+k)$$

$$= \nabla_{a+k-2}^{-(\alpha-2)}\nabla^{k-2} f(t) - \frac{(t-a-k+2)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}\nabla^{k-2} f(a+k-1) - \frac{(t-a-k+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\nabla^{k-1}f(a+k).$$

Continuing in this manner and summing by parts $k-2$ more times yields

$$\nabla_{a+k}^{-\alpha}\nabla^k f(t) = \nabla_a^{k-\alpha} f(t) - \sum_{j=0}^{k-1}\frac{\nabla^j f(a+j+1)}{\Gamma(\alpha-k+j+1)}(t-a-j)^{\overline{\alpha-k+j}}.$$

Theorem [15]:(Nabla Difference Composed with a Whole Order Nabla Difference)

Let $v > 0$ and $k \in \mathbb{N}_0$ be given and choose $N \in \mathbb{N}$ such that $N-1 < v \leq N$. Then

$$\nabla_{a+k}^{v}\nabla^k f(t) = \nabla_{a+k}^{k+v} f(t). \tag{3. 18}$$

Proof [15]: Consider

$$\nabla_{a+k}^{v}\nabla^{k} f(t) = \nabla^{N}\left(\nabla_{a+k}^{-(N-v)}\nabla^{k} f(t)\right) - \sum_{j=0}^{k-1}\nabla^{j} f(a+k)\frac{(t-a-k)^{\overline{-v-k+j}}}{\Gamma(-v-k+j+1)} \qquad (3.19)$$

$$= \nabla^{N}\left(\nabla_{a+k}^{k-(N-v)} f(t) - \sum_{j=0}^{k-1}\nabla^{j} f(a+k)\frac{(t-a-k)^{\overline{N-v-k+j}}}{\Gamma(N-v-k+j+1)}\right)$$

$$= \nabla_{a+k}^{k+v} f(t) - \sum_{j=0}^{k-1}\nabla^{j} f(a+k)\nabla^{N}\frac{(t-a-k)^{\overline{N-v-k+j}}}{\Gamma(N-v-k+j+1)}$$

$$= \nabla_{a+k}^{k+v} f(t) - \sum_{j=0}^{k-1}\nabla^{j} f(a+k)\nabla^{N-1}\frac{(t-a-k)^{\overline{N-v-k+j}}}{\Gamma(N-v-k+j+1)}$$

$$= \nabla_{a+k}^{k+v} f(t) - \sum_{j=0}^{k-1}\nabla^{j} f(a+k)\nabla^{N-1}\frac{(t-a-k)^{N\overline{-v-k+j}}}{\Gamma(N-v-k+j+1)}.$$

Taking the difference inside the summation $N-1$ more times, we get

$$\nabla_{a+k}^{v}\nabla^{k} f(t) = \nabla_{a+k}^{k+v} f(t) - \sum_{j=0}^{k-1}\nabla^{j} f(a+k)\nabla^{N-1}\frac{(t-a-k)^{\overline{-v-k+j}}}{\Gamma(-v-k+j+1)}.$$

CHAPTER 4

DELTA FRACTIONAL LAPLACE TRANSFORM

4.1 Definitions

Definition 1[11]: The Laplace transform of a function $f: \mathbb{N}_a \to \mathbb{R}$, can be written as

$$\mathcal{L}_a\{f\}(s) = \sum_{k=0}^{\infty} \frac{f(k+a)}{(s+1)^{k+1}}, \qquad (4.1)$$

for each $s \in \mathbb{R}$ for which the above series converges.

Definition 2 [11]: We say that a function $f: \mathbb{N}_a \to \mathbb{R}$ is of exponential order r $(r > 0)$ if there exists a constant $A > 0$ such that

$$|f(t)| \le Ar^t, for\ \ sufficiently\ large\ t \in \mathbb{N}_a. \qquad (4.2)$$

Suppose that a given function $f: \mathbb{N}_a \to \mathbb{R}$ is of some exponential order $r > 0$. Then there exist a constant $A > 0$ and a natural number $m \in \mathbb{N}_0$ such that for each $t \in \mathbb{N}_{a+m}$. We have $|f(t)| \le Ar^t$. We may write, therefore, for any $s \in \mathrm{R}$ outside of the ball $\overline{B - 1(r)}$,

$$\mathcal{L}_a\{f\}(s) = \sum_{k=0}^{\infty} \left|\frac{f(k+a)}{(s+1)^{k+1}}\right| = \sum_{k=0}^{m-1} \left|\frac{f(k+a)}{(s+1)^{k+1}}\right| + \sum_{k=m}^{\infty} \left|\frac{f(k+a)}{(s+1)^{k+1}}\right|$$

$$\leq \sum_{k=0}^{m-1} \left|\frac{f(k+a)}{(s+1)^{k+1}}\right| + \sum_{k=m}^{\infty} \frac{\mathrm{Ar}^{k+a}}{|s+1|^{k+1}}$$

$$= \sum_{k=0}^{m-1} \left|\frac{f(k+a)}{(s+1)^{k+1}}\right| + \frac{Ar^{a}}{|s+1|} \sum_{k=m}^{\infty} \left(\frac{\mathrm{r}}{|s+1|}\right)^{k}$$

$$= \sum_{k=0}^{m-1} \left|\frac{f(k+a)}{(s+1)^{k+1}}\right| + \frac{Ar^{a}}{|s+1|} \frac{\left(\frac{r}{|s+1|}\right)^{m}}{1-\left(\frac{r}{|s+1|}\right)}$$

$$= \sum_{k=0}^{m-1} \left|\frac{f(k+a)}{(s+1)^{k+1}}\right| + \frac{A}{|s+1|^{m}} \frac{r^{a+m}}{|s+1|-r} < \infty.$$

Lemma 1 [11]: (Existence of the Laplace Transform)

Suppose $f : \mathbb{N}_a \rightarrow \mathbb{R}$ are of exponential order $\mathrm{r} > 0$. Then

$$\mathcal{L}_a\{f\}(s) \text{ exists for } s \in \mathbb{R}.$$

Lemma 2 [11]: (Linearity and Uniqueness of the Laplace Transform)

Suppose $f,\ g:\ \mathbb{N}_a \rightarrow \mathbb{R}$ *are of exponential order* $r > 0$ *and let* $c_1, c_2 \in \mathbb{R}$.

Then

$$\mathcal{L}_a\{c_1 f + c_2 g\}(s) = c_1 \mathcal{L}_a\{f\}(s) + c_2 \mathcal{L}_a\{g\}(s), \text{ for } s \in \mathbb{R}$$

and

$$\mathcal{L}_a\{f\}(s) \equiv \mathcal{L}_a\{g\}(s) \text{ on } s \in \mathbb{R} \Leftrightarrow f(t) \equiv g(t) \text{ on } \mathbb{N}_a.$$

The relation between the shifted Laplace transform and the original one is important when solving difference equation. This is explained d in the upcoming Lemma.

Lemma 3 [11]: (Shifting Property)

Let $\mathrm{m} \in \mathbb{N}_0$ be given and suppose $\mathrm{f}: \mathbb{N}_{a-m} \rightarrow \mathbb{R}$ and $\mathrm{g}: \mathbb{N}_a \rightarrow \mathbb{R}$

are of exponentil order $r > 0$.

Then, for $s \in \mathbb{R}$

$$\mathcal{L}_{a-m}\{f\}(s) = \frac{1}{(s+1)^m}\mathcal{L}_a\{f\}(s) + \sum_{k=0}^{m-1}\frac{f(k+a-m)}{(s+1)^{k+1}} \tag{4.3}$$

$$\mathcal{L}_{a-m}\{g\}(s) = (s+1)^m\mathcal{L}_a\{g\}(s) - \sum_{k=0}^{m-1}(s+1)^{m-1-k}\, g(k+a). \tag{4.4}$$

Proof [11]: Let

f, g, r and m be as given in the ststement of the lemma. Then, for $s \in R$

$$\mathcal{L}_{a-m}\{f\}(s) = \sum_{k=0}^{\infty}\frac{f(k+a-m)}{(s+1)^{k+1}}$$

$$= \sum_{k=m}^{\infty}\frac{f(k+a-m)}{(s+1)^{k+1}} + \sum_{k=0}^{m-1}\frac{f(k+a-m)}{(s+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty}\frac{f(k+a)}{(s+1)^{k+m+1}} + \sum_{k=0}^{m-1}\frac{f(k+a-m)}{(s+1)^{k+1}}$$

$$= \frac{1}{(s+1)^m}\mathcal{L}_a\{f\}(s) + \sum_{k=0}^{m-1}\frac{f(k+a-m)}{(s+1)^{k+1}},$$

and

$$\mathcal{L}_{a+m}\{g\}(s) = \sum_{k=0}^{\infty}\frac{g(k+a-m)}{(s+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty}\frac{g(k+a)}{(s+1)^{k-m+1}}$$

$$= \sum_{k=0}^{\infty} \frac{g(k+a)}{(s+1)^{k-m+1}} - \sum_{k=0}^{m-1} \frac{g(k+a)}{(s+1)^{k-m+1}}$$

$$= (s+1)^m \mathcal{L}_a\{g\}(s) - \sum_{k=0}^{m-1} (s+1)^{m-1-k}\, g(k+a).$$

Now, one can easily verify that by applying formulas consecutively yields the identity

$\mathcal{L}_{(a+m)-m}\{f\}(s) = \mathcal{L}_{(a+m)-m}\{f\}(s) = \mathcal{L}_a\{f\}(s), for\ s \in \mathbb{R}.$

4.2 The convolution

In the following, we define the convolution of two functions [7].

Definition 3 [11]: *Define the convolution of two functions* $f, g : \mathbb{N}_a \to \mathbb{R}$ *by*

$$(f * g)(t) := \sum_{r=a}^{t} f(r) g(t-r+a), for\ t \in \mathbb{N}_a. \tag{4.5}$$

The following Lemma gives the Laplace transform of the convolution of two functions.

Lemma [11]: (The Laplace Transform of the Convolution)

Let $f, g: \mathbb{N}_a \to \mathbb{R}$ *be of exponential order* $r > 0$. *Then*

$$\mathcal{L}_a\{f * g\}(s) = (s+1)\mathcal{L}_a\{f\}(s)\mathcal{L}_a\{g\}(s), for\ s \in \mathbb{R}.$$

Proof [11]: Let f, g and r be as in the statement of the Lemma. Then

$$\mathcal{L}_a\{f * g\}(s) = \sum_{k=0}^{\infty} \frac{(f * g)(k+a)}{(s+1)^{k+1}}$$
$$= \sum_{k=0}^{\infty} \frac{(f * g)(k+a)}{(s+1)^{k+1}} \sum_{r=a}^{k+a} f(r) g(k+a-r+a)$$

$$=\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{f(r+a)g(k-r+a)}{(s+1)^{k+1}},$$

where by applying the change of variables $\tau = k - r$ yields the two independent summations:

$$\sum_{\tau=0}^{\infty}\sum_{r=0}^{\infty}\frac{f(r+a)g(\tau+a)}{(s+1)^{\tau+r+1}} = (s+1)\sum_{r=0}^{\infty}\frac{f(r+a)}{(s+1)^{r+1}}\sum_{\tau=0}^{\infty}\frac{g(\tau+a)}{(s+1)^{\tau+1}}$$

$$= (s+1)\mathcal{L}_a\{f\}(s)\mathcal{L}_a\{g\}(s), for\ s \in \mathbb{R}.$$

Corollary 1 [11]: Suppose that $f: \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 0$ and let $v > 0$ be given, with $N - 1 < v \leq N$. Then

$\mathcal{L}_{a+v-N}\{\Delta_a^{-v} f\}(s) and\ \mathcal{L}_{a+N-v}\{\Delta_a^{v} f\}(s)$

both exist for $s \in \mathbb{R}$.

Proof [11]: Let *f, r* and *v* be as given in the statement of the corollary. By Lemma 2, we know that for each $\epsilon \in 0$, both $\Delta_a^{-v} f$ and Δ_a^{v} are of exponential order $r + \epsilon$. Now, fix an arbitrary point $s_0 \in \mathbb{R}$. Since $dist\ (s_0, \overline{B_{-1}(r))} > 0$, there exists an $\epsilon_0 > 0$ small enough so that $s_0 \in \mathbb{R}$. Since $\Delta_a^{-v} f$ and $\Delta_a^{v} f$ are both exponential order $r + \epsilon_0$, it follows that both series $\mathcal{L}_{a+v-N}\{\Delta_a^{-v} f\}(s)$ and $\mathcal{L}_{a+v-N}\{\Delta_a^{-v} f\}(s)$ converges at $s = s_0$.

4.3 The Laplace Transform of Fractional Operators

Below we discuss the Laplace transforms of the delta fractional sums and differences.

Theorem [11]: (The Laplace Transform of a Delta Fractional Sum)

Suppose $f:\mathbb{N}_a \rightarrow$is *of exponential order* $r \geq 1$ *and let* $v > 0$ *be given, with* $N-1 < v \leq N$. *Then for* $s \in \mathbb{R}$,

$$\mathcal{L}_{a+v}\{\Delta_a^{-v}(s)\} = \frac{(s+1)^v}{s^v}\mathcal{L}_a\{f\}(s) \tag{4.6}$$

and

$$\mathcal{L}_{a+v-N}\{\Delta_a^{-v}f\}(s) = \frac{(s+1)^{v-N}}{s^v}\mathcal{L}_a\{f\}(s).$$

(4.7)

Proof [11]: *Let* f, r, v and N be as given in the statement of the theorem. Note that though we assume f is of exponential order $r \geq 1$, it does hold that f is of exponential order $r \in (0,1)$ *thus* f is of exponential order 1.

Therefore, the assumption $r \geq 1$ is not for excluding functions of exponential order $r \in (0,1)$, but rather for insuring that the previous Lemma applied below, will hold whenever $s \in \mathbb{R}$.

We have

$$\mathcal{L}_{a+v-N}\{\Delta_a^{-v} f\}(s) = \frac{1}{(s+1)^N}\mathcal{L}_{a+v}\{\Delta_a^{-v} f\}(s) + \sum_{k=0}^{N-1}\frac{\Delta_a^{-v}(k+a+v-N)}{(s+1)^{k+1}}$$

$$= \frac{1}{(s+1)^N}\mathcal{L}_{a+v}\{\Delta_a^{-v} f\}(s).$$

Taking the N zeros of $\Delta_a^{-v} f$ into account. Moreover, we conclude that

$$\mathcal{L}_{a+v}\{\Delta_a^{-v} f\}(s) = \sum_{k=0}^{\infty}\frac{\Delta_a^{-v}(k+a+v)}{(s+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty}\frac{1}{(s+1)^{k+1}}\sum_{r=a}^{k+a}\frac{(k+a+v-\sigma(r))^{\underline{v-1}}}{\Gamma(v)} f(r)$$

$$= \sum_{k=0}^{\infty} \frac{1}{(s+1)^{k+1}} \sum_{r=a}^{k+a} f(r)\, h_{v-1}((k+a)-r+a, a-(v-1))$$

$$= \sum_{k=0}^{\infty} \frac{\left(f * h_{v-1}(.,a-(v-1))\right)(k+a)}{(s+1)^{k+1}}$$

$$= \mathcal{L}_a\{f * h_{v-1}(.,a-(v-1))\}(s)$$

$$= (s+1)\mathcal{L}_a\{f\}(s)\mathcal{L}_a\ \{h_{v-1}\{f * h_{v-1}(.,a-(v-1))\}(s).$$

By using the Laplace transform of the convolution, we have

$$\mathcal{L}_{a+v}\{\Delta_a^{-v} f\}(s) = (s+1)\frac{(s+1)^{v-1}}{s^v}\mathcal{L}_a\{f\}(s), \qquad since\ r \geq 1,$$

$$= \frac{(s+1)^{v-1}}{s^v}\mathcal{L}_a\{f\}(s).$$

Thus proving (4.6) we obtain (4.7) as consequence

$$\mathcal{L}_{a+v-N}\{\Delta_a^{-v} f\}(s) = \frac{1}{(s+a)^N}\mathcal{L}_{a+v}\{\Delta_a^{-v} f\}(s)$$

$$= \frac{(s+1)^{v-N}}{s^v}\mathcal{L}_a\{f\}(s), for\ s \in \mathbb{R}.$$

Theorem [11]: (Laplace of Difference)

Suppose $f: \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$ and let $v > 0$ be given, with $N-1 < v \leq N$

. Then for $s \in \mathbb{R}$,

$$\mathcal{L}_{a+N-v}\{\Delta_a^{v} f\}(s) = \frac{s^v}{(s+1)^{v-N}}\mathcal{L}_a\{f\}(s) - \sum_{j=0}^{N-1} s^j \Delta_a^{v-1-j} f(a+N-v). \qquad (4.8)$$

Proof [11]: Let f, r, v and N be as given in the statement of the theorem. We already know that (4.8) holds when $v = N$. If $N-1 < v < N$, on the other hand, then $0 < N - v < 1$ and we may apply (4.7) and composition rule in succession as follows:

$$\mathcal{L}_{a+N-v}\{\Delta_a^v f\}(s) = \mathcal{L}_{a+N-v}\left\{\Delta^N \Delta_a^{-(N-v)} f\right\}(s)$$

$$= s^N \mathcal{L}_{a+N-v}\left\{\Delta_a^{-(N-v)} f\right\}(s) - \sum_{j=0}^{N-1} s^j \, \Delta^{N-1-j} \Delta_a^{-(N-v)} f(a+N-v)$$

$$= s^N \frac{(s+1)^{N-v}}{s^{N-v}} \mathcal{L}_a\{f\}(s) - \sum_{j=0}^{N-1} s^j \, \Delta^{N-1-j} \Delta_a^{-(N-v)} f(a+N-v)$$

$$= \frac{s^v}{(s+1)^{N-v}} \mathcal{L}_a\{f\}(s) - \sum_{j=0}^{N-1} s^j \, \Delta^{N-1-j} \Delta_a^{-(N-v)} f(a+N-v).$$

Theorem [11]: We may certainly compose the results from the previous theorem. In particular, observe that under the same assumption as in these two theorems, we have for $s \in \mathbb{R}$

$$\mathcal{L}_{(a+N-v)+N-v}\{\Delta_{a+v-N}^v \Delta_a^{-v} f\}(s) = \mathcal{L}_a\{\Delta_{a+v-N}^v(\Delta_a^{-v} f)\}(s)$$

$$= \frac{s^v \mathcal{L}_{a+v-N}\{\Delta_a^{-v} f\}(s)}{(s+1)^{v-N}} - \sum_{j=0}^{N-1} s^j \Delta_{a+v-N}^{v-1-j} \Delta_a^{-v} f((a+v-N)+N-v)$$

$$= \frac{s^v}{(s+1)^{v-N}} \left[\frac{(s+1)^{v-N}}{s^v} \mathcal{L}_{(a)}\{f\}(s)\right] - \sum_{j=0}^{N-1} s^j \Delta_a^{-(j|+1)} f(a)$$

$$= \mathcal{L}_{(a)}\{f\}(s).$$

Proof [11]: The proof is similar to the proof of the previous theorem.

4.4 A Power Rule and Composition Rule

In this subsection we prove the power rule and composition rule using the Laplace transform.

Theorem [11]: (Power rule)

Let $v, \mu > 0$ *be given. Then for* $t \in \mathbb{N}_{a+\mu+v}$, we have

$$\Delta_{a+\mu}^{-v}\left((t-a)^{\underline{\mu}}\right) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+v)}(t-a)^{\underline{\mu+v}}. \tag{4.9}$$

Proof [11]: With the previous Remark in hand, we have for $s \in \mathbb{R}$,

$$\mathcal{L}_{a+\mu+v}\{\Delta_{a+\mu}^{-v}(t-a)^{\underline{\mu}}\}(s) = \frac{(s+1)^{v}}{s^{v}}(t-a)^{\underline{\mu}}\,(s)\}$$

$$= \frac{(s+1)^{v}}{s^{v}}\Gamma(\mu+1)\mathcal{L}_{a+\mu}\{h_{\mu}(.,a)\}(s)$$

$$= \frac{(s+1)^{v}}{s^{v}}\Gamma(\mu+1)\frac{(s+1)^{\mu}}{s^{\mu+1}}$$

$$= \Gamma(\mu+1)\frac{(s+1)^{\mu+1}}{s^{\mu+v+1}}$$

$$= \Gamma(\mu+1)\mathcal{L}_{a+\mu+v}\{h_{\mu+v}(.,a)\}(s)$$

$$= \mathcal{L}_{a+\mu+v}\left\{\frac{\Gamma(\mu+1)}{\Gamma(\mu+v+1)}\,(t-a)^{\underline{\mu+v}}\right\}(s).$$

By the one-to-one property of the Laplace transform, it follows that

$$\Delta_{a+\mu}^{-v}\,(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+v)}(t-a)^{\underline{\mu+v}}, for\ t \in \mathbb{N}_{a+\mu+v}.$$

Theorem [7]: (Composition of a Fractional Sum with a Fractional Sum)

Suppose that $f: \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 0$ and let $v, \mu > 0$ be given. Then for $t \in \mathbb{R}_{a+\mu+v}$,

$$\Delta_a^{-v}\Delta_a^{-\mu} f(t) = \Delta_a^{-v-\mu} f(t) = \Delta_a^{-\mu}\Delta_a^{-v} f(t).$$

Proof [7]: Let f, r, μ and v be as given in the statement of the theorem. It follows from the Corollary 1 that

$$\mathcal{L}_{a+\mu+v}\{\Delta_a^{-v}\Delta_a^{-\mu} f\},\ \mathcal{L}_{a+\mu}\{\Delta_a^{-\mu} f\}\ and\ \mathcal{L}_{a+(v+\mu)}\{\Delta_a^{-(v+\mu)} f\}$$

all exist on $\mathbb{R}$. Therefore, we may apply (4.6) multiple times to write for $s \in \mathbb{R}$,

$$\mathcal{L}_{a+\mu+v}\{\Delta_a^{-v}\Delta_a^{-\mu} f\}(s) = \frac{(s+1)^v}{s^v}\mathcal{L}_{a+\mu}\{\Delta_a^{-\mu} f\}(s)$$

$$= \frac{(s+1)^v}{s^v}\frac{(s+1)^\mu}{s^\mu}\mathcal{L}_a\{f\}(s)$$

$$= \frac{(s+1)^{v+\mu}}{s^{v+\mu}}\mathcal{L}_a\{f\}(s) = \mathcal{L}_{a+(v+\mu)}\left\{\Delta_a^{-(v+\mu)} f\right\}(s)$$

$$= \mathcal{L}_{a+\mu+v}\{\Delta_a^{-v-\mu} f\}(s).$$

The result then follows from symmetry and the one-to-one property of the Laplace transform.

4.5 A Fractional Initial Value Problem

By the far most substantial of the Laplace transform is presented in theorem below. Note that the fractional initial value problem solved below by the *fractional Laplace transform method* is identical to problem solved in previous chapter using the fractional composition rules.

Theorem [11]: *Suppose $f: \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$ and let $v > 0$ be given with $N - 1 < v \leq N$. The unique solution to the fractional initial value problem*

$$\begin{cases} \Delta^{v}_{a+v-N}\, y(t) = f(t), & t \in \mathbb{N}_a, \\ \Delta^{i} y(a+v-N) = A_i, & i \in \{0, 1, \ldots, N-1\}; A_i \in \mathbb{R} \end{cases}$$

is given by

$$y(t) = \sum_{i=0}^{N-1} \alpha_i (t-a)^{\underline{i+v-N}} + \Delta_a^{-v} f(t), for\ t \in \mathbb{N}_{a+v-N},$$

where

$$\alpha_i := \sum_{p=0}^{i} \sum_{k=0}^{i-p} \frac{(-1)^k}{i!} (i-k)^{\underline{N-v}} \binom{i}{p} \binom{i-p}{k} A_p,$$

$for\ i \in \{0, 1, \ldots, N-1\}$.

Proof: The proof is presented in [11].

CHAPTER 5

NABLA FRACTIONAL LAPLACE TRANSFORM

The Laplace transform of the Nabla fractional discrete calculus is similar that one of the Delta fractional discrete calculus. But all the results in here look easier to be obtained when this transform is performed.

5.1 Definitions and Properties

Definition 1 [13]: For a function $f: \mathbb{N}_a \to \mathbb{R}\ and\ s \in \mathbb{R},$ we define the Laplace transform of f by

$$\mathcal{L}_a\{f\} := \int_a^\infty e^p_{\ominus s}(t,a) f(t) \nabla\, t, \tag{5.1}$$

or in its sum form

$$\mathcal{L}_a\{f\} := \sum_{k=1}^{\infty} (1-s)^{k-1} f(a+k), \tag{5.2}$$

which is easily verified using the techniques explained in time scales books.

The linearity of this transform follows from its definition. However, we still need to discuss the uniqueness and existence of such transform.

Definition 2 [15]: A function $f: \mathbb{N}_a \to \mathbb{R}$ is said to be exponential order α if there exist a constant $M > 0$ and a number $T \in \mathbb{N}_a$ such that $|f(t) \leq M|e_\alpha(t,a)|$ for all $t \in \mathbb{N}_T$.

Theorem (Existence of the Laplace Transform) [16]: The Laplace transform of any function of exponential order α exists for $\left|\frac{1-s}{1-\alpha}\right| < 1$.

Proof [15]: Let f be a functional of exponential order α, so there are constants $M > 0$ and $T \in \mathbb{N}_a$ such that $|f(t)| \leq M|e_\alpha(t,a)|$ for all $t \in \mathbb{N}_T$. Fix an integer N such that it is both greater than one and greater than $T - a$, so $a + N > T$. Then for all $k \geq N$, we have that

$$|f(a+k)| \leq M|e_\alpha(a+k,a) = \frac{M}{|1-\alpha|}.$$

Multiply both sides by $|1-s|^{k-1}$ and taking the sum from $k = N$ to infinity, it follows that for $\left|\frac{1-s}{1-\alpha}\right| < 1$,

$$\sum_{k=N}^{\infty} |1-s)^{k-1} f(a+k) \leq M \frac{1}{|1-s|} \sum_{k=N}^{\infty} (\frac{|1-s|}{|1-\alpha)})^k$$

$$= M \frac{1}{|1-s|} (\frac{|1-s|}{|1-\alpha|})^N \sum_{k=0}^{\infty} (\frac{|1-s|}{|1-\alpha|})^k < \infty.$$

It follows that $\mathcal{L}_a\{f\}(s)$ converges absolutely for $\left|\frac{1-s}{1-\alpha}\right| < 1$.

Theorem [13]: (Uniqueness of the Laplace Transform).

$\mathcal{L}_a\{f\}(s) = 0 \; if \; and \; only \; if \;\; f(t) = 0 \; for \; t \in \mathbb{N}_{a+1}$.

Proof [13]: The backward direction is trivial. Now we consider the forward direction. Assume that $\mathcal{L}_a\{f\}(s) = 0$. This means that $\sum_{k=1}^{\infty}(1-s)^{k-1} f(a+k) = 0$. Let $z = 1 - s$, shift the index of the sum down by 1 and let $b_k = f(a+k+1)$. So now we have $\sum_{k=0}^{\infty} b_k z^k = 0$. Suppose for a contradiction that there is an integer α such that

$b_\alpha \neq 0$,and without loss of generality, suppose α is the smallest such integer. Since $s \neq 1$, then $z \neq 0$ for all n. By definition of p, it follows that $\lim_{n\to\infty} p(z_n) = b_\alpha$. However, this implies that $b_\alpha = 0$, which is a contradiction. We therefore have that $b_k = 0$ for all k, so $f(t) = 0\ for\ t \in \mathbb{N}_{a+1}$.

Theorem [13]:(Laplace Transform of Taylor Monomials with Integer Index)

For all non-negative integers n, we have

$$\mathcal{L}_a\{h_n(.,a)\}(s) = \frac{1}{s^{n+1}}\ for\ |1-s| < 1. \tag{5.3}$$

Proof [13]: The proof is by induction on n. By definition $h_0(t,a) = 1$, so

$$\mathcal{L}_a\{1\}(s) = \sum_{k=1}^{\infty}(1-s)^{k-1} = \frac{1}{1-(1-s)} = \frac{1}{s}, \text{ for } |1-s| < 1.$$

Suppose now that $\mathcal{L}_a\{h_n(.,a)\}(s) = \frac{1}{s^{n+1}}$ for some $n \geq 0$ and

for $|1-s| < 1$. Then, consider $\mathcal{L}_a\{h_{n+1}(.,a)\}(s) = \int_a^\infty e_{\ominus s}^{p}(t,a)h_{n+1}(t,a)\nabla t$. We will apply the integration by parts formula (1) with $u(t) = h_{n+1}(t,a)$,

$\nabla ut = e_{\ominus s}^{p}(t,a)$. It then follows that $\nabla ut = h_n(t,a)$ and it can be shown that

$v(t) = -\frac{1}{s}e_{\ominus s}(t,a)$ is a (Nabla) antidifference of $e_{\ominus s}(t,a)$. This means that

$$\mathcal{L}_a\{h_{n+1}(.,a)\}(s) = -\frac{1}{s}e_{\ominus s}(t,a)h_{n+1}(t,a)|_a^\infty + \frac{1}{s}\int_a^\infty e_{\ominus s}(t,a)h_n(t,a)\nabla t$$

$$= -\frac{1}{s}(1-s)^{t-a}h_{n+1}(t,a)|_a^\infty + \frac{1}{s}\mathcal{L}_a\{h_n(.,a)\}(s).$$

Evaluating the first terms as $t \to \infty$, given the assumption that $|1-s| < 1$, it means that the term goes to zero. Likewise, it is easy to show that $h_n(a,a) = 0$ for all $n \geq 1$, thus we have that $\mathcal{L}_a\{h_{n+1}(.,a)\}(s) = \frac{1}{s^{n+2}}$ completing the proof.

Now we present the Laplace transform the factional order Nabla Taylor monomials.

Theorem [13]: (Laplace Transform of Taylor Monomials with Non-Integer Index)

For $a\ noninteger\ real\ number\ v\ , we\ have$

$$\mathcal{L}_a\{h_v(.,a)\}(s) = \frac{1}{s^{v+1}}, for\ |1-s| < 1. \tag{5.4}$$

Proof [13]:

We have

$$\mathcal{L}_a\{h_v(.,a)\}(s) = \sum_{k=1}^{\infty}(1-s)^{k-1}h_v(a+k,a)$$

$$= \sum_{k=1}^{\infty}(1-s)^{k-1}\frac{\Gamma(k+v)}{\Gamma(k)\Gamma(v+1)}$$

$$= \sum_{k=1}^{\infty}(1-s)^{k-1}\frac{(1+v)^{\overline{k}}}{\Gamma(k+1)}$$

$$= \sum_{k=1}^{\infty}(1-s)^{k}\frac{\Gamma(-v)}{\Gamma(k+1)\Gamma(-v-k)}$$

$$= \sum_{k=1}^{\infty}\binom{-(v+1)}{k}(-(1-s))^{k} = \frac{1}{s^{v}+1}.$$

5.2 The Convolution of Two Functions

In order to find the Laplace transform of the Nabla fractional sums and differences we have to define the convolution of two functions in the frame of Nabla discrete calculus.

Definition 3 [13]: *For* $f, g: \mathbb{N}_a \to \mathbb{R}$ and all $t \in \mathbb{N}_{a+1}$, *we define the convolution of* f *and* g *by*

$$(f * g)(t) := \int_a^t f(t - \rho(s) + a) g(s) \nabla s. \tag{5.5}$$

Theorem [13]: (Fractional Sum as Convolution)

Let $v \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$ *and* $f: \mathbb{N}_a \to \mathbb{R}$, then

$$\nabla_a^{-v} f(t) = (h_{v-1}(., a) * f(t). \tag{5.6}$$

Proof [13]:

We have

$$(h_{v-1}(., a) * f(.)(t) \quad = \int_a^t h_{v-1}(t - \rho(s) + a, a) f(s) \nabla s$$

$$= \frac{1}{\Gamma(v)} \int_a^t h_{v-1}(t - \rho(s))^{\overline{v-1}} f(s) \nabla s = \nabla_a^{-v} f(t).$$

Theorem [13]: (Convolution Theorem)

$$For\ f, g: \mathbb{N}_a \to \mathbb{R}, we\ have\ that \mathcal{L}_a\{f(* g\}(s) = \mathcal{L}_a\{f\}(s) \mathcal{L}_a\{g\}(s). \tag{5.7}$$

Proof [13]:

$$\mathcal{L}_a\{f * g\}(s) = \sum_{k=1}^{\infty} (1 - s)^{k-1} (f * g)\ (a + k)$$

$$= \sum_{k=1}^{\infty} (1 - s)^{k-1} \sum_{r=a+1}^{a+k} f(a + k - \rho(r) + a) g(r)$$

$$= \sum_{k=1}^{\infty} \sum_{r=1}^{k} (1 - s)^{k-1}\ f(a + k - \rho(r) + a) g(r)$$

$$= \sum_{k=1}^{\infty} \sum_{r=1}^{k} (1-s)^{k-1} \; f(a+k-\rho(r)+a) g(r)$$

$$= \sum_{r=1}^{\infty} (1-s)^{r-1} \; g(r+a) \sum_{k=1}^{\infty} (1-s)^{k-1} \; f(k+a) = \mathcal{L}_a\{g\}(s) \mathcal{L}_a\{f\}(s).$$

5.3 The Laplace Transforms of Nabla Fractional Sums and Differences

In order to be able to solve initial value problems for Nabla fractional difference equations we need to extend the properties of the Laplace transfom. So that it contains The Laplace transforms of Nabla fractional sums and differences.

Theorem [13]: (Transformation of Fractional Sums)

$For f: \mathbb{N}_a \to \mathbb{R} \; and \; v \in \mathbb{R}^+, we \; have$

$$\mathcal{L}_a\{\nabla_a^{-v} f\}(s) = \frac{1}{s^v} \mathcal{L}_a\{f\}(s). \tag{5.8}$$

Proof [13]:

$$\mathcal{L}_a\{\nabla_a^{-v} f\}(s) = \mathcal{L}_a\{h_{v-1}(.,a) * f\}(s) = \mathcal{L}_a\{h_{v-1}(.,a)\}(s) \mathcal{L}_a\{f\}(s)$$

$$= \frac{1}{s^v} \mathcal{L}_a\{f\}(s).$$

We want to establish similar properties of fractional differences; however, we must first establish integer-order difference properties. In order to do this, we need to find the Laplace transforms of the Nabla difference of integer orders..

Theorem [15]: (Transform of Nabla Difference)

$For f: \mathbb{N}_a \to \mathbb{R} \; and \; v \in \mathbb{R}^+, we \; have$

$$\mathcal{L}_{a+1}\{\nabla f\}(s) = s \mathcal{L}_{a+1}\{f\}(s) - f(a+1). \tag{5.9}$$

Proof [15]:

$$\mathcal{L}_{a+1}\{\nabla f\}(s) = \sum_{k=1}^{\infty}(1-s)^{k-1}\nabla f(a+1+k)$$

$$= \sum_{k=1}^{\infty}(1-s)^{k-1}(\mathrm{f}(\mathrm{a}+1+\mathrm{k}) - \mathrm{f}(\mathrm{a}+\mathrm{k})$$

$$= \mathcal{L}_{a+1}\{\nabla f\}(s) - \sum_{k=0}^{\infty}(1-s)^{k} f(a+1+k)$$

$$= \mathcal{L}_{a+1}\{\nabla f\}(s) - f(a+1) - (1-s)\sum_{k=1}^{\infty}(1-s)^{k} f(a+1+k)$$

$$= \mathcal{L}_{a+1}\{\nabla f\}(s)\big(1 - f(a+1) - (1-s)\big) - f(a+1)$$

$$= \mathcal{L}_{a+1}\{\nabla f\}(s) - f(a+1).$$

Theorem [13]: (The Laplace Transform of Fractional Differences)

$$\mathcal{L}_{a+n}\{\nabla^{n} f\}(s) = s^{n}\mathcal{L}_{a+n}\{\nabla^{n} f\}(s) - \sum_{k=1}^{n} s^{n-k}\nabla^{k-1} f(a+n).$$

Proof [13]: The results follow from induction on n with the previous theorem as a base case. The inductive step is omitted. Then, we want to find the Laplace transform of a v^{th} order difference where $0 < v < 1$. First, however, a useful lemma will be necessary.

Lemma [13]: (Shifting Rule)

Given $f: \mathbb{N}_a \to \mathbb{R}$, we have

$$\mathcal{L}_{a+n}\{f\}(s) = \frac{1}{1-s}\mathcal{L}_{a}\{f\}(s) - \frac{1}{1-s} f(a+1).$$

Proof [13]:

$$= \frac{1}{1-s}\mathcal{L}_a\{f\}(s) + \frac{1}{1-s}\sum_{k=1}^{\infty}(1-s)^{k-1}f(a+k+1)$$

$$= \frac{1}{1-s}f(a+1) + \mathcal{L}_{a+1}\{f\}(s).$$

The desired results follows from this form.

With this, we are ready to provide the general from of the Laplace transform of a v^{th} order, $0 < v < 1$, fractional order difference.

Theorem [13]: Given $f: \mathbb{N}_a \to \mathbb{R}$ and $0 < v < 1$, then for $t \in \mathbb{N}_{a+1}$, we have

$$\mathcal{L}_{a+1}\{\nabla_a^v f\}(s) = s^v \mathcal{L}_{a+1}\{f\}(s) - \frac{1-s^v}{1-s} f(a+1).$$

Proof [13]: Consider the following:

$$\mathcal{L}_{a+1}\{\nabla_a^v f\}(s) = \mathcal{L}_{a+1}\left\{\nabla \nabla_a^{-(1-v)} f\right\}(s)$$

$$= s\mathcal{L}_{a+1}\left\{\nabla_a^{-(1-v)} f\right\}(s) - \nabla_a^{-(1-v)} f(a+1).$$

We observe that

$$\nabla_a^{-(1-v)} f(a+1) = f(a+1),$$

therefore, we obtain:

$$\mathcal{L}_{a+1}\{\nabla_a^v f\}(s) = s\left(\frac{1}{1-s}\mathcal{L}_a\left\{\nabla_a^{-(1-v)} f\right\}(s) - \frac{1}{1-s} f(a+1)\right) - f(a+1 = \frac{s^v}{1-s}\mathcal{L}_a\{f\}(s) - \frac{1}{1-s} f(a+1)$$

.

5.4 Generalized Power Rule

Theorem [13]: (Generalized Power Rule)

Let $v \in \mathbb{R}^+$ and $\in \mathbb{R}$ such that μ and $v + \mu$ are not negative integers, then for

$t \in \mathbb{N}_a$, we have

(i.) $$\nabla_a^{-v}(t-a)^{\overline{\mu}} = \left(\frac{\Gamma(\mu+1)}{\Gamma(\mu+v+1)}\right)(t-a)^{\overline{\mu+v}}$$

,

(ii.) $$\nabla_a^{v}(t-a)^{\overline{\mu}} = \left(\frac{\Gamma(\mu+1)}{\Gamma(\mu+v+1)}\right)(t-a)^{\overline{\mu-v}}.$$

Proof [13]: We will first establish the result for (i.), after which (ii.) will follow. Consider the following:

$$\mathcal{L}_a\{\nabla_a^{-v} h_\mu(.,a)\}(s) = \frac{1}{s^v}\mathcal{L}_a\{h_\mu(.,a)\}(s) = \frac{1}{s^{v+\mu+1}}$$

$$= \mathcal{L}_a\{h_{v+\mu}(.,a)\}(s).$$

By definiton 3 and the linearity of the Laplace transform, (i.) holds for $t \in \mathbb{N}_{a+1}$. Obderving that for $t = a$ and stated equality holds, hence (i.) follows for all $t \in \mathbb{N}_a$.

Now for (ii.), choose N such that $N - 1 < v \leq N$ and consider the following:

$$\nabla_a^{v}(t-a)^{\overline{\mu}} = \nabla^N \nabla_a^{-(N-v)}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+N-v+1)}\nabla^N(t-a)^{\overline{\mu+N-v}}.$$

5.5 Solutions to Initial Value Problems

We now will consider a general v^{th}-order fractional nabla initial-value problem and give a formula for its solution when, $1 < v \leq 2$. Then, the fractional initial value problem.

$$\nabla_a^v f(t) = 0, \qquad t \in N_{a+2},$$

$$f(a+2) = A_0, \quad A_0 \in R,$$

$$\nabla f(a+2) = A_1, \; A_1 \in \mathbb{R},$$

has the unique solution

$$f(t) = [(2-v)A_0 + (v-1)A_1]h_{v-1}(t,a) + [(v-1)A_0 - v\,A_1]h_{v-2}(t,a). \qquad (5.10)$$

Proof [15]: Taking the Laplace transform (based at $a+2$) of both sides of

$\nabla_a^v f(t) = 0$, we have

$$\frac{s^v}{(1-s)^s}\mathcal{L}_a\{f\}(s) - \sum_{k=0}^{1}\left[\frac{s^2}{(1-s)^{2-k}}\nabla_a^{-(2-v)} f(a+k+1) + \Delta^{2-k-1}\nabla_a^{-(2-v)} f(a+2)s^k\right] = 0.$$

Expanding this out, we get:

$$\frac{s^v}{(1-s)^2}\mathcal{L}_a\{f\}(s) - \left(\frac{s}{1-s}\right)^2\nabla_a^{-(2-v)} f(a+1) - \nabla\nabla_a^{-(2-v)} f(a+2)$$

$$-\left(\frac{s}{1-s}\right)^2\nabla_a^{-(2-v)} f(a+2) - s\nabla_a^{-(2-v)} f(a+2) = 0.$$

Now, we substitute $f(a+2) = A_0$ and $\nabla f(a+2) = A_1$, where

$$f(a+1) = f(a+2) - \nabla f(a+2) = A_0 - A_1.$$

Also, note (by Lemma 2) that:

$$\nabla_a^{-(2-v)} f(a+2) = [(2-v)[A_0 - A_1] + A_0]$$

and

$$\nabla\nabla_a^{-(2-v)} f(a+2) = \nabla_a^{-(1-v)} f(a+2) = [(1-v)[A_0 - A_1] + A_0].$$

$\mathcal{L}_{a+2}\{\nabla_a^v f\}(s) = \frac{s^v}{(1-s)^2}\mathcal{L}_a\{f\}(s) - (\frac{s}{1-s})^2[A_0 - A_1] - [(1-v)[A_0 - A_1] + A_0] - (\frac{s}{1-s})[(2-v)[A_0 - A_1] + A_0 = 0$

.

Next, we combine all terms with respect to A_0 and A_1 with common denominator of $(1-s)^2$.

$$= \frac{s^v}{(1-s)^s}\mathcal{L}_a\{f\}(s) - \frac{vs - s - v + 2}{(1-s)^2}A_0 + \frac{v - vs - 1}{(1-s)^2}A_1 = 0$$

$$= \frac{s^v}{(1-s)^s}\mathcal{L}_a\{f\}(s) = \frac{vs - s - v + 2}{(1-s)^2}A_0 - \frac{v - vs - 1}{(1-s)^2}A_1 = 0.$$

We now rearrange the terms and solve for the Laplace transform of $f(t)$.

$$\mathcal{L}_a\{f\}(s) = [(2-v)A_0 + (v-1)A_1]\frac{1}{s^v} + [(v-1)A_0 + v\,A_1]\frac{1}{s^{v-1}}.$$

Finally, we take the inverse Laplace transform to get the desired result

$$f(t) = [(2-v)A_0 + (v-1)A_1]h_{v-1}(t,a) + [(v-1)A_0 - v\,A_1]h_{v-2}(t,a)$$

$for\ t \in N_{a+2}.$

Next, we look at the non-homogenous equation with zero initial conditions.

Theorem [15]: Let $f, g: N_a \to R\ and\ 1 < v \le 2$. Then, for $t \in N_{a+2}$, the fractional initial value problem:

$$\nabla_a^v f(t) = g(t), \qquad t \in N_{a+2}$$

$$f(a+2) = 0,$$

$$\nabla f(a+2) = 0,$$

has the solution:

$$f(t) = \nabla_a^{-v} g(t) - [g(a+1) + g(a+2))]h_{v-1}(t,a) + g(a+2)h_{v-2}(t,a). \qquad (5.11)$$

Proof [15]: We take the Laplace transform based at $a+2$ of both sides of the equation.

$\mathcal{L}_a\{f\}(s) = \mathcal{L}_{a+2}\{g\}(s)$.

Next, we use the Corollary 1 on the left hand side and the Laplace transform shifting theorem on the right hand side of the equation. We have

$$\frac{s^v}{(1-s)^2}\mathcal{L}_a\{f\}(s) - \left(\frac{s}{1-s}\right)^2 \nabla_a^{-(2-v)} f(a+1) - \nabla\nabla_a^{-(2-v)} f(a+2) - \left(\frac{s^2}{1-s}\right)\nabla_a^{-(2-v)} f(a+2) - s\,\nabla_a^{-(2-v)} f(a+2) = \frac{1}{(1-s)^2}\mathcal{L}_a\{g\}(s) - \frac{1}{(1-s)^2}\, g(a+1) - \frac{1}{1-s}\, g(a+1)$$

.

Now, we plug in $f(a+2) = 0$. Also, all of the fractional sums and $f(a+1)$ and $\nabla f(a+2)$. Thus, we can plug in zero for them as well.

$$\frac{s^v}{(1-s)^2}\mathcal{L}_a\{f\}(s) = \frac{1}{(1-s)^2}\mathcal{L}_a\{g\}(s) - \frac{1}{(1-s)^2}\, g(a+1) - \frac{1}{1-s}\, g(a+1).$$

Next, we solve for the Laplace transform of $f(t)$ to get

$$\mathcal{L}_a\{f\}(s) = \frac{1}{s^v}\mathcal{L}_a\{g\}(s) - \frac{1}{s^v}\, g(a+1) - \frac{(1-s)}{s^v}\, g(a+2)$$

$$= [\mathcal{L}_a\{h_{v-1}(t,a)\}(s)\mathcal{L}_a\{g\}(s)] = \frac{1}{s^v}\, g(a+1) - \frac{1}{s^v}\, g(a+2) + \frac{1}{s^{v-1}}\, g(a+2).$$

Finally, we take the inverse Laplace transform and note that

$\nabla_a^{-v} g(t) = h_{v-1}(t,a) * g(t)$ to get:

$$f(t) = [h_{v-1}(.,a) * g(.)] - [g(a+1) + g(a+2)]h_{v-1}(t,a) + g(a+2)h_{v-2}(t,a)$$

$$= \nabla_a^{-v} g(t) - [g(a+1) + g(a+2)]g_{v-1}(t,a) + g(a+2)h_{v-2}(t,a)$$

$for\ t \in \mathbb{N}_{a+2}$.

CHAPTER 6

CONCLUSION

The fractional calculus is a field of applied mathematics that studies the integration and differentiation of functions of any order. And it turned out that this calculus is a strong tool that can be used when scientists want to mathematically model physical phenomena happening in our real world.

Fractional calculus has a discrete version. The question is whether the discrete version of this calculus will also have its role in the mathematical modeling. Indeed, many papers have reported that the discretized fractional calculus is significant in mathematical modeling as well (see for example Refs.[30-33] and the references therein).

In this thesis, we discussed the discrete fractional calculus in the frame of two operators which are the forward operator (Delta) and the backward operator (Nabla). Even though they are similar in a sense, each of this calculus has its advantage and is superior to the other in other senses.

Due to need of a method to solve fractional difference equations, we discussed the Laplace transform in both frames, the Delta and the Nabla operators.

This thesis can be considered as a survey on discrete fractional calculus and the discrete fractional Laplace transform. And we hope that this thesis will be a useful reference for the next generation of young mathematicians in case they want to work on discrete fractional difference equations.

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