ORTHOGONALITY OF STURM-LIOUVILLE PROBLEMS AND SOME ASYMPTOTIC BEHAVIOURS

AHMED ALJUMAILI

JANUARY 2015

ORTHOGONALITY OF STURM-LIOUVILLE PROBLEMS
AND SOME ASYMPTOTIC BEHAVIOURS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES OF
ÇANKAYA UNIVERSITY

BY
AHMED ALJUMAILI

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCES
IN
THE DEPARTMENT OF
MATHEMATICS AND COMPUTER SCIENCE

JANUARY 2015

Title of the Thesis : **Orthogonality of Sturm-Liouville Problems and Some Asymptotic Behaviours.**

Submitted by **Ahmed ALJUMAILI**

Approval of the Graduate School of Natural and Applied Sciences, Çankaya University.

Prof. Dr. Taner ALTUNOK
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Billur KAYMAKÇALAN
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Prof. Dr. Kenan TAŞ
Supervisor

Examination Date: 30.01.2015

Examining Committee Members

Prof. Dr. Billur KAYMAKÇALAN	(Çankaya Univ.)	
Prof. Dr. Kenan TAŞ	(Çankaya Univ.)	
Assoc. Prof. Dr. Fahd JARAD	(THKU)	

STATEMENT OF NON-PLAGIARISM PAGE

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: Ahmed, ALJUMAILI

Signature :

Date : 30.01.2015

ABSTRACT

ORTHOGONALITY OF STURM-LIOUVILLE PROBLEMS AND SOME ASYMPTOTIC BEHAVIOURS

ALJUMAILI, Ahmed

M.Sc., Department of Mathematics and Computer Science

Supervisor: Prof. Dr. Kenan TAŞ

January 2015, 44 pages

In this thesis, we discussed some important aspects of Sturm-Liouville theory such as orthogonality, Fourier series and asymptotic formulas for eigenvalues and eigenfunctions. After giving an introduction about the history of Sturm-Liouville Theory. We gave the way to convert any problem into regular Sturm-Liouville problem by finding suitable weight function. We included a method of finding asymptotic formulas for eigenvalues and eigenfunctions of Sturm-Liouville Problems. We also obtain an important formula about the solutions of a specific SL-problem and obtain formulas of eigenvalues and eigenfunctions of this problem.

Keywords: Sturm-Liouvile Problem, Eigenvalues and Eigenfunctions, Orthogonality, Weight Function, Fourier Series, Asymptotic Behaviours.

ÖZ

STURM-LIOUVILLE PROBLEMLERİNDE DİKLİK VE BAZI ASİMTOTİK DAVRANIŞLAR

ALJUMAILI, Ahmed
Yüksek Lisans, Matematik-Bilgisayar Anabilim Dalı
Tez Yöneticisi: Prof. Dr. Kenan TAŞ
Ocak 2015, 44 sayfa

Bu tezde Sturm-Liouville teorisi ile ilgili olarak diklik ve Fourier serileri kavramları ile özdeğer ve özvektörler için asimtotik formüller gibi bazı önemli kavramları incelenmiştir.

Sturm-Liouville teorisi hakkında genel bir bilgilendirme yapılmış ve, herhangi bir problemin uygun bir ağırlık fonksiyonu tanımlanarak nasıl düzenli bir Sturm-Liouville problemine dönüştürülebileceği incelenmiştir.

Sturm-Liouville problemlerinin özdeğer ve özvektörleri için asimtotik formüllerin nasıl bulunabileceği hakkında bir yöntem verilmiştir. Ayrıca özel bir SL-problemi için özdeğer ve özvektörleri ve dolayısıyla çözümleri veren önemli bir formül elde edilmiştir.

Anahtar Kelimeler: Sturm-Liouville Problemi, Özdeğer ve Özvektörler, Diklik, Ağırlık Fonksiyonu, Fourier Serileri, Asimtotik Davranışlar.

ACKNOWLEDGEMENTS

I thank Allah, The Almighty for His help and endless support in enlightening my path of knowledge, of this humble work.

I would like to take this opportunity to show my sincere gratitude to all people who helped me and gave me the possibility along the way to achieve this dissertation.

Here, I would like to extend my thanks and warm appreciation to my supervisor Prof. Dr. Kenan TAŞ, who supported, encouraged, and paved my way through the different stages of study. I would also like to thank the Head of Department in the Graduate School of Natural and Applied Sciences at Cankaya University Prof. Dr. Billur KAYMAKÇALAN for her precious help and support during my study.

Finally, I have to thank my family and all my faithful friends, who supported and motivated me to continue this research.

To them all I am very grateful.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Background

Sturm-Liouville theory was established by two mathematicians. The first one is the Swiss mathematician Charles François Sturm, who was born in 1803 in Geneva and also he took his education in this city. He put his remark on Geneva by his scientific activities. After that in 1825 he transferred to Paris and he settled there until his death in 1855. He did most of his scientific research in Paris. The second scientist is the French mathematician Joseph Liouville (1809-1882), who became a close friend with Sturm and his cooperator [1].

Charles François Sturm and Joseph Liouville published in 1836-1837 many papers about second order linear ordinary differential equations, which included boundary value problems. As a result of their scientific efforts during this period, what is known as Sturm-Liouville Theory came into existence. The effect of these papers exceeds their topic to general linear and nonlinear differential equations and analysis which includes functional analysis. They were the first to feel its importance to find directly the properties of solutions from the equation even when no analytic expressions for solutions are available [2].

A Sturm-Liouville Problem is a special kind of boundary value problem, which consists of a second order of linear differential equation and two complementary conditions. A second order differential operator which is a self-edjoint in Sturm-Liouville problems has orthogonal sequence of eigenfuncitons which leads to essential theory of Fourier series [3].

In spite of the fact that the topic of Sturm-Liouville problem is more than 170 years old, many Mathematicians, Physicists and engineers have written thousands of papers about it. However, this topic is active field of research today. And also every year many papers are published on Sturm-Liouville problems.

1.2 Organization of the Thesis

This thesis contains five chapters. It covers some important aspects of Sturm-Liouville theory such as orthogonality, Fourier series and asymptotic formulas for eigenvalues and eigenfunctions .

Chapter 1 is an introduction to the history of Sturm-Liouville Theory.

Chapter 2 includes a definition of Sturm-Liouville Problem and some essential theorems of Sturm-Liouville Problem and its solutions (eigenfuntions), which leads to essential theories of Fourier series.

Chapter 3, includes the way to convert any problem into regular Sturm-Liouville problem by finding suitable weight function.

Chapter 4, includes finding asymptotic formulas for eignvalues and eigenfunctions of Sturm-Liouville Problems.

Chapter 5 includes the conclusion.

CHAPTER 2

ORTHOGONALITY OF STURM-LIOUVILLE PROBLEMS

In this chapter we will provide boundary value problem known as Sturm-Liouville problem and several important notions including :

- Eigenvalues and eigenfunctions.

- Orthogonality.

- Expansion of function.

- Fourier series.

These notions are often used in the applications of differential equations in engineering and physics.

Sometimes we will denote to Sturm-Liouville problem as SL-problem.

2.1 Sturm-Liouville Problem :

2.1.1 Definition of regular and singular SL-problem:

1. A second-order homogenous linear differential equation which is in the form :

$$\frac{d}{dx}\left[c_0(x)\frac{dy}{dx}\right]+\left[c_2(x)+\lambda s(x)\right]y=0 \qquad (2.1)$$

where both $c_0(x)$ and $s(x)$ are continuous and positive on the closed interval $[\alpha,\beta]$, and $c_0'(x)$ esists and is continuous on $[\alpha,\beta]$, and $c_2(x)$ is real and continous on $[\alpha,\beta]$, and λ is an independent parameter of x .

2. two complementary conditons

$$\left.\begin{array}{r} M_1 y(\alpha)+M_2 y'(\alpha)=0 \\ N_1 y(\beta)+N_2 y'(\beta)=0 \end{array}\right\} \qquad (2.2)$$

where M_1, M_2, N_1 and N_2 are real constants and at least one of M_1 and M_2 are not zero , and at least one of the N_1 and N_2 are not zero .

The equation (2.1) with two complementary conditons (2.2) is called a regular Sturm-Liouville problem [4].

And the equation (2.1) with two complementary conditons (2.2) on the interval $[\alpha,\beta]$ is called singuler Sturm-Liouville problem if at least one of these hold:

a) $\alpha=-\infty$ or $\beta=\infty$ or both of them, it means the interval $[\alpha,\beta]$ is unbounded.

b) $c_0(x)=0$ or $s(x)=0$ for some x belongs to the closed interval $[\alpha,\beta]$.

c) Both the absolute values of $c_0(x)$ and $s(x)$ or one of them gose to infinty when x gose to α or x gose to β or both to of them [5].

The complementary conditions if they are given by

$$y(\alpha)=0 \quad , \quad y(\beta)=0$$

or

$$y'(\alpha)=0 \quad , \quad y'(\beta)=0$$

are consider two exceptionally important cases [6].

2.1.2 Definitions of eigenvalues and eigenfunctions :

The values of the parameter λ in the homogenous linear differential equation

$$\frac{d}{dx}\left[c_0(x)\frac{dy}{dx}\right]+\left[c_2(x)+\lambda s(x)\right]y=0 \qquad (2.1)$$

of the Sturm-Liouville problem with the two complementary conditons

$$\left.\begin{array}{r} M_1 y(\alpha)+M_2 y'(\alpha)=0 \\ N_1 y(\beta)+N_2 y'(\beta)=0 \end{array}\right\} \qquad (2.2)$$

for which there exist nontrivial solutions of the problem are called the eigenvalues of the problem. The nontrivial solutions which correspond to these eigenvalues are called the eigenfunctions of the problem [6].

2.1.3 Theorem for eigenvalues and eigenfuntctions of SL- problem [7]:

For the regular Sturm-Liouville problem which consists of:

1. A second-order homogenous linear differential equation which is written as :

$$\frac{d}{dx}\left[c_0(x)\frac{dy}{dx}\right]+\left[c_2(x)+\lambda s(x)\right]y=0 \qquad (2.1)$$

where $c_0(x)$, $c_2(x)$ and $s(x)$ are real and continuous functions on $[\alpha,\beta]$, $s(x)$is differentialble, both $c_0(x)$and $s(x)$ are larger than zero for all $x\in[\alpha,\beta]$, and λ is a parameter independent of x .

2. Two complementary conditons

$$\left.\begin{aligned} M_1 y(\alpha)+M_2 y'(\alpha)=0 \\ N_1 y(\beta)+N_2 y'(\beta)=0 \end{aligned}\right\} \qquad (2.2)$$

where M_1, M_2, $N_1$and N_2 are real constants and at least one of $M_1$and $M_2$are not zero , and at least one of the $N_1$and $N_2$are not zero .

Conclusions

1. There are infinite number of eigenvalues $\lambda_r,(r=1,2,3,...)$.These eigenvalues λ_r can be arranged as a monotonic increasing sequence in the form

   $$\lambda_1<\lambda_2<\lambda_3<.........$$

   such that

   $$\lambda_r\to\infty \quad as \quad r\to\infty.$$

2. The eigenfunctions $\phi_r(x),(r=1,2,3,...)$ which correspond to the eigenvlaues λ_r have exactly $(r-1)$ zeros in the open interval (α,β).

2.1.4 Example for finding eigenvalues and eigenfunctions of regular SL-problem:

Let us suppose that we have this system of Sturm-Liouville problem:

$$\left(x^2 y'\right)' + \lambda y = 0 \tag{2.3}$$

and

$$\left.\begin{array}{l} y(1) = 0 \\ y(2) = 0 \end{array}\right\} \tag{2.4}$$

for all $x \in [1,2]$.

To solve this system , we can write the equation (2.3) as :

$$x^2 y'' + 2xy' + \lambda y = 0 \tag{2.5}$$

the characteristic equation for (2.5) is :

$$n^2 + n + \lambda = 0 \tag{2.6}$$

the roots of (2.6) are :

$$n_1 = \frac{-1 - \sqrt{1 - 4\lambda}}{2}$$

and

$$n_2 = \frac{-1 + \sqrt{1 - 4\lambda}}{2} \quad .$$

Now there are three cases which depend on the sign of $(1 - 4\lambda)$:

First case, if $\lambda < 1/4$, then $1 - 4\lambda > 0$. That means the two roots n_1 and n_2 of characteristic eqution (2.6) are real. Therefore the general solution of (2.5) is :

$$y = j_1 x^{n_1} + j_2 x^{n_2}$$

now by Applying the complementery conditons (2.4) we have:

$$j_1 + j_2 = 0$$

and

$$j_1 2^{n_1} + j_2 2^{n_2} \ .$$

Since $n_1 \neq n_2$, the solution $j_1 = j_2 = 0$ is the only solution .Therefore $\lambda < 1/4$ is not eigenvalue.

Second case, if $\lambda = 1/4$, that means $1-4\lambda = 0$, then the roots of characteristic equtation (2.6) are :

$$n_1 = n_2 = -1/2.$$

Therefore the general solution of (2.3) is :

$$y = Wx^{-1/2} + Zx^{-1/2}\ln x.$$

Now by applying the complementery condtions (2.4) we have:

$$W = 0$$

and

$$Z2^{-1/2}\ln 2 = 0.$$

Therefore $y = 0$ and also in this case λ is not an eigenvalue.

Third case, if $\lambda > 1/4$, that means $1-4\lambda < 0$. Set $1-4\lambda = -4q^2$ with $q > 0$. In this case the roots of characteristic equtation (2.6) are :

$$n_1 = -\frac{1}{2} - iq$$

and

$$n_1 = -\frac{1}{2} + iq.$$

The general solutions of the equation (2.3) is :

$$y(x) = x^{-\frac{1}{2}}\left[j_1 \cos(qlinx) + j_2 \sin(qlinx)\right]$$

more simplified :

$$y(x) = \frac{\left[j_1 \cos(q \ln x) + j_2 \sin(q \ln x)\right]}{\sqrt{x}}.$$

Now by applying the complementey conditions (2.4) we get :

$$y(1) = \frac{\left[j_1 \cos(q \ln 1) + j_2 \sin(q \ln 1)\right]}{\sqrt{1}} = 0$$

that means:

$$y(1) = j_1 = 0$$

and

$$y(2) = \frac{\left[j_1 \cos(q \ln 2) + j_2 \sin(q \ln 2)\right]}{\sqrt{2}} = 0$$

thus,

$$j_2 \sin(q \ln 2) = 0 .$$

To obtain a nontrivial solution $y(x)$, we must put :

$$\sin(q \ln 2) = 0$$

thus,

$$q = \frac{r\pi}{\ln 2} \quad , \qquad r \in Z^+$$

now from $1 - 4\lambda = -4q^2$, we have the eigenvalues :

$$\lambda_r = \frac{1}{4} + q_r^2 = \frac{1}{4} + \left(\frac{r\pi}{\ln 2}\right), \qquad r \in Z^+$$

and the corresponding eigenfunctions are :

$$y_r(x) = \left(\frac{1}{\sqrt{x}}\right) \sin \frac{r\pi \ln x}{\ln 2} .$$

2.2 Orthogonality of Eigenfunctions of SL-problem:

2.2.1 Definition of orthogonal [8]:

If we have the two functions K and Q, these functions are called orthogonal with respect to the weight function $s > 0$ on the interval $[\alpha, \beta]$ if and only if

$$\int_{\alpha}^{\beta} K(x) Q(x) s(x) \; dx = 0 .$$

2.2.1.1 Example for orthogonal:

Let $K(x) = \sin(x)$ and $Q(x) = \sin(2x)$, these two funtions are orthogonal with $s(x) = 1$ (weight function) on the interval $[0, \pi]$, for :

$$\int_0^{\pi} \sin(x).\sin(2x)(1) \; dx = \int_0^{\pi} \sin(x).2\sin(x).\cos(x) \; dx$$

$$= 2\int_0^{\pi} \sin^2(x).\cos(x) \; dx$$

$$= \left[\frac{2\sin^3(x)}{3}\right]_0^{\pi} = 0$$

2.2.2 **Definition of orthogonal system[7]:**

Let $\{\phi_r(x)\},(r=1,2,....)$, be an infinite set of functions defined on the interval $\alpha \le x \le \beta$. The set $\{\phi_r(x)\}$ is called an orthogonal system with respect to the function s (weight function) on $\alpha \le x \le \beta$ if every two distinct functions of the set are orthogonal with respect to s on $\alpha \le x \le \beta$. That is, the set $\{\phi_r(x)\}$ is orthogonal with respect to s on the interval $\alpha \le x \le \beta$ if

$$\int_{\alpha}^{\beta} \phi_r(x)\phi_i(x)s(x) \; dx = 0, \quad for \quad r \ne i.$$

2.2.2.1 Example for orthogonal system :

Let we have $\phi_r(x)=\sin(rx)\ (r=1,2,....)$, on $[0,\pi]$. This infinit set of functions $\{\phi_r\}$ is orthogonal system with $s(x)=1$ (weight function) on $[0,\pi]$, for

$$\int_0^{\pi} \sin(rx)\sin(ix)(1) \; dx = \int_0^{\pi}\left[\frac{\cos(rx-ix)-\cos(rx-ix)}{2}\right] \; dx$$

$$= \left[\frac{\sin\,(r-i)x}{2(r-i)} - \frac{\sin\,(r-i)x}{2(r-i)}\right]_0^{\pi}$$

$$= 0, \quad \text{for} \quad r \ne i.$$

2.2.3 Theorem for orthogonality of eigenfunctions of SL-problem [5-6-9]:

For the regular Sturm-Liouville problem which consists of

1. A second-order homogenous linear differential equation which is written as :

$$\frac{d}{dx}\left[c_0(x)\frac{dy}{dx}\right]+[c_2(x)+\lambda s(x)]y=0 \qquad (2.1)$$

where $c_0(x)$, $c_2(x)$ and $s(x)$ are real and continuous functions on $[\alpha,\beta]$, $s(x)$ is differentialble, both $c_0(x)$ and $s(x)$ are larger than zero for all $x \in [\alpha,\beta]$, and λ is a parameter independent of x .

2. Two complementary conditons

$$\left.\begin{aligned} M_1 y(\alpha) + M_2 y'(\alpha) = 0 \\ N_1 y(\beta) + N_2 y'(\beta) = 0 \end{aligned}\right\} \qquad (2.2)$$

where M_1, M_2, N_1 and N_2 are real constants and at least one of M_1 and M_2 is not zero , and at least one of the N_1 and N_2 is not zero .

Let λ_r and λ_i be any two distinct eigenvalues of this problem. Let ϕ_r be a eigenfunction corresponding to λ_r and let ϕ_i be a eigenfunction corresponding to λ_i . Then, The eigenfunctions ϕ_r and ϕ_i are orthogonal with respect to the weight function s on the interval $\alpha \le x \le \beta$.

Proof :

Since ϕ_r is an eigenfunction corresponding to λ_r , the function ϕ_r satisfies the equation (2.1) with $\lambda = \lambda_r$.

And since ϕ_i is an eigenfunction corresponding to λ_i , the function ϕ_i satisfies the equation (2.1) with $\lambda = \lambda_i$.

So we substitute the derivatives of ϕ_r and ϕ_i by ϕ_r' and ϕ_i' respectively, we get

$$\frac{d}{dx}\left[c_0(x)\phi_r'(x)\right] + \left[c_2(x) + \lambda_r s(x)\right]\phi_r(x) = 0 \qquad (2.7)$$

and

$$\frac{d}{dx}\left[c_0(x)\phi_i'(x)\right] + \left[c_2(x) + \lambda_i s(x)\right]\phi_i(x) = 0 \qquad (2.8)$$

for all x , $x \in [\alpha, \beta]$.

Multiply the equation (2.7) by ϕ_i, we get

$$\phi_i(x)\frac{d}{dy}\left[c_0(x)\phi_r'(x)\right] + c_2(x)\phi_r(x)\phi_i(x) + \lambda_r\phi_r(x)\phi_i(x)s(x) = 0 \qquad (2.9)$$

multiply the equation (1.8) by ϕ_r , we get

$$\phi_r(x)\frac{d}{dy}\left[c_0(x)\phi_i'(x)\right]+c_2(x)\phi_r(x)\phi_i(x)+\lambda_i\phi_r(x)\phi_i(x)s(x)=0 \qquad (2.10)$$

by subtract the equation (2.10) from the equation (2.9), we get

$$\phi_i(x)\frac{d}{dy}\left[c_0(x)\phi_r'(x)\right]+\lambda_r\phi_r(x)\phi_i(x)s(x)-\phi_r(x)\frac{d}{dy}\left[c_0(x)\phi_i'(x)\right]-\lambda_i\phi_r(x)\phi_i(x)s(x)=0$$

and thus

$$(\lambda_r-\lambda_i)\phi_r(x)\phi_i(x)s(x)=\phi_r(x)\frac{d}{dy}\left[c_0(x)\phi_i'(x)\right]-\phi_i(x)\frac{d}{dy}\left[c_0(x)\phi_r'(x)\right]$$

now, integrate each part of the last equation from α to β to have:

$$(\lambda_r-\lambda_i)\int_\alpha^\beta \phi_r(x)\phi_i(x)s(x)dx=\int_\alpha^\beta \phi_r(x)\frac{d}{dy}\left[c_0(x)\phi_i'(x)\right]dx-\int_\alpha^\beta \phi_i(x)\frac{d}{dy}\left[c_0(x)\phi_r'(x)\right]dx \qquad (2.11)$$

now, we use integration by parts for the right hand of the equation (2.11) by assuming:

$$\text{first part}\left[u=\phi_r(x),\;\; dv=\frac{d}{dx}\left[c_0(x)\phi_i'(x)\right]dx\right],$$

and

$$\text{second part}\left[u=\phi_i(x),\;\; dv=\frac{d}{dx}\left[c_0(x)\phi_r'(x)\right]dx\right]$$

to get,

$$(\lambda_r-\lambda_i)\int_\alpha^\beta \phi_r(x)\phi_i(x)s(x)dx=\left[\phi_r(x)c_0(x)\phi_i'(x)\right]_\alpha^\beta-\int_\alpha^\beta c_0(x)\phi_i'(x)\phi_r'(x)dx$$

$$-\left[\phi_i(x)c_0(x)\phi_r'(x)\right]_\alpha^\beta+\int_\alpha^\beta c_0(x)\phi_i'(x)\phi_r'(x)dx$$

and thus

$$(\lambda_r-\lambda_i)\int_\alpha^\beta \phi_r(x)\phi_i(x)s(x)dx=\left[\phi_r(x)c_0(x)\phi_i'(x)\right]_\alpha^\beta-\left[\phi_i(x)c_0(x)\phi_r'(x)\right]_\alpha^\beta$$

$$(\lambda_r-\lambda_i)\int_\alpha^\beta \phi_r(x)\phi_i(x)s(x)dx=\left[c_0(x)\left[\phi_r(x)\phi_i'(x)-\phi_i(x)\phi_r'(x)\right]\right]_\alpha^\beta$$

$$(\lambda_r-\lambda_i)\int_\alpha^\beta \phi_r(x)\phi_i(x)s(x)dx=c_0(\beta)\left[\phi_r(\beta)\phi_i'(\beta)-\phi_i(\beta)\phi_r'(\beta)\right]-c_0(\alpha)\left[\phi_r(\alpha)\phi_i'(\alpha)-\phi_i(\alpha)\phi_r'(\alpha)\right] \qquad (2.12)$$

Since ϕ_r and ϕ_i are eigenfunctions of the equation (2.1) and satisfy the complimentary conditions (2.2), and now if

$$M_2 = N_2 = 0$$

in the complimentary conditions (2.2), these conditions will be

$$y(\alpha) = 0, \quad y(\beta) = 0$$

then in this case

$$\phi_r(\alpha) = 0, \quad \phi_r(\beta) = 0$$

and

$$\phi_i(\alpha) = 0, \quad \phi_i(\beta) = 0$$

so the right hand of the equation (2.12) will be equal to zero .

And if $M_2 = 0$ but $N_2 \neq 0$ in the complimentary conditions (2.2), these conditions will be:

$$y(\alpha) = 0, \quad Cy(\beta) + y'(\beta) = 0$$

where

$$C = \frac{C_1}{C_2}$$

then the second bracket in the right hand of the equation (2.12) will equal to zero.

Now add and subtract $[C\phi_i(\beta)\phi_r(\beta)]$ for the first bracket on the right hand of equation (2.12),

$$[C\phi_i(\beta)\phi_r(\beta) + \phi_r(\beta)\phi_i'(\beta) - C\phi_i(\beta)\phi_r(\beta) - \phi_i(\beta)\phi_r'(\beta)]$$

more simplifed :

$$[(C\phi_i(\beta) + \phi_i'(\beta))\phi_r(\beta) - (C\phi_r(\beta) + \phi_r'(\beta))\phi_i(\beta)]$$

and also it is equal to zero. Therefore the right hand of the equation (2.8) will be zero.

$$(\lambda_r - \lambda_i)\int_\alpha^\beta \phi_r(x)\phi_i(x)s(x)dx = 0$$

since λ_r and λ_i are distinct eigenvalues, their difference $(\lambda_r - \lambda_i) \neq 0$, therefore we must have:

$$\int_\alpha^\beta \phi_r(x)\phi_i(x)s(x)dx = 0$$

and so ϕ_r and ϕ_i are orthogonal with respect to s for all $x, x \in [\alpha, \beta]$.

2.2.3.1 Example for orthogonal system of eigenfunctions for SL-problem :

Let we have the system of SL-problem by :

$$y'' + \lambda y = 0$$

and

$$\left.\begin{array}{l} y(0) = 0 \\ y(\pi) = 0 \end{array}\right\},$$

for all $x \in [0, \pi]$.

The eigenvalues of of this system of SL-problem are :

$$\lambda_r = r^2, \qquad (r = 1,2,3,......)$$

and the corresponding eigenfunctions are :

$$y_r(x) = j_r \sin(rx), \qquad (r = 1,2,3,......)$$

and

$$j_r, (r = 1,2,3,......)$$

is arbitrary constant not equal zero.

Let $\{\phi_r\}$ refer to the infinit set of eigenfunctions with $j_r = 1, (r = 1,2,3,......)$. That is,

$$\phi_r(x) = \sin(rx), \qquad (r = 1,2,3,......)$$

then by the previous theorem, the set $\{\phi_r\}$ is an orthogonal system with respect to $s(x) = 1$ (weight function) for all $x \in [0, \pi]$.

That is, the previous theorem shows that :

$$\int_0^{\pi} \sin(rx)\sin(ix)(1) \; dx = \int_0^{\pi} \left[\frac{\cos(rx - ix) - \cos(rx - ix)}{2} \right] dx$$

$$= \left[\frac{\sin\,(r - i)x}{2(r - i)} - \frac{\sin\,(r - i)x}{2(r - i)} \right]_0^{\pi}$$

$$= 0, \quad for \quad r \neq i \quad (r, i = 1,2,....)$$

2.3 The Expansion of Function :

The expansion of function is very important topic because it made a significant development in the advaced mathematical analysis and it had many applications.
The form of expansion of function f is :

We will find the expansion for function f from an orthonormal system of eigenfunctions $\{\phi_r(x)\},(r=1,2,.....)$ of SL-problem. This function and eigenfuctions must satisfy some restrictive conditions as we will see that in the next theorom. Before that we will provide the definition of normalized and orthonormal system.

2.3.1 Definition of normalized [7]:

A function ϕ is called normalized with respect to the function s (weight function) on the interval $\alpha \le x \le \beta$ if and only if

$$\int_{\alpha}^{\beta} [\phi(x)]^2 \; s(x) \; dx = 1 .$$

2.3.1.1 Example for normalized:

This function $\phi(x)=\sqrt{2/\pi}\,\sin(x)$, is normalized with respect to $s(x)=1$ (weight function) on the interval $0 \le x \le \pi$,

$$\int_0^{\pi} \left(\sqrt{\frac{2}{\pi}}\sin(x)\right)^2 .(1) \; dx = \frac{2}{\pi}\int_0^{\pi} \sin^2(x) \; dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= 1$$

2.3.2 Definition of orthonormal system [7]:

The infinit set functions $\{\phi_r(x)\},(r=1,2,....)$, which are defined on the interval $\alpha \le x \le \beta$, are called an orthonormal system with respect to s (weight function) on $\alpha \le x \le \beta$ if :

(1) it is an orthogonal system with respect to s for all $x \in [\alpha, \beta]$.

(2) every function of the system is normalized with respect to **s** for all $x \in [\alpha, \beta]$.

That is, the set $\{\phi_r(x)\}$ is orthonormal with respect to **s** for all $x \in [\alpha, \beta]$ if :

$$\int_\alpha^\beta \phi_r(x)\phi_i(x)s(x)dx = \begin{cases} 0 & for \quad r \neq i \quad orthogonal \\ 1 & for \quad r = i \quad normalized \end{cases}$$

2.3.2.1 Example for orthonormal eigenfunctions of SL-problem :

Let we have the system of SL-problem by :

$$y'' + \lambda y = 0$$

and

$$\left.\begin{array}{l} y(0)=0 \\ y(\pi)=0 \end{array}\right\},$$

for all $x \in [0, \pi]$.

We proved in example (1.2.3.1) that this system of SL-problem has set of orthogonal eigenfunctions $\{\phi_r(x)\}$, when :

$$\phi_r(x) = j_r \sin(rx), \quad (r = 1,2,3,......)$$

Now we need to prove that it is also normalized to have orthonormal system for eigenfunctions of this SL-problem.

Thus must hold that :

$$\int_0^\pi [\phi_r(x)]^2 s(x)dx = 1, \quad (r = 1,2,....)$$

now if $\{\phi_r(x)\}$ is not normalized it will be :

$$\int_0^{\pi} [\phi_r(x)]^2 s(x)dx = V_r , \quad (r = 1,2,....)$$

and so

$$\int_0^{\pi} \left[\frac{1}{\sqrt{V_r}}\phi_r(x)\right]^2 s(x)dx = 1 , \quad (r = 1,2,....)$$

therefore the set

$$\left\{\frac{1}{\sqrt{V_r}}\phi_r(x)\right\}, \quad (r = 1,2,.....)$$

is normalized. and we will denote

$$v_r = \frac{1}{\sqrt{V_r}}, \quad (r = 1,2,.....)$$

now for our example we will find v_r by :

$$V_r = \int_0^{\pi} [j_r \sin(rx)]^2 (1)dx = j_r^2 \cdot \frac{\pi}{2}$$

$$v_r = \frac{1}{\sqrt{V_r}} = \frac{1}{j_r}\sqrt{\frac{2}{\pi}}$$

now,

$$v_r.\phi_r(x) = \left[\frac{1}{j_r}\sqrt{\frac{2}{\pi}}\right].j_r \sin(rx) = \sqrt{\frac{2}{\pi}}.\sin(rx), \quad (r = 1.2.....)$$

therefore the eigenfunctions $\mathrm{T}_r(x) = \sqrt{2/\pi}.\sin(rx), (r = 1,2,......)$ are orthonormal system for this Sturm-Liouville problem .

2.3.3 Theorem for expansion function from SL-problem [6]:

1. Let a regular SL-problem:

$$\frac{d}{dx}\left[c_0(x)\frac{dy}{dx}\right] + [c_2(x) + \lambda s(x)]y = 0 \tag{2.1}$$

$$\left.\begin{aligned} M_1 y(\alpha) + M_2 y'(\alpha) = 0 \\ N_1 y(\beta) + N_2 y'(\beta) = 0 \end{aligned}\right\} \tag{2.2}$$

where $c_0(x)$, $c_2(x)$ and $s(x)$ are real functions, $c_0(x)$ has a continuous derivative , $c_2(x)$ and $s(x)$ are continuous, both $c_0(x)$ and $s(x)$ are larger than zero for all $x \in [\alpha, \beta]$.

Let $\{\lambda_r\}, (r = 1,2,...)$ be the infinit set of eigenvalues in the form,

$$\lambda_1 < \lambda_2 < \lambda_3 < ....$$

and let $\{\phi_r\}, (r = 1,2,...)$ be the corresponding set of orthonormal eigenfunctions of this problem .

2. Let f be continuous on the interval $\alpha \le x \le \beta$ and has a piecewise continuous derivative f' on the interval $\alpha \le x \le \beta$, and is such that

$$f(\alpha) = 0 \text{ if } \phi_1(\alpha) = 0 \text{ and } f(\beta) = 0 \text{ if } \phi_1(\beta) = 0$$

then the series

$$\sum_{r=1}^{\infty} j_r \phi_r(x) \tag{2.13}$$

where

$$j_r = \int_{\alpha}^{\beta} f(x)\phi_r(x)s(x)\ dx, \quad (r = 1,2,.....), \tag{2.14}$$

converges uniformly and completely to f on the interval $\alpha \le x \le \beta$.

2.3.3.1 Example for finding expansion of function from orthonormal eigenfunctions of SL-problem:

Let

$$f(x) = \pi x - x^2, x \in [0, \pi],$$

and

$$\phi_r(x) = \sqrt{2/\pi}.\sin(rx), 0 \le x \le \pi \qquad (r = 1,2,......)$$

are orthonormal eigenfunctions of SL- problem in example (1.3.2.1). We will use previous theorm (2.3.3) to find the expansion of f by the serise :

$$\sum_{r=1}^{\infty} j_r \phi_r, \qquad (2.13)$$

where

$$j_r = \int_{\alpha}^{\beta} f(x)\phi_r(x)s(x)\ dx, \qquad (r = 1,2,.....), \qquad (2.14)$$

after simplifying, we will get :

$$j_r = \sqrt{\frac{2}{\pi}}.\frac{4}{n^3}(1-\cos(r\pi))$$

After subsitituting the value of j_r in the series (2.13), it will become :

$$\frac{8}{\pi}\sum_{r=1}^{\infty} \frac{\sin\ (2r-1)x}{(2r-1)^3}. \qquad (2.15)$$

That means f will be :

$$\left(\pi x - x^2\right) \approx \frac{8}{\pi}\sum_{r=1}^{\infty} \frac{\sin\ (2r-1)x}{(2r-1)^3}, \quad \textit{for} \quad x \in [0,\pi]$$

We can write (2.15) as :

$$\frac{8}{\pi}\left[\frac{\sin\ (x)}{1^3} + \frac{\sin\ (3x)}{3^3} + \frac{\sin\ (5x)}{5^5} + ...\right]$$

Now, note that $c_0(x)$, $c_2(x)$ and $s(x)$ in the differential equations of Sturm-Liouville problem of this example satisfy the first condition of the previous theorem (2.3.3).

Since the function f is a polynomail function, so it satisfy the condition of continuity and a piecewised continuous derivative for all $x \in [0,\pi]$, also we note :

$$\phi_1(0) = \sqrt{\frac{2}{\pi}}\sin(0) = 0, \qquad f(0) = \pi.0 - (0)^2 = 0,$$

$$\phi_1(0) = \sqrt{\frac{2}{\pi}}\sin(\pi) = 0, \qquad f(\pi) = \pi.\pi - (\pi)^2 = 0,$$

therefore all the conditions of the previous theorem (2.3.3) are achieved. That means the series (2.15) converges uniformly and completely to $f(x)$, $x \in [0,\pi]$ Thus we can write :

$$\left(\pi x - x^2\right) = \frac{8}{\pi}\sum_{r=1}^{\infty} \frac{\sin\ (2r-1)x}{(2r-1)^3}, \quad \textit{for} \quad x \in [0,\pi]$$

2.4 Trigonometric Fourier Series:

In the previous section (1.3), we found the expansion of function f from an orthonormal eigenfunctions $\{\phi_r\},(r=1,2,...)$ of SL-problem with respect to the function s (weight function) on the closed interval $[\alpha\ ,\ \beta\ ]$, by the form of series :

$$\sum_{r=1}^{\infty} j_r\phi_r, \qquad (2.13)$$

where

$$j_r=\int_{\alpha}^{\beta} f(x)\phi_r(x)s(x)\ \ dx, \qquad (r=1,2,.....). \qquad (2.14)$$

Accurately, this expansion is formal expansion and with some restrictive conditions for f and $\{\phi_r\}$ leads to make the series (2.13) converges to f for all $x\in[\alpha\ ,\ \beta\ ]$. In the next definition we will assume that the functions on (2.14) are integrable and we will give a name for the expansion of f.

2.4.1 Defintion of Fourier series [10]:

Let we have an orthonormal system $\{\phi_r(x)\}(r=1,2,.....)$ with respect to the function s (weight function) on the interval $[\alpha,\beta]$. Let f be a function such that for each $(r=1,2,.....)$ the product $f\phi_r s$ is integrable on $[\alpha,\beta]$. Then the series :

$$\sum_{r=1}^{\infty} j_r\phi_r, \qquad (2.13)$$

where

$$j_r=\int_{\alpha}^{\beta} f(x)\phi_r(x)s(x)\ \ dx, \qquad (r=1,2,.....). \qquad (2.14)$$

is called the Fourier series of f relative to the system $\{\phi_r\}$ and the coefficients j_r are called the Fourier constants of f relative to $\{\phi_r\}$ and its written by :

$$f(x)\ \sim\ \sum_{r=1}^{\infty} j_r\phi_r, \qquad \alpha\le x\le\beta\ .$$

There is an important kind of Fourier series. Therefore to introduce this purpose, let we have the systems of functions $\{T_r(x)\}$:

$$\left.\begin{aligned} T_1(x) &= 1, \\ T_{2r}(x) &= \cos\left(\frac{r\pi x}{h}\right) \qquad (r = 1,2,.....), \\ T_{2r+1}(x) &= \sin\left(\frac{r\pi x}{h}\right) \qquad (r = 1,2,.....), \end{aligned}\right\} \tag{2.16}$$

for $x \in [-h,h]$and $h > 0$. When we apply the definition of orthonormal system for $\{T_r(x)\}$ with respect to $s(x) = 1$(weight funtion), the result will be orthonormal system $\{\phi_r(x)\}$ by :

$$\left.\begin{aligned} \phi_1(x) &= \frac{1}{\sqrt{2h}}, \\ \phi_{2r}(x) &= \frac{1}{\sqrt{h}}\cos\left(\frac{r\pi x}{h}\right) \qquad (r = 1,2,.....), \\ \phi_{2r+1}(x) &= \frac{1}{\sqrt{h}}\sin\left(\frac{r\pi x}{h}\right) \qquad (r = 1,2,.....), \end{aligned}\right\} \tag{2.17}$$

for $x \in [-h,h]$.

Now we will apply the definition of Fourier series on the set funtions $\{\phi_r(x)\}$ as orthonormal system (2.17) for $x \in [-h,h]$, we will have a special kind of the Fourier series as in the next definition.

2.4.2 Definition of trigonometric Fourier series [6]:

Let function f is defined on the interval $-h < x < h$ and the intergrals

$$\int_{-h}^{h} f(x) cox \frac{r\pi x}{h} dx \quad \text{and} \quad \int_{-h}^{h} f(x) \sin \frac{r\pi x}{h} dx, \qquad (r = 1,2,.....)$$

exist.Then the trigonomatric Fourier series of f on the interval $-h \le x \le h$ is :

$$f(x) \sim \frac{1}{2}A_0 + \sum_{r=1}^{\infty} \left(A_r \cos\frac{r\pi x}{h} + B_r \sin\frac{r\pi x}{h} \right), \qquad -h \le x \le h \tag{2.18}$$

where

$$A_r = \frac{1}{h}\int_{-h}^{h} f(x) cox \frac{r\pi x}{h} dx, \qquad (r = 0,1,2,.....)$$

$$B_r = \frac{1}{h}\int_{-h}^{h} f(x)\sin \frac{r\pi x}{h} dx, \qquad (r = 1,2,.....)$$

the constants A_r and B_r are called the Fourier coefficients of f **.**

According to determination of Fourier coefficients, there are two significant cases when f is even function or odd function.

2.4.3 Definition of Fourier sine series :

Let function f be defined on the interval $0 \le x \le h$ and the intergrals

$$\int_{0}^{h} f(x)\sin \frac{r\pi x}{h} dx, \qquad (r = 1,2,.....)$$

exsist.Then the Fourier sine series of f on interval $0 \le x \le h$ is :

$$f(x) \sim \sum_{r=1}^{\infty} B_r \sin \frac{r\pi x}{h} \quad 0 \le x \le h$$

where

$$B_r = \frac{2}{h}\int_{0}^{h} f(x)\sin \frac{r\pi x}{h} dx, \qquad (r = 1,2,.....).$$

We note that the Fourier sine series is similar to the trigonometric Fourier series (2.18) of the odd function which is defined on $-h \le x \le h$ [11].

2.4.4 Definition of Fourier cosine series:

Let function f be defined on the interval $0 \le x \le h$ and the intergrals

$$\int_{0}^{h} f(x)\cos \frac{r\pi x}{h} dx, \qquad (r = 0,1,2,.....)$$

exsist. Then the Fourier cosine series of f on interval $0 \le x \le h$ is :

$$f(x) \sim \frac{1}{2}A_0 + \sum_{r=1}^{\infty} A_r \cos\frac{r\pi x}{h} \quad 0 \le x \le h$$

where

$$A_r = \frac{2}{h}\int_0^h f(x)\cos\frac{r\pi x}{h}\,dx, \qquad (r = 0,1,2,.....).$$

We note that the Fourier cosine series is similar to the trigonometric Fourier series (2.18) of the even function which is defined on $-h \le x \le h$ [11].

2.4.5 Convergence of trigonometric Fourier series:

In the previous section we found the trigonometric Fourier series of f for all x which belongs to the closed interval $[-h,h]$ in the form :

$$f(x) \sim \frac{1}{2}A_0 + \sum_{r=1}^{\infty}\left(A_r \cos\frac{r\pi x}{h} + B_r \sin\frac{r\pi x}{h}\right), \quad -h \le x \le h \qquad (2.18)$$

where

$$A_r = \frac{1}{h}\int_{-h}^{h} f(x) cox\frac{r\pi x}{h}\,dx, \qquad (r = 0,1,2,.....)$$

$$B_r = \frac{1}{h}\int_{-h}^{h} f(x)\sin\frac{r\pi x}{h}\,dx, \qquad (r = 1,2,.....)$$

and this expansion of function f is formal expansion becasue we did not talk about the convergence of this expansion .

Now we will provide two theorems for convergence of the trigonometric Fourier series and the Fourier sine series and the Fourier cosine series.

2.4.5.1 Theorem for convergence of trigonometric Fourier series[6]:

Let function f be periodic of period $2h$ and piecewise smooth on the closed interval $[-h,h]$, then the trigonometric Fourier series of f ,

$$\frac{1}{2}A_0 + \sum_{r=1}^{\infty}\left(A_r \cos\frac{r\pi x}{h} + B_r \sin\frac{r\pi x}{h}\right), \quad -h \le x \le h \qquad (2.18)$$

where

$$A_r = \frac{1}{h}\int_{-h}^{h} f(x) cox \frac{r\pi x}{h} dx, \qquad (r = 0,1,2,.....)$$

$$B_r = \frac{1}{h}\int_{-h}^{h} f(x)\sin \frac{r\pi x}{h} dx, \qquad (r = 1,2,.....)$$

converges at every point x to the value

$$\frac{f(x+)+f(x-)}{2}$$

where $f(x-)$ is the left-hand limit of f at x and $f(x+)$ is the right-hand limit of f at x.

And, the trigonometric Fourier series of f at x converges to $f(x)$ when f is continuous at x, because the average of limils for the right-hand and left hand for f at x will equale $f(x)$.

2.4.5.2 Theorem for convergence of Fourier sine and cosine series[6]:

Let function f be piecewise smooth on the closed interval for all $x \in [0,h]$, then

1. The Fourier sine series of f,

$$\sum_{r=1}^{\infty} B_r \sin \frac{r\pi x}{h} \quad 0 \le x \le h$$

where

$$B_r = \frac{2}{h}\int_{0}^{h} f(x)\sin \frac{r\pi x}{h} dx, \qquad (r = 1,2,.....)$$

converges at every point x to the value

$$\frac{f(x+)+f(x-)}{2},$$

for every x belongs to the open interval $(0,h)$.

And, the Fourier sine series of f at x converges to $f(x)$ when f is continuous at x, for all x belongs to the open interval $(0,h)$.

And, at the two end points of the closed interval $[0,h]$, the Fourier sine series of f converges to zero.

And, the Fourier sine series of f converges at point x to the value

$$\frac{u(x+)+u(x-)}{2}$$

where u is odd function and periodic function of period $2h$ which coincide with function f in the open interval for all x belongs to $(0,h)$ and is such that $u(0)=u(h)=0$.

2. The Fourier cosine series of f ,

$$\frac{1}{2}A_0+\sum_{r=1}^{\infty} A_r \cos\frac{r\pi x}{h},$$

where

$$A_r=\frac{2}{h}\int_0^h f(x)\cos\frac{r\pi x}{h}dx, \qquad (r=0,1,2,.....)$$

converges to the value

$$\frac{f(x+)+f(x-)}{2},$$

for every x belongs to the open interval $(0,h)$.

And, the Fourier cosine series of f at x converges to $f(x)$ when f is continuous at x, for every x belongs to the open interval $(0,h)$.

And, the Fourier cosine series of f converges to $f(0+)$ [the right-hands limit of f] at $x=0$ and to $f(h-)$ [the left-hands limit of f] at $x=h$.

And, the Fourier cosine series of f converges at every point x to the value

$$\frac{z(x+)+z(x-)}{2}$$

where z is even function and periodic function of period $2h$ which coincide with function f in the closed interval for every x belongs to $[0,h]$.

CHAPTER 3

THE CONVERSION TO STURM-LIOUVILLE EQUATION

In this chapter we will convert any problem into regular Sturm-Liouville problem by finding suitable weight function .

3.1 Theorem of Self-adjoint Operator [12]:

Let L be the operator

$$L(y)=c_o(x)y''+c_1(x)y'+c_2(x)y$$

for $x\in[\alpha,\beta]$.

Suppose

$$p(x)=e^{\int\frac{c_1(x)-c_0'(x)}{c_0(x)}dx}=\frac{1}{|c_0(x)|}e^{\int\frac{c_1(x)}{c_0(x)}dx}<\infty$$

over $I=[\alpha,\beta]$, and C is a linear space of functions which the following hold:

(i) C is invariant under L.

(ii) $\langle y_1,y_1\rangle=\int_\alpha^\beta p{y_1}^2dx<\infty$ (finite), $\forall\, y_1\in C$.

(iii) $\forall\quad y_1,y_2\in C$

$$\{\,p(x)c_0(x)[y_1(x)y_2'(x)-y_2(x)y_1'(x)]\,\}_\alpha^\beta=[pc_0(y_1y_2'-y_2y_1')]_\alpha^\beta=0$$

then the operator L is self-adjoint on the interval I with respect to the inner product:

$$\langle y_1,y_2\rangle=\int_\alpha^\beta p(x)y_1(x)y_2(x)dx$$

Proof:

Starting with the conditions of self-adjoint of operator L,

$$\langle Ly_1, y_2\rangle = \int_\alpha^\beta p(x)Ly_1(x)y_2(x)dx$$

$$= \int_\alpha^\beta p(x)[c_0(x)y_1'' + c_1(x)y_1' + c_2(x)y_1]y_2\,dx$$

$$= \int_\alpha^\beta pc_0 y_1'' y_2 dx + \int_\alpha^\beta pc_1 y_1' y_2 dx + \int_\alpha^\beta pc_2 y_1 y_2 dx\,.$$

Now using integration by parts for the first part and the second part:

$$\int_\alpha^\beta pc_0 y_1'' y_2 dx = [(pc_0 y_2)y_1']_\alpha^\beta - \int_\alpha^\beta y_1'(pc_0 y_2)'\,dx \quad \begin{Bmatrix} u = pc_0 y_2 & \to & du = (pc_0 y_2)'\,dx \\ dv = y_1'' dx & \to & v = y_1' \end{Bmatrix}$$

$$\int_\alpha^\beta pc_1 y_1' y_2 dx = [(pc_1 y_2)y_1]_\alpha^\beta - \int_\alpha^\beta y_1(pc_1 y_2)'\,dx \quad \begin{Bmatrix} u = pc_1 y_2 & \to & du = (pc_1 y_2)'\,dx \\ dv = y_1' dx & \to & v = y_1 \end{Bmatrix}$$

so,

$$\langle Ly_1, y_2\rangle = [(pc_0y_2)y_1']_\alpha^\beta - \int_\alpha^\beta y_1'(pc_0y_2)'\,dx + [(pc_1y_2)y_1]_\alpha^\beta - \int_\alpha^\beta y_1(pc_1y_2)'\,dx + \int_\alpha^\beta pc_2y_1y_2dx$$

now,

$$\int_\alpha^\beta y_1'(pc_0y_2)'\,dx = \left[(pc_0y_2)'\,y_1\right]_\alpha^\beta - \int_\alpha^\beta y_1(pc_0y_2)''\,dx \quad \begin{Bmatrix} u = (pc_0y_2)' & \to & du = (pc_0y_2)''\,dx \\ dv = y_1'dx & \to & v = y_1 \end{Bmatrix}$$

$$\langle Ly_1, y_2\rangle = [(pc_0y_2)y_1']_\alpha^\beta - \left[(pc_0y_2)'\right]_\alpha^\beta + \int_\alpha^\beta y_1(pc_0y_2)''\,dx + [(pc_1y_2)]_\alpha^\beta - \int_\alpha^\beta y_1(pc_1y_2)'\,dx$$
$$+ \int_\alpha^\beta pc_2y_1y_2dx$$

$$\langle Ly_1, y_2\rangle = \left[(pc_0y_2)y_1' - (pc_0y_2)'\,y_1 + (pc_1y_2)y_1\right]_\alpha^\beta + \int_\alpha^\beta\left[y_1(pc_0y_2)'' - y_1(pc_1y_2)' + y_1(pc_2y_2)\right]dx$$

with more simplification, we get:

$$\langle Ly_1, y_2\rangle = \left[(pc_0y_2)y_1' - (pc_0y_2)'\,y_1 + (pc_1y_2)y_1\right]_\alpha^\beta + \int_\alpha^\beta y_1\left[(pc_0y_2)'' - (pc_1y_2)' + (pc_2y_2)\right]dx \qquad (3.1)$$

we have

$$\langle y_1, Ly_2\rangle = \int_\alpha^\beta p(x)y_1(x)Ly_2(x)dx$$

$$\langle Ly_1, y_2\rangle = \int_\alpha^\beta p(x)y_1[c_0(x)y_2'' + c_1(x)y_2' + c_2(x)y_2]dx$$

$$\langle Ly_1, y_2\rangle = \int_\alpha^\beta p\,y_1[c_0y_2'' + c_1y_2' + c_2y_2]dx \qquad (3.2)$$

now (3.1) and (3.2) must be equal , if the boundary terms

$$\left[(pc_0 y_2) y_1' - (pc_0 y_2)' y_1 + (pc_1 y_2) y_1 \right]_\alpha^\beta = 0 \tag{3.3}$$

and

$$\int_\alpha^\beta y_1 \left[(pc_0 y_2)'' - (pc_1 y_2)' + (pc_2 y_2) \right] dx = \int_\alpha^\beta p\, y_1 \left[c_0 y_2'' + c_1 y_2' + c_2 y_2 \right] dx \tag{3.4}$$

in the equation (3.4), $(pc_0 y_2)'' - (pc_1 y_2)' = pc_0 y_2'' + pc_1 y_2'$ must hold:

$$[(pc_0) y_2]'' - [(pc_1) y_2]' = pc_0 y_2'' + pc_1 y_2'$$

$$\left[(pc_0) y_2' + (pc_0)' y_2 \right]' - \left[(pc_1) y_2' + (pc_1)' y_2 \right] = pc_0 y_2'' + pc_1 y_2'$$

$$[(pc_0) y_2']' + \left[(pc_0)' y_2 \right]' - (pc_1) y_2' - (pc_1)' y_2 = pc_0 y_2'' + pc_1 y_2'$$

$$(pc_0) y_2'' + (pc_0)' y_2' + (pc_0)' y_2' + (pc_0)'' y_2 - (pc_1) y_2' - (pc_1)' y_2 = pc_0 y_2'' + pc_1 y_2'$$

$$(pc_0) y_2'' + 2(pc_0)' y_2' + (pc_0)'' y_2 - (pc_1) y_2' - (pc_1)' y_2 = pc_0 y_2'' + pc_1 y_2'$$

$$(pc_0)'' y_2 - (pc_1)' y_2 + 2(pc_0)' y_2' = (pc_1) y_2' + pc_1 y_2'$$

$$\left[(pc_0)'' - (pc_1)' \right] y_2 + 2(pc_0)' y_2' = 2(pc_1) y_2' .$$

Now by making the coefficient of y_2 and y_2' on the both sides equal, we will get:

$$(pc_0)'' - (pc_1)' = 0 \tag{3.5}$$

and

$$(pc_0)' = pc_1 \tag{3.6}$$

the equation (3.3):

$$\left[(pc_0 y_2) y_1' - (pc_0 y_2)' y_1 + (pc_1 y_2) y_1 \right]_\alpha^\beta = 0 \tag{3.3}$$

can be simplified to get:

$$\left[(pc_0 y_2) y_1' - \left[(pc_0) y_2' + (pc_0)' y_2 \right] y_1 + (pc_1 y_2) y_1 \right]_\alpha^\beta = 0$$

$$\left[(pc_0 y_2) y_1' - (pc_0) y_2' y_1 - (pc_0)' y_2 y_1 + (pc_1 y_2) y_1 \right]_\alpha^\beta = 0$$

we will substitute $(pc_0)' = pc_1$ and we will get,

$$\Big[(pc_0)y_2y_1' - (pc_0)y_2'y_1 - (pc_0)'y_2y_1 + (pc_0)'y_2y_1\Big]_\alpha^\beta = 0$$

$$[(pc_0)y_2y_1' - (pc_0)y_2'y_1]_\alpha^\beta = 0$$

$$[pc_0(y_2y_1' - y_2'y_1)]_\alpha^\beta = 0 \qquad (3.7)$$

after multiplying the equation (3.7) by (-1), we will get the boundary condition (iii),

$$[pc_0(y_1y_2' - y_2y_1')]_\alpha^\beta = 0$$

now we will find the weight function $p(x)$ from the equation (3.6):

$$(pc_0)' = pc_1 \qquad (3.6)$$

$$pc_0' + p'c_0 = pc_1$$

$$pc_0' = pc_1 - pc_0'$$

$$[p'c_0 = pc_1 - pc_0'] \div (pc_0)$$

$$\frac{p'}{p} = \frac{(c_1 - c_0')}{c_0}$$

by integral both sides , we will get :

$$\int \frac{p'}{p}dx = \int \frac{(c_1 - c_0')}{c_0}dx$$

$$\ln|p| = \int \frac{(c_1 - c_0')}{c_0}dx$$

$$p(x) = e^{\int \frac{(c_1(x) - c_0'(x))}{c_0(x)}dx}$$

or

$$p(x) = e^{\int \left(\frac{c_1}{c_0} - \frac{c_0'}{c_0}\right)dx} = e^{\int \frac{c_1}{c_0}dx}.e^{-\int \frac{c'}{c_0}dx}$$

$$p(x) = e^{\int \frac{c_1}{c_0}dx}.e^{-\ln|c_0|} = e^{\int \frac{c_1}{c_0}dx}.|c_0|^{-1}$$

$$p(x) = \frac{1}{|c_0(x)|}e^{\int \frac{c_1(x)}{c_0(x)}dx}.$$

We can also find the weight function by another way, If we consider the equation :

$$c_0(x)y'' + c_1(x)y' + c_2(x)y = -\lambda w(x)y \qquad (3.8)$$

for $x \in [\alpha, \beta]$, with $c_1(x) \neq c_0'(x)$.

Now multiply both sides by some function $p(x)$, we will get:

$$p(x)c_0(x)y'' + p(x)c_1(x)y' + p(x)c_2(x)y = -\lambda p(x)w(x)y \tag{3.9}$$

form the definition of sturm-liouville equation , we need to hold :

$$(pc_0)' = pc_1$$

to make equation (3.9) satisfy Sturm-Liouville equation ,

$$(pc_0)' = pc_1$$

$$pc_0' + p'c_0 = pc_1$$

$$pc_0' = pc_1 - pc_0'$$

$$[p'c_0 = pc_1 - pc_0'] \div (pc_0)$$

$$\frac{p'}{p} = \frac{(c_1 - c_0')}{c_0}$$

by integral both sides:

$$\int \frac{p'}{p} dx = \int \frac{(c_1 - c_0')}{c_0} dx$$

$$\ln|p| = \int \frac{(c_1 - c_0')}{c_0} dx$$

we get the weight function :

$$p(x) = e^{\int \frac{(c_1(x) - c_0'(x))}{c_0(x)} dx}.$$

3.2 Examples for Converting Some Famous Equtions to SL-equation:

3.2.1 Legendre polynomials [13]:

Consider this Legendre equation,

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \tag{3.10}$$

we must find the weight function to convert the equation (3.10) to the form of SL-equation**,**

$$p(x) = \frac{1}{|c_0(x)|} e^{\int \frac{c_1(x)}{c_0(x)} dx} = \left[\frac{1}{|1 - x^2|} \right] e^{\int \frac{-2x}{|1-x^2|} dx}$$

$$p(x) = \left[\frac{1}{|1 - x^2|} \right] e^{\ln |1-x^2|} = \left[\frac{1}{|1 - x^2|} \right] . |1 - x^2|$$

$$p(x) = 1 .$$

Since the weight function $p(x)=1$, we get :

$$\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx} \right] + \lambda y = 0 \qquad (3.11)$$

so the equation (3.11) becomes in the form of SL- equation .

3.2.2 Laguerre polynomials [14]:

Consider this Laguerre equation:

$$x y'' + (1 - x)y' - \lambda y = 0 \qquad (3.12)$$

since $c_0(x)=x$, $c_1(x)=(1-x)$ and $c_0'(x)=1$.

We note $c_0'(x)\neq c_1(x)$, that means the equation(3.12) is not in the form of SL-equation.

Now we must find the weight function $p(x)$,

$$p(x) = \frac{1}{|c_0(x)|} e^{\int \frac{c_1(x)}{c_0(x)} dx} = \left[\frac{1}{|x|} \right] .e^{\int \frac{1-x}{x} dx}$$

$$p(x) = \left[\frac{1}{|x|} \right] .e^{\int \left(\frac{1}{x} - 1 \right) dx} = \left[\frac{1}{|x|} \right] .e^{\int \frac{1}{x} dx} .e^{-\int 1 \, dx}$$

$$p(x) = \left[\frac{1}{|x|} \right] .e^{\ln |x|} .e^{-x} = \left[\frac{1}{|x|} \right] .|x| .e^{-x} = e^{-x}$$

by multiplying the equation (3.12) by the weight function $p(x) = e^{-x}$, we will get :

$$xe^{-x} y'' + (1 - x)e^{-x} y' - \lambda e^{-x} y = 0 \qquad (3.13)$$

now , we note in the equation (3.13), $c_0(x) = xe^{-x}$ and $c_0'(x) = (1 - x)e^{-x} = c_1(x)$

we get:

$$\frac{d}{dx}\left(xe^{-x}\frac{dy}{dx}\right) - \lambda e^{-x} y = 0 \qquad (3.14)$$

the equation (3.14) becomes as the form of SL- equation .

3.2.3 Hermite polynomials [15]:

Consider the following Hermite equation:

$$y'' - 2xy' + 2\lambda y = 0 \qquad (3.15)$$

since $c_0(x)=1$, $c_1(x)=-2x$ and $c_0'(x)=0$.

We note $c_0'(x)\neq c_1(x)$, that means the equation (3.15) is not in the form of SL-equation. Therefore we will find $p(x)$:

$$p(x) = \frac{1}{|c_0(x)|} e^{\int \frac{c_1(x)}{c_0(x)}dx} = \left[\frac{1}{1}\right].e^{\int \frac{-2x}{1}dx}$$

$$p(x) = e^{-2\int xdx} = e^{-2\left(\frac{x^2}{2}\right)} = e^{-x^2}$$

multiply the equation (3.15) by the weight function $p(x)=e^{-x^2}$, we will get :

$$xe^{-x^2} y'' + (1-x)e^{-x^2} y' + 2\lambda e^{-x^2} y = 0 \qquad (3.16)$$

now , we note in the equation $(3,12)$, $c_0(x) = xe^{-x^2}$ and $c_0'(x) = (1-x)e^{-x^2} = c_1(x)$

we get :

$$\frac{d}{dx}\left(xe^{-x^2}\frac{dy}{dx}\right) + 2\lambda e^{-x^2} y = 0 \qquad (3.17)$$

the equation (3.17) becomes in the form of SL-equation .

3.2.4 Confluent hypergeometric equation [16]:

Here the differential equation is,

$$xy'' + (z-x)y' - \lambda y = 0 \qquad (3.18)$$

since $c_0(x)=x$, $c_1(x)=(z-x)$ and $c_0'(x)=1$.

We note $c_0'(x) \neq c_1(x)$, that means the equation (3.18) is not in the form of SL-equation .

$$p(x) = \frac{1}{|c_0(x)|} e^{\int \frac{c_1(x)}{c_0(x)} dx} = \left[\frac{1}{|x|}\right] . e^{\int \frac{z-x}{x} dx} = \left[\frac{1}{|x|}\right] . e^{\int \left(\frac{z}{x} - 1\right) dx}$$

$$p(x) = \left[\frac{1}{|x|}\right] . e^{z \int \frac{1}{x} dx} . e^{-\int 1\, dx} = \left[\frac{1}{|x|}\right] . e^{z \ln |x|} . e^{-x} = \left[\frac{1}{|x|}\right] . |x|^z . e^{-x}$$

$$p(x) = |x|^{z-1} . e^{-x}$$

multiply the equation $(3.18)$by the weight function $p(x) = |x|^{z-1} . e^{-x}$, we will get:

$$|x|^{z-1} e^{-x} x y'' + |x|^{z-1} e^{-x} (z - x) y' + \lambda |x|^{z-1} e^{-x} y = 0 \qquad (3.19)$$

Now, we note in the equation(3.19),

$$c_0(x) = |x|^{z-1} x e^{-x} \text{ and } c_0'(x) = (z-x)|x|^{z-1} e^{-x^2} = c_1(x)$$

we will get:

$$\frac{d}{dx}\left(|x|^{z-1} x e^{-x} \frac{dy}{dx} \right) + \lambda |x|^{z-1} e^{-x} y = 0 \qquad (3.20)$$

the equation (3.20) becomes in the form of SL-equation .

3.2.5 Chebyshev polynomials [8]:

Consider the following Chebyschev equation:

$$(1 - x^2) y'' - x y' - \lambda y = 0 \qquad (3.21)$$

now we will discuss the equation (3.21),

$$c_0(x) = (1 - x^2) \ , \quad c_1(x) = -x \quad \text{and} \quad c_0'(x) = -2x$$

We note $c_0'(x) \neq c_1(x)$, that means the equation (3.21) is not in the form of SL-equation**.**

$$p(x) = \frac{1}{|c_0(x)|} e^{\int \frac{c_1(x)}{c_0(x)} dx} = \left[\frac{1}{|1 - x^2|}\right] . e^{\int \frac{-x}{1-x^2} dx}$$

$$p(x)=\left[\frac{1}{\left|1-x^2\right|}\right].e^{\frac{1}{2}\int\left(\frac{-2x}{1-x^2}\right)dx}=\left[\frac{1}{\left|1-x^2\right|}\right].e^{\frac{1}{2}\ln\left|1-x^2\right|}$$

$$p(x)=\left[\frac{1}{\left|1-x^2\right|}\right].\left[1-x^2\right]^{\frac{1}{2}}=\frac{1}{\sqrt{\left(1-x^2\right)}}$$

when multiplying the equation (3.21) by the weight function $p(x)=\frac{1}{\sqrt{\left(1-x^2\right)}}$, we will get:

$$\left(1-x^2\right)^{\frac{1}{2}}y''-\frac{x}{\sqrt{1-x^2}}y'-\frac{\lambda}{\sqrt{1-x^2}}y=0 \qquad (3.22)$$

now, we note in the equation (3.22),

$$c_0(x)=\left(1-x^2\right)^{\frac{1}{2}} \text{ and } c_0'(x)=-\frac{x}{\sqrt{1-x^2}}=c_1(x)$$

we will get:

$$\frac{d}{dx}\left(\sqrt{1-x^2}\,\frac{dy}{dx}\right)+\frac{\lambda}{\sqrt{1-x^2}}y=0 \qquad (3.23)$$

the equation (3.23) becomes in the form of Sturm- Liouville equation .

CHAPTER 4

ASYMPTOTIC FORMULAS FOR EIGENVALUES AND EIGENFUNCTIONS OF STURM-LIOUVILLE PROBLEMS

In this chapter we will find asymptotic formulas for eignvalues and eigenfunctions of Sturm-Liouville problems. Firstly we will talk about asymptotic notation (Big O notation):

4.1 Asymptotic Notation (Big O Notation) :

Big O notation, which contains a capital letter O (not zero), was invented in 1892 by Paul Bachmann (1873-1920), who is a German mathematician [17]. And then in 1909, its use was popularized by Edmund Landau (1877-1938). Therefore, sometimes they named this symbol Bachmann-Landau symbol [18]. Big O notation is used in computer science, complexity theory and mathematics to describe the asymptotic behavior of functions. Essentially, it is used to inform us how fast a function grows or declines. The letter O is used because the rate of growth of function is also called its order. Big O is more common in use than other four notations for comparing functions which are little o, big Omega (Ω), little Omega (ω) and Theta (Θ) [19].

4.1.1 Definition of big O notation [19]:

$f(x) \in O(g(x))$ if there exist positive constant c, x_0 such that :

$$|f(x)| \le c|g(x)| \quad \text{for all} \quad x \ge x_0,$$

that means $g(x)$ is an asymptotic upper bond for $f(x)$ or $f(x)$ is Big O of $g(x)$.

$$f(x) \in O(g(x)) \text{ equivalent to } f(x) = O(g(x))$$

which is more common in use and we must be careful because it is not the same if we write,

$$O(g(x)) = f(x) .$$

4.1.2 Properties of big O notation [19-20]:

1. $x + K = O(x), \ \forall \quad K \in \Re$ (ignoring constant) .
2. If $f(x) = O(g(x))$, then $Kf(x) = O(g(x))$.
3. If

$$f(x) = j_n x^n + j_{n-1} x^{n-1} + \ldots\ldots.. + j_1 x^1 + j_0 ,$$

where

$$j_0, j_1, \ldots\ldots, j_{n-1}, j_n$$

are real numbers, then

$$f(x) = O(x^n)$$

4. If $f(x) = O(g(x))$ and $g(x) = O(h(x))$, then

$$f(x) = O(h(x)).$$

5. If $f(x) = O(g(x))$ and $g(x) = O(h(x))$, then

$$f(x) + g(x) = O(h(x)).$$

That means, If $f_r(x) = O(g(x))$ for all $r \in N$, then

$$\sum_{r=0}^{K} f_r(x) = O(g(x))$$

for any $K \in N$.

6. If $f_1(x) = O(g_1(x))$ and $f_2(x) = O(g_2(x))$, then

$$f_1(x) + f_2(x) = O(\max(|g_1(x)|.|g_2(x)|)).$$

7. If $f_1(x) = O(g(x))$ and $f_2(x) = O(g(x))$, then

$$f_1(x) + f_2(x) = O(g(x)).$$

8. If $f_1(x) = O(g_1(x))$ and $f_2(x) = O(g_2(x))$, then

$$f_1(x).f_2(x) = O(g_1(x).g_2(x)).$$

9. If $f(x) + g(x) = O(h(x))$ and $f(x)$ and $g(x)$ are asymptotic nonnegative, then

$$f(x) = O(h(x)).$$

10. If $f(x).g(x) = O(h(x))$ and $g(x)$ has a positive asymptotic lower bound, then

$$f(x) = O(h(x)).$$

11. If $f(x) = O(g(x))$ and $f'(x) = O(g'(x))$, then

$$f'(x) = O(g(x).g'(x)).$$

4.2 Asymptotic Formulas for Eignvalues and Eigenfunctions of SL- problem:

Consider the Sturm-Liouville problem:

$$Ly = \frac{d^2 y}{dx^2} + q(x)y = \lambda y$$

and

$$\begin{aligned} y(a)\cos\alpha + y'(a)\sin\alpha = 0 \\ y(b)\cos\beta + y'(b)\sin\beta = 0 \end{aligned}$$

if we divide boundary conditons into $\sin\alpha \neq 0$ and $\sin\beta \neq 0$ respectively , we obtain :

$$\begin{aligned} y(a)\frac{\cos\alpha}{\sin\alpha} + y'(a)\frac{\sin\alpha}{\sin\alpha} = 0 \\ y(b)\frac{\cos\beta}{\sin\beta} + y'(b)\frac{\sin\beta}{\sin\beta} = 0 \end{aligned}$$

that means:

$$\begin{aligned} y(a)\cot\alpha + y'(a) = 0 \\ y(b)\cot\beta + y'(b) = 0 \end{aligned}$$

let $\cot\alpha = -h$ and $\cot\beta = H$, then we obtain the condtions :

$$\begin{aligned} y'(a) - hy(a) = 0 \\ y'(b) + Hy(b) = 0 \end{aligned}$$

if $q(x)$ is continuous and the numbers h and H are finite, then this problem is called regular Sturm-Liuoville problem. Otherwise it is called singular Strum-Liuoville problem.

4.2.1 Theorem for solutions of the SL-problem and formulas of eigenvalues and eigenfunctions [21]:

Consider Sturm-Liouville problem:

$$-y'' + q(x)y = \lambda y \tag{4.1}$$

and

$$\left.\begin{aligned} y'(0) - hy(0) = 0 \\ y'(\pi) + Hy(\pi) = 0 \end{aligned}\right\} \tag{4.2}$$

where $x \in [0,\pi]$, and $q(x): x \in [0,\pi] \to R$ continuous.

Let $\varphi(x,\lambda)$ be the solution of (4.1) which satisfies the conditions $\varphi(0,\lambda) = 1$ and $\varphi'(0,\lambda) = h$.

Also, suppose that $\psi(x,\lambda)$ is the solution of (4.1) which satisfies the conditions $\psi(0,\lambda) = 0$ and $\psi'(0,\lambda) = 1$.

If $\lambda = s^2$ then

$$\varphi(x,\lambda) = \cos(sx) + \frac{h}{s}\sin(sx) + \frac{1}{s}\int_0^x \sin[s(x-\tau)]q(t)\varphi(\tau,\lambda)d\tau \tag{4.3}$$

and

$$\psi(x,\lambda) = \frac{1}{s}\sin(sx) + \frac{1}{s}\int_0^x \sin[s(x-\tau)]q(t)\psi(\tau,\lambda)d\tau \tag{4.4}$$

Proof:

Firstly, we will prove equation (4.3). Since $\varphi(x,\lambda)$ is the solution of (4.1), then we have,

$$\int_0^x \sin[s(x-\tau)]q(t)\varphi(\tau,\lambda)d\tau = \int_0^x \sin[s(x-\tau)][\varphi''(\tau,\lambda) + \lambda\varphi(\tau,\lambda)]d\tau$$

$$\int_0^x \sin[s(x-\tau)]q(t)\varphi(\tau,\lambda)d\tau = \int_0^x \sin[s(x-\tau)]\varphi''(\tau,\lambda)d\tau$$

$$+ s^2\int_0^x \sin[s(x-\tau)]\varphi(\tau,\lambda)d\tau \tag{4.5}$$

now, integration by parts for the first part of the right hand of the last equation is done by using:

$$u = \sin[s(x-\tau)] \to du = -s\cos[s(x-\tau)]d\tau$$
$$dv = \varphi''(\tau,\lambda)d\tau \quad \to \quad v = \varphi'(\tau,\lambda)$$

we obtain:

$$\int_0^x \sin[s(x-\tau)]\varphi''(\tau,\lambda)d\tau = \{\varphi'(\tau,\lambda)\sin[s(x-\tau)]\}_0^x$$
$$+ s\int_0^x \cos[s(x-\tau)]\varphi'(\tau,\lambda)d\tau$$

$$\int_0^x \sin[s(x-\tau)]\varphi''(\tau,\lambda)d\tau = \varphi'(x,\lambda)\sin[s(x-x)] - \varphi'(0,\lambda)\sin[s(x-0)]$$
$$+ s\int_0^x \cos[s(x-\tau)]\varphi'(\tau,\lambda)d\tau \ .$$

From the initial condition we have $\varphi'(0,\lambda) = h$, we obtain:

$$\int_0^x \sin[s(x-\tau)]\varphi''(\tau,\lambda)d\tau = -h\sin(sx) + s\int_0^x \cos[s(x-\tau)]\varphi'(\tau,\lambda)d\tau$$

integration by pasts again by using :

$$u = \cos[s(x-\tau)] \to du = s\sin[s(x-\tau)]d\tau$$
$$dv = \varphi'(\tau,\lambda)d\tau \to v = \varphi(\tau,\lambda)$$

we obtain:

$$\int_0^x \sin[s(x-\tau)]\varphi''(\tau,\lambda)d\tau = -h\sin(sx) + s\{\varphi(\tau,\lambda)\cos[s(x-\tau)]\}_0^x$$
$$- s^2\int_0^x \sin[s(x-\tau)]\varphi(\tau,\lambda)d\tau$$

$$\int_0^x \sin[s(x-\tau)]\varphi''(\tau,\lambda)d\tau = -h\sin(sx) + s\{\varphi(x,\lambda)\cos[s(x-x)] - \varphi(0,\lambda)\cos[s(x-0)]\}$$
$$- s^2\int_0^x \sin[s(x-\tau)]\varphi(\tau,\lambda)d\tau$$

from the initial condition we have $\varphi(0,\lambda) = 1$, we obtain:

$$\int_0^x \sin[s(x-\tau)]\varphi''(\tau,\lambda)d\tau = -h\sin(sx) + s\varphi(x,\lambda) - s\cos(sx)$$
$$- s^2\int_0^x \sin[s(x-\tau)]\varphi(\tau,\lambda)d\tau \qquad (4.6)$$

we substitute (4.5) on (4.6), we obtain:

$$\int_0^x \sin[s(x-\tau)]q(t)\varphi(\tau,\lambda)d\tau = -h\sin(sx) + s\varphi(x,\lambda) - s\cos(sx) - s^2\int \sin[s(x-\tau)]\varphi(\tau,\lambda)d\tau$$
$$+ s^2\int_0^x \sin[s(x-\tau)]\varphi(\tau,\lambda)d\tau$$

$$\int_0^x \sin\left[s(x-\tau)\right]q(t)\varphi(\tau,\lambda)d\tau = -h\sin(sx) + s\varphi(x,\lambda) - s\cos(sx) \qquad (4.7)$$

thus:

$$\varphi(x,\lambda) = con(sx) + \frac{h}{s}\sin(sx) + \frac{1}{s}\int_0^x \sin\left[s(x-\tau)\right]q(t)\varphi(\tau,\lambda)d\tau \qquad (4.8)$$

from (4.7) and (4.8) we obtain:

$$\varphi(x,\lambda) = con(sx) + \frac{h}{s}\sin(sx) + \frac{1}{s}\left[-h\sin(sx) + s\varphi(x,\lambda) - s\cos(sx)\right]$$

$$\varphi(x,\lambda) = con(sx) + \frac{h}{s}\sin(sx) - \frac{h}{s}\sin(sx) + \varphi(x,\lambda) - con(sx)$$

$$\varphi(x,\lambda) = \varphi(x,\lambda).$$

The proof of equation (4.3) is complete**.**

Secondly, we will prove the equation(4.4)**. S**ince $\psi(x,\lambda)$ is the solution of(4.1), then, we have:

$$\int_0^x \sin\left[s(x-\tau)\right]q(t)\psi(\tau,\lambda)d\tau = \int_0^x \sin\left[s(x-\tau)\right]\left[\psi''(\tau,\lambda) + \lambda\psi(\tau,\lambda)\right]d\tau$$

$$\int_0^x \sin\left[s(x-\tau)\right]q(t)\psi(\tau,\lambda)d\tau = \int_0^x \sin\left[s(x-\tau)\right]\psi''(\tau,\lambda)d\tau$$

$$+ s^2\int_0^x \sin\left[s(x-\tau)\right]\psi(\tau,\lambda)d\tau \qquad (4.9)$$

now, integration by parts for the first part of the right hand for the equation (4.9) is done by using:

$$u = \sin\left[s(x-\tau)\right] \rightarrow du = -s\cos\left[s(x-\tau)\right]d\tau$$

$$dv = \psi''(\tau,\lambda)d\tau \rightarrow v = \psi'(\tau,\lambda)$$

we obtain :

$$\int_0^x \sin\left[s(x-\tau)\right]\psi''(\tau,\lambda)d\tau = \left\{\psi'(\tau,\lambda)\sin\left[s(x-\tau)\right]\right\}_0^x$$

$$+ s\int_0^x \cos\left[s(x-\tau)\right]\psi'(\tau,\lambda)d\tau$$

$$\int_0^x \sin\left[s(x-\tau)\right]\psi''(\tau,\lambda)d\tau = \psi'(x,\lambda)\sin\left[s(x-x)\right] - \psi'(0,\lambda)\sin\left[s(x-0)\right]$$

$$+ s\int_0^x \cos\left[s(x-\tau)\right]\psi'(\tau,\lambda)d\tau$$

from the conditions we have $\psi'(0,\lambda) = 1$, we obtain:

$$\int_0^x \sin\left[s(x-\tau)\right]\psi''(\tau,\lambda)d\tau = -\sin(sx) + s\int_0^x \cos\left[s(x-\tau)\right]\psi'(\tau,\lambda)d\tau$$

integration by pasts again by using :

$$u = \cos[s(x-\tau)] \rightarrow du = s\sin[s(x-\tau)]d\tau$$

$$dv = \psi'(\tau,\lambda)d\tau \rightarrow v = \psi(\tau,\lambda)$$

we obtain:

$$\int_0^x \sin\left[s(x-\tau)\right]\psi''(\tau,\lambda)d\tau = -\sin(sx) + s\{\psi(\tau,\lambda)\cos\left[s(x-\tau)\right]\}_0^x$$

$$-s^2\int_0^x \sin[s(x-\tau)]\psi(\tau,\lambda)d\tau$$

$$\int_0^x \sin[s(x-\tau)]\psi''(\tau,\lambda)d\tau = -\sin(sx) + s\{\psi(x,\lambda)\cos[s(x-x)] - \psi(0,\lambda)\cos[s(x-0)]\}$$

$$-s^2\int_0^x \sin[s(x-\tau)]\psi(\tau,\lambda)d\tau$$

from the conditions we have $\psi(0,\lambda)=0$, we obtain:

$$\int_0^x \sin[s(x-\tau)]\psi''(\tau,\lambda)d\tau = -\sin(sx) + s\psi(x,\lambda) - s^2\int_0^x \sin[s(x-\tau)]\psi(\tau,\lambda)dx \qquad (4.10)$$

we substitute (4.10) on (4.9), we obtain:

$$\int_0^x \sin[s(x-\tau)]q(t)\psi(\tau,\lambda)d\tau = -\sin(sx) + s\psi(x,\lambda) - s^2\int \sin[s(x-\tau)]\psi(\tau,\lambda)d\tau$$

$$+s^2\int_0^x \sin[s(x-\tau)]\psi(\tau,\lambda)d\tau$$

$$\int_0^x \sin\left[s(x-\tau)\right]q(t)\psi(\tau,\lambda)d\tau = -\sin(sx) + s\psi(x,\lambda) \qquad (4.11)$$

thus**:**

$$s\psi(x,\lambda) = \sin(sx) + \int_0^x \sin\left[s(x-\tau)\right]q(t)\psi(\tau,\lambda)d\tau$$

$$\psi(x,\lambda) = \frac{1}{s}\sin(sx) + \frac{1}{s}\int_0^x \sin\left[s(x-\tau)\right]q(t)\psi(\tau,\lambda)d\tau \qquad (4.12)$$

now we substitute (4.11) on (4.12), we obtain:

$$\psi(x,\lambda) = \frac{1}{s}\sin(sx) + \frac{1}{s}\left[-\sin(sx) + s\psi(x,\lambda)\right]$$

$$\psi(x,\lambda) = \frac{1}{s}\sin(sx) - \frac{1}{s}\sin(sx) + \psi(x,\lambda)$$

$$\psi(x,\lambda) = \psi(x,\lambda)\textbf{.}$$

By this way we proved the equation (4.4)**.**

Remark1:

Suppose $s=\sigma+it=\sqrt{\lambda}$ **,** where it is customary to take the branch of the square root on the positive real λ**-** axis. However, it should be in mind that $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are entire functions of λ so that no need to make use of the branch of s **,** all asymptotic formulas involving combinations of elementary functions for which the branch actually disappears [21].

Remark2 [21]:

If $s=\sigma+it=\sqrt{\lambda}$ **,** then $\exists s_0>0$ s.t. $\forall |s|>s_0$, we have :

$$\left.\begin{aligned}\varphi(x,\lambda)&=O\left(e^{|t|.x}\right)\\ \psi(x,\lambda)&=O\left(|s|^{-1}.e^{|t|.x}\right)\end{aligned}\right\} \qquad (4.13)$$

$$\varphi(x,\lambda)=\cos(sx)+O\left(|s|^{-1}.e^{|t|x}\right) \qquad (4.14)$$

$$\psi(x,\lambda)=\frac{1}{s}\sin(sx)+O\left(|s|^{-1}.e^{|t|x}\right). \qquad (4.15)$$

where $x\in[0,\pi]$.

4.2.2 Evaluate asymptotic formulas for eignvalues and eigenfunctions of SL-problem:

Now, we can evaluate asymptotic formulas for the eigenvalues and eigenfunctions corresponding to the SL-problem (4.1) and (4.2).

By theorem (**4.2.1)** and remark 1, we have:

$$\varphi(x,\lambda)=\cos(sx)+\frac{h}{s}\sin(sx)+\frac{1}{s}\int_0^x\sin[s(x-\tau)]q(t)\varphi(\tau,\lambda)d\tau \qquad (4.3)$$

and

$$\varphi(x,\lambda)=\cos(sx)+O\left(|s|^{-1}.e^{|t|x}\right). \qquad (4.14)$$

Suppose $h \neq \infty$ and $H \neq \infty$. Since $\varphi(x,\lambda)$ is the solution, then if we write the value of this function at point (π) into condition (2) we will find eigenvalues.

Because of Remark1, all eigenvalues are real and a number of negative eigenvalues is finite. Therefore, from equation (4.14) we have:

$$\varphi(x,\lambda) = \cos(sx) + O\left(\frac{1}{s}\right) \tag{4.16}$$

now, differentiating the equation (4.3) with respect to x and using the equation (4.16), we obtain:

$$\varphi'(x,\lambda) = -s\sin(sx) + h\cos(sx) + O\left(\frac{1}{s}\right)$$

since $y'(\pi) + Hy(\pi) = 0$ and by substitution we get :

$$-s\sin(s\pi) + h\cos(s\pi) + O\left(\frac{1}{s}\right) + H\left[\cos(s\pi) + O\left(\frac{1}{s}\right)\right] = 0$$

thus

$$-s\sin(s\pi) + (h+H)\cos(s\pi) + O\left(\frac{1}{s}\right) = 0 \tag{4.17}$$

for large values of s, the equation (4.17) has solutions. This means that we have infinitley many eigenvalues for the specified Sturm-Liouville problem.

Now, our claim is to show that, for a sufficiently large number of n, the equation (4.17) has only one solution . That is the unique solution .

For this, differentiate the equation (4.17) with respect to s . We obtain :

$$-\sin(s\pi) - s\pi\cos(s\pi) - (h+H)\pi\sin(s\pi) + O(1) = 0 \tag{4.18}$$

For sufficiently large values of s, we can easily see that left side of equation (4.18) is not zero .

That means, for a sufficiently large number of λ_n , the equation (4.17) has unique eigenvalue, which is the unique solution.

4.3 Future Work:

We will study the zero of eigenvalues (nodal points) and the cases in which at least one of h and H equal to ∞.

CHAPTER 5

CONCLUSION

In this thesis, we provided in chapter two some properties of SL-problem and we focused on the property of orthogonal of SL-problem. By using theorem (2.2.3), we found that the eigenfunctions of regular SL-problem are orthogonal with respect to the weight function s on the closed interval $[\alpha, \beta]$. And then from this result we found that these eigenfunctions of SL-problem can form a set of orthonormal eigenfunctions of SL-problem. And with some conditions on function f in theorem (2.3.3), the set of orthonormal eigenfunctions of SL-problem can form the series (2.13) which converges to the function f. The series (2.13) which is an expansion of function f is called the Fourier series. And after that Fourier series guides us to special and important kind of Fourier series called trigonometric Fourier series. And also there are two significant cases when f is even function or odd function according to determination of Fourier coefficients, which are Fourier sine and cosine series.

In chapter three, by using theorem of Self-adjoint Operator, we found the suitable weight function to convert any problem into regular Sturm-Liouville problem.

In chapter four, we discussed Asymptotic Formulas for Eigenvalues and Eigenfunctions of Sturm-Liouville problems. We also gave an important formula about the solutions of a specific SL-problem and obtain formulas of eigenvalues and eigenfunctions of this problem.

REFERENCES

1. **Amrein W. O., Hinz A. N., Pearson D. B., (2005),** "*Sturm-Liouville Theory: Past and Present*", Birkhäuser Verlag, Germany, pp. vii-xviii.

2. **Zettl A., (2005),** "*Sturm-Liouville Theory*", The American Mathematical Society, The United States of America, pp.ix-xi.

3. **Al-Gwaiz M. A., (2008)**, "*Sturm-Liouville Theory and its Applications*", Springer-Verlag, London, pp. v-vii.

4. **Atkinson F. V., Mingarelli A. B., (2011),** "*Multiparameter Eigenvalue Problems: Sturm-Liouville Theory*", CRC Press Taylor & Francis Group, Florida, pp. 1-4.

5. **Adzievski K., Siddiqi A. H., (2014),** "*Introduction to Partial Differential Equations for Scientist and Engineers*", CRC Press Taylor & Francis Group, LLC, Florida, p. 161, 169.

6. **Ross S. L., (1984),** "*Differential Equations*", 3th Edition, John Wiley & Sons, Inc., Singapore, pp. 588-630.

7. **Agarwal R. P., O'Regan D., (2008),** "*An Introduction to Ordinary Differential Equations*", Springer Science + Business Media LLC, New York, pp. 268, 271.

8. **Birkhoff G., Rota G., (1989),** "*Ordinary Differential Equations*", 4th Edition, John Wiley & Sons, The United States of America, pp. 302, 368.

9. **Zill D. G., Wright W. S., (2013),** "*Differential Equations with Boundary Value Problems*", Book Cole, Cengage Learning, Boston, pp. 441-442.

10. **Brown J. W., Churchill R. V., (1993),** "*Fourier Series and Boundary Value Problems*", 5th Edition, McGraw-Hill, the United States of America, pp. 48-49.

11. **Hanna J. R., Rowland J. H., (1990),** "*Fourier Series, Transform, and Boundary Value Problems*", 2th Edition, John Wiley & Sons, New York, pp.78.

12. **Azad H., Laradji A., Mustafa M. T**., (2010), "Polynomial Solutions of Differential Equations", Technical Report Series, TR 415, pp.1-16.

13. **Courant R., Hilbert D.**, **(1966),** "*Methods of Mathematical Physics Volume1*", New York, pp. 325.

14. **Chakrabarti A., (1996),** "*Elements of Ordinary Differential Equations and Special Functions*", 2th Edition, New Age International (P) Ltd Publisher, New Dalhi, pp. 86.

15. **Sharma S. K., Gupta V. G., (2010),** "*Differential Equations*", 45th Edition, Satyendra Rastogi (Mita), India, pp. 537.

16. **Andrews G. E., Askey R., Roy R., (1999),** "*Special Functions*", The Press Syndicate of The University of Cambridge, Cambridge, pp. 188.

17. **Dasgupata S.**, **(2014),** "*It Began with Babbage: The Genesis of Computer Science*", Oxford University Press, Oxford, pp. 253.

18. **Holmes M. H., (2013),** "*Introduction to Perturbation Methods*", 2th Edition, Springer Science + Business Media, New York, pp. 4.

19. **Rosen K. H.**, **(2012),** "*Discrete Mathematics and Its Applications*", 7th Edition, The McGraw-Hill Compaines, New York, pp. 205-213.

20. **Drozdek A., (2013),** "Data Structures and Algorithms in C^{++}", 4th Edition, Cengage Learing, Boston, USA, pp. 54-55.

21. **Fulton C. T., (1994),** "Eigenvalue and Eigenfunction Asymptotics for Regular Sturm-Liouville Problems", Journal of Mathematical Analysis and Applications, pp. 188, 297-340.

APPENDICES A

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Aljumaili, Ahmed

Date and Place of Birth: 29 March 1971, Baghdad

Marital Status: Single

Phone: +9647702975050

Email: 71ahmedhashem@gmail.com

EDUCATION

Degree	**Institution**	**Year of Graduation**
M.Sc.	Çankaya University, Mathematics and Computer Science	2015
B.Sc.	Baghdad University, Mathematics	1993
High School	Al-Mansour Preparatory School	1988

WORK EXPERIENCE

Year	**Place**	**Enrollment**
1997-2001	Libya	Teacher
2007	Ministry of Higher Education, Iraq	Employee

FOREIN LANGUAGES

English.

HOBBIES

Reading.