IMPROVEMENTS IN FINITE ELEMENT METHOD AND METHOD OF MOMENTS FOR THE SOLUTION OF ELECTROMAGNETIC PROBLEMS

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IMPROVEMENTS IN FINITE ELEMENT METHOD AND METHOD OF MOMENTS FOR THE SOLUTION OF ELECTROMAGNETIC PROBLEMS

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ABSTRACT

IMPROVEMENTS IN FINITE ELEMENT METHOD AND METHOD OF MOMENTS FOR THE SOLUTION OF ELECTROMAGNETIC PROBLEMS

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In this thesis, two numerical techniques, the finite element method (FEM) and the method of moments (MoM) are improved for solving electromagnetic boundary value problems governed by differential and integral equations. The FEM is a numerical technique used to solve differential equations. The solution domain is divided into small intervals (elements), and the unknown function is calculated by finding an approximate solution by a truncated series of different shape functions. This approximate solution can be replaced with the original function in the differential equation. As a result, we get the matrix of equations that can be solved to obtain separate solution to the problem. In this thesis, we used the sigmoid function as a novel

application, and compare the accuracy of FEM with the sigmoid function against the well-known linear and the step functions. Two examples of electromagnetic problems are presented. It is concluded that the sigmoid function under specific conditions yields the most accurate results. For MoM, this thesis presents a new mathematical algorithm for the solution of electromagnetic problems. This algorithm is schemed to be suitable for solving the singularity that exists in the solution matrix. This adaptive integration algorithm aims to avoid the singularity in the evaluation of the integral so called the Cauchy Principal Value integral. An example electrostatic problem is presented and it is proved that the novel method is accurate.

Keywords: Finite Element Method, Method of Moments, Shape Functions, Singularity Matrix, Sigmoid Function, Cauchy Principal Value.

ÖZ

ELEKTROMANYETİK PROBLEMLERİN ÇÖZÜMÜNDE SONLU ELEMAN METODU VE MOMENTLER METODUNDA GELİŞMELER

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Bu tezde sonlu eleman metodu (FEM) ve momentler metodundan (MoM) oluşan iki sayısal teknik diferansiyel ve integral denklemler tarafından yürütülen elektromanyetik sınır değeri problemlerinin çözülmesi için geliştirilmiştir. Sonlu eleman metodu diferansiyel denklemlerin çözülmesi için kullanılan sayısal bir tekniktir. Çözüm bölgesi küçük aralıklara (öğeler) bölünmüştür ve bilinmeyen fonksiyonun üstü kesik farklı şekil fonksiyonları serisi ile yaklaşık bir çözüm bulunarak hesaplanmaktadır. Bu yaklaşık çözüm diferansiyel denklemdeki orijinal fonksiyon ile yer değiştirebilir. Bunun sonucunda probleme ayrı çözüm elde etmek

için çözülebilecek denklemler matrisi elde etmekteyiz. Bu tezde sigmoid fonksiyonu yeni bir uygulama olarak kullandık ve FEM'in sigmoid fonksiyon ile doğruluğunu iyi bilinen lineer ve basamak fonksiyonu ile karşılaştırdık. İki elektromanyetik problem örneği sunulmuştur. Sigmoid fonksiyonun belirli koşullar altında en doğru sonuçları verdiği bulunmuştur. Bu tez MoM için elektromanyetik problemlerin çözümünde yeni bir matematiksel algoritma sunmaktadır. Bu algoritma çözüm matrisinde mevcut tekilliği çözmeye uygun olacak şekilde tasarlanmıştır. Bu uyumlu entegrasyon algoritması Cauchy Esas Değeri entegrali adlı entegralin değerlendirilmesinde tekillikten kaçınmayı hedeflemektedir. Örnek bir elektrostatik problem sunulmuş olup yeni metodun doğru olduğu kanıtlanmıştır.

Anahtar kelimeler: Sonlu Eleman Metodu, Momentler Metodu, Şekil Fonksiyonları, Tekillik Matrisi, Sigmoid Fonksiyon, Cauchy Esas Değeri

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LIST OF ABBREVIATIONS

EM	Electromagnetic
FEM	Finite Element Method
FEA	Finite Element Analysis
FDM	Finite Difference Method
MoM	Method of Moments
BEM	Boundary Element Method
BVP	Boundary Value Problem
PDE	Partial Differential Equation
BC	Boundary Condition
CPV	Cauchy Principal Value
IE	Integral Equation
MSE	Mean Square Error

CHAPTER 1

INTRODUCTION

Numerical analysis is the study of algorithms used for solving problems of mathematical analysis through finding the numerical approximation. Consequently, finding easiest approximated solution coincide with the exact analysis [1]. Many boundary value problems (BVP's) in electromagnetics require the solution of first order ordinary differential equations. Examples of such problems are the solution of the Laplace and the Poisson's equations in one-dimension geometry, or the electromagnetic wave solutions in a bounded medium [2]. There are many public numerical techniques like boundary element method (BEM), method of moments (MoM), finite difference method (FDM) and finite element method (FEM) [3]. This thesis considers the finite element method and the method of moments that are used for solving the electromagnetic boundary value problems. The FEM is a numerical technique that is used to solve differential equations and the MoM for the solution of integral equations. The accuracy of FEM is investigated for the solution of electromagnetic boundary value problems having first order ordinary differential equations [4]. The solution domain is divided into small intervals (elements), and the unknown function is approximated by a truncated series of various functions (shape functions**)** [5] [6]. The approximated solution of the unknown function is then replaced into the original differential equation, and the resulting matrix equation is solved to get the discrete solution of the problem [7]. The most familiar shape functions that

have been used in FEM applications are the linear (hat) and step functions. In this thesis, we use the sigmoid function as a novel application, and compare the accuracy of FEM with the sigmoid function against the well-known linear and the step functions. Two examples of electromagnetic problems are presented, the first one calculates the electrostatic potential of the parallel plate capacitor at specific separated distance, maintained at constant voltage across its terminals. The governing equation of the problem is the" Poisson's equation. The second example is the solution of electrostatic potential inside a spherical cloud of electrons with a uniform charge density. According to these examples, it is concluded that the sigmoid function under specific conditions yields the most accurate results.

This thesis also presents a new mathematical algorithm for the solution of electromagnetic problems by using MoM. This algorithm is schemed to be suitable for solving the singularities along the diagonal matrix of method of moment solution. The adaptive integration algorithm depends on the Cauchy Principal Value integration. This algorithm, splits the interval of the method of moment integration into subintervals of different lengths, and transforms the integration to non-singular integrals. One example of electromagnetic problem is presented. This example computes the charge density along a thin conductor wire at a given radius and length with a uniform applied voltage. This technique shows a superior solution of this problem with high efficiency.

In summary, the aim of this thesis is to present new algorithms, to modify these numerical methods, for solving electromagnetic BVP's. These techniques are important for all engineering disciplines because differential equations are encountered in many engineering problems. The contents of chapters are listed below.

Chapter 1 is an introduction and objectives of this thesis.

Chapter 2 includes a fundamental concept.

numerical methods like reviewing electromagnetic theory, electromagnetic boundary conditions, classification electromagnetic boundary conditions, classification partial differential equations and the classification of electromagnetic problems.

Chapter 3 contains an introduction and brief history about FEM, description and formulation of the finite element method, numerical analysis and results

Chapter 4 includes an introduction to MoM description, formulation of the method, numerical analysis and results.

Chapter 5 includes the conclusion part.

CHAPTER 2

FUNDAMENTAL CONCEPTS

2.1 Numerical methods

The numerical analysis is considered to be one of the most powerful and important sections of mathematics, which serves to produce and develop an effective algorithm to hold a numerical solution to complex scientific problems. Most of the problems that appeared in science and engineering are very difficult and sometimes impossible to solve for an accurate solution. Due to the extensive development in the computing technology, numerical approximation has become more general and popular to use for different fields. Consequently, there are many scientific software's like, Matlab, Mathematica, Maple…etc, to solve problems in an effective and easy way. These software's contain satisfactory capacities that utilize standard numerical techniques. Choosing an appropriate numerical method for solving problems is very important for producing an accurate result in less time. Also, one should know what is going wrong when outcomes are not as predictable [8][9]. Furthermore, the numerical methods can be used in various fields of physics and engineering, as well as art entered into scientific calculations [10] [11].

The methods to solve partial differential equations are divided into two groups: analytical and numerical methods, the analytic method is a unique solution for the problem, while the numerical method is an approximated approach for solving problems. The numerical approximation of partial differential equation is the essence

of the mathematical modelling of the real word. Besides, the numerical approximation of the PDE does not request a learning of a large information about the problem [12]. Generally, these methods can be sorted into two types as

- Experimental techniques,
- Exact solution method (analytical method), or numerical techniques.

Experimental techniques often consume so much time, sometimes, be dangerous and usually not given sufficient flexibility in changing variables. Nevertheless, every numerical method includes analytic simplifications to certain degree, for this reason it is easy to apply in different fields of sciences [13].

The general analytic and numerical methods that used in solving electromagnetic problems are

A. exact solutions methods (Analytical methods)

- method of expansion series.
- method of variable separation.
- method of conformal mapping.
- methods of perturbation.
- method of integrated solutions, like Fourier and Laplace transforms.

B. approximate solution methods (Numerical methods)

- finite element method.
- method of moments.
- weighted residuals method.
- finite difference method.
- method of lines.
- Monte Carlo method.
- transmission-line modeling method.

We will take into consideration two of the numerical analysis, the first one is the finite element method and the second one is the method of moments, we study them in detail in the subsequent chapters.

Till the 1940s of the previous century, a majority of electromagnetic problems was resolved by using the analytical methods [14]. Hence, a high level of skills, experience, and overwork were wanted to apply those techniques, so, there is a limited use of some practical problems to be resolved because of the complexity of geometric shapes that define them. In the mid-1960s, the numerical solution of electromagnetic problems began more broadly with the aid of current fast advanced computers. This enabled the specialists in finding solutions to the many complex problems, quickly and efficiently and very close to the real solution.

Before starting to studies the numerical methods under consideration, it is necessary to understand the physical laws that control the fundamentals of electromagnetism [15]. This accomplished in the following sections.

2.2 Electromagnetic Theory Review

Everything about the topic of electromagnetics can be followed through eight equations. These equations are four field equations of Maxwell, and the other four equations for the independent medium [16] – [19]. It might be useful to state important theorems usually utilized as a part of EM before we briefly survey these equations. The first one is the divergence (or Gauss's) theorem. This theory manages the investigation of the vector field over the surface to the behavior of the vector field inside the surface. Alternatively, the divergence theorem states the outward flux of a closed surface for the vector field is equal to the integral volume of the divergence inside the surface [20]. In the physical and engineering mathematics, the divergence

theorem plays very serious role in particular electrostatics problems. Nevertheless, this theory can be circulated to any number of dimensions. In one dimension, it is proportional to the essential theorem of calculus. In two dimensions, it is proportional to Green's theorem, where it is a special case of the general Stokes' theorem [21] given as

$$\oint_S \bar{F}.\overline{dS} = \int_v (\nabla.\ \bar{F}\ )dv \tag{2.1}$$

where F is the vector field, ∇ is a Del. The second theorem is the Stokes's theorem, where it can be expressed as

$$\oint_L \bar{F}.dI = \int_S \nabla \times \bar{F}\ dS \tag{2.2}$$

Electromagnetic theory can be reviewed through the use of the basic concept of electric charges. The electromagnetic theory is concerned with the electric and the magnetic fields produced by electrical charges that are at rest and motion, where the electrostatic fields are produced by static charges and magnetostatic fields are generated by charges at a constant velocity. As far as it relates to our study, we will consider only the electrostatic fields.

2.3 Electrostatic Fields

The electrostatic field can be controlled by two essential laws. The first law is the Gauss's law, which is a direct result of Coulomb's force law.

$$\oint \overline{D}.\overline{dS} = \int \rho_v\, dv \tag{2.3}$$

The second law which depicts the electrostatic fields as

$$\oint \bar{E}\ .\overline{dI} = 0 \tag{2.4}$$

where $\bar{D}$ is the electric flux density (C /m^2), ρ_v is the volume charge density (C /m^3) and $\bar{E}$ is the electric field (V/m).

The differential form of (2.3) and (2.4) can be can be expressed in the different form by applying (2.1) to (2.3) and (2.2) to (2.4).

we obtain

$$\nabla . \bar{D} = \rho_v \tag{2.5}$$

and

$$\nabla \times \bar{E} = 0 , \tag{2.6}$$

The terms $\bar{D}$ and $\bar{E}$ are linked as

$$\bar{D} = \epsilon \ \bar{E} \tag{2.7}$$

where ϵ is the dielectric permittivity (F /m) of the medium. In terms of the electric potential V (in volts) and $\bar{E}$ can be written as follows

$$\bar{E} = -\bar{\nabla} V \tag{2.8}$$

or

$$V = -\int \bar{E} . \overline{dI}, \tag{2.9}$$

Combining (2.5), (2.7), and (2.8) gives Poisson's equation:

$$\nabla . \epsilon \nabla V = -\rho_v \tag{2.10}$$

or, if ϵ is constant,

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} , \tag{2.11}$$

when $\rho_v = 0$, the (2.11) becomes Laplace's equation:

$$\nabla . \epsilon \nabla V = 0 , \tag{2.12}$$

when ϵ constant [15] [22] we get

$$\nabla^2 V = 0, \tag{2.13}$$

where ∇^2 is the Laplacian operator.

2.4 Electrostatic boundary conditions

The electromagnetic field inside a material can be described by its constitutive parameters: electric permittivity ϵ, conductivity σ, and magnetic permeability μ. If these parameters are independent of E and H, the medium can be called linear, otherwise it is nonlinear. The medium can be homogeneous if the constitutive parameters are not functions of space variables, otherwise it is inhomogeneous. It is isotropic if the constitutive parameters are independent of direction, otherwise it is anisotropic.

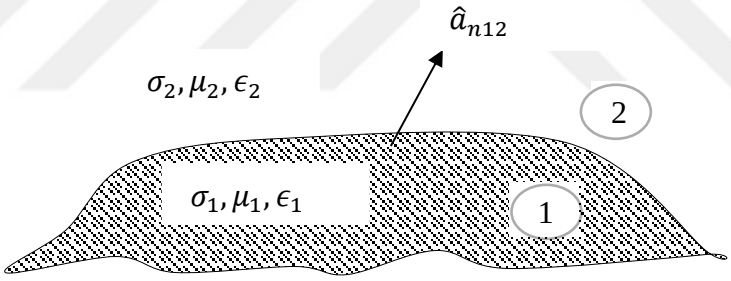


Figure 1. Crossing point between two mediums.

The boundary conditions at interface area that separate two different mediums 1 and 2 with different parameters $(\sigma_1, \epsilon_1, \mu_1)$ and $(\sigma_2, \epsilon_2, \mu_2)$ respectively are shown in Fig.1. The formulations are given as

$$\bar{E}_{1t} = \bar{E}_{2t} \; or \; (\bar{E}_1 - \bar{E}_2) \times \hat{a}_{n12} = 0 \tag{2.14}$$

$$\bar{H}_{1t} - \bar{H}_{2t} = \bar{K} \; or \; (\bar{H}_1 - \bar{H}_2) \times \hat{a}_{n12} = \bar{J} \tag{2.15}$$

$$\bar{D}_{1n} - \bar{D}_{2n} = \rho_S \; or \; (\bar{D}_1 - \bar{D}_2) \cdot \hat{a}_{n12} = \rho_S \tag{2.16}$$

$$\bar{B}_{1n} - \bar{B}_{2n} = 0 \; or \; (\bar{B}_2 - \bar{B}_1) \cdot \hat{a}_{n12} = 0 \tag{2.17}$$

where $\hat{a}_{n12}$ is the unit normal vector directed from medium 1 to medium 2. 1 and 2 are the subscripts that refer to fields in areas 1 and 2. t and n are the subscripts respectively that denote the tangential and normal components of the fields. $\overline{H}$ is the magnetic field (A/m). $\bar{B}$ is the magnetic flux density (tesla or W /m^2), μ is the permeability (H /m) and σ is the conductivity (1/Ω. m) of the medium.

As noted from the above equations, the normal components of $\bar{B}$ and the tangential components of $\bar{E}$of the (2.14) and (2.17) are continuous across the boundary. The tangential components of $\overline{H}$ for (2.15) is discontinuous on the boundary by the surface current density $\bar{J}$. Also, in (2.16), $\overline{D}$ is discontinuous on the boundary. Maxwell's equations are given as

$$\overline{\nabla} \cdot \overline{D} = \rho_v \tag{2.18}$$

$$\overline{\nabla} \cdot \bar{B} = 0 \tag{2.19}$$

$$\overline{\nabla} \times \overline{\mathrm{E}} = -\frac{\partial \bar{B}}{\partial t} - \bar{J}_m \tag{2.20}$$

$$\overline{\nabla} \times \overline{H} = \frac{\partial \overline{D}}{\partial t} + \bar{J}_e \tag{2.21}$$

where $\bar{J}_m = \rho_m * \overline{H}$ is the magnetic current density (V/ m^2), ρ_m or ($\mathcal{R}$) is the magnetic resistivity (Amp-turns per weber) and $\bar{J}_e$ is the electric current density (A /m^2). The above four equations are pointed to as Maxwell's equations in the public form when a medium is source free ($\bar{J} = 0$, $\rho_v = 0$) [15] [23] [24].

When we are dealing with the electrostatic problems, we consider the charge distribution as adjacent point over the mathematical surface like a spherical shell. Over such a surface, electric field results in discontinuity. Let us assume these charges are a positive charge, the electric field will be away from the surface on both sides. To know how much the electric field is discontinuous over the surface, we can use the Gauss's law.

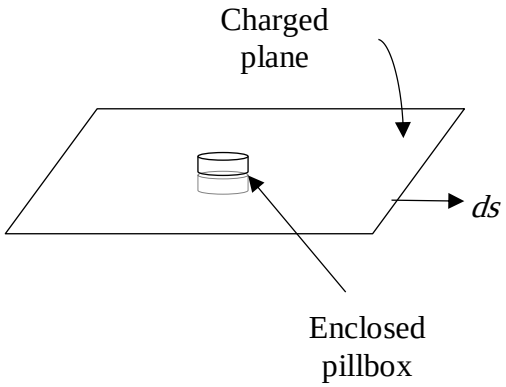


Figure 2. Surface enclosing charge.

We have a charge density over a surface, where the density distribution depends on the location and on the shape of the surface, the surface may be curved. But if we consider a small area of size A as shown in Fig.2. and built a small pillbox surrounding the small surface area and make a small extend distance above and below this surface, we can apply the Gauss's law as

$$\oint \bar{E}.\,\overline{da} = \frac{q}{\epsilon_0} \tag{2.22}$$

Where the integral on the left side of equation is the integral over the surface of the pillbox and q represents the charge enclosed by the pillbox. If we make a small enough area, the ρ and $\bar{E}$ become constants. So, the whole charge enclosed by the pillbox is ρA, this way, we get a constant on both sides of the equation, so that

$$\oint \bar{E}.\,\overline{da} = \left(E_{\perp}^{above} - E_{\perp}^{below}\right)A \tag{2.23}$$

The negative symbol appears because the $\overline{da}$ is referred to reverse ended areas of the pillbox (above and below cross point medium). We therefore get

$$E_{\perp}^{above} - E_{\perp}^{below} = \frac{\rho}{\epsilon_0} \tag{2.24}$$

This way, the perpendicular components of the field over the surface has discontinuity of ρ/ϵ_0. Also, the thickness of the pillbox can be made as small as we want by making it's above and below surface lie on the surface itself for decreasing the

contribution of the integral surface of the pillbox's two sides. Additionally, the tangential components of $\bar{E}$ parallel to the surface can be obtained from the Stock's theorem.

Since the $\bar{\nabla} \times \bar{E} = 0$, in electrostatics, we can get the following integral as

$$\oint \bar{E}.\overline{dI} = 0 \tag{2.25}$$

Suppose we have a rectangle whose plane is vertical to the surface such that one side of it lies over the surface and the invers side lies beneath the surface, then

$$\oint E.dI = E_{\|}^{above} l - E_{\|}^{below} l \tag{2.26}$$

where l is the rectangle side length.

The negative sing refers to the opposite integral around the rectangle *dI*, to the opposite directions of rectangle sides. Once again, we can make the other two sides of the rectangle vertical to the surface, as small as we prefer, by this there is no effect from them, so the integral is zero, we deduce

$$E_{\|}^{above} = E_{\|}^{below} \tag{2.27}$$

That means, the component of E parallel to the surface is continued over the surface. So, the difference of potential between two points can be computed by

$$V(b) - V(a) = -\int_a^b E.dI \tag{2.28}$$

By making the distance between two selected points above and below the surface as small as possible, the integration decreases to zero, since the integration for the parallel components of $\bar{E}$ is continued. The potential is continuous across the surface [25**].**

2.5 General types of Boundary Conditions

Generally, through the discretization process of poison's equation we can deduce three types of boundary conditions. Typically, these boundary conditions are of Neumann,

Dirichlet and the boundary that has both previous types, called mixed boundary condition. These boundary conditions can be expressed in the following forms

1. Dirichlet boundary condition: can be defined in terms of the potential as

$$\Phi(r) = 0, \quad r\ on\ S \tag{2.29}$$

2. Neumann boundary condition: is defined on the derivative of the potential, i.e. the electric field.

$$\frac{\partial \Phi(r)}{\partial n} = 0, \quad r\ on\ S \tag{2.30}$$

3. Mixed boundary condition: combination of Dirichlet and Neumann boundary conditions

$$\frac{\partial \Phi(r)}{\partial n} + h(r)\Phi(r) = 0\,, \quad r\ on\ S \tag{2.31}$$

where $h\,(r)$ is defined function and $\frac{\partial \Phi}{\partial n}$ is the derivative of the directional lengths in the normal direction outside the limits of S, that's mean

$$\frac{\partial \Phi}{\partial n} = \overline{\nabla}\Phi.\,\hat{a}_n \tag{2.32}$$

where, $\hat{a}_n$ is the unit normal vector point out of R, as shown in Fig. 3.

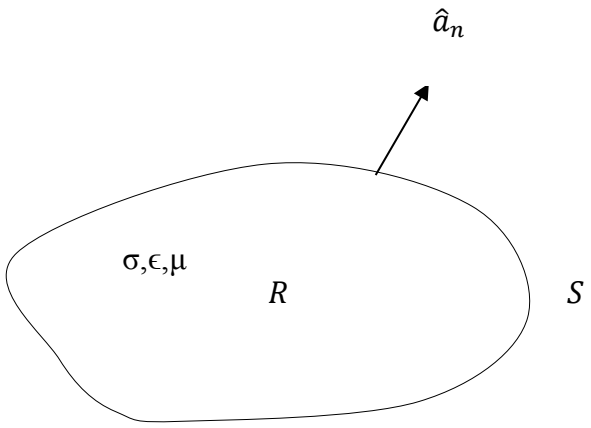


Figure 3. The solution medium R with boundary S.

2.6 Electromagnetic Problems

The most appropriate method for finding the solution to electromagnetic problems can be determined by the following steps

- What is the solution area of the problem to be solved?
- The next step is the nature and the type of the equation which describes the problem.
- The types of the boundary conditions regarding the problem.

The above steps are not independent of each other and can be described according to the problem.

2.7 Solution Regions

There are many types of solution regions, it is possible that the problem considers an internal region, then it is called inner, closed or bounded. Similarly, when the problem considers an outer region, it is called outer, open or unbounded region. Let us assume the solution area identified in R with boundary *S* is shown in Fig.3. If whole or part of *S* is at infinity, we can say that R is exterior and open region, otherwise R is interior and closed region. For example, the spread of the wave inside the waveguide is an internal problem, while the scattering of electromagnetic waves as a result of raindrops when spread in free space and the emission of dipole antenna are exterior problems. The solution regions can be categorized in terms of electrical constitutive properties, like conductivity (σ) and permeability (μ), where one can say that the solution region is linear or non-linear, homogeneous or inhomogeneous, and isotropic or anisotropic. We will be keen on our review, generally, with straight, homogeneous, isotropic media.

2.8 Differential Equations

The mathematical modelling means that any practical problem or real world can be formulated by mathematical equations. Often, this leads to ignoring some unimportant details which relate to the solution of the model required to the problem. The subject of partial differential equations (PDEs) takes an important role in mathematical modelling. Mathematical model depends heavily on partial differential equations as a result of its importance and effective in expressing various problems. Electromagnetic problems can be classified in terms of mathematical equations describing them, these equations are differential equations and integral equations, or both. Most of electromagnetic problems can be expressed in terms of an operator equation.

$$L = g \tag{2.33}$$

where L is an operator, g is the known excitation (source function). The known equation for this type of electromagnetic problems is the Poisson's equation. (2.33) can be expressed in differential form as

$$-\nabla^2 V = \frac{\rho_v}{\epsilon} \tag{2.34}$$

where $L = -\nabla^2$, Laplacian operator, $g = \rho_v/\epsilon$, source term and $\mathrm{V} = \Phi$, electric potential. In integral form, Poisson's equation is of the form

$$V = \int \frac{\rho_v dv}{4\pi\epsilon r^2} \tag{2.35}$$

where $L = \int \frac{dv}{4\pi r^2}$, $g = V$, and $\Phi = \rho_v/\epsilon \Rightarrow V = L.\Phi$

Generally, a second-order partial differential equation can be expressed as follows

$$a\frac{\partial^2\Phi}{\partial^2 x} + b\frac{\partial^2\Phi}{\partial\, x\partial\, y} + c\frac{\partial^2\Phi}{\partial^2 y} + d\frac{\partial\Phi}{\partial x} + e\frac{\partial\,\Phi}{\partial\, y} + f\Phi = g \tag{2.36}$$

or simply

$$a\Phi_{xx} + b\Phi_{xy} + c\Phi_{yy} + d\Phi_x + e\Phi_y + f\Phi = g \quad (2.37)$$

The coefficients, *a*, *b* and *c*, are functions of *x* and y, they may also depend on themselves, in which case the partial differential equation is said to be nonlinear. If the term g (x, y) in equation (2.37) is equal to zero, the PDE is called homogeneous.

If g (x, y) ≠ 0 then inhomogeneous. It is noted that (2.37) has the same style of the equation (2.33). So, L can be stated by

$$L = a\frac{\partial^2}{\partial^2 x} + b\frac{\partial^2}{\partial\, x\partial\, y} + c\frac{\partial^2}{\partial^2 y} + d\frac{\partial}{\partial x} + e\frac{\partial}{\partial\, y} + f \quad (2.38)$$

In general, a PDE can have both boundary values and initial values. The partial differential equations can be called steady state equations when boundary conditions are specified. If partial differential equations have only initial condition they are called transient equations [26]. A common case of this kind of PDE involves the Laplace's equation

$$\frac{\partial^2\Phi}{\partial^2 x} + \frac{\partial^2\Phi}{\partial^2 y} = 0 \quad (2.39)$$

and Poisson's equation

$$\frac{\partial^2\Phi}{\partial^2 x} + \frac{\partial^2\Phi}{\partial^2 y} = g(x, y) \quad (2.40)$$

2.9 Integral Equations

The integral equations (IE) are consider one of the most and important subject in applied mathematics and physics, they can be used as mathematical models in different fields. Also, the integral equations have the ability to modify several other mathematical problems, especially when there are no analytical solutions [27].

Integral equations can be defined as those equations that contained under integral sign the unknown function Φ. A typical and simple example of these types of integrals are

Fourier, Laplace, and Hankel transforms. The most common type is a linear integral equation that divided into two types named after Fredholm and Volterra. One set is the Fredholm's equations of the first, second, and third kind, as

$$f(x) = \int_a^b K(x,t)\Phi(t)dt \tag{2.41}$$

$$f(x) = \Phi(x) - \lambda \int_a^b K(x,t)\Phi(t)dt \tag{2.42}$$

$$f(x) = a(x)\Phi(x) - \lambda \int_a^b K(x,t)\Phi(t)dt \tag{2.43}$$

where λ is a scalar, sometimes equal to unity or possibly complex parameter, *K (x, t)* is the kernel of the integral equation, $\Phi(x)$ is unknown function, *a* and *b* are the known limits of integral.

The second set of integral equations is the Volterra equations of the first, second, and third kind, this group of integrations have an upper limit with variable, as

$$f(x) = \int_a^x K(x,t)\Phi(t)dt \tag{2.44}$$

$$f(x) = \Phi(x) - \lambda \int_a^x K(x,t)\Phi(t)dt \tag{2.45}$$

$$f(x) = a(x)\Phi(x) - \lambda \int_a^x K(x,t)\Phi(t)dt \tag{2.46}$$

All the previous integral equations can be homogeneous if their results are equal to zero, and they can be called linear equations if the Φ is linear. If Φ shows up in the power of n > 1 under the integral sign the equation is nonlinear as shown in the following integral

$$f(x) = \Phi(x) - \int_a^b K(x,t)\Phi^2(t)dt \qquad (2.47)$$

Also, when the upper or lower limit *a* or *b* or the kernel *K (x, t)* goes to infinite, an integral equation becomes singular. Lastly, when *K (x, t)* is equal to *K (t, x)* the kernel *K (x, t)* is called to be symmetric [28] [29].

2.10 Shape Function

Most functions that are difficult to find a solution by traditional mathematical methods can be solved easily by numerical methods, where the numerical methods approximate the solutions by known functions with unknown coefficients. The domain of these unknown functions is sliced into sub - domains and it can be solved. The nodes at the ends of the main domain are called the global nodes while the nodes that connect between each two inner segments are called internal nodes. Generally, to calculate values at positions other than the global nodes we interpolate them with what is called the shape functions between the internal nodes. Thus, the shape function is used to approximate quantities like temperature, stress, strain, elasticity, voltage, electric charge density, electric field… etc. A one-dimensional element with length L is shown in Fig.4. It has two nodes, one at each end, denoted i and j, and known nodal function N_i and N_j.

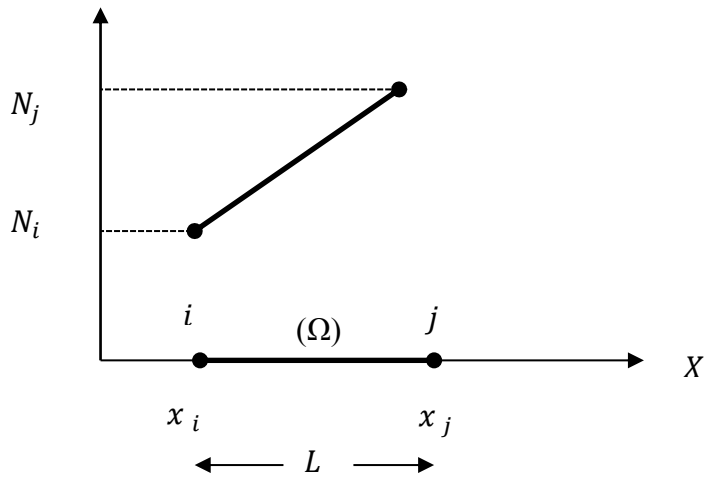


Figure 4. One-dimensional element with length L.

We can deduce automatically that the element is first order (linear) since it contains no 'midside' nodes. Generally, for linear elements, the polynomial interpolation function is first order. If the element is second order, the polynomial function would be second order (quadratic), and so on [30].

2.11 Isoparametric Mapping of Element

When we want to find a solution to irregular domain by a numerical analysis, the domain is partitioned into sub- domains to construct a mesh across the main domain, and the elements must be allowed to take more general shapes. This is done by using the original elements and transforming them by some mapping. The basic idea is included in this process depends on the mapping of the simple geometric shape in the local coordinate system into partition shapes in the global Cartesian coordinate system. The mapping from local to global coordinates will take the form

$$x = \sum_{k=1}^{m} F_k(\xi_k, \eta_k) x_k, \qquad y = \sum_{k=1}^{m} F_k(\xi_k, \eta_k) y_k \tag{2.48}$$

where m is the number of points defining the geometry of the element. The functions $F_k(\xi_k, \eta_k)$ clearly satisfy the following relation

$$F_k(\xi_{kl}, \eta_l) = \delta_{kl} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases} \tag{2.49}$$

where δ_{kl} is the Kronecker delta. Hence there is a one-to-one correspondence between the nodes in the parent element and the distorted element in the global coordinate system. This mapping can be expressed in three-dimensions as

$$x = \hat{x}(\xi, \eta, \zeta), \qquad y = \hat{y}(\xi, \eta, \zeta), \qquad z = \hat{z}(\xi, \eta, \zeta) \tag{2.50}$$

where (ξ, η, ζ) and (x, y, z) are the coordinates in the natural and the physical domain, respectively. The mapping of equations (2.50) can be more represented in vector form as

$$x = \varphi(\xi), \tag{2.51}$$

where (x, y, z) and (ξ, η, ζ) are the cartesian components of x and ξ, respectively. Here, φ maps the regular-shaped domain Ωe□ to the irregular-shaped domain Ωe [31] as shown in Fig.5.

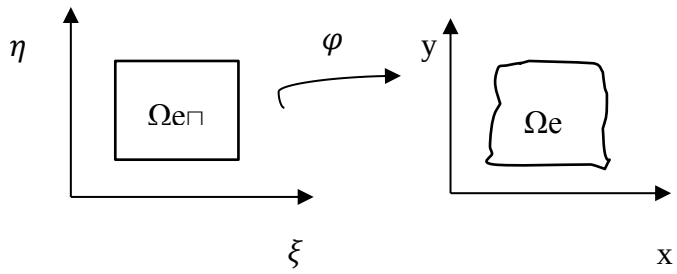


Figure 5. Schematic of a parametric mapping from Ωe□ to Ωe**.**

CHAPTER 3

FINITE ELEMENT METHOD

3.1 Introduction

The finite element method is a numerical analysis that uses an approximated approach to facilitate the solution of problems in different fields such as engineering and physical sciences. The analysis of this numerical technique for a finite element called a finite element analysis (FEA). Generally, the analytical solution is needed to solve the BVP's of the partial differential equations. The resulting equations from this analysis can be solved by finite element method, by building a system of mathematical equations to find an approximate value for the unknowns for a certain number of points over the main domain [32] [33]. That is why the large problems are divided into small parts.

Thus, the simple mathematical equations that model these finite elements are then assembled into a larger system of mathematical equations that model the complete problem, in order to reduce the error function associated with this solution as much as possible, the solution of these mathematical equations can be computed by means of various calculus techniques [34] [35]. The finite element method can be seen as a powerful tool and the success of this method depends on the regarded procedure that used in the formulation of the problem [36] [37]. This method was developed by many scientists, in 1909 Ritz develop a very effective way in approximate solutions to the problems in the mechanics of distortion solids. This method works to approximate the

energy function in terms of known functions with unknown coefficients. This leads to a system of equations by which the unknown coefficients could be found, but this method faced many of encountered constraints where one of these restrictions was that, the function used in the solution had to agree with the boundary conditions of the problem [38]. By the same token, in 1943 the Courant used the Ritz method and tried to develop it by adding more opportunities to this method through inserting special linear functions defined over the triangular areas, and used this method to find a solution of unknown torsion problems, at a known function value at the node points of triangular area. This way, the main constraint to the Ritz on function related to boundary conditions have been solved [39]. Many years later, in particular in 1960, Clough suggest the method of Ritz and formulation of Courant independently and inserted the term finite element and use this method in analysis of plane stresses [40]. The widespread use of the finite element method is the possibility to use a computer to calculate the large size of computations that are required by the finite element method.

After that, many developments are carried out on the FEM like, Argyris [41], Turner and Martin in [42], Hrennikov [43] and many others. In 1967 Zienkiewicz and Cheung published the first book called the finite element method in structural and continuum mechanics [44]. This book is broadly used, it shows the wide explanation of the technique and its applicability to any field of problems. In spite of the fact that, the technique has been generally utilized already in the field of basic mechanics, it has been effectively used now for the resolution of a diverse sorts of engineering [45].

Briefly, any problem can be solved by the numerical technique named finite element method essentially include four steps.

- dividing the solution area (space) into a finite number of sub districts (sub-space) or elements.
- driving a controlling equation for the specified element.
- collecting of all elements in the solution area yield a system of equations.
- solving the obtained equation system.

3.2 Numerical Steps of the Finite Element Method

Finite element method is a numerical approach to solve (approximately) BVP's governed by partial differential equation PDE's

$$\mathcal{L}\, u = f \; in\; \Omega, \tag{3.1}$$

where $\mathcal{L}$ is the partial differential operator, u is the unknown function, f is the known function (source function) and Ω = region subset of R^1, R^2, R^3. In addition, a boundary condition is given by

$$Bu = g, \quad on\; \partial\Omega \tag{3.2}$$

where B is the partial differential operator, g is given and $\partial\Omega$ is the boundary of Ω. The (3.1) and (3.2) is the definition of BVP. For example

$$\nabla^2 u = f, \qquad in\; \Omega \tag{3.3}$$

$$u = 0, \qquad on\; \partial\Omega \tag{3.4}$$

$$\nabla^2 = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + \frac{\partial^2}{\partial^2 z} \tag{3.5}$$

To solve the BVP, the region Ω (the computation domain) is partitioned into simple shaped sub domain called an element. This of solution called mesh generation as shown in Fig.6. In 1-D applications, $\Omega \subset R$ and the elements are line segments

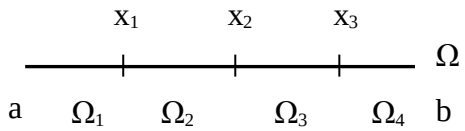


Figure 6. Line segments domain.

where $\Omega = [a, b]$, $\Omega_1 = [a, x_1]$, $\Omega_2 = [x_1, x_2]$, $\Omega_3 = [x_2, x_3]$ and $\Omega_4 = [x_3, b]$.

In 2-D application, $\Omega \subset R^2$, and the elements generally triangles or quadrilaterals as shown in Fig.7

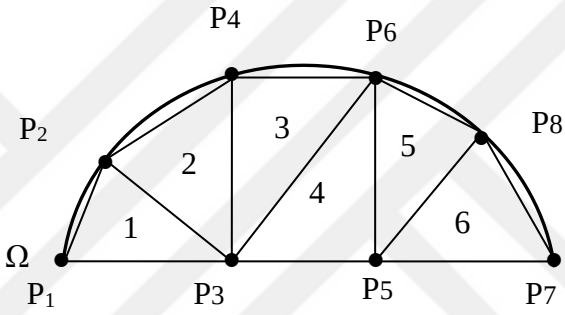


Figure 7. Discretization of the continuum domain region into sub- domains.

where the Ω – is a semicircle domain in $R^2$and P are the nodes. The process of the partition of the considered main domain into sub- domains, called finite elements.

In 1-D application, the BVP that will be studied is the differential equation can be expressed as

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = f(x), \quad a < x < b \tag{3.6}$$

$$u(a) = u_1, u(b) = u_n$$

where $a(x)$ is a given function, $f(x)$ is the non-zero source function, $u(x)$ is the unknown function, $u(a)\ and\ u(b)$ are Dirichlet boundary conditions.

The numerical analysis of the FEM is improved by the shape functions, it can be explained in details in the following steps

The first step:

The numerical analysis process begins by dividing the main domain into N elements and arbitrary element is denoted by $\Omega_e$as in Fig.8.

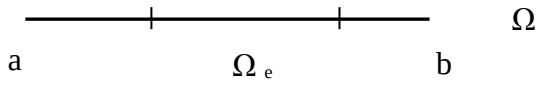


Figure 8. Arbitrary element Ω_e.

The second step:

Over the Ω_e, we defined shape function N_1 and N_2, such that N_1 is one at node 1 and drop to zero linearly at node 2 and vice versa for N_2 as shown in Fig.9, Fig.10, and Fig.11. for linear, step and sigmoid functions respectively.

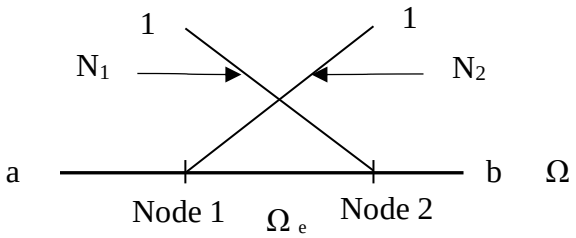


Figure 9. Define linear (hat) shape function over sub-domain Ω_e.

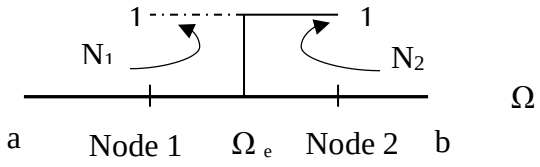


Figure 10. Define step function over sub-domain Ω_e.

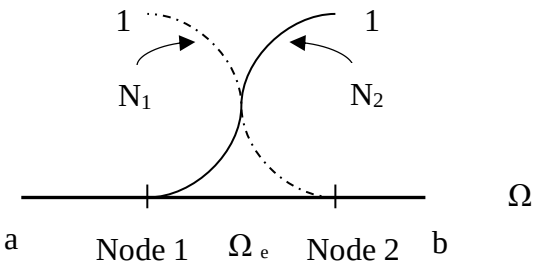


Figure 11. Define sigmoid function over sub-domain Ω_e.

In this work, we adopted three types of shape functions, linear (hat), step and a novel one is the sigmoid functions, we will discuss them in detail later. Within Ω_e, the unknown function $u(x)$ is expressed as

$$u(x) \approx \sum_{j=1}^{2} u_j \ N_j(x) \tag{3.7}$$

Note that, $N_j(x)$ are the known functions with the unknown coefficients u_j of the node values of $u(x)$. This approximated function is substituted into the differential equation (3.6) as

$$\sum_{j=1}^{2} u_j \ \left[-\frac{d}{dx}\left(a(x)\frac{dN_j(x)}{dx}\right)\right] = f(x), \qquad x \in \Omega_e \tag{3.8}$$

The third step:

Choosing the weight function equal to the shape function $N_1(x)$ and $N_2(x)$ and form the inner production

$$\sum_{j=1}^{2} u_j \int_{\Omega e} \left[-\frac{d}{dx}\left(a(x)\frac{dN_j(x)}{dx}\right)\right] N_i(x)dx = \int_{\Omega e} f(x)N_i(x)dx, x \in \Omega_e,$$

$$i = 1,2 \tag{3.9}$$

The process of choosing the weight equation and the shape function from the same set of function is known as " Rayleigh -Ritz Method "[46].

The forth step:

The integration processes. From the previous step we have obtained a 2×2 matrix equation called the local FEM matrix " A^e_{local}", where the $i\,j^{th}$ entry is

$$a_{ij} = \int_{\Omega e} \left[-\frac{d}{dx}\left(a(x)\frac{dN_j(x)}{dx} \right) \right] N_i(x)\,dx, \qquad i,j = 1,2 \tag{3.10}$$

The right-hand side is a 2×1 vector, *b* has the entry

$$b_i = \int_{\Omega e} f(x) N_i(x) dx, x \in \Omega e, \qquad i = 1,2 \tag{3.11}$$

The integration is not carried out directly in term of (x) variable but the element is mapped to the local element defined by the region $-1 \le \xi \le 1$, "ξ" is called local coordinate and " X " called global coordinate as shown in Fig.12.

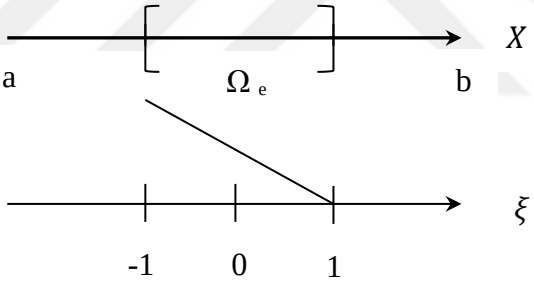


Figure12. Element mapping from global to local coordinate.

Now, the N_i's can be express in term of local coordinates

$$N_i(\xi) = \alpha\xi + \beta \tag{3.12}$$

To find α and β, we use the limits of interval (sub-domain) where it's value at (-1) equal to '1' and at (1) equal to '0'.

$$N_1(-1) = -\alpha + \beta = 1 \tag{3.13}$$

and

$$N_1(1) = \alpha + \beta = 0 \tag{3.14}$$

This leads to $\alpha = -0.5$ and $\beta = 0.5$

$$N_1(\xi) = 0.5(1 - \xi) \tag{3.15}$$

Similarly, we can obtain

$$N_2(\xi) = 0.5(1 + \xi) \tag{3.16}$$

We need $\frac{dN_1(x)}{dx}$ and $\frac{dN_2(x)}{dx}$, where the" X " can be expressed in terms of " ξ" as follows

$$X(\xi) = \sum_{k=1}^{2} X_{k,e} \; N_k(\xi). \tag{3.17}$$

where $X_{1,e}$ and $X_{2,e}$ are the X – coordinates of the nodes. The expression of the global word in terms of the shape functions is called "Isoparametric mapping". isoparametric means both $X(\xi)$ and $N(\xi)$ are expressed in term of the same shape function

$$X(\xi) = X_{1,e}N_1(\xi) + X_{2,e}N_2(\xi) \tag{3.18}$$

$$X(\xi) = X_{1,e}0.5(1 - \xi) + X_{2,e}0.5(1 + \xi) \tag{3.19}$$

So, the calculation of $\frac{dN_1}{dx}$, $\frac{dN_2}{dx}$ can be done as follows

$$\frac{dN_1(\xi)}{d\xi} = \frac{dN_1}{dx} \cdot \frac{dx}{d\xi} \tag{3.20}$$

$$\frac{dN_1(\xi)}{d\xi} = \frac{dN_1}{dx} \; 0.5\big(X_{2,e} - X_{1,e}\big), \tag{3.21}$$

Similarly,

$$\frac{dN_2(\xi)}{d\xi} = \frac{dN_2}{dx} \cdot \frac{dx}{d\xi} \tag{3.22}$$

$$\frac{dN_2(\xi)}{d\xi} = \frac{dN_2}{dx} \; 0.5\big(X_{2,e} - X_{1,e}\big), \tag{3.23}$$

By this way

$$\frac{dN_1(\xi)}{d\xi} = 0.5 \text{ and } \frac{dN_2(\xi)}{d\xi} = 0.5,$$

This leads to

$$\frac{dN_1}{dx} = \frac{-1}{X_{2,e}-X_{1,e}} \quad \text{and} \quad \frac{dN_2}{dx} = \frac{1}{X_{2,e}-X_{1,e}}$$

In the integral, we have the differential element dx, which can be related to $d\xi$, as

$$dx = 0.5\left(X_{2,e} - X_{1,e}\right)d\xi \tag{3.24}$$

Let the interval length be $h_e = X_{2,e} - X_{1,e}$, so, the ij^{th} entry of the local matrix of (3.10) can be written in the local word system, using integral by parts

$$a_{ij} = \int_{\Omega e} a(x)\frac{dN_j(x)}{dx} \cdot \frac{dN_i(x)}{dx}\, dx - a(x)\frac{dN_j(x)}{dx}N_i \Big|_{X_{1,e}}^{X_{2,e}} \tag{3.25}$$

The second term very small, can be neglected as compared with the first term. So, for interior element, we have

$$a_{ij} = \int_{\Omega e} a(x)\frac{dN_j(x)}{dx} \cdot \frac{dN_i(x)}{dx}\, dx \tag{3.26}$$

Since the extent is small $a(x)$ can be assumed constant on sub-domain Ωe, and can be called a constant a_e. Thus, the ij^{th} entry of the local element can be expressed in the local coordinate system as

$$a_{ij} = a_e \int_{-1}^{1} \frac{-(0.5)}{{h_e}^2}\, h_e\, d\xi = \frac{-a_e}{h_e} \tag{3.27}$$

Thus, we can find all entries of all elements such as.

If $i = j$, that is $a_{11} = a_{22} = \frac{a_e}{h_e}$. So, the local FEM matrix is

$$A^e_{local} = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.28}$$

After all the local FEM matrices are found, they must be rearranged to find the global FEM matrix. This phase of the solution is called " matrix assembly"

The fifth step:

The matrix assembly. The global FEM matrix can be done by assembling the local matrices. For the element e, let $a_e = 1$, thus, the local matrix is

$$A^e_{local} = \begin{bmatrix} \frac{1}{h_e} & \frac{-1}{h_e} \\ \frac{-1}{h_e} & \frac{1}{h_e} \end{bmatrix} \tag{3.29}$$

Let us consider the matrix assembly 4 elements (h) mesh with 5 nodes as in Fig.13.

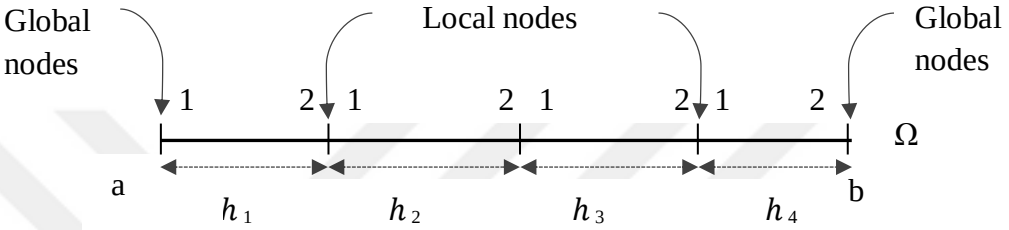


Figure 13. Assembly pieces for the whole domain.

For the 1st element

$$A^1_{local} = \frac{1}{h_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.30}$$

This local matrix must be insert in the global 5×5 matrix

$$A = \begin{bmatrix} \frac{1}{h_1} & \frac{-1}{h_1} & 0 & 0 & 0 \\ \frac{-1}{h_1} & \frac{1}{h_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.31}$$

For the second element

$$A^2_{local} = \frac{1}{h_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.32}$$

Since, when inserting the second element the global matrix becomes as

$$A = \begin{bmatrix} \frac{1}{h_1} & \frac{-1}{h_1} & 0 & 0 & 0 \\ \frac{-1}{h_1} & \frac{1}{h_1}+\frac{1}{h_2} & \frac{-1}{h_2} & 0 & 0 \\ 0 & \frac{-1}{h_2} & \frac{1}{h_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.33}$$

For other elements, it can be written as

$$A = \begin{bmatrix} \frac{1}{h_1} & \frac{-1}{h_1} & 0 & 0 & 0 \\ \frac{-1}{h_1} & \frac{1}{h_1}+\frac{1}{h_2} & \frac{-1}{h_2} & 0 & 0 \\ 0 & \frac{-1}{h_2} & \frac{1}{h_2}+\frac{1}{h_3} & \frac{-1}{h_3} & 0 \\ 0 & 0 & \frac{-1}{h_3} & \frac{1}{h_3}+\frac{1}{h_4} & \frac{-1}{h_4} \\ 0 & 0 & 0 & \frac{-1}{h_4} & \frac{1}{h_4} \end{bmatrix}, \tag{3.34}$$

The sixth step:

The final step is the solution. As shown in the previous step we obtain a system of linear equations can be resolved easily, as

$$Ax = b \tag{3.35}$$

where $A = 5 \times 5$ matrix, $x = 5 \times 1$ vector and $b = 5 \times 1$ vector. The vector b can be in the following form when the right-hand side. of the differential equation is zero (Laplace equation).

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{3.36}$$

Now, we have to insert the boundary conditions into the matrix equation.

$$u(a) = u_1, \;\; u(b) = u_2 \tag{3.37}$$

By modifying the 1st and the 5th row of matrix equation as

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{-1}{h_1} & \frac{1}{h_1}+\frac{1}{h_2} & \frac{-1}{h_2} & 0 & 0 \\ 0 & \frac{-1}{h_2} & \frac{1}{h_2}+\frac{1}{h_3} & \frac{-1}{h_3} & 0 \\ 0 & 0 & \frac{-1}{h_3} & \frac{1}{h_3}+\frac{1}{h_4} & \frac{-1}{h_4} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.38}$$

When the right hand side non-zero (Poison's equation) *b* vector will become

$$b = \begin{bmatrix} u_1 \\ b_2 \\ b_3 \\ b_4 \\ u_2 \end{bmatrix}, \tag{3.39}$$

Practically, we can verify the accuracy of the finite element method by using multiple shape functions. The references [46] [47] contains more details about the finite element method.

3.3 Formulation of the Finite Element Method with the Shape Functions

In this thesis, we're studying the finite element method by using three types of shape functions, these functions are, linear (hat), step function and the sigmoid function. The first two functions are commonly used in this field and the third function is used for the first time. In order to verify the efficiency and the accuracy of this numerical technique, we are using two examples. In the first example, the electrostatic potential is saved inside a charged dielectric parallel plate capacitor by solving the 2nd order ordinary differential equation (Poisson's equation). In the second example, the electrostatic potential is solved inside a 3-D spherical region of electron cloud with a uniform charge density. The results show that the use of sigmoid function with specific constants yields more accurate results than either of the hat function or the step function.

3.3.1 Accuracy Verification of FEM with Sigmoid Function

The sigmoid function can be expressed as

$$N(x) = \frac{1}{1 + e^{-a_1(x-b_1)}}, \qquad x \in \Omega_e \tag{3.40}$$

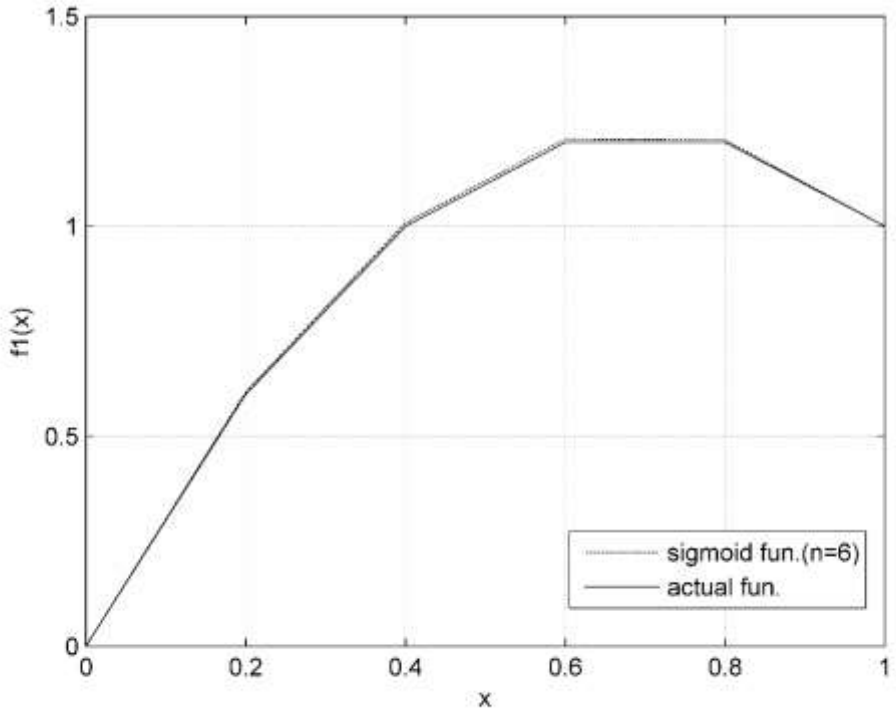
where a_1, b_1 are constants to be determined in a given solution interval $[a, b]$. The accuracy of the sigmoid function can be verified by using optimization function like discrete mean square error function (MSE) as

$$E(n) = \sum \left| f(n) - u(n, a_1, b_1) \right|^2, \qquad n \in [a, b] \tag{3.41}$$

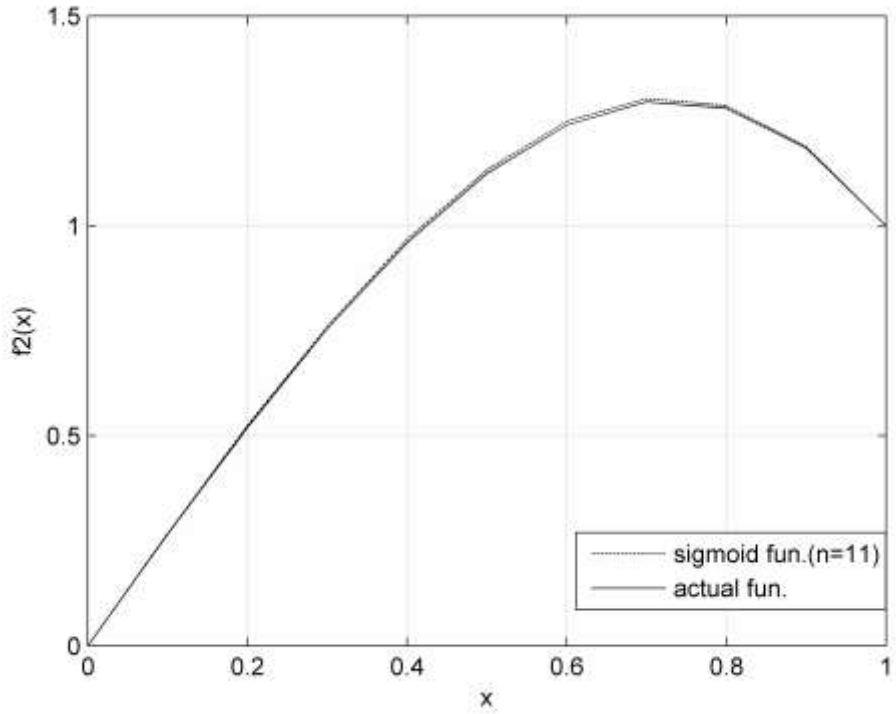
where n is the sampling points, $f(n)$ is the discrete samples of the exact solution and u is the approximated solution. The aim is to minimize (3.41) to find the optimum values for the constants a and b. The minimization process can be explained by using the following boundary value problem BVP

$$\frac{-d^2 f_1(x)}{d^2 x} = kx, \qquad 0 < x < 1 \tag{3.42}$$

with boundary conditions $u(0) = 0,\ u(1) = 1$. where $f_1(x)$ is the unknown function and k is a constant. The considered domain [0,1] can be divided into any number of n nodes with $(n-1)$ segments. We use 2-D algorithm starting with arbitrary values of a_1 and b_1 to get an optimal values for them to make MSE as small as possible. The graphical results are shown in Fig.14. This figure shows how much of the sigmoid function identical with the exact solution of two boundary value problems. The examples of BVP's with the corresponding constants a_1 and b_1 at different segment lengths are shown in table (1). We note that the table contains two examples of BVP's and the constants a_1 and b_1 for the sigmoid function are functions of the segment length. Also, the matrix A in FEM analysis has an increasing condition number with increasing number of segments.



a) $f_1(x)$



b) $f_2(x)$

Figure 14. Solution of the (BVP's) in Table 1 for a) n = 6, b) n =11.

Table 1. Examples of optimization for the constants a_1 and b_1 in the interval $x \in [0,1]$

BVP's	Exact solution	Number of nodes	Segment length	a1	b1	Error	Matrix A condition number
$-\frac{d^2 f_1(x)}{dx^2} = 5.$ $f_1(0) = 0. f_1(1) = 1$	$f_1(x) = \frac{-x(5x-7)}{2}$	6	0.2	-30	0.1	1.82 ×10−4	32.2
$\frac{d^2 f_1(x)}{dx^2} = 5.$ $f_1(0) = 0. f_1(1) = 1$	$f_1(x) = \frac{-x(5x-7)}{2}$	11	0.1	-60	0.05	3.65 × 10−4	96.9
$-\frac{d^2 f_1(x)}{dx^2} = 5.$ $f_1(0) = 0. f_1(1) = 1$	$f_1(x) = \frac{-x(5x-7)}{2}$	101	0.01	-610	0.005	4.31 × 10−4	4841
$-\frac{d^2 f_2(x)}{dx^2} = 10x.$ $f_2(0) = 0. f_2(1) = 1$	$f_2(x) = \frac{8x}{3} - \frac{5x^3}{3}$	6	0.2	-30	0.1	1.85 × 10−4	32.2
$-\frac{d^2 f_2(x)}{dx^2} = 10x.$ $f_2(0) = 0. f_2(1) = 1$	$f_2(x) = \frac{8x}{3} - \frac{5x^3}{3}$	11	0.1	-60	0.05	3.71 × 10−4	96.9
$-\frac{d^2 f_2(x)}{dx^2} = 10x.$ $f_2(0) = 0. f_2(1) = 1$	$f_2(x) = \frac{8x}{3} - \frac{5x^3}{3}$	101	0.01	-610	0.005	4.38 × 10−4	4841

Under these conditions, the sigmoid function with minimal error is obtained to be symmetric and passing through the points ($a_{\Omega e}$, 0.05) and ($b_{\Omega e}$, 0.95) where $\Omega_e \in [a_{\Omega e}, b_{\Omega e}]$ as shown in Fig.15. Also, this function has asymptotes $y = 0$ and $y = 1$ at $x = \mp\infty$ respectively. The values for the constants a_1 and b_1 are plotted in logarithmic scale against the segment length Ω_e in Fig.16. In the following examples, we used these constants to solve the BVP's from electrostatics in order to verify the theory.

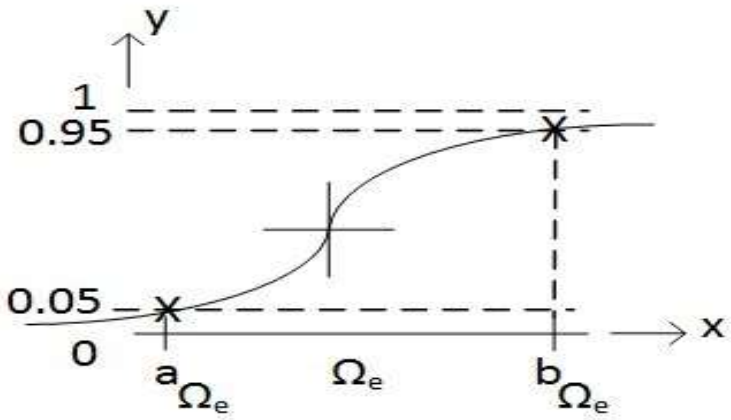


Figure 15. The sigmoid function that yields the best accuracy with (FEM).

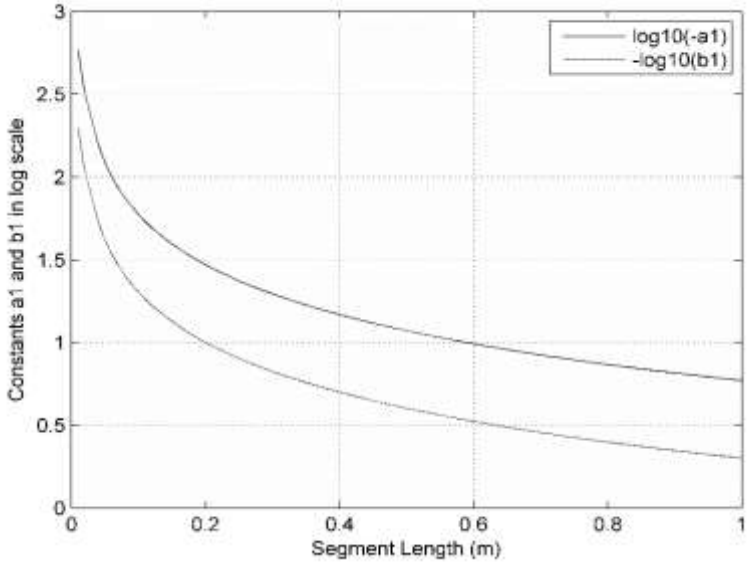


Figure16. Optimum values for the constants a_1 and b_1.

3.3.2 Examples

3.3.2.1 Electrostatic potential inside a charged parallel plate capacitor.

We considered solving the electrostatic potential inside a parallel plate capacitor maintained at constant voltage V_0 across its terminals. Fig.17 shows the physical layout of the problem. The capacitor is filled with a dielectric material whose electric permittivity (relative) is ϵ_r =1. The dielectric is assumed to have a charge density ρ = $10^{-6} C/m^3$, and the plate separation distance $d = 0.01m$.

The governing equation of the problem is the" Poisson's equation" given by

$$\frac{d^2\,V(y)}{dy^2} = \frac{-\rho}{\epsilon} \tag{3.40}$$

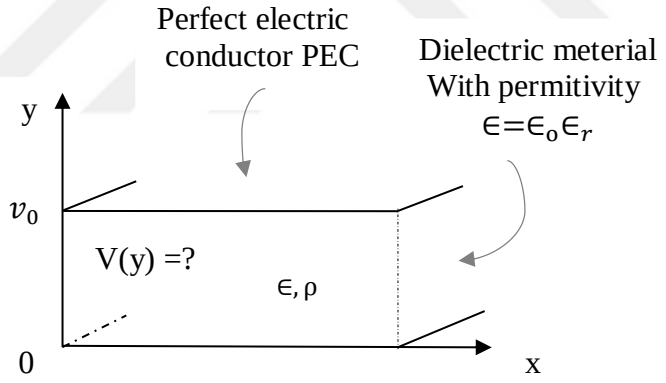


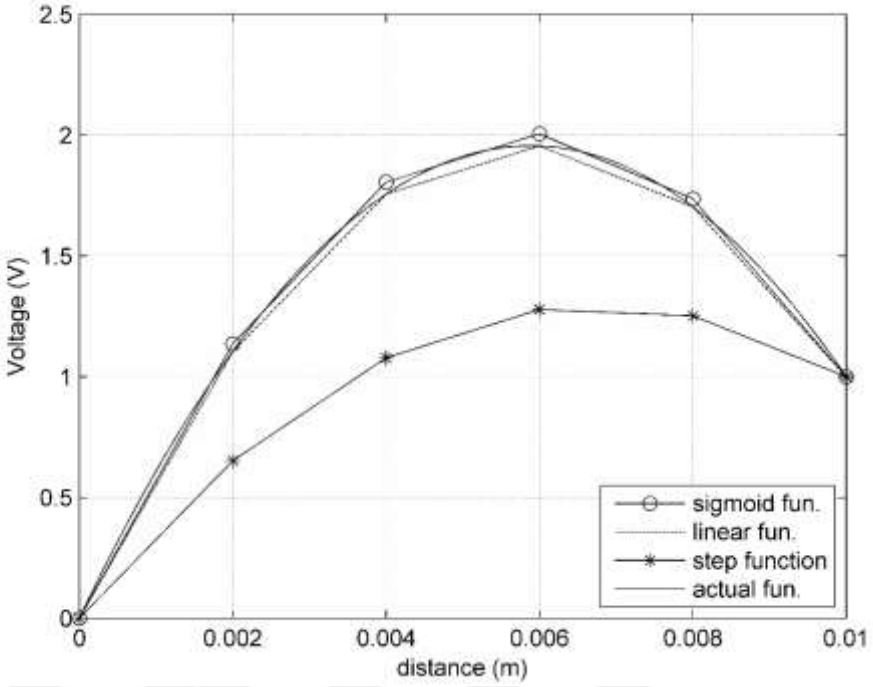
Figure 17. The parallel plate capacitor.

Assuming that the voltage varies only along the y direction. The boundary conditions are

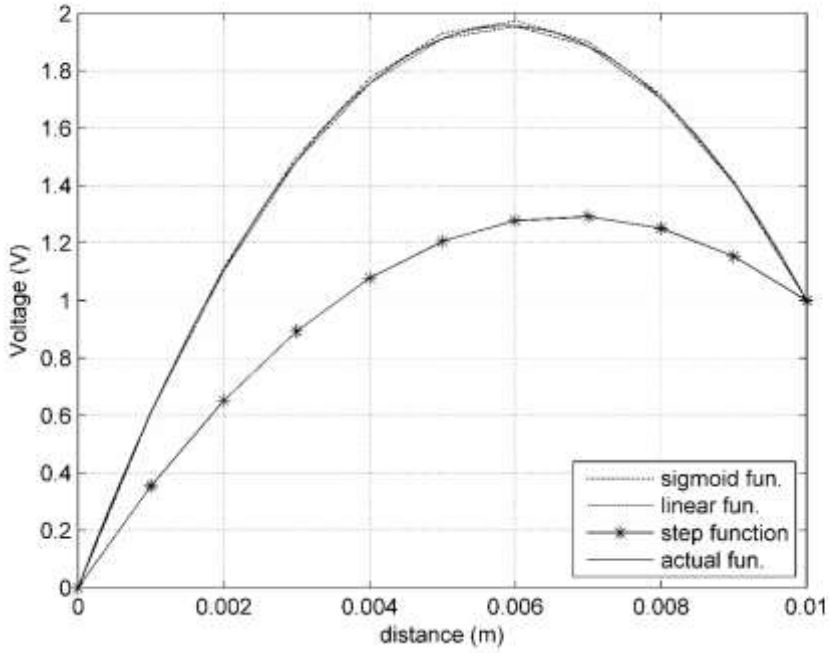
$$V\,(0) = 0, V(d) = 1V \tag{3.41}$$

The FEM simulation is carried out in Matlab for 5 and 10 elements using three different shape functions, linear, step and sigmoid. The results are shown in Fig.18. It is observed that the FEM with a sigmoid function gives the most accurate results when

the predetermined constants $a_1 = -3000$ and $b_1 = 0.001$ are used. As well, 5 and 10 elements are used in simulations for the purpose of bringing out the differences in accuracy.



a) $n = 6$



b) $n = 11$

Figure 18. The solution of example-1 for a) n=6, b) n=11 nodes.

The following mathematical expressions are used for each shape function during the simulation.

For linear function:

$$N_1(y) = \frac{1}{h_e} y, \qquad N_2(y) = \frac{-1}{h_e} y + \frac{1}{h_e}\ , y \in \Omega_e\ , \tag{3.42}$$

where h_e is the length for each equally spaced element.

Step function:

Step function is approximated as piecewise linear function:

$$N_1(y) = \begin{cases} 0, & 0 \le y \le h_e/4 \\ (2/h_e)y - \left(\frac{1}{2}\right), & h_e/4 < y \le 3h_e/4 \\ 1, & 3h_e/4 < y \le h_e \end{cases} \tag{3.43}$$

$$N_2(y) = \begin{cases} 1, & 0 \le y \le h_e/4 \\ (-2/h_e)y + \left(\frac{h_e}{4}\right), & h_e/4 < y \le 3h_e/4 \\ 0, & 3h_e/4 < y \le h_e \end{cases}$$

The sigmoid function:

The choice of the shape function is one of the important parameters that affects the performance of FEM In a given solution interval, [a , b]. For this reason, we consider the sigmoid function defined as [48].

$$N_1(y) = \frac{1}{1 + e^{-a_1(y-b_1)}}\ , \qquad N_2(y) = \frac{1}{1 + e^{a_1(y-b_1)}}, y \in \Omega e \tag{3.44}$$

where: a_1 and b_1 are constants.

The exact solution of the problem is given by [49].

$$V(y) = \frac{-\rho}{2\epsilon} y^2 + \left(\frac{\rho d}{2\epsilon} + \frac{V_o}{d}\right) y, \tag{3.45}$$

3.3.2.2 Electrostatic Potential Inside a Spherical Region of Electrons with a Uniform Charge Density

In this example, we considered the solution of electrostatic potential inside a spherical cloud of electrons with a uniform charge density $\rho = -\rho_0$ (where ρ_0 is a positive quantity). The physical depiction of the problem is shown in Fig.19.

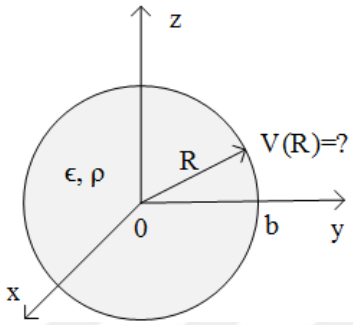


Figure 19. Electrostatic potential inside spherical cloud of electrons.

The charge exists for $0 \leq R \leq b$ where R is the radial distance in spherical coordinates. We assumed b = 2m, and the region is of free space with a uniform volume charge density $\rho = -\rho_0 = -1 \times 10^{-12} C/m^3$. Under these conditions, the governing equation to find the electric potential for $0 \leq R \leq b$ is given by [49].

$$\frac{d}{dR}\left(R^2 \frac{dV(R)}{dR}\right) = \frac{\rho_o}{\epsilon_o} R^2, \tag{3.46}$$

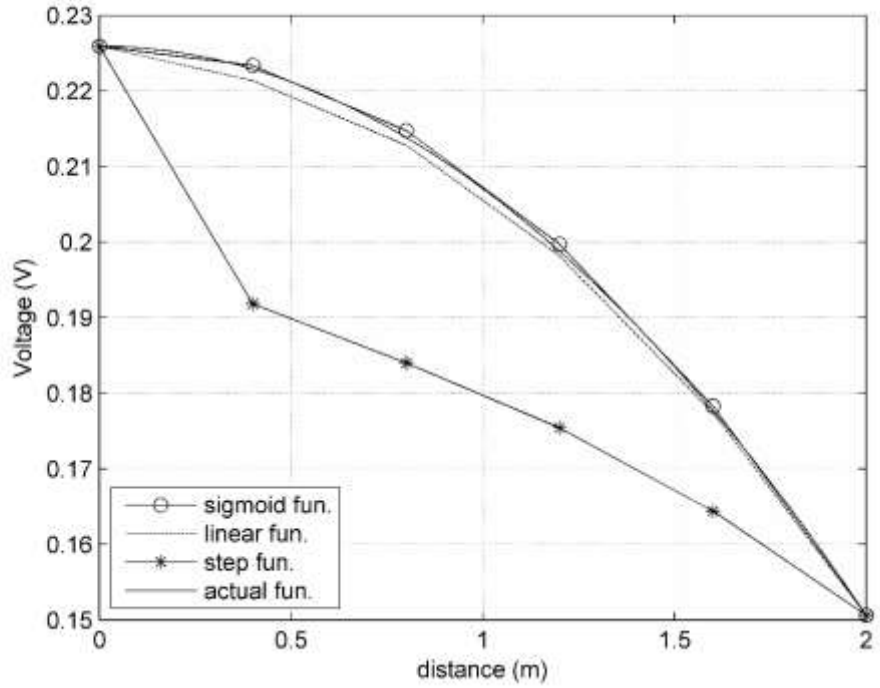
Subjected to the boundary conditions

$$V(0) = 0.226V\,, \;\; V(b = 2) = 0.15V \tag{3.47}$$

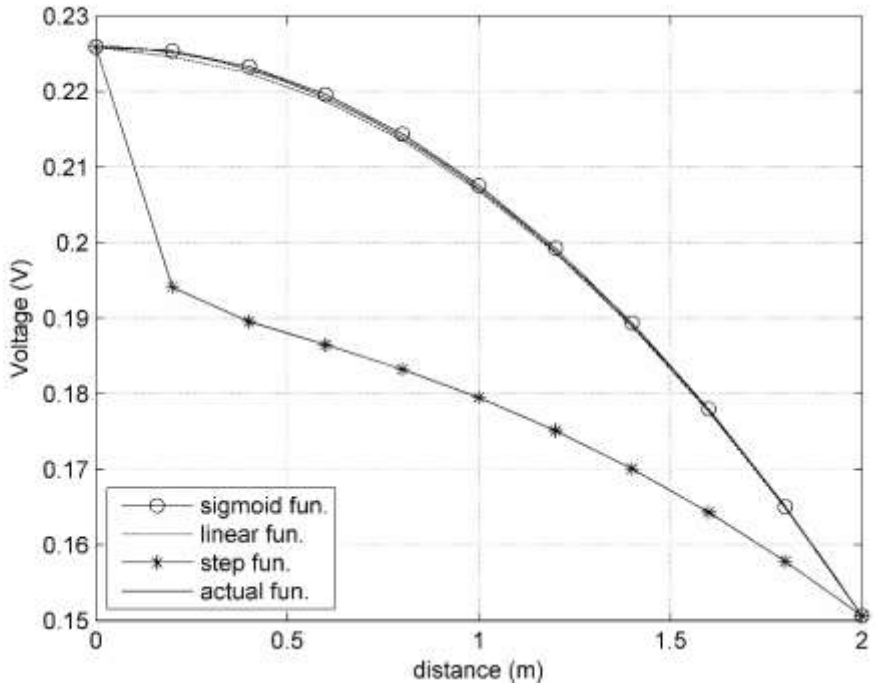
The actual formulation of this problem is as follows

$$V(R) = -\frac{\rho_o}{3\,\epsilon_o}\left(\frac{3b^2}{2} - \frac{R^2}{2}\right), \tag{3.48}$$

The results of the numerical analysis are shown in Fig.20. It is observed that the FEM with the sigmoid function is the most accurate all the other shape functions. The response is obtained by using the predetermined constants $a_1 = -14.72$ and $b_1 = 0.2$.



(a) $n = 6$



(b) $n = 11$

Figure 20. The solution of example-2 for a) n=6, b) n=11 nodes.

CHAPTER 4

METHOD OF MOMENTS

4.1 Introduction

The method of moments can be considered to be one of the most significant numerical techniques that has the ability in solving problems in the form of boundary linear integral [50]. This method can be used in different fields of engineering and science, such as fluid mechanics, fracture mechanics, acoustics and electromagnetism… etc. Since the 1980s, MoM has become more spread when computer started to be used more frequently than analytic calculations. Theoretically, it works by producing a "mesh" over the modeled surface [51].

4.2 General description of the method of moments

The method of moments MoM is frequently used in analytic electromagnetic problems. Usually, the application of this method to electromagnetic problems involves four steps:

- determining the suitable integral equation.
- convert the integral equation system into a matrix equation.
- assessing the matrix elements, and
- dissolving the resulting set of equations simultaneously.

Let as consider the inhomogeneous equation

$$L(f) = g \qquad (4.1)$$

where L is a linear operator, g is known function and f unknown function of L. It can be expressed as

$$f = \sum_n \alpha_n f_n \qquad (4.2)$$

where α_n are constants, f_n are the expansion functions or basis functions.

The equation (4.2) is the exact solution, in the form of an unlimited summation and the f_n is the set of basis functions. For approximate resolutions, (4.2) is typically a limited summation. Compensating (4.2) in (4.1) and using the linearity of L, we have

$$\sum_n \alpha_n L(f_n) = g \qquad (4.3)$$

Now, the weighted function or testing terms $w_1, w_2, w_3, \ldots$, are placed in the set of L and take the internal product of (4.3) with each w_m the result is

$$\sum_n \alpha_n \langle w_m, Lf_n \rangle = \langle w_m, g \rangle, m = 1,2,3, \ldots \qquad (4.4)$$

This set of equations can be expressed in term of matrix form as

$$[l_{mn}][\alpha_n] = [g_m] \qquad (4.5)$$

where

$$[l_{mn}] = \begin{bmatrix} \langle w_1, Lf_1 \rangle & \langle w_1, Lf_2 \rangle & \ldots \\ \langle w_2, Lf_1 \rangle & \langle w_n, Lf_2 \rangle & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix} \qquad (4.6)$$

$$[\alpha_n] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{bmatrix}, \qquad [g_m] = \begin{bmatrix} \langle w_1, g \rangle \\ \langle w_2, g \rangle \\ \vdots \end{bmatrix} \qquad (4.7)$$

The inverse $\left[l^{-1}\right]$ of the matrix$[l]$ can be found if it is not singular. Whereupon, the α_n are expressed by

$$[\alpha_n] = \left[l^{-1}{}_{mn}\right][g_m] \qquad (4.8)$$

Alternatively, the solution of f given by (4.2), can be defined by the matrix of functions as

$$[\tilde{f_n}] = [f_1 \quad f_2 \quad f_3 \quad \cdots] \tag{4.9}$$

Then, the f becomes as

$$f = [\tilde{f_n}][\alpha_n] = [\tilde{f_n}][l^{-1}{}_{mn}][g_m] \tag{4.10}$$

The analytical solution (exact solution) might be compatible with the approximate solution, based on the choice of the f_n and w_n . The selection of $f_n = w_n$ is known Galerkin's method [52] [53]. The choice of the f_n and w_n is one of the serious task when we want to find numerical solutions for any particular problem. The $w_n$should be linearly independent and can be chosen so that the product $\langle w_n, g \rangle$ depend on independent properties of g. Similarly, the f_n must be straightly independent and chosen so that, followed the same approximation of (4.2) for f. There are some extra factors that effects on the selection of f_n and w_n are

- the accuracy of the wanted resolution.
- the suitable assessment of matrix elements.
- the size of the matrix that can be inverted.
- the accomplishment of a well-conditioned matrix $[l]$.

4.3 Numerical Steps of the Method of Moments

Let us consider the Fredholm's integral equation

$$\int_a^b k(t,s)f(s)ds = g(t), t \in [a,b] \tag{4.11}$$

where f is the unknown function, g is a known function, t, s are independent variable and k is the kernel of integral.

Alternatively, the Fredholm's integral equation can be express as

$$\mathcal{L} f = g \ , \tag{4.12}$$

where $\mathcal{L}$ is the integral operator. Thus, the numerical steps of the method of moments are

The first step:

Express the unknown function in terms of known function with unknown coefficients as

$$f \cong \sum_{j=1}^{N} c_j f_j \, , j = 1,2, \dots , N \tag{4.13}$$

where f_j's are known function (basis function) and c_j's are unknown coefficients

The second step:

Substitute the approximated (4.13) into (4.12) operator equation (Fredholm integral)

$$\mathcal{L}\left\{\sum_{j=1}^{N} c_j f_j\right\} = g \tag{4.14}$$

using linearity

$$\sum_{j=1}^{N} c_j \, \mathcal{L} \, f_j \, = g \tag{4.15}$$

This equation contains N unknown $(c_1, c_2, c_3, \dots , c_N)$, we need N equations to solve c_j's.

The third step:

Define the residue

$$R = g - \mathcal{L} \hat{f} \tag{4.16}$$

where $\hat{f}$ approximated function. Minimize *R* to get N more equations, for this purpose, we take the inner product of residual with some functions w_i , *i* = 1, 2,...,*N*, and equate the result to zero.

$$< R, w_i > = \int_{\Omega} R(t).w_i(t)\, dt \tag{4.17}$$

where Ω is a domain and $< R, w_i >$ is the inner product integral. This approach is called "method of weighted residuals" and solve

$$\sum_{j}^{N} c_j < \mathcal{L}.f - g, w_i >= 0 \tag{4.18}$$

The fourth step:

Choice of the weight function

$$w_i = \delta(t - t_i), \qquad t \in (a, b) \tag{4.19}$$

where w_i is the delta function. Then this method is called point match or "collocation"

The fifth step:

Choice of the basis functions.

1. Delta function: $\delta(t - t_i)$.

2. Unit step function (stair case approximation).

The basis function can be expressed as in (4.20) with sliced domain as in Fig.21.

$$f(x) = \begin{cases} 1, & on\ s_i \\ 0, & elsewhere \end{cases} \tag{4.20}$$

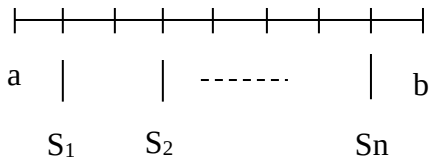


Figure 21. Sliced domain, (s) is the midpoint of each sub-domain.

4.4 A Novel algorithm for the Solution of the Singularity of Electrostatic Problem by Using Method of Moments

The method of moments can be considered to be one of the best numerical technique, which has achieved a great successes and high efficiency in answering electromagnetic problems known to the researchers, found to have its way in this field. It can be used in solving boundary value problems with differential [54].

The majority of the electromagnetic problems can be expressed in terms of integral equations or in terms of partial differential equations in multidimensions. The method of moments has the ability to solve the integral equations easily. This technique resolves the integral equations. As a result, forming N linear equation system, with N unknown coefficients, then it works to find the solution of this system of equations by technical matrix algebra.

Often, the classical boundary integral methods have the singularity at a kernel of Greens function. The traditional way followed for the solution of this problem is that, it has transformed the Maxwell's equation system to bounded integral, then doing the partitioning. Some works solve the singularity as in [55], but in this reference the process is reversed, where the differential equations are first partitioned on a uniform grid, secondly, transforming to boundary difference equations. This way, it eliminates the singularity of the Greens function. In [56], the numerical solution of singular integral equations is analysed by utilizing Chebyshev polynomials in the first, second, third and fourth kind to get a system of linear algebraic equations, the resulting systems are resolved numerically. The procedure of the current effort predictable to be helpful for disband singular integral equations of the first kind, including partially singular and partially uniform kernels. The [57] offers a solution to the static charge distribution

on a thin conductor wire using the method of moments. Because of the interfere between the test and source area, as a result a singularity along the diagonal components of A (matrix that present a solution to the static charge distribution on a thin conductor wire) will appear. This thesis presents a novel robust algorithm to solve this problem. In this algorithm, we have considered the problem of numerical evaluation of Cauchy principal value integration [58]. The strategy of this algorithm is that it divides the singularity integral into two intervals and modifies the limits of integrals to overcome the diagonal singularity of the matrix. This is will be explained in the next sections.

4.4.1 Method of Moments for the Solution the Static Charge Distribution on a Thin Wire

4.4.1.1 Segmentation of the finite straight thin wire

The mathematical modelling of electromagnetic problems and the calculations of the charge density systems become a widespread research in the last three decades. Today it plays an important role in spacecrafts [57] [59-61]. Assume a straight conducting wire of length L and radius a , where a << L, held at constant v_o and naturally exist charge density ρ spread on the surface. The aim is to solve ρ given v_o. For approximating the solution, let us assume that electrical charges are spread completely along the x-axis. Therefore, the electric potential resulting from these electrical charges at some point on the x- axis can be expressed as

$$v(x) = \frac{1}{4\pi\epsilon_o}\int_0^L \frac{\rho(x')}{|x - x'|}dx' \qquad (4.21)$$

where ρ (x') is the distributed charge, x is the arbitrary observation point along the x-axis, x' is the location of the arbitrary point charge that lie somewhere away from the

origin, and $(x - x')$ the distance of observation point to the source point at the center. The thin wire is sliced to subsections and taken one charge at midpoint of each segment, where each slice with length $\Delta_x = L/N$ and x' as a midpoint of segment $x'_n = n\,\Delta_x/2$ as shown in Fig.22

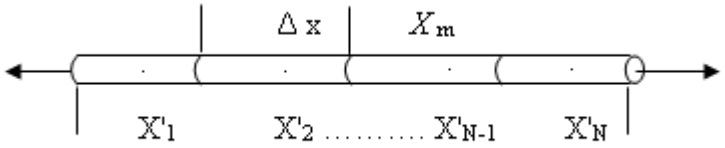


Figure 22. The partitioned solution domain.

The unknown function ρ, can be approximated by an estimate function $\hat{\rho}$ of a linear combination of discrete basis function as

$$\hat{\rho}(x) \approx \sum_{1}^{N} \alpha_n \, u_n(x'), \tag{4.22}$$

where u_n is the basis function, α_n is the weighted coefficients (unknown coefficients) and $u_n(x')$ can be chosen as a delta or step functions.

$$u_n(x') = \delta(x' - x'_n), \tag{4.23}$$

The complete charge distribution on a thin conductor wire (rod) can be approximated as if it was a series of points located at the midpoints x'_n. The position of x'_n (match point) is at the center of each sub-domain. So, the next goal would be to find the solutions for unknown expansion coefficients α_n. This is the best approximation to the true solution for charge distribution ρ. Now replacing ρ in (4.21) by approximation function $\hat{\rho}$ in (4.22) yields

$$v(x) \approx \frac{1}{4\pi\epsilon_o} \int_{0}^{L} \frac{1}{|x - x'|} \sum_{n=1}^{N} \alpha_n \, \delta(x - x'_n)\, dx' \tag{4.24}$$

$$v(x) \approx \frac{1}{4\pi\epsilon_o} \sum_{n=1}^{N} \alpha_n \int_0^L \frac{\delta(x - x'_n)}{|x - x'|}\, dx' \tag{4.25}$$

$$v(x) \approx \frac{1}{4\pi\epsilon_o} \sum_{n=1}^{N} \frac{\alpha_n}{|x - x'|} \tag{4.26}$$

After this approximation, the error in integral equation arises. This error can be represented by residual. Consequently, this error should be minimized, this can be done by solving a specific set of expansion weighted coefficients. All these expressions are a sequence of voltage patterns selected along the conductor so that the expansion coefficient leads the residual of every point to zero. Therefore, we write the residual as

$$R(x) = v(x) - \frac{1}{4\pi\epsilon_o} \sum_{n=1}^{N} \frac{\alpha_n}{|x - x'|}, \tag{4.27}$$

4.4.1.2 Forming the matrix of the charge distribution

We have the unknown coefficients $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ in (4.27). To get a solution for this problem, let us take the voltage at the position $x = L/2$, where the whole conductor is at the identical voltage potential. Now, to minimize the error R, it must be equal to zero. We assume, a pattern of an arbitrary collection of M voltage along the wire such that $v(x_m) = v_0$. We get a system of *M* linear of functions with *N* undefined coefficients.

$$v(x) - \frac{1}{4\pi\epsilon_o} \sum_{n=1}^{N} \frac{\alpha_n}{|x_1 - x'_n|} = 0,$$

$$v(x) - \frac{1}{4\pi\epsilon_o} \sum_{n=1}^{N} \frac{\alpha_n}{|x_2 - x'_n|} = 0,$$

$$v(x) - \frac{1}{4\pi\epsilon_o}\sum_{n=1}^{N}\frac{\alpha_n}{|\, x_m - x'_n \,|} = 0, \tag{4.28}$$

Let $N = M$. In this way, we can ensure a unique solution for all values of weighted extension coefficients [57]. The above system of equations can be express as matrix-vector $A\,x{=}b$ as shown in (4.29).

$$\begin{bmatrix} \frac{1}{|x_1-x'_1|} & \frac{1}{|x_1-x'_2|} & \cdots & \frac{1}{|x_1-x'_{N-1}|} & \frac{1}{|x_1-x'_N|} \\ \frac{1}{|x_2-x'_1|} & \frac{1}{|x_2-x'_2|} & \cdots & \frac{1}{|x_2-x'_{N-1}|} & \frac{1}{|x_2-x'_N|} \\ \frac{1}{|x_3-x'_1|} & \frac{1}{|x_3-x'_2|} & \cdots & \frac{1}{|x_3-x'_{N-1}|} & \frac{1}{|x_3-x'_N|} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{|x_{N-1}-x'_1|} & \frac{1}{|x_{N-1}-x'_2|} & \cdots & \frac{1}{|x_{N-1}-x'_{N-1}|} & \frac{1}{|x_{N-1}-x'_N|} \\ \frac{1}{|x_N-x'_1|} & \frac{1}{|x_N-x'_2|} & \cdots & \frac{1}{|x_N-x'_{N-1}|} & \frac{1}{|x_N-x'_N|} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{N-1} \\ \alpha_N \end{bmatrix} = \begin{bmatrix} 4\pi\epsilon_o V_o \\ 4\pi\epsilon_o V_o \\ 4\pi\epsilon_o V_o \\ \vdots \\ 4\pi\epsilon_o V_o \\ 4\pi\epsilon_o V_o \end{bmatrix} \tag{4.29}$$

Thus, the unknown coefficients(α_n) represented by vector x can be solved as

$$x = A^{-1}\, b \tag{4.30}$$

This permits to write every matrix element (sub-domain) as

$$A_{mn} = \frac{1}{x_m - x'_n}, \tag{4.31}$$

where $x_m = m\frac{\Delta_x}{2}$, and $x_n = n\frac{\Delta_x}{2}$, and they represent test and source locations respectively. Since, $n = m$ this leads to interfere amongst test and source areas. In this case, the singularity over the diagonal elements of A occurs. So, to solve this singularity, it can be eliminated by assuming the charge density spread over the hollow tube. Thus, the expression of the diagonal matrix A_{mm} of each element as [57]

$$A_{mm} = \frac{1}{\Delta_x}\ln\left|\, \frac{\Delta_x + \sqrt{{\Delta_x}^2 + 4a^2}}{\Delta_x - \sqrt{{\Delta_x}^2 + 4a^2}} \,\right| \tag{4.32}$$

In the next section, we deal with a new algorithm in solving this problem

4.4.2 Analytical Transformation for Solving the Singularity

Recently, the hyper singular integral plays a serious role in the development. It is one of bounded integral types using for singularity cancelation [62]. The formulation of this integral has been applied to solve boundary value problems. In this specialization, it can be observed two main tactics, one of them uses of Cauchy principal value CPV and the other uses formulation in term of finite parts [63]. Our work has been adopted the Cauchy principal value and modify a new algorithm for solving the diagonal singularity matrix of the charge distribution on thin wire [57]. The general form of Cauchy principal value integral is [58].

$$I = \int_{a}^{b} \frac{f(x)}{(x-\tau)} dx, \quad (a < \tau < b) \tag{4.33}$$

In numerical form

$$I_{a,b,\tau} = \int_{a}^{b} \frac{f(x)}{(x-\tau)} dx, \quad (a < \tau < b) \tag{4.34}$$

let $a = -1$ and $b = 1$, the integration becomes

$$I_{\tau}(f) = I_{\tau,-1,1}(f) = \int_{-1}^{1} \frac{f(x)}{(x-\tau)} dx \tag{4.35}$$

Alternatively, by splitting the integral limits

$$I_{a,b,\tau} = \lim_{x \to 0} \int_{a}^{\tau-\mu} \frac{f(x)}{(x-\tau)}\, dx + \int_{\tau+\mu}^{b} \frac{f(x)}{(x-\tau)}\, dx \tag{4.36}$$

For simplicity let us assume

$$\delta = \min(1+\tau, 1-\tau) \tag{4.37}$$

We get

$$I_\tau(f) = \int_{|x-\tau|\geq\delta} \frac{f(x)}{x-\tau}dx + \int_{\tau-\delta}^{\tau+\delta} \frac{f(x)}{x-\tau}\ dx \tag{4.38}$$

According to our assumption as in (4.37), the first term of equation (4.38) can be solved as

$$\int_{|x-\tau|\geq\delta} \frac{f(x)}{x-\tau}dx \equiv \int_{\tau+\delta}^{1} \frac{f(x)}{x-\tau}\ dx\,, if\ \delta = 1+\tau \tag{4.39}$$

and

$$\int_{|x-\tau|\geq\delta} \frac{f(x)}{x-\tau}dx \equiv \int_{-1}^{\tau-\delta} \frac{f(x)}{x-\tau}\ dx,\ if\ \delta = 1-\tau \tag{4.40}$$

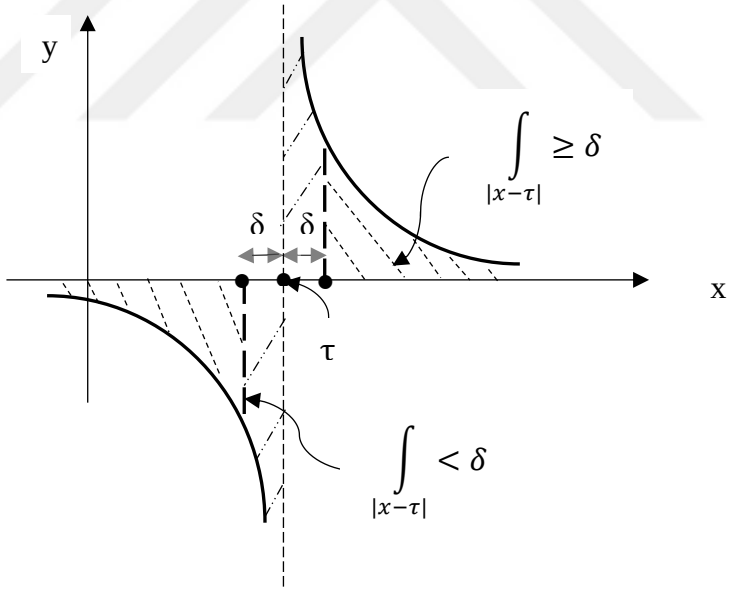


Figure 23. Plot the function (3.38).

The first term of (4.38) will be out of the singularity. The second term will have the singularity represented by δ region as shown in Fig.23. it can be derived as follows

$$\int_{\tau-\delta}^{\tau+\delta} \frac{f(x)}{x-\tau}\ dx\ , \tag{4.41}$$

Let $u = x - \tau \Longrightarrow u + \tau = x$, in this case we can substitute each *x* by $u + \tau$ and $du = dx$, since $x = \tau - \delta$ that is lead $u = \tau - \delta - \tau \Longrightarrow u = -\delta$,

$$\int_{\tau-\delta}^{\tau+\delta} \frac{f(x)}{x-\tau}\ dx\ = \int_{-\delta}^{+\delta} \frac{f(u+\tau)}{u}\ du \tag{4.42}$$

By splitting the term equation (4.42) we get

$$= \int_{-\delta}^{0} \frac{f(u+\tau)}{u}\ du + \int_{0}^{+\delta} \frac{f(u+\tau)}{u}\ du \tag{4.43}$$

$$= \int_{0}^{-\delta} -\frac{f(u+\tau)}{u}\ du + \int_{0}^{+\delta} \frac{f(u+\tau)}{u}\ du \tag{4.44}$$

$$= \int_{0}^{+\delta} \frac{f(u+\tau) - f(u-\tau)}{u}\ du \tag{4.45}$$

The equation (4.38) will becomes as

$$I_\tau(f) = \int_{|x-\tau|\geq\delta} \frac{f(x)}{x-\tau} dx + \int_{0}^{+\delta} \frac{f(u+\tau) - f(u-\tau)}{u}\ du \tag{4.46}$$

The second term of equation (4.46) represent the singularity region as shown in Fig.23, it can be neglected if we consider δ very small. The first term can be solved if we consider the convention $\delta = \min(1-\tau, 1+\tau)$, where this term is solved in matlab code as shown in appendix A (matlab code). Finally, we get an integral that can eliminate the singularities along the matrix A. The alternative transformation of $I_\tau(f)$ called the subtracting out the singularity [58] as

$$I_\tau(f) = \int_{-1}^{1} \frac{f(x)}{x-\tau}dx + \int_{-1}^{1} \frac{f(x)-f(\tau)}{x-\tau}dx$$

$$= f(\tau) log \frac{1-\tau}{1+\tau} + \int_{-1}^{1} \frac{f(x)-f(\tau)}{x-\tau}dx \qquad (4.47)$$

But this equation has a problem if the splitting integral or the quadrature node very near to τ. So, using equations (4.46) and (4.47) we get

$$I_\tau(f) = f(\tau) log \frac{1-\tau}{1+\tau} + \int_{|x-\tau|\geq\delta} g(x)\,dx + \int_{0}^{+\delta} h(x)\,dx \qquad (4.48)$$

where $g(x) = \frac{f(x)-f(\tau)}{x-\tau}$, $h(x) = \frac{f(\tau+x)-f(\tau-x)}{x}$. The third term of (4.48) represent the singularity, which is the same equation (4.45).

4.5 Example: Computation Charge Density a Long Thin Conductor Wire

Consider a thin conductive wire with length L = 1.0 m and radius a = L/40. The thin conductor wire is subjected to a uniform potential of V0 = 1.0 V with step functions as the basis for the charge distribution and N = 40 subdivisions as shown in Fig.24.

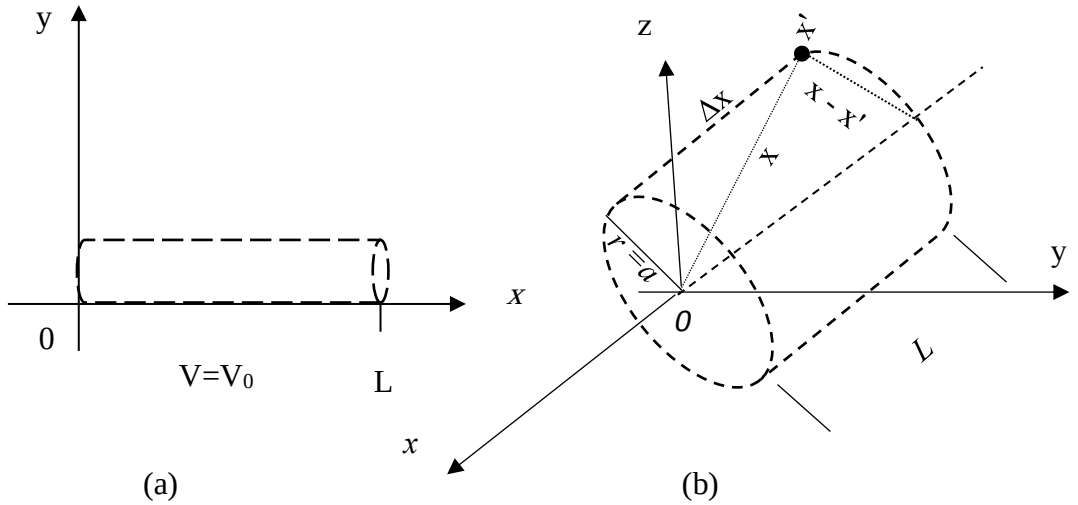


Figure 24. Thin conductor wire, a) charged segment with applied voltage, b) segment with source and test point charge.

The following figures shows the charge distribution along the conductor without singularity. Fig.25 shows the charge distribution according hollow tube assumption in (4.32) and Fig. 26 according to a new algorithm as in equation (4.46).

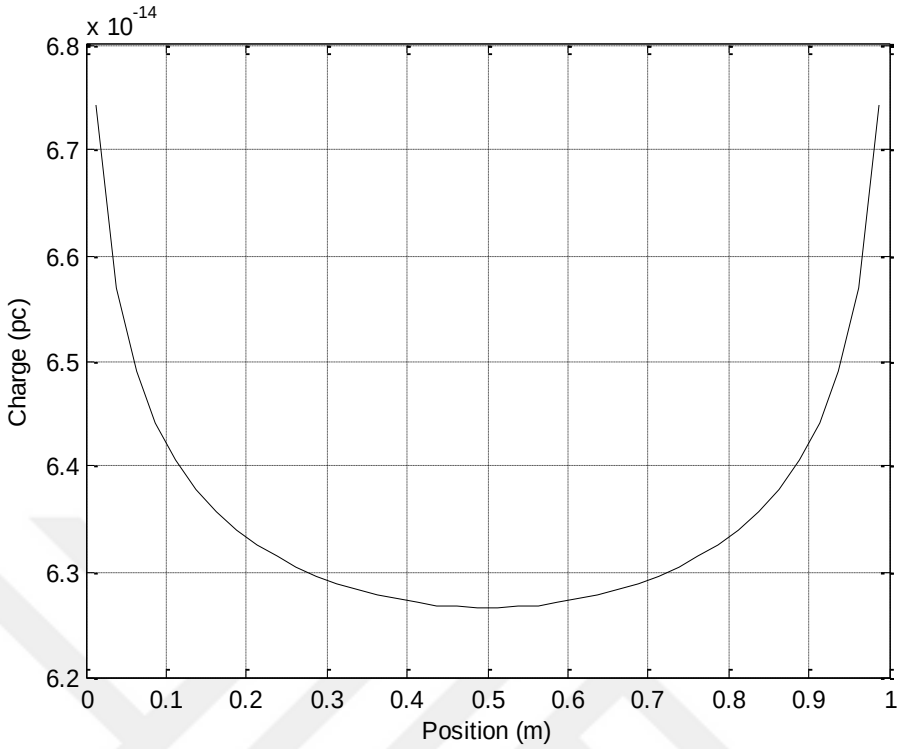


Figure 25. Charge distribution along the hollow tube according to the equation (4.32).

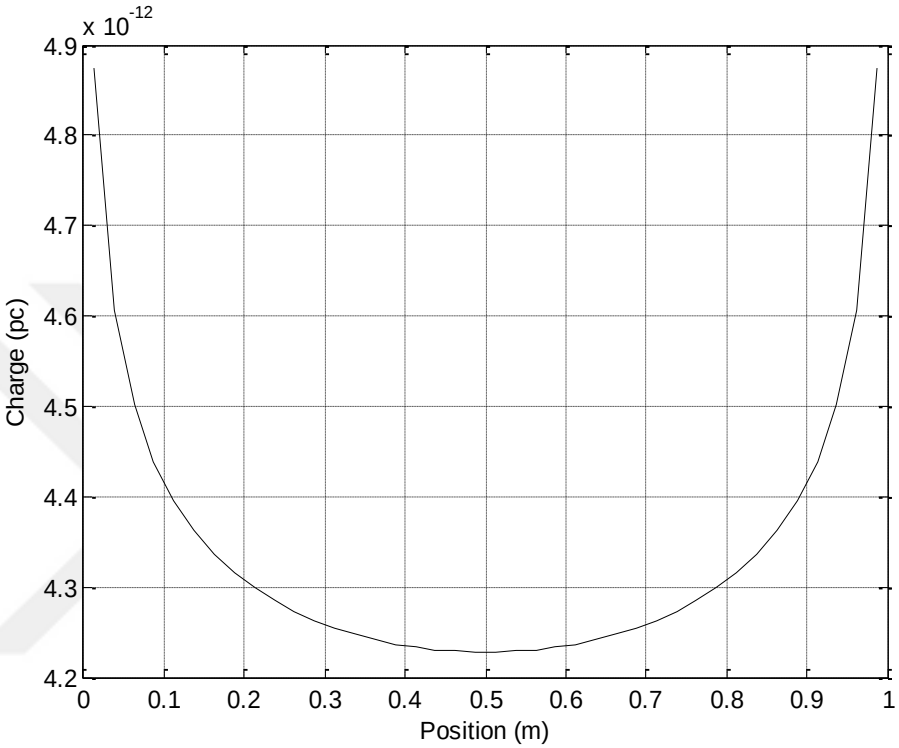


Figure 26. Compute the charge distribution by a novel algorithm along thin conductor wire.

CHAPTER 5

CONCLUSION

This thesis has considered two numerical techniques to solve some boundary value problems in electromagnetics. The first numerical method is the finite element method and the second one is the method of moments. In this thesis, we have analysed the accuracy of FEM with three shape functions, linear, step and the sigmoid function in solving electrostatic BVP's. We have concluded the following results:

The sigmoid function yields more accurate results than the linear (hat) function, especially for small number of elements, but the solutions become closer for high number of elements. The sigmoid function yields much more accurate results than the step function. During the FEM analysis, the constant a_e described in (3.27) is evaluated by using its value at the center point of each local FEM element. As a result, an error is introduced in the FEM analysis. However, this does not affect the comparison of the FEM results in terms of the selection of the shape functions since the same constant a_e has been used in all examples.

This thesis also introduced a new mathematical technique for numerical solution of electromagnetic problems governed by integral equations. The" Method of Moment" is a common numerical technique to solve such problems. In MoM, the solution domain is discretised by samples, and for each sample, the unknown function is approximated by a truncated series approximation. This way, the integral equation is transformed into a matrix equation whose solution gives an approximate solution to

the problem. At the samples where the source and the observation locations overlap (diagonals of the solution matrix) singularities exist. Thus, for many problems, the solution does not exist because of this matrix singularity. Several remedies have been applied to overcome singularities in literature. The proposed algorithm is schemed to solve the singularity problem of the solution matrix based on the Cauchy Principal Value theorem. An example of finding electrostatic charge distribution on a linear conducting rod is used to prove the accuracy and applicability of the proposed method. The results have shown that the method is indeed effective and accurate for solving such problems.

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APPENDIX A

A.1 MATLAB CODE

```
% Charge distribution problem by MoM by
the split method using step
% function using the absolute value
clear; clc;
L=1;
V0=1;
N=40; % 40 divisions
he=L/(N);
e0=8.854*1e-12;
mu0=4*pi*1e-7;
syms x
% Forming the matrix A
mu=1e-6;
for m=1:N
    for n=1:N
        xm=m*he-(he/2); % center points
        xn=n*he-(he/2);
        fx=1/abs(xm-x);
        if xm==xn

          A(m,n)=-log((xn-mu)-xm)+log((xn-
he/2)-xm)+log((xn+he/2)-xm)-log((xn+mu)-
xm);
        elseif xm < xn
                A(m,n)=log((xn+he/2)-xm)-
log((xn-he/2)-xm);
            elseif xm > xn
                A(m,n)=-log((xn+he/2)-
xm)+log((xn-he/2)-xm);
            end
```

```
        end

    end
end

 % Solving the unknown charge distribution
for m=1:N
    xm=m*he-(he/2); % center points
    b=(4*pi*e0*V0)*ones(N,1);
end
u=inv(A)*b;
for m=1:N
    xm(m,1)=m*he-(he/2);
end
plot(xm,u,'k'); grid on
xlabel('Position (m)');
ylabel('Charge (pc)');
display(sum(u));
```

A.2 CURRICULUM VITAE

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