THE DOUBLE LAPLACE TRANSFORM

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THE DOUBLE LAPLACE TRANSFORM

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Title of the Thesis : **The Double Laplace Transform**

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ABSTRACT

THE DOUBLE LAPLACE TRANSFORM

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In this thesis, we present the formal definition of the double Laplace transform and calculate the Double Laplace transforms of some elementary functions directly from the definition. The existence conditions for the double Laplace transform and the basic properties of the double Laplace transforms are stated . applications of the double Laplace transforms to the solutions of certain integral equations and boundary value problems are also discussed in this work.

Keywords: Double Laplace Transform, Exponential Order, Convolution, Partial Derivatives.

ÖZ

ÇİFT LAPLACE DÖNÜŞÜMÜ

BASHEER, Husam
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Tez Yöneticisi: Prof. Dr. Kenan TAŞ

Ocak 2015, 30 sayfa

Bu tezde, çift Laplace dönüşümünün tanımı sunulmuş ve bazı elementer fonksiyonların çift Laplace dönüşümlerinin tanımı kullanılarak doğrudan hesaplanmıştır. Çift Laplace dönüşümünün varlığı için gerekli koşullar tartışılmış ve bu dönüşümün temel özellikleri belirtilmiştir. Herhangi bir mertebeden kısmi türevin de çift Laplace dönüşümü elde edilmiştir. İki fonksiyonun çift konvolüsyonu tanımlanmış ve bu konvolüsyonunun cift Laplace dönüşümü hesaplanmıştır. Başlangıç ve sınır koşullu belirli Kısmi diferansiyel denklemler için çift Laplace dönüşümlerinin uygulamaları da tartışılmıştır.

Anahtar Kelimeler: Çift Laplace Dönüşümü, Üstel Mertebe, Konvolüsyon, Kısmi Türevler.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Background

Many physical processes in nature are described by differential and integral equation with initial conditions or boundary conditions. Integral transforms not only helped in developing the theory of such equations but also provided methods to solve these equations [1-2]. One of the most important integral transforms is the laplace transform.this well known integral transform was first used by laplace in 1812 when he was working on probability theory[1-2]. Since that time many works have been devoted to the study of the properties of the laplace transform and its various applications in many fields of science. Bateman in 1910 used the modern Laplace transform followed by Bemotien in 1920. This transform gained a more modern approach. In 1920 when doeth applied this transform on differential integro-differential equation[3]. Since most of the physical evolve in time in semi finite or infinite domains, Laplace transform conspired with Fourier transform has shown a strong analytical method to solve partial differential equations obtained when dealing with there processes. But the one variable Laplace transform is not capable of solving this equations alone. Thus it is very important to generalize the Laplace transform to function of multi-variables. the properties of double Laplace transform were discussed in [4-5]. Applications of this transform to partial differential equation were done by many authors(see [6-7] and the references therein). In this work, a surrey on double Laplace transform is made definitions and properties of the double Laplace transform are discussed and many of its applications to different kinds of partial differential equation are presented.

1.2 Organization of the Thesis

This thesis contains four chapters. All the necessary information about the double Laplace transform including all definitions and theorems and properties of different applications.

Chapter 1, is an introduction to the history of double Laplace transform and objectives of this thesis.

Chapter 2, includes an introduction basic definitions of double Laplace transform .

In Chapter 3, the study basic properties and formulas and the double Laplace transform of derivatives and the double convolution.

In Chapter 4, touching for some applications of double Laplace in partial differential equation and solve some important problems for partial differential equations.

Chapter 5, includes the conclusion part.

CHAPTER 2

BASIC DEFINITIONS AND FORMULAS

2.1 Basic Definitions of Double Laplace Transform

2.1.1. Definition of the double Laplace transform

Let f be a function of two variables x and t, where $x\ ,\ t\ >\ 0$ The double Laplace transform of f is defined by

$$L_t\, L_x\left\{f(x,t)\right\}(s_1,s_2)=\int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x}\ f(x,t)dxdt \qquad (2.1)$$

whenever the improper integral converges. Here s_1, s_2 are complex numbers where L_x and L_t represent the Laplace transforms with respect to the variables x and t respectively [4].

Below, we present the double Laplace transform of some function.

2.1.2. Example (the double Laplace transform of 1)

Let $f(x,t)=1$ a continuous function the Laplace transform is easily found to be as follows:

$$L_t\, L_x\left\{f(x,t)\right\}(s_1,s_2)=\int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x}\ f(x,t)dxdt$$

$$=\int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x}\ dxdt$$

$$=\int_0^\infty e^{-s_2 t}\left\{\ \int_0^\infty e^{-s_1 x}\ dx\ \right\}dt$$

$$= \int_0^\infty e^{-s_2 t}\left\{[\frac{e^{-s_1 x}}{-s_1}]_0^\infty\right\}dt = \int_0^\infty \frac{1}{s_1} e^{-s_2 t}dt$$

$$= \frac{1}{s_1}\int_0^\infty e^{-s_2 t}dt = \frac{1}{s_1}\left[\frac{e^{-s_2 t}}{-s_2}\right]_0^\infty$$

$$= \frac{1}{s_1}\left[\frac{1}{s_2}\right] = \frac{1}{s_1 s_2}$$

Where s_1 and s_2 are positive.

2.1.3. Definition of an exponential order

Let $f(x,t)$ be a continuous function on $[0,\infty) \times [0,\infty)$. Then f is said to be of exponential order, if

$$SUP_{x,t\geq 0} \frac{|f(x,t)|}{e^{ax+bt}} < \infty \qquad (2.2)$$

for some $a,b \in R$. [4]

The following theorem shows the existence of the double Laplace transforms and afunction of exponential order.

2.1.4. Theorem (existence of the double Laplace transform)

If $f(x,t)$ is of exponential order, then double Laplace transform of f exists.

Proof: Suppose f is of exponential order, that is

$$SUP_{x,t\geq 0} \frac{|f(x,t)|}{e^{ax+bt}} = M$$

for some $a,b,M \in R$. Then

$$|f(x,t)| \leq Me^{(ax+bt)}$$

for all $x,t \geq 0$. Thus

$$\left| \int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x} f(x,t)dxdt \right|$$

$$\leq \int_0^\infty \left|e^{-s_2 t}\right| \left|\int_0^\infty\right| \left|e^{-s_1 x}\right| \left|f(x,t)\right| dxdt$$

$$= \int_0^\infty e^{-s_2 t} \int_0^\infty e^{-s_1 x} \left|f(x,t)\right| dxdt$$

$$\leq \mathrm{M} \int_0^\infty e^{-s_2 t} \int_0^\infty e^{-s_1 x} e^{(ax+bt)} dxdt$$

$$= M \int_0^\infty e^{-(s_2-b)t} \int_0^\infty e^{-(s_1-a)x} dxdt$$

thus the integral in question converges for $s_1 > a$ and $s_2 > b$.

we have to rote that not all the function of two variable one of exponential order. For instance the function $e^{-x^3-t^3}$ is not of exponential order and thus it does not have the Laplace transform.

from now on, we consider functions for which double Laplace transformation exist without specifying of convergence of the integrals in question, but keep in mind broadest the'regions' possible .

2.2. The Inverse of Double Laplace Transform

2.2.1. Definition of the inverse of double Laplace transform

Suppose $f(x,t)$ possesses first order partial derivatives $\frac{\partial \mathrm{f}}{\partial \mathrm{x}}$ and $\frac{\partial f}{\partial t}$ and second order derivative $\frac{\partial^2 f}{\partial x \partial t}$ and there exist positive constants M, γ_1, γ_2 such that for all $0 < x$, $t < \infty$

$$\left|f(x,t)\right| < Me^{\gamma_1 x + \gamma_2 t}, \left|\frac{\partial^2 f}{\partial x \partial t}\right| < e^{\gamma_1 x + \gamma_2 t} \tag{2.3}$$

then if

$$\overline{f}(s_1, s_2) = \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 t} f(x,t) dxdt \tag{2.4}$$

we have [8]

$$f(x,t) = \lim_{T_1,T_2\to\infty} \frac{1}{(2\pi i)^2} \int_{c_1-iX}^{c_1+iX} \int_{c_2-iT}^{c_2+iT} e^{-s_1x-s_2t}\,\overline{f}(s_1,s_2)ds_1ds \tag{2.5}$$

or [2]

$$f(t_1,t_2) = \frac{1}{(2\pi i)^2} \int_{c_1-i\infty}^{c_1+i\infty} \int_{c_2-i\infty}^{c_2+i\infty} e^{-s_1x-s_2t}\,\overline{f}(s_1,s_2)ds_1ds_2 \tag{2.6}$$

Where $c_1 > \gamma_1$ and $c_2 > \gamma_2$.

The next theorem shows that if $\overline{f}(s_1,s_2)$ is known then $f(x,t)$ can be uniquely obtained from $\overline{f}(s_1,s_2)$

2.2.2. Theorem for uniqueness of the double Laplace transform

Let $f(x,t)$ and $g(x,t)$ be continuous functions defined for $x,t \geq 0$ and having Laplace transforms $\overline{f}(s_1,s_2)$ and $\overline{g}(s_1,s_2)$ respectively

if

$$\overline{f}(s_1,s_2) = \overline{g}(s_1,s_2) \tag{2.7}$$

then

$$f(x,t) = g(x,t) \tag{2.8}$$

Proof [9] If α and β are sufficiently large, then the integral representation given by

$$f(x,t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{s_2x}[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{s_1t}\,\bar{f}(s_1,s_2)\,ds_1]ds_2$$

for the inverse double Laplace transform, can be used to obtain

$$f(x,t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{s_2x}[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{s_1t}\,\bar{f}(s_1,s_2)\,ds_1]ds_2$$

$$=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{s_2x}[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{s_1t}\,\overline{g}(s_1,s_2)\,ds_1]ds_2$$

$$= g(x,t)$$

CHAPTER 3

BASIC PROPERTIES AND CONVOLUTION FORMULA

3.1 Basic Properties and Formulas

In this section we consider some of the properties of the double Laplace Transform that will enable us to find further transform pairs { $f(x,t)$, $\overline{f}(s_1,s_2)$ } without having to compute consider the following.

3.1.1 Theorem (linearity property)

if $$L_t L_x\{f(x,t)\} = \overline{f}(s_1,s_2) \tag{3.1}$$

for $s_1 > a$ and $s_2 > b$, and $L_t L_x\{g(x,t) = \overline{g}(s_1,s_2)$, for $s_1 > c$ and $s_2 > d$

and $\alpha, \beta \in R$

then

$$L_t L_x\{\alpha f(x,t) + \beta g(x,t)\} = \alpha L_t L_x\{f(x,t)\} + \beta L_t L_x\{g(x,t)\} \tag{3.2}$$

for $s_1 >$ max $\{a,c\}$ and $s_2>$ max $\{b,d\}$.

Proof : This follows easily from the linearity of the integral [4].

3.1.2 Theorem (division by xt)

if

$$L_t L_x\{f(x,t)\} = \overline{f}(s_1,s_2)$$

then

$$L_t L_x\left\{\frac{f(x,t)}{xt}\right\} = \int_{s_1}^{\infty}\int_{s_2}^{\infty} \overline{f}(u,v)\,dudv$$

Proof :[10] Assume that

$$\int_{s_1}^{\infty}\int_{s_2}^{\infty}\bar{f}(u,v)\,dudv \quad \text{exists}$$

integrating both sides of

$$\bar{f}(u,v)=\int_0^{\infty}\int_0^{\infty}e^{-ux-vt}f(x,t)dxdt$$

with respect to u from s_1 to ∞ and with respect to v from s_2 to ∞ ,we get

$$\int_{s_1}^{\infty}\int_{s_2}^{\infty}\bar{f}(u,v)\,dudv=\int_{s_1}^{\infty}\int_{s_2}^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-s_1x}\ e^{-s_2t}\ f(x,t)\,dxdtdudv$$

$$=\int_{s_2}^{\infty}\int_0^{\infty}\int_0^{\infty}\left[\frac{e^{-s_1x}}{-x}\right]_{u=s_1}^{\infty}\ e^{-s_2t}\ f(x,t)\,dxdtdv$$

$$=\int_0^{\infty}\int_0^{\infty}\left[0+\frac{e^{-s_1x}}{x}\right]\left[\frac{e^{-s_2t}}{-t}\right]_{v=s_2}^{\infty}\quad f(x,t)\,dxdt$$

$$=\int_0^{\infty}\int_0^{\infty}\ e^{-s_1x}\ e^{-s_2t}\ \frac{f(x,t)}{xt}dxdt$$

$$=L_tL_x\left\{\frac{f(x,t)}{xt}\right\}$$

thus

$$L_tL_x\left\{\frac{f(x,t)}{xt}\right\}=\int_{s_1}^{\infty}\int_{s_2}^{\infty}\bar{f}(u,v)\,dudv$$

3.1.3 Theorem (change of scale property)

if

$$L_t\,L_x\left\{\,f(x,t)\right\}=\bar{f}(s_1,s_2)$$

then

$$L_t\,L_x\left\{\,f(ax,bt)\right\}=\frac{1}{ab}\,\bar{f}\,(\frac{s_1}{a},\frac{s_2}{b})$$

where a and b are non zero constants [10].

Proof: from (2.1), we have

$$L_t\, L_x\left\{ f(ax,bt)\right\} = \int_0^\infty e^{-s_2 t} \int_0^\infty e^{-s_1 x}\, f(ax,bt)dxdt \tag{3.3}$$

Putting $ax = u$ and $bt = v$ in the integral of (3.3), where u and v takes the limit from 0 to ∞. Hence, we get

$$L_x\, L_t\left\{ f(ax,bt)\right\} = \int_0^\infty e^{-s_2(\frac{v}{b})} \int_0^\infty e^{-s_1\frac{u}{a}}\, f(u,v)\frac{du}{a}\frac{dv}{b}$$

$$= \frac{1}{ab}\int_0^\infty e^{-s_2(\frac{v}{b})} \int_0^\infty e^{-s_1\frac{u}{a}}\, f(u,v)\; dudv$$

$$= \frac{1}{ab}\bar{f}(\frac{s_1}{a}\,,\,\frac{s_2}{b})$$

thus

$$L_x\, L_t\left\{ f(ax,bt)\right\} = \frac{1}{ab}\bar{f}(\frac{s_1}{a}\,,\,\frac{s_2}{b})$$

3.1.4 Theorem (multiplication by $x^m t^n$)

if

$$L_t L_x\left\{f(x,t)\right\} = \bar{f}\,(s_1\,, s_2)$$

then

$$L_t L_x\left\{x^m t^n\; f(x,t)\right\} = (-1)^{m+n}\frac{\partial^{m+n}}{\partial s_1^m \partial s_2^n}\bar{f}\,(s_1\,, s_2)$$

Proof : [10] from

$$\bar{f}(s_1, s_2) = \int_0^\infty\int_0^\infty e^{-s_1 x - s_2 t}\, f(x,t)dxdt$$

we get

$$\frac{\partial^{m+n}}{\partial s_1^m\,\partial s_2^n}\bar{f}\,(s_1\,, s_2) = \frac{\partial^{m+n}}{\partial s_1^m\,\partial s_2^n}\int_0^\infty \int_0^\infty e^{-s_1 x - s_2 t}\;\; f(x,t)\, dxdt$$

$$= \int_0^\infty\int_0^\infty \frac{\partial^{m+n}}{\partial s_1^m\,\partial s_2^n}\left[e^{-s_1 x - s_2 t}\, f(x,t)\right]dxdt$$

$$= \int_0^\infty\int_0^\infty (-x)^m(-t)^n\, e^{-s_1 x - s_2 t}\, f(x,t)dxdt$$

$$= (-1)^{m+n} \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 t} x^m t^n f(x,t) dxdt$$

$$= (-1)^{m+n} L_t L_x \{x^m t^n f(x,t)\}$$

3.1.5 Theorem (first shifting property)

if

$$L_t\, L_x \left\{ \{ f(x,t) \} = \bar{f}\,(s_1, s_2) \right\}$$

then

$$L_t\, L_x \left\{ e^{ax+bt}\; f(x,t) \right\} = \bar{f}\,(s_1 - a\;,\; s_2 - b)$$

where a and b are non zero constants [10] .

Proof: From (2.1), we have

$$L_t\, L_x \left\{ e^{ax+bt}\; f(x,t) \right\} \;= \int_0^\infty e^{-s_2 t} \int_0^\infty e^{-s_1 x} e^{ax+bt}\; f(x,t) dxdt$$

$$= \int_0^\infty e^{-(s_2-b)t} \int_0^\infty e^{-(s_1-a)x}\; f(x,t)\, dxdt = \bar{f}\,(s_1 - a\;,\; s_2 - b)$$

thus

$$L_t\, L_x \left\{ e^{ax+bt}\; f(x,t) \right\} = \bar{f}\,(s_1 - a\;,\; s_2 - b)$$

3.1.6 Theorem

if

$$g(x,t) = \begin{cases} 0 & If \quad x < a \quad or \quad t < b \\ f(x-a\;,\; t-b) & x > a \quad,\quad t > b \end{cases} \tag{3.4}$$

Then if $\bar{f}(s_1, s_2)$ exists

$$\bar{g}(s_1, s_2) = e^{-s_1 a - s_2 b}\, \bar{f}(s_1, s_2) \tag{3.5}$$

Proof : From (2.1), we have

$$\bar{g}(s_1,s_2)=\int_0^\infty\int_0^\infty e^{-s_1x-s_2t}f(x,t)dxdt$$

$$=\int_b^\infty\int_a^\infty e^{-s_1x-s_2t}f(x-a,t-b)dxdt$$

Let

$$u=x-a$$
$$v=t-b$$

$$=\int_0^\infty\int_0^\infty e^{-s_1(u+a)}e^{-s_2(v+b)}f(u,v)dudv$$

thus

$$\bar{g}(s_1,s_2)=e^{-s_1a-s_2b}\bar{f}(s_1,s_2)$$

3.1.7 Theorem

if

$$g(x,t)=\begin{cases} f(x) & x<t \\ f(t) & t>x \end{cases} \tag{3.6}$$

and $\bar{f}(s_1,s_2)$ exists, then

$$\bar{g}(s_1,s_2)=\frac{s_1+s_2}{s_1s_2}\bar{f}(s_1,s_2) \tag{3.7}$$

Proof: From (2.1), we have

$$\bar{f}(s_1,s_2)=\int_0^\infty\int_0^\infty e^{-s_1x-s_2t}g(x,t)dxdt$$

$$=\int_0^\infty\int_t^\infty e^{-s_1x-s_2t}f(t)dxdt+\int_0^\infty\int_x^\infty e^{-s_1x-s_2t}f(x)dxdt$$

$$=\int_0^\infty e^{-s_2t}f(t)\left[\int_t^\infty e^{-s_1x}dx\right]dt+\int_0^\infty e^{-s_1x}f(x)\int_x^\infty e^{-s_2t}dxdt$$

$$=\frac{1}{s_1}\int_0^\infty e^{-(s_1+s_2)t}f(t)dt+\frac{1}{s_2}\int_0^\infty e^{-(s_1+s_2)x}f(x)dx$$

thus

$$\bar{g}(s_1, s_2) = \frac{(s_1 + s_2)}{s_1 s_2} \bar{f}(s_1 + s_2)$$

3.1.8 Theorem

if

$$g(x,t) = \begin{cases} f(x) & x < t \\ 0 & x > t \end{cases} \tag{3.8}$$

then

$$L_t L_x \{g(x,t)\} = \frac{1}{s_2} \bar{f}(s_1 + s_2) \tag{3.9}$$

Proof: From (2.1), we have

$$\begin{aligned} L_t L_x \{g(x,t)\} &= \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 t} g(x,t) dx dt \\ &= \int_0^\infty \int_x^\infty e^{-s_1 x - s_2 t} f(x) dx dt \\ &= \int_0^\infty e^{-s_1 x} f(x) \int_x^\infty e^{-s_2 t} dt dx \\ &= \left\{ \int_0^\infty e^{-s_1 x} f(x) \left[-\frac{e^{-s_2 t}}{s_2} \right]_{t=x}^\infty \right\} dx \\ &= \int_0^\infty e^{-s_1 x} f(x) \frac{e^{-s_2 x}}{s_2} dx \\ &= \frac{1}{s_2} \int_0^\infty e^{-(s_1 + s_2)x} f(x) dx \end{aligned}$$

thus

$$L_t L_x \{g(x,t)\} = \frac{1}{s_2} \bar{f}(s_1 + s_2)$$

3.1.9 Corollary

if

$$g(x,t) = \begin{cases} f_1(x) & x < t \\ f_2(t) & x > t \end{cases} \tag{3.10}$$

and the Laplace $\bar{f}_1$ and $\bar{f}_2$ exists, then

$$\bar{g}(s_1, s_2) = \frac{1}{s_2}\bar{f}_1(s_1+s_2) + \frac{1}{s_1}\bar{f}_2(s_1+s_2) \tag{3.11}$$

Proof: straight forward.

3.1.10 Theorem

If $\bar{f}(s_1, s_2)$ exists then, the double Laplace transform of

$$g(x,t) = \begin{cases} f(x, t-x) & t > x \\ 0 & t < x \end{cases} \quad \text{is} \tag{3.12}$$

$\bar{f}(s_1+s_2, s_2)$

Proof: From (2.1), we have

$$L_t L_x\{g(x,t)\} = \int_0^\infty \int_x^\infty e^{-s_1 x - s_2 t} f(x, t-x)\,dx\,dt$$

let

$$u = t - x \Rightarrow t = u + x$$

we get

$$= \int_0^\infty \int_0^\infty e^{-s_1 x - s_2(u+x)} f(x,u)\,dx\,du$$

$$= \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 u - s_2 x} f(x,u)\,dx\,du$$

$$= \int_0^\infty \int_0^\infty e^{-(s_1+s_2)x - s_2 u} f(x,u)\,dx\,du$$

$$= \bar{f}(s_1+s_2, s_2)$$

Similarly, it can be proved that the double Laplace transform of

$$g(x,t) = \begin{cases} f(x-t, t) & x > t \\ 0 & x < t \end{cases}$$

is $\bar{f}(s_1, s_1+s_2)$

3.2. The Double Laplace Transform of Derivatives.

In this section we present the double Laplace transform of the partial derivatives of a functions.

3.2.1 Theorem (the double Laplace transform of the first order partial derivatives)

If $f(x,t)$ be a continuous function and its first order partial derivatives are of exponential order [11], then

$$L_t L_x\{\frac{\partial f(x,t)}{\partial x}\}(s_1,s_2)=s_1 L_t L_x\{f(x,t)\}-L_t\{f(0,t)\} \tag{3.13}$$

$$L_t L_x\{\frac{\partial f(x,t)}{\partial t}\}(s_1,s_2)=s_2 L_t L_x\{f(x,t)\}-L_x\{f(x,0)\} \tag{3.14}$$

respectively, where $x,t>0$

Proof: (3.13) by using the definition of the double Laplace transform

$$L_t\, L_x\left\{f(x,t)\right\}=\int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x}\; f\;(x,t)dxdt$$

we get

$$L_t\, L_x\left\{f(x,t)\right\}=\int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x}\; f_x(x,t)dxdt$$

$$=\int_0^\infty e^{-s_2 t}\left\{\int_0^\infty e^{-s_1 x}\; f_x(x,t)dx\right\}dt$$

$$=\int_0^\infty e^{-s_2 t}\left\{\int_0^\infty [\,e^{-s_1 x}\; f(x,t)\;\Big]_{x=0}^{\infty}-\int_0^\infty(-s_1)e^{-s_1 x}\; f(x,t)dx\right\}dt$$

$$=\int_0^\infty e^{-s_2 t}\left\{0-f(0,t)+s_1\int_0^\infty e^{-s_1 x}\; f(x,t)dx\right\}dt$$

$$=-\int_0^\infty e^{-s_2 t}f(0,t)dt+s_1\int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x}\; f(x,t)dxdt$$

$$=-L_t\{f(0,t)\}+s_1 L_t L_x\{f(x,t)\}$$

$$=s_1 L_t L_x\{f(x,t)\}-L_t\{f(0,t)\}$$

thus

$$L_t L_x\{\frac{\partial f(x,t)}{\partial x}\}(s_1,s_2) = s_1 L_t L_x\{f(x,t)\} - L_t\{f(0,t)\}\ .$$

Now I will prove (3.14) by using the definition of the double Laplace transform

$$L_t\, L_x\left\{f(x,t)\right\} = \int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x}\ f\ (x,t)dxdt$$

we get

$$\begin{aligned}
L_t\, L_x\left\{f(x,t)\right\} &= \int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x}\ f_t(x,t)dxdt \\
&= \int_0^\infty e^{-s_1 x}\left\{\int_0^\infty e^{-s_2 t} f_t(x,t)dt\right\}dx \\
&= \int_0^\infty e^{-s_1 x}\left\{\left[e^{-s_2 t} f(x,t)\ \right]_{t=0}^{\infty} - \int_{t=0}^{\infty}(-s_2)e^{-s_2 t} f(x,t)dt\right\}dx \\
&= \int_0^\infty e^{-s_1 x}\{0 - f(x,0) + s_2\int_{t=0}^{\infty} e^{-s_2 t} f(x,t)dt\}dx \\
&= -\int_0^\infty e^{-s_1 x} f(x,0)dx + s_2\int_{x=0}^{\infty} e^{-s_1 x}\int_{t=0}^{\infty} e^{-s_2 t} f(x,t)dtdx \\
&= -L_x\{f(x,0)\} + s_2 L_t L_x\{f(x,t)\} \\
&= s_2 L_t L_x\{f(x,t)\} - L_x\{f(x,0)\}
\end{aligned}$$

thus

$$L_t L_x\{\frac{\partial f(x,t)}{\partial t}\}(s_1,s_2) = s_2 L_t L_x\{f(x,t)\} - L_x\{f(x,0)\}.$$

3.2.2 Theorem(the double Laplace transform of the second order partial derivatives)

Let $f(x,t)$ be a continuous function of xponential order such that its second partial derivatives are continuous function of xponential order as well,[12] then

$$L_t L_x\{\frac{\partial^2 f(x,t)}{\partial^2 x}\} = s_1^{\ 2}\ L_t L_x\{f(x,t)\} - s_1 L_t\ \{f(0,t)\} - L_t\{\frac{\partial f(0,t)}{\partial x}\} \qquad (\ 3.15)$$

$$L_t L_x[\frac{\partial^2 f(x,t)}{\partial^2 t}] = s_2^{\ 2} L_t\ L_x\{f(x,t) - s_2 L_t\ \{f(x,0)\} - L_x\{\frac{\partial f(x,0)}{\partial t}\} \qquad (3.16)$$

$$L_t\, L_x\left\{\frac{\partial^2}{\partial x\,\partial t}\, f(x,t)\right\} = s_1 s_2\ \bar{f}(s_1, s_2) - s_1\ \bar{f}(s_1, 0) - s_2\ \bar{f}(0, s_2) + f(0,0) \qquad (3.17)$$

Proof: from (3.13) we get:-

$$L_t L_x\{f_{xx}(x,t)\} = s_1 L_t L_x\{f_x(x,t)\} - L_t\{f_x(0,t)\}$$

using (3.13) we get:

$$L_t L_x\{f_{xx}(x,t)\} = s_1[s_1 L_t L_x\{f(x,t) - L_t\{f(0,t)\}] - L_t\{f_x(0,t)\}$$

$$= s_1^2 L_t L_x\{f(x,t)\} - s_1 L_t\{f(0,t)\} - L_t\{f_x(0,t)\}$$

thus:-

$$L_t L_x\{\frac{\partial^2 f(x,t)}{\partial^2 x}\} = s_1{}^2\ L_t L_x\{f(x,t)\} - s_1 L_t\ \{f(0,t)\} - L_t\{\frac{\partial f(0,t)}{\partial x}\}.$$

From (3.14) , we get

$$L_t L_x\{f_{tt}(x,t)\} = s_2 L_t L_x\{f_t(x,t)\} - L_x\{f_t(x,0)\}$$

using (3.14) , we get

$$L_t L_x\{f_{tt}(x,t)\} = s_2[s_2 L_t L_x\{f(x,t)\} - L_x\{f(x,0)\}] - L_x\{f_t(x,0)\}$$

$$= s_2^2 L_t L_x\{f(x,t)\} - s_2 L_x\{f(x,0)\} - L_x\{f_t(x,0)\}$$

thus

$$L_t L_x[\frac{\partial^2 f(x,t)}{\partial^2 t}] = s_2^2 L_t\ L_x\{f(x,t) - s_2 L_x\ \{f(x,0)\} - L_x\{\frac{\partial f(x,0)}{\partial t}\}.$$

For the proof 0f (3.17) we will use the definition of the double Laplace transform of the mixd partial derivatives

$$L_t\ L_x\left\{\frac{\partial^2}{\partial x\,\partial t}\ f(x,t)\right\} = L_t\ L_x\left\{f_{xt}\ (x,t)\right\}$$

$$= \int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x}\ f_{xt}(x,t)dxdt$$

$$= \int_0^\infty e^{-s_1 x}\{\int_0^\infty e^{-s_2 t}\ f_{xt}(x,t)dt\}dx$$

$$= \int_{x=0}^\infty e^{-s_1 x}\left\{\ \left[e^{-s_2 t}\ f_x(x,t)\right]_{t=0}^\infty - \int_{t=0}^\infty e^{-s_2 t}(-s_2)\ f_x(x,t)dt\right\}dx$$

$$= \int_{x=0}^\infty e^{-s_1 x}\left\{0 - f_x(x,0) + s_2\int_{t=0}^\infty e^{-s_2 t}\ f_x(x,t)dt\right\}dx$$

$$= -\int_{x=0}^{\infty} e^{-s_1 x} f_x(x,0)dx + s_2 \int_{x=0}^{\infty} e^{-s_1 x} \int_{t=0}^{\infty} e^{-s_2 t} f_x(x,t)dtdx$$

$$= -\left\{ \left[e^{-s_1 x} f(x,0) \right]_{x=0}^{\infty} - \int_{x=0}^{\infty} e^{-s_1 x}(-s_1) f(x,0)dx \right\} + s_2 \int_{t=0}^{\infty} e^{-s_2 t} \left\{ \int_{x=0}^{\infty} e^{-s_1 x} f_x(x,t)dx \right\} dt$$

$$- \int_{x=0}^{\infty} e^{-s_1 x}(-s_1) f(x,t)dx \quad \Big\} dt$$

$$= f(0,0) - s_1 \bar{f}(s_1,0) - s_2 \int_{t=0}^{\infty} e^{-s_2 t} f(0,t)dt + s_1 s_2 \int_{t=0}^{\infty} e^{-s_2 t} \int_{x=0}^{\infty} e^{-s_1 x} f(x,t)dxdt$$

$$= f(0,0) - s_1 \bar{f}(s_1,0) - s_2 \bar{f}(0,s_2) + s_1 s_2 \bar{f}(s_1,s_2)$$

thus

$$L_t L_x \{ f_{xt}(x,t) \} = s_1 s_2 \bar{f}(s_1,s_2) - s_1 \bar{f}(s_1,0) - s_2 \bar{f}(0,s_2) + f(0,0) .$$

The previous theorem can be generalized as follows :

3.2.3 Theorem (the double Laplace transform of a general partial derivatives)

Let $f(x,t)$ and all of its partial derivatives $\frac{\partial^{i+j} f}{\partial t^i \partial x^j}$, $i = 0,1,....,m$, $j = 0,1,....,n$ be of exponential order then

$$L_t L_x \{ \frac{\partial^n}{\partial x^n} f(x,t) \} = s_1^n L_x L_t \{ f(x,t) \} - \sum_{i=0}^{n-1} s_1^{n-1-i} L_t \{ \frac{\partial^i}{\partial x^i} f(0,t) \} \qquad \text{for } n \geq 1 \qquad (3.18)$$

$$L_t L_X \{ \frac{\partial^m}{\partial t^m} f(x,t) \} = s_2^m L_x L_t \{ f(x,t) \} - \sum_{j=0}^{m-1} s_2^{m-j-1} L_x \{ \frac{\partial^j}{\partial t^j} f(x,0) \} \qquad \text{for } m \geq 1 \qquad (3.19)$$

$$L_t L_x \{ \frac{\partial^{m+n}}{\partial t^m \partial x^n} f(x,t) \} = s_1^n s_2^m [L_t L_x \{ f(x,t) \} - \sum_{j=0}^{m-1} s_2^{-j-1} L_x \{ \frac{\partial^j}{\partial t^j} f(x,0) \}$$

$$- \sum_{i=0}^{n-1} s_1^{-1-i} L_t \{ \frac{\partial^i}{\partial x^i} f(0,t) \} + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} s_1^{-1-i} s_2^{-j-1} \frac{\partial^{i+j}}{\partial t^j \partial x^i} f(0,0)] \qquad (3.20)$$

Proof: [11] we will use the mathematical induction to proof (3.18). (3.19) can be proved similarly for $n=1$,the formula is true from proposilion 3.2.1 suppose that the formula true for $p \leq n-1$

$$L_t L_x\left\{\frac{\partial^n}{\partial x^n} f(x,t)\right\} = L_t L_x\left\{\frac{\partial}{\partial x}\left(\frac{\partial^{n-1}}{\partial x^{n-1}} f(x,t)\right)\right\}$$

by on using proposilion 3.2.1 we have

$$L_t L_x\left\{\frac{\partial^n}{\partial x^n} f(x,t)\right\} = s_1 L_t L_x\left\{\frac{\partial^{n-1}}{\partial x^{n-1}} f(x,t)\right\} - L_t\left\{\frac{\partial^{n-1}}{\partial x^{n-1}} f(0,t)\right\}$$

$$= s_1\left[s_1^{n-1} L_t L_x\{f(x,t)\} - \sum_{n=0}^{n-2} s_1^{n-2-k} L_t\left\{\frac{\partial^k}{\partial x^k} f(0,t)\right\}\right] - L_t\left\{\frac{\partial^{n-1}}{\partial x^{n-1}} f(0,t)\right\}$$

$$= s_1^n L_t L_x\{f(x,t) - \sum_{n=0}^{n-1} s_1^{n-1-k} L_t\left\{\frac{\partial^k}{\partial x^k} f(0,t)\right\} .$$

$$L_t L_x\{\frac{\partial^{m+n}}{\partial t^m x^m} f(x,t)\} = L_t L_x\{\frac{\partial^m}{\partial t^m}(\frac{\partial^n}{\partial x^n} f(x,t))\}$$

by using (3.19) we get

$$L_t L_x\{\frac{\partial^{m+n}}{\partial t^m x^m} f(x,t)\} = s_2^m L_t L_x\{\frac{\partial^n}{\partial x^n} f(x,t)\} - \sum_{j=0}^{m-1} s_2^{m-j-1} L_x\{\frac{\partial^{j+n}}{\partial t^{j+n}} f(x,0)\} .$$

By using (3.18) now , we get

$$L_t L_x\{\frac{\partial^{m+n}}{\partial t^m x^m} f(x,t)\} = s_2^m [s_1^n L_t L_x\{f(x,t)\} - \sum_{i=0}^{n-1} s_1^{n-1-i} L_t\{\frac{\partial^i}{\partial x^i} f(0,t)\}]$$
$$- s_1^n \sum_{j=0}^{m-1} s_2^{m-j-1} L_x\{\frac{\partial^j}{\partial t^j} f(x,0)\} + \sum_{j=0}^{m-1} s_2^{m-j-1} \sum_{i=0}^{n-1} s_1^{n-i-1}\{\frac{\partial^{j+i}}{\partial t^j \partial x^i} f(0,0)\}$$

3.2.4 Theorem (the double Laplace transform of an integral)

if

$$L_t L_x\{f(x,t)\} = \bar{f}(s_1, s_2)$$

then

$$L_t L_x\left\{\int_0^x \int_0^t f(u,v)\,du dv\right\} = \frac{\bar{f}(s_1, s_2)}{s_1 s_2} \quad s_1 > 0 \text{ , } s_2 > 0$$

Proof: [10] Let

$$g(x,t)=\int_0^x \int_0^t f(u,v)\,dudv$$

hence we have

$$g_{xt}(x,t)=f(x,t)\,and\ g(0,0)=0$$

$$L_t L_x\{g_{xt}(x,t)\}=L_t L_x\{f(x,t)\}=\bar{f}(s_1,s_2)$$

From**(** theorem 3.2.3 **)** we have

$$L_t\,L_x\left\{\frac{\partial^2}{\partial x\,\partial t}g(x,t)\right\}=L_t\,L_x\{g_{xt}\,(x,t)\}$$

$$=s_1 s_1\,\bar{g}\,(s_1,s_2)-s_1\bar{g}\,(s_1,0)-s_2\,\bar{g}\,(0,s_2)+g(0,0)$$

thus we have

$$\bar{f}(s_1,s_2)=s_1 s_2\,\bar{g}(s_1,s_2)-s_1\bar{g}\,(s_1,0)-s_2\,\bar{g}(0,s_2)$$

$$\bar{g}\,(s_1,s_2)=\frac{1}{s_1 s_2}\bar{f}\,(s_1,s_2)+\frac{1}{s_2}\bar{g}(s_1,0)+\frac{1}{s_1}\bar{g}(0,s_2)$$

$$\bar{g}\,(s_1,s_2)=\frac{1}{s_1 s_2}\bar{f}\,(s_1,s_2)+\frac{1}{s_2}\,L\{g(x,0)\}+\frac{1}{s_1}L\{g(0,t)\}$$

but

$$L\{g(x,0)\}=0 \quad and \quad L\{g(0,t)\}=0$$

there fore

$$\bar{g}\,(s_1,s_2)=\frac{1}{s_1 s_2}\bar{f}\,(s_1,s_2)$$

hence

$$L_t L_x\left\{\int_0^x \int_0^t f(u,v)\,dudv\right\}=\frac{\bar{f}\,(s_1,s_2)}{s_1,s_2}$$

3.3 The Double Convolution

3.3.1 Definition (the double convolution)

The double convolution between two continuous functions $f(x,t)$ and $g(x,t)$ is defined by [13]

$$f(x,t)**g(x,t)=\int_0^x\int_0^t f(\upsilon,\tau)g(x-\upsilon,t-\tau)\,d\tau d\upsilon \qquad (3.21)$$

3.3.2 Theorem (commutativity)

$$f(x,t)**g(x,t) = g(x,t)**f(x,t) \tag{3.22}$$

Proof: [4]

$$f(x,t)**g(x,t) = \int_0^x \int_0^t f(\upsilon,\tau) g(x-\upsilon, t-\tau) d\tau d\upsilon$$

Let

$$u = x - \upsilon \Rightarrow \upsilon = x - u$$
$$w = t - \tau \Rightarrow \tau = t - w$$

$$\frac{\partial(\upsilon,\tau)}{\partial(u,w)} = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1$$

$$f(x,t)**g(x,t) = \int_0^x \int_0^t f(x-u, t-w) g(u,w) dw du$$
$$= g(x,t)**f(x,t)$$

The following theorem gives the double Laplace transform of the convolution of two functions .

3.3.3 Theorem (the double Laplace transform of convolution)

$$L_t L_x\{f(x,t)**g(x,t)\} = L_t L_x\{f(x,t)\} L_t L_x\{g(x,t)\}$$
$$= \overline{f}(s_1,s_2)\overline{g}(s_1,s_2) \tag{3.23}$$

Proof :[4]

$$L_t L_x\{f(x,t)**g(x,t)\}$$

$$= \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 t} \int_0^x \int_0^t f(v,\tau) g(x-v, t-\tau) d\tau dv dx dt$$

$$= \int_0^\infty \int_0^\infty \int_0^x \int_0^t e^{-s_1 x - s_s t} f(v,\tau) g(x-v, t-\tau) d\tau dv dx dt$$

If one uses the transformation

$$x = u + v$$

$$t = w + \tau$$

$$v = \upsilon$$

$$\tau = \tau$$

we get

$$L_t L_x\{f(x,t)**g(x,t)\} = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1(u+v)}e^{-s_2(w+\tau)} f(v,\tau)g(u,w)d\tau dvdudw$$

$$= \int_0^\infty\int_0^\infty e^{-s_1u-s_2w} g(u,w)dudw \int_0^\infty\int_0^\infty e^{-s_1v-s_2\tau} dvd\tau$$

$$= \overline{f}(s_1,s_2).g(s_1,s_2)$$

CHAPTER 4

APPLICATION OF DOUBLE LAPLACE TRANSFORM IN PARTIAL DIFFERENTIAL EQUATION

In this chapter we are going to solve some important problems for partial differential equations which is one of the most important subjects in mathematics and other sciences by using double Laplace transform**.**

4.1 Examples for Partial Differential Equations

It is well known that in order to obtain the solution of partial differential equations by integral transform methods we need the following two steps:

1- We transform the partial differential equations to algebraic equations by using double integral transform methods.

2- On using the double inverse transform to get the solution of PDEs [7].

4.1.1 Example (solving non-homogeneous wave equation with convolution term)

Apply double Laplace transform to solve non-homogeneous wave equation with convolution term, where the non-homogeneous term is double convolution in general case consider non-homogeneous wave equation in the form

$$k(x,t) ** (u_{tt} - u_{xx}) = f(x,t) ** g(x,t) \qquad (t,x) \in R_+^2 \tag{4.1}$$

where $k(x,t)$ is polynomial defined by $K(x,t) = \sum_{j=1}^{m} \sum_{i=1}^{n} x^j t^i$ and the symbol $**$ means the double convolution with respect to x and t under the conditions

$$u(x,0) = r_1(x) \qquad u_t(x,0) = r_1'(x) \tag{4.2}$$

$$u(0,t) = h_1(t) \qquad u_x(0,t) = h_1'(t) \tag{4.3}$$

[14] where the non-homogeneous term of Eq. (4.1) is double convolution terms and non-homogeneous initial condition are single convolution. If one applies the double Laplace transform to Eq. (4.1) and single Laplace transform to Eqs. (4.2) and (4.3) we obtain

$$U(s_2,s_1)=\frac{s_1R_1(s_2)}{(s_1^{\,2}-s_2^{\,2})}-\frac{s_2H_1(s_1)}{(s_1^{\,2}-s_2^{\,2})}+\frac{s_2R_1(s_2)-R_1(0)}{(s_1^{\,2}-s_2^{\,2})}-\frac{s_1H_1(s_1)-H_1(0)}{(s_1^{\,2}-s_2^{\,2})}$$

$$+\frac{\overline{f}(s_1,s_2)\overline{g}(s_1,s_2)}{(s_1^{\,2}-s_2^{\,2})K(s_2,s_1)} \tag{4.4}$$

where $R_1(s_2),H_1(s_1)$ are single Laplace transform of initial condition with respect to x,t respectively and $\overline{f}(s_1,s_2),\overline{g}(s_1,s_2)$ is double Laplace transform of f(t,x)** g(t,x). By taking inverse double Laplace transform of Eq. (4.4) we obtain the solution of Eq. (4.1) as follows

$$u(t,x)=L_{S_1}^{-1}L_{S_2}^{-1}\left[\frac{s_1R_1(s_2)}{(s_1^{\,2}-s_2^{\,2})}-\frac{s_2H_1(s_1)}{(s_1^{\,2}-s_2^{\,2})}+\frac{s_2R_1(s_2)-R_1(0)}{(s_1^{\,2}-s_2^{\,2})}\right]+L_{S_1}^{-1}L_{S_2}^{-1}\left[-\frac{s_1H_1(s_1)-H_1(0)}{(s_1^{\,2}-s_2^{\,2})}\right.$$

$$\left.+\frac{\overline{f}(s_1,s_2)\overline{g}(s_1,s_2)}{(s_1^{\,2}-s_2^{\,2})K(s_2,s_1)}\right] \tag{4,5}$$

4.1.2 Example (solving partial integrodifferential equation with boundary conditions)

Consider the following partial integrodifferential equation

$$f(x,t)=u_{tt}-u_{xx}+u+\int_0^x\int_0^t g(x-\alpha,t-\beta)u(\alpha,\beta)d\alpha d\beta \tag{4.6}$$

with boundary conditions

$$u(0,t)=f_1(t) \qquad u_x(0,t)=f_2(t) \tag{4.7}$$

and initial conditions

$$u(x,0)=g_1(x) \qquad u_t(x,0)=g_2(x) \tag{4.8}$$

[9] .By taking double Laplace transform for (4.6) and single Laplace transform for (4.7) and (4.8), we get

$$u(s_1,s_2)=\frac{s_1/G_1(s_1)+1/G_2(s_1)}{s_1^{\ 2}-s_2^{\ 2}+1+\overline{g}(s_1,s_2))}-\frac{s_1/F_1(s)-1/F_2(s)+\overline{f}(s_1,s_2)}{s_1^{\ 2}-s_2^{\ 2}+1+\overline{g}(s_1,s_2))} \tag{4.9}$$

by applying double inverse Laplace transform for (4.9), we obtain the solution of (4.6) in the following form

$$u(x,t)=L_t^{-1}L_x^{-1}[\frac{s_1/G_1(s_1)+1/G_2(s_1)}{s_1^{\ 2}-s_2^{\ 2}+1+\overline{g}(s_1,s_2))}-\frac{s_1/F_1(s)-1/F_2(s)+\overline{f}(s_1,s_2)}{s_1^{\ 2}-s_2^{\ 2}+1+\overline{g}(s_1,s_2))}] \tag{4.10}$$

We provide the double inverse Laplace transform existing for each terms in the right side of (4.10). In particular,consider the following example.

4.1.3 Example (solving the partial integrodifferential equation with conditions)

Consider the partial integro-differential equation

$$u_{tt}-u_{xx}+u+\int_0^x\int_0^t e^{x-\alpha+t-\beta}u(\alpha,\beta)d\alpha d\beta=e^{x+t}+xte^{x+t} \tag{4.11}$$

with conditions

$$u(x,0)=e^x \qquad u_t(x,0)=e^x$$
$$u(0,t)=e^t \qquad u_x(0,t)=e^t \tag{4.12}$$

[9]. By taking double Laplace transform for (4.11) and single Laplace transform for (4.12), we have

$$(s_2^{\ 2}-s_1^{\ 2}+1+\frac{1}{(s_1-1)(s_2-1)})u(s_1,s_2)$$
$$=\frac{s_2}{s_1-1}+\frac{1}{s_1-1}-\frac{s_1}{s_2-1}-\frac{1}{s_2-1}+\frac{1}{(s_1-1)(s_2-1)}+\frac{1}{(s_1-1)^2(s_2-1)^2} \tag{4.13}$$

by simplifying (4.13), we obtain

$$u(s_1,s_2)=\frac{1}{(s_1-1)(s_2-1)} \tag{4.14}$$

by using double inverse Laplace transform for (4.14),we obtain the solution of (4.11) as follows

$$u(x,t)=e^{x+t} \tag{4.15}$$

4.1.4 Example (solving the non-homogeneous wave equation)

Let us consider the non-homogeneous wave equation in the form

$$u_{tt}-u_{xx}=\frac{1}{2}e^{x+t}-\frac{1}{2}\cos(x)e^{t}-\frac{1}{2}e^{x}\cos(t)+\frac{1}{2}\cos(x+t) \tag{4.16}$$

$$u(0.x)=\alpha(x) \qquad u_{t}(0,x)=\alpha'(x) \tag{4.17}$$

$$u(t,0)=\alpha(t) \qquad u_{x}(t,0)=\alpha'(t) \tag{4.18}$$

where all the initial conditions have singularity at $x=0$ and $t=0$ and $(t,x)\in R_{+}^{2}$ Then it is easy to see that the non-homogeneous term of Eq. In the form of

$$u_{tt}(t,x)-u_{xx}(t,x)=f(t,x)**g(t,x)+h(x,t) \tag{4.19}$$

$(t,x)\in R_{+}^{2}$

can be written in the form

$$\sin(x+t)**e^{x+t}=\frac{1}{2}e^{x+t}-\frac{1}{2}\cos(x)e^{t}-\frac{1}{2}e^{x}\cos(t)+\frac{1}{2}\cos(x+t) \tag{4.20}$$

[14]. Now we apply the Laplace transform technique for Eq(4.19) and we obtain the solution of Eq.(4.19) in the form of

$$u(x,t)=\frac{1}{4}e^{x}\cos(t)-\frac{1}{4}e^{x}\cos(x)+\frac{1}{8}\sin(t-x)+\frac{1}{4}t\sin(x+t)+\frac{1}{4}\cos(x+t)+\frac{1}{4}e^{x+t}$$

$$+\frac{1}{4}e^{x+t}t+\frac{1}{8}\sin(x+t) \tag{4.21}$$

Now, if we consider to multiply the left-hand side equation of (4.19) by the non-constant coefficient $t^{3}x^{2}**$then Eq.(4.19) becomes

$$x^{2}t^{3}**(u_{tt}-u_{xx})=\frac{1}{2}e^{x+t}-\frac{1}{2}\cos(x)e^{t}-\frac{1}{2}e^{x}\cos(t)+\frac{1}{2}\cos(x+t) \tag{4.22}$$

$$u(0,x)=\alpha(x) \qquad u_{t}(0,x)=\alpha'(x)$$

$$u(t,0)=\alpha(t) \qquad u_{x}(t,0)=\alpha'(t) \tag{4.23}$$

by using the similar technique as above, we obtain the solution of Eq.(4.18) as

$$v(x,t)=\frac{1}{48}e^{x}\cos(t)-\frac{1}{48}e^{t}\sin(x)+\frac{1}{96}\cos(-x+t)+\frac{1}{16}e^{x+t}-\frac{1}{96}\cos(x+t)$$

$$-\frac{1}{16}\sin(x+t)-\frac{1}{48}\cos(x+t)t+\frac{1}{48}e^{x+t}t \tag{4.24}$$

if we take second derivatives with respect to t and x for Eq.(4.22), and taking the difference between the second derivatives and multiply the result by convolution $x^2t^3 **$ we obtain the non-homogeneous term plus a function $h(t,x)$ as

$$x^2t^3 **(v_{tt}-v_{xx}) = \sin(x+t)**e^{x+t}$$

$$+\frac{1}{2}t\cos(x)+\frac{1}{2}t^2\cos(x)+\frac{1}{2}t\sin(x)-\frac{1}{12}t^3e^x-\frac{1}{2}te^x-\frac{1}{2}t^2e^x+\frac{1}{12}t^3x^2+\frac{1}{6}t^3x$$

$$+\frac{1}{12}t^3\cos(x)-\frac{1}{12}t^3\sin(x)+\frac{1}{2}tx^2+\frac{1}{2}t^2x^2+\frac{1}{2}xt^2-\frac{1}{2}\cos(x+t)+\frac{1}{2}x\sin(t)$$

$$-\frac{1}{2}x^2e^t+\frac{1}{2}x\cos(t)+\frac{1}{2}x^2\cos(t)-\frac{1}{2}xe^t \tag{4.25}$$

That is in the form of $x^2t^3 **(v_{tt}-v_{xx}) = u_{tt}-u_{xx}+h(x,t)$ and one can easily obtain the function $h(x,y)$.

4.2. Some Boundary Value Problems

In this section we going to solve some boundary value problem by using double Laplace transform.

4.2.1 Example (solving a boundary value problem)

Let

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - \sin \pi x \ , \quad 0 < x < 1 \ , \ t > 0$$

for :-

1- $y(x,0^+) = 0 \qquad 0 < x < 1$

2- $y(0,t) = 0 \qquad t > 0$

3- $y(1,t) = 0 \qquad t > 0$

4- $y_t(x,0^+) = 0 \qquad 0 < x < 1$

[15]. Solution: Taking double Laplace transform

$$s^2\bar{y}(s_1,s_2)-s\bar{y}(s_1,0)-\bar{y}_t(s_1,0)$$

$$=s_1^2\bar{y}(s_1,s_2)-s_1\,\bar{y}(0,s_2)-\bar{y}_x(0,s_2)-\frac{\pi}{s_1^2+\pi^2}\frac{1}{s_2}$$

but

$$\bar{y}(s_1,0)=0 \quad , \quad \bar{y}(0,s_2)=0 \quad , \quad \bar{y}_x(0,s_2)=0$$

$$s^2\bar{y}(s_1,s_2)=s_1^2\bar{y}(s_1,s_2)-\bar{y}_x(0,s_2)-\frac{\pi}{s_1^2+\pi^2}\frac{1}{s_2}$$

$$\bar{y}(s_1,s_2)=\frac{1}{s_1^2-s_2^2}\bar{y}_x(0,s_2)-\frac{1}{s_1^2-s_2^2}\frac{\pi}{s_1^2+\pi^2}\frac{1}{s_2}$$

$$\bar{y}(s_1,s_2)=\frac{1}{s_1^2-s_2^2}\bar{y}_x(0,s_2)-\frac{\pi}{(s_1^2+\pi^2)(s_2^2+\pi^2)}\frac{1}{s_2}+\frac{\pi}{(s_1^2-s_2^2)(s_2^2+\pi^2)}\frac{1}{s_2}$$

$$\bar{y}(s_1,s_2)=\frac{1}{s_1^2-s_2^2}\{\bar{y}_x(0,s_2)+\frac{\pi}{(s_2^2+\pi^2)}\frac{1}{s_2}\}-\frac{\pi}{(s_1^2+\pi^2)(s_2^2+\pi^2)}\frac{1}{s_2}$$

$$\bar{y}(s_1,s_2)=\frac{1}{2s_2}\{\frac{1}{s_1-s_2}+\frac{1}{s_1+s_2}\}\{\bar{y}_x(0,s_2)+\frac{\pi}{(s_2^2+\pi^2)}\frac{1}{s_2}\}-\frac{\pi}{(s_1^2+\pi^2)(s_2^2+\pi^2)}\frac{1}{s_2}$$

using L_x^{-1}

$$\bar{y}(x,s_2)=\frac{1}{2s_2}\{e^{s_2x}+e^{-s_2x}\}\{\bar{y}_x(0,s_2)+\frac{\pi}{(s_2^2+\pi^2)}\frac{1}{s_2}\}-\sin\pi x\frac{1}{s_2(s_2^2+\pi^2)} \qquad (4.26)$$

as $x\to 1$, $\bar{y}(1,s_2)\to 0$

$$\bar{y}(1,s_2)=\frac{1}{2s_2}\{e^{s_2}+e^{-s_2}\}\{\bar{y}_x(0,s_2)+\frac{\pi}{(s_2^2+\pi^2)}\frac{1}{s_2}\}-\sin\pi x\frac{1}{s_2(s_2^2+\pi^2)}$$

$$0=\frac{1}{2s_2}\{e^{s_2}+e^{-s_2}\}\{\bar{y}_x(0,s_2)+\frac{\pi}{(s_2^2+\pi^2)}\frac{1}{s_2}\}-0$$

$$\bar{y}_x(0,s_2)=-\frac{\pi}{(s_2^2+\pi^2)}\frac{1}{s_2}$$

then (4.26) Become

$$\bar{y}(x,s_2)=-\sin\pi x\frac{1}{s_2(s_2^2+\pi^2)}$$

$$\bar{y}(x,s_2)=\sin\pi x\frac{1}{\pi^2}\{\frac{s_2}{s_2^2+\pi^2}-\frac{1}{s_2}\}$$

using L_t^{-1} we get

$$y(x,t) = (\sin \pi x)\frac{1}{\pi^2}\{\cos \pi t - 1\}$$

4.2.2 Example (finding the temperature at any point of a bar at any time)

A semi-infinite insulated bar which coincides with the x-axis, $x > 0$ is initially at temperature zero At $t = 0$, a quantity of heat is instantaneously generated at the point $x = a$ where $a > 0$. Find the temperature at any point of the bar at any time $t > 0$

Solution: The equation for heat conduction in the bar is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad x > 0, t > 0 \tag{4.27}$$

The fact that a quantity of heat is instantaneously generated at the point $x = a$ can be represented by the boundary condition

$$u(a,t) = QH(t) \tag{4.28}$$

where Q is a constant and $H(t)$ is the dirac delta function. Also, since the initial temperature is zero and since the temperature must be bounded [15], we have

$$u(x,0) = 0 \qquad , \qquad |u(x,t)| < M$$

Taking double Laplace transform on (4.27)

$$s_2\bar{u}(s_1,s_2) - \bar{u}(s_1,0)$$

$$= k\{s_1^2\bar{u}(s_1,s_2) - s_1\bar{u}(0,s_2) - \bar{u}_x(0,s_2)\}$$

since

$$u(x,0) = 0 \Rightarrow \bar{u}(s_1,0) = 0$$

$$s_2\bar{u}(s_1,s_2) = k\{s_1^2\bar{u}(s_1,s_2) - s_1\bar{u}(0,s_2) - \bar{u}_x(0,s_2)\}$$

$$(ks_1^2 - s_2)\bar{u}(s_1,s_2) = ks_1\bar{u}(0,s_2) + k\bar{u}_x(0,s_2)$$

$$\bar{u}(s_1,s_2) = \frac{ks_1}{(ks_1^2 - s_2)}\bar{u}(0,s_2) + \frac{k}{(ks_1^2 - s_2)}\bar{u}_x(0,s_2)$$

$$\bar{u}(s_1,s_2) = \frac{s_1}{\left(s_1 + \sqrt{\frac{s_2}{k}}\right)\left(s_1 - \sqrt{\frac{s_2}{k}}\right)}\bar{u}(0,s_2) + \frac{1}{\left(s_1 + \sqrt{\frac{s_2}{k}}\right)\left(s_1 - \sqrt{\frac{s_2}{k}}\right)}\bar{u}_x(0,s_2)$$

$$\bar{u}(s_1,s_2) = \frac{1}{2}\left\{\frac{1}{s_1 + \sqrt{\frac{s_2}{k}}} + \frac{1}{s_1 - \sqrt{\frac{s_2}{k}}}\right\}\bar{u}(0,s_2) + \frac{1}{2}\sqrt{\frac{k}{s_2}}\left\{\frac{1}{s_1 - \sqrt{\frac{s_2}{k}}} - \frac{1}{s_1 + \sqrt{\frac{s_2}{k}}}\right\}\bar{u}_x(0,s_2)$$

using L_x^{-1}

$$\bar{u}(x,s_2)=\frac{1}{2}\left\{e^{-\sqrt{\frac{s_2}{k}}x}+e^{\sqrt{\frac{s_2}{k}}x}\right\}\bar{u}(0,s_2)+\frac{1}{2}\sqrt{\frac{k}{s_2}}\left\{e^{\sqrt{\frac{s_2}{k}}x}-e^{-\sqrt{\frac{s_2}{k}}x}\right\}\bar{u}_x(0,s_2)$$

$$\bar{u}(x,s_2)=\frac{1}{2}\left\{\bar{u}(0,s_2)-\sqrt{\frac{k}{s_2}}\bar{u}_x(0,s_2)\right\}e^{-\sqrt{\frac{s_2}{k}}x}+\frac{1}{2}\left\{\bar{u}(0,s_2)+\sqrt{\frac{k}{s_2}}\bar{u}_x(0,s_2)\right\}e^{\sqrt{\frac{s_2}{k}}x} \qquad (4.29)$$

since $u(x,t)$ is bounded as $x\to\infty$ then $\bar{u}(x,s_2)$ is bounded as $x\to\infty$

from boundedness condition, we require

$$\frac{1}{2}\left\{\bar{u}(0,s_2)+\sqrt{\frac{k}{s_2}}\bar{u}_x(0,s_2)\right\}=0$$

$$\bar{u}_x(0,s_2)=-\sqrt{\frac{s_2}{k}}\bar{u}(0,s_2)$$

then (4.29) become

$$\bar{u}(x,s_2)=\frac{1}{2}\left\{\bar{u}(0,s_2)+\sqrt{\frac{k}{s_2}}\sqrt{\frac{s_2}{k}}\bar{u}(0,s_2)\right\}e^{-\sqrt{\frac{s_2}{k}}x}$$

$$\bar{u}(x,s_2)=\bar{u}(0,s_2)e^{-\sqrt{\frac{s_2}{k}}x} \qquad (4.30)$$

since

$$u(a,t)=QH(t)\Rightarrow\bar{u}(a,s_2)=Q$$

As $x\to a$ In (4.30) then (4.30) Become

$$\bar{u}(a,s_2)=\bar{u}(0,s_2)e^{-\sqrt{\frac{s_2}{k}}a}$$

$$Q=\bar{u}(0,s_2)e^{-\sqrt{\frac{s_2}{k}}a}$$

$$\bar{u}(0,s_2)=Qe^{\sqrt{\frac{s_2}{k}}a}$$

then (4.30) become

$$\bar{u}(x,s_2)=Qe^{-(x-a)\sqrt{\frac{s_2}{k}}}$$

using L_t^{-1} we find the required temperature

$$u(x,t)=\frac{Q}{2\sqrt{\pi kt}}e^{\frac{-(x-a)^2}{4kt}}$$

The point source $x=a$ is some times called a heat source of strength Q.

CHAPTER 5

CONCLUSION

Many physical processes in nature evolve with time in semi infinite or infinite domains. Since these processes are described by both ordinary and partial differntial equations, solving such equations is of great importance. Since the Laplace transforms transform a differential equation to an algebraic one, the double Laplace transform is considered to be a strong tool for solving partial differential equations that appear in various fields of science and engineering. Besides,the Laplace transforms method is considered to be the easiest methods used to solve such equations because unlike the other methods used less and uncomplicated calculations are needed.

This thesis can be considered as a survey on double Laplace transform. and we believe that this thesis will be a reference for all scientists who want to use double Laplace transform to solve linear partial differential equation which they encounter in their scientific researches.

REFERENCES

1. **Debnath L., Bhatta D., (2007),** "*Integral Transforms and Their Applications*", Taylor & Francis Group, LLC, London, pp. 207-209, 212-215.

2. **Duffy D. G., (2004),** "*Transform Methods for Solving Partial Differential Equations*", Chapman and Hall/CRC, Boca Raton, pp. 7-12.

3. **Davies B., (2001)**, "*Integral Transforms and Their Applications*", Springer-Verlag New York, USA, vol. 4, pp. 133-151.

4. **Ditkin V. A., Prudnikov A. P., (1962),** "*Operational Calculus in Two Variables and its Applications*", Pergamon Press LTD , Oxford, London W.I., pp. 3-28., 30-31.

5. **Brychkov Y. A., Glaeske H. J., Prudnikov A. P., Tuan V. K., (1992),** "*Multidimensional Integral Transformations*", Gordon and Breach Science Publishers, Paris, pp. 10-13.

6. **Kilicman A., Eltayeb H., (2007),** "*An Application of Double Laplace Transform to Partial Differential Equations*", Symposium Kebangsaan Sains Matematik, Universiti Teknologi, Shah Alam, Malaysia, pp. 335-336.

7. **Kılıcman A., Eltayeb H., (2010),** "*A Note on Integral Transforms and Partial Differential Equations*", Applied Mathematical Sciences, vol. 4, no. 3, pp. 109-118.

8. **Aghili A., Moghaddam B. P., (2011),** "*Certain Theorems on two Dimensional Laplace Transform and non Homogeneous Partial Differential Equations*", Surveys in Mathematics and its Applications, vol. 6, pp. 166-167.

9. **Eltayeb H., Kiliçman A.**, **(2013)**, "*A Note on Double Laplace Transform and Telegraphic Equations*", Hindawi Publishing Corporation Abstract and Applied Analysis, vol. 2013, pp. 2-5.

10. **Dhunde R. R., Bhondge N. M., Dhongle P. R., (2013),** "*Some Remarks on the Properties of Double Laplace Transforms*", International Journal of Applied Physics and Mathematics, vol. 3, no. 4, pp. 293-295.

11. **Anwar A. M., Jarad F., Baleanu D., Ayaz F., (2013),** "*Fractional Caputo Heat Equation Within the Double Laplace Transform*", Romanian Journal of Physics., vol. 58, no. 1-2, Bucharest, pp. 18-20.

12. **Elzaki T. M., Hilal E. M., (2012),** "*Solution of Telegraph Equation by Modified of Double Sumudu Transform "Elzaki Transform*", Mathematical Theory and Modeling ISSN, vol. 2, no. 4, pp. 95-96.

13. **Kashuri A., Fundo A., Liko R., (2013),** "*On Double New Integral Transform and Double Laplace Transform*", European Scientific Journal, vol. 9, no. 33, pp. 83-84.

14. **kilicman A., Eltayeb H.**, **(2009),** "*A note on Defining Singular Integral as Distribution and Partial Differential Equations with Convolution Term*", Mathematical and Computer Modelling 49, pp. 335-336.

15. **Dhunde R. R,. Waghmare G. L., (2013),** "*Applications of Double Laplace Transform to Boundary Value Problems*", IOSR Journal of Mathematics (IOSR-JM), vol. 9, pp. 58-60.

APPENDICES A

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