SOME SPECTRAL METHODS WITH APPLICATIONS FOR THE NUMERICAL SOLUTIONS OF PARTIAL DIFFERERENTIAL EQUATIONS

TUĞBA ALTUNALAN

AUGUST 2016

SOME SPECTRAL METHODS WITH APPLICATIONS FOR THE NUMERICAL SOLUTIONS OF PARTIAL DIFFERERENTIAL EQUATIONS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES OF
ÇANKAYA UNIVERSITY

BY
TUĞBA ALTUNALAN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
IN
THE DEPARTMENT OF
MATHEMATICS AND COMPUTER SCIENCE

AUGUST 2016

Title of the Thesis: **Some Spectral Methods with Applications for the Numerical Solutions of Partial Differential Equations**

Submitted by **Tuğba ALTUNALAN**

Approval of the Graduate School of Natural and Applied Sciences, Çankaya University.

Prof. Dr. Halil Tanyer EYYÜBOĞLU
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Özlem DEFTERLİ
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Özlem DEFTERLİ
Supervisor

Examination Date: 01.08.2016

Examining Committee Members

Prof. Dr. Gerhard-Wilhelm WEBER	(METU)	
Assist. Prof. Dr. Özlem DEFTERLİ	(Çankaya Univ.)	
Assist.Prof. Dr. Dumitru BALEANU	(Çankaya Univ.)	

STATEMENT OF NON-PLAGIARISM PAGE

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name	:	Tuğba ALTUNALAN
Signature	:	
Date	:	01.08.2016

ABSTRACT

SOME SPECTRAL METHODS WITH APPLICATIONS FOR THE NUMERICAL SOLUTIONS OF PARTIAL DIFFERERENTIAL EQUATIONS

ALTUNALAN, Tuğba
M.Sc., Department of Mathematics and Computer Science
Supervisor: Assist. Prof. Dr. Özlem DEFTERLİ

August 2016, 85 pages

This thesis examines how some classes of partial differential equations are solved numerically by using a set of spectral methods, namely Spectral Collocation method and Spectral Galerkin method. It begins with a brief reminder about orthogonal polynomials and fundamentals of partial differential equations. Then, the description of the considered spectral methods are given in the following part of the thesis. The use of spectral methods for solving real-world problems which are modelled by partial differential equations, is given through some illustrative examples which appears in different fields of science and engineering. Therefore, the usability, efficiency and the importance of the studied spectral methods are emphasized. In these recently studied representative examples, which are reviewed from various application areas, some new and very performant mathematical tools were involved, namely fractional differential operators. The obtained numerical results from the mentioned spectral methods were presented through simulations with a comparative study by some other numerical methods in the literature.

Keywords: Partial Differential Equations, Spectral Methods, Orthogonal Polynomials, Approximate Solutions

ÖZ

KISMİ TÜREVLİ DENKLEMLERİN SAYISAL ÇÖZÜMLERİ İÇİN BAZI SPEKTRAL YÖNTEMLER VE UYGULAMALARI

ALTUNALAN, Tuğba
Yüksek Lisans, Matematik-Bilgisayar
Tez Yöneticisi: Yrd. Doç. Dr. Özlem DEFTERLİ

Ağustos, 2016, 85 sayfa

Bu tezde, kısmi türevli denklemlerin bazı sınıflarının bir dizi spektral yöntemler kullanılarak nasıl sayısal olarak çözüldüğü incelenmiştir. Bu yöntemler sırasıyla; Spektral Sıralama yöntemi ve Spektral Galerkin yöntemidir. İlk olarak, ortogonal polinomlar ve kısmi türevli denklemlerin temelleri hakkında kısa bir hatırlatma yapılmıştır. Tezin bir sonraki kısmında, ele alınan spektral yöntemlerin tanımlamaları verilmiştir. Bilim ve mühendisliğin farklı alanlarında ortaya çıkan ve kısmi türevli denklemlerle modellenebilen gerçek dünya problemlerinin çözümünde, spektral yöntemlerin kullanımı ve önemi açıklayıcı örnekler kullanılarak verilmiştir. Böylelikle, çalışılan spektral yöntemlerin kullanımı, verimi ve önemi vurgulanmıştır. Çeşitli uygulama alanlarından derlenen ve yakın zamanda çalışılan bu önemli örneklerde, kesirli türevli operatörler olarak adlandırılan yeni ve çok kullanışlı matematiksel araçlar dahil edilmiştir. Bahsi geçen spektral yöntemlerin kullanımından elde edilmiş olan sayısal sonuçlar grafiklerle verilmiş, literatürdeki diğer bazı sayısal yöntemlerle yapılan karşılaştırılması sunulmuştur.

Keywords: Kısmi Türevli Denklemler, Spektral Yöntemler, Ortogonal Polinomlar, Yaklaşık Çözümler

ACKNOWLEDGEMENTS

I would like to express my special thanks to my supervisor Assist. Prof. Dr. Özlem Defterli for her expertise, guidance and very useful suggestions during my thesis study. She was very patient and supportive through the research process.

I had an opportunity to take some of my master courses from Prof. Dr. Billur Kaymakçalan and Prof. Dr. Dumitru Baleanu, therefore I deeply thank them for sharing their experience and knowledge with me during my master education. I also would like to take this opportunity to thank to the thesis defense committe members, for their valuable remarks and corrections.

I am very glad to have the full support and understanding of my family members; İlknur Altunalan, Mustafa Altunalan and Tolga Altunalan. I always appreciate their efforts to provide me the opportunity to have this graduate study. Finally, I would like to thank to my dear friends Selin, Kardelen, Zeynep, İzgen, Şirin, Birdem Ayfer and Elif for their care, love, patience and moral support.

TABLE OF CONTENTS

CHAPTERS:

LIST OF FIGURES

FIGURES

LIST OF TABLES

TABLES

LIST OF ABBREVIATIONS

PDE	Partial Differential Equation
BVP	Boundary Value Problem
IVP	Initial Value Problem
ODE	Ordinary Differential Equation
S-FFPE	Space-Fractional Fokker-Planck Equation
INAM	Implicit Numerical Approximation Method
MAE	Maximum Absolute Error
T-FFPE	Time-Fractional Fokker-Planck Equation
FFVP	Fractional Final Value Problem
FIVP	Fractional Initial Value Problem
FSLP	Fractional Sturm-Liouville Problem
RFSLP	Regular Fractional Sturm-Liouville Problem
FFT	Fast Fourier Transform
LGL	Legendre Gauss Lobatto
PG	Petrov Galerkin
FE	Finite-Element
FD	Finite-Difference

CHAPTER 1

INTRODUCTION

Numerical analysis is the collection of methods for computing numerical data. In some cases, this results producing a sequence of approximations by repeating the procedure several times. Numerical methods for Partial Differential Equations (PDEs) can be classified into the local and global cases. The Finite-Difference (FD) and Finite-Element (FE) methods are depend on local arguments, while the spectral methods can give much higher accuracy, at the expense of domain flexibility.

Spectral methods were in common use during the last couple decades in order to find the numerical solutions of PDEs since they provide better accuracy in comparison to FD and FE methods. The rate of convergence of spectral approximations based on the smoothness of the solution which gives as the possibility to reach high accuracy with a small number of data. This property is called as ‘spectral accuracy’. On the other hand, in FD and FE methods, the order of accuracy is based on some fixed negative power of N, where N is the number of considered grid points. Spectral methods have various meanings depending on the application areas like Functional Analysis, Signal Processing, etc.

The numerical solutions obtained by the use of Spectral methods are denoted as a finite expansion of some set of basis functions. When the PDEs is written in terms of the coefficients of this expansion, the method is known as a Galerkin Spectral Method. Spectral Collocation Methods (which also known as pseudospectral methods), is another subclass of spectral methods and they are similar to Finite Differences Methods because of the direct use of grid points. These grid points are called ‘collocation points’. The other type of method class is the Tau Spectral Methods which are similar to the Galerkin Spectral Methods, however the expanding basis does not have to obey the boundary conditions requiring additional equations.

Recently, many researchers have started to study the spectral methods. The purpose of their studies is to find the numerical solutions close to exact solutions of partial differential equations. Although there are several methods for this manner, the selection of a suitable method for the numerical solutions of partial differential equation can be difficult since method differs in itself.

In this thesis study, our aim is to give a fundamental knowledge about some spectral methods which are widely in use, and also to present some new results indicating their performance through four different new applications from various fields by using fractional operators as a new mathematical aspect. The structure of this thesis is as follows. In Chapter 1, the main definitions for orthogonal polynomials are reviewed. Based on this knowledge, two main spectral methods are presented in Chapter 2. Moreover, a preliminary about PDEs is given in Appendix part. Approximate methods for certain PDEs are reviewed in details, containing Spectral Collocation Method and Spectral Galerkin Methods. In Chapter 3, some recent results in the literature are presented by the study of four different real-world applications using the new aspect of fractional calculus and the corresponding numerical comparative study is given among two of them. Chapter 4 is dedicated to my concluding remarks.

1.1 Preliminaries About Orthogonal Polynomials

1.1.1 Definition and Construction

Orthogonal Polynomials are essential polynomials in the construction and analysis of spectral methods. Therefore, they are important to investigate and comprehend some fundamental specifications of orthogonal polynomials in the weighted Sobolev space $L^2_w(a,b)$ if [1]

$$< f, g >:= (f, g)_w := \int_a^b w(x) f(x) \overline{g(x)} dx = 0, \quad \textbf{(1.1)}$$

where w is a fixed positive weight function in (a, b). It manage to substantiated that $< \; , \; >$ described above is an inner product in $L_w^2(a, b)$.

A sequence of orthogonal polynomials is a sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials with $\deg(p_n) = n$, and having the property

$$< p_i, p_j >= 0 \qquad \text{for} \qquad i \neq j. \quad \textbf{(1.2)}$$

Since orthogonality was not shifted by multiplying a nonzero constant it has been normalized the polynomial p_n so that the coefficient of x^n is one, i.e. $p_n(x) = x^n + a_{n-1}^{(n)} x^{n-1} + \cdots + a_0^{(n)}$ which is called monic polynomial.

The orthogonal polynomials can be produced by the following repetition relation [1]:

$$\begin{cases} p_0 = 1 \\ p_1 = x - \alpha_1 \\ \dots . \\ p_{n+1} = (x - \alpha_{n+1}) p_n - \beta_{n+1} \, p_{n-1} \,, n \geq 1 \end{cases}$$

where

$$\alpha_{n+1} = \int_a^b xwp_n^2 \, dx \Big/ \int_a^b wp_n^2 dx \quad \text{and} \quad \beta_{n+1} = \int_a^b xwp_n p_{n-1} dx \Big/ \int_a^b wp_{n-1}^2 dx . \quad \textbf{(1.3)}$$

Example 1.1: As an example consider $w(x) = 1$ and $(a, b) = (0,1)$. Using the Gram-Schmidt process the orthogonal polynomials can be produced as follows: Start with the sequence $\{1, x, x^2\}$. Choose $p_0(x) = 1$. Then $p_1(x)$ can be derived as

$$p_1 = x - \frac{<x,1>}{<1,1>} = x - \frac{1}{2} \quad \text{by} \quad p_1(x) = \frac{<x, p_0(x)>}{<p_0(x), p_0(x)>} \;, \quad \textbf{(1.4)}$$

since

$$< 1,1 >= \int_0^1 dx = 1 \qquad \text{and} \qquad < x, 1 > = \int_0^1 xdx = \frac{1}{2} \quad \textbf{(1.5)}$$

Futher, from the following recurrence relation $p_2(x)$ is obtained as,

$$p_2(x) = x^2 \frac{<x^2,p_0(x)>}{<p_0(x),p_0(x)>}\ p_0(x) - \frac{<x^2,p_1(x)>}{<p_1(x),p_1(x)>}\ p_1(x) \qquad \textbf{(1.6)}$$

$$= x^2 - \frac{<x^2,1>}{<1,1>} - \frac{<x^2,x-\frac{1}{2}>}{<x-\frac{1}{2},x-\frac{1}{2}>}\left(x-\frac{1}{2}\right) = x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right)$$

$$= x^2 - x - \frac{1}{6},$$

since

$$< x^2, 1 >= \int_0^1 x^2 dx = \frac{1}{3}\ , \qquad < x^2, x - \frac{1}{2} > = \int_0^1 x^2(x-\frac{1}{2})dx \qquad \textbf{(1.7)}$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}\ ,$$

and

$$< x - \frac{1}{2}, x - \frac{1}{2} > = \int_0^1 (x-\frac{1}{2})^2 dx = \frac{1}{12}. \qquad \textbf{(1.8)}$$

The polynomials $p_0(x) = 1$, $p_1 = x - 1/2$ and $p_2(x) = x^2 - x + 1/6$ are three main monic orthogonal polynomials on the interval (0,1) considering weight function $w(x) = 1$. By repeating this process, it is acquired that

$$p_3(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\ ,$$

$$p_4(x) = x^4 - 2x^3 + \frac{9}{7}x^2 - \frac{2}{7}x + \frac{1}{70}\ ,$$

$$p_5(x) = x^5 - \frac{5}{2}x^4 + \frac{20}{9}x^3 - \frac{5}{6}x^2 + \frac{5}{42}x - \frac{1}{252}\ , \text{ and so on.}$$

The orthonormal polynomials would be $q_0 = p_0(x)/\sqrt{h_0} = 1$, like

$$q_3 = \frac{p_3(x)}{\sqrt{h_3}} = 20\sqrt{7}(x^3 - \frac{3}{2}x^2 + \frac{3}{5}\text{x} - \frac{1}{20}), \quad \text{etc.}$$

Gram-Schmidt Algorithm

$$\text{W} = \frac{<w,\text{v}_1>}{\|\text{v}_1\|}\text{v}_1 + \cdots + \frac{<w,\text{v}_\text{n}>}{\|\text{v}_\text{n}\|^2}\text{v}_\text{n}. \qquad \textbf{(1.9)}$$

Given an arbitrary basis $\{u_1, u_2, .., u_n\}$ for an n - dimensional inner product space V, the Gram-Schmidt Algorithm constructs an orthogonal basis $\{v_1, v_2, .., v_n\}$ for v.

→Let $v_1 = u_1$,

→Let $v_2 = u_2 - proj_{w1}^{u2} = u_2 - \frac{<u_2,v_1>}{\|v_1\|^2} v_1$ where w_1 is the space spanned by v_1 , and $proj_{w_1}^{u_2}$ is the orthogonal projection of u_2 on w_1

Theorem 1 .1:

A sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfies [1]

$$p_{n+1}(x) = (A_n x + B_n) p_n(x) + C_n p_{n-1}(x), (n = 1,2,3 \dots), \quad \textbf{(1.10)}$$

where

$$A_n = \frac{k_{n+1}}{k_n}, \quad n = 0,1,2 \dots \quad \text{and} \quad C_n = \frac{-A_n}{A_{n-1}} - \frac{h_n}{h_{n-1}}, \ (n = 1,2,3,..). \quad \textbf{(1.11)}$$

Theorem 1.2:

A sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfies [1]

$$\sum_{k=0}^{n} \frac{p_k(x) p_k(y)}{h_k} = \frac{k_n}{h_n k_{n+1}} \times \frac{p_{n+1}(x) p_n(y) - p_{n+1}(y) p_n(x)}{x-y}, \quad \textbf{(1.12)}$$

$(n = 0,1,2, \dots)$ and

$$\sum_{k=0}^{n} \frac{\{p_k(x)\}^2}{h_k} = \frac{k_n}{h_n k_{n+1}} \times (p'_{n+1}(x) p_n(x) - p_{n+1}(x) p'_n(x)). \quad \textbf{(1.13)}$$

Zeros of Orthogonal Polynomials

The zeros of the orthogonal polynomials play a highly significant role in the construction of spectral methods.

Theorem 1.3:

If $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials on the interval (a, b) with compared to the weight function $w(x)$, then the polynomial $p_n(x)$ has completely n real simple zeros in the interval (a, b) [1].

Theorem 1.4:

If $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials on the interval (a, b) with compared to the weight function $w(x)$, the the zeros of $p_n(x)$ and $p_{n+1}(x)$ separate each other [1].

Gauss Quadrature

Let $x_0, x_1, \ldots, x_n$ be the zeros of p_{N+1} .Then, the linear system [2]

$$\sum_{j=0}^{N} p_k(x_j)\, w_j = \int_a^b p_k(x) w(x)\, dx, \quad \text{for all} \quad (p \in P_{2N+1}). \qquad \textbf{(1.14)}$$

The Gauss quadrature is the most accurate formula not possible to find x_j, w_j such that Eq.(**1.14)** holds for all polynomials $p \in P_{2N+2}$. Although, considering the well known result **(*)** it sets off collocation points $\{x_i\}$ does not contain the endpoint a or b, so it may create difficulties for boundary value problems. The second one is the Gauss-Radau quadrature related to the roots of the polynomial

$$g(x) = p_{N+1}(x) + \alpha\, p_N(x), \qquad \textbf{(1.15)}$$

where α is a fixed and $q(\omega) = 0$. It can be ascertained that $q(x)/(x-\omega)$ is orthogonal to all polynomials of degree less than or equal to $N-1$ in $L^2 w(a, b)$ with $\widetilde{w} = w(x)(x-\omega)$. Hence, the N roots of $g(x)/(x-\omega)$ are all real, simple lying in (a, b).

Gauss-Radau Quadrature

Let $x_0 =$ ω and $x_1, \ldots, x_N$ be the zeroes of $g(x)/(x-\omega)$, where $g(x)$ is described by Eq.**(1.15)**. Then, the linear system Eq.**(1.14)** approves a unique solution $(w_0, w_1, \ldots, w_N)^t$ with $w_j > 0$ for $(j = 0,1, .. N)$ [2].

Furthermore,

$$\sum_{j=0}^{N} p(x_j) w_j = \int_a^b p(x) w(x)\, dx \quad \text{for all} \quad (p \in P_{2N}). \qquad \textbf{(1.16)}$$

(*): When $\{p_n(x)\}_{n=0}^{\infty}$ is a orthogonal monic polynomials then $p_{n+1}(x)$ be also orthogonal to any polynomial p_q,with $q \leq n$.

Suchlike, a Gauss-Radau quadrature can be executed by fixing $X_N = b$. Herewith, the Gauss-Radau quadrature is appropriate for problems with one boundary point. Since the set of collocation points included the two end points, the Gauss-Lobatto quadrature the most prevalently has been used in spectral approximations. Here, considering the polynomial

$$g(x) = p_{N+1} + \alpha\, p_N(x) + \beta\, p_{N-1}(x), \tag{1.17}$$

where α and β are chosen so that $g(\omega) = g(b) = 0$. One can verify that $g(x)/((x-a)(x-b))$ is orthogonal to all polynomials of degree less than or equal to $N-2$ in $L^2\hat{w}(a,b)$ with $\hat{w}(x) = w(x)(x-a)(x-b)$. Hence, the $N-1$ zeroes of $g(x)/((x-a)(x-b))$ are all real simple and lie in (a,b) [3].

Gauss-Lobatto Quadrature

Let $x_0 = a$, $x_N = b$ and $x_1, \dots, x_{N-1}$ be the $N-1$ roots of $g(x)/((x-a)(x-b))$, where $g(x)$ is described by Eq.(**1.17**). Then the linear system Eq.**(1.14)** approves a unique solution $(w_0, w_1, \dots, w_N)^t$, with $w_j > 1$ for $(j = 0,1, \dots N)$ [3].

Furthermore,

$$\sum_{j=0}^{N} p(x_j) w_j = \int_a^b p(x) w(x)\, dx \ \text{ for all } \ (p \in P_{2N-1}). \tag{1.18}$$

If f is a continuous function on (a,b) and $x_1 < x_2 < \cdots < x_n$ are n distinct points in (a,b), also there exist exactly one polynomial P with degree $\leq n-1$ such that $P(x_i) = f(x_i)$ for all $(i = 1,2, \dots, n)$.

This polynomial P can uncomplicatedly be found by using Lagrange interpolation as follows:

Describe [3]

$$p(x) = (x-x_1)(x-x_2) \dots (x-x_n) \ \text{ and consider}$$

$$p(x) = \sum_{i=1}^{n} f(x_i) \frac{p(x)}{(x-x_i)p'(x_i)},$$

$$= \sum_{i=1}^{n} f(x_i) \frac{(x-x_1) \dots \dots (x-x_{i-1})(x-x_{i+1}) \dots (x-x_n)}{(x_i-x_1) \dots \dots (x_i-x_{i-1})(x_i-x_{i+1}) \dots (x_i-x_n)}. \tag{1.19}$$

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of orthogonal polynomials on the interval (a, b) with compared to the weight function $w(x)$. Then for $x_1 < x_2 < \cdots < x_n$ take the n distinct real zeros of the polynomial $p_n(x)$. If f is a polynomial of degree $\leq 2n - 1$, then $f(x) - p(x)$ is a polynomial of degree $\leq 2n - 1$ with at least the zeros $(x_1 < x_2 < \cdots < x_n)$.

Now, defining to this,

$$f(x) = p(x) + r(x)p_n(x), \tag{1.20}$$

where $r(x)$ is a polynomial of degree $\leq n - 1$. It can be also written as

$$f(x) = \sum_{i=1}^{n} f(x_i) \frac{p_n(x)}{(x - x_i)p_n'(x_i)} + r(x)p_n(x), \tag{1.21}$$

It result that

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{n} f(x_i) \int_a^b \frac{w(x)p_n(x)}{(x - x_i)p_n'(x_i)} dx + \int_a^b w(x)r(x)p_n(x)dx. \tag{1.22}$$

Since degree $[r(x)] \leq n - 1$ the orthogonality specialty states that the latter integral equals the zero. Indicating that

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{n} \lambda_{n,i} f(x_i), \qquad \text{with} \tag{1.23}$$

$$\lambda_{n,i} = \int_a^b \frac{w(x)p_n(x)}{(x - x_i)p_n'(x_i)} dx, \quad (i = 1,2, \dots, n) \tag{1.24}$$

This is the *Gauss Quadrature* formula. It gives the value of the integral supposing that f is a polynomial of degree $\leq 2n - 1$ if the value of $f(x_i)$ is known for the n zeros $x_1 < x_2 < \cdots < x_n$ of the polynomial $p_n(x)$. If f is not a polynomial of degree $\leq 2n - 1$ it results in an approximation of the integral [3]:

$$\int_a^b w(x)f(x)dx \approx \sum_{i=1}^{n} \lambda_{n,i} f(x_i) \text{ , with} \tag{1.25}$$

$$\lambda_{n,i} := \int_a^b \frac{w(x)p_n(x)}{(x - x_i)p_n'(x_i)} dx. \tag{1.26}$$

The coefficients $\{\lambda_{n,i}\}_{i=1}^{n}$ are called as Christoffel Numbers. Considering that these numbers does not hold on the function f. These Christoffel numbers are all positive and this can be indicated as follows. Having to

$$\lambda_{n,i} = \int_a^b w(x) l_{n,i}(x) dx \quad \text{with} \quad l_{n,i}(x) := \frac{P_n(x)}{(x - x_i) p_n'(x_i)}\ ,\ i = 1,2,\dots n, \qquad \textbf{(1.27)}$$

then $l_{n,i}^2(x) - l_{n,i}(x)$ is a polynomial of degree $\leq 2n - 1$ which vanishes at the zeros $\{x_{n,k}\}_{k=1}^{n}$ of $p_n(x)$. Hence

$l_{n,i}^2(x) - l_{n,i}(x) = p_n(x)\, q(x)$ for some polynomial q of degree $\leq 2n - 1$

This indicate that

$$\int_a^b w(x)(l_{n,i}^2(x) - l_{n,i}(x))dx = \int_a^b w(x) p_n(x)\, q(x)\, \mathrm{dx} = 0, \qquad \textbf{(1.28)}$$

by orthogonality. Hence we have

$$\lambda_{n,i} = \int_a^b w(x) l_{n,i}(x) dx = \int_a^b w(x)\{l_{n,i}(x)\}^2\, \mathrm{dx} > 0. \qquad \textbf{(1.29)}$$

Theorem 1.5:

Let $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials on the interval (a, b) with compared to the weight function $w(x)$ and let $m < n$. Then, between any two zeros of $p_m(x)$ there is at least one zero of $p_n(x)$ [3].

Advantages and Disadvantages of Gaussian Quadratures

Select both these weights and locations so that a higher order polynomial can be integrated

- Functional values must now be calculated at non-uniformly spanned points in order to have better accuracy,
- Weights are no more simple numbers,
- Generally obtained at $[-1,1]$, for other intervals [a,b] a mapping to [-1,1] can be considered,
- The process is usually non-linear itself,
- An appropriate multidimensional nonlinear solver can be used or a step-by-step procedure can be applied if possible,

A function is approximated with a Gaussian Quadrature formula it is caused an error proportional to $(2n)^{th}$ derivative [4].

Gaussian Quadrature on $[a, b]$

A coordinate transformation from $[a, b]$ to $[-1,1]$ is needed as

$$t = \frac{b-a}{2}x + \frac{b+a}{2},$$

$$x = -1 \rightarrow t = a,$$

$$x = 1 \rightarrow t = b$$

then $\int_a^b f(t)\, dt = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)(\frac{b-a}{2})dx = \int_{-1}^{1} g(x)dx$. **(1.30)**

Gaussian Quadrature on [-1,1]

$$n = 2: \int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2)$$ **(1.31)**

Having the exact integral for $f = x^0, x^1, x^2 x^3$ yields the following.

-four equations for four unknows

$$f = 1 \rightarrow \int_{-1}^{1} 1\, dx = 2 = c_1 + c_2,$$

$$f = x \rightarrow \int_{-1}^{1} x\, dx = 0 = c_1 x_1 + c_2 x_2,$$

$$f = x^2 \rightarrow \int_{-1}^{1} x^2 dx = \frac{2}{3} = c_1 x_1^2 + c_2 x_2^2,$$

$$f = x^3 \rightarrow \int_{-1}^{1} x^3 dx = 0 = c_1 x_1^3 + c_2 x_2^3,$$

solving them results that

$$I = \int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f(\frac{1}{\sqrt{3}}).$$ **(1.32)**

Example 1.2 : Gaussian Quadrature

Evaluate $I = \int_0^4 t\, e^{2t}\, dt = 5216{,}926477$

Performing the coordinate transformation

$$t = \frac{b-a}{2}x + \frac{b+a}{2} = 2x + 2\ ;\ d_t = 2dx$$ gives **(1.33)**

$$I = \int_0^4 te^{2t} d_t = \int_{-1}^{1}(4x + 4)\, e^{4x+4} dx = \int_{-1}^{1} f(x)\, dx.$$ **(1.34)**

Calculating the final integral by using different approximations produces the following results:

»By two point formula

$$I = \int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = \left(4 - \frac{4}{\sqrt{3}}\right)e^{4-\frac{3}{\sqrt{3}}} + \left(4 + \frac{4}{\sqrt{3}}\right)e^{4+\frac{3}{\sqrt{3}}} \quad \textbf{(1.35)}$$

$$= 9.167657324 + 3468.376279 = 3477.5433936 \quad (\varepsilon = 33.34\ \%)$$

» By three- point formula

$$I = \int_{-1}^{1} f(x)dx = \frac{5}{9}f\left(-\sqrt{0.6}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{0.6}\right) \quad \textbf{(1.36)}$$

$$= \frac{5}{9}\left(4 - 4\sqrt{0.6}\right)e^{4-\sqrt{0.6}} + \frac{8}{9}4e^{4} + \frac{5}{9}(4 + 4\sqrt{0.6})e^{4+\sqrt{0.6}}$$

$$= \frac{5}{9}(2.221191545) + \frac{8}{9}(218.3926001) + \frac{5}{9}(8589.142689)$$

$$= 4967.106689 \quad (\varepsilon = 4.79\ \%).$$

» By four- point formula

$$I = \int_{-1}^{1} f(x)dx = 0.34785\ [\,f(-0.861136) + f(0.861136)] +$$

$$0.6522145\ [f(-0.339981) + f(0.339981)] \quad \textbf{(1.37)}$$

$$= 5197.54375 \quad (\varepsilon = 0.37\ \%).$$

1.1.2. Types of Orthogonal Polynomials

The Classical Orthogonal Polynomials are classified as follows:

Name	$\boldsymbol{P_n(x)}$	$\boldsymbol{w(x)}$	$\boldsymbol{(a,b)}$
Hermite	$H_n(x)$	e^{-x^2}	(-∞, ∞)
Laquerre	$L_n^{(\alpha)}(x)$	$e^{-x}\, x^{\alpha}$	(0, ∞)
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$(1-x)^{\alpha}(1+x)^{\beta}$	(-1,1)
Legendre	$P_n(x)$	1	(-1,1)

Table 1.1: *List of the classical orthogonal polynomials with weight functions and their intervals.*

The Hermite polynomials are orthogonal on the interval$(-\infty,\infty)$ with compared to the normal distribution $w(x) = e^{-x^2}$, the Laquerre polynomials are orthogonal on the interva $(0,\infty)$ with compared to the Gamma distribution $e^{-x}\, x^{\alpha}$ and the Jacobi polynomials are orthogonal on the $(1,-1)$ with the compared to the beta distribution $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$.

The Legendre polynomials form a special case $(\alpha = \beta = 0)$ of the Jacobi polynomials.

These classical orthogonal polynomials provide an orthogonality relation a three term recurrence relation, a second order linear differential equation and so-called Rodriquez formula. Additionally, for each family of classical orthogonal polynomials it has a generating function [5].

In the sequel it will often use the Kronecker delta which is described by

$$\delta_{m,n} := \begin{cases} 0 & , m \neq n \\ 1 & \ \ m = n \end{cases},$$

for $m,n \in \{0,1,2,\ldots\}$ and the notation $D = \frac{d}{d_x}$ states for the differentiation operator.

Then the Leibniz' rule is

$$D^n[f(x)g(x)] = \sum_{k=0}^{n}\binom{n}{k} D^k f(x) D^{n-k} g(x), \qquad (n = 0,1,2\ldots). \qquad \textbf{(1.38)}$$

The proof is by mathematical induction and by use of Pascal's triangle identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \qquad (k = 1,2,..,n). \qquad \textbf{(1.39)}$$

(i) Hermite Polynomial

The Hermite polynomials are orthogonal on the interval $(-\infty, \infty)$ regarding normal distribution $w(x) = e^{-x^2}$. They can be described by way of their Rodriquez Formula

$$H_n(x) = \frac{(-1)^n}{w(x)} D^n w(x) = (-1)^n e^{x^2} D^n e^{-x^2},\ n = 0, 1, 2...,\qquad \textbf{(1.40)}$$

where the differentiation operator D is described by Eq.**(1.38)**. Since $D^{n+1} = D^n \times D$ acquired to

$$D^{n+1}w(x) = D[D^n w(x)] = (-1)^n D[w(x)H_n(x)]$$

$$= (-1)^n [w'(x)H_n(x) + w(x)H_n'(x)]$$

$$= \quad (-1)^{n+1} w(x)[2xH_n(x) - H_n'(x)], \qquad (n = 0,1,2,...), \qquad \textbf{(1.41)}$$

which implies that

$$H_{n+1}(x) = 2xH_n(x) - H_n'(x), \quad (n = 0,1,2,...), \qquad \textbf{(1.42)}$$

The definition Eq.**(1.40)** implies that $H_0(x) = 1$. Then the relation Eq.**(1.42)** indicate by induction that $H_n(x)$ is a polynomial of degree n. Moreover, $H_{2n}(x)$ is even and $H_{2n+1}(x)$ is odd and the spearheading coefficient of the polynomial $H_n(x)$ equals $k_n = 2^n$. The Hermite polynomials satisfy the orthogonolity relation

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = 2^n n!\, \delta_{mn} \quad , \quad (m, n \in \{0,1,2..\}) \qquad \textbf{(1.43)}$$

To prove this, use definition Eq.**(1.40)** to obtain

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = (-1)^n \int_{-\infty}^{\infty} e^{-x^2} H_m(x) D^n dx\ . \qquad \textbf{(1.44)}$$

Now, using the integration by parts n times to result in that the integral disappeares for $m < n$. To $m = n$ the following results by;

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_n(x)dx = (-1)^n \int_{-\infty}^{\infty} e^{-x^2} H_n(x) D^n dx \qquad \textbf{(1.45)}$$

$$= \int_{-\infty}^{\infty} e^{-x^2} H_n(x) D^n dx$$

$$= k_n\, n! \int_{-\infty}^{\infty} e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

This proves the orthogonality relation Eq.**(1.40)**. In order that find the three term repetition formula starting with

$$w(x) = e^{-x^2} \;\rightarrow\; w'(x) = -2x\, w(x), \tag{1.46}$$

By using Leibniz' rule

$$D^{n+1}w(x) = D^n w'(x) = D^n[-2x\, w(x)] = -2xD^n w(x) - 2nD^{n-1}w(x), \tag{1.47}$$

which indicate that

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)\;,\; (n = 1,2,3,\dots),$$

combining Eq.**(1.42)** and Eq.**(1.48)** it finds that

$$H'_{n+1}(x) = 2n\, H_{n-1}(x), \qquad (\, n = 1,2,3,\dots), \tag{1.49}$$

Differentiation of Eq.**(1.42)** gives

$$H'_{n+1}(x) = 2xH'_n(x) + 2H_n(x) - H''_n(x)\;, \qquad (n = 0,1,2,\dots), \tag{1.50}$$

Now using to Eq.**(1.49)** to conclude that

$$2(n+1)H_n(x) = 2xH'_n(x) + 2H_n(x) - H''_n(x) \quad , \quad (n = 0,1,2,\dots), \tag{1.51}$$

which implies that $H_n(x)$ satisfies the second order linear differential equation

$$y''(x) - 2xy' + 2ny(x) = 0\,, (\, n \in \{0,1,2,\dots\}). \tag{1.52}$$

Finally, to prove the generating function

$$e^{2xt-t^2} = \sum_0^\infty \frac{H_n(x)}{n!} t^n, \tag{1.53}$$

start with

$$f(t) = e^{-(x-t)^2} = e^{-x^2} \times e^{2xt-t^2}, \tag{1.54}$$

then Taylor series for $f(t)$ is

$f(t) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} t^n$ by using the substitution $x - t = u$

$$f^n(0) = \left[\frac{d^n}{dt^n}\, e^{-(x-t)^2}\right]\; t = 0, = \;(-1)^n \left[\frac{d^n}{du^n}\, e^{-u^2}\right]\; u = x \tag{1.55}$$

$$= (-1)^n D^n e^{-x^2} = e^{-x^2} H_n(x) \text{ for } \quad (n = 0,1,2,\dots),$$

Hence it has

$$e^{-x^2} \times e^{2xt-t^2} = \; e^{-(x-t)^2} f(t) = f(t) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} t^n, \tag{1.56}$$

$$= \; e^{-x^2} \sum_0^\infty \frac{H_n(x)}{n!} t^n,$$

$$e^{2xt-t^2} = \sum_0^\infty \frac{H_n(x)}{n!} t^n \;. \tag{1.57}$$

This proves the generating function Eq.**(1.57)** *(ii) Legendre Polynomail*

The Legendre polynomials are orthogonal on the interval $(-1,1)$ with compared to the weight function $w(x) = 1$. It is acquired regarding their Rodriques Formula that:

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{1}{w(x)} D^n[w(x)\,(1-x^2)^n] = \frac{(-1)^n}{2^n n!} D^n[\,(1-x^2)^n], \quad \mathbf{(1.58)}$$

This is the special case $\alpha = \beta = 0$ of the Jacobi polynomials:

$$P_n(x) = P_n^{(0,0)}(x) = {}_2F_1\left({-n, n+1 \atop 1}; \frac{1-x}{2}\right), (n = 0,1,2,\dots), \quad \mathbf{(1.59)}$$

further it has

$$P_n(-x) = (-1)^n P_n(x)\,,\ P_n(1) = 1 \ \text{ and}$$
$$P_n(-1) = (-1)^n, (n = 0,1,2,\dots). \quad \mathbf{(1.60)}$$

The leading coefficient of the Polynomial $P_n(x)$ equals

$$k_n = \frac{(-n)_n (n+1)_n}{(1)_n n!} \frac{(-1)^n}{2^n} = \frac{(2n)!}{2^n (n!)^2}, \quad (n = 0,1,2,\dots), \quad \mathbf{(1.61)}$$

The orthogonality relation is

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{mn}, \ (m, n \in \{0,1,2,..\}), \quad \mathbf{(1.62)}$$

It is shown by using the Rodriques Formula Eq.**(1.58)** and integration by parts

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} P_m(x) D^n[(1-x^2)^n] dx \quad \mathbf{(1.63)}$$
$$= \frac{1}{2^n n!} \int_{-1}^{1} P_m(x) D^n[(1-x^2)^n] dx\,,$$

which vanishes for $m < n$. For $m = n$;

$$\int_{-1}^{1} P_m(x) D^n[(1-x^2)^n] dx = k_n n!\, dx = \frac{(2n)!}{2^n n!} \int_{-1}^{1} (1-x^2)^n dx \quad \mathbf{(1.64)}$$

Finally, by using the substitution $1 - x = 2$t one can have for

$$\int_{-1}^{1} (1-x^2)^n dx = \int_{-1}^{1} (1-x)^n (1+x)^n d_x = \int_{0}^{1} (2t)^n (2-2t)^n 2dx \quad \mathbf{(1.65)}$$
$$=2^{2n+1} B(n+1, n+1) = 2^{2n+1} \frac{\Gamma(n+1)\,\Gamma(n+1)}{\Gamma(2n+2)}$$
$$= \frac{2^{2n+1}(n!)^2}{(2n+1)!},\ n = 0,1,2,\dots$$

Hence,

$$\int_{-1}^{1} \{P_n(x)\}^2\, dx = \frac{(2n)!}{2^{2n}(n!)^2} \frac{2^{2n+1}(n!)^2}{(2n+1)!} = \frac{2}{2n+1}, \ (n = 0,1,2,\dots). \quad \mathbf{(1.66)}$$

This proves the orthogonality relation Eq.**(1.62)**. In order find a generating function for the Legendre polynomials one can use the hypergeometric representation Eq.**(1.58)** to find

$$\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-n)_k(n+1)_k}{(1)_k\, k!}(\frac{1-x}{2})^k t^n \quad \textbf{(1.67)}$$

$$= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{(-n)_k(n+1)_k}{k!\ k!}(\frac{1-x}{2})^k t^n$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-n-k)_k(n+k+1)_k}{k!\ k!}(\frac{1-x}{2})^k t^{n+k}$$

$$=\sum_{k=0}^{\infty} \frac{(2k)!}{k!k!}(\frac{x-1}{2})^k t^k \sum_{n=0}^{\infty} \frac{(2n+1)_n}{n!} t^n$$

$$= \sum_{k=0}^{\infty} \frac{(2k)!}{k!k!}(\frac{x-1}{2})^k t^k (1-t)^{-2k-1}$$

$$= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{k!}\ [2(x-1)t]^k (1-t)^{-2k-1} = (1-t)^{-2k-1}[1-\frac{2(x-1)t}{(1-t)^2}]^{-1/2}$$

$$= [(1-t)^2 - 2(x-1)t]^{-1/2} = (1-2xt+t^2)^{-1/2} = \frac{1}{\sqrt{1-2xt+t^2}}\,.$$

This proves the generating function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n. \quad \textbf{(1.68)}$$

Calling $F(x,t) = (1-2xt+t^2)^{-1/2}$ then it has

$$\frac{\partial}{\partial_t}F(x,t) = \frac{-1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) = \frac{x-t}{(1-2xt+t^2)^{\frac{3}{2}}}, \quad \textbf{(1.69)}$$

which this implies that

$$(1-2xt+t^2)\frac{\partial}{\partial_t}F(x,t) = (x-t)F(x,t) \quad \textbf{(1.70)}$$

Now use Eq.**(1.68)** to obtain

$$(1-2xt+t^2)\ \sum_{n=1}^{\infty} nP_n(x)t^{n-1} = (x-t)\sum_{n=0}^{\infty} P_n(x)t^n, \quad \textbf{(1.71)}$$

which can also be written as

$$\sum_{n=1}^{\infty} nP_n(x)t^{n-1} - 2x\ \sum_{n=1}^{\infty} n\,P_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1}$$

$$= x\sum_{n=0}^{\infty} P_n(x)\,t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1}$$ or equivalenty

$$\sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - x\ \sum_{n=0}^{\infty}(2n+1)\,P_n(x)t^n + \sum_{n=0}^{\infty}(n+1)P_n(x)t^{n+1} = 0.$$

This leads to $P_1(x) = xP_0(x)$ and the three term recurrence formula;

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0,\ \ (n = 1,2,3\ldots). \quad \textbf{(1.72)}$$

(iii) Chebysev Polynomial

For $x \in [1,1]$ the Chebysev polynomials $T_n(x)$ of the first kind and the Chebysev polynomials $U_n(x)$ of the second kind can be described by

$$T_n(x) = \cos(n\theta) \ , U_n(x) = \frac{\sin(n+1)\theta}{sin\theta} \ , \ x = cos\theta \ , (n = 0,1,2, \dots), \quad \textbf{(1.73)}$$

with the orthogonality property

$$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}T_m(x)T_n(x)dx = \int_0^{\pi}\cos(m\theta)\cos(n\theta)\,d\theta = 0 \ , m \neq n \quad \textbf{(1.74)}$$

and

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}U_m(x)U_n(x)dx = \int_0^{\pi}\sin(m+1)\theta\sin(n+1)\,\theta d\theta = 0 \ , m \neq n \quad \textbf{(1.75)}$$

Both families of orthogonal polynomials satisfies the three term recurrence relation

$$P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x), \ \ (n = 1,2,3 \dots),$$

since having to

$$T_{n+1}(x) + T_{n-1}(x) = \cos(n+1)\,\theta + \cos(n-1)\,\theta$$
$$= 2cos\theta\cos(n\theta) = 2xT_n(x), \quad \textbf{(1.76)}$$

and

$$U_{n+1}(x) + U_{n-1}(x) = \frac{\sin(n+2)\theta + \sin(n\theta)}{sin\theta} = \frac{2cos\theta\sin(n+1)\theta}{sin\theta} = 2xU_n(x), \quad \textbf{(1.77)}$$

Note that

$$T_0(x) = U_0(x) = 1 \ , T_1(x) = x \text{ and } \quad U_1(x) = 2x, \quad \textbf{(1.78)}$$

also ,

$$T_n(x) = U_n(x) - xU_{n-1}(x), \qquad (n = 1,2,3, \dots), \quad \textbf{(1.79)}$$

since

$$U_n(x) - xU_{n-1}(x) = \frac{\sin(n+1)\theta - cos\theta\sin(n\theta)}{sin\theta} = \frac{sin\theta\cos(n\theta)}{sin\theta} = \cos(n\theta) = T_n(x).$$

(1.80)

In order to find a generating function for the Chebysev polynomials $T_n(x)$ of the first kind, multiply the repetitive relation by t^{n+1} and take the sum to get

$$\sum_{n=1}^{\infty} T_{n+1}(x)t^{n+1} = 2x\sum_{n=1}^{\infty} T_n(x)t^{n+1} - \sum_{n=1}^{\infty} T_{n-1}(x)t^{n+1}, \quad \textbf{(1.81)}$$

If,

$$F(x,t) = \sum_{n=0}^{\infty} T_n(x)t^n, \quad |t|<1 \tag{1.82}$$

is defined, then it has

$$F(x,t) - T_1(x)t - T_0(x) = 2xt[F(x,t) - T_0(x)] - t^2\, F(x,t), \tag{1.83}$$

which implies that

$$(1 - 2xt + t^2)F(x,t) = T_0(x) + T_1(x)t - 2xtT_0(x) \tag{1.84}$$

$$= 1 + xt - 2xt = 1 - xt.$$

Hence the generating function is obtained as

$$\sum_{n=0}^{\infty} T_n(x)t^n = F(x,t) = \frac{1-xt}{1-2xt+t^2}, \quad |t| < 1. \tag{1.85}$$

In the same way having for the Chebysev polynomials $U_n(x)$ of the second kind

$$G(x,t) = \sum_{n=0}^{\infty} U_n(x)t^n, |t| < 1 \tag{1.86}$$

Then,

$$\sum_{n=0}^{\infty} U_n(x)t^n = G(x,t) = \frac{1}{1-2xt+t^2}, \quad |t| < 1. \tag{1.87}$$

This can be used, for instance to prove that

$$\sum_{k=0}^{n} T_k(x)x^{n-k} = U_n(x), \quad (n = 0,1,2, \dots), \tag{1.88}$$

In fact, having for $|t| < 1$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} T_k(x)x^{n-k}\right)t^n = \sum_{k=0}^{\infty}\sum_{n=k}^{n} T_k(x)x^{n-k}t^n \tag{1.89}$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} T_k(x)x^n t^{n+k}$$

$$= \sum_{k=0}^{\infty} T_k(x)t^n \sum_{n=0}^{\infty}(xt)^n = \frac{1-xt}{1-2xt+t^2} \times \frac{1}{1-xt}$$

$$= \frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n.$$

In a similar way, one can prove that

$$\sum_{k=0}^{n} P_k(x)P_{n-k}(x) = U_n(x), (n = 0,1,2, \dots), \tag{1.90}$$

where $P_n(x)$ indicate the Legendre polynomial. In fact, it has for $|t| < 1$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} P_k(x)P_{n-k}(x)\right)t^n = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} P_k(x)P_{n-k}(x)t^n, \tag{1.91}$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} P_k(x)P_n(x)t^{n+k},$$

$$= \sum_{k=0}^{\infty} P_k(x)t^k \sum_{n=0}^{\infty} P_n(x)t^n,$$

$$= \frac{1}{\sqrt{1-2xt+t^2}} \times \frac{1}{\sqrt{1-2xt+t^2}},$$

$$= \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n.$$

(iv) Jacobi Polynomial

The Jacobi polynomials are orthogonal on the interval $(-1, 1)$ with compared to the Beta distribution $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$.They can be acquired regarding their Rodriques Formula [6]:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{1}{w(x)}D^n[(w(x)(1-(1-x^2)^n] \qquad \textbf{(1.92)}$$

$$= \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}D^n\left[((1-x)^{n+\alpha})\left((1+x)^{n+\beta}\right)\right],$$

for $(n = 0,1,2,\dots)$. By using Leibniz' rule,

$$D^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right] = \sum_{k=0}^{n}\binom{n}{k}D^k(1-x)^{n+\alpha}D^{n-k}(1+x)^{n+\beta}$$

$$= \sum_{k=0}^{n}\binom{n}{k}(-1)^k(n+\alpha)(n+\alpha-1)\dots(n+\alpha-k+1)(1-x)^{n+\alpha-k}$$

$$x(n+\beta)(n+\beta-1)\dots(\beta+k+1)\dots(1+x)^{\beta+k}$$

$$= n!\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(1-x)^{n+\alpha-k}(1+x)^{\beta+k}, (n = 0,1,2,\dots).$$

(1.93)

This implies that

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(1-x)^{n-k}(1+x)^k, \ (n = 0,1,2,\dots).$$

(1.94)

It shows that $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n and have the properties

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \quad (n = 0,1,2\dots), \qquad \textbf{(1.95)}$$

and

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{k} \text{ and } P_n^{(\alpha,\beta)}(-1) = (-1)^n\binom{n+\beta}{k}, \ (n = 0,1,2,\dots)$$

In order to find a hypergeometric representation, writing for $x \neq 1$

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{x-1}{2}\right)^n\sum_{k=0}^{n}\binom{n+\alpha}{n}\binom{n+\beta}{n-k}\left(\frac{x+1}{x-1}\right)^k \ \left(n = 0,1,2,\dots\right). \qquad \textbf{(1.96)}$$

and now for $x = 1$, the following equation is obtained.

$$\left(\frac{x+1}{x-1}\right)^k = \left(1+\frac{2}{x-1}\right)^k = \sum_{i=0}^{k}\binom{k}{i}\left(\frac{2}{x-1}\right)^i \qquad (k = 0,1,2,\dots). \qquad \textbf{(1.97)}$$

Then by changing the order of summations for $x \neq 1$ and letting $(n = 0,1,2,\dots)$ it becomes

$$P_n^{(\alpha,\beta)}(x) = (\tfrac{x-1}{2})^n \sum_{i=0}^n \sum_{k=i}^n \binom{n+\alpha}{n}\binom{n+\beta}{n-k}\binom{k}{i}\left(\tfrac{2}{x-1}\right)^i, \quad \textbf{(1.98)}$$

Now keeping the order in the first sum to find for $x \neq 1$ and for $(n = 0,1,2,\dots)$ one can have

$$P_n^{(\alpha,\beta)}(x) = (\tfrac{x-1}{2})^n \sum_{i=0}^n \sum_{k=0}^n \binom{n+\alpha}{n-i+k}\binom{n+\beta}{i-k}\binom{n-i+k}{n-i}\left(\tfrac{2}{x-1}\right)^{n-i} \quad \textbf{(1.99)}$$

$$= \sum_{i=0}^n \sum_{k=0}^n \binom{n+\alpha}{n-i+k}\binom{n+\beta}{i-k}\binom{n-i+k}{n-i}\left(\tfrac{x-1}{2}\right)^i$$

$$= \sum_{i=0}^n \sum_{k=0}^n \frac{\Gamma(n+\alpha+1)}{(n-i+k)!\,\Gamma(i-k+\alpha+1)} \times \frac{\Gamma(n+\beta+1)}{(i-k)!\,\Gamma(n-i+k+\beta+1)} \times \frac{(n-i+k)!}{(n-i)!k!}\frac{x-1^i}{2},$$

Since $i \in \{0,1,2,\dots,n\}$ having by using the Chu-Vandermonde summation formula

$$\sum_{k=0}^n \frac{(-i)_k(-i-\alpha-1)_k}{(n-i+\beta+1)_k\, k!} = {}_2F_1\left(\begin{matrix}-i,-i-\alpha-1\\ n-i+\beta+1\end{matrix};1\right) = \frac{(n+\alpha+\beta+1)_i}{(n-i+\beta+1)_i}, \quad \textbf{(1.100)}$$

and using $\Gamma(n-i+\beta+1)(n-i+\beta+1)_i = \Gamma(n+\beta+1)$, it can be rewritten as

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{n!}\sum_{i=0}^n \frac{(-n)_i(n+\alpha+\beta+1)_i}{\Gamma(i+\alpha+1)\, i!}(\tfrac{1-x}{2})^i \quad \textbf{(1.101)}$$

$$= \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}\sum_{i=0}^n \frac{(-n)_i(n+\alpha+\beta+1)_i}{(\alpha+1)_i\, i!}(\tfrac{1-x}{2})^i, \ (n = 0,1,2,\dots).$$

This proves the hypergeometric representation

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} F\left(\begin{matrix}-n,n+\alpha+\beta+1\\ \alpha+1\end{matrix};\tfrac{1-x}{2}\right) \ (n = 0,1,2,..), \quad \textbf{(1.102)}$$

Note that this result also rely for $x = 1$. Based on the symmetry property Eq **(1.95)**, above equation can be rewritten as;

$$P_n^{(\alpha,\beta)}(x) = (-1)^n\binom{n+\beta}{n}\, {}_2F_1\left(\begin{matrix}-n,n+\alpha+\beta+1\\ \beta+1\end{matrix};\tfrac{1+x}{2}\right), \ (n = 0,1,2 \dots). \quad \textbf{(1.103)}$$

Note that the hypergeometric representation implies that [6]

$$\frac{d}{d_x}P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\frac{(-n)(n+\alpha+\beta+1)}{\alpha+1}(-\tfrac{1}{2}) \times {}_2F_1\left(\begin{matrix}-n+1,n+\alpha+\beta+2\\ \alpha+2\end{matrix};\tfrac{1-x}{2}\right) \quad \textbf{(1.104)}$$

$$= \frac{n+\alpha+\beta+1}{2}\binom{n+\alpha}{n-1}\, {}_2F_1\left(\begin{matrix}-n+1,n+\alpha+\beta+2\\ \alpha+2\end{matrix};\tfrac{1-x}{2}\right)$$

$$= \frac{n+\alpha+\beta+1}{2}\, P_{n-1}^{(\alpha+1,\beta+1)}(x), \ (n = 1,2,3 \dots).$$

Hence, the hypergeometric representation says that the leading coefficient of the polynomial $P_n^{(\alpha,\beta)}(x)$ equals

$$k_n = \binom{n+\alpha}{n} \frac{(-n)_n (n+\alpha+\beta+1)_n}{(\alpha+1)_n \, n!} \frac{(-1)^n}{2^n} = \frac{(n+\alpha+\beta+1)_n}{2^n \, n!} \ , \ (n = 0,1,2,\dots). \quad \textbf{(1.105)}$$

Now, it is shown that the Jacobi polynomials provide the orthogonality relation

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \, P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) dx = \frac{2^{\alpha+\beta+1} \, \Gamma(n+\alpha+1) \, \Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1) \, \Gamma(n+\alpha+\beta+1)} \times \delta_{mn},$$

(1.106)

for $\alpha > -1$, $\beta > -1$ and $m, n \in \{0,1,2,\dots\}$. By using the definition in Eq.**(1.92)**, this formula is indicated and integration by parts. The value of the integral in the case $m = n$ can be estimated by using the leading coefficient and then writing the integral in terms of a Beta integral:

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \left\{P_n^{(\alpha,\beta)}(x)\right\}^2 dx,$$

$$= \frac{(-1)^n}{2^n n!} \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) \, D^n[\,(1-x)^{\alpha}(1+x)^{\beta}]dx,$$

$$= \frac{1}{2^n n!} \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) \, D^n \, (1-x)^{\alpha}(1+x)^{\beta} dx,$$

$$= \frac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1) 2^{2n} n!} \int_{-1}^{1} (1-x)^{n+\alpha}(1+x)^{n+\beta} dx \, , (n = 0,1,2,\dots),$$

(1.107)

and by using the substitution $1 - x = 2t$,

$$\int_{-1}^{1} (1-x)^{n+\alpha}(1+x)^{n+\beta} dx \qquad \textbf{(1.108)}$$

$$= \int_{0}^{1} (2t)^{n+\alpha}(2-2t)^{n+\beta} 2dt \ 1,$$

$$= 2^{2n+\alpha+\beta+1} \int_{0}^{1} (t)^{n+\alpha}(1-t)^{n+\beta} dt,$$

$$= 2^{2n+\alpha+\beta+1} B(n+\alpha+1, n+\beta+1),$$

$$= 2^{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1) \, \Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)},$$

$$= 2^{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1) \, \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1) \, \Gamma(2n+\alpha+\beta+1)} \ , \ (n = 0,1,2,\dots).$$

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy the second order linear differential equation

$$(1-x^2)y''(x) + [\beta - \alpha - (\alpha+\beta+2)_x]y'(x) + n(n+\alpha+\beta+1)y(x) = 0.$$

(1.109)

A generating function for the Jacobi polynomials is given by

$$\frac{2^{\alpha+\beta}}{(1+R-t)^{\alpha}(1+R+t)^{\beta}R} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n \,, \quad R = \sqrt{1-2xt+t^2} \;. \qquad \textbf{(1.110)}$$

CHAPTER 2
TWO SPECTRAL METHODS FOR APPROXIMATE SOLUTIONS OF CERTAIN PDE's

Some Basic Ideas Of Spectral Methods

It is well known that the FD methods approximate derivatives of a function by local arguments. These methods give exact results for low order polynomials and the approach is reasonable since the derivative is a local property of a function. Inversely, spectral methods are global. The usual way for introducing them is given as follows:

$$u(x) \approx \sum_{k=0}^{N} b_k \sigma_k(x) \quad , \qquad \textbf{(2.1)}$$

where the basis functions $\sigma_k(x)$ can be polynomials or trigonometric functions. There exist different choices of selecting the basis functions, for instance [7]:

- $\sigma_k(x) = e^{ikx}$ (The Fourier Spectal Method);
- $\sigma_k(x) = C_k(x)$ where $C_k(x)$ are the Chebysev polynomials (The Chebysev Spectral Method);
- $\sigma_k(x) = L_k(x)$ where $L_k(x)$ are the Legendre polynomials (The Legendre Spectral Method).

In the procedure of finite element method the concluding stiffness matrix is sparse, but in the spectral method it is full which cause more time consuming calculations rather than finite differences or finite elements. After all, the discovery of the Fast Fourier Transform by Cooley and Tukey solves this problem.

2.1 Spectral-Collocation Method

The collocation method is the most famous form of the spectral methods among practitioners. It is very easy to execute in particular for one-dimensional problems, even for very complex nonlinear equations, and the method generally reach to satisfactory result as long as the problems keep being appropriate [7].

2.1.1. Obtaining Differentiation Matrices

(i)Differentiation Matrices for Polynomial Basis Functions

Differentiation matrices play a significant role in the application of spectral method. To define the idea of the differentiation matrix, let us consider the following problem as an example

$$u_{xx} = f\ ,\ \ x \in (-1,1)\ ;\quad u(\pm 1) = 0 \qquad \textbf{(2.2)}$$

Denoting $x_j = -1 + jh\ , 0 \le j \le N$, with $h = 2/N$. Appliying the finite difference method for Eq.**(2.2)** is to approximate u_{xx} by the central difference formula:

$$u_{xx} \approx \frac{1}{h^2}[u(x+h) - 2u(x) + u(x-h)]\ . \qquad \textbf{(2.3)}$$

Since the solution of the continuous problem Eq.**(2.2)** and the discrete problem are different, using the U to denote the solution of the discrete problem. One can easily verify that the discrete solution satisfies, the following matrix equation:

$$\begin{pmatrix} \frac{-2}{h^2} & \frac{1}{h^2} & 0 & \cdots & 0 \\ \frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} & \cdots & 0 \\ 0 & \frac{1}{h^2} & \frac{-2}{h^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \frac{-2}{h^2} \end{pmatrix} \begin{pmatrix} U(x_1) \\ U(x_2) \\ U(x_3) \\ \vdots \\ U(x_{N-1}) \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{N-1}) \end{pmatrix} . \qquad \textbf{(2.4)}$$

The matrix above is the so called *differentiation matrix (DM)* of the finite-difference method for second order derivative. In general, for a problem involving the the m^{th}derivative $u^{(m)}$, the differentiation matrix is defined by

$$D^m = (d_{ij}^{(m)})_{i,j=0}^{N} \quad , \qquad \textbf{(2.5)}$$

and it satisfies

$$\begin{pmatrix} U^{(m)}(x_0) \\ U^{(m)}(x_1) \\ \vdots \\ U^{(m)}(x_N) \end{pmatrix} = D^m \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_N) \end{pmatrix}. \qquad \textbf{(2.6)}$$

The differentiation matrix is dependent on the collocation points and the chosen basis functions.

Polynomial Basis Function

If the basis functions $\sigma_k(x)$ are polynomials, the spectral approximation is of the form $u^N(x) = \sum_{k=0}^{N} a_k \sigma_k(x)$, where the coefficients a_k can be specific from a given set of collocation points $\{x_j\}_{j=0}^{N}$ and the function values $u^N(x_j)$. Since $u^N(x)$ is a polynomial, it can also be written in the form

$$u^N(x) = \sum_{k=0}^{N} u^N(x_k) F_k(x) , \qquad \textbf{(2.7)}$$

where the $F_k(x)$ are called Lagrange polynomials which satisfies

$$F_k(x_j) = \begin{cases} 0 \; ; \; k \neq j \\ 1 \; ; \; k = j \end{cases} .$$

In this part, it has used Eq.**(2.7)**, the equivalent form of $u^N(x)$, to obtain the differentiation matrices for polynomial basis function. If the basis functions are not polynomials (e.g. trigonometric functions), then some changes must be made by using parameters and finally following equations are obtained [7]

$$F_{n,v}(x) = \sum_{m=0}^{n} \frac{c_{n,v}^m}{m!} x^m, \qquad \textbf{(2.8)}$$

where $m = 0,1, \dots, M \; ; n = m, m+1, \dots, N \; ; M \geq 0$ and

$$c_{n,v}^m = \left[\frac{d^m}{{d_x}^m} F_{n,v}(x)\right]_{x=0},$$

$$F_{n,v}(x) = \frac{\gamma_n(x)}{\gamma_n'(\alpha_v)(x-\alpha_v)} , \qquad \gamma_n(x) := \prod_{k=0}^{n} (x - \alpha_k). \; , \qquad \textbf{(2.9)}$$

It follows from Eq.**(2.8)** that

$$F_{n,v}(x) = \frac{x-\alpha_n}{(\alpha_v - \alpha_n)} \quad F_{n-1,v}(x) , \text{ for } v < n \; ; \qquad \textbf{(2.10)}$$

$$F_{n,n}(x) = \frac{\gamma_{n-1}(x)}{\gamma_{n-1}(\alpha_n)} = \frac{\gamma_{n-2}(\alpha_{n-1})}{\gamma_{n-1}(\alpha_n)} \; (x - \alpha_{n-1}) F_{n-1,n-1}(x) \;\; (n > 1) . \qquad \textbf{(2.11)}$$

By replacing Eq.**(2.8)** in the equations Eq.**(2.10)** and Eq.(**2.11)**, and by equating powers of x, the essential relations for the weights are obtained:

$$c_{n,v}^{m} = \frac{1}{\alpha_n - \alpha_v}(\alpha_n c_{n-1,v}^{m} - m c_{n-1,v}^{m-1}) \quad \text{for} < n \;, \tag{2.12}$$

$$c_{n,v}^{m} = \frac{\gamma_{n-2}(\alpha_{n-1})}{\gamma_{n-1}(\alpha_n)}(m c_{n-1,n-1}^{m-1} - \alpha_{n-1} c_{n-1,n-1}^{m}) \;, \tag{2.13}$$

Notice that the relation

$$\sum_{v=0}^{n} c_{n,v}^{m} = 0 \text{ for } m > 0 \;; \sum_{v=0}^{n} c_{n,v}^{0} = 1 \tag{2.14}$$

can be used on behalf of Eq.**(2.13)** to obtain $c_{n,n}^{m}$. However, this would increase the operation count and might also cause a growth of errors in the case of floating arithmetic. It is obvious that $c_{0,0}^{0} = 1$. Using this fact, together with Eq.**(2.12)**, it is obtained that [7]

$$c_{1,0}^{0}, c_{1,0}^{1}, \ldots . c_{1,0}^{M}. \tag{2.15}$$

Then using $c_{0,0}^{0} = 1$ and Eq.**(2.13)** cause to

$$c_{1,1}^{0}, c_{1,1}^{1}, \ldots . c_{1,1}^{M}. \tag{2.16}$$

The above information, together with Eq.**(2.12)** give

$$c_{2,0}^{0}, c_{2,0}^{1}, \ldots . c_{2,0}^{M} \;;$$

$$c_{2,1}^{0}, c_{2,1}^{1}, \ldots . c_{2,1}^{M} \;. \tag{2.17}$$

Using Eq.**(2.13)** or Eq.**(2.14)** one can find

$$c_{2,2}^{0}, c_{2,2}^{1}, \ldots . c_{2,2}^{M} \;.$$

Repeating the above process will generate all the coefficients $c_{n,v}^{m}$, for ($m \leq n \leq N$, $0 \leq v \leq n$).

(ii)Differentiation Matrices for Fourier Collocation Methods

Fourier Series and Differentiation

It is well known that if f is 2π-periodic and has a continuous first derivative then its Fourier Series converges uniformly to f. In applications, the infinite Fourier Series

$$F(x) \sim \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik}, \tag{2.18}$$

is shorten to the following finite series

$$F(x) = \sum_{k=-N/2}^{\frac{N}{2}-1} \alpha_k e^{ikx} \ . \quad \textbf{(2.19)}$$

Assume that $F(x_j) = f(x_j)$,where $x_j = \frac{2\pi j}{N}$, $(0 \le j \le N-1)$.It can be shown that

$$f(x_j) = \sum_{k'=0}^{N-1} \alpha_{k' - \frac{N}{2}} (-1)^j e^{ik' x_j} \quad (0 \le j \le N-1) \ . \quad \textbf{(2.20)}$$

This, together with $P(x) = \sum_{k=0}^{N-1} c_k E_k(x)$ with $c_k =< f\ , E_k > N$ and $E_k = e^{ikx}$

$$\alpha_{k'-N/2} = \frac{1}{N} \sum_{j=0}^{N-1} (-1)^j F(x_j)\, e^{-ik' x_j} \quad (0 \le k' \le N-1). \quad \textbf{(2.21)}$$

Now, differentiating the interrupted Fourier Series $F(x)$ termwise to get the approximate derivatives gives the following from **(2.19)**

$$F^{(m)}(x) = \sum_{k=-N/2}^{N/2} \alpha_k (ik)^m e^{ikx} \ , \quad \textbf{(2.22)}$$

where m is a positive integer $F^{(m)}(x)$ can be written in the following equivalent form:

$$F^{(m)}(x) = \sum_{k'=0}^{N-1} \alpha_{k' - \frac{N}{2}} (i\left(k' - \frac{N}{2}\right))^m e^{i(k' - \frac{N}{2})x} \ . \quad \textbf{(2.23)}$$

Using Eq.**(2.21)** and Eq.**(2.23)**, it is obtained that

$$\begin{pmatrix} F^{(m)}(x_0) \\ F^{(m)}(x_1) \\ \vdots \\ F^{(m)}(x_{N-1}) \end{pmatrix} = ((-1)^j e^{ikxj} (i(k - \frac{N}{2}))^m)_{j,k=0}^{N-1} \begin{pmatrix} \alpha_{-\frac{N}{2}} \\ \alpha_{-\frac{N}{2}+1} \\ \vdots \\ \alpha_{\frac{N}{2}-1} \end{pmatrix}, \quad \textbf{(2.24)}$$

$$= \frac{1}{N} ((-1)^j e^{ikxj} (i(k - \frac{N}{2}))^m)_{j,k=0}^{N-1} ((-1)^k e^{-ijx_k})_{j,k=0}^{N-1} \begin{pmatrix} F(x_0) \\ F(x_1) \\ \vdots \\ F(x_{N-1}) \end{pmatrix}.$$

This states that the m^{th} order differentiation matrix has to do with Fourier Spectral-Collocation methods is given by

$$D^m = \frac{1}{N} ((-1)^j e^{ikxj} (i(k - \frac{N}{2}))^m)_{j,k=0}^{N-1} ((-1)^k e^{-ijx_k})_{j,k=0}^{N-1} \ . \quad \textbf{(2.25)}$$

Differentiation Matrices Using Direct Formulas

First choosing the $x_j = \frac{2\pi j}{N}$, $0 \le j \le N-1$. The corresponding interpolant is given by

$$t_N(x) = \sum_{j=1}^{N} \emptyset_j(x) F_j \ , \quad \textbf{(2.26)}$$

where the Lagrange polynomials $F_j(x)$ are of the form

$$F_j(x) = \frac{1}{N} \sin \frac{N}{2}\left(x - x_j\right) cot \frac{1}{2}\left(x - x_j\right), \ \ (N \text{ even}) \quad \textbf{(2.27)}$$

$$F_j(x) = \frac{1}{N} \sin \frac{N}{2}\left(x - x_j\right) csc \frac{1}{2}\left(x - x_j\right), \ \ (N \text{ odd}) \quad \textbf{(2.28)}$$

It can be shown that an equivalent form of $t_N(x)$ (barycentric form of interpolant) is

$$t_N(x) = \sum_{j=1}^{N} (-1)^j f_j cot \frac{1}{2}(x - x_j) \Big/ \sum_{j=1}^{N} (-1)^j cot \frac{1}{2}(x - x_j) \, , (N \text{ even}) \quad \textbf{(2.29)}$$

$$t_N(x) = \sum_{j=1}^{N} (-1)^j f_j csc \frac{1}{2}(x - x_j) \Big/ \sum_{j=1}^{N} (-1)^j csc \frac{1}{2}(x - x_j) \, , (N \text{ odd}) \quad \textbf{(2.30)}$$

The differentiation matrix $D^{(m)} = \left(F_j^{(m)}(x_k)\right)$ is obtained by Gottlieb for N even, $1 \leq k \, , j \leq N$:

$$D_{kj}^1 = \begin{cases} 0 \ \ if \ \ k = j \, , \\ 1/2(-1)^{k-j} \cot \frac{(k-j)h}{2} \ \ \ if \ \ k \neq j \, ' \end{cases} \quad \textbf{(2.31)}$$

$$D_{kj}^2 = \begin{cases} \frac{-\pi}{3h^2} - \frac{1}{6} \ \ \ if \ \ \ k = j \, , \\ -(-1)^{k-j} \frac{1}{2} csc^2 \frac{(k-j)h}{2} \ \ if \ \ k \neq j \, , \end{cases} \quad \textbf{(2.32)}$$

Similarly, for N odd,$1 \leq k \, , j \leq N$:

$$D_{kj}^1 = \begin{cases} 0 \ \ if \ \ k = j \, , \\ 1/2(-1)^{k-j} \csc \frac{(k-j)h}{2} \ \ \ if \ \ k \neq j \, ' \end{cases} \quad \textbf{(2.33)}$$

$$D_{kj}^2 = \begin{cases} \frac{-\pi}{3h^2} - \frac{1}{6} \ \ \ if \ \ \ k = j \, , \\ -(-1)^{k-j} \frac{1}{2} csc \frac{(k-j)h}{2} cot \frac{(k-j)h}{2} \ if \ \ k \neq j \, , \end{cases} \quad \textbf{(2.34)}$$

It can be shown that if N is odd then

$$D^m = (D^1)^m \ \ . \quad \textbf{(2.35)}$$

If N is even, the above formula only holds for odd m.

2.1.2 Chebysev Collocation Method for Two-Point BVPs

The Chebyshev collocation method below for the following linear second-order two-point boundary-value problem (BVP) is described,

$$\epsilon u'''(x) + p(x)u'(x) + g(x)u(x) = f(x)\ , \quad (x \in I \coloneqq (-1,1)) \qquad \textbf{(2.36)}$$

where ϵ is a (fixed) parameter that controls the singular behavior of the problem, and p, q and f are given functions. If ϵ sequential of 1, the problem is non-singular , while for adequately small ϵ, the problem can display singular behavior such as sharp boundary and interior layers. In the next state Eq.**(2.36)** is called a singularly perturbed BVP. The boundary condition for Eq**.(2.36)** is given by

$$a_{-}u(-1) + b_{-}u'(-1) = c_{-} \quad , \quad a_{+}u(1) + b_{+}u'(1) = c_{+} \qquad \textbf{(2.37)}$$

Loss of generality, assume that $a \pm \geq 0$ also assume that

$$a_{-}^{2} + b_{-}^{2} \neq 0 \text{ , and } a_{-}b_{-} \leq 0 \text{ ; } a_{+}^{2}b_{+}^{2} \neq 0 \text{ , and } a_{+}b_{+} \geq 0 \text{ ;} \qquad \textbf{(2.38)}$$

$$g(x) - \frac{1}{2}p'(x) \geq 0 \text{ , } \forall_{x} \in (I) \text{ ;} \qquad \textbf{(2.39)}$$

$$p(1) > 0 \ \text{ if } b_{+} \neq 0 \text{ , } \ p(-1) < 0 \ \text{ if } b_{-} \neq 0 \qquad \textbf{(2.40)}$$

It is easy to control that the above conditions to provide the well-posedness of Eq **(2.38)** and Eq**.(2.37)**.

(i)BVPs with Dirichlet Boundary Conditions

First taken in consideration the simplest boundary conditions:

$$u(-1) = c_{-} \ \text{ , } \ u(1) = c_{+} \ \text{ ,} \qquad \textbf{(2.41)}$$

The Chebyshev interpolation polynomial can be written as

$$u^{N}(x) = \sum_{j=0}^{N} U_{j}F_{j}(x) \ \text{ ,} \qquad \textbf{(2.42)}$$

where $x_j = \cos(\frac{j\pi}{N})$, $(0 \leq j \leq N)$ are the Chebyshev-Gauss Lobatto collocation points, $\{U_j\}_{j=1}^{N-1}$ are the coefficients to be identified and $F_j(x)$ is the Lagrange interpolation polynomial associated with $\{x_j\}$. The Chebyshev collocation method aims to search u^Nin the form of Eq**.(2.42)** such that $u^N(-1) = c_{-}$, $u^N(1) = c_{+}$, and that the equation holds at the interior collocation points:

$$\varepsilon u_{xx}^N(x_j) + p(x_j)u_x^N(x_j) + q(x_j)u^N(x_j) = f(x_j)\,,\ (1 \le j \le N-1). \qquad \textbf{(2.43)}$$

Now using the definition of the differentiation matrix presented in the Eq.**(2.1.1)** it take a system of linear equations of the form,

$$\sum_{j=1}^{N-1}\left[\epsilon(D^2)_{ij} + p(x_i)(D^1)_{ij} + q(x_i)\delta_{ij}\right]U_j =$$

$$f(x_i) - [\epsilon(D^2)_{i0} + p(x_i)(D^1)_{i0}]c_+ - [\varepsilon(D^2)_{iN} + p(x_i)(D^1)_{iN}]c_-\,, \qquad \textbf{(2.44)}$$

for the $\{U_j\}_{j=1}^{N-1}$, where δ_{ij} is the Kronecker delta. In the above equations, it has used the boundary conditions $U_0 = c_+\,, U_N = c_-$ (notice that $x_0 = 1$ and $x_N = -1$).

To summarize: the spectral-collocation solution for the BVP Eq.**(2.36)** with the Dirichlet boundary conditions Eq.**(2.41)** keep up with the linear system

$$A\overline{U} = \overline{b\,,} \qquad \textbf{(2.45)}$$

where $\overline{U} = [U_1, \ldots\ldots, U_{N-1}]^T$; the matrix $A = (a_{ij})$ and the vector $\bar{b}$ are given

$$a_{ij} = \varepsilon(D^2)_{ij} + p(x_i)(D^1)_{ij} + q(x_i)\delta_{ij}\,, \quad (1 \le i,j \le N-1) \qquad \textbf{(2.46)}$$

$$b_{\boldsymbol{i}} = f(x_i) - [\varepsilon(D^2)_{i0} + p(x_i)(D^1)_{i0}]_{c_+} - [\varepsilon(D^2)_{iN} + p(x_i)(D^1)_{iN}]_{c_-}\,,$$
$$(1 \le i \le N-1).$$

(2.47)

The solution to the above system gives the approximate solution to the Eq.**(2.36)** at the collocation points. The approximation solution in the total space is defined by Eq. **(2.42)**.

(ii) BVPs with General Boundary Conditions

Now the general boundary conditions in Eq.**(2.40)** is considered with the assumption $b_- \neq 0$ and $b_+ \neq 0$. It follows from Eq.**(2.40)** that

$$a_-U_N + b_-\sum_{j=0}^{N}(D^1)_{N_j}U_j = c_-\,, a_+U_0 + b_+\sum_{j=0}^{N}(D^1)_{oj}U_j = c_+\,, \qquad \textbf{(2.48)}$$

which causes

$$b_-(D^1)_{N_0}U_0 + (a_- + b_-(D^1)_{NN})U_N = c_- - b_-\sum_{j=1}^{N-1}(D^1)_{N_j}U_j\,, \qquad \textbf{(2.49)}$$

$$(a_+ + b_+(D^1)_{00}U_0 + b_+(D^1)_{0_N}U_N = c_+ - b_+\sum_{j=1}^{N-1}(D^1)_{0_j}U_j\,, \qquad \textbf{(2.50)}$$

$$U_0 = \tilde{c}_+ - \sum_{j=1}^{N-1} \tilde{\alpha}_{0j} U_j \ , \quad U_N = \tilde{c}_- - \sum_{j=1}^{N-1} \tilde{\alpha}\, N_j U_j \qquad , \qquad (2.51)$$

where the parameters $\tilde{c}_+, \tilde{\alpha}_{0j}, \tilde{c}_-, \tilde{\alpha}_{N_j}$ are defined by

$$\tilde{c}_+ = (\tilde{d}c_- - \tilde{b}c_+)/(\tilde{a}\tilde{d} - \tilde{c}\tilde{b}) \quad \tilde{c}_- = (\tilde{a}c_+ - \tilde{c}c_-)/(\tilde{a}\tilde{d} - \tilde{c}\tilde{b}) \ , \qquad (2.52)$$

$$\tilde{\alpha}_{0j} = (\tilde{d}b_-(D^1)_{Nj} - \tilde{b}b_+(D^1)_{0j})/(\tilde{a}\tilde{d} - \tilde{c}\tilde{b}) \quad , \qquad (2.53)$$

$$\tilde{\alpha}_{N_j} = (\tilde{a}b_+(D^1)_{0j} - \tilde{c}b_-(D^1)_{Nj})/(\tilde{a}\tilde{d} - \tilde{c}\tilde{b}) \ , \qquad (2.54)$$

$$\tilde{a} \coloneqq b_-(D^1)_{N0} \qquad \tilde{b} \coloneqq a_- + b_-(D^1)_{NN} \ , \qquad (2.55)$$

$$\tilde{c} \coloneqq a_+ + b_+(D^1)_{00} \qquad \tilde{d} \coloneqq b_+(D^1)_{0N} \quad , \qquad (2.56)$$

To summarize [6]: let the constants $b_-$and b_+ in Eq.**(2.37)** be nonzero.The spectral collocation solution for the BVP in Eq.**(2.36)** with the general boundary condition in Eq. **(2.37)** results the linear system.

$$A\overline{U} = \overline{b} \ , \qquad (2.57)$$

where $A = (a_{ij})$ is a $(N-1) \times (N-1)$ matrix and $\overline{b} = (b_j)$ is $a\ (N-1)$ dimensional vector with components

$$a_{ij} = \epsilon(D^2)_{ij} + p(x_i)(D^1)_{ij} + q_i\delta_{ij} - [\varepsilon(D^2)_{i0} + p(x_i)(D^1)_{i0}]\tilde{\alpha}_{0j} - [\varepsilon(D^2)_{iN} + p(x_i)(D^1)_{iN}]\tilde{\alpha}_{Nj} \ , \qquad (2.58)$$

$$b_i = f(x_i) - [\varepsilon(D^2)_{i0} + p(x_i)(D^1)_{i0}]\tilde{c}_+ - [\varepsilon(D^2)_{iN} + p(x_i)(D^1)_{iN}]\tilde{c}_- \ . \qquad (2.59)$$

2.1.3 Collocation Method in the Weak Form and Preconditioning

Collocation Methods in the Weak Form

A changeable method usually preserves important properties of the continuos problem such as coercivity, continuity and symmetry of the bilinear form, and causes optimal error estimates.

Consider Eq.**(2.36)** and Eq.**(2.37)**. It can be assumed that $c_\pm = 0$ without loss of generality.

It can be shown that

$$H_*^1(I) = \{v \in H^1(I) : u(-1) = 0 \ \text{ if } b_- = 0\ ;\ u(1) = 0 \text{ if } \ b_+ = 0\}, \qquad (2.60)$$

and

$$h_- = \begin{cases} 0 \ \ if \ \ a_-b_- = 0 \\ a_-/b_- \ \ if \ \ a_-b_- = 0 \end{cases} \quad , \quad h_+ = \begin{cases} 0 \ if \ a_+b_+ = 0 \\ a_+/b_+ \ \ if \ \ a_+b_+ \neq 0 \end{cases} . \quad \textbf{(2.61)}$$

Then the Galerkin method with numerical integral for Eq.**(2.36)** and Eq.**(2.37)** with $c_\pm = 0$ works as:

Find $u_N \in X_N = P_N \cap H_*^1(I)$ such that

$$b_N(u_N, v_N) = < f, v >_N \ , (\forall_{v_N} \in X_N) \ , \quad \textbf{(2.62)}$$

where

$$b_N(u_N, v_N) := \epsilon < u_N', v_N' >_N + \epsilon h_+ u_N(1) v_N(1) - \epsilon h_- u_N(-1) v_N(-1) \quad \textbf{(2.63)}$$

$$+ < p(x) u_N', v_N >_{\boldsymbol{N}} + < q(x) u_N, v_N >_{\boldsymbol{N}} , \quad \textbf{(2.64)}$$

with $< -, - >_N$ specifying the discrete inner product related to the Legendre-Gauss-Lobatto quadrature.Noting that the main difficulty appears at the boundaries with complicated boundary conditions if main the Chebyshev-Gauss-Lobatto quadrature is preferred to be used. This difficulty can be overcomed by replacing X_N by $\tilde{X}_N = \{u \in P_N : a_\pm u(\pm 1) + b_\pm(\pm 1) = 0\}$.

Now attempting to re-interpret Eq.**(2.62)** into a collocation form. To fix the idea, assume that the $b_\pm \neq 0$ and denote

$$u_N(x) = \sum_{k=0}^{N} u_N(x_k) h_k(x) \ , \ \overline{w} = (u_N(x_0), u_N(x_1), , , u_N(x_N))^T \quad \textbf{(2.65)}$$

$$a_{kj} = b_N(h_j, h_k) \ \ , \quad A = (a_{kj})_{k,j=0}^{N} \quad \textbf{(2.66)}$$

$$\bar{f} = (f(x_0), f(x_1), \ldots f(x_N))^T \ \ , W = diag(w_0, w_1, \ldots, w_N) \quad \textbf{(2.67)}$$

where $\{w_k\}_{k=0}^{N}$ are the weights in the Legendre-Gauss-Lobatto quadrature. Then, Eq. **(2.62)** is equivalent to the linear system

$$A\overline{w} = W\bar{f}, \quad \textbf{(2.68)}$$

The entries a_{kj} can be determined as follows using

$\sum_{j=0}^{N} p(x_j)\, w_j = \int_a^b p(x) w(x) d_x$, for all $p(x) \in P_{2N-1}$, and integration by parts, having to

$$< h_j', h_k' >_N = (h_j', h_k') = -(h_j'', h_k) + h_j' h_k|_{\pm 1} \quad \textbf{(2.69)}$$

$$= -(D^2)_{kj} w_k + d_{0j}\delta_{0k} - d_{Nj}\delta_{Nk} \ .$$

Consequently,

$$a_{kj} = [-\varepsilon(D^2)_{kj} + p(x_k)d_{kj} + q(x_k)\delta_{kj}]w_k + \varepsilon(d_{0j} + h_+\delta_{0j})\delta_{0k} - \varepsilon(d_{Nj} + h_-\delta_{Nj})\delta_{Nk} \qquad 0 \le k,j \le N \quad . \qquad \textbf{(2.70)}$$

Note that here the matrix A is of order $(N+1) \times (N+1)$ on behalf of order $(N-1) \times (N-1)$ as in the pure collocation case. When analyzed

$$< u'_N, h'_K >_N = -u''_N(x_k)w_k + u'_N(1)\delta_{0k} - u'_N(-1)\delta_{Nk} \quad . \qquad \textbf{(2.71)}$$

Thus, taking $v_N = h_j(x)$ in Eq.**(2.62)** for $(j = 0,1,..N)$, and observing that $w_0 = w_N = 2/N(N+1)$, one can find that

$$-\varepsilon u''_N(x_j) + p(x_j)u'_N(x_j) + q(x_j)u_N(x_j) = f(x_j) \; , \; 1 \le j \le N-1 \; , \qquad \textbf{(2.72)}$$

$$a_{\pm}u_N(\pm 1) + b_{\pm}u'_N(\pm 1) = \frac{b_{\pm}}{\varepsilon}\frac{2}{N(N+1)}\tau_{\pm} \quad , \qquad \textbf{(2.73)}$$

where

$$\tau_{\pm} = f(\pm 1) - \{-\varepsilon u''_N(\pm 1) + p(\pm 1)u'_N(\pm 1) + q(\pm 1)u_N(\pm 1)\} \, . \qquad \textbf{(2.74)}$$

As it is seen Eq.**(2.62)** it exactly satisfies Eq.**(2.36)** at the interior collocation points $\{x_j\}_{j=1}^{N-1}$ but the boundary conditions Eq.**(2.37)** (with $c_{\pm} = 0$) is only valid approximately with an error scaled to the residue of the equation Eq.**(2.36)**,with u changed by the approximate solution u_N, at the boundary. So; Eq.**(2.62)** does not correspond exactly to a collocation method and is called as *collocation method in the weak form*, where Eq.**(2.70)** is equivalent to the usual collocation method.

The collocation methods, both in the strong form or weak form, cause to full and in suitable linear system. Therewith, a direct solution method such as Gaussian elimination is only suitable for one-dimensional problems with a small number of unknown. For multi-dimensional problems and problems with large number of unknowns, an iterative method with a suitable preconditioner can be used. To this end, it is preferable to first transform the problem Eq.**(2.36)** and Eq. **(2.37)** into a self-adjoint form. It is observed first that it can be assumed $c_{\pm} = 0$ by modifying the right-hand side function$f$. Then, multiplying the function

$$a(x) = \exp(-\frac{1}{\varepsilon}\int p(x)dx). \qquad \textbf{(2.75)}$$

to Eq.**(2.36)** and noting that $-\varepsilon a'(x) = a(x)p(x)$, Eq.**(2.36)** and Eq. **(2.37)** with $c_{\pm} = 0$ can be written as

$$-(a(x)u'(x))' + b(x)u = g(x)\ , \qquad x \in (-1,1)$$
$$a_{-}u(-1) + b_{-}u'(-1) = 0\ , \qquad a_{+}u(1) + b_{+}u'(1) = 0 \qquad \textbf{(2.76)}$$

where $b(x) = a(x)g(x)/\varepsilon$ and $g(x) = a(x)f(x)/\epsilon$.

Finite Difference Preconditioning

The collocation method in the stable form for Eq.**(2.76)** is:

Find $u_N \in P_N$ such that

$$-(au_N')'(x_j) + b(x_j)u_N(x_j) = g(x_j)\ , \quad 1 \le j \le N-1\ ,$$
$$a_{-}u_N(-1) + b_{-}u_N'(-1) = 0, \quad a_{+}u_N(1) + b_{+}u_N'(1) = 0, \qquad \textbf{(2.77)}$$

As demonstrated earlier, Eq.**(2.77)** can be rewritten as an $(N-1) \times (N-1)$ linear system

$$A\bar{w} = \bar{f}\ , \qquad \textbf{(2.78)}$$

where the unknowns are $\{w_j = u_N(x_j)\}_{j=1}^{N-1}$,$\bar{w} = (w_1, \dots \dots, w_{N-1})^T$ and $\bar{w} = (f(w_1), \dots, f(w_{N-1}))^T$. As suggested by Orszag, it has builded a preconditioner for A by using a finite difference approximation to Eq.**(2.77)**. Let it define

$$h_k = x_{k-1} - x_k\ , \quad \bar{h}_k = \frac{1}{2}(x_{k-1} - x_{k+1}),$$
$$x_{k+\frac{1}{2}} = \frac{1}{2}(x_{k+1} + x_k)\ , \ a_{k+\frac{1}{2}} - a(x_{k+\frac{1}{2}})\ . \qquad \textbf{(2.79)}$$

2.2 Spectral Galerkin Methods

An alternative technique to spectral collocation is the so called spectral Galerkin method which depends on a changeable formulation and uses, on behalf of Lagrange polynomials, compact combinations of orthogonal polynomials as basis functions. It will be shown that by choosing suitable basis functions, the Spectral-Galerkin Method can cause to be conditioned linear system with rare matrices for problems with constant or polynomial coefficients.

Now, proof of the ideas of Spectral-Galerkin Methods for the two point BVP:

$$-\varepsilon U'' + p(x)U' + g(x)U = F \ , \ x \in I = (-1,1), \quad \textbf{(2.80)}$$

(where ε is a control parameter for singularity) with the general condition

$$a_- U(-1) + b_- U'(1) = c_- \ , a_+ U(1) + b_+ U'(1) = c_+, \quad \textbf{(2.81)}$$

satisfying

$$a_-^2 + b_-^2 \neq 0 \text{ and } a_- b_- \leq 0 \ ;$$
$$a_+^2 + b_+^2 \neq 0 \text{ and } a_+ b_+ \geq 0 \ ;$$
$$g(x) - \frac{1}{2} p'(x) \geq 0 \ , \forall x \in (I);$$
$$p(1) > 0 \text{ if } b_+ \neq 0 \ , p(-1) < 0 \text{ if } b_- \neq 0 \ . \quad \textbf{(2.82)}$$

This includes in particular the Dirichlet ($a_\pm = 1$ and $b_\pm = 1$), the Neumann ($a_\pm = 0$ and $b_\pm = 1$), and the mixed ($a_- = b_+ = 0$ or $a_+ = b_- = 0$) boundary conditions. Assume that $a_\pm$, $b_\pm$ and $c_\pm$ satisfy Eq.**(2.39)** so that the problem Eq.**(2.80)** and Eq. **(2.81)** is well-posed.

Unlike the pseudospectral or collocation methods which need to have approximate solution to satisfy Eq.**(2.80)** the Galerkin Method depends on permutational formulation. Therefore, it is desirable, whenever probable, to reformulate the problem Eq.**(2.80)** and Eq.**(2.81)** into a self-adjoint form [5].

Reformulation of the Problem

First reduce Eq.**(2.80)** and Eq.**(2.81)** to a problem with homogeneous boundary conditions [5]:

- *Case 1*: $a_\pm = 0$ and $b_\pm \neq 0$. Setting to $\tilde{u} = \beta x^2 + \gamma x$, where β and γ are uniquely specified by asking $\tilde{u}$ to satisfy Eq.**(2.81)** namely

$$-2b_- \beta + b_- \gamma = c_- \ \ , 2b_+ \beta + b_+ \gamma = c_+, \quad \textbf{(2.83)}$$

- *Case 2*: $a_-^2 + a_+^2 \neq 0$. Setting $\tilde{u} = \beta x^2 + \gamma x$, where β and γ can again be uniquely determined by asking $\tilde{u}$ to suffice Eq.**(2.81)**. Actually, one has

$$(-a_- + b_-)\beta + a_- \gamma = c_- \ , \ (a_+ + b_+)\beta + a_+ \gamma = c_+ \quad \textbf{(2.84)}$$

whose determinant is

$$DET = -a_- a_+ + b_- a_+ - a_- a_+ - b_+ a_- \ . \quad \textbf{(2.85)}$$

Thus Eq.**(2.39)** implies that $b_- \leq 0$ and $b_+ \geq 0$ which result $DET < 0$. Now, setting the $u = U - \tilde{u}$ and $f = F - (-\varepsilon\tilde{u}'' + p(x)\tilde{u}' + g(x)\tilde{u})$. Then u satisfies the equation

$$\varepsilon u'' + p(x)u' + g(x)u = f \text{ , in } I = (-1,1) \quad \textbf{(2.86)}$$

with the homogeneous boundary conditions

$$a_- u(-1) + b_- u'(1) = 0 \ , \ a_+ u(1) + b_+ u'(1) = 0. \quad \textbf{(2.87)}$$

Next, the above equation is transformed into a self-adjoint form which is more advantages for error analysis and for progressing effective numerical schemes. For this purpose, multiplying the $a(x) = \exp(-\frac{1}{\varepsilon}\int p(x)dx$ – Eq.**(2.86)** and noting $-\varepsilon d(x) = a(x)p(x)$ then Eq.**(2.86)** is found to be equivalent to [5]

$$-(a(x)u'(x))' + b(x)u = g(x) \ , \quad \textbf{(2.88)}$$

where $b(x) = a(x)g(x)/\varepsilon$ and $g(x) = a(x)f(x)/\varepsilon$.

2.2.1 Galerkin Method

Here, we are looking for the approximate solutions of Eq.**(2.88)** and Eq.**(2.87)** in the space

$$X_N = \{v \in P_N : a_\pm v(\pm 1) + b_\pm v'(\pm 1) = 0\} \ . \quad \textbf{(2.89)}$$

Verifiying the exact boundary condition. This is different from a usual finite element approach where only the Dirichlet boundary conditions are imperative while the general boundary conditions Eq.**(2.87)** are processed as natural boundary conditions. The essential advantage of this approach is that it cause sparse matrices for problems with constant or polynomial coefficients, while the disadvantage is that a powerful order on the solution is essential for convergence [5].

Let $w(x)$ be a positive weight function and $I_N = C(-1,1) \rightarrow P_N$ be the interpolating operator associated with Gauss-Lobatto points. Then, the Spectral Galerkin Method for Eq.**(2.88)** and Eq.**(2.87)** is to look for $u_N \in X_N$ such that

$$-([I_N(a(x))u_N']', v_N)_w + (I_N(b(x))u_N), v_N)_w = (I_N f, v_N)_w \ , \forall v_N \in X_N. \quad \textbf{(2.90)}$$

Given a set of basis function $\{\varphi_k\}_{k=0,1,..,N-2}$ for X_N, define

$$u_N(x) = \sum_{n=0}^{N-2} \hat{u}_n \, \emptyset_n(x) \, , \, \overline{u} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{N-2})^T, \quad \textbf{(2.91)}$$

$$s_{kj} = -([I_N(a(x)\emptyset_j')]', \emptyset_k)_w \ , \ m_{kj} = (I_N(b(x)\emptyset_j), \emptyset_k)_w, \quad \textbf{(2.92)}$$

Thereupon, the stiffness and mass matrices are

$$S = (s_{kj})_{0\le k,j \le N-2} \ , \ M = (m_{kj})_{0\le k,j\le N-2} \, , \quad \textbf{(2.93)}$$

By setting $u_N(x) = \sum_{n=0}^{N-2} \hat{u}_n \, \emptyset_n(x)$ and $v_N(x) = \emptyset_j(x)$, $0 \le j \le N-2$ in Eq. **(2.90)**, it is observed that the equation Eq.**(2.90)** is equivalent to linear system

$$(S + M)\bar{u} = \bar{f}. \quad \textbf{(2.94)}$$

2.2.2 Legendre-Galerkin Method

To identify the important specifications of the Spectral-Galerkin Methods, consider the model problem

$$-u'' + \alpha u = f \text{ , in } I = (-1,1), \quad \textbf{(2.95)}$$

$$a_{\pm} u(\pm 1) + b_{\pm} u'(\pm 1) = 0 \ , \quad \textbf{(2.96)}$$

where α is a non-negative constant. In this case, the Spectral-Galerkin Method becomes [5]:

Find $u_N \in X_N$ such that

$$\int_I \ u_N' v_N' d_x + \alpha \int_I \ u_N v_N d_x = \int_I \ I_N f v_N d_x \ , \forall v_N \in X_N \quad \textbf{(2.97)}$$

where refering to as the Legendre-Galerkin Method for Eq.**(2.95)** and Eq.**(2.96)**.

Basis Function, Stiffness and Mass Matrices

The linear system Eq.**(2.97)** contains dependently the basis functions of X_N. While in FE methods the close points are used to construct basis functions in a way to reduce their interactions in the physical space, the close orthogonal polynomials can be used to construct the basis functions in Spectral Galerkin Method to minize their interactions in the interval space. Correspondingly, the basis functions of the following form can be searched;

$$\emptyset_k(x) = L_k(x) + a_k L_{k+1}(x) + b_k L_{k+2}(x). \quad \textbf{(2.98)}$$

Lemma 2.1: For all $k \geq 0$, there exist unique $\{a_k, b_k\}$ such that $\emptyset_k(x)$ of the form Eq.**(2.101)** satisfies the boundary condition Eq.**(2.96)** [5].

Lemma 2.2: The stiffness matrix S is a diagonal matrix with

$$s_{kk} = -(4k+6)_{b_k}\ ,\ \ (k = 0,1,2,\dots). \quad \textbf{(2.99)}$$

The mass matrix M is a symmetric penta-diagonal matrix whose nonzero elements are [9]

$$m_{jk} = m_{kj} = \begin{cases} \frac{2}{2k+1} + a_k^2 \frac{2}{2k+3} + b_k^2 \frac{2}{2k+5}, & j = k \\ a_k \frac{2}{2k+3} + a_{k+1} b_k \frac{2}{2k+5}, & j = k+1 \\ b_k \frac{2}{2k+5}\ , & j = k+2 \end{cases} \quad \textbf{(2.100)}$$

Algorithm

In summary: given the values of f at LGL points $\{x_i\}$, solution to Eq **(2.97)** , at these LGL points can be acquired as follows:

- Compute LGL points, $\{a_k, b_k\}$ and nonzero elements of S and M,
- Calculate the Legendre coefficients of $I_N f(x)$ from $\{f(x_i)\}_{i=0}^{N}$, (backward Legendre transform) and evaluate $\bar{f}$ in

$$\tilde{f}_k = (I_N f, \psi_k)\ ,\ \ \bar{f} = (\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{N-2})^T\ \ ,$$

- Solve $\bar{u}$ from $(\alpha M + S)\bar{u} = \bar{f}$,
- Determine $\{\hat{u}_j\}_{j=0}^{N}$ such that $\sum_{j=0}^{N-2} \hat{u}_j\ \emptyset_j(x) = \sum_{j=0}^{N} \hat{u}_j\ L_j(x)$,
- Evaluate $u_N(x_j) = \sum_{i=0}^{N} \hat{u}_i\ L_i(x_j)\ \ , j = 0,1,\dots,N$ (forward Legendre transform).

2.2.3 Chebyshev-Galerkin Method

Set $w(x) = (1-x^2)^{-1/2}$ and $f_N = I_N f$ which is the Chebyshev interpolation polynomial of f concerned to the Chebyshev-Lobatto points. Then Eq **(2.90)** have the followings form[5]:

$$\int_I\ u_N''\, v_N\, w\, d_x + \alpha \int_I\ u_N v_N w(x) d_x = \int_I\ I_N f v_N w(x) d_x\ , (\forall v_N \in X_N) \quad \textbf{(2.101)}$$

which is called as the Chebyshev-Galerkin method for the problem defined as

$$-u'' + \alpha u = f \text{ , in } I = (-1,1),$$
$$\alpha_{\pm} u(\pm 1) + b_{\pm} u'(\pm 1) = 0 \quad (\alpha \text{ is a non-negative constant}) . \qquad \textbf{(2.102)}$$

Basis function, stiffness and mass matrices

Like former in the case, there is a need to search the basis functions of X_N of the form

$$\emptyset_k(x) = T_k(x) + a_k T_{k+1}(x) + b_k T_{k+2}(x) . \qquad \textbf{(2.103)}$$

Lemma 2.3:

$$a_k = -\{(a_+ + b_+(k+2)^2)(-a_- + b_- k^2) - \frac{(a_- - b_-(k+2)^2)(-a_+ - b_+ k^2)}{DET_k},$$

$$b_k = \{(a_+ + b_+(k+1)^2)(-a_- + b_- k^2) + \frac{(a_- - b_-(k+1)^2)(-a_+ - b_+ k^2)}{DET_k},$$

$$DET_k = 2a_+ a_- + (k+1)^2 (k+2)^2 (a_- b_+ - a_+ b_- - 2b_- b_+), \qquad \textbf{(2.104)}$$

then

$$\emptyset_k(x) = T_k(x) + a_k T_{k+1} + b_k T_{k+2} , \qquad \textbf{(2.105)}$$

satisfies the boundary condition $\{a_{\pm} u(\pm 1) + b_{\pm} u'(\pm 1) = 0\}$ [5].

So, having by a dimension argument that X_N =span $\{\emptyset_k(x): k = 0,1,\ldots N-2\}$. One can easiliy show from that the mass matrix M is a symmetric positive definite penta-diagonal matrix which has the following

$$m_{jk} = m_{kj} = \begin{cases} c_k \frac{\pi}{2}(1 + a_k^2 + b_k^2) , \ j = k \\ \frac{\pi}{2}(a_k + a_{k+1} b_k) , \ j = k+1 , \\ \frac{\pi}{2} b_k \ , \ j = k+2 \end{cases} \qquad \textbf{(2.106)}$$

where $c_0 = 2$ and $c_k = 1$ for $k \geq 1$.

Algorithm

The Chebyshev-Galerkin Method for $\{(-u'' + \alpha u = f \text{ , in } I = (-1,1)\}$ and $\{a_{\pm} u(\pm 1) + b_{\pm} u'(\pm 1) = 0\}$ includes the following steps [5]:

Step 1: Compute $\{a_k, b_k\}$ and nonzero elements of S and M ;

Step 2: Determine the Chebyshev coefficients of $I_N f(x)$ from $\{f(x_i)\}_{i=0}^{N}$ and evaluate $\bar{f}$;

Step 3: Solve $\bar{u}$ from $(S+M)\bar{u} = \bar{f}$;

Step 4: Calculate $u_N(x_j) = \sum_{i=0}^{N-2} \hat{u}_i \emptyset_i\,(x_j)$, $j = 0,1,..,N$.

2.2.4 Chebyshev-Legendre Galerkin Method

The fundamental benefit of using Chebshev polynomials is that the separated Chebyshev transforms can be implemented in $\sigma(Nlog_2^N)$ operations by using Fast Fourier Transform. But, the Chebyshev-Galerkin method causes to non-symmetric and full stiffness matrices. Besides, the Legendre-Galerkin Method causes to symmetric sparse matrices, but the discrete Legendre transforms are expensive ($\sigma(N^2)$operations). In order to that benefit from the advantages and minimize the difficulties of both the Legendre and Chebyshev polynomials, one may use the so called Chebyshev-Legendre Galerkin Method which can be defined as [5]:

$$\alpha \int_I u_N v_N d_x + \int_I u_N' v_N' d_x = \int_I I_N^c f v_N d_x, \qquad \textbf{(2.107)}$$

where I_N^c specify the interpolation operator connected to the Chebyshev-Gauss Lobatto points.

For the given the values of f at the Chebyshev-Gauss Lobatto points $\left\{x_i = \cos\left(\frac{i\pi}{N}\right)\right\}$ $0 \le i \le N$, determining the values of u_N (solution of Eq.**(2.90)**) at the CGL points can be performed by the following steps [5]:

Step 1: Compute $\{a_k, b_k\}$ and nonzero element of S and M;

Step 2: From an estimate of the Legendre coefficients of $I_N^c f(x)$ from $\{f(x_i)\}_{i=0}^{N}$,

Step 3: Evaluate $\bar{f}$ from $f_k = \int_I I_N^c f \emptyset_k d_x$, $\bar{f} = (f_0, f_1, \dots, f_{N-2})^T$solve $\bar{u}$ from $(S+M)\bar{u} = \bar{f}$;

Step 4: Evaluate $u_N(x_j) = \sum_{i=0}^{N-2} \hat{u}_i \emptyset_i\,(x_j)$, $(j = 0,1,\dots,N)$

CHAPTER 3

APPLICATIONS

3.1 Fractional Sturm-Liouville Eigen-Problems: Theory and Numerical Solutions

In paper [8] the authors examine the regular fractional Sturm-Liouville problem of two types, with respect to the combinations of left-sided and right sided fractional derivatives of order $\mu = \frac{v}{2} \in (0,1)$ both in Riemann-Liouville and Caputo sense. They obtained the analytical eigensolutions to both types of problems as non polynomial functions called Jacobi poly-fractionamials. In both problems the eigenvalues are shown to be explicit, real, separated and simple. The corresponding eigenfunctions are orthogonal according to the related weight functions.

Moreover, some useful relations for these eigenfunctions and some important properties for poly-fractionomials were obtained in that study [8]. Sturm-Liouville problem for the some cases mentioned earlier were studied and they derived the corresponding eigenfunctions and numerical solutions. [See the reference [8] also for the results of singular case].

Basic Definitions

The left-sided and right-sided Riemann-Liouville integrals of order μ, when $0 < \mu < 1$ are described, as follows [10]

$$\left({}^{RL}_{x_L}I^{\mu}_{x} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_{x_L}^{x} \frac{f(s)ds}{(x-s)^{1-\mu}}, \quad x > x_L \qquad \textbf{(3.1)}$$

and

$$\left({}^{RL}_{x}I^{\mu}_{x_R} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{x_R} \frac{f(s)ds}{(s-x)^{1-\mu}}, \quad x < x_R \qquad \textbf{(3.2)}$$

where Γ stands for the Euler Gamma function. The, left-sided and right-sided fractional derivatives of order μ are described, depending on Eq.**(3.1)** and Eq.**(3.2)** , as

$$\left({}_{x_L}^{RL}D_x^{\mu} f\right)(x) = \frac{d}{dx}\left({}_{x_L}^{RL}I_x^{1-\mu} f\right)(x) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_{x_L}^{x} \frac{f(s)ds}{(x-s)^{\mu}}\ ,\ \ x > x_L \tag{3.3}$$

and

$$\left({}_{x}^{RL}D_{x_R}^{\mu} f\right)(x) = -\frac{d}{dx}\left({}_{x}^{RL}I_{x_R}^{1-\mu} f\right)(x) = \frac{1}{\Gamma(1-\mu)} - \frac{d}{dx}\int_{x}^{x_R} \frac{f(s)ds}{(s-x)^{\mu}}\ ,\ x < x_R \tag{3.4}$$

Additionally, the corresponding left-sided and right-sided Caputo derivatives of order $\mu \in (0,1)$ are acquired as

$$\left({}_{x_L}^{C}D_x^{\mu} f\right)(x) = \left({}_{x_L}^{RL}I_x^{1-\mu}\frac{df}{dx}\right)(x) = \frac{1}{\Gamma(1-\mu)}\int_{x_L}^{x} \frac{f'(s)ds}{(s-x)^{\mu}}\ ,\ \ x > x_L \tag{3.5}$$

and

$$\left({}_{x}^{C}D_{x_R}^{\mu} f\right)(x) = \left({}_{x}^{RL}I_{x_R}^{1-\mu}\frac{-df}{dx}\right)(x) = \frac{1}{\Gamma(1-\mu)}\int_{x_L}^{x} \frac{-f'(s)ds}{(x-s)^{\mu}}\ ,\ \ x < x_R \tag{3.6}$$

The two definitions of the left-sided and right-sided fractional derivatives of both types are connected by the following relationship,

$$\left({}_{x_L}^{RL}D_x^{\mu} f\right)(x) = \frac{f(x_L)}{\Gamma(1-\mu)(x-x_L)^{\mu}} + \left({}_{x_L}^{C}D_x^{\mu} f\right)(x), \tag{3.7}$$

and

$$\left({}_{x}^{RL}D_{x_R}^{\mu} f\right)(x) = \frac{f(x_R)}{\Gamma(1-\mu)(x-x_L)^{\mu}} + \left({}_{x}^{C}D_{x_R}^{\mu} f\right)(x). \tag{3.8}$$

Additionally, integration by parts in the fractional sense is acquired for the aforesaid fractional derivatives as follows;

$$\int_{x_L}^{x_R} f(x)\,{}_{x}^{RL}D_{x_R}^{\mu} g(x)dx = \int_{x_L}^{x_R} g(x)\,{}_{x_L}^{C}D_x^{\mu} f(x)dx - f(x)\,{}_{x}^{RL}I_{x_R}^{\mu} g(x)\ |_{x=x_L}^{x_R}, \tag{3.9}$$

and

$$\int_{x_L}^{x_R} f(x)\,{}_{x_L}^{RL}D_x^{\mu} g(x)dx = \int_{x_L}^{x_R} g(x)\,{}_{x}^{C}D_{x_R}^{\mu} f(x)dx + f(x)\,{}_{x_L}^{RL}I_x^{\mu} g(x)\ |_{x=x_L}^{x_R}. \tag{3.10}$$

As a result, the Riemann-Liouville fractional derivatives useful speciality is remembered. Let's suppose $0 < p < 1$ and $0 < q < 1$ and $f(x_L) = 0\ \ x > x_L$, then

$${}_{x_L}^{RL}D_x^{p+q} f(x) = \left({}_{x_L}^{RL}D_x^{p}\right)\left({}_{x_L}^{RL}D_x^{q}\right) f(x) = \left({}_{x_L}^{RL}D_x^{q}\right)\left({}_{x_L}^{RL}D_x^{p}\right) f(x). \tag{3.11}$$

Description of the regular fractional Sturm-Liouville problems of 1st and 2nd kind

A regular fractional Sturm-Liouville problem (RFSLP) of order $v \in (0,2)$ is taken in consideration, where the differential part contains both left-and right-sided fractional derivatives, each of order $\mu = \frac{v}{2} \in (0,1)$ as [11]

$$ {}^{RL}D^{\mu}\Big[p_i(x)\ {}^{C}D^{\mu}\Phi_{\lambda}^{(i)}(x)\Big] + q_i(x)\Phi_{\lambda}^{(i)}(x) + \lambda w_i(x)\Phi_{\lambda}^{(i)}(x) = 0 \quad x \in [x_L, x_R] \quad \textbf{(3.12)} $$

where $i \in \{1,2\}$ with $i = 1$ symbolising the RFSLP of first kind, where ${}^{RL}D^{\mu} \equiv {}^{RL}_{\ x}D^{\mu}_{x_R}$ and ${}^{C}D^{\mu} \equiv {}^{\ C}_{x_L}D^{\mu}_{x}$ and $i = 2$ related to the RFSLP of second kind in which ${}^{RL}D^{\mu} \equiv {}^{RL}_{x_L}D^{\mu}_{x}$ and ${}^{C}D^{\mu} \equiv {}^{C}_{x}D^{\mu}_{x_R}$. In this case $\mu \in (0,1)$, $p_i(x) \neq 0$, $w_i(x)$ is non-negative weight function, and $q_i(x)$ is a probable function, also p_i, q_i and w_i are real-valued continuos functions in the interval $[x_L, x_R]$.

The boundary-value problem in Eq.**(3.12)** provides the boundary conditions

$$ a_1\Phi_{\lambda}^{(i)}(x_L) + a_2\, {}^{RL}I^{1-\mu}\Big[p_i(x)\ {}^{C}D^{\mu}\Phi_{\lambda}^{(i)}(x)\Big]\ |_{x=x_L} = 0\ , \quad \textbf{(3.13)} $$

$$ b_1\Phi_{\lambda}^{(i)}(x_R) + b_2\, {}^{RL}I^{1-\mu}\Big[p_i(x)\ {}^{C}D^{\mu}\Phi_{\lambda}^{(i)}(x)\Big]\ |_{x=x_R} = 0 \quad , \quad \textbf{(3.14)} $$

where $a_1^2 + a_2^2 \neq 0$, $b_1^2 + b_2^2 \neq 0$, and with $i = 1$ denoting the RFSLP of first kind, having ${}^{RL}I^{1-\mu} \equiv {}^{RL}_{\ x}I^{1-\mu}_{x_R}$ while ${}^{RL}I^{1-\mu} \equiv {}^{RL}_{x_L}I^{1-\mu}_{x}$ when $i = 2$ for RFSLP of second kind.[8]

Theorem 3.1

The eigenvalues are real in Eq.**(3.12)**, and the eigenfunctions, related to explicit eigenvalues in each problem are orthogonal with compared to the weight functions $w_i(x)$ [8].

The following fractional derivative operator is described [8]

$$ L_i^{\mu} \coloneqq {}^{RL}D^{\mu}[K\, {}^{c}D^{\mu}(.)], \quad \textbf{(3.15)} $$

where K is constant, $L_1^{\mu} \coloneqq {}^{RL}_{\ x}D^{\mu}_{x_R}[K\, {}^{\ C}_{x_L}D^{\mu}_{x}(.)]$ in RFSLP-1 and for the case of RFSLP-2, $L_2^{\mu} \coloneqq {}^{RL}_{x_L}D^{\mu}_{x}[K\, {}^{C}_{x}D^{\mu}_{x_R}(.)]$, where $\mu \in (0,1)$. Actually, $p_i(x) = K$, continuous non-zero constant function $\forall x \in [-1,1]$ and K is a stiffness constant. Then,

$$L_i^{\mu}\Phi_{\lambda}^{(i)}(x) + \lambda(1-x)^{-\mu}(1+x)^{\mu}\Phi_{\lambda}^{(i)}(x) = 0,\ i \in \{1,2\}, x \in [-1,1] \quad \textbf{(3.16)}$$

For the problems RFSLP-1 and RFSLP-2, Eq.**(3.16)** is solved and it is subjected to a homogeneous Dirichlet and a homogeneous fractional integro-differential boundary condition [8].

$$\Phi_{\lambda}^{(1)}(-1) = 0,$$
$$ {}^{RL}_{x}I_{1}^{1-\mu}\left[K\,{}_{-1}^{C}D_{x}^{\mu}\,\Phi_{\lambda}^{(1)}(x)\right]|_{x=+1} = 0, \quad \textbf{(3.17)}$$

and

$$\Phi_{\lambda}^{(2)}(+1) = 0,$$
$$ {}^{RL}_{-1}I_{x}^{1-\mu}\left[K\,{}_{x}^{C}D_{1}^{\mu}\,\Phi_{\lambda}^{(2)}(x)\right]|_{x=-1} = 0. \quad \textbf{(3.18)}$$

In non-local calculus and fractional differential equations, the boundary conditions Eq.**(3.17)** and Eq.(**3.18)** are natural and they are activated by the fractional integration by parts in Eq.(**3.9)** and Eq.**(3.10)**. Actually, the essential specifications of eigensolutions in the theory of classical Sturm-Liouville problems are related with the integration-by-parts formula and the selected boundary conditions. In the setting chosen here in [8, 25], it was shown that the eigen-spectra of RFSLP-1 and RFSLP-2 are simple and fully discrete.

Theorem 3.2

The exact eigenfunctions to Eq.**(3.16)**, when $i = 1$, thus RFSLP-1 subject to Eq. **(3.17)** are given as [8]

$$\Phi_{n}^{(1)}(x) = (1+x)^{\mu}\,P_{n-1}^{-\mu,\mu}(x), \quad (\forall n \geq 1), \quad \textbf{(3.19)}$$

and the suitable distinct eigenvalues are

$$\lambda_{n}^{(1)} = -\frac{K\,\Gamma(n+\mu)}{\Gamma(n-\mu)}\ ,\ (\forall n \geq 1). \quad \textbf{(3.20)}$$

Additionally, the exact eigenfunctions to Eq.**(3.16)**, when $i = 2$, thus RFSLP-2 subject to Eq.**(3.18)**, are given as [8]

$$\Phi_{n}^{(2)}(x) = (1-x)^{\mu}P_{n-1}^{\mu,-\mu}(x),\ (\forall n \geq 1), \quad \textbf{(3.21)}$$

where the suitable distinct eigenvalues are

$$\lambda_n^{(2)} = \lambda_n^{(1)} = -\frac{K\,\Gamma(n+\mu)}{\Gamma(n-\mu)}\ , \forall n \geq 1\ . \qquad \textbf{(3.22)}$$

Proof 3.1

The proof is divided to three parts. In Part1, Eq.**(3.19)** and Eq.**(3.20)** is proved firstly. From Eq.**(3.19)**, it is explicit that $\Phi_n^{(1)}(-1) = 0$. For checking the other boundary conditions since $\Phi_n^{(1)}(x) = 0$, by formula Eq.**(3.7)**, ${}_{-1}^{C}D_x^{\mu}$ is switched by ${}_{-1}^{RL}D_x^{\mu}$. Then by doing some work on RL fractional derivative and integration, it is shown that Eq. **(3.19)** actually provide Eq.**(3.16)**, when $i = 1$, with the eigenvalues Eq.**(3.20)**. Lastly, the fractional derivative on the left-hand side and fractional integration on the right side are performed by using ${}_{-1}^{RL}D_x^{\mu}\{(1+x)^{\mu}\, P_n^{-\mu,\mu}(x)\} = \frac{\Gamma(n+\mu+1)}{\Gamma(n+1)} P_n(x)$ and ${}_{x}^{RL}I_1^{\mu}\{(1-x)^{-\mu} P_n^{-\mu,+\mu}(x)\} = \frac{\Gamma(n+\mu+1)}{\Gamma(n+1)} P_n(x)$ to get the eigenvalues $\lambda_n^{(1)}$of problem type RFSLP-1 which are shown to be real and discrete (see [8] for details).

In Part2, for the problem type RFSLP-2 eigenfunctions and eigenvalues can be obtained by following the same steps as in Part1. It is explicit that $\Phi_n^{(2)}(1) = 0$. Since $\Phi_n^{(2)}(1) = 0$, to control the other boundary condition in Eq**.(3.18)**, by Eq. **(3.8)**, ${}_{x}^{C}D_1^{\mu}$ is switched by ${}_{x}^{RL}D_1^{\mu}$. Finally the orthogonal of the eigenfunctions in Eq.**(3.29)** with compared to $w_2(x) = (1-x)^{-\mu}(1+x)^{-\mu}$ is shown where $c_k^{\mu,-\mu}$ represents the orthogonality constant of the family of Jacobi polynomials with parameters $\mu, -\mu$.

In Part3, its shown that $\sum_{n=1}^{N} a_n \Phi_n^{i}(x) \xrightarrow{L_w^2} f(x)$, by using Cauchy-Schwarts inequality and Weierstrass Theorem, specifying that$\{\Phi_n^{(i)}(x)\colon n = 1,2,..\}$ is dense in the Hilbert space forming a basis for $L_w^2[-1,1]$. Moreover, it is proved by contradiction technique that the $\lambda_n^{(i)}$ are simple: (see reference [8] for detailed steps of the proof).

Example 3.1

The eigenfunctions of RFSLP-1,$\Phi_n^{(1)}(x)$,of different orders and related to different values of $\mu = 0.35$, $\mu = 0.5, \mu = 0.99$ is plotted in Fig1. In these figures, the eigensolutions are compared with the corresponding standard Jacobi polynomials $P_n^{+\mu,-\mu}(x)$. Similarly, it is plotted the eigenfunctions of RFSLP-2, $\Phi_n^{(2)}(x)$ of different orders that are compared to $P_n^{+\mu,-\mu}(x)$ in Fig2.

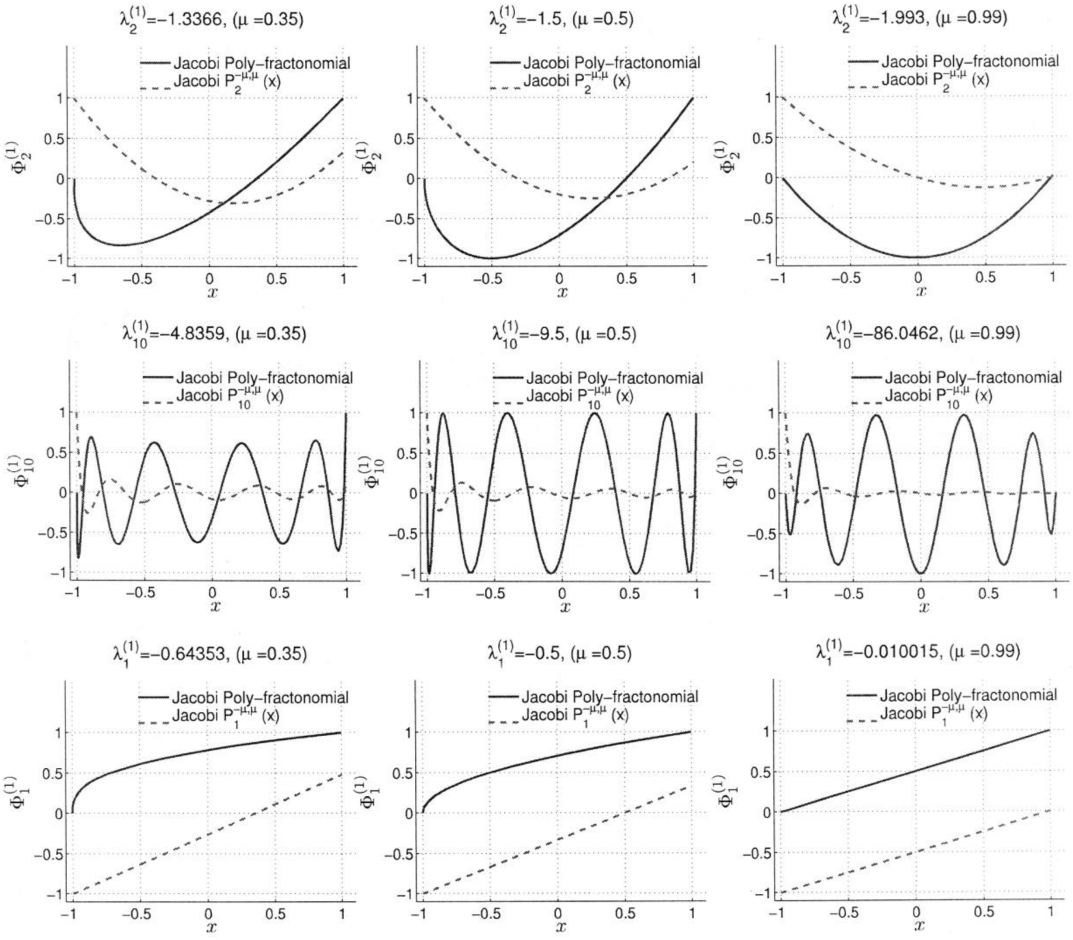

Fig1: *Eigenfunctions $\Phi_n^{(1)}(x)$ of RFSLP-1 versus x, for values n=2,10,1 and for values pf fractional order* $\mu = \frac{v}{2} = 0.35,0.5,0.99$ [8].

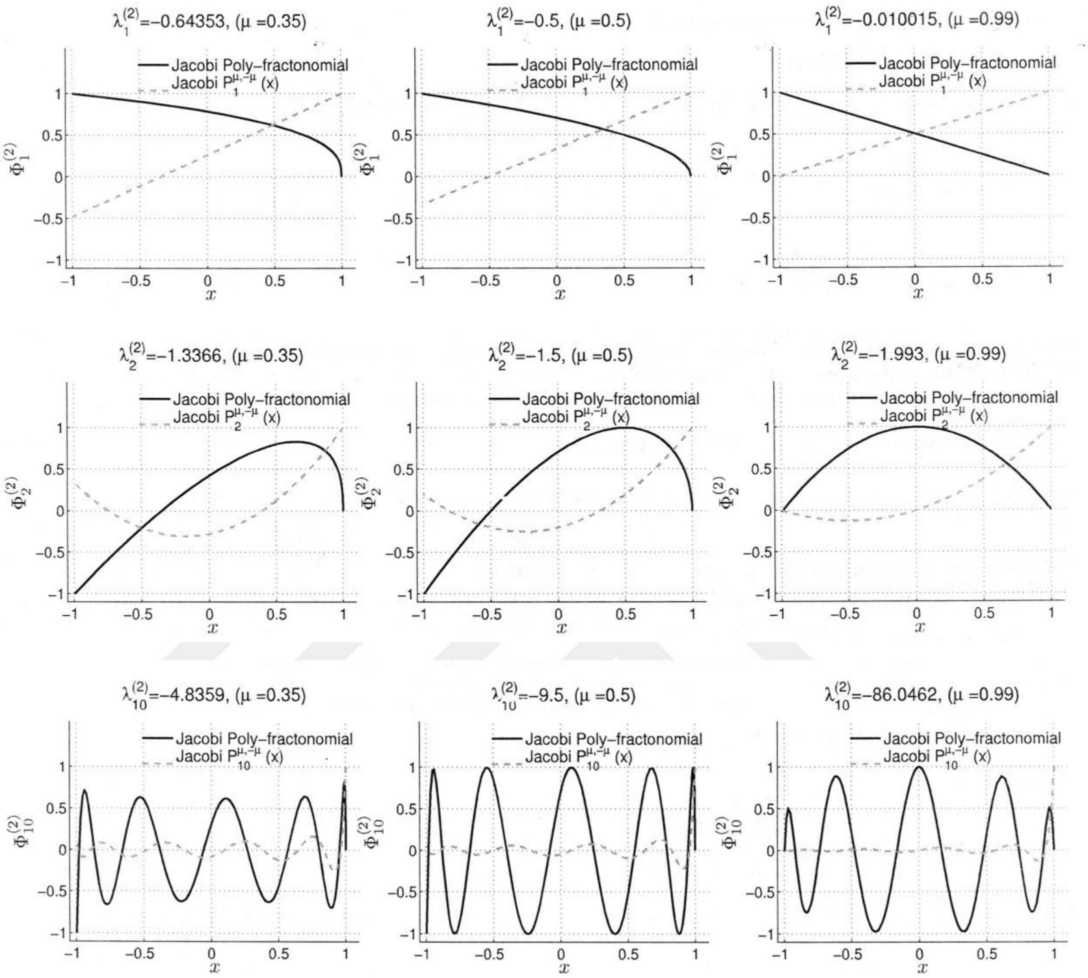

Fig2: *Eigenfunctions $\Phi_n^{(2)}(x)$ of RFSLP-2 versus x, for values n=2,10,1 and for values pf fractional order* $\mu = \frac{v}{2} = 0.35, 0.5, 0.99$ [8].

Definition 3.1

A fractonomial is described as a function $f: \mathrm{c} \rightarrow \mathrm{c}$ of non-integer power (expressed as $t^{k+\mu}$, where k$\in \mathbb{Z}^+$ and $\mu \in (0,1)$ in which the power can be written as a sum of an integer and non-integer number. In addition to this, $F_e^{n+\mu}$ denotes the fractal expansion set, which is described as the set of all fractonomials of order less than equal $n + \mu$ and given in [8] as

$$F_e^{n+\mu} = span\{t^{k+\mu} : \mu \in (0,1),\ k = 0,1,..,n\}. \qquad \textbf{(3.23)}$$

Definition 3.2

A poly-fractonomial of order $n+\mu<\infty, n\in\{0,1,2,..,N<\infty\}$ and $\mu\in(0,1)$, is described as

$$F_{n+\mu}(t)=a_0t^{\mu}+a_1t^{1+\mu}+\cdots+a_nt^{n+\mu},$$

where $a_j\in$ c, are constants and $t^{j+\mu}\in F_e^{n+\mu}$ for $j\in\{0,1,..,n\}$. $\mathcal{F}^{n+\mu}$ denotes here the space of all poly-fractonomials up to order $n+\mu$ [8].

Theorem 3.3

The altered eigensolutions to Eq.**(3.16)**, $\widetilde{\Phi}_n^{(i)}(t)$, $n\in N$ and $n<\infty$ form a consummate hierarchical basis for the finite dimensional space of poly-fractonomials $F_{n-1+\mu}$, where $\mu\in(0,1)$ [8].

For RFSLP-1& RFSLP-2; the properties of the solutions of (3.16) [8]*:*

- Non-polynomial nature:

  Because of the multiplier $(1\pm x)^{\mu}$ of fractional power, the eigenfunctions display a non-polynomial behavior. As a consequence, to differentiate as Jacobi poly-fractonomial of order $n+\mu$.

- Asymptotic eigenvalues $\lambda_n^{(i)}$:

  The asymptotic values are summarized as

$$|\lambda_n^{(i)}|=\begin{cases}Kn^2, \mu\to 1\\ Kn,\ \mu\to 1/2\\ K\ \ \mu\to 0\end{cases},$$

- Orthogonality:

$$\int_{-1}^{1}(1-x)^{-\mu}(1+x)^{-\mu}\Phi_k^{(i)}(x)\Phi_m^{(i)}(x)dx=$$

$$\int_{-1}^{1}(1-x)^{\alpha_i}(1+x)^{\beta_i}P_{m-1}^{\alpha_i,\beta_i}(x)-P_{m-1}^{\alpha_i,\beta_i}(x)dx=J_k^{\alpha_i,\beta_i}\delta_{kj} \quad \textbf{(3.24)}$$

$$J_k^{\alpha_i,\beta_i}=\frac{2}{2k-1}\frac{\Gamma(k+\alpha_i)\,\Gamma(k+\beta_i)}{(k-1)!\,\Gamma(k)}\ , \quad \textbf{(3.25)}$$

  where $(\alpha_1,\beta_1)=(-\mu,\mu)$ and $(\alpha_2,\beta_2)=(\mu,-\mu)$.

- Fractional Derivatives:

$$ {}^{RL}_{-1}D^{\mu}_{x}\Phi^{(1)}_{n} = {}^{C}_{-1}D^{\mu}_{x}\Phi^{(1)}_{n} = {}^{RL}_{x}D^{\mu}_{1}\Phi^{(2)}_{n} = {}^{C}_{x}D^{\mu}_{1}\Phi^{(2)}_{n} = \frac{\Gamma(n+\mu)}{\Gamma(n)} P_{n-1}(x), \qquad \textbf{(3.26)} $$

where $P_{n-1}(x)$ state that standard Legendre polynomial of order $n-1$.

- Orthogonality property for fractional derivative:

$$ \int_{-1}^{1} D^{\mu}\Phi^{(i)}_{k} D^{\mu}\Phi^{(i)}_{j} dx = (\frac{\Gamma(k+\mu)}{\Gamma(k)})^{2} \frac{2}{2k-1} \delta_{kj}, \qquad \textbf{(3.27)} $$

where D^{μ} can be either ${}^{RL}_{-1}D^{\mu}_{x}$ or ${}^{C}_{-1}D^{\mu}_{x}$, when $i = 1$, and can be either ${}^{RL}_{x}D^{\mu}_{1}$ or ${}^{C}_{x}D^{\mu}_{1}$ when $i = 2$ [12,13].

- First derivatives:

$$ \frac{d\Phi^{(1)}_{n}(x)}{dx} = \mu(1+x)^{\mu-1}P^{-\mu,\mu}_{n-1}(x) + \frac{n}{2}(1+x)^{\mu}P^{1-\mu,1+\mu}_{n-2}(x), \qquad \textbf{(3.28)} $$

$$ \frac{d\Phi^{(2)}_{n}(x)}{dx} = -\mu(1+x)^{\mu-1}P^{\mu,-\mu}_{n-1}(x) + \frac{n}{2}(1-x)^{\mu}P^{1+\mu,1-\mu}_{n-2}(x). \qquad \textbf{(3.29)} $$

- Special values:

$$ \Phi^{(1)}_{n}(-1) = 0, \qquad \textbf{(3.30)} $$

$$ \Phi^{(1)}_{n}(+1) = 2^{\mu}\binom{n-1-\mu}{n-1}, \qquad \textbf{(3.31)} $$

$$ \Phi^{(2)}_{n}(+1) = 0, \qquad \textbf{(3.32)} $$

$$ \Phi^{(2)}_{n}(-1) = (-1)^{n-1}\Phi^{(1)}_{n}(+1). \qquad \textbf{(3.33)} $$

3.2 Exponentially Accurate Spectral and Spectral Element Methods for Fractional ODEs

In this study [17] the authors, Petrov-Galerkin (PG) spectral method is improved for Fractional Initial-Value Problems (FIVPs) and Fractional Final Value Problems (FFVPs) of the fractional derivative order $\nu \in (0,1)$, with Dirichlet initial/final conditions. Petrov-Galerkin spectral method is initially developed where the corresponding stiffness matrix is diagonal. The solutions for FIVPs and FFVPs are calculated in terms of the newly defined Jacobi poly fractonomials [17].

Basic Definition

The Fractional Initial-Value Problem (FIVP) is described in [17] as follows

$$ {}_0D_t^v u(t) = f(t)\,,\ t \in (0,T], \tag{3.34}$$

$u(0) = u_0$, where ${}_0D_t^v$ specify the left-sided Riemann-Liouville fractional derivative of order $v \in (0,1)$ and given as

$$ {}_0D_t^v u(t) = \frac{1}{\Gamma(1-v)}\frac{d}{dt}\int_0^t \frac{u(s)ds}{(t-s)^v}\ \ t > 0. \tag{3.35}$$

The Fractional Final-Value Problem (FFVP) is described in [18] with the fractional derivative order $v \in (0,1)$, as

$$ {}_tD_T^v u(t) = f(t)\,,\ t \in (0,T], \tag{3.36}$$

$u(T) = u_T$, where ${}_tD_T^v$ denotes the right-sided Riemann-Lioville fractional derivative of order $v \in (0,1)$ and described as

$$ {}_tD_T^v u(t) = \frac{1}{\Gamma(1-v)}\frac{-d}{dt}\int_t^T \frac{u(s)ds}{(s-t)^v}\ \ t < T. \tag{3.37}$$

In Eq.**(3.35)** and Eq.**(3.37)**, the Caputo fractional derivatives ${}_0^C D_t^v$ and ${}_t^C D_T^v$ can also be used. These above definitions are connected as [18]

$$ {}_0D_t^v u(t) = \frac{u(0)}{\Gamma(1-v)t^v} + {}_0^C D_t^v u(t), \tag{3.38}$$

$$ {}_tD_T^v u(t) = \frac{u(t_T)}{\Gamma(1-v)(T-t)^v} + {}_0^C D_t^v u(t)\ . \tag{3.39}$$

Description of Petrov-Galerkin Spectral Method

First, it is searched for the solution of the FIVPs in terms of the new fractional basis functions, called Jacobi polyfractonomials, which are the eigenfunctions of the FSLP of first kind, explicity acquired as

$$ {}^{(1)}P_n^{\alpha,\beta,\mu}(x) = (1+x)^{-\beta+\mu-1} P_{n-1}^{\alpha-\mu+1,-\beta+\mu-1}(x),\ \ x \in [-1,1], \tag{3.40}$$

where $P_{n-1}^{\alpha-\mu+1,-\beta+\mu-1}(x)$ are the standart Jacobi polynomials in which $\mu \in (0,1)$, $-1 \le \alpha < 2-\mu$, and $-1 \le \beta < \mu - 1$, and ${}^{(1)}P_n^{\alpha,\beta,\mu}(x)$ symbolise the eigenfunctions of the regular FSLP of first kind [18]. For $\alpha = \beta = -1$, the fractional eigenfunctions are presented as [17].

$$^{(1)}P_n^{\mu}(x) = (1+x)^{\mu} P_{n-1}^{-\mu,\mu}(x),\ x \in [-1,1] \quad \textbf{(3.41)}$$

Taking $u_0 = 0$ and $t \in [0,T]$ gives,

$$^{1}\tilde{P}_n^{\mu}(t) = (\tfrac{2}{T})^{\mu} t^{\mu} P_{n-1}^{-\mu,\mu}\left(x(t)\right), \quad \textbf{(3.42)}$$

which specify the changed basis functions of fractional order $(n-1+\mu)$ that is acquired through the affine mapping $x = \frac{2t}{T-1}$ transforming the standard interval $[-1,1]$ to $[0,T]$. The left-sided Riemann Liouville fractional derivative of Eq **(3.42)** is given as

$$_0D_t^{\mu}(\,^{1}\tilde{P}_n^{\mu}\left(x(t)\right) = \left(\frac{2}{T}\right)^{\mu}\ _{-1}D_x^{\mu}\left(\,^{1}\tilde{P}_n^{\mu}(x)\right) = \left(\frac{2}{T}\right)^{\mu} \frac{\Gamma(n+\mu)}{\Gamma(n)} P_{n-1}\left(x(t)\right), \quad \textbf{(3.43)}$$

where the μ^{-th} order fractional derivative of such fractional basis functions of order $(n-1+\mu)$ is a standard Legendre polynomials of integer order $(n-1)$.

Now, Eq.**(3.34)** is tested with respect to a various set of test functions which are in fact the eigenfunctions of the FSLP of second kind

$$^{(2)}P_k^{\alpha,\beta,\mu}(x) = (1-x)^{-\alpha+\mu-1} P_{k-1}^{-\alpha+\mu-1,\beta-\mu+1}(x) \quad (x \in [-1,1]), \quad \textbf{(3.44)}$$

that are in the family of the Jacobi polyfractonomials where this time $-1 \leq \alpha < 1-\mu$ and $-1 \leq \beta \leq 2\mu - 1$. The following fractional test functions are employed.

$$^{(2)}P_k^{\mu}(x) = (1-x)^{\mu} P_{k-1}^{\mu,-\mu}(x) \quad (x \in [-1,1]). \quad \textbf{(3.45)}$$

By executing the same affine mapping $x = \frac{2t}{T-1}$, the changed test functions are obtained [17].

$$^{(2)}\tilde{P}_k^{\mu}\left(x(t)\right) = \left(\frac{2}{T}\right)^{\mu} (T-t)^{\mu} P_{k-1}^{\mu,-\mu}(x(t)), \quad \textbf{(3.46)}$$

relative to the interval $[0,T]$. The right-sided Riemann-Liouville fractional derivate of Eq **(3.46)** is attained as

$$_tD_T^{\mu}(^{(2)}\tilde{P}_k^{\mu}(t)) = \left(\frac{2}{T}\right)^{\mu}\ _xD_{+1}^{\mu}\left(\,^{2}P_k^{\mu}(x)\right) = \left(\frac{2}{T}\right)^{\mu} \frac{\Gamma(k+\mu)}{\Gamma(k)} P_{k-1}\left(x(t)\right), \quad \textbf{(3.47)}$$

where the last equality support to Eq.**(3.43)**. The relation in Eq.**(3.47)** also be connected with the Caputo fractional derivatives due to Eq.**(3.39)**.

In FIVP in Eq.**(3.34)**, it is searched for the approximate solution of the form

$$u(t) \approx u_N(t) = \sum_{n=1}^{N} a_n\, ^{(1)}\tilde{P}_n^{\mu}(t). \quad \textbf{(3.48)}$$

Consequently, the expansion coefficients are obtained in [18] as

$$a_k = \frac{1}{\gamma_k} \int_0^t f(t)\, {}^{(2)}\tilde{P}_k^{\mu}\big(x(t)\big)dt, \tag{3.49}$$

where $\gamma_k = \left(\frac{2}{T}\right)^{2\mu-1} (\frac{k+\mu}{k})^2 \frac{2}{2k-1}$.

The technique of lifting a known solution is applied for the case of non-homogeneous initial conditions by dividing the solution $u(t)$ into two parts as

$$u(t) = u_H(t) + u_D, \tag{3.50}$$

in which $u_H(t)$ is compatible to the homogeneous solution and $u_D \equiv u_0$ is the non-zero initial condition. Then, Eq.**(3.50)** is substituted into Eq.**(3.34)** and the fractional derivative is taken on the known u_D to the right-hand side to get

$$_0D_t^V u_H(t) = h(t), \ \ t \in (0,T], \tag{3.51}$$

$u_H(0) = 0$ where $h(t) = f(t) - \frac{u_D}{\Gamma(1-v)t^v}$ .In the problem given by **(3.51)**, the Caputo fractional derivative can also be used with $h(t) = f(t)$.

Example 3.2

The Numerical results obtained by the use of the PG spectral method to solve the fractional initial-value problem $_0D_t^V u(t) = f(t)$, $t \in [0,1]$, corressponding to $v = 1/10$ and $9/10$ were studied in the following example. Here, four different types of exact solutions are considered: (i) monomial $u^{ext}(t) = t^{10}$, (ii) smooth function $u^{ext}(t) = t^6 \sin(\pi t)$, (iii) fractional function $u^{ext}(t) = t^{13/2}\sin(\pi t^{\frac{4}{3}})$ and finally (iv) combination of fractonomials and a smoot function $u^{ext}(t) = t^6 \exp(\mathrm{t}^2) + \mathrm{t}^{8+\frac{5}{7}} + t^{10+1/3}$ [17].

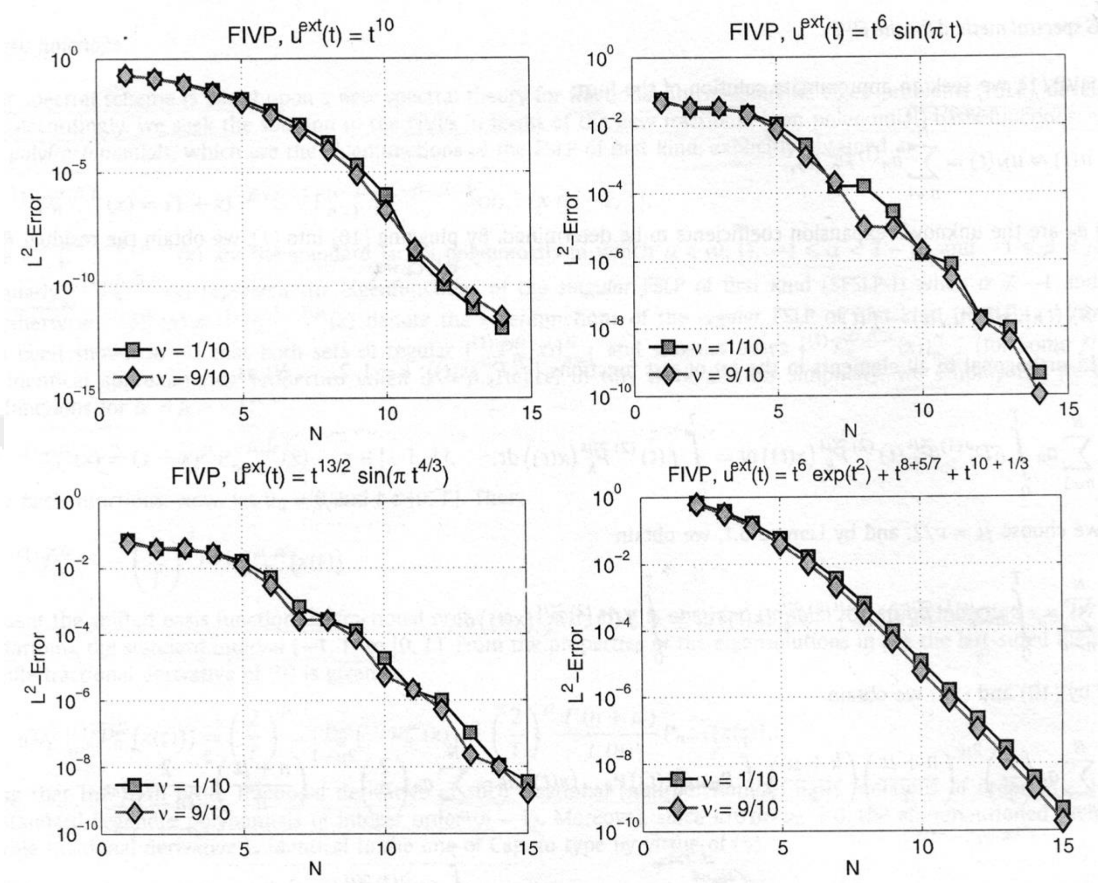

Fig 3: *PG spectral method for FIVP: log-linear L^2- error of the numerical solution to* ${}_0D_t^{\nu}u(t) = f(t),\ t \in [0,1]$, *versus N, corresponding to* $v = 1/10$ *and* $9/10$ where $u^{ext}(t) = t^{10}$ *is the exact solution* [17].

Now, it is searched for the approximate solution to Eq.**(3.36)** of the form

$$u(t) \approx u_N(t) = \sum_{j=1}^{N} b_j{}^{(2)}\tilde{P}^{\mu}\big(x(t)\big)dt, \tag{3.52}$$

where b_j are the unknown expansion coefficients. By plugging Eq.**(3.52)** into Eq. **(3.36)** and forcing the residual function $R_N(t)$ to be L^2-orthogonal to each element in the set of the test functions, the unknown coefficients is obtained [17] as

$$b_k = \frac{1}{\gamma_k}\int_0^T f(t)^{(1)}\tilde{P}_k^{\mu}\big(x(t)\big)dt. \tag{3.53}$$

$u(t)$ is decomposed as in Eq.**(3.50)** and substituted into Eq.**(3.36)** to get the equivalent finite-value problem as in below finite-value problem

$${}_tD_T^{\nu}u_H(t) = w(t),\ \ t \in [0,T), u_H(T) = 0, \tag{3.54}$$

where $(t) = f(t) - \frac{u_T}{\Gamma(1-v)(T-t)^v}$.

Similarly, the numerical results of PG spectral method for FFVP is carried out in [17], for various types of exact solutions like $u^{ext}(t) = (T-t)^{10}, u^{ext}(t) = (T-t)^6 \sin(\pi(T-t))$, $u^{ext}(t) = (T-t)^{13/2} \sin(\pi(T-t)^{\frac{4}{3}})$ and $u^{ext}(t) = (T-t)^6 \exp(\mathrm{T}-\mathrm{t})^2 + (\mathrm{T}-\mathrm{t})^{8+\frac{5}{7}} + (T-t)^{10+1/3}$. Finally, it is observed that [17] in all these test cased the exponential convergence is obtained.

3.3 Legendre Spectral Collocation Method for the Solution of the Model Describing Biological Species Living Together

In the study [21], Legendre spectral collocation method and Bernstein polynomial collocation method are executed together for solving the system of integro-differential equations, which may generally arise in fluid dynamics, biological models and chemical kinetics, representing the model of some biological species living together. The existing techniques reduced the system of integro-differential equations to a system of nonlinear algebraic equations. The solution of this nonlinear system is obtained [21] numerically by Newton's method.

First of all, it is taken notice of the general mathematical model of the mentioned problem consists of integro-differential equation in the following form

$$\frac{dx(t)}{dt} = x(t)[k_1 - \gamma_1 y(t) - \int_{t-T_0}^{t} f_1(t-\tau)y(\tau)d\tau] + g_1(t), \quad 0 \le t \le l\, , k_1, \gamma_1 > 0 \tag{3.56}$$

$$\frac{dy(t)}{dt} = y(t)[-k_2 - \gamma_2 x(t) - \int_{t-T_0}^{t} f_2(t-\tau)x(\tau)d\tau] + g_2(t), 0 \le t \le l, k_2, \gamma_2 > 0 \tag{3.57}$$

with initial conditions $x(0) = \alpha_1, y(0) = \alpha_2$ where g_1, g_2, f_1, f_2 are given functions and $x(t), y(t)$ are unknown functions.

Now, $x(t)$ and $y(t)$ are number of two separate species at time t where first species increases and the second one decreases. When a steady state condition is exchanged between these two species, it is defined by the following system of two integro-differential equations

$$\frac{dx(t)}{dt} = x(t)[k_1 - \gamma_1 y(t) - \int_{t-T_0}^{t} f_1(t-\tau)y(\tau)d\tau] + g_1(t), \quad k_1 > 0 \qquad \textbf{(3.58)}$$

$$\frac{dy(t)}{dt} = y(t)[-k_2 - \gamma_2 x(t) - \int_{t-T_0}^{t} f_2(t-\tau)x(\tau)d\tau] + g_2(t), \quad k_2 > 0 \qquad \textbf{(3.59)}$$

where k_1 and $-k_2$ are the coefficients for increasing and decreasing behaviour of the first and second species. The parameters γ_1 , f_1 and γ_2, f_2 are based on the respective species. Here, in Eqns.**(3.58-3.59)** the function values of $g_1(t)$ and $g_2(t)$ are taken to be zero as a special case of Eqns**.(3.56-3.57)**. The above model is solved numerically with Legendre spectral collocation method and Bernstein polynomial collocation method with a comparative study [21].

Description of the Legendre Spectral Collocation Method

In order to use Legendre spectral collocation method, it is taken notice of the Legendre-Gauss points $\{t_j\}_{j=0}^{M} \in [a,b]$ i.e. , the roots of $L_{M+1}(t) = 0$ where L_{M+1} is the $(M+1)^{\text{th}}$ Legendre Polynomial.

By using the Lagrange interpolation polynomials for the unknown function $x(t)$ and $y(t)$ as

$$x(t) = \sum_{k=0}^{M} x_k F_k(t), \qquad \textbf{(3.60)}$$

$$y(t) = \sum_{k=0}^{M} y_k F_k(t), \qquad \textbf{(3.61)}$$

the system of integro-differential equations in Eqns.**(3.56-3.57)** is approximated andreduced to

$$\sum_{k=0}^{M} x_k F_k'(t) = \sum_{k=0}^{M} x_k F_k(t)[\, k_1 - \gamma_1 \sum_{k=0}^{M} y_k F_k(t) - \int_{t-T_0}^{t} f_1(t-\tau) (\sum_{k=0}^{M} y_k F_k(\tau)d\tau] + \, g_1(t), \qquad \textbf{(3.62)}$$

$$\sum_{k=0}^{M} y_k F_k'(t) = \sum_{k=0}^{M} y_k F_k(t)[\, k_2 - \gamma_2 \sum_{k=0}^{M} x_k F_k(t) - \int_{t-T_0}^{t} f_2(t-\tau) (\sum_{k=0}^{M} x_k F_k(\tau)d\tau] + \, g_2(t). \qquad \textbf{(3.63)}$$

In order to use Gauss-Legendre quadrature rule, is changed the interval $[t - T_0, t]$ to $[-1,1]$ in the integral part of Eqns.**(3.62-3.63)** by

$$s = 1 + 2\left(\frac{\tau - t}{T_0}\right).$$

The integrands of Eqns.**(3.62-3.63)** are reduced to [21]

$$\int_{t-T_0}^{t} f_1(t-\tau)\left(\sum_{k=0}^{M} y_k F_k(\tau)\right)d\tau$$

$$= \frac{T_0}{2}\int_{-1}^{1} f_1\left(-\frac{T_0}{2}(s-1)\right)\left(\sum_{k=0}^{M} y_k F_k(t+\frac{T_0}{2}(s-1)\right)ds$$

$$= \frac{T_0}{2}\sum_{j=0}^{M} w_j f_1(-\frac{T_0}{2}(s_j-1)\left(\sum_{k=0}^{M} y_k F_k(t+\frac{T_0}{2}(s_j-1)\right), \quad \textbf{(3.64)}$$

$$\int_{t-T_0}^{t} f_2(t-\tau)\sum_{k=0}^{M} x_k F_k(\tau))d\tau$$

$$= \frac{T_0}{2}\int_{-1}^{1} f_2\left(-\frac{T_0}{2}(s-1)\right)\left(\sum_{k=0}^{M} x_k F_k(t+\frac{T_0}{2}(s-1)\right)ds$$

$$= \frac{T_0}{2}\sum_{j=0}^{M} w_j f_2(-\frac{T_0}{2}(s_j-1))\left(\sum_{k=0}^{M} x_k F_k(t+\frac{T_0}{2}(s_j-1)\right), \quad \textbf{(3.65)}$$

where s_j , $j = 0,1, \ldots, M$ are Legendre-Gauss points, the roots of $L_{M+1}(t) = 0$ and the weights are $w_j = \frac{2}{\left(1-s_j^2\right)[L'_{M+1}(s_j)]^2}$, $(\, 0 \le j \le M)$. Then using Eqns.**(3.64-3.65)** and Eqns.**(3.62-3.63)** [21], the following expression are obtained [21]

$$\sum_{k=0}^{M} x_k F'_k(t) = \sum_{k=0}^{M} x_k F_k(t)[\, k_1 - \gamma_1 \sum_{k=0}^{M} y_k F_k(t) - \frac{T_0}{2}\sum_{j=0}^{M} w_j f_1(-\frac{T_0}{2}(s_j-1))$$
$$\left(\sum_{k=0}^{M} y_k F_k(t+\frac{T_0}{2}(s_j-1)\right) + g_1(t), \quad \textbf{(3.66)}$$

$$\sum_{k=0}^{M} y_k F'_k(t) = \sum_{k=0}^{M} y_k F_k(t)[\, k_2 - \gamma_2 \sum_{k=0}^{M} x_k F_k(t) - \frac{T_0}{2}\sum_{j=0}^{M} w_j f_2(-\frac{T_0}{2}(s_j-1))$$
$$\left(\sum_{k=0}^{M} x_k F_k(t+\frac{T_0}{2}(s_j-1)\right) + g_2(t). \quad \textbf{(3.67)}$$

The collocation points as Gauss-Legendre points t_i , $i = 0,1, \ldots, M-1$ is performed for Eqns.**(3.66-3.67)** [21]

$$\sum_{k=0}^{M} x_k F'_k(t_i) = \sum_{k=0}^{M} x_k F_k(t_i)[\, k_1 - \gamma_1 \sum_{k=0}^{M} y_k F_k(t_i) -$$
$$\frac{T_0}{2}\sum_{j=0}^{M} w_j f_1(-\frac{T_0}{2}(s_j-1))\left(\sum_{k=0}^{M} y_k F_k(t_i+\frac{T_0}{2}(s_j-1)\right) + g_1(t_i) \quad \textbf{(3.68)}$$

$$\sum_{k=0}^{M} y_k F'_k(t_i) = \sum_{k=0}^{M} y_k F_k(t_i)[\, k_2 - \gamma_2 \sum_{k=0}^{M} x_k F_k(t_i) -$$
$$\frac{T_0}{2}\sum_{j=0}^{M} w_j f_2(-\frac{T_0}{2}(s_j-1))\left(\sum_{k=0}^{M} x_k F_k(t_i+\frac{T_0}{2}(s_j-1)\right) + g_2(t_i). \quad \textbf{(3.69)}$$

The Eq.**(3.68)** and Eq.**(3.69)** is formed a system of $(2M)$ number of nonlinear algebraic equations with $(2M+2)$, number of unknowns [21]. Imposing the initial conditions,

$$\sum_{k=0}^{M} x_k F_k(0) = \alpha_1; \tag{3.70}$$

$$\sum_{k=0}^{M} y_k F_k(0) = \alpha_2; \tag{3.71}$$

$(2M+2)$ equations can be totally be obtained for $(2M+2)$ unknowns in order to get a unique solution for x_k and y_k $(k = 0,1,2\ldots,M)$. Thus it is acquired the approximate solutions of the integro-differential equations Eqns.**(3.56)-(3.57)** by using Eq.**(3.60)** and Eq.**(3.61)** [21].

Description of the Bernstein Polynomials Collocation Method

The n[th] degree Bernstein polynomials on the interval $[a,b]$ is [22, 23]

$$B_{i,n}(t) = \binom{n}{i}\frac{(t-a)^i(b-t)^{n-i}}{(b-a)^n}\ ,\ i = 0,1,\ldots,n.\,, \tag{3.72}$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$.

Each of these $(n+1)$ polynomials having degree n satisfies the following expressions:

$$B_{i,n}(t) = 0 \text{ if } i < 0 \text{ or } i > n\,,$$

$$B_{i,n}(a) = B_{i,n}(b) = 0\,, \text{ for } (1 \le i \le n-1)\,,$$

$$\sum_{i=0}^{n} B_{i,n}(t) = 1.$$

A function $u(t)$ described over $[a,b]$ can be approximated by Bernstein polynomials basis functions of degree n as

$$u(t) \approx \sum_{i=0}^{n} c_i B_{i,n}(t) = C^T B(t), \tag{3.73}$$

where C and $B(t)$ are $(n+1)\times 1$ vectors described as

$$C = [c_0, c_1, \ldots, c_n]^T,$$

$$B(t) = [B_{0,n}(t), B_{1,n}(t), \ldots, B_{n,n}(t)]^T.$$

First, it is approximated the unknown functions $x(t)$ and $y(t)$ for solving equations Eq.**(3.56)** and Eq.**(3.57)**.

$$x(t) = C^T B(t),\ \ y(t) = D^T B(t), \tag{3.74}$$

The equations Eq.**(3.56)** and Eq.**(3.57)** are reduced as

$$C^T B'(t) = \left(C^T B(t)\right)[k_1 - \gamma_1\left(D^T B(t)\right) - \int_{t-T_0}^{t} f_1(t-\tau)\left(D^T B(\tau)\right)d\tau] + g_1(t) \,, \tag{3.75}$$

$$D^T B'(t) = \left(D^T B(t)\right)[-k_2 - \gamma_2\left(C^T B(t)\right) - \int_{t-T_0}^{t} f_2(t-\tau)(C^T B(\tau))d\tau] + g_2(t). \tag{3.76}$$

The collocation points $t_1 = t_0 + lh$ where $t_0 = a,\ h = \frac{b-a}{n}$ and $l = 0,1,\dots,n-1$, are placed in the Eq.**(3.75)** and Eq.**(3.76)**

$$C^T B'(t_1) = \left(C^T B(t_1)\right)[k_1 - \gamma_1\left(D^T B(t_1)\right) - \int_{t-T_0}^{t} f_1(t_1-\tau)\left(D^T B(\tau)\right)d\tau] + g_1(t_1) \,, \tag{3.77}$$

$$D^T B'(t_1) = \left(D^T B(t_1)\right)[-k_2 - \gamma_2\left(C^T B(t_1)\right) - \int_{t-T_0}^{t} f_2(t_1-\tau)(C^T B(\tau))d\tau] + g_2(t_1). \tag{3.78}$$

Corresponding boundary conditions becomes;

$$C^T B(0) = \alpha_1,$$
$$D^T B(0) = \alpha_2. \tag{3.79}$$

In this way, it is acquired $(2n+2)$ number of nonlinear algebraic equations with same number of unknowns as C and D from the Eq.**(3.77)** and Eq.**(3.79)**. These systems are solved numerically, to get the approximate values of C and D and hence the approximate solutions of the integro-differential Eq.(**3.56)** and Eq.**(3.57**) by using the Eq.**(3.74)** can be obtained (see the reference [21] for the figures and tables of numerical results).

3.4 A Jacobi Gauss-Lobatto and Gauss-Radau Collocation Algorithm for Solving Fractional Fokker-Planck Equations

In the paper [24], the authors developed a new effective spectral algorithm for approximating the solutions of the time-fractional Fokker-Planck equation and space-fractional Fokker-Planck equation. This numerical algorithm based especially on shifted Jacobi spectral collocation method where the shifted Jacobi Gauss-Lobatto scheme is used for the spatial discretization and the shifted Jacobi Gauss-Radau scheme is used for temporal approximation. In order to examine the performance of the constructed algorithm, several numerical examples are studied and their numerical results are compared with two other methods [28] in the literature.

Basic definition

The basic speciality of the Jacobi polynomials [29] is that they are the eigenfunctions to the singular Sturm-Liouville problem:

$$(1-x^2)\psi''(x)+[\vartheta-\theta+(\theta+\vartheta+2)x]\psi'(x)+k(k+\theta+\vartheta+1)\psi(x)=0. \tag{3.80}$$

The Jacobi polynomials are produced by using the three term recurrence relation:

$$P_{k+1}^{(\theta,\vartheta)}(x)=\left(a_k^{(\theta,\vartheta)}x-b_k^{(\theta,\vartheta)}\right)(x)-c_k^{(\theta,\vartheta)}P_{k-1}^{(\theta,\vartheta)}(x),\ k\geq 1, \tag{3.81}$$

$$P_0^{(\theta,\vartheta)}(x)=1,\quad P_1^{(\theta,\vartheta)}(x)=\frac{1}{2}(\theta+\vartheta+2)x+\frac{1}{2}(\theta-\vartheta), \tag{3.82}$$

where

$$a_k^{(\theta,\vartheta)}=\frac{(2k+\theta+\vartheta+1)(2k+\theta+\vartheta+2)}{2(k+1)(k+\theta+\vartheta+1)},$$

$$b_k^{(\theta,\vartheta)}=\frac{(\vartheta^2-\theta^2)(2k+\theta+\vartheta+1)}{2(k+1)(k+\theta+\vartheta+1)(2k+\theta+\vartheta)},$$

$$c_k^{(\theta,\vartheta)}=\frac{(k+\theta)(k+\vartheta)(2k+\theta+\vartheta+2)}{(k+1)(k+\theta+\vartheta+1)(2k+\theta+\vartheta)}.$$

The Jacobi polynomials have the following relations:

$$P_k^{(\theta,\vartheta)}(-x)=(-1)^kP_k^{(\theta,\vartheta)},\ P_k^{(\theta,\vartheta)}(-1)=\frac{(-1)^k\,\Gamma(k+\vartheta+1)}{k!\,\Gamma(\vartheta+1)}. \tag{3.83}$$

Additionally, the q^{th} derivative of $P_k^{(\theta,\vartheta)}(x)$ can be obtained as

$$D^{(q)}P_k^{(\theta,\vartheta)}(x)=\frac{\Gamma(j+\theta+\vartheta+q+1)}{2^q\,\Gamma(j+\theta+\vartheta+1)}P_{k-q}^{(\theta+q,\vartheta+q)}(x). \tag{3.84}$$

If the shifted Jacobi polynomial of degree k is obtained by $P_{L,k}^{(\theta,\vartheta)}(x) = P_k^{(\theta,\vartheta)}\left(\frac{2x}{L} - 1\right), L > 0$, and with the help of Eq.**(3.83)** and Eq.**(3.84)**, then its q^{th} −order derivative can be written as [24]

$$D^q P_{L,k}^{(\theta,\vartheta)}(0) = \frac{(-1)^{k-q}\,\Gamma(k+\vartheta+1)(k+\theta+\vartheta+1)_q}{L^q\,\Gamma(k-q+1)\,\Gamma(q+\vartheta+1)}, \quad \textbf{(3.85)}$$

$$D^q P_{L,k}^{(\theta,\vartheta)}(L) = \frac{\Gamma(k+\theta+1)(k+\theta+\vartheta+1)_q}{L^q\,\Gamma(k-q+1)\,\Gamma(q+\theta+1)}, \quad \textbf{(3.86)}$$

$$D^q P_{L,k}^{(\theta,\vartheta)}(x) = \frac{\Gamma(q+k+\theta+\vartheta+1)}{L^q\,\Gamma(k+\theta+\vartheta+1)} P_{L,k-q}^{(\theta+q,\vartheta+q)}(x), \quad \textbf{(3.87)}$$

where q is a positive integer number. The relation between q^{th} −order derivative of shifted Jacobi polynomial and the polynomial itself can be written as

$$D^q P_{L,k}^{(\theta,\vartheta)}(x) = \sum_{i=0}^{k-q} C_q(k,i,\theta,\vartheta)\, P_{L,i}^{(\theta,\vartheta)}(x), \quad \textbf{(3.88)}$$

where

$$C_q(k,i,\theta,\vartheta) = \frac{(k+\lambda)_q (k+\lambda+q)_i (i+\theta+q+1)_{k-i-q}\,\Gamma(i+\lambda)}{L^q (k-i-q)!\,\Gamma(2i+\lambda)} \times \, {}_3F_2\left(\begin{matrix} -k+i+q, & k+i+\lambda+q, & i+\theta+1 \\ i+\theta+q+1, & 2i+\lambda+1 & \end{matrix}; 1\right),$$

where $\lambda = \theta + \vartheta + 1$ and ${}_3F_2$ is a hypergeometric series [30].

Decription of the Time-fractional Fokker Planck Equation and its Numerical Solution

Take into consideration Time-fractional Fokker Planck equation

$$D_t u = D_i^{1-\beta}(d_0(x)D_{xx} + d_1(x)D_x)u + f(x,t), \ \ 0 < \beta \le 1, \quad \textbf{(3.89)}$$

with the initial condition

$$u(x,0) = \psi(x), \ x \in [0,L], \quad \textbf{(3.90)}$$

and boundary conditions

$$u(0,t) = \varphi_0(t), \ \ u(L,t) = \varphi_1(1), \ t \in [0,T], \quad \textbf{(3.91)}$$

where $f(x,t), d_0(x), d_1(x), \varphi_0(t), \varphi_1(1), \psi(x)$ are given functions. The function

$u(x,t)$ is supposed to be the casual function of time and space, having the zero value for $t < 0$ and $x < 0$, and

$$D_t^{\beta} u(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^1 (t-z)^{-\beta} u^{(t)}(x,z) dz, \qquad \textbf{(3.92)}$$

is the time-fractional derivative of the function $u(x,t)$ in the Caputo sense. To construct the numerical scheme for the given problem, the shifted Jacobi Gauss-Radau points [31] are used for the temporal approximation and the set of shifted Jacobi Gauss-Lobatto points [31] are used for spatial approximation. For solving the T-FFPE given by Eq.**(3.89)** or Eq.**(3.91)** the main step of the shifted Jacobi collocation consider method is outlined as

$$u(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} P_{L,i}^{(\theta_1,\vartheta_1)}(x) P_{T,j}^{(\theta_2,\vartheta_2)}(t) \qquad \textbf{(3.93)}$$

and so, the approximation of the temporal partial derivative $D_t u(x,t)$ can be calculated as

$$D_t u(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} P_{L,i}^{(\theta_1,\vartheta_1)}(x) D_1(P_{T,j}^{(\theta_2,\vartheta_2)}(t)) = \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_1^{i,j}(x,t) \textbf{(3.94)}$$

where $f_1^{i,j}(x,t) = P_{L,i}^{(\theta_1,\vartheta_1)}(x) D_1(P_{T,j}^{(\theta_2,\vartheta_2)}(t))$. Besides;

$$D_t^{1-\beta}(d_0(x)D_{xx} + d_1(x)D_x)_u = d_0(x) \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_2^{i,j}(x,t)$$
$$+ d_1(x) \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_3^{i,j}(x,t), \qquad \textbf{(3.95)}$$

where

$$f_2^{i,j}(x,t) = D_{xx}\left(P_{L,i}^{(\theta_1,\vartheta_1)}(x)\right) D_t^{1-\beta}(P_{T,j}^{(\theta_2,\vartheta_2)}(t)) \text{ and}$$
$$f_3^{i,j}(x,t) = D_x\left(P_{L,i}^{(\theta_1,\vartheta_1)}(x)\right) D_t^{1-\beta}(P_{T,j}^{(\theta_2,\vartheta_2)}(t)).$$

Hence, the Eq.**(3.89)** and Eq. **(3.91)** can be rewritten in the following form with the advantage given by Eq.**(3.93)** and Eq.**(3.95)**:

$$\sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_1^{i,j}(x,t) =$$
$$d_0(x) \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_2^{i,j}(x,t) + d_1(x) \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_3^{i,j}(x,t) + f(x,t), \quad (x,t) \in$$
$$[0,L] \times [0,T], \qquad \textbf{(3.96)}$$

correspondingly, the numerical discretizations for initial and boundary conditions are

$$u(x,0) = \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} P_{L,i}^{(\theta_1,\vartheta_1)}(x) P_{T,j}^{(\theta_2,\vartheta_2)}(0) = \psi(x),$$

$$u(0,t)=\sum_{i=0}^{N}\sum_{j=0}^{M}a_{i,j}\,P_{L,i}^{(\theta_1,\vartheta_1)}(0)P_{T,j}^{(\theta_2,\vartheta_2)}(t)=\psi_0(t),$$
$$u(L,t)=\sum_{i=0}^{N}\sum_{j=0}^{M}a_{i,j}\,P_{L,i}^{(\theta_1,\vartheta_1)}(L)P_{T,j}^{(\theta_2,\vartheta_2)}(t)=\psi_1(t)\ .$$

(3.97)

Applying the spectral collocation method, the resudial of Eq.**(3.89)** is set to zero at $(N-1)M$ of collocation points. In addition to that, in Eq.**(3.97)** the initial-boundary conditions will be collocated at collocation points. For $(M+1)(N+1)$ unknown of $a_{i,j}$, there are $M(N-1)$ equations of the form

$$\sum_{i=0}^{N}\sum_{j=0}^{M}a_{i,j}f_1^{i,j}\left(x_{L,N,r}^{(\theta_1,\vartheta_2)},t_{T,M,s}^{(\theta_2,\vartheta_2)}\right)d_0\left(x_{L,N,r}^{(\theta_1,\vartheta_2)}\right)$$
$$\sum_{i=0}^{N}\sum_{j=0}^{M}a_{i,j}f_2^{i,j}\left(x_{L,N,r}^{(\theta_1,\vartheta_2)},t_{T,M,s}^{(\theta_2,\vartheta_2)}\right)+d_1\left(x_{L,N,r}^{(\theta_1,\vartheta_2)}\right)$$
$$\sum_{i=0}^{N}\sum_{j=0}^{M}a_{i,j}f_3^{i,j}\left(x_{L,N,r}^{(\theta_1,\vartheta_2)},t_{T,M,s}^{(\theta_2,\vartheta_2)}\right)+f\left(x_{L,N,r}^{(\theta_1,\vartheta_2)},t_{T,M,s}^{(\theta_2,\vartheta_2)}\right)$$
$$r=1,\dots,N-1\,;s=1,\dots,M,$$

(3.98)

resulting a system of $M\times(N-1)$ algebraic equations in the unknowns $a_{i,j}$ ($i=0,1,2\dots,N; j=0,1,2\dots,M$). The remaining system is acquired by using the initial condition Eq.**(3.90)**, as

$$\sum_{i=0}^{N}\sum_{j=0}^{M}a_{i,j}\,P_{L,i}^{(\theta_1,\vartheta_1)}\left(x_{L,N,r}^{(\theta_1,\vartheta_2)}\right)P_{T,j}^{(\theta_2,\vartheta_2)}(0)=\psi\left(x_{L,N,r}^{(\theta_1,\vartheta_2)}\right),\ r=1,2\dots.,N-1$$

(3.99)

and by using the boundary conditions Eq.**(3.91)**, as

$$\sum_{i=0}^{N}\sum_{j=0}^{M}a_{i,j}\,P_{L,i}^{(\theta_1,\vartheta_1)}(0)P_{T,j}^{(\theta_2,\vartheta_2)}\left(t_{T,M,s}^{(\theta_2,\vartheta_2)}\right)=\psi_0\left(t_{T,M,s}^{(\theta_2,\vartheta_2)}\right),$$
$$\sum_{i=0}^{N}\sum_{j=0}^{M}a_{i,j}\,P_{L,i}^{(\theta_1,\vartheta_1)}(L)P_{T,j}^{(\theta_2,\vartheta_2)}\left(t_{T,M,s}^{(\theta_2,\vartheta_2)}\right)=\psi_1\left(t_{T,M,s}^{(\theta_2,\vartheta_2)}\right)\ ,s=0,\dots,M.$$

(3.100)

where $t_{T,M,s}^{(\theta_2,\vartheta_2)},s=0,1,\dots,M$ are the roots of $P_{T,M+1}^{(\alpha,\beta)}(t)$, while $x_{L,N,r}^{(\theta_1,\vartheta_2)},r=1,2,\dots,N-1$ are the roots of $P_{L,N}^{(\alpha,\beta)}(x)$. All these systems of the algebraic equations in the unknown $a_{i,j}$ can be solved by an appropriate iterative method. Finally, the approximate solution $u(x,t)$ given in Eq.**(3.92)** can be obtained[24].

Description of Space-fractional Fokker-Planck equation and its numerical solution

Take into consideration Space-Fractional Fokker-Planck equation

$$D_t u = \left(d_0(x) D_x^{2\beta} + d_1(x) D_x^{\beta}\right) u + f(x,t), \; 0 < \beta \leq 1, \tag{3.101}$$

with the initial equation Eq.**(3.90)** and boundary conditions Eq.**(3.91)**, where

$$D_x^{\beta} u(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^x (x-z)^{-\beta} u^{(1)}(z,t) dz,$$
$$D_x^{2\beta} u(x,t) = \frac{1}{\Gamma(2-2\beta)} \int_0^x (x-z)^{-2\beta} u^{2}(z,t) dz$$

is the space-fractional derivatives of the function $u(x,t)$ in the Caputo sense. The approximate solution is described as in Eq.**(3.92)** , then spatial derivatives are

$$D_x^{\beta} u(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_4^{i,j}(x,t),$$

$$D_x^{2\beta} u(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_5^{i,j}(x,t), \tag{3.102}$$

where

$$f_4^{i,j}(x,t) = D_x^{\beta}\left(P_{L,i}^{(\theta_1,\vartheta_1)}(x)\right) P_{T,j}^{(\theta_2,\vartheta_2)}(t),$$

$$f_5^{i,j}(x,t) = D_x^{2\beta}\left(P_{L,i}^{(\theta_1,\vartheta_1)}(x)\right) P_{T,j}^{(\theta_2,\vartheta_2)}(t).$$

So rewriting Eq.**(3.94)** by using Eq.**(3.102)**, the Eq.**(3.101)** is converted into

$$\sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_1^{i,j}(x,t) = d_0(x) \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_5^{i,j}(x,t) + d_1(x) \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_4^{i,j}(x,t) + f(x,t), (x,t) \in [0,L] \times [0,T]. \tag{3.103}$$

The Eq.**(3.104)** is collocated at $M \times (N-1)$ points, as follows

$$\sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_1^{i,j}\left(x_{L,N,r}^{(\theta_1,\vartheta_2)}, t_{T,M,s}^{(\theta_2,\vartheta_2)}\right) d_0\left(x_{L,N,r}^{(\theta_1,\vartheta_2)}\right) \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_5^{i,j}\left(x_{L,N,r}^{(\theta_1,\vartheta_2)}, t_{T,M,s}^{(\theta_2,\vartheta_2)}\right) + d_1\left(x_{L,N,r}^{(\theta_1,\vartheta_2)}\right) \sum_{i=0}^{N} \sum_{j=0}^{M} a_{i,j} f_4^{i,j}\left(x_{L,N,r}^{(\theta_1,\vartheta_2)}, t_{T,M,s}^{(\theta_2,\vartheta_2)}\right) + f\left(x_{L,N,r}^{(\theta_1,\vartheta_2)}, t_{T,M,s}^{(\theta_2,\vartheta_2)}\right), \; 1 \leq r \leq N-1; 1 \leq s \leq M \tag{3.104}$$

By the same procedure given for T-SFFPE, a system of $(M+1) \times (N+1)$

algebraic equations are obtained [25,26,27] in the unknown $a_{i,j}$ ($i = 0,1, \dots, N; j = 0,1, \dots, M$) considering together with the initial and boundary conditions. As a result, the approximate solution $u(x,t)$ given in Eq.**(3.92)** can be derived.

Numerical Results

Example 3.4

Take into consideration the following T-FFPE

$$\frac{\partial u(x,t)}{\partial t} = D_t^{1-\beta} (\frac{\partial}{\partial x}\left(d(x)\frac{\partial}{\partial x}u(x,t)\right) + f(x,t), (x,t) \in [0,1] \times [0,1],$$

with the initial and boundary conditions $u(x,0) = 0$, $u(0,t) = t^{\beta+2}, u(1,t) = et^{\beta+2}$ where $d(x) = e^x$ and $f(x,t) = \frac{\Gamma(\beta+3)e^x t^2}{2-2e^x t^{\beta+2}}$. The exact solution is $u(x,t) = e^x t^{\beta+2}$.

In [23], the comparative study between INAM (the implicit numerical approximation method) [27] and constructed method in [24] is done in terms of maximum absolute errors (MAEs) at $\theta_1 = \vartheta_1 = 0 , \theta_2 = \vartheta_2 = \frac{1}{2}$ with different values of β and different numbers of collocation points[24] for detailed numerical comparisons.

β	INAS ($N = M = 200$)	Our method ($N = M = 8$)
0.4	5.999×10^{-4}	8.818×10^{-5}
0.7	1.231×10^{-3}	4.071×10^{-5}
0.9	1.502×10^{-3}	1.073×10^{-5}

Table 3.1:*Comparing between the maximum absolute erors for $\theta_1 = \vartheta_1 = 0 , \theta_2 = \vartheta_2 = \frac{1}{2}$ of example 3.4* [24].

Example 3.5

Take into consideration the following S-FFPE

$$\frac{\partial u(x,t)}{\partial t} = \left(-x\frac{\partial^\beta}{\partial x^\beta} + \frac{x^2}{2}\frac{\partial^{2\beta}}{\partial x^{2\beta}}\right)u(x,t) + f(x,t), (x,t) \in [0,3] \times [0,3],$$ with the

initial and boundary conditions $u(0,t) = 0,\ u(3,t) = 3e^t, u(x,0) = x$, where

$f(x,t) = \frac{1}{2}xe^t(2 - \frac{(x-2)x^{1-\beta}}{\Gamma(2-\beta)}$. The exact solution is $u(x,t) = xe^t$.

θ_1	ϑ_1	θ_2	ϑ_2	4	8	12	16
0	0	0	0	7.727×10^{-1}	1.774×10^{-3}	6.233×10^{-7}	1.364×10^{-10}
0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	4.470×10^{-1}	7.346×10^{-4}	2.074×10^{-7}	2.738×10^{-11}
$\frac{1}{2}$	$\frac{1}{2}$	0	0	7.346×10^{-1}	1.589×10^{-3}	4.180×10^{-7}	4.410×10^{-11}

Table 3.2: *The maximum errors using their method for example 3.5 with* $N = M = 4(4)16$ *and* $\beta = 0.5$ [24].

x	t	E	x	t	E	x	t	E
0.1	0.1	1.776×10^{-15}	0.1	1.5	2.664×10^{-15}	0.1	3	5.258×10^{-13}
0.5		1.776×10^{-15}	0.5		2.664×10^{-15}	0.5		4.973×10^{-14}
0.9		2.220×10^{-16}	0.9		7.105×10^{-15}	0.9		4.725×10^{-13}
1.3		4.440×10^{-16}	1.3		2.664×10^{-15}	1.3		5.080×10^{-13}
1.7		0	1.7		7.105×10^{-15}	1.7		7.531×10^{-13}
2.1		1.776×10^{-15}	2.1		1.776×10^{-15}	2.1		8.668×10^{-13}
2.5		1.332×10^{-15}	2.5		7.105×10^{-15}	2.5		5.684×10^{-13}
2.9		2.220×10^{-15}	2.9		5.568×10^{-13}	2.9		2.477×10^{-11}

Table 3.3:*Absolute errors their method for example 3.5 with* $\theta_1 = \vartheta_1 = \theta_2 = \vartheta_2 = \frac{1}{2}$ *,* $N = M = 16$ *and* $\beta = 0.4$ [24].

CHAPTER 4

CONCLUSION

We recall that spectral methods have wide applications for the numerical solutions of partial differential equations which model various real-world problems appearing in areas of science and engineering. In my thesis, I presented a review on some class of spectral methods and their useful applications from different areas reflecting new insights of some useful mathematical tools.

At first, I reviewed some basic definitions of the orthogonal polynomials and fundamental PDEs also particular examples were given for both. In the later chapters, the two main spectral methods, namely Spectral Collocation Method and Spectral Galerkin Method, are presented in details together with some examples. In order to point out the importance and the benefit of the examined spectral methods, four studied applications from different areas are reviewed in the last chapter. Among these illustrative applications, the solution of a biological model about the species living together is presented by using Legendre Spectral Collocation method. The remaining three applications used the very new and promising notion of fractional derivative in studying Strum-Liouville problem of ordinary differential equations, and Fokker-Planck equation. The problems with these new insights are studied under the consideration of spectral methods and previously obtained numerical results are presented with some comparative studies.

As a future work, we are planning to apply the considered spectral methods and also some new classes of spectral methods for the approximate solutions of complicated nonlinear models which may appear in the fields of bioengineering, computational biology and so on. Some useful and important extensions of both the model class and the spectral methods can be considered with the help of fractional calculus.

Additionaly, necessary improvements for the convergence and accuracy can be studied for further applications.

REFERENCES

1. **S. Gabor (2003),** *"Orthogonal Polynomials"*, Rhode Island 02904-2294, USA.
2. **R.L. Burden, J.D. Faires (1993)**, "*Numerical analysis*", Boston: PWS Publishing Company.
3. **H.L. Krall, I.M. Sheffer (1967),** "*Orthogonal Polynomials in two variables",* Ann. Matt. Pure Appl. Ser. 4, (76) 325-376.
4. **Y.J. Kim, K.H. Kwon, J.K. Lee (1997),** *"Orthogonal polynomials in two variables and second order Pdes"*, J. Comput. Appl. Math., (82).
5. **I. N. Sneddon (1957),** *"Elements of Partial Differential Equations"* McGraw-Hill Book Company, Inc., NewYork.
6. **Y.J. Kim, K.H. Kwon, J.K. Lee (1998),** *"Partial Differential Equations having Orthogonal Polynamial Solutions",* J. Comput. Appl. Math. ,(99)
7. **J. Shen, T.Tao, Li-Lang (2010),** *"Spectral Methods; Analysis, Algorithms and Applications",* Newyork.
8. **M. Zayernouri, G.E. Karniadakis (2013),** "*Fractional Sturm-Liouville eigen-problems: Theory and numerical approximation"*J. Comput. Phys., (252) 495-517.
9. **B. Jin, R. William (2012),** *"Eigenvalue problems of the model with a fractional derivative"* J. Comput. Phys.,, (231) 4954-4966.
10. **K.S. Miller, B. Ross (1993),** "*Introduction to the Fractional Calculus and Fractional Differential Equations"* Inc., New York, NY.
11. **K. Diethelm, N.J. Ford, (2002)** *"Analysis of fractional differential equations",* J.Math.Anal.Appl 265 (2) 229-248.
12. **A.A. Killbass, H.M. Srivastava, (2006),** *"Theory and Applications of Fractional Differential Equations",* Elsevier, Amsterdam.
13. **M. Klimek, O.P. Agrawal (2012),***"On a regular fractional Sturm-Liouville problem with derivatives of order in (0, 1), in: Proceedings of the 13th International Carpathian Control Conference (ICCC)"*, 978-1-4577-1868.

14. **E. Barkai, R. Metzler. J. Klafter (2000),***" From continuous time random walks to fractional Fokker-Planck equation"*, Phys.Rev.E 61 (1) 132.
15. **N. Sugimoto (1991),** *"Burgers equation with a fractional derivative; hereditary effects on nonlinear acoustic waves"*, J.Fluid Mech. (225) 631-653.
16. **B.J. Henry, S.L.Wearne (2000),** *"Fractional reaction-diffusion"*, Physica A 276 (3) 448-456.
17. **M. Zayernouri, G.E. Karniadakis (2014),** *"Exponentially accurate spectral and spectral element methods for fractional ODEs"*, J. Comput. Phys., (257) 460-480.
18. **J. Abdul (1999),** *"Introduction to Integral Equations with Applications"*, Wiley, NewYork.
19. **E. Babolian, J. Biazar (2002),** *"Solving the problem of biological species living together by Adomian decomposition method"*, Appl.Math.Comput.(129) 339-343.
20. **J. Biazar, Z. Ayati, M. Partovi (2011),** *"Homotopy perturbation method for biological species living together"*, Int.J.Appl.Math.Res.2 (1) 44-48.
21. **P.K. Sahu, S.S. Ray (2016),** *"Legendre spectral collocation method for the solution of the model describing biological species living together"*, J.Comput.Appl. Math. (296) 47-55.
22. **P.K. Sahu, S.Saha Ray (2015),** *"A new numerical approach for the solution of nonlinear Fredholm integral equations system of second kind by using Bernstein collocation method",* Math. Meth. Appl. Sci. (38) 274-280.
23. **M.I. Bhatti, P. Bracken (2007),** *"Solutions of differential equations in a Bernstein polynomial basis"*, J.Comput .Appl. Math. (205) 272-280.
24. **R.M. Hafez, Ezz-Eldien S.S., A.H. Bhrawy, E.A. Ahmed, D. Baleanu (2015),** *"A Jacobi Gauss-Lobatto and Gauss-Radau collocation algorithm for solving fractional Fokker-Planck equations",* Nonlinear Dyn. (82) 1431-1440.
25. **R.L. Magin, (2006),** *"Fractional Calculus in Bioengineering"*Begell House Publishers", Newyork.

26. **A.A. Kilbass, H.M. Srivastava, J.J. Trujillo, (2006),** *“Theory and Applications of Fractional Differential Equations”,* Elsevier, San Diego
27. **I. Podlubnly, (1999),** *“Mathematics in Science and Engineering”*, Academic Press Inc., San Diego, CA.
28. **C. Wu, L. Lu, (2010),** *“Implicit numerical approximation scheme for the Fokker-Planck equation”*, Appl.Math.Comput.(216), 1945-1955.
29. **G. Szegö, (1939),** *“Orthogonal Polynomials”*, Colloquium Publications, American Mathematical Society.ISBN 978-08218-1023-1, MR 0372517G.
30. **Y. Luke, (1969),** *“The Special Functions and Their Approximations”*, vol. 2. Academic Press, NewYork.
31. **C. Canuto, Hussaini, M.Y., Quarteroni, A. Zang, T.A. (2006),** *“Spectral Methods: Fundementals in Single Domains.”* Springer, NewYork.
32. **D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo (2012),** “*Fractional Calculus: Models and Numerical Methods”* ISBN-13:978-9814355209, World Scientific Publishing Co. Pte. Ltd. Singapure
33. **J. Shen, T. Tang (2006),** *“Spectral and High- Order Methods with Applications”,* 16 Donghuangchenggen North Street, China.
34. **I. Podlubny (1999),** “*Fractional Differential Equations*”, AcademicPress, SanDiego, CA, USA.

APPENDIX

A- INTRODUCTION TO FIRST ORDER AND SECOND ORDER PDEs

A.1 Basics of First Order PDEs

Solving the Transport Equation

Determine every function $u(x,t)$ that solves

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0\ , \tag{A.1}$$

where v is a fixed constant.

Perform a linear change of variables to eliminate one partial derivative

$$\alpha = ax + bt\ , \tag{A.2}$$

$$\beta = cx + dt\ , \tag{A.3}$$

where x, t: original independent variables, α, β: new independent variables, a, b, c, d: constants to be chosen "conveniently" must satisfy $ad - bc \neq 0$. Multivariable chain rule is used to convert α and β derivatives as [5,6]

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta}\ , \tag{A.4}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta}\ , \tag{A.5}$$

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = \left(b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta}\right) + v\left(a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta}\right) = (b + av)\frac{\partial u}{\partial \alpha} + (d + cv)\frac{\partial u}{\partial \beta}\ . \tag{A.6}$$

Choosing $a = 0, b = 1, c = 1, d = -v$, the original PDE becomes $\frac{\partial u}{\partial \alpha} = 0$.This tell that $u = f(\beta) = f(cx + dt) = f(x - vt)$ for any (differentiable) function f.

Theorem A.1

The general solution to the transport equation $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$ is given by

$$u(x,t) = f(x - vt)\ . \tag{A.7}$$

where f is any differentiable function of one variable

Example A.1

Solve the transport equation $\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0$ given the initial condition $u(x,0) = xe^{-x^2}, -\infty < x < \infty$.

Solution: It's known that the general solution is given by $u(x,t) = f(x-3t)$. It is used the initial condition $f(x) = f(x-3{,}0) = u(x,0) = xe^{-x^2}$ for finding a solution. Thus, $(x,t) = (x-3t)e^{-(x-3t)^2}$.

Example A.2

Solve $5\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = x$ given the initial condition $u(x,0) = sin2\pi x, \quad -\infty < x < \infty$.

Solution: The linear change of variables is performed as $\alpha = ax + bt, \beta = cx + dt$ and $5\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 5\left(b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta}\right) + v\left(a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta}\right) = (a+5b)\frac{\partial u}{\partial \alpha} + (c + 5d)\frac{\partial u}{\partial \beta}$. $a = 1, b = 0, c = 5, d = -1$ is choosen then $\alpha = x$ and so the PDE becomes $\frac{\partial u}{\partial \alpha} = \alpha$.

Integrating yields; $u = \frac{\alpha^2}{2}f(\beta) = \frac{x^2}{2} + f(5x-t)$ and with the initial conditions; $\frac{x^2}{2} + f(5x) = u(x,0) = sin2\pi x$. It is replaced x with $x/5$; $f(x) = \sin\frac{2\pi x}{5} - \frac{x^2}{50}$. So, $u(x,t) = \frac{x^2}{2} + sin\frac{2\pi(5x-t)}{5} - \frac{(5x-t)^2}{50} = \frac{xt}{5} - \frac{t^2}{50} + sin\frac{2\pi(5x-t)}{5}$.

In general, a linear change of variables can always be used to convert a PDE of the form

$$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = C(x,y,u) \qquad \textbf{(A.8)}$$

into an 'ODE' which is actually a PDE containing only one partial derivative.

Characteristics Curves

To develop a technique to solve the first order PDE of the form [5,6]

$$\frac{\partial u}{\partial x} + p(x,y)\frac{\partial u}{\partial y} = 0, \qquad \textbf{(A.9)}$$

look for characteristic curves in the xy- plane along which the solution u satisfies an ODE. Consider u along a curve $y = y(x)$. On this curve,

$$\frac{d}{dx}u\big(x, y(x)\big) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \quad \textbf{(A.10)}$$

Comparing Eq.**(A.9)** and Eq.**(A.10)**, if it is required

$$\frac{dy}{dx} = p(x, y) \quad \textbf{(A.11)}$$

then the PDE becomes the ODE of the form

$$\frac{d}{dx}u\big(x, y(x)\big) = 0 \quad \textbf{(A.12)}$$

These are the characteristic ODEs of the original PDE. If it is expressed the general solution to Eq.**(A.11)** in the form $\rtimes\,(x, y) = C$ each value of C gives a characteristic curve. Eq.**(A.12)** says that u is constant along the characteristic curves, so that

$$u(x, y) = f(c) = f(\rtimes\,(x, y)) \quad \textbf{(A.13)}$$

The Method of Characteristic-Special Case

Theorem A.2

The general solution to $\frac{\partial u}{\partial x} + p(x, y)\frac{\partial u}{\partial y} = 0$ is given by $u(x, y) = f(\rtimes\,(x, y))$ where $(\rtimes\,(x, y) = C$ gives the general solution to $\frac{dy}{dx} = p(x, y)$ and f is any differentiable function of one variable [5].

Example A.3

Solve $2y\frac{\partial u}{\partial x} + 3x^2 - 1\frac{\partial u}{\partial y} = 0$ by the method of characteristics.

Solution: First, it is divided the PDE by $2y$ obtaining

$$\frac{dy}{dx} = \frac{3x^2-1}{2y},$$

This is a separable equation

$$2y\,dy = 3x^2 - 1\ \ dx,$$

$$y^2 = x^3 - x + c.$$

Then, the solution is $y^2 - x^3 + x = c$ and hence $u(x, y) = f(y^2 - x^3 + x)$.

This technique can be generalized to PDEs of the form

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = C(x,y,u)\ . \qquad \textbf{(A.14)}$$

Example A.4

Solve $\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = u.$

As above, along a curve $y = y(x)$, $\frac{d}{dx}u\big(x,y(x)\big) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$. Comparison with the original PDE gives the characteristic ODEs $\frac{dy}{dx} = x.\ \frac{d}{dx}u\big(x,y(x)\big) = u\big(x,y(x)\big)$. First, $y = \frac{x^2}{2} + y(0)$ and second that $u\big(x,y(x)\big) = u\big(0,y(0)\big)e^x = f(y(0)e^x$. Combining these gives $u(x,y) = f(y - \frac{x^2}{2})e^x$.

Summary

Consider a first order PDE of the form

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = C(x,y,u)\ . \qquad \textbf{(A.15)}$$

When $A(x,y)$ and $B(x,y)$ are constants, a linear change of variables can be used to convert Eq.**(A.15)** into an 'ODE'. In general, the method of characteristics yields a system of ODEs equivalent to Eq.**(A.15)**. In principle, these ODEs can always be solved completely to give the general solution to Eq.**(A.15)**.

A.2 Second Order PDE in Two Variables

The general second order PDE in two variables is of the form

$$F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial x \partial y},\frac{\partial^2 u}{\partial y^2}\right) = 0 \qquad \textbf{(A.16)}$$

The equations is quasi-linear if it is linear in the highest order derivatives (second order) that is of the form

$$a\big(x,y,u,u_x,u_y\big)u_{xx} + 2b\big(x,y,u,u_x,u_y\big)u_{xy} + c\big(x,y,u,u_x,u_y\big)u_{yy} = d\big(x,y,u,u_x,u_y\big).$$

(A.17)

The equation is semi-linear if the coefficients a, b, c are independent of u. That is if it takes the form

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = d\big(x,y,u,u_x,u_y\big). \qquad \textbf{(A.18)}$$

Finally, if the equation is semi-linear and d is a linear function of u, ux and uy, the equation is linear. That is when F is linear in u and all its derivatives. The semi-linear equation above and attempt a change of variable to obtain a more convenient form for the equation

Let $\xi = \emptyset(x,y)$, $\eta = \psi(x,y)$ be an invertible transformation of coordinates. That is

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} \end{vmatrix} \neq 0. \qquad \textbf{(A.19)}$$

By the chain rule,

$$U_x = u_\xi \emptyset_x + u_\eta \psi_x , \qquad \textbf{(A.20)}$$

$$U_y = u_\xi \emptyset_y + u_\eta \psi_y , \qquad \textbf{(A.21)}$$

$$U_{xx} = U_\xi \emptyset_{xx} + \ \emptyset_x\big(U_{\xi\xi}\emptyset_x + U_{\xi\eta}\psi_x\big) + U_\eta \psi_{xx} + \psi_x\big(U_{\eta\xi}\emptyset_x + U_{\eta\eta}\psi_x\big).$$

$$= U_{\xi\xi}\emptyset_x^2 + 2U_{\xi\eta}\emptyset_x\psi_y + U_{\eta\eta}\psi_x^2 + \text{first order derivatives of } u. \qquad \textbf{(A.22)}$$

Similarly,

$$U_{yy} = U_{\xi\xi}\emptyset_y^2 + 2U_{\xi\eta}\emptyset_y\psi_y + U_{\eta\eta}\psi_y^2 + \text{ first order derivatives of } u. \qquad \textbf{(A.23)}$$

$$U_{xy} = U_{\xi\xi}\emptyset_x\emptyset_y + U_{\xi\eta}\big(\emptyset_x\psi_y + \emptyset_y\psi_x\big) + U_{\eta\eta}\psi_x\psi_y + \text{ first order derivatives}$$

of u. **(A.24)**

Substituting into the PDE it is acquired that,

$$A(\xi,\eta)u_{\xi\xi} + 2B(\xi,\eta)u_{\xi\eta} + C(\xi,\eta)u_{\eta\eta} = D\big(\xi,\eta,u,u_\xi,u_\eta\big), \qquad \textbf{(A.25)}$$

where

$$A(\xi,\eta) = a\emptyset_x^2 + 2b\emptyset_x\emptyset_y + c\emptyset_y^2 , \qquad \textbf{(A.26)}$$

$$B(\xi,\eta) = a\emptyset_x\psi_x + b\big(\emptyset_x\psi_y + \psi_x\emptyset_y\big) + c\big(\emptyset_y\psi_y\big) , \qquad \textbf{(A.27)}$$

$$C(\xi,\eta) = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2 \ , \qquad \textbf{(A.28)}$$

It is easily follows that

$$B^2 - AC = (b^2 - ac)\Big(\frac{\partial(\xi,\eta)}{\partial(x,y)}\Big)^2. \qquad \textbf{(A.29)}$$

Therefore $B^2 - AC$ has the same sign as $b^2 - ac$.Now, it is choosed the new coordinates $\xi = \emptyset(x,y)$, $\eta = \psi(x,y)$ to simplify the PDE. $\emptyset(x,y) =$ constant, $\psi(x,y) =$ constant defines two families of curves in R^2. On a member of the family $\emptyset(x,y) =$ constant then

$$\frac{d\emptyset}{dx} = \emptyset_x + \emptyset_y y' = 0 \, , \quad \textbf{(A.30)}$$

Therefore, substituting in the expression for $A(\xi,\eta)$ it is acquired that

$$A(\xi,\eta) = a\emptyset_y^2 y'^2 - 2b\emptyset_y y' + c\emptyset_y^2 = \emptyset_y^2[ay'^2 - 2by' + c], \quad \textbf{(A.31)}$$

It is choosen that the two families of curves given by the two families of solution of the ordinary differential equation

$$ay'^2 - 2by' + c = 0 \, . \quad \textbf{(A.32)}$$

This nonlinear ODE is called the characteristic equation of the PDE and provided that $a \neq 0$, $b^2 - ac > 0$ it can be written as

$$y' = \frac{b \mp \sqrt{b^2 - ac}}{a} \, . \quad \textbf{(A.33)}$$

For this choice of coordinates $A(\xi,\eta) = 0$ and similarly it can be shown that $C(\xi,\eta) = 0$. The PDE becomes

$$2B(\xi,\eta)u_{\xi\eta} = D\left(\xi,\eta,u_\xi,u_\eta\right) , \quad \textbf{(A.34)}$$

where it is easy to show that $B(\xi,\eta) \neq 0$. It can write the PDE in the normal form

$$u_{\xi\eta} = D\left(\xi,\eta,u_\xi,u_\eta\right). \quad \textbf{(A.35)}$$

The two families of curves $\emptyset(x,y) =$constant, $\psi(x,y) =$constant acquired as solution of the characteristic equation are called characteristic and the semi-linear partial differential equation is called hyperbolic if $b^2 - ac > 0$ so it has two families of characteristics and a normal form as given above if $b^2 - ac > 0$, then the characteristic equation has complex solution and there are no real characteristic. The functions $\emptyset(x,y)$, $\psi(x,y)$ are now complex conjugates. A change of variable to the real coordinates

$$\xi = \emptyset(x,y) + \psi(x,y), \quad \eta = -i(\emptyset(x,y) - \psi(x,y)) \, , \quad \textbf{(A.36)}$$

results in the partial differential equation where the mixed derivative term vanishes,

$$u_{\xi\xi} + u_{\eta\eta} = D(\xi,\eta,u,u_\xi,u_\eta) \, . \quad \textbf{(A.37)}$$

In this case the semi-linear partial differential equation is called elliptic if $b^2 - ac < 0$. Notice that the left hand side of the normal form is the Laplacian. Thus Laplaces equation is a spectral case of an elliptic equation (with $D = 0$).

If $b^2 - ac = 0$, the characteristic equation $y' = \frac{b}{a}$ has only one family of solutions $\psi(x, y)$ =constant. Apply change of the variables

$$ξ = x\,, \qquad η = \psi(x, y), \tag{A.38}$$

then

$$A(ξ, η) = a,\;\; B(ξ, η) = a\psi_x + b\psi_y, \tag{A.39}$$

$$C(ξ, η) = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2 = \frac{(a\psi_x + b\psi_y)^2 - (b^2 - ac)\psi^2}{a} = \frac{B(ξ,η)^2}{a}, \tag{A.40}$$

also since $\psi(x, y)$ =constant,

$$0 = \psi_x + \psi_y y' = \psi_x + \psi y^{\frac{a}{b}} = \frac{a\psi_x + b\psi_y}{a} = \frac{B(ξ,η)}{a}, \tag{A.41}$$

therefore, $B(ξ, η) = 0$, $C(ξ, η) = 0$, $A(ξ, η) \neq 0$ and the normal form in the case $b^2 - ac = 0$ is

$$A(ξ, η)u_{ξξ} = D(ξ, η, u, u_ξ, u_η) \text{ or finally } u_{ξξ} = D(ξ, η, u, u_ξ, u_η)\,. \tag{A.42}$$

The partial differential equation is called parabolic in the case$b^2 - ac = 0$. An example of a parabolic PDE is the equation of heat conduction

$$\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} = 0 \text{ where } u = u(x, t). \tag{A.43}$$

The wave equation $u_{tt} - u_{xx} = 0$ is a hyperbolic equation. The Laplace equation $u_{xx} = u_{yy} = 0$ is a elliptic equation [5,6].

Example A.5

Classify the following linear second order PDE and find its general solution $xyu_{xx} + x^2u_{xy} - yu_x = 0$.

In this example $b^2 - ac = (\frac{x^2}{2})^2 = 0$ the PDE is hyperbolic provided $x \neq 0$ and the characteristic equations are

$$y' = \frac{b \mp \sqrt{b^2 - ac}}{a} = \frac{\frac{x^2}{2} \pm \frac{x^2}{2}}{xy} = 0 \text{ or } \frac{x}{y},$$

If $y' = 0$ then y =constant,

If $y' = \frac{x}{y}$, $x^2 - y^2$ =constant. Therefore two families of characteristic are

$\xi = x^2 - y^2$, $\eta = y$. Using the chain rule a number of times, it is calculated that the partial derivatives

$$u_x = u_\xi 2x + u_\eta \times 0 = 2xu_\xi ,$$

$$u_{xx} = 2u_\xi + 2x\big(u_{\xi\xi} \times 2x + u_{\xi\eta} \times 0\big) = 2u_\xi + 4x^2 u_{\xi\xi} ,$$

$$u_{xy} = 2x\big(u_{\xi\xi}(-2y) + u_{\xi\eta} \times 1\big) = -4xyu_{\xi\xi} + 2xu_{\xi\eta} .$$

Substituting into the PDE the normal form is obtained as

$$u_{\xi\eta} = 0 \text{ (provided } x \neq 0).$$

Integrating this equation with respect to η

$$u_\xi = f(\xi),$$

where f is an arbitrary function of one real variable. Integrating again with respect to ξ

$$U(\xi, \eta) = \int f(\xi) d\xi + G(\eta) = F(\xi) + G(\eta),$$

where F, G are arbitrary functions of one real variable. Reverting to the original coordinates the general solution is found as

$$u(x, y) = F(x^2 - y^2) + G(y).$$

Example A.6

Classify, and reduce to the normal form and obtain the general solution of the PDE

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 4x^2.$$

For this equation $b^2 - ac = xy^2 - x^2y^2 = 0$ the equation is parabolic everywhere in the plane(x, y). The characteristic equation is

$$y' = \frac{b}{a} = \frac{xy}{x^2} = \frac{y}{x}.$$

Therefore there is one family of characteristics $\frac{y}{x}$ =constant. Let $\xi = x$ and $\eta = \frac{y}{x}$ then using chain rule

$$u_x = u_\xi \times 1 + u_\eta \left(\frac{-y}{x^2}\right) = u_\xi - \frac{y}{x^2} u_\eta ,$$

$$u_y = u_\xi \times 0 + u_\eta \left(\frac{1}{x}\right) = \frac{1}{x} u_\eta ,$$

$$u_{xx} = u_{\xi\xi} - \frac{2y}{x^2}u_{\xi\eta} + \frac{y^2}{x^4}u_{\eta\eta} + \frac{2y}{x^3}u_{\eta} \,,$$

$$u_{yy} = \frac{1}{x^2}u_{\eta\eta} \,,$$

$$u_{yx} = \frac{1}{x}u_{\xi\eta} - \frac{y}{x^3}u_{\eta\eta} - \frac{1}{x^2}u_{\eta} \,,$$

Substituting into the PDE it is acquired that the normal form

$$u_{\xi\xi} = 4 \,,$$

Integrating with respect to ξ

$u_{\xi} = 4\xi + f(\eta)$ where f is an arbitrary function of a real variable. Integrating again with respect to ξ gives

$$U(\xi, \eta) = 2\xi^2 + \xi f(\eta) + g(\eta) \,,$$

Therefore the general solution is given by

$u(x, y) = 2x^2 + xf\left(\frac{y}{x}\right) + g(\frac{y}{x})$ where f, g are arbitrary functions of a real variable.

Example A.7

Consider $yu_{xx} + u_{yy} = 0$ in the region where $y > 0$, the equation is elliptic. Solving $y\mu^2 + 1 = 0$ one finds two complex solutions

$\mu_1 = \frac{i}{y^{1/2}}$ and $\mu_2 = \frac{-i}{y^{1/2}}$ then looking for two complex families of characteristics,

$$\frac{dy}{dx} + \mu_1 = 0 \text{ and } \frac{dy}{dx} + \mu_2 = 0 \,, \frac{dy}{dx} + \frac{i}{y^{1/2}} = 0 \text{ and }, \frac{dy}{dx} + \frac{-i}{y^{1/2}} = 0 \,,$$

The solutions of the equations are

$$\frac{2}{3}y^{3/2} + ix = c_1 \text{ and } \frac{2}{3}y^{3/2} - ix = c_2,$$

Therefore,

$\emptyset = \frac{2}{3}y^{3/2} + x$ and $\psi = \frac{2}{3}y^{3/2} - x$ and then

$$\xi = \frac{1}{2}(\emptyset + \psi) = \frac{2}{3}y^{3/2} \text{ and } \eta = \frac{1}{2i}(\emptyset - \psi),$$

The derivatives of ξ and η are,

$\xi_x = 0,\ \xi_y = y^{1/2}, \eta_x = 1, \eta_y = 0$ with $u(x,y) = v(\xi(x,y), \eta(x,y))$ it gets $u_x = v_\eta, u_y = y^{1/2}v_\xi,\ u_{xx} = v_{\eta\eta},\ u_{yy} = yv_{\xi\xi} + 1/2y^{-1/2}v_\xi$.

Substituting into the equation, one can obtain

$$v_{\xi\xi} + v_{\eta\eta} + \frac{1}{2y^{\frac{3}{2}}}v_\xi = 0 .$$

Finally, since $\xi = \frac{2}{3}y^{3/2}$, the equation becomes $v_{\xi\xi} + v_{\eta\eta} + \frac{2}{\xi}v_\xi = 0$.

A.2.1 Boundary Value and Initial Value Problems for 2nd order PDEs

PDEs can have several solutions, for instance, the following Laplace equation

$$u_{xx} + u_{yy} = 0 \qquad \textbf{(A.44)}$$

is solved by $u = x^2 - y^2,\ u = e^x cosy,\ u = \ln(x^2 + y^2)$. It is usually difficult to find the general solutions for higher-order PDEs therefore a unique solution can be obtained by imposing extra constraints on the boundary of the region in space variable or at time variable.

If there are too many constraints, the problem is called overdetermined and does not have any solutions. If there are very few constraints, the problem has more than one solution. A differential equation is called a boundary value or initial value problem if it has been given with all necessary boundary and/or initial conditions. Most usual IVPs, in a region $IR^n \times [0, +\infty)$ defined as follows [5,6];

The Cauchy Problem

Find a function u such that

$$\begin{cases} PDE\ \ in\ R^n \times [0, +\infty) \\ u\ |_{t=0} = \rtimes(x) \end{cases} \quad \text{or} \quad \begin{cases} PDE\ in\ R^n \times [0, +\infty) \\ u\ |_{t=0} = \rtimes(x) \\ u_t\ |_{t=0} = \psi(x) \end{cases} \qquad \textbf{(A.45)}$$

where $\rtimes$ and ψ are given functions, defined in R^n. Mostly used BVPs, in a region Ω in space, are given as in below;

The Dirichlet Problem: Determine a function u such that

$\begin{cases} PDE \ in \ \Omega \\ u = \rtimes \ on \ \partial\Omega \end{cases}$ where $\rtimes$ is a given function defined on $\partial\Omega$ **(A.46)**

The Neumann Problem: Find a function u such that

$\begin{cases} PDE \ in \ \Omega \\ \frac{\partial U}{\partial n} = \psi \ on \ \partial\Omega \end{cases}$ where ψ is a given function defined on $\partial\Omega$ and $\frac{\partial U}{\partial n}$ is the normal derivative. **(A.47)**

A BVP is called *well posed* if it has a solution, this solution is unique and the solution depends continuously on the data given in the problem.

Examples for Different types of Initial and Boundary Value PDE Problems

(*i*) *For Hyperbolic Equation*

$$u_{tt}(x,t) = u_{xx}(x,t) \ , \quad \textbf{(A.48)}$$

$$u(x,0) = f(x) \ , \quad \textbf{(A.49)}$$

$$u_t(x,0) = g(x) \ , \quad \textbf{(A.50)}$$

$$u(0,t) = u(l,t) = 0 \ . \quad \textbf{(A.51)}$$

Fourier's Method

To find the solution of differential equation $u_{tt}(x,t) = u_{xx}(x,t)$ on $(0,l)$ apply the separation of variables ansatz

$$u(x,t) = v(x)w(t) \ , \quad \textbf{(A.52)}$$

and inserting the ansatz into $u_{tt}(x,t) = u_{xx}(x,t)$ on $(0,l)$ gives that

$$v(x)w''(t) = v''(x)w(t) \ . \quad \textbf{(A.53)}$$

or, if $v(x)w(t) \neq 0$, $\frac{w''(t)}{w(t)} = \frac{v''(x)}{v(x)}$. It follows, provided $v(x)w(t)$ is a solution of differential equation Eq.**(A.48)** and $(x)w(t) \neq 0$, $\frac{w''(t)}{w(t)}$ =const.=:$-\lambda$ and $\frac{v''(x)}{v(x)} = -\lambda$ since x,t are independent variables. Assume $v(0) = v(l) = 0$ then $v(x)w(t)$ satisfies the boundary condition Eq.**(A.51)**. Thus looking for solution of the eigenvalue problem

$$-v''(x) = \lambda v(x) \text{ in } (0,l) \ , \quad \textbf{(A.54)}$$

$$v(0) = v(l) = 0 \ . \quad \textbf{(A.55)}$$

results has the eigenvalues $\lambda_n = (\frac{\pi}{l}n)^2$, $(n = 1,2, \dots)$ and associated eigenfunctions as $v_n = \sin(\frac{\pi}{l}nx)$. The solutions of

$$-w''(t) = \lambda_n w(t) \text{ are } \sin\left(\sqrt{\lambda_n}t\right), cos\left(\sqrt{\lambda_n}t\right).$$

can be get by setting $w_n(t) = \alpha_n cos\left(\sqrt{\lambda_n}t\right) + \beta_n \sin\left(\sqrt{\lambda_n}t\right)$ where $\alpha_n, \beta_n \in R$. It is easily seen that $w_n(t)v_n(t)$ is a solution of differential equation Eq.**(A.48)** and since Eq.**(A.48)** is linear and homogeneous, also $u_N = \sum_{n=1}^{N} w_n(t)v_n(x)$ which provides the differential equation Eq.**(A.48)** and the boundary conditions Eq.**(A.51)**. Consider the formal solution of Eq.**(A.48)** and Eq.**(A.51)**.

$$u(x,t) = \sum_{n=1}^{\infty}(\alpha_n cos\left(\sqrt{\lambda_n}t\right) + \beta_n \sin\left(\sqrt{\lambda_n}t\right) \sin\left(\sqrt{\lambda_n}x\right)) \quad \textbf{(A.56)}$$

Here "formal" means that it is known that here neither that the right hand side converges nor that is a solution of the IBVP. Formally, the mystery coefficients can be estimated from initial conditions Eq.**(A.49)** and Eq.**(A.50)** as follows;

$$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\sqrt{\lambda_n}x\right) = f(x) , \quad \textbf{(A.57)}$$

Multiplying this equation by $\sin\left(\sqrt{\lambda_k}x\right)$ and integrate over $(0, l)$,

$$\alpha_n \int_0^l sin^2\left(\sqrt{\lambda_k}x\right)dx = \int_0^l f(x) \sin\left(\sqrt{\lambda_k}x\right) dx, \quad \textbf{(A.58)}$$

Recall that; $\int_0^l \sin\left(\sqrt{\lambda_n}x\right) \sin\left(\sqrt{\lambda_k}x\right) dx = \frac{l}{2}\delta_{nk}$ then

$$\alpha_k = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{\pi k}{l}x\right) dx. \quad \textbf{(A.59)}$$

By the same argument it follows from

$$u_t(x,0) = \sum_{n=1}^{\infty} \beta_n \sqrt{\lambda_n} \sin\left(\sqrt{\lambda_n}x\right) = g(x) , \quad \textbf{(A.60)}$$

that

$$\beta_k = \frac{2}{k\pi}\int_0^l g(x) \sin\left(\frac{\pi k}{l}x\right) dx . \quad \textbf{(A.61)}$$

Under additional assumptions $f \in C_0^4(0, l)$, $g \in C_0^3(0, l)$ it follows that the right hand side of Eq.**(A.56)**, where α_n, β_n are given by Eq.**(A.59)** and Eq.**(A.61)**, respectively, defines a classical solution of Eq.**(A.48)** and Eq.**(A.51)** since under these assumptions the series for u and the formal differentiate series for u_t, u_{tt}, u_x, u_{xx} converges uniformly on $0 \le x \le l$, $0 \le t \le T$, $0 < T < \infty$ fixed .

Example A.8

Rectangle membrane

Let $\Omega = (0, a) \times$ (0,b). Using the method of separation of variables it is found all eigenvalues of $(-\Delta v = \lambda v \ in \ \Omega), (v = 0 \ on \ \partial\Omega)$ which are given by

$\lambda_{kl} = \sqrt{\frac{k^2}{a^2} + \frac{l^2}{b^2}}$, $k, l = 1,2, ..$ and associated eigenvalues, not normalized are

$$u_{kl}(x) = \sin\left(\frac{\pi k}{a} x_1\right) \sin\left(\frac{\pi k}{b} x_2\right) \quad \textbf{(A.62)}$$

(ii)For Parabolic Equations

Consider the initial-boundary value problem for $c = c(x, t)$

$$c_t = D\Delta_c \ in \ \Omega \times (0, \infty) \ , \quad \textbf{(A.63)}$$

$$c(x, 0) = c_0(x) \ \ x \in \overline{\Omega} \ , \quad \textbf{(A.64)}$$

$$\frac{\partial c}{\partial n} = 0 \ \ on \ \ \partial\Omega \times (0, \infty) \ \ , \quad \textbf{(A.65)}$$

Here is $\Omega \subset R^n, n$ the exterior unit normal at the smooth parts of $\partial\Omega, D$ a positive constant and $c_0(x)$ a given function. In practice to diffusion problems, $c(x, t)$ is the concertiation and D the coefficient of diffusion. First Fick's rule says that $w = D \ \partial c/\partial\Omega$, where w is the flow of the substance through the boundary $\partial\Omega$. Thus, it is assume that there is no flow through the boundary, according to the Neumann boundary condition Eq.**(A.65)**,

(iii) For Elliptic Equations

Assume $\Omega \subset R^n$ is a connected domain.

Dirichlet Problem

The Dirichlet Problem (first boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\Delta u = 0 \ in \ \Omega \ \ , \quad \textbf{(A.66)}$$

$$u = \Phi \ on \ \partial\Omega \ , \quad \textbf{(A.67)}$$

Assume Ω is bounded, then a solution to the Dirichlet problem is uniquely specified.

Neumann Problem

The Neumann Problem (second boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of

$$\Delta u = 0 \; in \; \Omega \,, \tag{A.68}$$

$$u = \Phi \; on \; \partial\Omega \,, \tag{A.69}$$

where Φ is given and continuous on $\partial\Omega$. Assume Ω is bounded, then a solution to the Dirichlet problem is in the class $u \in C^2(\overline{\Omega})$ and uniquely determind up to a constant.

Mixed Boundary Value Problem

The mixed boundary value problem (third boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\Delta u = 0 \; in \; \Omega \;, \tag{A.70}$$

$$\frac{\partial u}{\partial n} + hu = \Phi \;\; on \; \partial\Omega \;\;, \tag{A.71}$$

where Φ and h are given and continuous on $\partial\Omega$.Assume Ω is bounded and adequately regular, then a solution to the mixed problem is uniquely specified in the class $u \in C^2(\overline{\Omega})$ provided $h(x) \geq 0$ on $\partial\Omega$ and $h(x) > 0$ for at least one point $x \in \partial\Omega$.

Green's Function for Δ

The harmonic function satisfies

$$u(x) = \int_{\partial\Omega} (\gamma(y,x) \frac{\partial u(y)}{\partial n_y} - u(y) \frac{\partial \gamma(y,x)}{\partial n_y}) dS_y \;\;, \tag{A.72}$$

where $\gamma(y,x)$ is a main solution. Usually, u does not provide the boundary condition in the above boundary value problems. It is tried the find a $\emptyset$ such that u provide also the boundary condition, since $\gamma = s + \emptyset$ where $\emptyset$ is an arbitrary harmonic function for each fixed x. Consider the Dirichlet problem, then looking for a $\emptyset$ for instance,

$$\gamma(y,x) = 0, \; y\epsilon\partial\Omega \,, \; x\epsilon\Omega \tag{A.73}$$

then

$$u(x) = -\int_{\partial\Omega} \frac{\partial\gamma(y,x)}{\partial n_y} u(y) \, dS_y \,, \qquad x \in \partial\Omega \tag{A.74}$$

Suppose that u takes its boundary values Φ of the Dirichlet problem, then

$$u(x) = -\int_{\partial\Omega} \frac{\partial\gamma(y,x)}{\partial n_y} \Phi(y)\, dS_y \ . \qquad \textbf{(A.75)}$$

It is claimed that this function solves the Dirichlet problem Eq.**(A.66)** and Eq.**(A.67)**. A function $\gamma(y,x)$ which satisfies Eq.**(A.72)**, and some additional assumptions, is called Green's function. Rather, it is defined a Green function as follows;

Definition

A function $G(x,y),\ y,x \in \overline{\Omega}\,, x \neq y$, is called *Green function* related to Ω and to the Dirichlet problem Eq.**(A.66)** and Eq.**(A.67)** if for fixed $x \in \Omega$, it is considered $G(x,y)$ as a function of y, with the following specifications hold:

1) $G(y,x) \in C^2(\Omega/\{x\}) \cap C(\overline{\Omega}\,/\{x\})\ ,\ \Delta y G(x,y) = 0,\ x \neq y,$

2) $(y,x) - s(|\ x - y\ |) \in C^2(\Omega) \cap C(\Omega)\ ,$

3) $(y,x) = 0\ if\ y \in \partial\Omega,\ x \neq y\ .$

For more details see the reference [5,6].