ON SOME APPLICATIONS OF LOCAL FRACTIONAL CALCULUS

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ON SOME APPLICATIONS OF LOCAL FRACTIONAL CALCULUS

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ABSTRACT

ON SOME APPLICATIONS OF LOCAL FRACTIONAL CALCULUS

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In this thesis, some basic definitions and theorems for the local fractional calculus are given. Based on these definitions and theorems, the applications are presented within the local fractional calculus. It is shown that the applications of the local fractional calculus give very good results on the solution of physical and mathematical equations.

Keywords: Local Fractional Calculus, Local Fractional Derivative, Local Fractional Integral, Local Fractional Differential Equations, Wave Equation, Local Fractional Sumudu Transform.

ÖZ

YEREL KESİRLİ ANALİZİN UYGULAMALARI ÜZERİNE

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Bu tezde yerel kesirli analiz hakkında bazı temel tanımlar ve teoremler verilmiştir. Bu tanım ve teoremlerden yola çıkılarak yerel kesirli analiz ile ilgili uygulamalar sunulmuştur. Yerel fiziksel ve matematiksel denklemlerin çözümünde kesirli analiz uygulamaları oldukça iyi sonuçlar vermiştir.

Anahtar Kelimeler: Yerel Kesirli Analiz, Yerel Kesirli Türev, Yerel Kesirli İntegral, Yerel Kesirli Diferansiyel Denklemler, Dalga Denklemleri, Yerel Kesirli Sumudu Dönüşümü.

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TABLE OF CONTENTS

LIST OF SYMBOLS

${}_{x_0}D_x^{\alpha}$	: Local fractional derivative operator
d^{α}	: Local fractional differential operator
${}_aI_b^{(\alpha)}$	: Local fractional integral operator
Γ	: Gamma function
$E_{\alpha}(x^{\alpha})$	: Mittag-Leffler function
C_{α}	: Local fractional continuous
L_{α}	: Laplace transform operator
L_{α}^{-1}	: The inverse of Laplace transform
$sinh_{\alpha}$	: Hyperbolic sine function
$cosh_{\alpha}$	: Hyperbolic cosine function
LFS_{α}	: Local fractional Sumudu transform
LFS_{α}^{-1}	: The inverse of local fractional Sumudu transform

CHAPTER 1

INTRODUCTION

1.1. Background

In the last years, the local fractional calculus has taken a lot of attention andit has been investigated intensivelyby many researchers [1]. The local fractional calculus (LFC) is defined on fractals [2] which were suggested by Mandelbrot [3]. The local fractional calculus has been applied to the real world problems [4,17, 58, 68, 69].

The classical calculus cannot properly deal with non-differentiable functions. However, the local fractional calculus is one of the best candidates to solve this problem and it has been applied to model several practical problems in engineering [5].

Below, we give examples of studies that have been done in the recent years.
The Maxwell theory on Cantor sets was studied in [6]; the Heisenberg uncertainty principle within local fractional Fourier Series was discussed in [7]. In [8], it was developed a new Neumann series method for solving a family of local fractional Fredholm and Volterra integral operation; Through the studying in [9], the mappings for some special functions on Cantor sets were investigated. In [10], Helmhotz and the diffusion equations involving the local fractional derivative operations are presented on Cantor sets. In [11], the discrete wavelet transform via local fractional operations was structured and applied to process the signals on Cantor sets.The local fractional variational iteration method was given to handle the damped wave equation and dissipative wave equation in fractal strings [12]. A new model of the

scale conservation equation in the mathematical theory of vehicular traffic flow is suggested in [13]. A comparison is applied between the fractional iteration and decomposition methods which can be applied to the wave equation on Cantor sets [14]. In [15], the applications of local fractional variational iteration method are given to handle the local fractional Laplace equations. The local fractional variational iteration method is used [16] to solve the subdiffusion and wave equations. In [18], by utilizing the fractional complex transform method, the transport equations in fractal porous media are investigated. Also, a new wavelet transform is introduced within the framework of the local fractional calculus [19].

The thesis consist of five chapters. A review of the LFC and its applications is presented. In the first chapter, the theorems for local fractional derivative are presented.

Chapter two deals with the properties and theorems of local fractional integral and local fractional Taylor's theorem.

In Chapter three, the local fractional differential equation, the local fractional Fourier series and the Laplace transform are briefly mentioned.

In Chapters four and five, some applications of the local fractional calculus are given.

1.2. Importance of Local Fractional Calculus

Fractional calculus is a branch of mathematics dealing with arbitrary order of derivatives and integrals [21,32,34,36,70]. Many physical systems were modeled more accurately with this type of calculus. So, the fractional calculus has played a significant role in different fields such as mechanics [41], physics [45,51,56], nanotechnology [54], bioengineering [53], signal processing [61,63], economics [60], control theory [65], viscoelastic [57] and other fields of engineering [59].

The local fractional derivatives and integrals,defined on fractals [3,67], hold an important place in the fractional dynamic theory.

We recall that there are many definitions of local fractional derivatives and integrals. Firstly, we focus on the notations suggested by Kolwankar and Gangal. They suggested the formula [17], namely

$$ {}_{x_o}D_x^{\alpha} g(x) = \frac{d^{\alpha} g(x)}{dx^{\alpha}}\Big|_{x=x_0} =: \lim_{x \to x_o} \frac{d^{\alpha}(g(x)-g(x_0))}{d(x-x_0)^{\alpha}}, \; 0<\alpha \leq 1. \tag{1.1} $$

Here, α is precisely the Hölder exponent offunction defined in Cantor's set [49].

Kolwankar and Gangal introduced the local fractional integral as [24, 69]

$$ {}_aI_b^{(\alpha)} g(x) = \lim_{M \to \infty} \sum_{i=0}^{M-1} g(x_i) \frac{d^{-\alpha} 1_{dx_i(x)}}{d(x_{i+1}-x_i)} \quad ,i=0,1,...,M-1, \tag{1.2} $$

with $1_{dx_i(x)}$ being the unit function defined on $[x_i, x_{i+1}]$.

Jumarie arrived at [25,26], using the generalization of Taylor series,

$$ {}_{x_o}D_x^{\alpha} g(x) = \frac{d^{\alpha} g(x)}{dx^{\alpha}}\Big|_{x=x_0} =: \lim_{x\to x_0} \frac{\Delta^{\alpha}[g(x)-g(x_0)]}{h^{\alpha}}, \tag{1.3}$$

where $\Delta^{\alpha} g(x) = \sum_{t=0}^{\infty} (-1)^t \binom{\alpha}{t} g(x+(\alpha-t)h), \quad 0<\alpha\le 1.$

Jumarie suggested the fractional integral as follows [25,26]

$$ {}_0I_x^{(\alpha)} g(x) = \int_0^x g(t)(dt)^{\alpha} =: \alpha\int_0^x (x-t)^{\alpha-1} g(t)dt\,, \quad 0<\alpha\le 1. \tag{1.4}$$

This notation of the fractional derivative and integral deals with the non-differentiable functions. This calculus is called the modified fractional calculus.

Parvate and Gangal introduced the local fractional derivative as [27, 28]

$$ {}_{x_o}D_x^{\alpha} g(x) = \frac{d^{\alpha} g(x)}{dx^{\alpha}}\Big|_{x=x_0} =: G-\lim_{x\to x_0} \frac{g(x)-g(x_0)}{S_G^{\alpha}(x)-S_G^{\alpha}(x_0)}, \tag{1.5}$$

where $G-\lim_{x\to x_0}$ is the notion of the limit of $g(x)$ through the points of fractal set G.

Parvate and Gangal introduced a local fractional integral [27, 28], namely

$$ {}_aI_b^{(\alpha)} g(x) = \int_a^b g(x) d_G^{\alpha} x = \sum_{j=0}^{M-1} g(x_j)(S_G^{\alpha}(x_{j+1}) - S_G^{\alpha}(x_j)),\ 0<\alpha\le 1. \tag{1.6}$$

Adda and Cresson proposed a local fractional derivative [29,30] as

$$ {}_{x_o}D_x^{\alpha} f(x) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}\Big|_{x=x_0} =: \lim_{x\to x_0^{\sigma}} D_{y,-\sigma}^{\alpha}\left[\sigma(f-f(x_0))(x)\right], \tag{1.7}$$

with $\sigma=\mp$, where $D_{y,-\sigma}^{\alpha}$ is the Riemann-Liouville derivative operator.

Gao, Yang and Kang went through these definitions and obtained the notation of the local fractional derivative [31,33,55,58] as given below

$$ {}_{x_o}D_x^{\alpha} g(x) = \frac{d^{\alpha} g(x)}{dx^{\alpha}}\Big|_{x=x_0} =: \lim_{x \to x_0} \frac{\Delta^{\alpha}[g(x) - g(x_0)]}{(x - x_0)^{\alpha}}, \tag{1.8} $$

such that $\Delta^{\alpha}\left[g(x) - g(x_0)\right] \cong \Gamma(1+\alpha)\Delta\left[g(x) - g(x_0)\right]$.

Gao, Yang and Kang introduced a local fractional integral [27,31,37,58,62] as

$$ {}_aI_b^{(\alpha)} g(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b g(t)(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{i=0}^{M-1} g(t_i)(\Delta t_i)^{\alpha},\ 0 < \alpha \le 1, $$

where $\Delta t_i = \mathrm{t}_{\mathrm{i}+1} - \mathrm{t}_{\mathrm{i}}$ and $\Delta t = max\left\{\Delta t_1, \Delta t_2, ..., \Delta t_i\right\}$ for $0 < \alpha \le 1$ and $i = 0,1,2,3...,M-1$, $t_0 = a < t_1 < t_2 < ... < t_M = b$ is a partition of $\left[a,b\right]$.

If $a = b$, then ${}_aI_a^{(\alpha)} g(x) = 0$ and if $a < b$ then ${}_aI_b^{(\alpha)} g(x) = -\,{}_bI_a^{(\alpha)} g(x)$.

1.3. Local Fractional Derivative

Suppose that $g(x) \in C_{\alpha}\left[a,b\right]$ for $0 < \alpha \le 1$, $\delta > 0$ and $x \in \left(x_0 - \delta, x_0 + \delta\right)$ the limit [49],

$$ {}_{x_o}D_x^{\alpha} g(x) =: \lim_{x \to x_0} \frac{\Gamma(1+\alpha)[g(x) - g(x_0)]}{(x - x_0)^{\alpha}}, \tag{1.9} $$

is finite, then $g(x)$ has the local fractional derivative of order α at $x = x_0$,

$$ {}_{x_o}D_x^{\alpha} g(x) = \frac{d^{\alpha} g(x)}{dx^{\alpha}}\Big|_{x=x_0} = g^{(\alpha)}\left(x_0\right). $$

1.3.1. Left - Right Local Fractional Derivatives

If $g(x) \in C_{\alpha}\left[a,b\right]$ for $0 < \alpha \leq 1$, $\delta > 0$ and $x \in \left(x_0 - \delta, x_0\right)$ the limit [38]

$$ {}_{x_o^-}D_x^{\alpha} g(x) =: \lim_{x \to x_0^-} \frac{\Gamma(1+\alpha)[g(x) - g(x_0^-)]}{(x - x_0^-)^{\alpha}}, \tag{1.10}$$

is finite, then $g(x)$ has the left local fractional derivative of order α at $x = x_0$

$$ {}_{x_o^-}D_x^{\alpha} g(x) = \frac{d^{\alpha} g(x)}{dx^{\alpha}} \Big|_{x=x_0^-} = g^{(\alpha)}\left(x_0^-\right).$$

If $g(x) \in C_{\alpha}\left[a,b\right]$ for $0 < \alpha \leq 1$, $\delta > 0$ and $x \in \left(x_0, x_0 + \delta\right)$ the limit

$$ {}_{x_o^+}D_x^{\alpha} g(x) =: \lim_{x \to x_0^+} \frac{\Gamma(1+\alpha)[g(x) - g(x_0^+)]}{(x - x_0^+)^{\alpha}}, \tag{1.11}$$

is finite then $g(x)$ has the right local fractional derivative of order α at $x = x_0$,

$$ {}_{x_o^+}D_x^{\alpha} g(x) = \frac{d^{\alpha} g(x)}{dx^{\alpha}} \Big|_{x=x_0^+} = g^{(\alpha)}\left(x_0^+\right).$$

Proposition 1.1. [49]

If ${}_{x_o^-}D_x^{\alpha} g(x)$ and ${}_{x_o^+}D_x^{\alpha} g(x)$ exist and ${}_{x_o^-}D_x^{\alpha}\, g(x) = {}_{x_o^+}D_x^{\alpha}\, g(x)$, then

$$ {}_{x_o^-}D_x^{\alpha}\, g(x) = {}_{x_o^+}D_x^{\alpha}\, g(x) = {}_{x_o}D_x^{\alpha}\, g(x). \tag{1.12}$$

1.3.2. The Increment of a Function [49]

The increment of $g(x)$ is

$$\Delta^{\alpha} g(x) = g^{(\alpha)}(x)(\Delta x)^{\alpha} + \chi(\Delta x)^{\alpha}, \tag{1.13}$$

where Δx is increment of x and $\chi \to 0$ as $\Delta x \to 0$ for $0 < \alpha \leq 1$.

1.3.3. The Local Fractional Differential [17]

The local fractional differential is

$$d^{\alpha} g = g^{(\alpha)}(x)(dx)^{\alpha} \qquad 0 < \alpha \leq 1.$$

If there exist any point $x_0 \in (a,b)$ such that

$$\frac{d^{\alpha} g(x)}{dx^{\alpha}}\Big|_{x=x_0} = g^{(\alpha)}(x_0), \tag{1.14}$$

$D_{\alpha}(a,b)$ is called the α- local fractional derivative set.

Proposition 1.2. [30]

If $g \in D_{\alpha}(a,b)$,then $g \in C_{\alpha}(a,b)$.

Proof. **[30]**

From (1.13) and (1.14), we obtain

$$\Delta^{\alpha} g(x) = g^{(\alpha)}(x)(\Delta x)^{\alpha} + \chi(\Delta x)^{\alpha},$$

$$g(x) - g(x_0) = g^{(\alpha)}(x)(x - x_0)^{\alpha} + \chi(x - x_0)^{\alpha},$$

$$|g(x)| = \left| g^{(\alpha)}(x)(x - x_0)^{\alpha} + \chi(x - x_0)^{\alpha} + g(x_0) \right|.$$

Take the limit of the both sides $x \to x_0$ we get $\lim_{x \to x_0} |g(x)| = |g(x_0)|$.

Proposition 1.3. [49]

Suppose that $g \in D_{\alpha}(a,b)$,then $g(x)$ is local fractional differentiable on (a,b).

Suppose that $h(x), t(x) \in D_{\alpha}(a,b)$,then the differentiation rules below are valid [38-39]

$$\frac{d^{\alpha}\left(h(x)+t(x)\right)}{dx^{\alpha}}=\frac{d^{\alpha}\left(h(x)\right)}{dx^{\alpha}}+\frac{d^{\alpha}\left(t(x)\right)}{dx^{\alpha}}, \qquad (1.15)$$

$$\frac{d^{\alpha}\left(h(x)t(x)\right)}{dx^{\alpha}}=t(x)\frac{d^{\alpha}\left(h(x)\right)}{dx^{\alpha}}+h(x)\frac{d^{\alpha}\left(t(x)\right)}{dx^{\alpha}}, \qquad (1.16)$$

$$\frac{d^{\alpha}\left(\frac{h(x)}{t(x)}\right)}{dx^{\alpha}}=\frac{t(x)\frac{d^{\alpha}(h(x))}{dx^{\alpha}}+h(x)\frac{d^{\alpha}(t(x))}{dx^{\alpha}}}{t^{2}(x)}, \text{ if } t(x)\neq 0, \qquad (1.17)$$

$$\frac{d^{\alpha}(kh(x))}{dx^{\alpha}}=k\frac{d^{\alpha}(h(x))}{dx^{\alpha}}, \text{ k is a constant.} \qquad (1.18)$$

If $g(x)=(hot)(x)$, then

$$\frac{d^{\alpha}g(x)}{dx^{\alpha}}=h^{\alpha}(t(x))(t^{(1)}(x))^{\alpha}. \qquad (1.19)$$

Some of the above results were discussed in [38,40]. We have

1) $E_{\alpha}(x^{\alpha})=\sum_{i=0}^{\infty}\frac{x^{\alpha i}}{\Gamma(1+i\alpha)}$ where $0<\alpha\leq 1$,

2) $\frac{d^{\alpha}(x^{i\alpha})}{dx^{\alpha}}=\frac{\Gamma\left(1+i\alpha\right)}{\Gamma\left(1+(i-1)\alpha\right)}x^{(i-1)\alpha}$,

3) $\dfrac{d^{\alpha}\left(E_{\alpha}(mx^{\alpha})\right)}{dx^{\alpha}}=mE_{\alpha}\left(x^{\alpha}\right)$, where *m* is a constant,

4) $\dfrac{d^{\alpha}(E_{\alpha}(x^{\alpha}))}{dx^{\alpha}}=E_{\alpha}(x^{\alpha}),$

5) $\dfrac{d^{\alpha}(sin_{\alpha}x^{\alpha})}{dx^{\alpha}}=cos_{\alpha}x^{\alpha},$

6) $\dfrac{d^{\alpha}(cos_{\alpha}x^{\alpha})}{dx^{\alpha}}=-sin_{\alpha}x^{\alpha}.$

1.3.4. The 2α-Local Fractional Derivative and Higher-Order Derivative [49]

The 2α-local fractional derivative of $g(x)$ for $0<\alpha\le 1$

$$D_x^{2\alpha}\left[g(x)\right]=(D_x^{\alpha}.D_x^{\alpha})g(x)=\frac{d^{\alpha}}{dx^{\alpha}}\left[\frac{d^{\alpha}}{dx^{\alpha}}g(x)\right]=g^{(2\alpha)}(x). \qquad (1.20)$$

Similarly, we have $k\alpha$-local fractional derivative

$$D_x^{k\alpha}\left[g(x)\right]=(\underbrace{D_x^{\alpha}.D_x^{\alpha}.D_x^{\alpha}...D_x^{\alpha}}_{k\,times})g(x)$$

$$=g^{(k\alpha)}(x). \qquad (1.21)$$

1.3.5. Theorems for Local Fractional Derivatives

Theorem 1.4. (Local fractional Rolle's theorem)[50]

Suppose that $g\in C_{\alpha}\left[a,b\right]$ and $g\in D_{\alpha}(a,b)$. If $g(a)=g(b)$, then there exists a point $t\in(a,b)$ with

$$g^{(\alpha)}(t)=0, \qquad (1.22)$$

where $\alpha\in\left(0,1\right]$.

Proof. **[50]**

Case 1 : If $g(x)=0$ in $\left[a,b\right]$, then for all $x\in(a,b)$ we have $g^{(\alpha)}(x)=0$.

Case 2 : If $g(x)\neq 0$ in $\left[a,b\right]$, because $g(x)$ is continuous there are points at which $g(x)$ gets its maximum and minimum values, denoted by M and m respectively.

Since $g(x)\neq 0$, at least one of the values M and m is not zero.

Suppose, for instance, $m\neq 0$ and that $g(t)=m$. For this case

$$g(t+\Delta x)\geq g(t).$$

If $\Delta x>0$, then we arrive at the relations

$$\frac{\Gamma(1+\alpha)[g(t+\Delta x)-g(t)]}{(\Delta x)^{\alpha}}\geq 0,$$

and

$$\lim_{(\Delta x)\to 0^{+}}\frac{\Gamma(1+\alpha)[g(t+\Delta x)-g(t)]}{(\Delta x)^{\alpha}}\geq 0.$$

Similarly, if $\Delta x<0$, then we have

$$\frac{\Gamma(1+\alpha)\left[g(t+\Delta x)-g(t)\right]}{(\Delta x)^{\alpha}}\leq 0,$$

and

$$\lim_{(\Delta x)\to 0^{-}}\frac{\Gamma(1+\alpha)\left[g(t+\Delta x)-g(t)\right]}{(\Delta x)^{\alpha}}\leq 0.$$

Since $g(x)\in D_{\alpha}(a,b)$, then ${}_{x_o^-}D_x^{\alpha}g(x)={}_{x_o^+}D_x^{\alpha}g(x)$.

It happen only if the right and left derivatives are both equal to zero.

$g^{(\alpha)}(t)=0$ as required. Similarly, we take $M\neq 0$ we arrive at the formula (1.22).

Theorem 1.5. [17] (Local Fractional Mean Value Theorem)

Suppose that $g \in C_\alpha[a,b]$ and $g \in D_\alpha(a,b)$, then [17] there exists $\varphi \in (a,b)$ with

$$g(b) - g(a) = \frac{g^{(\alpha)}(\varphi)(b-a)^\alpha}{\Gamma(1+\alpha)}, \tag{1.23}$$

where $\alpha \in (0,1]$.

Proof. **[17]**

We define the $G(x)$ function

$$G(x) = \Gamma(1+\alpha)\left\{[g(x) - g(a)] - [g(b) - g(a)]\frac{(x-a)^\alpha}{(b-a)^\alpha}\right\}, \tag{1.24}$$

with $\alpha \in (0,1]$.

We have $G(a)=0$ and $G(b)=0$.

Appliying the Theorem 1.4 to the function $G(x)$,

$$G^{(\alpha)}(\varphi) = g^{(\alpha)}(\varphi) - \frac{\Gamma(1+\alpha)[g(b) - g(a)]}{(b-a)^\alpha} = 0,\ a < \varphi < b, \tag{1.25}$$

$$g^{(\alpha)}(\varphi) = \frac{\Gamma(1+\alpha)[g(b) - g(a)]}{(b-a)^\alpha},$$

then we get (1.23).

Theorem 1.6. [50] (Cauchy's Generalized Mean Value Theorem)

Suppose that $h(x),t(x) \in C_\alpha[a,b]$ and $h(x),t(x) \in D_\alpha(a,b)$. If $t(b) \neq t(a)$ then there exists a point $m \in (a,b)$

$$\frac{h(b)-h(a)}{t(b)-t(a)}=\frac{h^{(\alpha)}(m)}{t^{(\alpha)}(m)}. \tag{1.26}$$

Proof. **[50]**

We define

$$G(x)=\Gamma(1+\alpha)\left[h(x)-h(a)\right]-\frac{\Gamma(1+\alpha)\left[h(b)-h(a)\right]}{t(b)-t(a)}\left[t(x)-h(a)\right]. \tag{1.27}$$

Then, we have $G(a)=0$ and $G(b)=0$.

Appliying *the Theorem 1.4* to the function $G(x)$ in (1.27)

we have the relation $G^{(\alpha)}(m)=0$, $a<m<b$,

$$G^{(\alpha)}(m)=h^{(\alpha)}(m)-\frac{\left[h(b)-h(a)\right]}{\left[t(b)-t(a)\right]}t^{(\alpha)}(m)=0.$$

Thus, we arrive at the formula (1.26).

Theorem 1.7. (Local Fractional L'Hospital's Rule) [38]

Suppose that $g(x),h(x)\in C_\alpha\left[a,b\right]$ and $g(x),h(x)\in D_\alpha(a,b)$. $\lim\limits_{x\to x_0} g(x)=0$ and $\lim\limits_{x\to x_0} h(x)=0$, furthermore K denotes either a real number or one of the symbols $-\infty,+\infty$.

If $\lim\limits_{x\to x_0}\dfrac{g^{(\alpha)}(x)}{h^{(\alpha)}(x)}=K$, then it is also true that $\lim\limits_{x\to x_0}\dfrac{g(x)}{h(x)}=K$.

CHAPTER 2

2.1. The Local Fractional Integral

Let $g(x)\in C_\alpha[a,b]$, the local fractional integral of the function $g(x)$ is given by [42]

$$ {}_aI_b^{(\alpha)}g(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b g(t)(dt)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{k=0}^{M-1}g(t_k)(\Delta t_k)^\alpha , \tag{2.1}$$

with $0<\alpha\le 1,\Delta t_k=t_{k+1}-t_k$ and $\Delta t=max\{\Delta t_1,\Delta t_2,...,\Delta t_k,...\}$, where $[t_k,t_{k+1}]$, $k=0,1,2...,M-1$ and $t_0=a<t_1<t_2<...<t_M=b$, is partition of the interval $[a,b]$.

We assume that ${}_aI_a^{(\alpha)}g(x)=0$ and ${}_aI_b^{(\alpha)}g(x)=-{}_bI_a^{(\alpha)}g(x)$ if $a<b$.

2.1.1. Properties of the Local Fractional Integral

Property 2.1. [17]

Suppose that $g(x),h(x)\in C_\alpha[a,b]$, then we have

$$ {}_aI_b^{(\alpha)}[g(x)+h(x)]={}_aI_b^{(\alpha)}g(x)+{}_aI_b^{(\alpha)}h(x). \tag{2.2}$$

Proof. **[17]**

$$ {}_aI_b^{(\alpha)}[g(x)+h(x)]=\frac{1}{\Gamma(1+\alpha)}\int_a^b[g(t)+h(t)](dt)^\alpha $$

$$= \frac{1}{\Gamma(1+\alpha)} \int_a^b g(t)(dt)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_a^b h(t)(dt)^\alpha$$

$$= {}_aI_b^{(\alpha)} g(x) + {}_aI_b^{(\alpha)} h(x).$$

The proof of this property is completed.

Property 2.2. [49]

Suppose that $g(x) \in C_\alpha[a,b]$ and K is a constant, then we have

$${}_aI_b^{(\alpha)}[Kg(x)] = K\, {}_aI_b^{(\alpha)}[g(x)] \ . \tag{2.3}$$

***Proof.* [49]**

We take $g(x) \in C_\alpha[a,b]$ and K is a constant, then

$${}_aI_b^{(\alpha)}[Kg(x)] = \frac{1}{\Gamma(1+\alpha)} \int_a^b [Kg(t)](dt)^\alpha$$

$$= K \frac{1}{\Gamma(1+\alpha)} \int_a^b [g(t)](dt)^\alpha = K\, {}_aI_b^{(\alpha)}[g(x)].$$

Thus, the proof of the property was established.

Property 2.3. [50]

Suppose that $g(x) = c$, then

$${}_aI_b^{(\alpha)} c = \frac{c(b-a)^\alpha}{\Gamma(1+\alpha)} \ .$$

***Proof.* [50]**

Let we take $g(x) = c$ in (2.1),

$$ {}_aI_b^{(\alpha)}c = \frac{1}{\Gamma(1+\alpha)}\int_a^b c(dt)^\alpha = \frac{c}{\Gamma(1+\alpha)}(b-a)^\alpha $$
$$ = \frac{c(b-a)^\alpha}{\Gamma(1+\alpha)}. $$

Thus, we proved the statement.

Property 2.4. [17]

If $g(x) \in C_\alpha[a,b]$ and $g(x) \geq 0$, then we have

$$ {}_aI_b^{(\alpha)}g(x) \geq 0 \text{ with } b > a. \tag{2.4} $$

Proof. **[17]**

Let $g(x) \in C_\alpha[a,b]$ and $g(x) \geq 0$, then we have

$g(x_k) \geq 0, \quad k = 0,1,2,\ldots,M-1$.

We take the partition of $[a,b]$ is $[x_k, x_{k+1}]$ for $k = 0,1,2,\ldots,M-1$,

$x_0 = a < x_1 < x_2 < \ldots < x_{M-1} < x_M = b$.

Because $(\Delta x_i)^\alpha \geq 0$, we have

$$ {}_aI_b^{(\alpha)}g(x) = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta x \to 0}\sum_{k=0}^{M-1} g(x_k)(\Delta x_k)^\alpha \geq 0. $$

Hence, the proof of the property is finished.

Property 2.5. [50]

If $h(x), g(x) \in C_\alpha[a,b]$ and $h(x) \geq g(x)$, then we have

$$ {}_aI_b^{(\alpha)}h(x) \geq {}_aI_b^{(\alpha)}g(x) \text{ with } b > a. \tag{2.5} $$

Proof. **[50]**

Let $h(x) \geq g(x)$, then we have the relation

$h(x) - g(x) \geq 0$ and $h(x) - g(x) \in C_\alpha[a,b]$.

We take into account *the Property 2.4* and *the Property 2.1* , we get

$$
\begin{aligned}
&{}_aI_b^{(\alpha)}\big(h(x) - g(x)\big) \geq 0,\\
&{}_aI_b^{(\alpha)} h(x) - {}_aI_b^{(\alpha)} g(x) \geq 0,\\
&{}_aI_b^{(\alpha)} h(x) \geq {}_aI_b^{(\alpha)} g(x).
\end{aligned}
$$

Thus, the result is achieved.

Property 2.6. [49]

Let $g(x) \in C_\alpha[a,b]$ and let M_g and m_g are the maximum and minimum values of $g(x)$ in $[a,b]$. Then, we have

$$
M_g \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \geq {}_aI_b^{(\alpha)} g(x) \geq m_g \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \quad \text{with} \quad b > a. \tag{2.6}
$$

Proof. **[49]**

Let $g(x) \in C_\alpha[a,b]$ and we know $m_g \leq g(x) \leq M_g$.

In this inequality, let's integrate all sides for $b > a$. Moreover, we get inequality below by using *the Property 2.3*,

$$
\begin{aligned}
&{}_aI_b^{(\alpha)} m_g \leq {}_aI_b^{(\alpha)} g(x) \leq {}_aI_b^{(\alpha)} M_g,\\
&\frac{1}{\Gamma(1+\alpha)} \int_a^b m_g (dt)^\alpha \leq {}_aI_b^{(\alpha)} g(x) \leq \frac{1}{\Gamma(1+\alpha)} \int_a^b M_g (dt)^\alpha,\\
&m_g \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \leq {}_aI_b^{(\alpha)} f(x) \leq M_g \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}.
\end{aligned}
$$

The proof of this property is finished.

Property 2.7. [17]

If $g(x) \in C_{\alpha}[a,b]$, then we get

$$\left| {}_{a}I_{b}^{(\alpha)} g(x) \right| \le {}_{a}I_{b}^{(\alpha)} \left| g(x) \right|, \tag{2.7}$$

with $b > a$.

Proof. **[17]**

We know that $-\left|g(x)\right| \le g(x) \le \left|g(x)\right|$.

Taking the integration for $b > a$ and taking into account *the Property 2.5*,we conclude

$$-{}_{a}I_{b}^{(\alpha)} \left|g(x)\right| \le {}_{a}I_{b}^{(\alpha)} g(x) \le {}_{a}I_{b}^{(\alpha)} \left|g(x)\right|.$$

Then, we obtain

$$\left| {}_{a}I_{b}^{(\alpha)} g(x) \right| \le {}_{a}I_{b}^{(\alpha)} \left| g(x) \right|.$$

So, we finished the proof.

Property 2.8. [50]

If $g(x) \in C_{\alpha}[a,b]$ and $a < k < b$, then we have

$${}_{a}I_{b}^{(\alpha)} g(x) = {}_{a}I_{k}^{(\alpha)} g(x) + {}_{k}I_{b}^{(\alpha)} g(x). \tag{2.8}$$

Proof. **[50]**

Let $g(x) \in C_{\alpha}[a,b]$ and $a < k < b$ then $g(x)$ is the local fractional integral on $C_{\alpha}[a,b], C_{\alpha}[a,k]$ and $C_{\alpha}[k,b]$.

Let the partition of $[a,b]$ is $[x_j, x_{j+1}]$, where

$x_0 = a < x_1 < x_2 < ... < x_{M-1} < x_M = b$ and $j = 0,1,2...,M-1$.

Because to the definition of integration in (2.1),

$$ {}_aI_b^{(\alpha)}g(x)=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{M-1}g(x_j)(\Delta x_j)^{\alpha} . $$

Let the partition of $\left[a,k\right]$ be $\left[x_i,x_{i+1}\right]$, where

$x_0 = a < x_1 < x_2 < ... < x_t = k$ and $i = 0,1,2...,t-1$.

Because to the definition of integration in (2.1), we have

$$ {}_aI_k^{(\alpha)}g(x)=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{i=0}^{t-1}g(x_i)(\Delta x_i)^{\alpha} . $$

Let the partition of $\left[k,b\right]$ be $\left[x_i,x_{i+1}\right]$, where

$x_t = k < x_{t+1} < ... < x_{M-1} < x_M = b$ and $i = t,...,M-1$.

Due to (2.1), we conclude

$$ {}_kI_b^{(\alpha)}g(x)=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{i=t}^{M-1}g(x_i)(\Delta x_i)^{\alpha} . $$

Hence, it implies that

$$ \begin{aligned} \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{i=0}^{M-1}g(x_i)(\Delta x_i)^{\alpha} &= \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{i=0}^{t-1}g(x_i)(\Delta x_i)^{\alpha} \\ &+ \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{i=t}^{M-1}g(x_i)(\Delta x_i)^{\alpha}. \end{aligned} $$

Thus, we obtain (2.8).

2.1.2. Theorems for Local Fractional Integral

Theorem 2.9. (The Mean Value Theorem for Local Fractional Integrals) [50]

If $g(x)\in C_{\alpha}\left[a,b\right]$, then there exist a point η in (a,b) such that

$$ {}_aI_b^{(\alpha)}\, g(x)=g(\eta)\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)} . \tag{2.9} $$

***Proof.* [50]**

Let $g(x) \in C_\alpha[a,b]$ and let M_g and m_g are the maximum and minimum values of $g(x)$ in $[a,b]$, due to (2.6)

$$M_g \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \geq {}_aI_b^{(\alpha)} g(x) \geq m_g \frac{(b-a)^\alpha}{\Gamma(1+\alpha)},$$

and therefore

$$m_g \leq \underbrace{\frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^{(\alpha)} g(x)}_{g(\eta)} \leq M_g, \quad \eta \in (a,b).$$

There exists a point η in (a,b) providing the above inequality.

and thus, we obtain (2.9).

Theorem 2.10. [17]

Suppose that $g(x) \in C_\alpha[a,b]$, then there is a function,

$$\prod(x) = {}_aI_x^{(\alpha)} g(x).$$

The function has its derivative with respect to $(dx)^\alpha$, namely,

$$\frac{d^\alpha \prod(x)}{dx^\alpha} = g(x), \quad a < x < b. \tag{2.10}$$

***Proof.* [17]**

Let $x \in [a,b]$. There exists $x + \Delta x \in [a,b]$ such that

$$\prod(x) = {}_aI_{x+\Delta x}^{(\alpha)} g(x), \tag{2.11}$$

$$\Delta^{\alpha}\prod(x)=\Gamma(1+\alpha)\left[\prod(x+\Delta x)-\prod(x)\right]$$
$$=\Gamma(1+\alpha)\left[\int_{a}^{x+\Delta x} g(t)(dt)^{\alpha}-\int_{a}^{x} g(t)(dt)^{\alpha}\right]$$
$$=\Gamma(1+\alpha)\left[\int_{x}^{x+\Delta x} g(t)(dt)^{\alpha}\right]$$
$$=\Gamma(1+\alpha)\,{}_{x}I_{x+\Delta x}{}^{(\alpha)}g(x). \tag{2.12}$$

Applying the Theorem 2.9, there exists a point η, such that

$${}_{x}I_{x+\Delta x}{}^{(\alpha)}\,g(x)=g(\eta)\frac{(\Delta x)^{\alpha}}{\Gamma(1+\alpha)},$$

$$g(\eta)=\frac{\Gamma(1+\alpha)\,{}_{x}I_{x+\Delta x}{}^{(\alpha)}\,g(x)}{(\Delta x)^{\alpha}}.$$

From (2.12), we obtain

$$g(\eta)=\frac{\Delta^{\alpha}\prod(x)}{(\Delta x)^{\alpha}}. \tag{2.13}$$

Taking the limit $\frac{\Delta^{\alpha}\prod(x)}{(\Delta x)^{\alpha}}$ as $\Delta x\to 0$ it implies that

$$\lim_{\Delta x\to 0}\frac{\Delta^{\alpha}\prod(x)}{(\Delta x)^{\alpha}}=g(x). \tag{2.14}$$

Here, there exists $x=a$ and $\Delta x>0$ such that

$$\frac{d^{\alpha}\Pi(x)}{dx^{\alpha}}\Big|_{x=a^{+}}=g(a^{+}). \tag{2.15}$$

Thus, there exists $x=b$ and $\Delta x<0$ such that

$$\frac{d^{\alpha}\Pi(x)}{dx^{\alpha}}\Big|_{x=b^{-}}=g(b^{-}). \tag{2.16}$$

Combing (2.15) and (2.16), the proof is completed.

Theorem 2.11. [17]

If $h(x)\in C_\alpha[a,b]$, then there exists $\Pi \in C_\alpha[a,b]$ such that

$$\prod(x) = {}_aI_x^{(\alpha)}h(x). \tag{2.17}$$

***Proof.* [17]**

Taking *the Theorem 2.10* into account we deduce the desired result.

Theorem 2.12. (The Local Fractional Integration is Anti-Differentiation [49])

If $g(x)=h^{(\alpha)}(x)\in C_\alpha[m,n]$, then we have

$$ {}_mI_n^{(\alpha)}g(x)=h(n)-h(m). \tag{2.18}$$

***Proof* . [49]**

Let $\prod(x)= {}_mI_x^{(\alpha)}g(x)$ and $g(x)=h^{(\alpha)}(x)\in C_\alpha[m,n]$.

Using the Theorem 2.10 to $\Pi(x)-h(x)$,we get

$$\frac{d^\alpha\left(\Pi(x)-h(x)\right)}{dx^\alpha}=\frac{d^\alpha\left(\Pi(x)\right)}{dx^\alpha}-\frac{d^\alpha\left(h(x)\right)}{dx^\alpha}$$
$$=g(x)-g(x)=0.$$

Thus, we conclude

$$\Pi(x)-h(x)=c,$$

$$\left.\begin{array}{l}\Pi(n)=h(n)+c\\ \Pi(m)=h(m)+c\end{array}\right\}\Pi(n)-\Pi(m)=h(n)-h(m),$$

$$ {}_mI_n^{(\alpha)}g(x)=\Pi(n)-\Pi(m)=h(n)-h(m).$$

Thus, the proof is finished.

Theorem 2.13. [17]

Suppose that $h(x)\in C_1[a,b],(goh)(s)\in C_\alpha[h(a),h(b)]$, then we have

$$ {}_{h(a)}I_{h(b)}^{(\alpha)}g(x)={}_aI_b^{(\alpha)}(goh)(s)[h`(s)]^\alpha . \tag{2.19}$$

Proof. **[17]**

Let $G(x)={}_aI_x^{(\alpha)}g(x)$, then we arrive at the formula

$$ {}_{h(a)}I_{h(b)}^{(\alpha)}g(x)=G(h(b))-G(h(a)). \tag{2.20}$$

By using *the Theorem 2.11* in (2.20) , we have

$$\begin{aligned} G(h(b))-G(h(a)) &= {}_aI_b^{(\alpha)}\left[D^\alpha(Goh)\right](s) \\ &= {}_aI_b^{(\alpha)}G^{(\alpha)}(h(s))\left[h^{(1)}(s)\right]^\alpha = {}_aI_b^{(\alpha)}(goh)(s)\left[h^{(1)}(s)\right]^\alpha \end{aligned} \tag{2.21}$$

From (2.20) and (2.21), the proof of this theorem is provided.

Theorem 2.14. (Local Fractional Integration by Parts) [49]

Suppose that $g(x),m(x)\in D_\alpha(a,b)$ and $g^{(\alpha)}(x),m^{(\alpha)}(x)\in C_\alpha[a,b]$, then, we have [49]

$$ {}_aI_b^{(\alpha)}g(t)m^{(\alpha)}(t)=\left[g(t)m(t)\right]_a^b-{}_aI_b^{(\alpha)}g^{(\alpha)}(t)m(t). \tag{2.22}$$

Proof. **[49]**

We know that,

$$\frac{d^\alpha\left[g(t)m(t)\right]}{dt^\alpha}=g^{(\alpha)}(t)m(t)+g(t)m^{(\alpha)}(t), \tag{2.23}$$

therefore,

$$ {}_aI_b^{(\alpha)}\left\{\frac{d^\alpha\left[g(t)m(t)\right]}{dt^\alpha}\right\}=\left[g(t)m(t)\right]_a^b. \tag{2.24} $$

From (2.23) and (2.24), we obtain

$$ {}_aI_b^{(\alpha)}\left\{g^{(\alpha)}(t)m(t)+g(t)m^{(\alpha)}(t)\right\}=\left[g(t)m(t)\right]_a^b, $$

$$ {}_aI_b^{(\alpha)}\left\{g^{(\alpha)}(t)m(t)\right\}+{}_aI_b^{(\alpha)}\left\{g(t)m^{(\alpha)}(t)\right\}=\left[g(t)m(t)\right]_a^b. $$

Then, we conclude that

$$ {}_aI_b^{(\alpha)}\left\{g(t)m^{(\alpha)}(t)\right\}=\left[g(t)m(t)\right]_a^b-{}_aI_b^{(\alpha)}\left\{g^{(\alpha)}(t)m(t)\right\}. $$

So, we get the desired result.

Proposition 2.15. [43]

Suppose that for $0<\alpha\le 1,\ g^{(m\alpha)}(x)\in C_\alpha(a,b)$, thus

$$ \left({}_{x_0}I_x^{(m\alpha)}g(x)\right)^{(m\alpha)}=g(x), \tag{2.25} $$

where

$$ {}_{x_0}I_x^{(m\alpha)}g(x)=\underbrace{{}_{x_0}I_x^{(\alpha)}\dots{}_{x_0}I_x^{(\alpha)}}_{m\,times}g(x) \text{ and } g^{(m\alpha)}(x)=\underbrace{D_x^{(\alpha)}\dots D_x^{(\alpha)}}_{m\,times}g(x). $$

Proof. **[43]**

Taking *the Theorem 2.10* into account we deduce the result.

Proposition 2.16. [68]

Suppose that $h^{(n\alpha)}(x),h^{((n+1)\alpha)}(x)\in C_\alpha(a,b)$ for $0<\alpha\le 1$, then we have

$$ {}_{x_0}I_x^{(n\alpha)}\left[h^{(n\alpha)}(x)\right]-{}_{x_0}I_x^{((n+1)\alpha)}\left[h^{((n+1)\alpha)}(x)\right]=h^{(n\alpha)}(x_0)\frac{(x-x_0)^{n\alpha}}{\Gamma(n\alpha+1)}, $$

where (2.26)

$$ {}_{x_0}I_x^{((n+1)\alpha)}h(x) = \underbrace{{}_{x_0}I_x^{(\alpha)} \dots {}_{x_0}I_x^{(\alpha)}}_{n+1 times} h(x) $$ and

$$ h^{((n+1)\alpha)}(x) = \underbrace{D_x^{(\alpha)} \dots D_x^{(\alpha)}}_{n+1 times} h(x). $$

***Proof.* [68]**

Using *the Theorem 1.18*, we report that,

$$
\begin{aligned}
{}_{x_0}I_x^{((n+1)\alpha)}\left[h^{((n+1)\alpha)}(x)\right] &= {}_{x_0}I_x^{(n\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}\int_{x_0}^{x} h^{((n+1)\alpha)}(x)(dt)^{\alpha}\right] \\
&= {}_{x_0}I_x^{(n\alpha)}\left[h^{(n\alpha)}(x) - h^{(n\alpha)}(x_0)\right] \qquad (2.27) \\
&= {}_{x_0}I_x^{(n\alpha)}h^{(n\alpha)}(x) - {}_{x_0}I_x^{(n\alpha)}h^{(n\alpha)}(x_0).
\end{aligned}
$$

Considering the formula in (2.27), we have

$$
\begin{aligned}
{}_{x_0}I_x^{(n\alpha)}h^{(n\alpha)}(x_0) &= h^{(n\alpha)}(x_0)\, {}_{x_0}I_x^{(n\alpha)}1 \\
&= h^{(n\alpha)}(x_0)\, {}_{x_0}I_x^{((n-1)\alpha)}\left[\frac{1}{\Gamma(1+\alpha)}(x-x_0)^{\alpha}\right] \\
&= h^{(n\alpha)}(x_0)\, {}_{x_0}I_x^{((n-2)\alpha)}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}(x-x_0)^{2\alpha}\right] \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= h^{(n\alpha)}(x_0)\frac{(x-x_0)^{n\alpha}}{\Gamma(1+n\alpha)}.
\end{aligned}
$$

Applying this formula in (2.27), we obtain (2.26).

Theorem 2.17. [68] (Generalized Mean Value Theorem for Local Fractional Integrals)

Suppose that $g(x) \in C_{\alpha}[a,b], g^{(\alpha)}(x) \in C_{\alpha}(a,b)$, we have

$$g(x)-g(x_0)=g^{(\alpha)}(\eta)\frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)}\ \ a<x_0<\eta<x<b. \tag{2.28}$$

Proof. **[68]**

Taking n=1 in (2.26), the proof of this theorem is completed.

2.2. Local Fractional Taylor's Theorem

Theorem 2.18. [43]

Suppose that $g^{((t+1)\alpha)}(x)\in C_{\alpha}(a,b)$ for $t=0,1,...,n$ and $0<\alpha\le 1$, then we have,

$$g(x)=\sum_{t=0}^{n}\frac{g^{(t\alpha)}(x_0)}{\Gamma(1+t\alpha)}(x-x_0)^{t\alpha}+\frac{g^{((n+1)\alpha)}(\eta)}{\Gamma(1+(n+1)\alpha)}(x-x_0)^{(n+1)\alpha}, \tag{2.29}$$

with $a<x_0<\eta<x<b,\forall x\in(a,b)$, where $g^{((t+1)\alpha)}(x)=\underbrace{D_x^{(\alpha)}...D_x^{(\alpha)}}_{t+1 times}g(x)$.

Proof. **[43]**

From *the Proposition 2.16*, we arrive at this formula in (2.26)

$${}_aI_x^{(t\alpha)}\left[g^{(t\alpha)}(x)\right]-{}_aI_x^{((t+1)\alpha)}\left[g^{((t+1)\alpha)}(x)\right]=g^{(t\alpha)}(a)\frac{(x-a)^{t\alpha}}{\Gamma(t\alpha+1)}.$$

That is,

$$\underbrace{\sum_{t=0}^{n}{}_aI_x^{(t\alpha)}\left[g^{(t\alpha)}(x)\right]-{}_aI_x^{((t+1)\alpha)}\left[g^{((t+1)\alpha)}(x)\right]}=\sum_{t=0}^{n}g^{(t\alpha)}(a)\frac{(x-a)^{t\alpha}}{\Gamma(t\alpha+1)},$$

$$g(x)-{}_aI_x^{((n+1)\alpha)}\left[g^{((n+1)\alpha)}(x)\right] \tag{2.30}$$

Applying *the theorem 2.9* into (2.30).

$$
\begin{aligned}
{}_aI_x^{((n+1)\alpha)}\left[g^{((n+1)\alpha)}(x)\right] &= \frac{1}{\Gamma(1+\alpha)}\int_a^x {}_aI_x^{(n\alpha)} g^{((n+1)\alpha)}(x)(dt)^\alpha \\
&= \frac{{}_aI_x^{(n\alpha)}\left[g^{((n+1)\alpha)}(\eta)(x-a)^\alpha\right]}{\Gamma(1+\alpha)} \\
&= g^{((n+1)\alpha)}(\eta)\frac{{}_aI_x^{(n\alpha)}(x-a)^\alpha}{\Gamma(1+\alpha)} \\
&= \frac{g^{((n+1)\alpha)}(\eta)(x-a)^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)},
\end{aligned}
\tag{2.31}
$$

with $a<\eta<x, \forall x \in [a,b]$. Thus, we finish the proof.

Theorem 2.19. [49]

Suppose that $g^{((t+1)\alpha)}(x) \in C_\alpha(a,b)$ for $t=0,1,...,n$ and $0<\alpha\le 1$, then

$$
g(x)=\sum_{t=0}^{n}\frac{g^{(t\alpha)}(x_0)}{\Gamma(1+t\alpha)}(x-x_0)^{t\alpha}+R_{n\alpha}(x-x_0),
\tag{2.32}
$$

with $a<x_0<\eta<x<b, \forall x\in(a,b)$,

such that $g^{((t+1)\alpha)}(x)=\underbrace{D_x^{(\alpha)}...D_x^{(\alpha)}}_{t+1 times}g(x)$ and $R_{n\alpha}(x-x_0)=O\left((x-x_0)^{n\alpha}\right)$.

Proof. **[49]**

Applying *the Proposition 2.16*, we have

$$
\left|\frac{R_{n\alpha}(x-x_0)}{(x-x_0)^{n\alpha}}\right| = \left|\frac{g^{((n+1)\alpha)}(\eta)(x-x_0)^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)(x-x_0)^{n\alpha}}\right| = \left|\frac{g^{((n+1)\alpha)}(\eta)}{\Gamma(1+(n+1)\alpha)}(x-x_0)^{\alpha}\right|
$$

And that is ,

$$
\lim_{x\to x_0}\left|\frac{R_{n\alpha}(x-x_0)}{(x-x_0)^{n\alpha}}\right| = \lim_{x\to x_0}\left|\frac{g^{((n+1)\alpha)}(\eta)}{\Gamma(1+(n+1)\alpha)}(x-x_0)^{\alpha}\right| = 0 .
$$

Hence, we obtain the desired result.

Theorem 2.20. [43]

Suppose that $h^{((m+1)\alpha)}(x) \in C_{\alpha}(a,b)$ for $m = 0,1,\ldots,n$ and $0 < \alpha \leq 1$,

$$h(x) = \sum_{m=0}^{n} \frac{h^{(m\alpha)}(0)}{\Gamma(1+m\alpha)} x^{m\alpha} + \frac{h^{((n+1)\alpha)}(\theta x)}{\Gamma(1+(n+1)\alpha)} x^{(n+1)\alpha} \tag{2.33}$$

with $0 < \theta < 1, \forall x \in (a,b)$ where $h^{((m+1)\alpha)}(x) = \underbrace{D_x^{(\alpha)} \ldots D_x^{(\alpha)}}_{m+1 times} h(x)$.

***Proof.* [43]**

Applying *the Theorem 2.18* , for $x_0 = 0$ and $a < x_0 < \eta < x < b$, we obtain

$$h(x) = \sum_{m=0}^{n} \frac{h^{(m\alpha)}(0)}{\Gamma(1+k\alpha)} x^{m\alpha} + \frac{h^{((n+1)\alpha)}(\eta)}{\Gamma(1+(n+1)\alpha)} x^{(n+1)\alpha}. \tag{2.34}$$

If $\eta = \theta x$ in (2.34), then we have

$$\frac{h^{((n+1)\alpha)}(\eta)}{\Gamma(1+(n+1)\alpha)} x^{(n+1)\alpha} = \frac{h^{((n+1)\alpha)}(\theta x)}{\Gamma(1+(n+1)\alpha)} x^{(n+1)\alpha},$$

with $0 < \theta < 1$.

Thus, we obtain (2.33).

Theorem 2.21. [31] (Taylor's Series)

Suppose that $g^{((m+1)\alpha)}(x) \in C_{\alpha}(a,b)$ for $m = 0,1,\ldots,n$ and $0 < \alpha \leq 1$,

$$g(x) = \sum_{m=0}^{\infty} \frac{g^{(m\alpha)}(x_0)}{\Gamma(1+k\alpha)} (x - x_0)^{m\alpha}, \tag{2.35}$$

with $a < x_0 < x < b, \forall x \in (a,b)$ where $g^{((m+1)\alpha)}(x) = \underbrace{D_x^{(\alpha)} \ldots D_x^{(\alpha)}}_{m+1 times} g(x)$.

***Proof.* [31]**

According to the local fractional Taylor theorem ,we have

$$f(x)=\lim_{n\to\infty}\sum_{m=0}^{n}\frac{g^{(m\alpha)}(x_0)}{\Gamma(1+m\alpha)}(x-x_0)^{m\alpha}+\frac{g^{((n+1)\alpha)}(\mu)}{\Gamma(1+(n+1)\alpha)}(x-x_0)^{(n+1)\alpha}$$
$$=\lim_{n\to\infty}\sum_{m=0}^{n}\frac{g^{(m\alpha)}(x_0)}{\Gamma(1+m\alpha)}(x-x_0)^{m\alpha}$$
$$=\sum_{m=0}^{\infty}\frac{g^{(m\alpha)}(x_0)}{\Gamma(1+m\alpha)}(x-x_0)^{m\alpha}.$$

Thus, we get (2.35).

Theorem 2.22. [31] (Mc-Laurin's Series)

Suppose that $g^{((m+1)\alpha)}(x)\in C_\alpha(a,b)$ for $m=0,1,\dots,n,\dots$ and $0<\alpha\le 1$ then we have

$$g(x)=\sum_{m=0}^{\infty}\frac{g^{(m\alpha)}(0)}{\Gamma(1+m\alpha)}x^{m\alpha}, \tag{2.36}$$

with $a<x_0<x<b, \forall x\in(a,b)$ where $g^{((m+1)\alpha)}(x)=\underbrace{D_x^{(\alpha)}\dots D_x^{(\alpha)}}_{m+1 times}g(x)$.

Proof. **[31]**

In the Theorem 2.21, we take $x_0=0$, so the proof is completed.

An example for Mc-Laurin's series is the Mittag-Leffler function [51], namely

$$E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+k\alpha)}. \tag{2.37}$$

2.3. Local Fractional Indefinite Integral

2.3.1. Local Fractional Anti-Differentiation

Let $g(x)$ and $h(x)$ are two local fractional continuous functions defined on (a,b). If $h^{(\alpha)}(x)=g(x)$ for each x in (a,b), then $h(x)$ is called the local fractional anti-derivative of $g(x)$ on (a,b).

Theorem 2.23. [17]

If $h_1(x)$ and $h_2(x)$ are any local fractional anti-derivative of $g(x)$ on (a,b),then there is a constant C, $h_1(x) = h_2(x) + C$.

Proof. **[17]**

We take $t(x) = h_1(x) - h_2(x)$, then we have

$t^{(\alpha)}(x) = h_1^{(\alpha)}(x) - h_2^{(\alpha)}(x) = g(x) - g(x) = 0$,

for $\forall x \in (a,b)$.

Thus, there is a constant C for all x in (a,b), such that

$C = t(x) = h_1(x) - h_2(x)$,

Hence, we conclude that $h_1(x) = h_2(x) + C$.

2.3.2. Local Fractional Indefinite Integral

If $h(x)$ is an local fractional anti-derivative of $g(x)$on (a,b),then the set

$\{h(x) + C : C \, is \, constant\}$

is called a one-parameter family of local fractional anti-derivative of $g(x)$. We call this one-parameter family of local fractional anti-derivatives the local fractional indefinite integral of $g(x)$ on (a,b) and write it [17],

$$\frac{1}{\Gamma(1+\alpha)}\int g(x)(dx) = h(x) + C. \tag{2.38}$$

2.3.3. Local Fractional Indefinite Integral of Elementary Functions

For a constant C ,these formulas are valid [17];

$$\frac{1}{\Gamma(1+\alpha)}\int E_{\alpha}(x^{\alpha})(dx)^{\alpha}=E_{\alpha}(x^{\alpha})+C, \tag{1}$$

$$\frac{1}{\Gamma(1+\alpha)}\int x^{k\alpha}(dx)^{\alpha}=\frac{\Gamma(1+k\alpha)x^{(k+1)\alpha}}{\Gamma(1+(k+1)\alpha)}+C, \tag{2}$$

$$\frac{1}{\Gamma(1+\alpha)}\int sin_{\alpha}x^{\alpha}(dx)^{\alpha}=-cos_{\alpha}x^{\alpha}+C, \tag{3}$$

$$\frac{1}{\Gamma(1+\alpha)}\int cos_{\alpha}x^{\alpha}(dx)^{\alpha}=sin_{\alpha}x^{\alpha}+C. \tag{4}$$

CHAPTER 3

3.1. The Local Fractional Differential Equations

Suppose that $m(x)$ and $n(x)$ are defined on some interval (a,b), then the form of the following equation [17],

$$\frac{d^{\alpha}g(x)}{dx^{\alpha}}+m(x)g(x)=n(x), 0<\alpha\le 1 \tag{3.1}$$

is called α local fractional differential equation of $g(x)$.

Theorem 3.1. [17]

A model for Mittag-Leffler growth is the local fractional ordinary differential equation,

$$\frac{d^{\alpha}y}{dx^{\alpha}}+ty=0, t>0, y(0)=y_0\,. \tag{3.2}$$

The solution of this local fractional differential equation is given,

$$y(x)=y_0E_{\alpha}(-tx^{\alpha})\,. \tag{3.3}$$

Proof. **[17]**

In (3.2) we take the integration on both sides with respect to x,

$$\frac{d^{\alpha}y}{dx^{\alpha}}=-ty,$$

$$\frac{d^{\alpha}y}{y}=-tdx^{\alpha},$$

$$\frac{1}{\Gamma(1+\alpha)}\int\frac{d^{\alpha}y}{y}=\frac{1}{\Gamma(1+\alpha)}\int -t(dx)^{\alpha},$$

$$ln_{\alpha}y=-tx^{\alpha}+c,$$

$$y(x)=E_{\alpha}(-tx^{\alpha}+c),$$

$$y(x)=C.E_{\alpha}(-tx^{\alpha}).$$

Let $y_0=y(0)=C$, then we arrive at (3.3).

Similarly, a model for Mittag-Leffler growth is the local fractional differential equation

$$\frac{d^{\alpha}y}{dx^{\alpha}}=ty,t>0,y(0)=y_0. \tag{3.4}$$

Similarly, the solution of this local fractional differential equation is given by

$$y(x)=y_0E_{\alpha}(tx^{\alpha}).$$

Theorem 3.2. [17]

Suppose that $t>0$ and $m(x)$ is local fractional continuous on (a,b), then the local fractional equation

$$\frac{d^{\alpha}y}{dx^{\alpha}}+ty=m(x), \tag{3.5}$$

has the one-parameter of solutions, namely

$$y(x)=E_{\alpha}(-tx^{\alpha})\left[\frac{1}{\Gamma(1+\alpha)}\int m(x)E_{\alpha}(tx^{\alpha})(dx)^{\alpha}+c\right]. \tag{3.6}$$

***Proof.* [17]**

We multiply the given local fractional differential equation in (3.5) by $E_{\alpha}(tx^{\alpha})$ which is called the integration factor.

$$\underbrace{E_\alpha(tx^\alpha)\frac{d^\alpha y}{dx^\alpha}+tE_\alpha(tx^\alpha)y}_{\frac{d^\alpha}{dx^\alpha}\left[E_\alpha(tx^\alpha)y\right]}=m(x)E_\alpha(tx^\alpha). \tag{3.7}$$

By using the notation of the indefinite integral, we arrive at

$$E_\alpha(tx^\alpha)y=\frac{1}{\Gamma(1+\alpha)}\int m(x)E_\alpha(tx^\alpha)(dx)^\alpha+c. \tag{3.8}$$

From (3.8), we obtain

$$y(x)=E_\alpha(-tx^\alpha)\left[\frac{1}{\Gamma(1+\alpha)}\int m(x)E_\alpha(tx^\alpha)(dx)^\alpha+c\right].$$

The proof of theorem is completed.

If $m(x)$ and $n(x)$ are defined on (a,b), then the equation

$$\frac{d^{2\alpha}g(x)}{dx^{2\alpha}}+m(x)\frac{d^\alpha g(x)}{dx^\alpha}+n(x)g(x)=r(x), 0<\alpha\leq 1, \tag{3.9}$$

is called 2α local fractional differential equation in the variable $g(x)$, [17].

Theorem 3.3. [17]

Suppose that m and t are constant coefficients, then the local fractional equation,

$$\frac{d^{2\alpha}g(x)}{dx^{2\alpha}}+m\frac{d^\alpha g(x)}{dx^\alpha}+tg(x)=0 \tag{3.10}$$

has two-parameter family of solutions [17]

$$g(x)=KE_\alpha\left(\frac{m+\sqrt{m^2-4t}}{2}x^\alpha\right)+LE_\alpha\left(\frac{-m+\sqrt{m^2-4t}}{2}x^\alpha\right), m^2-4t\geq 0,$$

with two constants K and L. (3.11)

Proof. See [17] .

Theorem 3.4. [17]

The local fractional equation which has two constant coefficients [17]

$$\frac{d^{2\alpha}g(x)}{dx^{2\alpha}}+m\frac{d^{\alpha}g(x)}{dx^{\alpha}}+tg(x)=0, \tag{3.12}$$

In (3.12), m and t are coefficients.

$$g(x)=KE_{\alpha}\left(\frac{m+i^{\alpha}\sqrt{m^2-4t}}{2}x^{\alpha}\right)+LE_{\alpha}\left(\frac{-m+i^{\alpha}\sqrt{m^2-4t}}{2}x^{\alpha}\right), \tag{3.13}$$

$$m^2-4t<0,$$

with two constants K and L.

Proof. **[17]**

Suppose that $E_{\alpha}(kx^{\alpha})$ is a solution of 2α local fractional ordinary differential equation, namely

$$k^2+mk+t=0. \tag{3.14}$$

From (3.14) and $m^2-4t<0$, we have

$k_1=\dfrac{-m-i^{\alpha}\sqrt{m^2-4t}}{2}$ and $k_2=\dfrac{-m+i^{\alpha}\sqrt{m^2-4t}}{2}$, respectively.

Due to the fact that for any constant C, $CE_{\alpha}(kx^{\alpha})$ is a solution of the 2α local fractional ordinary differential equation, we show that,

$g(x)=KE_{\alpha}(k_1x^{\alpha})+LE_{\alpha}(k_2x^{\alpha})$ then

$$g(x)=KE_{\alpha}\left(\frac{m-i^{\alpha}\sqrt{m^2-4t}}{2}x^{\alpha}\right)+LE_{\alpha}\left(\frac{-m+i^{\alpha}\sqrt{m^2-4t}}{2}x^{\alpha}\right),$$

with two constants K and L.

3.2. The Total Local Fractional Differentials

3.2.1. Local Fractional Partial Derivative

Let a non-differentiable function $g(x,y)$ be defined in the domain D of the $xy-plane$. If y is fixed and x as variable are thought of, local fractional derivative of $g(x,y)$ with respect to x is called *the local fractional derivative with respect to x*, which is denoted as [17]

$$\frac{\partial^{\alpha} g(x,y)}{\partial x^{\alpha}}\Big|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}\left[g(x,y)-g(x_0,y)\right]}{(x-x_0)^{\alpha}}, \tag{3.15}$$

where $\Delta^{\alpha}\left[g(x,y)-g(x_0,y)\right] \cong \Gamma(1+\alpha)\Delta\left[g(x,y)-g(x_0,y)\right]$.

Similarly, the local fractional partial derivative of $g(x,y)$ with respect to y is called *the local fractional derivative with respect to y*, which is denoted by as [17],

$$\frac{\partial^{\alpha} g(x,y)}{\partial y^{\alpha}}\Big|_{y=y_0} = \lim_{y \to y_0} \frac{\Delta^{\alpha}\left[g(x,y)-g(x,y_0)\right]}{(y-y_0)^{\alpha}}, \tag{3.16}$$

where $\Delta^{\alpha}\left[g(x,y)-g(x,y_0)\right] \cong \Gamma(1+\alpha)\Delta\left[g(x,y)-g(x,y_0)\right]$.

3.2.2. Local Fractional Partial Derivative of Higher-Order [68]

Let $h(x,y)$ has partial derivatives at each points (x,y) in the domain D of the $xy-plane$, then

$$\frac{\partial^{\alpha} h(x,y)}{\partial x^{\alpha}} \text{ and } \frac{\partial^{\alpha} h(x,y)}{\partial y^{\alpha}},$$

are themselves functions of x and y, which may also have local fractional partial derivatives.

The 2α local fractional derivatives are denoted as given below,

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\alpha} h(x,y)}{\partial x^{\alpha}}=\frac{\partial^{2\alpha} h(x,y)}{\partial x^{\alpha}\partial x^{\alpha}}=h_{x^2}^{2\alpha}(x,y),$$

$$\frac{\partial^{\alpha}}{\partial y^{\alpha}}\frac{\partial^{\alpha} h(x,y)}{\partial y^{\alpha}}=\frac{\partial^{2\alpha} h(x,y)}{\partial y^{\alpha}\partial y^{\alpha}}=h_{y^2}^{2\alpha}(x,y),$$

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\alpha} h(x,y)}{\partial y^{\alpha}}=\frac{\partial^{2\alpha} g(x,y)}{\partial x^{\alpha}\partial y^{\alpha}}=h_{yx}^{2\alpha}(x,y),$$

$$\frac{\partial^{\alpha}}{\partial y^{\alpha}}\frac{\partial^{\alpha} h(x,y)}{\partial x^{\alpha}}=\frac{\partial^{2\alpha} h(x,y)}{\partial y^{\alpha}\partial x^{\alpha}}=h_{xy}^{2\alpha}(x,y).$$

Similarly, we have

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\alpha} h(x,y)}{\partial x^{\alpha}}=\frac{\partial^{3\alpha} h(x,y)}{\partial x^{\alpha}\partial x^{\alpha}\partial x^{\alpha}}=h_{x^3}^{3\alpha}(x,y),$$

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\alpha}}{\partial y^{\alpha}}\frac{\partial^{\alpha} h(x,y)}{\partial x^{\alpha}}=\frac{\partial^{3\alpha} h(x,y)}{\partial x^{\alpha}\partial y^{\alpha}\partial x^{\alpha}}=h_{xyx}^{3\alpha}(x,y).$$

If k is positive integer, then

$$\underbrace{\frac{\partial^{\alpha}}{\partial x^{\alpha}}\cdots\frac{\partial^{\alpha}}{\partial x^{\alpha}}}_{k\,times}h(x,y)=\frac{\partial^{k\alpha} h(x,y)}{\underbrace{\partial x^{\alpha}\ldots\partial x^{\alpha}}_{k\,times}}=h_{x^k}^{k\alpha}(x,y).$$

If k and t are positive integers, then

$$\underbrace{\frac{\partial^{\alpha}}{\partial x^{\alpha}}\cdots\frac{\partial^{\alpha}}{\partial x^{\alpha}}}_{k\,times}\underbrace{\frac{\partial^{\alpha}}{\partial y^{\alpha}}\cdots\frac{\partial^{\alpha}}{\partial y^{\alpha}}}_{t\,times}h(x,y)=\frac{\partial^{(k+t)\alpha} h(x,y)}{\underbrace{\partial x^{\alpha}\ldots\partial x^{\alpha}}_{k\,times}\underbrace{\partial y^{\alpha}\ldots\partial y^{\alpha}}_{t\,times}}=h_{y^t x^k}^{(k+t)\alpha}(x,y)$$

Theorem 3.5. [68]

If $g_{yx}^{2\alpha}(x,y)$ and $g_{xy}^{2\alpha}(x,y)$ are local fractional continuous on the domain D of the $xy-plane$, then we have

$$g_{yx}^{2\alpha}(x,y)=g_{xy}^{2\alpha}(x,y). \tag{3.17}$$

3.2.3. The Total Local Fractional Differentials

Let non-differentiable function $g=g(x,y)$ have the total increment

$$\Delta^{\alpha}g=\Gamma(1+\alpha)\left[g(x+\Delta x,y+\Delta y)-g(x,y)\right],$$

which is expressed as [17]

$$\Delta^{\alpha}g=K(\Delta x)^{\alpha}+L(\Delta y)^{\alpha}+O(\rho_{\alpha}),$$

where K and L are independent on $(\Delta x)^{\alpha}$ and $(\Delta y)^{\alpha}$, which are dependent on x and y and

$$\rho_{\alpha}=\sqrt{(\Delta x)^{2\alpha}+(\Delta y)^{2\alpha}}.$$

Then, $g(x,y)$ is the α local fractional differential at a point (x,y) and $K(\Delta x)^{\alpha}+L(\Delta y)^{\alpha}$ is the total local fractional differential at a point (x,y), denoted by $\Delta^{\alpha}g=K(\Delta x)^{\alpha}+L(\Delta y)^{\alpha}$.

Suppose that $g=g(x,y)$ have the α local fractional differential at a point $(x,y)\in D$, then $g=g(x,y)$ is the α local fractional differential in the region D. If $g=g(x,y)$ is the α local fractional differential at a point $(x,y)\in D$, then we have

$$\lim_{\rho_{\alpha}\to 0}\Delta^{\alpha}g=0. \tag{3.18}$$

Theorem 3.6. [17]

Suppose that $g = g(x,y)$ is the α local fractional differential at a point (x,y), then the partial derivatives are

$$\frac{\partial^{\alpha} g(x,y)}{\partial x^{\alpha}}, \frac{\partial^{\alpha} g(x,y)}{\partial y^{\alpha}},$$

exist and there is the total local fractional differential at the point (x,y), denoted by

$$d^{\alpha} g = \frac{\partial^{\alpha} g(x,y)}{\partial x^{\alpha}}(dx)^{\alpha} + \frac{\partial^{\alpha} g(x,y)}{\partial y^{\alpha}}(dy)^{\alpha}. \tag{3.19}$$

Proof. **[17]**

Assume that function $g = g(x,y)$ is the α local fractional differential at a point (x,y). Any points of interval $(x+\Delta x, y+\Delta y)$, the neighborhood of (x,y), is always satisfied below

$$\Delta^{\alpha} g = K(\Delta x)^{\alpha} + L(\Delta y)^{\alpha} + O(\rho_{\alpha}),$$

Suppose that, $\Delta y = 0$,

$$\Delta^{\alpha} g = K(\Delta x)^{\alpha} + L(\Delta y)^{\alpha} + O(\rho_{\alpha}),$$

exist and $\rho_{\alpha} = \sqrt{(\Delta x)^{2\alpha} + (\Delta y)^{2\alpha}} = \left|(\Delta x)^{\alpha}\right|$,

thus, we obtain the relation

$$\Delta^{\alpha} g = K(\Delta x)^{\alpha} + O(\left|(\Delta x)^{\alpha}\right|),$$

then

$$K = \lim_{(\Delta x)^{\alpha} \to 0} \frac{\Delta^{\alpha} g}{(\Delta x)^{\alpha}} = \frac{\partial^{\alpha} g(x,y)}{\partial x^{\alpha}}.$$

Similarly, if we take $\Delta x = 0$, then we obtain $L = \dfrac{\partial^{\alpha} g(x,y)}{\partial y^{\alpha}}$.

Hence, the proof of the Theorem is completed.

Theorem 3.7. [17]

Suppose that $h = h(x,y)$ is the local fractional partial derivatives

$$\frac{\partial^{\alpha} h(x,y)}{\partial x^{\alpha}}, \frac{\partial^{\alpha} h(x,y)}{\partial y^{\alpha}},$$

and both $\frac{\partial^{\alpha} h(x,y)}{\partial x^{\alpha}}$ and $\frac{\partial^{\alpha} h(x,y)}{\partial y^{\alpha}}$ are local fractional continuous at the point (x,y), then $d^{\alpha}h = \frac{\partial^{\alpha} h(x,y)}{\partial x^{\alpha}}(dx)^{\alpha} + \frac{\partial^{\alpha} h(x,y)}{\partial y^{\alpha}}(dy)^{\alpha}$.

Proof. **[17]**

Assume that $\frac{\partial^{\alpha} h(x,y)}{\partial x^{\alpha}}$ and $\frac{\partial^{\alpha} h(x,y)}{\partial y^{\alpha}}$ are local fractional continuous at the point (x,y).

At any point of the neighborhood, there is the total increment, denoted by

$$\begin{aligned}\Delta^{\alpha} h &= \Gamma(1+\alpha)\left[h(x+\Delta x, y+\Delta y) - h(x,y) \right] \\ &= \Gamma(1+\alpha)\left\{\left[h(x+\Delta x, y+\Delta y) - h(x,y+\Delta y)\right] + \left[h(x,y+\Delta y) - h(x,y)\right]\right\}.\end{aligned}$$

By using the mean value theorem we have the following identity,

$$h(x+\Delta x, y+\Delta y) - h(x,y+\Delta y) = \frac{h^{(\alpha)}(x,y+\Delta y)}{\Gamma(1+\alpha)}(\Delta x)^{\alpha},$$

which yields

$$h(x+\Delta x, y+\Delta y) - h(x,y+\Delta y) = \frac{h_x^{(\alpha)}(x+\Delta x, y+\Delta y)}{\Gamma(1+\alpha)}(\Delta x)^{\alpha}.$$

Suppose that $h_x^{(\alpha)}(x,y)$ is local fractional continuous at a point (x,y), we have

$$h(x+\Delta x, y+\Delta y) - h(x,y+\Delta y),$$

which is translated into

$$\Gamma(1+\alpha)\left[h(x+\Delta x,y+\Delta y)-h(x,y+\Delta y)\right]=h_x^{(\alpha)}(x+\Delta x,y+\Delta y)(\Delta x)^{\alpha}$$
$$=\left[h_x^{(\alpha)}(x,y)+\varepsilon_1\right](\Delta x)^{\alpha}.$$

Here, ε_1 is dependent on $(\Delta x)^{\alpha}$ and $(\Delta y)^{\alpha}$ and $\varepsilon_1 \to 0$ as $(\Delta x)^{\alpha} \to 0$ and $(\Delta y)^{\alpha} \to 0$.

Similarly, we have the following relation,

$$\Gamma(1+\alpha)\left[h(x,y+\Delta y)-h(x,y)\right]=\left[h_y^{(\alpha)}(x,y)+\varepsilon_2\right](\Delta y)^{\alpha},$$

where ε_2 is dependent on $(\Delta y)^{\alpha}$ and $\varepsilon_2 \to 0$ as $(\Delta y)^{\alpha} \to 0$.

Hence, the total increment of $h(x,y)$ is expressed as

$$\begin{aligned}\Delta^{\alpha}h &= h_x^{(\alpha)}(x,y)(\Delta x)^{\alpha}+h_y^{(\alpha)}(x,y)(\Delta y)^{\alpha}\\ &\quad+\varepsilon_1(\Delta x)^{\alpha}+\varepsilon_2(\Delta y)^{\alpha}\end{aligned} \tag{3.20}$$

From (3.20), we obtain

$$\left|\frac{\varepsilon_1(\Delta x)^{\alpha}+\varepsilon_2(\Delta y)^{\alpha}}{\rho_{\alpha}}\right| \le \varepsilon_1+\varepsilon_2,$$

Therefore, $\varepsilon_1(\Delta x)^{\alpha}+\varepsilon_2(\Delta y)^{\alpha} \to 0$ as $\rho_{\alpha} \to 0$ and

$$\Delta^{\alpha}h = h_x^{(\alpha)}(x,y)(\Delta x)^{\alpha}+h_y^{(\alpha)}(x,y)(\Delta y)^{\alpha}.$$

Considering $(\Delta x)^{\alpha}=(dx)^{\alpha}, (\Delta y)^{\alpha}=(dy)^{\alpha}$ and $\Delta^{\alpha}y=d^{\alpha}y$ we have

$$d^{\alpha}h = h_x^{(\alpha)}(x,y)(dx)^{\alpha}+h_y^{(\alpha)}(x,y)(dy)^{\alpha}. \tag{3.21}$$

Considering (1.86), $h=h(x,y)$ has the α local fractional differential at a point (x,y).

Theorem 3.8. [17]

Suppose that $h = h(x,y,z)$ has the α local fractional differential at a point (x,y,z), then the local fractional partial derivatives [17],

$$\frac{\partial^\alpha h(x,y,z)}{\partial x^\alpha}, \frac{\partial^\alpha h(x,y,z)}{\partial y^\alpha}, \frac{\partial^\alpha h(x,y,z)}{\partial z^\alpha}$$

exist and there is the total local fractional differential at the point (x,y,z), denoted by

$$d^\alpha h = \frac{\partial^\alpha h(x,y,z)}{\partial x^\alpha}(dx)^\alpha + \frac{\partial^\alpha h(x,y,z)}{\partial y^\alpha}(dy)^\alpha + \frac{\partial^\alpha h(x,y,z)}{\partial z^\alpha}(dz)^\alpha . \quad (3.22)$$

Proof. See [17].

Theorem 3.9. [17]

If $h = h(x,y,z)$ has the local fractional partial derivatives

$$\frac{\partial^\alpha h(x,y,z)}{\partial x^\alpha}, \frac{\partial^\alpha h(x,y,z)}{\partial y^\alpha}, \frac{\partial^\alpha h(x,y,z)}{\partial z^\alpha}$$

and if $\frac{\partial^\alpha h(x,y,z)}{\partial x^\alpha}, \frac{\partial^\alpha h(x,y,z)}{\partial y^\alpha}$ and $\frac{\partial^\alpha h(x,y,z)}{\partial z^\alpha}$ are local fractional continuous at the point (x,y), then [17]

$$d^\alpha h = \frac{\partial^\alpha h(x,y,z)}{\partial x^\alpha}(dx)^\alpha + \frac{\partial^\alpha h(x,y,z)}{\partial y^\alpha}(dy)^\alpha + \frac{\partial^\alpha h(x,y,z)}{\partial z^\alpha}(dz)^\alpha .$$

Proof. **[17]**

Taking the notation of local fractional differential into account, we conclude the result.

3.2.4. Local Fractional Derivative of Composite Function

Theorem 3.10. [17]

Suppose that $g = g(x,y)$ and its local fractional partial derivatives $g_x^{(\alpha)}(x,y)$ and $g_y^{(\alpha)}(x,y)$ are local fractional continuous, and $x = x(t)$ and $y = y(t)$ are themselves differentiable functions of t.

Let

$G(t) = g(x(t), y(t))$, then $\frac{d^\alpha G}{dt^\alpha}$ is local fractional differentiable and

$$\frac{d^\alpha G}{dt^\alpha} = \frac{d^\alpha g}{dx^\alpha}\left(\frac{dx}{dt}\right)^\alpha + \frac{d^\alpha g}{dy^\alpha}\left(\frac{dy}{dt}\right)^\alpha . \tag{3.23}$$

Proof. **[17]**

Assume that $g = g(x,y)$ and its local fractional partial derivatives $g_x^{(\alpha)}(x,y)$ and $g_y^{(\alpha)}(x,y)$ are local fractional continuous. We obtain the following formula

$$\Delta^\alpha g = g_x^{(\alpha)}(x,y)(\Delta x)^\alpha + g_y^{(\alpha)}(x,y)(\Delta y)^\alpha + \varepsilon_1(\Delta x)^\alpha + \varepsilon_2(\Delta y)^\alpha ,$$

when $\Delta x \to 0$ and $\Delta y \to 0$ as $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

When $\Delta x \to 0$ and $\Delta y \to 0$, we have

$$\left(\frac{\Delta x}{\Delta t}\right)^\alpha \to \left(\frac{dx}{dt}\right)^\alpha \text{ and } \left(\frac{\Delta y}{\Delta t}\right)^\alpha \to \left(\frac{dy}{dt}\right)^\alpha \text{ as } \Delta t \to 0,$$

$$\frac{d^\alpha G}{dt^\alpha} = \lim_{\Delta t \to 0} \frac{\Delta^\alpha G}{(\Delta t)^\alpha} = g_x^{(\alpha)}(x,y)(\frac{\Delta x}{dt})^\alpha + g_y^{(\alpha)}(x,y)(\frac{\Delta y}{dt})^\alpha$$

$$= \frac{d^\alpha g}{dx^\alpha}\left(\frac{dx}{dt}\right)^\alpha + \frac{d^\alpha g}{dy^\alpha}\left(\frac{dy}{dt}\right)^\alpha .$$

This finish the proof of the theorem.

3.3. Local Fractional Fourier Series

3.3.1. Fractional Trigonometric Forms of Local Fractional Fourier Series

Suppose that $f(x)$ is a periodic function which 2π is the period of $f(x)$. For $m \in Z$, the local fractional Fourier series of $f(x)$ is explained by the following expression as [17],

$$f(x)=\frac{a_0}{2}+\sum_{m=1}^{\infty}\left(a_m \cos_\alpha(mx)^\alpha + b_m \sin_\alpha(mx)^\alpha\right), \tag{3.24}$$

where the Fourier coefficients have the forms;

$$\begin{aligned} a_m &= \frac{1}{\pi^\alpha}\int_{-\pi}^{\pi} f(x)\cos_\alpha(mx)^\alpha (dx)^\alpha, \\ b_m &= \frac{1}{\pi^\alpha}\int_{-\pi}^{\pi} f(x)\sin_\alpha(mx)^\alpha (dx)^\alpha. \end{aligned} \tag{3.25}$$

3.3.2. Generalized Fractional Trigonometric Forms of Local Fractional Fourier Series

Suppose that $f(x)$ be a periodic function which $2t$ is the period of $f(x)$. For $m \in Z$, the local fractional Fourier series of $f(x)$ is explained by the following expression as [17]

$$f(x)=\frac{a_0}{2}+\sum_{m=1}^{\infty}\left(a_m \cos_\alpha \frac{\pi^\alpha (mx)^\alpha}{t^\alpha} + b_m \sin_\alpha \frac{\pi^\alpha (mx)^\alpha}{t^\alpha}\right), \tag{3.26}$$

where the Fourier coefficients are

$$\begin{aligned} a_m &= \frac{1}{t^\alpha}\int_{-t}^{t} f(x)\cos_\alpha \frac{\pi^\alpha (mx)^\alpha}{t^\alpha}(dx)^\alpha, \\ b_m &= \frac{1}{t^\alpha}\int_{-t}^{t} f(x)\sin_\alpha \frac{\pi^\alpha (mx)^\alpha}{t^\alpha}(dx)^\alpha. \end{aligned} \tag{3.27}$$

We consider (3.27), the weights of the fractional trigonometric functions are given below [44]

$$a_m = \frac{1/\Gamma(1+\alpha)\int_{-t}^{t} f(x) cos_\alpha m^\alpha (\frac{\pi x}{t})^\alpha (dx)^\alpha}{1/\Gamma(1+\alpha)\int_{-t}^{t} cos_\alpha^2 m^\alpha (\frac{\pi x}{t})^\alpha (dx)^\alpha},$$

$$b_m = \frac{1/\Gamma(1+\alpha)\int_{-t}^{t} f(x) sin_\alpha m^\alpha (\frac{\pi x}{t})^\alpha (dx)^\alpha}{1/\Gamma(1+\alpha)\int_{-t}^{t} sin_\alpha^2 m^\alpha (\frac{\pi x}{t})^\alpha (dx)^\alpha}. \tag{3.28}$$

3.4. The Local Fractional Laplace Transform

3.4.1. Definition of the Laplace Transform

The Laplace transform of $f(x)$ is given as [17,20]

$$L_\alpha \{ f(x) \} = f_s^{L,\alpha}(s) := \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x) (dx)^\alpha, \ 0 < \alpha \le 1. \tag{3.29}$$

3.4.2. Inverse of the Laplace Transforms [20]

We can define the inverse Laplace transform of $f(x)$ given in (3.29) as [20]

$$L_\alpha^{-1} \{ f_s^{L,\alpha}(s) \} =: \frac{1}{(2\pi)^\alpha} \int_{\beta - i\alpha}^{\beta + i\alpha} E_\alpha(s^\alpha x^\alpha) f_s^{L,\alpha}(s) (ds)^\alpha,$$

where $s^\alpha = \beta^\alpha + i^\alpha \infty^\alpha$ and $Re(s^\alpha) = \beta^\alpha > 0^\alpha$.

CHAPTER 4

4.1. Fractal Heat Conduction [52]

Our aim here is to show how the variational iteration method [66] can be applied to local fractional heat conduction equation.

The non-linear equation, which is

$$\frac{\partial^{2\alpha} N(x,t)}{\partial x^{2\alpha}} - \frac{\partial^{\alpha} N(x,t)}{\partial t^{\alpha}} = 0 \ , x \in [0,1] , \tag{4.1}$$

with a fractal boundary condition

$$\frac{\partial^{\alpha} N(0,t)}{\partial x^{\alpha}} = E_{\alpha}(t^{\alpha}) \ , N(0,t) = 0 ,$$

reads as a sum of linear K^{α} and non-linear M^{α} local fractional operators

$$K^{\alpha} N + M^{\alpha} N = 0 ,$$

which permits the following correction functional to be constructed. In [66], the given correction functional is

$$N_{n+1}(t) = N_n(t) + {}_{t_0}I_t^{(\alpha)} \left\{ \xi^{\alpha} \left[K^{\alpha} N_n(s) + M^{\alpha} \tilde{N}_n(s) \right] \right\} . \tag{4.2}$$

In (4.2), $\tilde{N}_n$ is a restricted local fractional variation and ξ^{α} is a fractal Lagrange multiplier. The determination of ξ^{α} needs the stationary conditions of the functional, which is $\delta^{\alpha} \tilde{N}_n = 0$.

In (4.2), the given equation becomes

$$N_{n+1}(x) = N_n(x) + {}_0 I_x^{(\alpha)} \left\{ \xi^{\alpha} \left[\frac{\partial^{2\alpha} N_n}{\partial x^{2\alpha}} - \frac{\partial^{\alpha} N_n}{\partial \tau^{\alpha}} \right] \right\} \tag{4.3}$$

and the stationary condition allow:

$$\delta^{\alpha} N_{n+1}(x) = \left[1-(\xi^{\alpha})^{(\alpha)}\right]\Big|_{\tau=x} \delta^{\alpha} N_n + \xi^{\alpha}\Big|_{\tau=x} \delta^{\alpha} \frac{\partial^{\alpha} N_n}{\partial x^{\alpha}} \\ + {}_0 I_x^{(\alpha)} \left\{ (\xi^{\alpha})^{(2\alpha)}\Big|_{\tau=x} \delta^{\alpha} N_n \right\} \tag{4.4}$$

In (4.4), we have

$$\left[1-(\xi^{\alpha})^{(\alpha)}\right]\Big|_{\tau=x} = 0 \ , \ \ \xi^{\alpha}\Big|_{\tau=x} = 0 \ , \ \ (\xi^{\alpha})^{(2\alpha)}\Big|_{\tau=x} = 0 ,$$

then, the Lagrange multiplier is obtained as

$$\xi^{\alpha} = \frac{(\tau - x)^{\alpha}}{\Gamma(1+\alpha)} . \tag{4.5}$$

Thus, the equation in (4.3) becomes

$$N_{n+1}(x) = N_n(x) + {}_0 I_x^{(\alpha)} \left\{ \frac{(\tau - x)^{\alpha}}{\Gamma(1+\alpha)} \left[\frac{\partial^{2\alpha} N_n(x,\tau)}{\partial x^{2\alpha}} - \frac{\partial^{\alpha} N_n(x,\tau)}{\partial \tau^{\alpha}} \right] \right\} . \tag{4.6}$$

Choosing an initial approximation $N(x,t) = x^{\alpha} E_{\alpha}(t^{\alpha}) / \Gamma(1+\alpha)$,we obtain

$$n_1(x,t) = n_0(x,t) + {}_0 I_t^{(\alpha)} \left\{ \frac{(\tau - t)^{\alpha}}{\Gamma(1+\alpha)} \left[\frac{\partial^{2\alpha} N_0(x,\tau)}{\partial x^{2\alpha}} - \frac{\partial^{\alpha} N_0(x,\tau)}{\partial \tau^{\alpha}} \right] \right\}$$
$$= E_{\alpha}(t^{\alpha}) \sum_{m=0}^{1} \frac{t^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)} ,$$

$$n_2(x,t) = n_1(x,t) + {}_0 I_t^{(\alpha)} \left\{ \frac{(\tau - t)^{\alpha}}{\Gamma(1+\alpha)} \left[\frac{\partial^{2\alpha} N_1(x,\tau)}{\partial x^{2\alpha}} - \frac{\partial^{\alpha} N_1(x,\tau)}{\partial \tau^{\alpha}} \right] \right\}$$
$$= E_{\alpha}(t^{\alpha}) \sum_{m=0}^{2} \frac{t^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)} ,$$

$$\vdots$$

Hereby, the local fractional series solution $N = \lim_{n \to \infty} N_n$ becomes

$$N_n(x,t) = E_\alpha(t^\alpha)\sum_{m=0}^{n}\frac{x^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)}. \tag{4.7}$$

Then, we obtain (4.7).

$$N(x,t) = \lim_{n\to\infty}\left[E_\alpha(t^\alpha)\sum_{m=0}^{n}\frac{x^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)}\right] = E_\alpha(t^\alpha)sinh_\alpha\left(x^\alpha\right), \tag{4.8}$$

where

$$sinh_\alpha\left(x^\alpha\right) = \frac{E_\alpha(x^\alpha)+E_\alpha(-x^\alpha)}{2}.$$

We know that the temperature field can be written in the form

$$\left|E_\alpha(t^\alpha)-E_\alpha(t_0{}^\alpha)\right| \le E_\alpha(t_0{}^\alpha)\left|t-t_0\right|\alpha < \varepsilon^\alpha$$

and

$$\left|sinh_\alpha\left(x^\alpha\right)-sinh_\alpha\left(x_0{}^\alpha\right)\right| < \left|cosh_\alpha\left(x_0{}^\alpha\right)\right|\left|x-x_0\right|^\alpha < \varepsilon^\alpha.$$

Therefore, the fractal dimensions of both $E_\alpha(t^\alpha)$ and $sinh_\alpha\left(x^\alpha\right)$ are equal to α. It is shown that the temperature describes transports processes in fractal media [52].

4.2 Solutions of Diffusion and Wave Equation on Cantor Sets

Below, we apply the local fractional variational iteration method [66] to the sub-diffusion and wave equation on Cantor sets.

4.2.1 Solution of Sub-Diffusion Equation on Cantor Sets [16]

Firstly, we can give the sub-diffusion equation on the Cantor sets

$$\frac{\partial^{2\alpha}N(x,t)}{\partial x^{2\alpha}} - \frac{1}{a^{2\alpha}}\frac{\partial^\alpha N(x,t)}{\partial t^\alpha} = 0 \tag{4.9}$$

with a fractal boundary value conditions,

$$\frac{\partial^{\alpha} N(0,t)}{\partial t^{\alpha}} = 0, \quad N(0,t) = a^{2\alpha} E_{\alpha}(t^{\alpha}). \tag{4.10}$$

We can take the initial value conditions by using (4.10), namely

$$N_0(x,t) = a^{2\alpha} E_{\alpha}(t^{\alpha}).$$

We structure a correction local fractional iteration functional as

$$N_{n+1}(x,t) = N_n(x,t) + {}_0I_x^{(\alpha)} \left\{ \frac{(\tau - x)^{\alpha}}{\Gamma(1+\alpha)} \left[\frac{\partial^{2\alpha} N_n(\tau,t)}{\partial \tau^{2\alpha}} - \frac{1}{a^{2\alpha}} \frac{\partial^{\alpha} N_n(\tau,t)}{\partial t^{\alpha}} \right] \right\}.$$

The first term has the form

$$N_1(x,t) = N_0(x,t) + {}_0I_x^{(\alpha)} \left\{ \frac{(\tau - x)^{\alpha}}{\Gamma(1+\alpha)} \left[\frac{\partial^{2\alpha} N_0(\tau,t)}{\partial \tau^{2\alpha}} - \frac{1}{a^{2\alpha}} \frac{\partial^{\alpha} N_0(\tau,t)}{\partial t^{\alpha}} \right] \right\}$$

$$= a^{2\alpha} E_{\alpha}(t^{\alpha}) + {}_0I_x^{(\alpha)} \left\{ \frac{(\tau - x)^{\alpha}}{\Gamma(1+\alpha)} (-E_{\alpha}(t^{\alpha})) \right\}$$

$$= a^{2\alpha} E_{\alpha}(t^{\alpha}) \left(\sum_{m=0}^{1} \frac{1}{a^{2m\alpha}} \frac{t^{2m\alpha}}{\Gamma(1+2m\alpha)} \right).$$

Similarly, the second approximation term can be calculated

$$N_2(x,t) = N_1(x,t) + {}_0I_x^{(\alpha)} \left\{ \frac{(\tau - x)^{\alpha}}{\Gamma(1+\alpha)} \left[\frac{\partial^{2\alpha} N_1(\tau,t)}{\partial \tau^{2\alpha}} - \frac{1}{a^{2\alpha}} \frac{\partial^{\alpha} N_1(\tau,t)}{\partial t^{\alpha}} \right] \right\}$$

$$= a^{2\alpha} E_{\alpha}(t^{\alpha}) \left(\sum_{m=0}^{2} \frac{1}{a^{2m\alpha}} \frac{t^{2m\alpha}}{\Gamma(1+2m\alpha)} \right).$$

The third approximation term is

$$N_3(x,t) = N_2(x,t) + {}_0I_x^{(\alpha)} \left\{ \frac{(\tau - x)^{\alpha}}{\Gamma(1+\alpha)} \left[\frac{\partial^{2\alpha} N_2(\tau,t)}{\partial \tau^{2\alpha}} - \frac{1}{a^{2\alpha}} \frac{\partial^{\alpha} N_2(\tau,t)}{\partial t^{\alpha}} \right] \right\}$$

$$= a^{2\alpha} E_{\alpha}(t^{\alpha}) \left(\sum_{m=0}^{3} \frac{1}{a^{2m\alpha}} \frac{t^{2m\alpha}}{\Gamma(1+2m\alpha)} \right).$$

$$\vdots$$

If we continue, we obtain the fractional series solution as

$$N_n(x,t) = a^{2\alpha} E_\alpha(t^\alpha)\left(\sum_{m=0}^{n} \frac{1}{a^{2m\alpha}} \frac{x^{2m\alpha}}{\Gamma(1+2m\alpha)}\right).$$

Hereby, the local fractional series solution $N = \lim_{n\to\infty} N_n$ is

$$\begin{aligned} N(x,t) &= \lim_{n\to\infty} N_n(x,t) \\ &= \lim_{n\to\infty} a^{2\alpha} E_\alpha(t^\alpha)\left(\sum_{m=0}^{\infty} \frac{1}{a^{2m\alpha}} \frac{x^{2m\alpha}}{\Gamma(1+2m\alpha)}\right) \\ &= a^{2\alpha} E_\alpha(t^\alpha) \cosh_\alpha\left(\frac{x^\alpha}{a^\alpha}\right). \end{aligned} \tag{4.11}$$

4.2.2 Solution of Wave Equation on Cantor Sets [16]

The wave equation can be written as

$$\frac{\partial^{2\alpha} W(x,t)}{\partial x^{2\alpha}} - \frac{1}{a^{2\alpha}} \frac{\partial^\alpha W(x,t)}{\partial t^{2\alpha}} = 0, \tag{4.12}$$

with a fractal value conditions given by

$$\frac{\partial^\alpha W(0,t)}{\partial t^\alpha} = a^{2\alpha} E_\alpha(t^\alpha),\ W(0,t) = 0. \tag{4.13}$$

We can take the initial value conditions by using (4.13)

$$W_0(x,t) = \frac{a^{2\alpha} E_\alpha(t^\alpha)}{\Gamma(1+\alpha)}.$$

We write a correction local fractional iteration functional, namely

$$W_{n+1}(x,t) = W_n(x,t) + {}_0 I_x^{(\alpha)}\left\{\frac{(\tau - x)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha} W_n(\tau,t)}{\partial \tau^{2\alpha}} - \frac{1}{a^{2\alpha}} \frac{\partial^\alpha W_n(\tau,t)}{\partial t^{2\alpha}}\right]\right\}$$

We find the first term as

$$W_1(x,t)=W_0(x,t)+{}_0I_x^{(\alpha)}\left\{\frac{(\tau-x)^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}W_0(\tau,t)}{\partial\tau^{2\alpha}}-\frac{1}{a^{2\alpha}}\frac{\partial^{\alpha}W_0(\tau,t)}{\partial t^{2\alpha}}\right]\right\}$$
$$=\frac{a^{2\alpha}x^{\alpha}E_{\alpha}(t^{\alpha})}{\Gamma(1+\alpha)}+{}_0I_x^{(\alpha)}\left\{\frac{(\tau-x)^{\alpha}}{\Gamma(1+\alpha)}(-\frac{x^{\alpha}E_{\alpha}(t^{\alpha})}{\Gamma(1+\alpha)})\right\}$$
$$=a^{3\alpha}E_{\alpha}(t^{\alpha})\left(\sum_{m=0}^{1}\frac{1}{a^{(2m+1)\alpha}}\frac{x^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)}\right).$$

Similarly, the second approximation term can deem, that is

$$W_2(x,t)=W_1(x,t)+{}_0I_x^{(\alpha)}\left\{\frac{(\tau-x)^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}W_1(\tau,t)}{\partial\tau^{2\alpha}}-\frac{1}{a^{2\alpha}}\frac{\partial^{\alpha}W_1(\tau,t)}{\partial t^{2\alpha}}\right]\right\}$$
$$=\frac{a^{2\alpha}x^{\alpha}E_{\alpha}(t^{\alpha})}{\Gamma(1+\alpha)}+\frac{x^{3\alpha}E_{\alpha}(t^{\alpha})}{\Gamma(1+3\alpha)}{}_0I_x^{(\alpha)}\left\{\frac{(\tau-x)^{\alpha}}{\Gamma(1+\alpha)}(-\frac{1}{a^{2\alpha}}\frac{x^{3\alpha}E_{\alpha}(t^{\alpha})}{\Gamma(1+3\alpha)})\right\}$$
$$=a^{3\alpha}E_{\alpha}(t^{\alpha})\left(\sum_{m=0}^{2}\frac{1}{a^{(2m+1)\alpha}}\frac{t^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)}\right).$$

In the same way, the third approximation term is reported,

$$W_3(x,t)=W_2(x,t)+{}_0I_x^{(\alpha)}\left\{\frac{(\tau-x)^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}W_2(\tau,t)}{\partial\tau^{2\alpha}}-\frac{1}{a^{2\alpha}}\frac{\partial^{\alpha}W_2(\tau,t)}{\partial t^{2\alpha}}\right]\right\}.$$
$$=a^{3\alpha}E_{\alpha}(t^{\alpha})\left(\sum_{m=0}^{3}\frac{1}{a^{(2m+1)\alpha}}\frac{t^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)}\right).$$

If we continue, we obtain the fractional series solution, namely

$$W_n(x,t)=a^{3\alpha}E_{\alpha}(t^{\alpha})\left(\sum_{m=0}^{n}\frac{1}{a^{(2m+1)\alpha}}\frac{x^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)}\right).$$

Thus, we can obtain the following local fractional series solution as

$$\begin{aligned}W(x,t)&=\lim_{n\to\infty}W_n(x,t)\\&=\lim_{n\to\infty}a^{3\alpha}E_{\alpha}(t^{\alpha})\left(\sum_{m=0}^{\infty}\frac{1}{a^{(2m+1)\alpha}}\frac{x^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)}\right)\\&=a^{3\alpha}E_{\alpha}(t^{\alpha})\sinh_{\alpha}(\frac{x^{\alpha}}{a^{\alpha}}).\end{aligned}\tag{4.14}$$

CHAPTER 5

5.1 Analysis of Fractal Wave Equations by Using Local Fractional Fourier Series

Here,our purpose investigate the following local fractional wave equation by using Local Fractional Fourier series [22]

$$\frac{\partial^{2\alpha}W(x,t)}{\partial t^{2\alpha}}-\frac{\partial^{\alpha}W(x,t)}{\partial t^{\alpha}}-\frac{\partial^{2\alpha}W(x,t)}{\partial x^{2\alpha}}=0, \tag{5.1}$$

where initial and boundary conditions are given as

$$\begin{aligned} &W(0,t)=W(r,t)=\frac{\partial^{\alpha}W(r,0)}{\partial x^{\alpha}}=0,\\ &W(x,0)=f(x),\\ &\frac{\partial^{\alpha}W(x,0)}{\partial t^{\alpha}}=g(x). \end{aligned} \tag{5.2}$$

If there is a particular solution of (5.1) written as

$$W(x,t)=\mu(x)U(t), \tag{5.3}$$

then, we obtain the equations

$$\begin{aligned} &\mu^{(2\alpha)}(x)+\lambda^{2\alpha}\mu=0,\\ &U^{(2\alpha)}+U^{(\alpha)}+\lambda^{2\alpha}U=0, \end{aligned} \tag{5.4}$$

where the boundary conditions are given by

$$\mu(0)=\mu^{(\alpha)}(r)=0.$$

Equation (5.1) has the following solution

$$\mu(x)=c_1\cos_{\alpha}\lambda^{\alpha}x^{\alpha}+c_2\sin_{\alpha}\lambda^{\alpha}x^{\alpha}, \tag{5.5}$$

where c_1 and c_2 are constant numbers [22].

In (5.5), for $x=0$ and $x=r$ we obtain

$$\mu(0)=c_1=0,$$

$$\mu(r)=\mu(x)\Big|_{x=r}=c_2 sin_\alpha \lambda^\alpha r^\alpha=0.$$

Clearly $c_2 \neq 0$, otherwise $\mu(x)=0$.

We attain

$$\lambda_n^\alpha r^\alpha = n^\alpha \pi^\alpha,$$

where n is an integer.

We get, the followings

$$\mu_n(x)=sin_\alpha \lambda_n^\alpha x^\alpha$$
$$=sin_\alpha n^\alpha (\frac{\pi x}{r})^\alpha, \quad (n=0,1,2,3,...).$$

For $\lambda^\alpha=\lambda_n^\alpha$ and $\vartheta>0$, (5.4) means that

$$\sum_{n=1}^{\infty}U_n(t)=\sum_{n=1}^{\infty}E_\alpha(-\frac{t^\alpha}{2})\times(A_n cos_\alpha \vartheta t^\alpha + B_n sin_\alpha \vartheta t^\alpha), \tag{5.6}$$

where

$$\vartheta=\frac{\sqrt{4(n\pi/r)^{2\alpha}-1}}{2}.$$

Thus, we have

$$W_n(x,t)=\mu(x)U_n(x)$$
$$=A_n cos_\alpha \vartheta(\frac{\pi x}{r})^\alpha E_\alpha(-\frac{1}{2}t^\alpha)+B_n sin_\alpha \vartheta(\frac{\pi x}{r})^\alpha E_\alpha(-\frac{1}{2}t^\alpha). \tag{5.7}$$

We suppose a local fractional Fourier series of (5.1) as

$$W(x,t)=\sum_{n=1}^{\infty}W_n(x,t)$$
$$=\sum_{n=1}^{\infty}E_\alpha(-\frac{1}{2}t^\alpha)\times(A_n cos_\alpha \vartheta t^\alpha + B_n sin_\alpha \vartheta t^\alpha)(\frac{\pi x}{r})^\alpha. \tag{5.8}$$

Thus, we get

$$\frac{\partial^\alpha W(x,t)}{\partial t^\alpha}=\sum_{n=1}^{\infty}\frac{\partial^\alpha W_n(x,t)}{\partial t^\alpha},$$

where

$$\frac{\partial^\alpha W_n(x,t)}{\partial t^\alpha}=-\frac{1}{2}E_\alpha(-\frac{1}{2}t^\alpha)(A_n\cos_\alpha \vartheta t^\alpha+B_n\sin_\alpha \vartheta t^\alpha)\sin_\alpha n^\alpha(\frac{\pi x}{r})^\alpha$$
$$+\vartheta E_\alpha(-\frac{1}{2}t^\alpha)(-A_n\sin_\alpha \vartheta t^\alpha+B_n\cos_\alpha \vartheta t^\alpha)\sin_\alpha n^\alpha(\frac{\pi x}{r})^\alpha$$

with

$$\vartheta=\frac{\sqrt{4(n\pi/r)^{2\alpha}-1}}{2}.$$

Take into account (5.8) and (5.2) we obtain [22]

$$W(x,0)=\sum_{n=1}^{\infty}W_n(x,0)$$
$$=\sum_{n=1}^{\infty}A_n\sin_\alpha n^\alpha(\frac{\pi x}{r})^\alpha=f(x),$$
$$\frac{\partial^\alpha W(x,0)}{\partial t^\alpha}=\sum_{n=1}^{\infty}(-\frac{1}{2}A_n+\vartheta B_n)\sin_\alpha n^\alpha(\frac{\pi x}{r})^\alpha=g(x). \tag{5.9}$$

Thus, we report

$$\sum_{n=1}^{\infty}\vartheta B_n\sin_\alpha n^\alpha(\frac{\pi x}{r})^\alpha=g(x)+\sum_{n=1}^{\infty}\frac{1}{2}A_n\sin_\alpha n^\alpha(\frac{\pi x}{r})^\alpha$$
$$=g(x)+\frac{1}{2}f(x). \tag{5.10}$$

We can take the function $F(x)$ as

$$F(x)=g(x)+\frac{1}{2}f(x).$$

Using (5.9) , we find that [22]

$$\sum_{n=1}^{\infty} A_n sin_\alpha n^\alpha (\frac{\pi x}{r})^\alpha = f(x),$$

$$\sum_{n=1}^{\infty} \vartheta B_n sin_\alpha n^\alpha (\frac{\pi x}{r})^\alpha = F(x).$$

We write the local fractional Fourier coefficients of this functions, respectively,

$$A_n = \frac{1/\Gamma(1+\alpha)\int_0^r f(x) sin_\alpha n^\alpha (\frac{\pi x}{r})^\alpha (dx)^\alpha}{1/\Gamma(1+\alpha)\int_0^r sin_\alpha^2 n^\alpha (\frac{\pi x}{r})^\alpha (dx)^\alpha},\ \ (n=0,1,2,3...),$$

$$\vartheta B_n = \frac{1/\Gamma(1+\alpha)\int_0^r F(x) sin_\alpha n^\alpha (\frac{\pi x}{r})^\alpha (dx)^\alpha}{1/\Gamma(1+\alpha)\int_0^r sin_\alpha^2 n^\alpha (\frac{\pi x}{r})^\alpha (dx)^\alpha},\ \ (n=0,1,2,3...), \tag{5.11}$$

and we can calculate as [22]

$$\frac{1}{\Gamma(1+\alpha)}\int_0^r sin_\alpha^2 n^\alpha (\frac{\pi x}{r})^\alpha (dx)^\alpha = \frac{r^\alpha}{2\Gamma(1+\alpha)}.$$

Then, we report

$$A_n = \frac{2\int_0^r f(x) sin_\alpha n^\alpha (\frac{\pi x}{r})^\alpha (dx)^\alpha}{r^\alpha},$$

$$B_n = \frac{2\int_0^r F(x) sin_\alpha n^\alpha (\frac{\pi x}{r})^\alpha (dx)^\alpha}{\vartheta r^\alpha}.$$

Therefore, we obtain the solution of (5.1) as

$$W(x,t) = \sum_{n=1}^{\infty} W_n(x,t)$$

$$= \sum_{n=1}^{\infty} E_\alpha(-\frac{1}{2}t^\alpha)(A_n cos_\alpha \vartheta t^\alpha + B_n sin_\alpha \vartheta t^\alpha) sin_\alpha n^\alpha (\frac{\pi x}{r})^\alpha$$

and

$$A_n = \frac{2\int_0^r f(x) \sin_\alpha n^\alpha (\frac{\pi x}{r})^\alpha (dx)^\alpha}{r^\alpha}, \quad (n = 1,2,3...)$$

$$B_n = \frac{2\int_0^r F(x) \sin_\alpha n^\alpha (\frac{\pi x}{r})^\alpha (dx)^\alpha}{\vartheta r^\alpha} \quad (n = 1,2,3...)$$

with

$$F(x) = g(x) + \frac{1}{2} f(x).$$

5.2 The Factal Models for the One Phase Problems of Discontinuous Heat Transfer

In [2], it was suggested a one phase fractal problem describes the melting of a fractal solid semi-infinite material at its melt temperature. This problem comprises the following equations:

$$\frac{\partial^\alpha m}{\partial t^\alpha} = \frac{\partial^{2\alpha} m}{\partial x^{2\alpha}} \quad , 0 < x < s \ , t > 0. \tag{5.12}$$

(5.12) states the flow of heat in the fractal liquid region [46] and (5.12) is derived from the local fractional one dimensional heat conduction equation, namely

$$\frac{\partial^\alpha m}{\partial x^\alpha} = \beta^\alpha \frac{d^\alpha s}{dt^\alpha} \ , \ x = s(t), t > 0. \tag{5.13}$$

In (5.13) the fractal Stefan condition is described and (5.13) expresses the absorption of heat , wherein β^α is Stefan number

$$m = 0 \ , x > 0 \ , \ t = 0, \tag{5.14}$$

$$m = 0 \ , \ x = s(t) \ , t > 0, \tag{5.15}$$

$$m = 1 \ , \ x = 0 \ , t \geq 0. \tag{5.16}$$

The condition (5.13) can be derived from the fact that the local fractional derivative of the temperature at $x = s(t)$ equal to zero. So, we have

$$\frac{D^{\alpha} m(x,t)}{Dt^{\alpha}} = 0 .$$

We obtain the following expression

$$\frac{\partial m}{\partial x}\frac{d^{\alpha} s}{dt^{\alpha}} + \frac{d^{\alpha} m}{dt^{\alpha}} = 0 . \quad (5.17)$$

From (5.12) and (5.17), we conclude

$$\frac{d^{\alpha} s}{dt^{\alpha}} = -\frac{\frac{d^{\alpha} m}{dt^{\alpha}}}{\frac{\partial m}{\partial x}} = -\frac{\frac{\partial^{2\alpha} m}{\partial x^{2\alpha}}}{\frac{\partial m}{\partial x}} . \quad (5.18)$$

We can use the expression in (5.13) and we get

$$\left(\frac{d^{\alpha} m}{dx^{\alpha}}\right)\left(\frac{\partial m}{\partial x}\right) = \beta^{\alpha}\frac{\partial^{2\alpha} m}{\partial x^{2\alpha}} \ , \ \ x = s(t), t > 0.$$

This result show that the fractal low is local fractional continuous at x. If m is local fractional continuous and m is continuous, we conclude that the fractal dimension is $\alpha = 1$.

(5.13) can be derived from the local fractional derivative of the temperature at $x = s(t)$equals to zero. So, we have

$$\frac{D^{\alpha} m(x,t)}{Dt^{\alpha}} = 0.$$

We obtain the following expression

$$\frac{\partial^{\alpha} m}{\partial x^{\alpha}}\left(\frac{ds}{dt}\right)^{\alpha} + \frac{d^{\alpha} m}{dt^{\alpha}} = 0. \quad (5.19)$$

From (5.12) and (5.19), we finally obtain

$$\left(\frac{ds}{dt}\right)^{\alpha} = -\frac{1}{\Gamma(1-\alpha)}\frac{d^{\alpha} s}{dt^{\alpha}} = -\frac{1}{\Gamma(1-\alpha)}\frac{\frac{\partial^{2\alpha} m}{\partial x^{2\alpha}}}{\frac{\partial m}{\partial x}} . \quad (5.20)$$

By using (5.13), (5.18) and (5.20), we get the final form,

$$\left(\frac{d^{\alpha}m}{dx^{\alpha}}\right)^{2}=\frac{\beta^{\alpha}}{\Gamma(1-\alpha)}\frac{\partial^{2\alpha}m}{\partial x^{2\alpha}}, x=s(t), t>0. \tag{5.21}$$

5.3 Fractional Complex Transform Method in Order to Wave Equations

In this application, it was considered [1] the fractional complex transform method for differential equations. Firstly, some propositions are presented concerning the fractional complex transform method below.

Proposition 5.1 [1]

$$\left.\begin{aligned} M &= \frac{x^{\alpha}}{\Gamma(1+\alpha)} \\ N &= \frac{y^{\alpha}}{\Gamma(1+\alpha)} \end{aligned}\right\} \tag{5.22}$$

and

$$\frac{\partial^{\alpha}T_1(x,y)}{\partial x^{\alpha}}+\frac{\partial^{\alpha}T_2(x,y)}{\partial y^{\alpha}}=0,$$

such that

$$\frac{\partial T_1(M,N)}{\partial M}+\frac{\partial T_2(M,N)}{\partial N}=0. \tag{5.23}$$

***Proof.* [1]**

Let us mention about the fractional complex transform in (5.22), then we can obtain,

$$\frac{\partial^{\alpha}T_1(x,y)}{\partial x^{\alpha}}=\frac{\partial T_1(M,N)}{\partial M}\frac{\partial^{\alpha}M}{\partial x^{\alpha}}+\frac{\partial T_1(M,N)}{\partial N}\frac{\partial^{\alpha}N}{\partial x^{\alpha}}=\frac{\partial T_1(M,N)}{\partial M},$$

$$\frac{\partial^{\alpha}T_2(x,y)}{\partial y^{\alpha}}=\frac{\partial T_2(M,N)}{\partial M}\frac{\partial^{\alpha}M}{\partial y^{\alpha}}+\frac{\partial T_2(M,N)}{\partial N}\frac{\partial^{\alpha}N}{\partial y^{\alpha}}=\frac{\partial T_2(M,N)}{\partial N}.$$

Thus, we get (5.23).

Proposition 5.2 [1]

We consider (5.22), then we can convert

$$\frac{\partial^{2\alpha}T_2(x,y)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}T_1(x,y)}{\partial x^{\alpha}\partial y^{\alpha}}+\frac{\partial^{2\alpha}T_2(x,y)}{\partial y^{\alpha}\partial x^{\alpha}}+\frac{\partial^{2\alpha}T_2(x,y)}{\partial y^{2\alpha}}=0$$

into

$$\frac{\partial^{2}T_2(M,N)}{\partial M^{2}}+\frac{\partial^{2}T_1(M,N)}{\partial M\partial N}+\frac{\partial^{2}T_2(M,N)}{\partial N\partial M}+\frac{\partial^{2}T_2(M,N)}{\partial M^{2}}=0. \quad (5.24)$$

Proof. **[1]**

Similarly with the previous proposition, we have;

$$\frac{\partial^{2\alpha}T_1(x,y)}{\partial x^{2\alpha}}=\frac{\partial T_1(M,N)}{\partial M^{2}}\frac{\partial^{\alpha}M}{\partial x^{\alpha}}+\frac{\partial T_1(M,N)}{\partial N^{2}}\frac{\partial^{\alpha}N}{\partial x^{\alpha}}=\frac{\partial^{2}T_1(M,N)}{\partial M^{2}},$$

$$\frac{\partial^{2\alpha}T_2(x,y)}{\partial y^{2\alpha}}=\frac{\partial T_2(M,N)}{\partial N^{2}}\frac{\partial^{\alpha}N}{\partial y^{\alpha}}+\frac{\partial T_2(M,N)}{\partial N^{2}}\frac{\partial^{\alpha}M}{\partial y^{\alpha}}=\frac{\partial^{2}T_2(M,N)}{\partial M^{2}},$$

$$\frac{\partial^{2\alpha}T_1(x,y)}{\partial y^{\alpha}\partial x^{\alpha}}=\frac{\partial^{2}T_1(M,N)}{\partial M\partial N}\frac{\partial^{\alpha}N}{\partial y^{\alpha}}+\frac{\partial^{2}T_1(M,N)}{\partial M\partial N}\frac{\partial N}{\partial x^{\alpha}}=\frac{\partial^{2}T_1(M,N)}{\partial N\partial M},$$

$$\frac{\partial^{2\alpha}T_2(x,y)}{\partial x^{\alpha}\partial y^{\alpha}}=\frac{\partial^{2}T_2(M,N)}{\partial N\partial M}\frac{\partial^{\alpha}M}{\partial x^{\alpha}}+\frac{\partial^{2}T_2(M,N)}{\partial M\partial M}\frac{\partial^{\alpha}M}{\partial y^{\alpha}}=\frac{\partial^{2}T_2(M,N)}{\partial M\partial N},$$

Thus, we obtain (5.24).

Proposition 5.3 [1]

We suppose that there is the following fractional complex transform

$$\begin{cases} M=\dfrac{x^{\alpha}}{\Gamma(1+\alpha)}, \\ N=\dfrac{y^{\alpha}}{\Gamma(1+\alpha)}, \\ K=\dfrac{z^{\alpha}}{\Gamma(1+\alpha)}, \end{cases} \tag{5.25}$$

and we have

$$\frac{\partial T_1(x,y,z)}{\partial x}+\frac{\partial T_2(x,y,z)}{\partial y}+\frac{\partial T_3(x,y,z)}{\partial z}=0,$$

such that

$$\frac{\partial^{\alpha} T_1(x,y,z)}{\partial x^{\alpha}}+\frac{\partial^{\alpha} T_2(x,y,z)}{\partial y^{\alpha}}+\frac{\partial^{\alpha} T_3(x,y,z)}{\partial z^{\alpha}}=0. \tag{5.26}$$

Proof. **[1]**

Let us use the fractional complex transform in (5.25), so we obtain

$$\begin{aligned} \frac{\partial^{\alpha} T_1(x,y,z)}{\partial x^{\alpha}} &= \frac{\partial T_1(M,N,K)}{\partial M}\frac{\partial^{\alpha} M}{\partial x^{\alpha}}+\frac{\partial T_1(M,N,K)}{\partial N}\frac{\partial^{\alpha} N}{\partial x^{\alpha}}+\frac{\partial T_1(M,N,K)}{\partial K}\frac{\partial^{\alpha} K}{\partial x^{\alpha}} \\ &= \frac{\partial T_1(M,N,K)}{\partial M}, \\ \frac{\partial^{\alpha} T_2(x,y,z)}{\partial y^{\alpha}} &= \frac{\partial T_2(M,N,K)}{\partial M}\frac{\partial^{\alpha} M}{\partial y^{\alpha}}+\frac{\partial T_2(M,N,K)}{\partial N}\frac{\partial^{\alpha} N}{\partial y^{\alpha}}+\frac{\partial T_2(M,N,K)}{\partial K}\frac{\partial^{\alpha} K}{\partial y^{\alpha}} \\ &= \frac{\partial T_2(X,Y,Z)}{\partial Y}, \\ \frac{\partial^{\alpha} T_3(x,y,z)}{\partial z^{\alpha}} &= \frac{\partial T_3(M,N,K)}{\partial M}\frac{\partial^{\alpha} M}{\partial z^{\alpha}}+\frac{\partial T_3(M,N,K)}{\partial N}\frac{\partial^{\alpha} N}{\partial z^{\alpha}}+\frac{\partial T_3(M,N,K)}{\partial K}\frac{\partial^{\alpha} K}{\partial z^{\alpha}} \\ &= \frac{\partial T_3(M,N,K)}{\partial K}. \end{aligned} \tag{5.27}$$

We directly obtain (5.26) by using above equations.

Proposition 5.4 [1]

When the fractional complex transform (5.25) is given,

$$\frac{\partial^2 T_1(x,y,z)}{\partial x^2}+\frac{\partial^2 T_2(x,y,z)}{\partial y^2}+\frac{\partial^2 T_3(x,y,z)}{\partial z^2}=0$$

thus, we get

$$\frac{\partial^{2\alpha} T_1(x,y,z)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha} T_2(x,y,z)}{\partial y^{2\alpha}}+\frac{\partial^{2\alpha} T_3(x,y,z)}{\partial z^{2\alpha}}=0\,.$$

Proof. **[1]**

Let us use the fractional complex transform in (5.25), so we obtain,

$$\begin{aligned}\frac{\partial^{2\alpha} T_1(x,y,z)}{\partial x^{2\alpha}}&=\frac{\partial^2 T_1(M,N,K)}{\partial M^2}\frac{\partial^\alpha M}{\partial x^\alpha}+\frac{\partial^2 T_1(M,N,K)}{\partial N^2}\frac{\partial^\alpha N}{\partial x^\alpha}+\frac{\partial^2 T_1(M,N,K)}{\partial K^2}\frac{\partial^\alpha K}{\partial x^\alpha}\\&=\frac{\partial^2 T_1(M,N,K)}{\partial M^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^{2\alpha} T_2(x,y,z)}{\partial x^{2\alpha}}&=\frac{\partial^2 T_2(M,N,K)}{\partial M^2}\frac{\partial^\alpha M}{\partial x^\alpha}+\frac{\partial^2 T_2(M,N,K)}{\partial N^2}\frac{\partial^\alpha N}{\partial x^\alpha}+\frac{\partial^2 T_2(M,N,K)}{\partial K^2}\frac{\partial^\alpha K}{\partial x^\alpha}\\&=\frac{\partial^2 T_2(M,N,K)}{\partial M^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^{2\alpha} T_3(x,y,z)}{\partial x^{2\alpha}}&=\frac{\partial^2 T_3(M,N,K)}{\partial M^2}\frac{\partial^\alpha M}{\partial x^\alpha}+\frac{\partial^2 T_3(M,N,K)}{\partial N^2}\frac{\partial^\alpha N}{\partial x^\alpha}+\frac{\partial^2 T_3(M,N,K)}{\partial K^2}\frac{\partial^\alpha K}{\partial x^\alpha}\\&=\frac{\partial^2 T_3(M,N,K)}{\partial M^2}.\end{aligned}$$

Take into account (5.25) and (5.27), we have completed this proof.

5.3.1. Wave Equations on Cantor sets

We mention the fractional complex transform method to operate three dimensional wave equations on Cantor sets.

Let's mention three dimensional wave equation. We write the fractional complex transform by using local fractional derivatives.

$$\begin{cases} T = \dfrac{t^{\alpha}}{\Gamma(1+\alpha)}, \\ M = \dfrac{(ax)^{\alpha}}{\Gamma(1+\alpha)}, \\ N = \dfrac{(ay)^{\alpha}}{\Gamma(1+\alpha)}, \\ K = \dfrac{(az)^{\alpha}}{\Gamma(1+\alpha)}, \end{cases}$$

such that

$$\frac{\partial^2 u(M,N,K,T)}{\partial T^2} + \nabla u(M,N,K,T) = 0,$$

where, $\nabla = \dfrac{\partial^2}{\partial M^2} + \dfrac{\partial^2}{\partial N^2} + \dfrac{\partial^2}{\partial K^2}$.

Let's mention two dimensional wave equation. We write the fractional complex transform by using local fractional derivatives [1]

$$\begin{cases} T = \dfrac{t^{\alpha}}{\Gamma(1+\alpha)}, \\ M = \dfrac{(ax)^{\alpha}}{\Gamma(1+\alpha)}, \\ N = \dfrac{(ay)^{\alpha}}{\Gamma(1+\alpha)}, \end{cases}$$

such that $\dfrac{\partial^2 u(M,N,T)}{\partial T^2} + \nabla u(M,N,T) = 0$,

where, $\nabla = \dfrac{\partial^2}{\partial M^2} + \dfrac{\partial^2}{\partial N^2}$.

If there is the mass function [47]

$$\gamma^{\alpha}\left[F,a,b\right]=\frac{1}{\Gamma(1+\alpha)}H^{\alpha}(F\cap(a,b))=\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)},$$

then, we obtain the following formula;

$$\gamma^{\alpha}\left[F,la,lb\right]=l^{\alpha}\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}, \text{ such that}$$

$$\begin{cases} T=\dfrac{t^{\alpha}}{\Gamma(1+\alpha)}=\gamma^{\alpha}\left[F,0,t\right], \\ M=\dfrac{(ax)^{\alpha}}{\Gamma(1+\alpha)}=\gamma^{\alpha}\left[F,0,ax\right], \\ N=\dfrac{(ay)^{\alpha}}{\Gamma(1+\alpha)}=\gamma^{\alpha}\left[F,0,ay\right], \\ K=\dfrac{(az)^{\alpha}}{\Gamma(1+\alpha)}=\gamma^{\alpha}\left[F,0,az\right]. \end{cases}$$

From [47], we conclude

$$\left|T(t_1)-T(t_2)\right|\le\varepsilon_1^{\alpha},\ \ \left|M(x_1)-M(x_2)\right|\le\varepsilon_2^{\alpha},$$
$$\left|N(y_1)-N(y_2)\right|\le\varepsilon_1^{\alpha},\ \ \left|K(z_1)-K(z_2)\right|\le\varepsilon_2^{\alpha},$$

for any $0<\varepsilon_i$ and $\varepsilon_i\in R$,which means that the fractal dimensions of transferring pairs are α.

5.4 Local Fractional Sumudu Transform

The Sumudu transform can be used to solve the differential equations[44-48]. The aims of this applications are to connect the Sumudu transform and (LFC).

We can take a new transform operator as

$$LFS_{\alpha}: f(x) \to F(t),$$

$$LFS_{\alpha}\left\{f(x)\right\} = LFS_{\alpha}\left\{\sum_{m=0}^{\infty} a_m x^{\alpha m}\right\} = \sum_{m=0}^{\infty} \Gamma(1+m\alpha) a_m z^{\alpha m}. \qquad (5.28)$$

We can give typical examples [20] as

$$LFS_{\alpha}\left\{E_{\alpha}(i^{\alpha} x^{\alpha})\right\} = \sum_{m=0}^{\infty} i^{\alpha m} z^{\alpha m},$$

$$LFS_{\alpha}\left\{\frac{x^{\alpha}}{\Gamma(1+\alpha)}\right\} = z^{\alpha}.$$

Definition 5.5 [20]

The local fractional Sumudu transform (LFST) of $g(x)$ of order α is defined as [20]

$$LFS_{\alpha}\left\{g(x)\right\} = G_{\alpha}(z) =: \frac{1}{\Gamma(1+\alpha)} \int_0^{\infty} E_{\alpha}(-z^{-\alpha} x^{\alpha}) \frac{g(x)}{z^{\alpha}} (dx)^{\alpha}, \quad 0 < \alpha \le 1 \qquad (5.29)$$

The inverse of LFST has the form

$$LFS_{\alpha}^{-1}\left\{G_{\alpha}(z)\right\} = g(x), \qquad 0 < \alpha \le 1. \qquad (5.30)$$

Theorem 5.6 (The Linearity of LFST) [20]

Suppose that

$$LFS_{\alpha}\left\{h(x)\right\} = H_{\alpha}(z),$$

$$LFS_{\alpha}\left\{g(x)\right\} = G_{\alpha}(z),$$

then we get [20]

$$LFS_{\alpha}\left\{h(x)+g(x)\right\}=H_{\alpha}(z)+G_{\alpha}(z). \tag{5.31}$$

Proof. **[20]**

We can get (5.31) by using the definition of LFST.

Theorem 5.7 [20] (Local Fractional Laplace-Sumudu Duality)

Suppose that

$L_{\alpha}\left\{h(x)\right\}=h_{s}^{L,\alpha}(s)$ and $LFS_{\alpha}\left\{h(x)\right\}=H_{\alpha}(z)$

then, we obtain

$$LFS_{\alpha}\left\{h(x)\right\}=\frac{1}{z^{\alpha}}L_{\alpha}\left\{h\left(\frac{1}{x}\right)\right\}, \tag{5.32}$$

$$L_{\alpha}\left\{h(x)\right\}=\frac{LFS_{\alpha}\left\{h(1/s)\right\}}{s^{\alpha}}. \tag{5.33}$$

Proof **. [20]**

The definitions of LFST and laplace transforms give directly (5.32) and (5.33).

Theorem 5.8 [20] (Local Fractional Sumudu Transform of Local Fractional Derivative)

Suppose that $LFS_{\alpha}\left\{g(x)\right\}=G_{\alpha}(z)$, then

$$LFS_{\alpha}\left\{\frac{d^{\alpha}g(x)}{dx^{\alpha}}\right\}=\frac{G_{\alpha}(z)-g(0)}{z^{\alpha}}. \tag{5.34}$$

Proof. **[20]**

We take $H(x)=\dfrac{d^{\alpha}g(x)}{dx^{\alpha}}$. From (5.32), we have

$$LFS_{\alpha}\left\{H(x)\right\}=\frac{L_{\alpha}\left\{H(1/x)\right\}}{z^{\alpha}}$$

and from [17]

$$L_{\alpha}\left\{H(1/x)\right\}=L_{\alpha}\left\{g(1/x)\right\}/z^{\alpha}-g(0),$$

$$\begin{aligned}LFS_{\alpha}\left\{H(x)\right\}&=\frac{L_{\alpha}\left\{H(1/x)\right\}}{z^{\alpha}}=\frac{L_{\alpha}\left\{g(1/x)\right\}/z^{\alpha}-g(0)}{z^{\alpha}}\\&=\frac{G_{\alpha}(z)-g(0)}{z^{\alpha}}.\end{aligned} \tag{5.35}$$

Theorem 5.9 [20] (Local Fractional Sumudu Transform of the Local Fractional Integral)

Suppose that $LFS_{\alpha}\left\{g(x)\right\}=G_{\alpha}(z)$, then we have

$$LFS_{\alpha}\left\{{}_{0}I_{x}^{(\alpha)}g(x)\right\}=z^{\alpha}G_{\alpha}(z). \tag{5.36}$$

Proof **.[20]**

We get $LFS_{\alpha}\left\{{}_{0}I_{x}^{(\alpha)}g(x)\right\}=\dfrac{1}{s^{\alpha}}L_{\alpha}\left\{g(x)\right\}$, from (5.32), namely

$$LFS_{\alpha}\left\{h(x)\right\}=\frac{1}{z^{\alpha}}L_{\alpha}\left\{h\left(\frac{1}{x}\right)\right\}=L_{\alpha}\left\{g(\frac{1}{x})\right\}=z^{\alpha}G_{\alpha}z, \tag{5.37}$$

where $h(x)={}_{0}I_{x}^{(\alpha)}g(x)$.

Thus, we complete this proof.

Theorem 5.10 [20,70] (Local Fractional Convolution)

Suppose that $LFS_{\alpha}\left\{h(x)\right\}=H_{\alpha}(z)$ and $LFS_{\alpha}\left\{g(x)\right\}=G_{\alpha}(z)$, then we have

$$LFS_{\alpha}\left\{g(x).h(x)\right\} = z^{\alpha}G_{\alpha}(z)H_{\alpha}(z), \tag{5.38}$$

where

$$g(x).h(x) = \frac{1}{\Gamma(1+\alpha)}\int_{0}^{\infty} f(t)g(x-t)(dt)^{\alpha}. \tag{5.39}$$

Proof. **[20]**

From (5.32), we conclude that

$$LFS_{\alpha}\left\{g(x).h(x)\right\} = \frac{L_{\alpha}\left\{g(x).h(x)\right\}}{z^{\alpha}}.$$

As [20], we can write

$$L_{\alpha}\left\{g(x).h(x)\right\} = L_{\alpha}\left\{g(1/x)\right\}.L_{\alpha}\left\{h(1/x)\right\}. \tag{5.40}$$

We obtain the followings

$$LFS_{\alpha}\left\{g(x).h(x)\right\} = \frac{L_{\alpha}\left\{g(1/x)\right\}.L_{\alpha}\left\{h(1/x)\right\}}{z^{\alpha}}$$
$$= z^{\alpha}G_{\alpha}(z)H_{\alpha}(z),$$

where $G_{\alpha}(z) = \frac{L_{\alpha}\left\{g(1/x)\right\}}{z^{\alpha}}$ and $H_{\alpha}(z) = \frac{L_{\alpha}\left\{h(1/x)\right\}}{z^{\alpha}}$.

Thus, we complete this proof.

CONCLUSION

The local fractional calculus is a new area of mathematics that studies the derivative and integration of functions of arbitrary order defined on fractals. So, it has attracted much attention of mathematicians, physicists and engineers.

In this thesis, we reviewed the basic definitions and theorems of local fractional calculus. Also we reviewed some recent applications and we showed that in both fields of physics and mathematics the local fractional calculus gives effective results.

We hope that, this thesis will be a useful tool for researchers who would like to work on the local fractional calculus and its applications.

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