FRACTIONAL DERIVATIVES WITH MITTAG-LEFFLER KERNEL AND THEIR APPLICATIONS

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FRACTIONAL DERIVATIVES WITH MITTAG-LEFFLER KERNEL AND THEIR APPLICATIONS

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STATEMENT OF NON-PLAGIARISM PAGE

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct. I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

FRACTIONAL DERIVATIVES WITH MITTAG-LEFFLER KERNEL AND THEIR APPLICATIONS

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Supervisor : Assist. Prof. Dr. Dumitru BALEANU

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In this thesis, we concentrate on studying the properties of a new fractional derivative with Mittag-Leffler kernel and presenting its discrete version. After that, we show some applications of these new fractional operators to couple real-world phenomena from the fields of engineering. In these applications, the results obtained from existing fractional derivatives are kept to express the performance of the considered new fractional derivative.

Keywords: Mittag-Leffler Function, Fractional Calculus, Atangana-Baleanu Fractional Derivative, Mittag-Leffler Kernel.

ÖZ

MITTAG-LEFFLER ÇEKİRDEĞİ İLE FRAKSİYONEL TÜREVLER VE UYGULAMALARI

CAN, Halil Sezgin

Yüksek Lisans, Matematik Anabilim Dalı

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Bu tez çalışmasında Mittag-Leffler çekirdeği kullanılarak yeni bir kesirli türevin özellikleri üzerinde durulmakta ve bunun ayrık versiyonu sunulmaktadır. Bunun ardından mühendislik alanlarından gerçek dünya olgularıyla eşleştirecek şekilde bu yeni kesirli operatörlerin bazı uygulamaları gösterilmektedir. Bu uygulamalarda, mevcut kesirli türevlerin sonuçlarıele alınan yeni kesirli türevin performansını vurgulamak için korunmuştur.

Anahtar Kelimeler: Mittag-Leffler Fonksiyonu, Kesirli Hesaplamalar, Atangana-Baleanu Kesirli Türevi, Mittag-Leffler Çekirdeği.

DEDICATION

Dedicated to my father, my wife and my daughter.

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LIST OF ABBREVIATIONS

M-L	:	Mittag-Leffler
L-T	:	Laplace Transform
C-F	:	Caputo-Fabrizio
L-C	:	Liouville-Caputo
E-L	:	Euler-Langrange
B-F-T	:	Bateman-Feshback-Tikochinsky
F-T	:	Fourier Transform

LIST OF SYMBOLS

E_α	Mittag-Leffler Function
$E_{\alpha,\beta}$	Generalized Mittag-Leffler Function
W	Wright Function
J^α	Fractional Operator
D^α	R-L Fractional Differential Operator.
${}^{C}\nabla_0^\alpha$	Discrete Caputo Fractional Derivative
${}^{CFR}\nabla_n^m$	Right Riemann Fractional Difference
${}_{m}^{CFR}\nabla^m$	Left Riemann Fractional Difference
${}^{CFC}\nabla^m$	Left Caputo Fractional Difference
${}^{CFC}\nabla_n^m$	Right Caputo Fractional Difference
H^f	Hamiltonian of Fractional Order

CHAPTER 1
INTRODUCTION

Fractional calculus deals with the derivatives and integrals of any real or complex order [1-40]. This type of calculus was initiated and developed by many great scientists like Leibniz, L'Hospital, Euler, Bernoulli, Riemann, Liouville, and many others [10-11]. Since its beginning many people have been working in this field to help its development and its applicability to various areas of mathematics, physics and engineering. It has been found that fractional calculus can be extensively used for many physical phenomena as a strong and effective tool to describe mathematical modeling [41-62].

Valuable uses for the Mittag-Leffler (M-L) functions have also been identified in various fields of the physical and applied sciences. In the last 20 years, about nine decades after being discovered by Swedish mathematician GM. Mittag-Leffler [3-8], these functions have grown in importance because they can potentially be utilized to solve problems in various applications, such as engineering, biological and physical sciences.

The aim of this master thesis is to introduce the new fractional derivative so called Atangana-Baleanu derivative as well as present some of its applications.

In this attempt to introduce readers of this thesis to the current state of research on M-L type functions and kernels, some particular articles are of key importance. The work by Atangana and Baleanu [1] made a significant contribution to the field of fractional derivatives and is valuable for its applications to models of heat transfer. In 2016, Abdeljawad and Baleanu [2] introduced discrete versions of fractional derivatives employing the M-L function together with proofs for formulas for the discrete integration by parts, further extending the field of applications for these functions. The following year, the same authors in [3] focused on monotonicity in the case of fractional difference operators with comparisons to the classical cases. Significant work has been done recently to establish the roles of Riemann-Liouville derivatives [2], Liouville-Caputo and Caputo-Fabrizio operators [3], and Bateman-Feshbach-Tikochinsky oscillators [5] in the context of M-L functions and kernels. In

2016 some work was conducted on applying such M-L function-based derivatives to finding the solutions for electrical RLC (resistor, inductor, capacitor) circuits, a type of circuit that happens to demonstrate fractality. Therefore its becomes an interesting example of the real-world demonstration for applications of these functions and derivatives [6].

The thesis is structured on five chapters. In the second section some basic tools regarding the Mittag-Leffler function, Riemann-Liouville and Caputo Fractional derivative and Atangana-Baleanu fractional derivative are given. Section 3 is devoted to the discrete Mittag-Leffler function and its properties. Section 4 deals with some properties of the discrete fractional derivatives with Mittag-Leffler kernel. Section 5 presents some illustrative applications. The conclusion part ends this master thesis.

CHAPTER 2
BASIC TOOLS

The Mittag-Leffler (M-L) function has a significant place in solving fractional-order differential equations. It can be applied in diverse fields, both theoretical and practical, including the flow of fluids, electric networks, probability, the theory of statistical distribution [8].

2.1 Mittag-Leffler Function and Its Properties

We denote by E_α (x) the Mittag-Leffler function and by $E_{\alpha,\beta}$ (x) its generalized version.

Definition 2.1 [11]:(Mittag-Leffler Function-one parameter)

$$E_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in C, \quad R(\alpha) > 0. \tag{2.1}$$

Definition 2.2[11]:(Generalized Mittag-Leffler Function-two parameter)

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in C, \quad R(\alpha) > 0, \quad R(\beta) > 0. \tag{2.2}$$

It follows from the definition that if we put α = 1, β = 1 we get $E_{1,1}(z) = e^z$.

For α = 1 and β = 2 or β = 3 , we conculude [11]

$$E_{1,2}(z) = \frac{e^z - 1}{z} \tag{2.3}$$

$$E_{1,3}(z) = \frac{e^z - 1 - z}{z^2}$$

Also we have [11]

$$E_{1,m}(z) = \frac{1}{z^{m-1}} \left\{ e^z - \sum_{k=0}^{m-1} \frac{z^k}{k!} \right\}. \tag{2.4}$$

Second, the hyperbolic sine and cosine are also specific cases of the M-L function,

$$E_{2,1}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(\alpha k + 1)} = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k)!} = \frac{cosh\ (z)}{z}, \tag{2.5}$$

$$E_{2,2}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(\alpha k+2)} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \frac{sinh\,(z)}{z}, \tag{2.6}$$

and we continue with the hyperbolic function. We look at the order of n [11]:

$$h_r(z,n) = \sum_{k=0}^{\infty} \frac{z^{nk+r-1}}{(nk+r-1)!} = z^{r-1}E_{n,r}(z^n), \tag{2.7}$$

which it is also a M-L function.

The other one is an error function. We take the general form of the M-L function and put a instead of ½ and b instead of 1 [11].

$$E_{\frac{1}{2},1}(z) = E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\left(\frac{k}{2}+1\right)} = e^{z^2} erfc(z). \tag{2.8}$$

The error function is also defined by the following:

$$erfc(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}.$$

Laplace transform (LT) of the Mittag-Leffler (M-L) Function

We will examine the relationship of the LT and the M-L function [10]. First, let us prove the following [11].

$$\int_0^{\infty} e^{-t}\, e^{\pm zt}dt = \frac{1}{1 \pm z} \qquad |z| < 0, \tag{2.9}$$

$$\int_0^{\infty} e^{-t}\, e^{\pm zt}dt = \frac{1}{1-z} = \sum_{z=0}^{\infty} \frac{(\pm z)^k}{k!}\int_0^{\infty} e^{-t}\, t^k dt = \sum_{k=0}^{\infty} (\pm z)^k$$
$$= \frac{1}{1 \pm z}, \tag{2.10}$$

$$\int_0^{\infty} e^{-t}\, t^k e^{h \pm zt} dt = \frac{k!}{(1-z)^{k+1}} \qquad |z| < 1. \tag{2.11}$$

Derivatives of the Mittag-Leffler Function

Using Riemann-Liouville (R-L) fractional-order differentiation ${}_0D_t^{\gamma}$ (γ is an arbitrary real number) of series demonstration, we obtain [11]

$${}_0D_t^{\gamma}\left(t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\gamma t^a)\right) = t^{\alpha k+\beta-1}E_{\alpha,\beta-\gamma}^{(k)}(\lambda t^a). \tag{2.12}$$

Take the particular case of the relationship for k=0, λ=1 and infer that γ is given [11]:

$$\left(\frac{d}{dt}\right)^m \left(t^{\beta-1}E_{m/n,\beta}(t^{m/n})\right) = t^{\beta-1}E_{\frac{m}{n},\beta}\left(t^{\frac{m}{n}}\right) + t^{\beta-1}\sum_{k=1}^{n}\frac{t\,\frac{-mk}{n}}{\Gamma\left(\beta-\frac{mk}{n}\right)}. \quad (2.13)$$

Performing the substitution $t = z^{\frac{n}{m}}$ and m =1 we obtain [11],

$$\frac{1}{n}\frac{d}{dz}\left(z^{(\beta-1)n}E_{1/n,\beta}(z)\right) = z^{\beta-1}E_{\frac{1}{n},\beta}(z) + z^{\beta n-1}\sum_{k=1}^{n}\frac{z^{-k}}{\Gamma\left(\beta-\frac{k}{n}\right)}. \quad (2.14)$$

Differential Equations for the Mittag-Leffler Function

Showing the M-L function and its corresponding equations differential equation, we indicate the following [11].

$$y_1(t) = t^{\beta-1}E_{\frac{m}{n},\beta}\left(t^{\frac{m}{n}}\right), \qquad y_2(t) = t^{\beta-1}E_{\frac{m}{n},\beta}(t^m),$$

$$y_3(t) = t^{\frac{(\beta-1)n}{m}}E_{\frac{m}{n},\beta}(t), \qquad y_4(t) = t^{(\beta-1)n}E_{\frac{m}{n},\beta}(t).$$

Then these functions satisfy the following respective differential equations[11]:

$$\frac{d^m y_1(t)}{dt^m} - y_1(t) = t^{\beta-1}\sum_{k=1}^{n}\frac{t^{-\frac{m}{n}k}}{\left(\beta-\frac{m}{n}k\right)}, \quad (2.15)$$

$$\frac{d^m y_g(t)}{dm^m} - y_2(t) = 0, \quad (2.16)$$

$$\left(\frac{m}{n}t^{1-\frac{n}{m}}\frac{d}{dt}\right)^m y_3(t) - y_3(t) = t^{\frac{(\beta-1)n}{m}}\sum_{k=1}^{n}\frac{t^{-k}}{\left(\beta-\frac{m}{n}k\right)}, \quad (2.17)$$

$$\frac{1}{n}\frac{dy_4(t)}{dt}y_4(t) - t^{n-1}y_4(t) = t^{\beta n-1}\sum_{k=1}^{n}\frac{t^{-k}}{\left(\beta-\frac{k}{n}\right)}. \quad (2.18)$$

Summation Formulas

We have [11],

$$\sum_{\nu=0}^{m-1} e^{i2\mu\nu k/m} = \begin{cases} m, & \text{if} \quad k \equiv 0 \quad (mod\ m), \\ 0, & \text{if} \quad k \not\equiv 0 \quad (mod\ m). \end{cases} \quad (2.19)$$

We start with the apparent formula.

$$\sum_{\nu=-m}^{m} e^{\frac{i2\mu\nu k}{2m}+1} = \begin{cases} 2m+1, & \text{if} \quad k \equiv 0 \quad (mod\ 2m+1), \\ 0, & \text{if} \quad k \not\equiv 0 \quad (mod\ 2m+1). \end{cases} \quad (2.20)$$

We then acquire the following [11].

$$E_{\alpha,\beta}(z) = \frac{1}{2m+1}\sum_{\nu=-m}^{m-1} E_{\frac{a}{2m+1},\beta}\left(z^{\frac{1}{2m+1}} e^{\frac{i2\mu\nu}{2m}+1}\right). \tag{2.21}$$

A generalization of the summation formula has thus been obtained.

$$\sum_{\nu=0}^{m-1} E_{\alpha,\beta}\left(\frac{z^{i2\mu\nu}}{m}\right) e^{\frac{i2\mu\nu(m-n)}{m}} = mz^n E_{m,\alpha,\beta+n\alpha}(z^m). \tag{2.22}$$

Apparently, for n=0, the relationship gives the summation formula.

Integration of the Mittag-Leffler Function

We have [11],

$$\int_0^z E_{\alpha,\beta}(\lambda t^\alpha) t^{\beta-1} dt = z^\beta E_{\alpha,\beta+1}(\lambda z^\alpha), \quad (\beta > 0). \tag{2.23}$$

We acquire a more general relationship.

$$\frac{1}{\Gamma(\nu)}\int_0^z (z-t)^{\nu-1} E_{\alpha,\beta}(\lambda t^\alpha) t^{\beta-1} dt = z^{\beta+\gamma-1} E_{2\alpha,\beta+\gamma}(\lambda z^\alpha), \tag{2.24}$$

$$\int_0^z t^{\beta-1} E_{\alpha,\beta}(\alpha\tau^\alpha)(t-\tau)^{\gamma-1} E_{\alpha,\gamma}(-a(t-\tau)^\alpha) dt = t^{\beta+\gamma-1} E_{2\alpha,\beta+\gamma}(a^2 t^{2\alpha}). \tag{2.25}$$

Asymptotic Expansions

Integration of the relationship gives [11],

$$\begin{aligned} E_{\frac{1}{n},\beta}(z) = z^{(1-\beta)n} e^{z^n} \Bigg\{ & z_0^{(1-\beta)n} e^{-z_0^n} E_{\frac{1}{n},\beta}(z_0), \\ & + n\int_{z_0}^z e^{-\tau^n}\left(\sum_{k=1}^{n} \frac{\tau^{-k}}{\left(\beta - \frac{k}{n}\right)} \tau^{3n-1}\right) dt \Bigg\}, \quad (n \le 1), \end{aligned} \tag{2.26}$$

which is valid for arbitrary $z_0 \neq 0$.

Then we put $\beta = 1$, $z_0 = 0$ [11],

$$E_{\frac{1}{n},1}(z) = e^{z^n}\left\{1 + n\int_0^z e^{-\tau^n}\left(\sum_{k=1}^{n-1} \frac{\tau^{k-1}}{\left(\frac{k}{n}\right)}\right)\right\}, \qquad (n \ge 2). \tag{2.27}$$

Taking n=2 as before, we acquire the following formula:

$$E_{\frac{1}{2},1}(z) = e^{z^2}\left\{1 + \frac{2}{\sqrt{\pi}}\int_0^z e^{-\tau^2} dt\tau\right\}. \tag{2.28}$$

from which the following asymptotic formula follows [11]:

$$E_{\frac{1}{2},1} \sim 2e^{z^2}, |\arg(z)| < \frac{\pi}{4}, |z| \to \infty. \tag{2.29}$$

Wright Function

The Wright function is significant for the explanation of particular linear fractional equations. This function is connected to the M-L function in two parameters, $E_{\alpha,\beta}$ (x) [10].

We give the definition of the Wright function [11]:

$$W(z;\alpha,\beta)=\sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(\alpha k+\beta)}. \tag{2.30}$$

We can also represent this function with the integral formula given below:

$$W(z;\alpha,\beta)=\frac{1}{2\pi i}\int_{H_a}\tau^{-\beta}\,e^{\tau+z\tau^{-a}}dt. \tag{2.31}$$

Finally, we can write the LT of the Wright function to also contain the M-L function.

$$\begin{aligned} L\{t;\alpha,\beta);s\} &= L\left\{\sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(\alpha k+\beta)}\right\} \\ &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}\frac{1}{s^{k+1}} \\ &= s^{-1}E_{\alpha,\beta}(s^{-1}). \end{aligned} \tag{2.32}$$

2.2 Basics of Riemann-Liouville and Caputo Fractional Operators

The Riemann-Liouville Fractional Differential Operator

First, it is helpful if the fractional integration operator is defined. Let us assume that a > 0, t > α, α, a, t ϵ R. Then the fractional operator [11]

$$J^a f(t):=\frac{1}{\Gamma(\alpha)}\int_0^t(\,t)(t-\tau)^{\alpha-1}dt. \tag{2.33}$$

is referred to as the R-L fractional integral of order a [12]. Next we show the R-L fractional differential operator.

Supposing that a > 0, t > α, α, a, tϵR [12]:

$$D^{\alpha}f(t):=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\dfrac{d^n}{dt^n}\displaystyle\int_0^t\frac{f(\tau)}{(t-\tau)^{\alpha+1-n}}dt, & n-1<\alpha<n\in N,\\ \dfrac{d^n}{dt^n}f(t), & \alpha=n\in N.\end{cases} \tag{2.34}$$

and this is referred to as the R-L fractional derivative or the R-L fractional differential operator of order a [12].

The Caputo Fractional Differential Operator

Let us assume that a>0, t>α, α, a, tϵR. The fractional operator

$$D_*^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \dfrac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} dt, & n-1<\alpha<n \in N, \\ \dfrac{d^n}{dt^n} f(t) & \alpha = n \in N. \end{cases} \tag{2.35}$$

is the Caputo fractional derivative or Caputo fractional differential operator of order a [13].

LT of the Basic Fractional Operator

Suppose that p > 0 and additionally that F(s) is the LT of f(t); then the following statements hold.

The LT of the R-L fractional differential operator of order a is given as shown here,

$$L\{D^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} \left[D^k I^{n-\alpha} f(t)\right]_{t=0}, \quad n-1<\alpha<n. \tag{2.36}$$

Let the LT of the Caputo fractional differential operator of order a be given by [10]

$$L\{D_*^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \quad , \quad n-1<\alpha<n \in N. \tag{2.37}$$

This can also be acquired in the form below, as well [10].

$$L\{D_*^{\alpha} f(t)\} = \frac{s^n F(s) - s^{n-1} f(0) - s^{n-1} f'(0) - \cdots - f^{(n-1)}(0)}{s^{n-\alpha}}. \tag{2.38}$$

Next, LT of the two-parameter function of M-L type can be given as follows [14].

$$L\left\{t^{\alpha m+\beta-1} E_{\alpha,\beta}(m)(\pm\lambda t^{\alpha})\right\} = \frac{m!\, s^{\alpha-\beta}}{(s^{\alpha}\lambda)^{m+1}}, \quad Re(s) > |\lambda|^{\frac{1}{\alpha}}. \tag{2.39}$$

2.3 Fractional Atangana-Baleanu Derivative

Definition 2.3[1]: If we take $f \in H^1(a,b), b>a, \alpha \in [0,1]$, we have the definition of the new fractional derivative as follows:

$$ {}_{b}^{ABC}D_t^{\alpha}\left(f(t)\right) = \frac{B(\alpha)}{1-\alpha} \int_b^t f'(x) E_{\alpha}\left[-\alpha \frac{(t-x)^{\alpha}}{1-\alpha}\right] dx. \tag{2.40}$$

Here, B(α) has properties similar to those in the Caputo and Fabrizio case. The above definition is important in discussing practical real-world problems. It will also be of great help when we use LT to solve some physical problems under the initial conditions. However, if alpha is 0, we do not regain the original function except when the function vanishes at the origin. The following definition is proposed with the goal of avoiding this situation.

Definition 2.4[1]: Letting $\mathrm{f} \in \mathrm{H}^1(\mathrm{a},\mathrm{b}), \mathrm{b} > a, \alpha \in [0,1]$, the new fractional derivative can be defined as follows:

$$ {}^{ABR}_{b}D_t^{\alpha}\left(f(t)\right) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_b^t f(x)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right]dx. \tag{2.41}$$

Equations (2.40) and (2.41) have a kernel that is nonlocal. Additionally, in equation (2.40), we obtain zero if the function is a fixed function.

Properties

We begin here with the relationship between both derivatives and the LT. It can be concluded by a straightforward calculation that [1]

$$\mathcal{L}\left\{{}^{ABR}_{0}D_t^{\alpha}\left(f(t)\right)\right\}(p) = \frac{B(\alpha)}{1-\alpha}\frac{p^\alpha \mathcal{L}\{f(t)\}(p)}{p^\alpha + \frac{\alpha}{1-\alpha}} \tag{2.42}$$

and

$$\mathcal{L}\left\{{}^{ABC}_{0}D_t^{\alpha}\left(f(t)\right)\right\}(p) = \frac{B(\alpha)}{1-\alpha}\frac{p^\alpha \mathcal{L}\{f(t)\}(p) - p^{\alpha-1}f(0)}{p^\alpha + \frac{\alpha}{1-\alpha}}, \tag{2.43}$$

respectively.

Proceeding from this, the following result is then reached.

Theorem 2.1[1]: Letting $f \in H^1(a,b), b > a, \alpha \in [0,1]$, the following relation is obtained.

$$ {}^{ABC}_{0}D_t^{\alpha}\left(f(t)\right) = {}^{ABR}_{0}D_t^{\alpha}\left(f(t)\right) + H(t). \tag{2.44}$$

Proof [1]: If we take definition (2.44) and apply the LT on both sides, the result below can be easily obtained:

$$\mathcal{L}\left\{{}^{ABC}_{0}D_t^{\alpha}\left(f(t)\right)\right\}(p) = \frac{B(\alpha)}{1-\alpha}\frac{p^\alpha \mathcal{L}\{f(t)\}(p)}{p^\alpha + \frac{\alpha}{1-\alpha}} - \frac{p^{\alpha-1}f(0)}{p^\alpha + \frac{\alpha}{1-\alpha}}\frac{B(\alpha)}{1-\alpha}. \tag{2.45}$$

Following equation (2.42), we have:

$$\mathcal{L}\left\{{}^{ABC}_{\ 0}D_t^{\alpha}\left(f(t)\right)\right\}(p) = \mathcal{L}\left\{{}^{ABR}_{\ 0}D_t^{\alpha}\left(f(t)\right)\right\}(p) - \frac{p^{\alpha-1}f(0)}{p^{\alpha}+\frac{\alpha}{1-\alpha}}\frac{B(\alpha)}{1-\alpha}. \tag{2.46}$$

If we apply the inverse Laplace on both sides of equation (2.46), the result will be as follows:

$$ {}^{ABC}_{\ 0}D_t^{\alpha}\left(f(t)\right) = {}^{ABR}_{\ 0}D_t^{\alpha}\left(f(t)\right) - c\, f(0)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}t^{\alpha}\right). \tag{2.47}$$

The proof is thus completed.

Theorem 2.2[1]: Taking a continuous function on the closed interval [a,b] gives the following inequality on [a,b]

$$\left\|{}^{ABR}_{\ 0}D_t^{\alpha}\left(f(t)\right)\right\| < \frac{B(\alpha)}{1-\alpha}K, \|h(t)\| = max_{a\le t\le b}|h(t)|. \tag{2.48}$$

Proof [1]:

$$\begin{aligned}\left\|{}^{ABR}_{\ 0}D_t^{\alpha}\left(f(t)\right)\right\| = &\left\|\frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t f(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right]dx\right\| \\ &< \frac{B(\alpha)}{1-\alpha}\left\|\frac{d}{dt}\int_0^t f\,(x)dx\right\| = \frac{B(\alpha)}{1-\alpha}\|f(x)\|.\end{aligned} \tag{2.49}$$

If we take K to be $\|f(x)\|$, we complete the proof.

Definition 2.5[4]: We can define the fractional integral associated with the derivative with the nonlocal kernel is as follows:

$$ {}^{AB}_{\ a}I_t^{\alpha}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f\,(y)(t-y)^{\alpha-1}dy. \tag{2.50}$$

CHAPTER 3

DISCRETE MITTAG-LEFFLER FUNCTION AND ITS PROPERTIES

Definition 3.1[4]: (∇ discrete M-L) For $\mu \in \mathbb{R}, |\mu| < 1$, and $\alpha, \beta, z \in \mathbb{C}$ with $Re(\alpha) > 0$, the nabla discrete M-L function can be given as:

$$E_{\overline{\alpha,\beta}}(\mu, z) = \sum_{t=0}^{\infty} \mu^t \frac{z^{\overline{t\alpha+\beta-1}}}{\Gamma(\alpha t+\beta)}. \tag{3.1}$$

For the case of $\beta = 1$, we have the following:

$$E_{\bar{\alpha}}(\mu, z) \triangleq E_{\overline{\alpha,1}}(\mu, z) = \sum_{t=0}^{\infty} \mu^t \frac{z^{\overline{t\alpha}}}{\Gamma(\alpha t+1)} \quad , \quad |\mu| < 1. \tag{3.2}$$

The generalized M-L function with three parameters was previously introduced as follows[4]:

$$E_{\alpha,\beta}^{\rho}(z) = \sum_{t=0}^{\infty} (\rho)_t \frac{z^t}{t!\,\Gamma(\alpha t+\beta)}. \tag{3.3}$$

Here $(\rho)_t = \rho(\rho+1)\cdots(\rho+t-1)$. Notice that $(1)_t = t!$ and so $E^1 N\alpha,\beta(z) = E_{\alpha,\beta},^1(z)$.

Moving on to the discrete process, it is necessary to define the version of the M-L function of three parameters as follows:

$$E_{\alpha,\beta}^{\rho}(\mu, z) = \sum_{t=0}^{\infty} \mu^t (\rho)_t \frac{z^{\alpha t+\beta-1}}{t!\,\Gamma(\alpha t+\beta)}. \tag{3.4}$$

Definition 3.2[4]: The (∇) discrete general M-L function having the three parameters α, β, and ρ is defined as shown here:

$$E_{\overline{\alpha,\beta}}^{\rho}(\mu, z) = \sum_{t=0}^{\infty} \mu^t (\rho)_t \frac{z^{\overline{t\alpha+\beta-1}}}{t!\,\Gamma(\alpha t+\beta)}. \tag{3.5}$$

Here, it can be noted that $E_{\overline{\alpha,\beta}}^{1}(\mu, z) = E_{\overline{\alpha,\beta}}(\mu, z)$.

Proposition 3.1[4]: (Summation and difference for the discrete M-L functions)

$$\nabla_t E_{\overline{\alpha}}(\mu, z) = E_{\overline{\alpha,\alpha}}(\mu, z),$$
$$\nabla_t E_{\overline{1,\beta}}(\mu, z) = E_{\overline{1,\beta+1}}(\mu, z),$$
$$\nabla_t E_{\overline{\alpha,\beta}}{}^{\gamma}(\mu, z) = E_{\overline{\alpha,\beta-1}}{}^{\gamma}(\mu, z), \tag{3.6}$$
$$\sum_{t=a+1}^{z} E_{\overline{\alpha,\beta}}(\mu, t-a) = E_{\overline{\alpha,\beta+1}}(\mu, z-a).$$

Definition 3.3[4]: Suppose that function f is defined on $\mathbb{N}_0$. Thus, for $0 < \alpha \leq 1$, we have its α-order Caputo fractional derivative as follows:

$$\begin{aligned} {}^{C}\nabla_0^{\alpha} f(k) &= \nabla_0^{-(1-\alpha)} \nabla f(k) \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{k} (k-\rho(s))^{\overline{-\alpha}} \nabla f(s), \end{aligned} \tag{3.7}$$

where $\rho(s) = s - 1$ and $\nabla_0^{-\alpha} f(k) = \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{k} (k-\rho(s))^{\overline{\alpha-1}} f(s)$ is seen to be the ∇ left fractional sum with the order of α.

If we take f as being defined on $\mathbb{N}$, it will follow that ${}^{C}\nabla_0^{\alpha} f(k)$ is defined on

$$\mathbb{N}_1 = \{1,2,3, \ldots\}.$$

For the Caputo fractional difference possessing the order of $m - 1 < \alpha \leq m$ beginning from $a(\alpha) = a + m - 1$, please see [46].

Example 3.1[4]: Take $0 < \alpha \leq 1, a \in \mathbb{R}$, as well as the ∇ left Caputo nonhomogeneous fractional difference equation as follows:

$${}^{C}\nabla_0^{\alpha} y(t) = \lambda y(t) + f(t), \qquad y(0) = a_0, \qquad t \in \mathbb{N}_0. \tag{3.8}$$

Now it is possible for one to write the solution for (3.8) in the following way:

$$y(t) = a_0 E_{\overline{\alpha}}(\lambda, t) + \sum_{s=1}^{t} E_{\overline{\alpha,\alpha}}\left(\lambda, t-\rho(s)\right) f(s). \tag{3.9}$$

Remark 3.1[4]: The solution of (3.12) with $\alpha = 1$ and $a_0 = 1$ is

$$y(t) = \sum_{k=0}^{\infty} \lambda^k \frac{t^{\overline{k}}}{k!} + \sum_{s=1}^{t} \sum_{k=0}^{\infty} \lambda^k \frac{(t-\rho(s))^{\overline{k}}}{k!} f(s). \tag{3.10}$$

The function $\hat{e}_{\lambda}(t,0) = (1-\lambda)^{-t}$, a ∇ discrete .exponential function, represents the first part of the solution above, with $|\lambda| < 1$. Please see [49] for more information.

Next we will review some facts about the discrete LT of the M-L type and convolution type functions (for detailed information, refer to [48]).

Definition 3.4[4]: Take $s \in \mathbb{R}, 0 < \alpha < 1$, and $f, g: \mathbb{N}_a \to \mathbb{R}$. Then the ∇ discrete convolution of f with g can be given according to the following:

$$(f * g)(k) = \sum_{s=a+1}^{k} g\left(k - \rho(s)\right) f(s). \tag{3.11}$$

Here, $\rho(s) = s - 1$ is understood to be a backward jumping operator employed in the case of ∇-analysis for the time scale of $\mathbb{Z}$. Such a backward jumping operator is required for showing the following discrete convolution theorem. It is also needed in order to acquire the dual relations existing between the left and right fractional sums and the differences by way of the Q-operator.

Proposition 3.2[4]: Let a function f be defined on $\mathbb{N}_a$. Then the nabla discrere Laplace transform has the form $\mathcal{N}f(z) = \sum_{t=1}^{\infty}(1-z)^{t-1}f(t)$, $\mathcal{N}_a f(z) = \sum_{t=a1}^{\infty}(1-z)^{t-1}f(t)$. In the case of any $\alpha \in \mathbb{R}\backslash\{\ldots,-2,-1,0\}, m \in \mathbb{R}$, and f, g defined on $\mathbb{N}_a$, the following can be written:

$$\left(\mathcal{N}_a(f * g)\right)(k) = (\mathcal{N}_a f)(k)(\mathcal{N} g)(k). \tag{3.12}$$

Proof [4]:

$$\begin{aligned}
(\mathcal{N}_a(f * g))(k) &= \sum_{t=a+1}^{\infty} (1-k)^{t-1} \sum_{m=a+1}^{t} f(m) g(t-\rho(m)) \\
&= \sum_{m=a+1}^{\infty} \sum_{t=m}^{\infty} (1-k)^{t-1} \sum_{m=a+1}^{t} f(m) g(t-\rho(m)) \\
&= \sum_{m=a+1}^{\infty} \sum_{r=1}^{\infty} (1-k)^{r-1}(1-k)^{m-1} f(m) g(r) \\
&= (\mathcal{N}_a f)(k)(\mathcal{N} g)(k).
\end{aligned} \tag{3.13}$$

where the alteration of variable $r = t - \rho(m)$ has been applied.

For more information about the situation in which $a = 0$ and $g(t) = t^{\overline{\alpha}}$, please consult [48].

Lemma 3.1[4]: For and $\alpha \in \mathbb{R}\{\dots,-2,-1,0\}$,

i. $\mathcal{N}\left(t^{\overline{\alpha-1}}\right)(k) = \frac{\Gamma(\alpha)}{k^{\alpha}}, |1-k| < 1,$ (3.14)

ii. $\mathcal{N}\left(t^{\overline{\alpha-1}} b^{-t}\right)(k) = \frac{b^{\alpha-1}\Gamma(\alpha)}{(k+b-1)^{\alpha}}, |1-k| < b.$ (3.15)

Lemma 3.2[4]: If we assume that f is a function that is defined on $\mathbb{N}$, we have:

$$\Big(\mathcal{N}\nabla\big(f(t)\big)\Big)(k) = k(\mathcal{N}f)(k) - f(0). \tag{3.16}$$

Lemma 3.2 can be generalized as follows.

Lemma 3.3[4]: Assume that the function f is defined on $\mathcal{N}_a$. For this case, the following result holds:

$$(\mathcal{N}_a\nabla(f(t)))(k) = k(\mathcal{N}_a f)(k) - (1-k)^a f(a). \tag{3.17}$$

To express this in a more general fashion,

$$(\mathcal{N}_{a(\alpha)}\nabla^n f)(k) = k^n(\mathcal{N}_{a(\alpha)}f)(k) - (1-k)^{a(\alpha)}\sum_{i=0}^{n-1} k^{n-a-i}\nabla_i f(a+1). \tag{3.18}$$

Lemma 3.4[4]: For any positive real number ν,

$$(\mathcal{N}_{a-1}\nabla_{a-1}^{-\nu} f)(s) = s^{-\nu}(\mathcal{N}_{a-1}f)(s). \tag{3.19}$$

A different proof can be provided for the lemma below, avoiding convolutions as can be seen in the work presented in [48].

Lemma 3.5[4]: If f is defined on $\mathbb{N}_0$ and $0 < \alpha \leq 1$, we have the following:

$$(\mathcal{N}\,{}^C\nabla_0^{\alpha} f)(k) = k^{\alpha}(\mathcal{N}f)(k) - k^{\alpha-1}f(0). \tag{3.20}$$

Proof: We get the following thanks to the definition and Lemma 3.2 that were established in [50]:

$$({}^C\nabla_0^{\alpha} f)(t) = \left(\nabla_0^{-(1-\alpha)}\nabla f\right)(t) = \nabla\left(\nabla_0^{-(1-\alpha)} f\right)(t) - \frac{t^{\overline{\bar{\alpha}}}}{\Gamma(1-\alpha)} f(0). \tag{3.21}$$

By employing the ∇ discrete LT and also using Lemma 3.1 and Lemma 3.2, the following is obtained:

$$(\mathcal{N}\,{}^C\nabla_0^{\alpha} f)(k) = k\left(\mathcal{N}\nabla_0^{-(1-\alpha)} f\right)(k) - \left(\nabla_0^{-(1-\alpha)} f\right)(0) - \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha)k^{1-\alpha}} f(0). \tag{3.22}$$

The result then follows by Lemma 3.4 with $a = 1$ and $(\nabla_0^{-(1-\alpha)})f(0) = 0$.

Remark 3.2[4]: It is possible to generalize Lemma 3.5 as follows. We get the following for f when it is defined on Na and $0 < \alpha \leq 1$:

$$(\mathcal{N}_a \quad {}^C\nabla_a^\alpha f)(k) = k^\alpha(\mathcal{N}_a f(k)) - (1-k)^a k^{\alpha-1} f(a). \tag{3.23}$$

Remark 3.1 allows this to be proved.

Lemma 3.6[4]: Supposing that $0 < \alpha \leq 1$ and letting f be defined on $\mathbb{N}_0$, we have:

i. $\big(\mathcal{N} E_{\overline{\alpha}}(\mu, t)\big)(k) = \frac{k^{\alpha-1}}{k^\alpha - \mu}$, (3.24)

ii. $\Big(\mathcal{N} E_{\overline{\alpha,\alpha}}(\mu, t)\Big)(k) = \frac{1}{k^\alpha-\mu}$. (3.25)

Proof [4]: The proof of (ii) is repeated because of an error in the calculation in Lemma 3.4 (ii) in [40]. (ii) Here it can be easily seen that $\nabla E_{\overline{\alpha}}(\mu, t) = \mu E_{\overline{\alpha,\alpha}}(\mu, t)$. In fact,

$$\nabla E_{\overline{\alpha}}(\mu, k) = \sum_{z=0}^{\infty} \mu^z \frac{z a k^{\overline{z\alpha-1}}}{\Gamma(\alpha z+1)}. \tag{3.26}$$

Division by balls of the gamma function results in zero, which subsequently results in

$$\nabla E_{\overline{\alpha}}(\mu, t) = \sum_{k=1}^{\infty} \mu^z \frac{k^{\overline{z\alpha-1}}}{\Gamma(\alpha z)} = \mu \sum_{k=0}^{\infty} \mu^z \frac{k^{\overline{z\alpha+a-1}}}{\Gamma(\alpha k+\alpha)} = \mu E_{\overline{\alpha,\alpha}}(\mu, t). \tag{3.27}$$

Using $\mathcal{N}$ along with (i) and Lemma 3.2, it can be concluded that

$$\begin{aligned}(\mathcal{N} E_{\overline{\alpha,\alpha}}(\mu, t)(k) &= \mu^{-1}[k\mathcal{N} E_{\overline{\alpha}}(\mu, t))(k) - E_{\overline{\alpha}}(\mu, 0)] \\ &= \mu^{-1}\left[\frac{k^\alpha}{k^\alpha-\mu} - 1\right] = \frac{1}{k^\alpha-\mu}.\end{aligned} \tag{3.28}$$

CHAPTER 4
PROPERTIES OF THE CONTINUOUS AND DISCRETE FRACTIONAL DERIVATIVES WITH MITTAG-LEFFLER KERNEL

4.1 Right Fractional Derivatives and the Formula for Integration by Parts

Defining f on the interval of [c,d], the Q-operator's action can then be written in the form of $(Qf)(t) = f(c + d - t)$ [2].

Definition 4.1[2]: Assuming that f ∈ H^1(c,d), c < d , α ∈ [0,1], the new (Left Caputo) fractional derivative in the way defined in the work of Atangana and Baleanu has the following definition:

$$({}^{ABC}_{\ \ c}D^{\alpha}f)(\mathrm{t}) = \frac{B(\alpha)}{1-\alpha}\int_c^t f'(x)E_\alpha[-\alpha\frac{(t-x)^\alpha}{1-\alpha}]dx. \tag{4.1}$$

In the left R-L sense, it is defined by:

$$({}^{ABR}_{\ \ c}D^{\alpha}f)(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_c^t f(x)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right]dx. \tag{4.2}$$

The associated fractional integral is given by:

$$({}^{AB}_{\ c}I^{\alpha}f)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\left({}_{a}I^{\alpha}f\right)(\mathrm{t}). \tag{4.3}$$

The new right R-L fractional derivative being proposed here can be indicated by ${}^{ABR}D_d^c$ and its corresponding integral by ${}^{AB}I_d^c$. Classical fractional calculus shows us that $({}_{c}I^{\alpha}Qf)(t) = Q(I_d^c f)(t)$ and $\left({}_{c}D^{\alpha}Qf\right)(t) = Q(D_d^c f)(t)$. This relation should be satisfied for the new left and right fractional derivatives as well as for the integrals [2].

$$\begin{aligned}({}_{a}{}^{ABR}D^{\alpha}Qf)(t) &= \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(c+d-x)E_\alpha[-\alpha\frac{(t-x)^\alpha}{1-\alpha}]dx \\ &= \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_{c+d-t}^{d} f(u)E_\alpha\left[-\alpha\frac{(u-(c+d-t)^\alpha}{1-\alpha}\right]dx.\end{aligned} \tag{4.4}$$

Here the alteration of the variable $u = c + d - x$ has been applied. Relation (4.4) leads to a definition for these new right fractional derivatives as follows.

Definition 4.2[2]: The new right fractional derivative having a M-L kernel of the order of $\alpha \in [0,1]$ can be defined in the manner given below:

$$\left({}^{ABR}D_b^{\alpha} f\right)(t) = -\frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_t^b f(x)\, E_{\alpha}\left[-\alpha\frac{(x-t)^{\alpha}}{1-\alpha}\right] dx. \tag{4.5}$$

Furthermore, we have [2]:

$$\begin{aligned}({}^{AB}_{\ a}I^{\alpha} Qf)(t) &= \frac{1-\alpha}{B(\alpha)} f(a+b-t) + \frac{\alpha}{B(\alpha)}({}_aI^{\alpha} Qf)(t)\\ &= \frac{1-\alpha}{B(\alpha)} f(a+b-t) + \frac{\alpha}{B(\alpha)} Q(\mathrm{I}_b^{\alpha} f)(t)\\ &= Q\left[\frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)}(\mathrm{I}_b^{\alpha} f)(t)\right].\end{aligned} \tag{4.6}$$

The equation $({}^{AB}D_b^{\alpha} f)(t) = u(t)$ can also be solved. Indeed,

$$\left({}^{AB}D_b^{\alpha} f\right)(t) = \left({}^{AB}D_b^{\alpha} QQf\right)(t) = (Q\,{}^{AB}_{\ a}D^{\alpha} Qf)(t) = u(t)$$

or (4.7)

$$({}^{AB}_{\ a}D^{\alpha} Qf)(t) = Qu(t),$$

and hence

$$Qf(t) = \frac{1-\alpha}{B(\alpha)} Qu(t) + \frac{\alpha}{B(\alpha)}\, {}_aI^{\alpha} Qu(t) = \frac{1-\alpha}{B(\alpha)} Qu(t) + \frac{\alpha}{B(\alpha)} QI_b^{\alpha} u(t). \tag{4.8}$$

If we apply Q to both of the sides as given above, the following result is obtained:

$$f(t) = \frac{1-\alpha}{B(\alpha)} u(t) + \frac{\alpha}{B(\alpha)} I_b^{\alpha} u(t). \tag{4.9}$$

Relations (4.6) and (4.9) yield a definition of the new right fractional integral as follows.

Theorem 4.1[2]: The functions $({}^{ABR}_{\ \ a}\mathrm{D}^{\alpha}\, \mathrm{f})(\mathrm{t})$ and $\left({}^{ABR}\mathrm{D}_b^{\alpha}\mathrm{f}\right)(\mathrm{t})$ are found to satisfy the following respective equations:

$$({}^{AB}_{\ a}I^{\alpha} g)(t) = f(t)\ , \qquad \left({}^{AB}I_b^{\alpha} g\right)(t) = f(t). \tag{4.10}$$

Proof [2]: Here only the left case is demonstrated. It is possible to demonstrate the right case using the Q-operator. According to the definition, it can be said that the first equation has equivalency with the following:

$$\frac{1-\alpha}{B(\alpha)} g(t) + \frac{\alpha}{B(\alpha)}\left({}_aI^{\alpha} g\right)(t) = f(t). \tag{4.11}$$

Applying the LT makes it possible to state that

$$\frac{1-\alpha}{B(\alpha)}G(s)+\frac{\alpha}{B(\alpha)}s^{-\alpha}G(s)=F(s), \tag{4.12}$$

from which it follows that

$$G(s)=\frac{B(\alpha)}{1-\alpha}\frac{F(s)s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}. \tag{4.13}$$

In the end, the Laplace inverse leads to $g(t)=({}^{ABR}_{\ a}D^{\alpha}f)(t)$.

Definition 4.3[2]: The definition of the new (right) Caputo fractional derivative with the order of $0<\alpha<1$ can be given as

$$\left({}^{ABC}D_b^{\alpha}f\right)(t)=-\frac{B(\alpha)}{1-\alpha}\int_t^b f'(x)E_{\alpha}\left[-\alpha\frac{(x-t)^{\alpha}}{1-\alpha}\right]dx. \tag{4.14}$$

Next, the right version of

$({}^{ABC}_{\ 0}D^{\alpha}f)(t)=({}^{ABR}_{\ 0}D^{\alpha}f)(t)-\frac{B(\alpha)}{1-\alpha}f(0)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t)^{\alpha}\right)$ can be given by employing the Q-operator that was discussed above.

Proposition 4.1[2]: The following identity is used in order to relate the new right R-L fractional derivative along with the new right Caputo fractional derivative:

$$\left({}^{ABC}D_b^{\alpha}f\right)(t)=\left({}^{ABR}D_b^{\alpha}f\right)(t)-\frac{B(\alpha)}{1-\alpha}f(b)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(b-t)^{\alpha}\right). \tag{4.15}$$

Proof [2]: Applying the Q-operator to identity $({}^{ABC}_{\ 0}D^{\alpha}f)(t)=({}^{ABR}_{\ 0}D^{\alpha}f)(t)-\frac{B(\alpha)}{1-\alpha}f(0)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t)^{\alpha}\right)$ and employing both $Q({}^{ABR}_{\ 0}D^{\alpha}f)(t)=\left({}^{ABR}D_b^{\alpha}Qf\right)(t)$ and $Q({}^{ABC}_{\ 0}D^{\alpha}f)(t)=\left({}^{ABC}D_b^{\alpha}Qf\right)(t)$ allows one to obtain that [2]

$$\left({}^{ABC}D_b^{\alpha}Qf\right)(t)=\left({}^{ABR}D_b^{\alpha}Qf\right)(t)-\frac{B(\alpha)}{1-\alpha}f(0)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(b-t)^{\alpha}\right). \tag{4.16}$$

Replacing $f(t)$ with $(Qf)(t)=f(b-t)$ makes it possible for the claim to be concluded.

Proposition 4.2[2]: (Integration by parts for the Caputo fractional derivative

$$\int_0^b({}^{ABC}_{\ a}D^{\alpha}f)(t)g(t)=\int_0^b f(t)\left({}^{ABR}D_b^{\alpha}g\right)(t)+\frac{B(\alpha)}{1-\alpha}f(t)E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},b-}g)(t)|_0^b, \tag{4.17}$$

$$\int_0^b({}^{ABC}D_b^{\alpha}f)(t)g(t)=\int_0^b f(t)\,({}^{ABR}_{\ 0}D^{\alpha}g)(t)+\frac{B(\alpha)}{1-\alpha}f(t)E^1_{\alpha,1,\frac{-\alpha}{1-\alpha},0^+}g)(t)|_0^b. \tag{4.18}$$

Proof [2]: $({}^{ABC}_{\ 0}D^{\alpha} f)(t) = ({}^{ABR}_{\ 0}D^{\alpha} f)(t) - \frac{B(\alpha)}{1-\alpha} f(0) E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t)^{\alpha}\right)$ and (4.16) allow for the first part to be proved, while proving the second part is further made possible.

Definition 4.4[3]: When $m \in (0,1)$ and f are defined on $\mathbb{N}_m$ or ${}_n\mathbb{N}$ in the right case, the following definitions can be obtained:

The new left (∇) Caputo fractional difference:

$$\begin{aligned}({}^{CFC}_{m}\nabla^{\alpha} f)(z) &= \frac{B(\alpha)}{1-\alpha} \sum_{k=m+1}^{z} (\nabla_k f)(k)(1-\alpha)^{t-\rho(k)} \\ &= B(\alpha) \sum_{k=m+1}^{z} (\nabla_k f)(k)(1-\alpha)^{z-k}.\end{aligned} \tag{4.19}$$

The new right (∇) Caputo fractional difference:

$$\begin{aligned}({}^{CFC}\nabla_n^{m} f)(z) &= \frac{B(\alpha)}{1-\alpha} \sum_{k=z}^{n-1} (-\nabla_k f)(k)(1-\alpha)^{k-\rho(z)} \\ &= B(\alpha) \sum_{k=z}^{n-1} (-\nabla_k f)(k)(1-\alpha)^{k-z}.\end{aligned} \tag{4.20}$$

The new left (∇) Riemann fractional difference:

$$\begin{aligned}({}^{CFR}_{m}\nabla^{\alpha} f)(z) &= \frac{B(\alpha)}{1-\alpha} \nabla_z \sum_{s=m+1}^{z} f(k)(1-\alpha)^{z-\rho(k)} \\ &= B(\alpha) \nabla_z \sum_{s=m+1}^{z} f(k)(1-\alpha)^{z-k}.\end{aligned} \tag{4.21}$$

The new right (∇) Riemann fractional difference:

$$\begin{aligned}({}^{CFR}\nabla_n^{m} f)(z) &= \frac{B(\alpha)}{1-\alpha} (-\Delta_z) \sum_{k=z}^{n-1} f(k)(1-\alpha)^{k-\rho(k)} \\ &= \mathrm{B}(\alpha)(-\Delta_{\mathrm{z}}) \sum_{k=z}^{n-1} f(\mathrm{k})(1-\alpha)^{\mathrm{k-z}}.\end{aligned} \tag{4.22}$$

Here, $B(\alpha)$ is a normalizing positive constant that is dependent on α and satisfies $B(0) = B(1) = 1$.

Remark 4.1[3]: In the limiting cases of $\alpha \to 0$ and $\alpha \to 1$, the following remarks can be made:

$$({}^{CFC}_{a}\nabla^{\alpha} f) \to f(k) - f(a) \quad as \quad \alpha \to 0,$$

and

$$({}_{a}^{CFC}\nabla^{\alpha}f)(k) \to \nabla f(k) \quad as\ \alpha \to 1,$$

$$({}^{CFC}\nabla_{b}^{\alpha}f)(k) \to f(k) - f(b)\ as \quad \alpha \to 0,$$

and

$$({}^{CFC}\nabla_{b}^{\alpha}f)(k) \to -\Delta f(k) \quad as \quad \alpha \to 1, \tag{4.23}$$

$$({}_{a}^{CFC}\nabla^{\alpha}f)(k) \to f(k) \quad as \quad \alpha \to 0,$$

and

$$({}_{a}^{CFR}\nabla^{\alpha}f)(k) \to \nabla f(k) \quad as \quad \alpha \to 1,$$

$$({}^{CFR}\nabla_{b}^{\alpha}f)(k) \to f(k) \quad as \quad \alpha \to 0,$$

and

$$({}^{CFR}\nabla_{b}^{\alpha}f)(k) \to -\Delta f(k) \quad as \quad \alpha \to 1.$$

Remark 4.2[3]: (the discrete Q-operator's action) The Q-operator presents regular action between the left and right new fractional differences, which can be shown as follows:

$$\begin{aligned} (Q{}_{a}^{CFR}\nabla^{\alpha}f)(k) &= ({}^{CFR}\nabla_{b}^{\alpha}Qf)(k), \\ (Q{}_{a}^{CFC}\nabla^{\alpha}f)(k) &= ({}^{CFC}\nabla_{b}^{\alpha}Qf)(k), \end{aligned} \tag{4.24}$$

where $(Qf)(k) = f(a+b-k)$.

Definition 4.5[3]: For $0 < \alpha < 1$ and $u: \mathbb{N}_a \to \mathbb{R}$, $a < b$, $a \equiv b$ (mod 1), it was found that the corresponding left fractional sum can be defined as:

$$({}_{a}^{CF}\nabla^{-\alpha}u)(k) = \frac{1-\alpha}{B(\alpha)}u(k) + \frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{k} u\,(s)ds, \tag{4.25}$$

the right fractional sum can be defined by

$$({}^{CF}\nabla_{b}^{-\alpha}u)(k) = \frac{1-\alpha}{B(\alpha)}u(k) + \frac{\alpha}{B(\alpha)}\sum_{s=k}^{b-1} u\,(s)ds. \tag{4.26}$$

In [29], it was shown that, $({}_{a}^{CF}\nabla^{-\alpha} \quad {}_{a}^{CF}\nabla^{\alpha})(k) = f(k)$ and $(\quad {}^{CF}\nabla_{b}^{-\alpha} \quad {}^{CF}\nabla_{b}^{\alpha}f)(k) = f(k)$ It was furthermore shown that $({}_{a}^{CF}\nabla^{\alpha} \quad {}_{a}^{CF}\nabla^{-\alpha}f)(k) = f(k)$ and $({}^{CF}\nabla_{b}^{\alpha} \quad {}^{CF}\nabla_{b}^{-\alpha}f)(k) = f(k)$.

Proposition 4.3[3]: (relation of Riemann- and Caputo-type fractional differences with exponential kernels)

$$({}_{a}^{CFC}\nabla^{\alpha} f)(k) = (({}_{a}^{CFR}\nabla^{\alpha} f)(k) - \frac{B(\alpha)}{1-\alpha} f(a)(1-\alpha)^{k-a},$$
$$({}^{CFC}\nabla_{b}^{\alpha})(k) = ({}^{CFR}\nabla_{b}^{\alpha})(k) - \frac{B(\alpha)}{1-\alpha} f(b)(1-\alpha)^{b-k}. \tag{4.27}$$

For us to be able to continue, the lemma below is required.

Lemma 4.1[3]: When $0 < \alpha < 1$ and for g defined on $\mathbb{N}_a$, the following is found:

i. $({}^{CF}{}_{a}\nabla^{-\alpha}(1-\alpha)^{t})(t) = \frac{(1-\alpha)^{a+1}}{B(\alpha)}$,

ii. $\nabla_s(1-\alpha)^{t-s} = \alpha(1-\alpha)^{t-s}$,

iii. $({}_{a}^{CF}\nabla^{-\alpha}\nabla g)(t) = (\nabla_{a}^{CF}\nabla^{-\alpha} g)(t) - \frac{\alpha}{B(\alpha)} g(a)$, (4.28)

iv. $\nabla(1-\alpha)^{t} = -\alpha(1-\alpha)^{t-1}$,

v. $({}_{a}^{CFR}\nabla^{\alpha})(1-\alpha)^{t}(t) = B(\alpha)(1-\alpha)^{t-1}[1-\alpha(t-a)]$,

vi. $({}_{a}^{CFR}\nabla^{\alpha} 1)(t) = B(\alpha)^{t-a-1}$.

Proof [13]: It is only necessary to provide the proofs (i), (iii), (v), and (vi) as the proofs of the others are straightforward.

Proof of (i) [13]:

$$\begin{aligned}
({}_{a}^{CF}\nabla^{-\alpha})(1-\alpha)^{t}(t) &= \frac{1-\alpha}{B(\alpha)}(1-\alpha)^{t} + \frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}(1-\alpha)^{s} \\
&= \frac{1-\alpha}{B(\alpha)}(1-\alpha)^{t} + \frac{\alpha}{B(\alpha)}(1-\alpha)^{a+1}\frac{1-(1-\alpha)^{t-a}}{1-(1-\alpha)} \\
&= \frac{1}{B(\alpha)}[(1-\alpha)^{t+1} + (1-\alpha)^{a+1} - (1-\alpha)^{t+1}] \\
&= \frac{1-\alpha^{a+1}}{B(\alpha)}.
\end{aligned} \tag{4.29}$$

Proof of (iii) [13]:

$$\begin{aligned}
({}_{a}^{CF}\nabla^{-\alpha}\nabla g)(t) &= \frac{1-\alpha}{B(\alpha)}\nabla g(t) + \frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}\nabla g(s) \\
&= \frac{1-\alpha}{B(\alpha)}\nabla g(t) + \frac{\alpha}{B(\alpha)}[g(t) - g(a)]
\end{aligned} \tag{4.30}$$

$$= \nabla\left[\frac{1-\alpha}{B(\alpha)}g(t) + \frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t} g\,(s)\right] - \frac{\alpha}{B(\alpha)}g(a)$$

$$= ({\nabla}_a^{CF}\nabla^{-\alpha}\nabla g)(t) - \frac{\alpha}{B(\alpha)}g(a).$$

Proof of (v): By (iv) we have [13]

$$\begin{aligned}({}_a^{CFR}\nabla^{\alpha}(1-\alpha)^t)(t) &= B(\alpha)\nabla\sum_{s=a+1}^{t}(\,1-\alpha)^{t-s}(1-\alpha)^s \\ &= B(\alpha)\nabla[(t-\alpha)(1-\alpha)^t] \\ &= B(\alpha)\nabla[(t-\alpha)^{t-1} - \alpha(t-a)(1-\alpha)^{t-1}] \\ &= B(\alpha)(1-\alpha)^{t-1}[1-\alpha(1-a)].\end{aligned} \tag{4.31}$$

Proof of (vi) [13]:

$$\begin{aligned}({}_a^{CFR}\nabla^{\alpha}1)(t) &= B(\alpha)\nabla_t\sum_{s=a+1}^{t}(\,1-\alpha)^{t-s} \\ &= B(\alpha)\left[1+\sum_{s=a+1}^{t}\nabla_t\,(1-\alpha)^{t-s}\right] \\ &= B(\alpha)\left[1-\alpha\sum_{s=a+1}^{t}(\,1-\alpha)^{t-1-s}\right] \\ &= B(\alpha)\left[1-\alpha\sum_{i=0}^{t-a-2}(\,1-\alpha)^{i}\right] \\ &= B(\alpha)\left[1-a\frac{1-(1-\alpha)^{t-a-1}}{1-(1-\alpha)}\right] \\ &= B(\alpha)(1-\alpha)^{t-a-1}.\end{aligned} \tag{4.32}$$

Definition 4.6[3]: If the function $y\colon \mathbb{N}_b \to \mathbb{R}$ satisfies $y(b) \geq 0$, then y is said to be an b-increasing function on $\mathbb{N}_b$ if

$$y(k+1) \geq by(k) \text{ for all k} \in \mathbb{N}_{b.} \tag{4.33}$$

Furthermore, if y is increasing on $\mathbb{N}_b$, then y is said to be an b-increasing function on $\mathbb{N}_b$, and if $b = 1$, we have coincidence between the concepts of increasing and b-increasing.

Definition 4.7[3]: If the function $y\colon \mathbb{N}_b \to \mathbb{R}$ satisfies $y(b) \leq 0$, then y is said to be an b-decreasing function on $\mathbb{N}_b$ if

$$y(k+1) \le by(k) \text{ for all } k \in \mathbb{N}_{b.} \tag{4.34}$$

Furthermore, if y is decreasing on $\mathbb{N}_b$, then y is said to be an α-decreasing function on $\mathbb{N}_b$, and if $b = 1$, we have coincidence between the concepts of decreasing and b-decreasing.

4.2 Monotonicity Results

Theorem 4.2[3]: Assume that $y: \mathbb{N}_{a-1} \to \mathbb{R}$. Further suppose that, for $0 < \alpha \le 1$,

$$({}_{a-1}^{CFR}\nabla^{\alpha} y)(k) \ge 0, \quad k \in \mathbb{N}_{a-1}. \tag{4.35}$$

Then $y(k)$ is α-increasing.

Proof [3]: It is possible to rewrite $({}_{a-1}^{CFR}\nabla^{\alpha} y)(k) = B(\alpha)\nabla S(k)$, where

$$S(k) = \sum_{s=a}^{k} y(s)(1-\alpha)^{k-s}.$$

By supposition, it is possible to write the following:

$$S(k) - S(k-1) = y(k) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{k-1} y(s)(1-\alpha)^{k-s} \ge 0. \tag{4.36}$$

When we substitute $k = a$ into (4.36), it can be seen that $y(a) \ge 0$, and when we substitute $k = a + 1$ into (4.36), we obtain the following:

$$y(a+1) - \frac{\alpha}{1-\alpha} y(a)(1-\alpha) = y(a+1) - \alpha y(a) \ge 0. \tag{4.37}$$

As a result, $y(a+1) \ge ay(a) \ge 0$. We can continue by induction on $k \in \mathbb{N}_a$. It can be assumed that $y(i+1) \ge ay(i) \ge 0$ for all $i < k$. It can also be shown that $y(k+1) \ge ay(k)$. If we replace k with $k+1$ in (4.36), it results in

$$\begin{aligned} y(k+1) \ge \frac{\alpha}{1-\alpha}[&(1-\alpha)^{k+1-a} y(a) + (1-\alpha)^{k-a} y(a+1) + \cdots \\ &+ (1-\alpha) y(k)] \end{aligned} \tag{4.38}$$

or

$$\begin{aligned} y(k+1) \ge [&\alpha(1-\alpha)^{k-a} y(a) + y(a) + (1-\alpha)^{k-a-1} y(a+1) + \cdots \\ &+\alpha y(k)] \ge \alpha y(k). \end{aligned} \tag{4.39}$$

Q.E.D.

The Caputo fractional difference monotonicity result can be stated as follows if Proposition 4.3 and Theorem 4.2 are used.

Theorem 4.3[3]: If the function $y\colon \mathbb{N}_{a-1} \to \mathbb{R}$ satisfies $y(a) \geq 0$ and we assume that, for $0 < \alpha \leq 1$,

$$({}_{a-1}^{CFC}\nabla^{\alpha} y)(t) \geq \frac{-B(\alpha)}{1-\alpha} f(a-1)(1-\alpha)^{t-a+1}, \quad t \in \mathbb{N}_{a-1}, \tag{4.40}$$

then we see that y(t) is α-increasing.

4.3 Discrete Fractional Differences Having Discrete Mittag-Leffler Kernels

Definition 4.8[4]: If we assume that f is defined on Na, $\mathbb{N}_a \cap_c \mathbb{N}, a < c, \alpha \in [0,1]$, then it follows that the ∇ discrete new (left Caputo) fractional difference as given in the course of the work recently presented by Atangana and Baleanu has the following definition:

$$\begin{aligned}({}_{a}^{ABC}\nabla^{\alpha})(k) &= \frac{B(\alpha)}{1-\alpha} \sum_{m=a+1}^{k} \nabla_m f(m) E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha}, k-\rho(m)\right) \\ &= \frac{B(\alpha)}{1-\alpha}\left[\nabla f(k) * E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha}, k\right)\right],\end{aligned} \tag{4.41}$$

while in the left Riemann sense it is defined by

$$\begin{aligned}({}_{a}^{ABR}\nabla^{\alpha})(k) &= \frac{B(\alpha)}{1-\alpha} \sum_{m=a+1}^{k} f\,(m) E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha}, k-\rho(m)\right) \\ &= \frac{B(\alpha)}{1-\alpha}\nabla_k\left[f(k) * E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha}, k\right)\right].\end{aligned} \tag{4.42}$$

It is to be noted that since for $0 < \alpha < \frac{1}{2}$ we have $-1 < \lambda = -\frac{\alpha}{1-\alpha} > 0$, and then $E_{\overline{\alpha}}(\lambda, k)$ is convergent for any $k \in \mathbb{N}$. As an example, $E_{\overline{\alpha}}(\lambda, 1) = (1-\alpha)$ if $0 < \alpha < \frac{1}{2}$. As a result, all of the AB types of fractional differences have convergence under the constraint $0 < \alpha < \frac{1}{2}$. Also note that since $t^{\overline{\alpha}}$ is increasing on $\mathbb{N}_0$, $E_{\overline{\alpha}}(\lambda, k)$ is found to be monotonically decreasing for $0 < \alpha < \frac{1}{2}$, $t > 0$, and $\lambda = \frac{-\alpha}{1-\alpha} > 0$ (for more detailed information about the continuous case $E_{\alpha}(-k^{\alpha})$ please refer to the work in [55]). We can show that $lim_{o-0}\frac{1}{\sigma}E_{\overline{\alpha}}(\frac{-1}{\sigma}, k-\rho(m)) = \delta_m(k) = \begin{cases}1, k=m \\ 0, k\neq m,\end{cases} = \alpha = 1$; this is the delta Dirac function on the time scale of $\mathbb{Z}$. Therefore, like it is possible to illustratively see in [56], it can be shown that, for $\alpha \to 0$, we find $({}_{a}^{ABR}\nabla^{\alpha} f)(k) \to f(k)$, and for $\alpha \to 1$, one finds $({}_{a}^{ABR}\nabla^{\alpha} f)(k) \to \nabla f(k)$. Notice that $E_{\overline{1}}(\frac{-1}{\alpha}, k-\rho(m)) = (1-\alpha)^{k-\rho(m)}, \sigma = \frac{1-\alpha}{\alpha}$, and hence, for example,

$$\lim_{\alpha\to 1}({}_{a}^{ABR}\,\nabla^{\alpha} f)(k) = \lim_{\alpha-1} B\,(\alpha)\nabla_k \textstyle\sum_{m=a+1}^{k} f\,(m)(1-\alpha)^{k-m} = \nabla f(k).$$

This is because the ∇ discrete exponential function possesses the following form: $e_\lambda(k,\rho(m)) = (\frac{1}{1-\lambda})^{k-\rho(m)}$ and $E_1(\lambda, k-\rho(m)) - e_\lambda(k,\rho(m))$.

For derivation of the appropriate fractional difference for the fractional difference that was explained above, the following equation must be considered:

$$({}_{a}^{ABR}\nabla^{\alpha} f)(k) = u(k). \tag{4.43}$$

With the application of $\mathcal{N}_a$ to (4.43) above, as well as by using Lemma 3.3, Proposition 3.2 with $g(k) = E_{\overline{\alpha}}(\lambda, k)$ with $\lambda = \frac{-\alpha}{1-\alpha}$, and Lemma 3.6,

$$\begin{gathered}\frac{B(\alpha)}{1-\alpha}\mathcal{N}_a(\nabla f(k) * E_{\overline{\alpha}}(\lambda,k))(k) = \frac{B(\alpha)}{1-\alpha} z(\nabla_a f(k) * E_{\overline{\alpha}}(\lambda,k))(0) \\ \frac{B(\alpha)}{1-\alpha} z\left[(\mathcal{N}_a f)(k)\cdot\frac{z^{\alpha-1}}{z^{\alpha}-\lambda}\right] = \big(\mathcal{N}_a u(k)\big)(z).\end{gathered} \tag{4.44}$$

That is,

$$(\mathcal{N}_a f)(z) = \frac{1-\alpha}{B(\alpha)}\big(\mathcal{N}_a u(k)\big)(z) - \frac{1-\alpha}{B(\alpha)}\frac{\lambda}{z^{\alpha}}\big(\mathcal{N}_a u(k)\big)(z). \tag{4.45}$$

With the application of the inverse of $\mathcal{N}_a$ and by using Proposition 3.2 and Lemma 3.1, the following can be concluded:

$$f(k) = \frac{1-\alpha}{B(\alpha)}u(k) + \frac{\alpha}{B(\alpha)}(\nabla_a^{-\alpha})(k). \tag{4.46}$$

From this, the definition below follows for the case of the fractional sum related to the fractional difference having a discrete M-L function kernel.

Definition 4.9[4]: The fractional sum that associates with $({}_{a}^{ABR}\nabla \alpha f)(t)$ with order $0 < \alpha < 1$ has the following definition:

$$({}_{a}^{AB}\nabla \alpha f)(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)}(\nabla_a^{-\alpha} f)(t). \tag{4.47}$$

From this, it can be understood that $\alpha = 0$ yields the function f; furthermore, $\alpha = 1$ yields $\sum_{s=a+1}^{t} f\,(s)$.

We have what follows due to the definition of the discrete fractional integral:

$$({}_{a}^{ABR}\nabla^{\alpha}\quad {}_{a}^{ABR}\nabla^{-\alpha} f)(t) = f(t). \tag{4.48}$$

In addition, the following can be stated:

Theorem 4.4[4]: For any $0 < \alpha \leq 1$ and f being defined on $\mathbb{N}_a$, $({}_{a}^{ABR}\nabla^{\alpha} f)(t)$ satisfies the equation

$$ {}_{a}^{ABR}\nabla^{-\alpha} g(t) = f(t). \tag{4.49} $$

Proof [4]: Taking the fractional sum's definition, the equation as stated in the theorem can now be taken as being equivalent to the following:

$$ \frac{1-\alpha}{B(\alpha)} g(t) + \frac{\alpha}{B(\alpha)} (\nabla_a^{-\alpha} g)(t) = f(t). \tag{4.50} $$

The discrete LT $\mathbb{N}_a$ can be applied and Lemma 3.4 can be used in order to acquire

$$ \frac{1-\alpha}{B(\alpha)} G(s) + \frac{\alpha}{B(\alpha)} s^{-\alpha} G(s) = F(s), \tag{4.51} $$

where $G(s) = (\mathcal{N}_a g)(s)$ and $F(s) = (\mathcal{N}_a f)(s)$. From this, it follows that

$$ G(s) = \frac{s^{\alpha} B(\alpha)}{(1-\alpha)s^{\alpha} + \alpha} F(s) = \frac{B(\alpha)}{f-\alpha} \frac{s^{\alpha}}{s^{\alpha} - \lambda} F(s), \tag{4.52} $$

where $\lambda = \frac{-\alpha}{1-\alpha}$. Finally, the inverse of $\mathbb{N}_a$ can be applied and the discrete convolution theorem, Proposition 3.2, or (4.44) can be used in order to reach the conclusion that $g(t) = ({}_{a}^{ABR}\nabla^{\alpha} f)(t)$.

Theorem 4.5[4]: The following can be written:

$$ ({}_{a}^{ABC}\nabla^{\alpha} f)(k) = ({}_{a}^{ABC}\nabla^{\alpha} f)(k) - f(a) \frac{b(\alpha)}{1-\alpha} E_{\overline{\alpha}}(\lambda, -a). \tag{4.53} $$

Proof [4]: From (4.44), we have

$$ (\mathcal{N}_a \quad {}_{a}^{ABR}\nabla^{\alpha} f)(z) = \frac{B(\alpha)}{1-\alpha} \left[(\mathcal{N}_a f)(z) \frac{z^{\alpha}}{z^{\alpha} - \lambda} \right], \tag{4.54} $$

where $\lambda = \frac{\alpha}{1-\alpha}$. Now we also have

$$ \begin{aligned} (\mathcal{N}_a \quad {}_{a}^{ABC}\nabla^{\alpha} f)(z) &= \frac{B(\alpha)}{1-\alpha} (\mathcal{N}_a \nabla f(k) * E_{\overline{\alpha}}(\lambda, k))(z) \\ &= \frac{B(\alpha)}{1-\alpha} (\mathcal{N}_a \nabla f)(k) (E_{\overline{\alpha}}(\lambda, k))(z) \\ &= \frac{B(\alpha)}{1-\alpha} [z(\mathcal{N}_a f)(z) - (1-z)^{\alpha} f(a)] \left[\frac{\alpha-1}{z^{\alpha} - 1} \right] \\ &= \frac{B(\alpha)}{1-\alpha} \left[(\mathcal{N}_a f)(z) \frac{z^{\alpha}}{z^{\alpha} - \lambda} \right] - (1-z)^{\alpha} f(a) \frac{B(\alpha)}{1-\alpha} \left[\frac{\alpha-1}{z^{\alpha} - 1} \right] \end{aligned} \tag{4.55} $$

From (4.54) and (4.55), we see that

$$ (\mathcal{N}_a \quad {}_{a}^{ABC}\nabla^{\alpha} f)(z) = (\mathcal{N}_a \quad {}_{a}^{ABR}\nabla^{\alpha} f)(z) - (1-z)^{\alpha} f(a) \frac{B(\alpha)}{\alpha} \left[\frac{\alpha-1}{z^{\alpha} - 1} \right]. \tag{4.56} $$

If the inverse of $\mathcal{N}_a$ is applied to (4.56), (4.53) can be concluded. Above,

$(\mathcal{N}_a f(k-a))(z) = (1-z)^{\alpha}(Nf(k))(z)$ has been used.

With the Q-operator's action on the left and right fractional sums and differences, it was found in recent work that the right fractional sums $({}^{AB}\nabla_b^{-\alpha} f)(k)$ as well as the differences $({}^{AB}\nabla_b^{-\alpha} f)(k)$ can then be defined as below.

Definition 4.10[4]: If we take $0 < \alpha < 1$, and if f is taken as being defined on ${}_b\mathbb{N}$, the right fractional difference of f can then be written as

$$({}^{ABR}\nabla_b^{\alpha} f)(t) = \frac{B(\alpha)}{1-\alpha}(-\Delta_t)\sum_{s=t}^{b-1} f(s)E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha},(s-\rho(t))\right), \tag{4.57}$$

that of the right Caputo one can be written as

$$({}^{ABC}\nabla_b^{\alpha} f)(t) = \frac{B(\alpha)}{1-\alpha}\sum_{s=t}^{b-1}(-\Delta_f)(t)E_{\overline{\alpha}}\left(\frac{-\alpha}{1-\alpha},(s-\rho(t))\right). \tag{4.58}$$

Definition 4.11[4]: (the new right fractional sum having M-L kernel) If we take $0 < \alpha < 1$, and if f is taken as being defined on ${}_b\mathbb{N}$, it follows that the definition of the right fractional sum of f can be written as

$$({}^{AB}\nabla_b^{\alpha} f)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}({}_b\nabla^{\alpha} f)(t). \tag{4.59}$$

Theorem 4.6[4]: Assuming that f is a function that is defined on ${}_b\mathbb{N}$ and $0 < \alpha < 1$, $({}^{ABR}\nabla_b^{\alpha}\ {}^{AB}\nabla_b^{-\alpha} f)(z) = f(z)$ and $({}^{AB}\nabla_b^{\alpha}\ {}^{ABR}\nabla_b^{-\alpha} f)(z) = f(z)$ are obtained.

Applying the Q-operator to both of the sides as well as replacing $f(z)$ with $(Qf)(z) = f(a+b-z)$, it is now possible for the following to be stated.

Theorem 4.7[4]: It is now possible to write the following:

$$({}^{ABC}\nabla_b^{\alpha} f)(t) = ({}^{ABR}\nabla_b^{\alpha} f)(t) - f(b)\frac{B(\alpha)}{1-\alpha}E_{\overline{\alpha}}(\lambda, b-t). \tag{4.60}$$

4.4 Integration by Parts for the Fractional Sums and Differences Having Discrete Mittag-Leffler Kernels

A formula for integration by parts for fractional sums needs to be stated and proved first.

Theorem 4.8[4]: Assuming that both f and g are able to be defined on $\mathbb{N}_a \cap {}_c\mathbb{N}$, $a \equiv c$ (mod1), and $0 < \alpha < 1$, we subsequently have the following:

$$\sum_{t=a+1}^{c-1} g\,(t)({}_{a}^{AB}\nabla^{\alpha} f)(t) = \frac{1-\alpha}{B(\alpha)} \sum_{t=a+1}^{c-1} g\,(t)f(t) + \frac{\alpha}{B(\alpha)} \sum_{t=a+1}^{c-1} f\,(t)({}_{c}\nabla^{-\alpha} g)(t)$$

$$= \sum_{t=a+1}^{c-1} f\,(t)({}^{AB}\nabla_a^{-\alpha} g)(t). \tag{4.61}$$

Similarly, we have

$$= \sum_{t=a+1}^{c-1} g\,(t)({}_{a}^{AB}\nabla^{\alpha} f)(t)$$

$$= \frac{1-\alpha}{B(\alpha)} \sum_{c=a+1}^{c-1} g\,(t)f(t) + \frac{\alpha}{B(\alpha)} \sum_{t=a+1}^{c-1} f\,(t)(\nabla_a^{-\alpha} g)(t) \tag{4.62}$$

$$= \sum_{t=a+1}^{c-1} f\,(t)({}_{a}^{AB}\nabla^{-\alpha} g)(t).$$

Proof [4]: Utilizing the new left fractional sum as it has been defined, the formula for integration by parts for the ∇ classical fractional sums [57], and finally the new right fractional sum as it has been defined, the proof follows.

Theorem 4.9[4]: Assuming that both g and f are able to be defined on $\mathbb{N}_a \cap {}_b\mathbb{N}, a \equiv c$ (mod1), and $0 < \alpha < 1$, one has the following:

$$\sum_{s=a+1}^{b-1} g\,(s)({}_{a}^{ABR}\nabla^{\alpha} g)(s) = \sum_{s=a+1}^{c-1} g\,(s)({}_{a}^{ABR}\nabla_c^{\alpha} f)(s). \tag{4.63}$$

Likewise:

$$\sum_{s=a+1}^{c-1} g\,(s)({}^{ABR}\nabla_c^{\alpha} g)(s) = \sum_{s=a+1}^{c-1} g\,(s)({}_{a}^{ABR}\nabla^{\alpha} f)(s). \tag{4.64}$$

Proof [4]: The proof can be accomplished by Theorem 4.9 and fact demonstrated earlier that these new fractional sums and differences can be said to be each other's inverses. It is possible to write:

$$\sum_{s=a+1}^{c-1} g\,(s)({}^{AB}\nabla_c^{\alpha} g)(s) = \sum_{s=a+1}^{c-1} ({}^{AB}\,\nabla_c^{-\alpha} \quad {}^{AB}\nabla_c^{-\alpha} f)(s)({}_{a}^{AB}\nabla^{\alpha} g)(s),$$

$$\sum_{s=a+1}^{c-1} ({}^{AB}\nabla_c^{-\alpha})(s)({}_a^{AB}\nabla^{-\alpha}\quad {}_a^{AB}\nabla^{\alpha} g)(s) = \sum_{s=a+1}^{c-1} g\,(s)({}^{AB}\nabla_c^{\alpha} f)(s). \tag{4.65}$$

For the presentation of a formula for the integration by parts intended for the Caputo type fractional differences having M-L kernels, it is now necessary for the discrete versions of the (left) generalized fractional integral operator to be defined. These were presented and discussed at length in [44]:

$$(\boldsymbol{E}_{\rho,\mu,\omega,a}^{\gamma}\varphi) = \int_a^x (x-t)^{\mu-1} E_{\rho,\mu}^{\gamma}[\omega(x-t)^{\rho}]\varphi(t)dt, \quad q > a. \tag{4.66}$$

Here, $E_{\rho,\mu}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\rho k+\mu)k!}$ is the generalized M-L function being defined for the complex $\rho, \mu\gamma$ $(Re(\rho) > 0)$ [10, 44]. In this thesis only the case of the discrete version for $\gamma = 1$ will be addressed.

Definition 4.12[4]: The discrete (left) generalized fractional integral operator can be written as follows:

$$\left(\boldsymbol{E}_{\overline{\rho,\mu},\omega,a}^{1}\varphi\right)(t) \sum_{k=a+1}^{t} (t-\rho(k))^{\overline{\mu-1}} E_{\overline{\rho,\mu}}\big(\omega, t-\rho(k)\big)\varphi(k), \quad t \in \mathbb{N}_a. \tag{4.67}$$

The discrete (right) generalized fractional integral operator can be written as follows:

$$\left(\boldsymbol{E}_{\overline{\rho,\mu},\omega,a}^{1}\varphi\right)(t) \sum_{k=s}^{b-1} (k-\rho(t))^{\overline{\mu-1}} E_{\overline{\rho,\mu}}\big(\omega, k-\rho(t)\big)\varphi(k), \quad t \in_b \mathbb{N}. \tag{4.68}$$

Theorem 4.10[4]: When both functions f and g are taken as being defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$, it is possible to write

$$\begin{aligned}\sum_{k=a}^{b-1} f\,(k)({}_{a-1}^{ABC}\nabla^{\alpha} g)(k) &= \sum_{k=a}^{b-1} g\,(k)({}^{ABR}\nabla_{b-1}^{\alpha} f)(k) \\ &\quad + g\big(\rho(t)\big)\frac{B(\alpha)}{1-\alpha}\left(\boldsymbol{E}_{\overline{\alpha,1}\lambda,a^-}^{1} f\right)(t)|_a^b.\end{aligned} \tag{4.69}$$

Similarly,

$$\begin{aligned}\sum_{k=a+1}^{b} f\,(k)({}^{ABC}\nabla_{b+1}^{\alpha} g)(k) &= \sum_{k=a+1}^{b} g\,(k)({}_{a+1}^{ABR}\nabla^{\alpha} f)(k) \\ &\quad - g\big(\sigma(t)\big)\frac{B(\alpha)}{1-\alpha}\left(\boldsymbol{E}_{\overline{\alpha,1}\lambda,a^+}^{1} f\right)(t)|_a^b,\end{aligned} \tag{4.70}$$

where $\lambda = \frac{-\alpha}{1-\alpha}$.

Proof [4]: By (4.53) and Theorem 4.9 we have

$$\sum_{k=a}^{b-1} f\,(k)({}_{a-1}^{ABC}\nabla^{\alpha} g)(k)$$

$$= \sum_{k=a}^{b-1} f\,(k)\left[({}_{a-1}^{ABR}\nabla^{\alpha} g)(k) - g(a-1)\frac{B(\alpha)}{1-\alpha}E_{\overline{\alpha}}(\lambda, k-\rho(a))\right] \tag{4.71}$$

$$= \sum_{k=a}^{b-1} g\,(k)({}^{ABR}\nabla_{b-1}^{\alpha} f)(k) - g(a-1)\frac{B(\alpha)}{1-\alpha}\sum_{k=a}^{b-1} f\,(k)E_{\overline{\alpha}}(\lambda, k-\rho(a))$$

$$= \sum_{k=a}^{b-1} g\,(k)({}^{ABR}\nabla_{b-1}^{\alpha} f)(k) + g(\rho(t))\frac{B(\alpha)}{1-\alpha}(\boldsymbol{E}_{\alpha,1,\lambda,b^-}^{1} f)(t)|_a^b.$$

The second part of the above proceeds from (4.60) as well as from the secondary part of Theorem 4.9.

CHAPTER 5
APPLICATIONS

5.1 Fractional Euler-Langrange Equations

Here the Euler-Langrange(E-L) equations for the case of the Lagrangian including a new Left Caputo derivative will be demonstrated [2].

Theorem 5.1[2]: Take $0 < \alpha \leq 1$ as a value that is a noninteger, $c \in R$, $0 < c$, and let us suppose that the functional J:

$C^2[0, c] \to \mathbb{R}$ taking the form

$$\mathrm{J(f)} = \int_0^{c} \mathrm{L}\left(\mathrm{k}, \mathrm{f(k)}, {}^{\mathrm{ABC}}_{\ \ \ 0}\mathrm{D}^{\alpha}\, \mathrm{f(k)}\right) \mathrm{dt}, \tag{5.1}$$

possesses a local extremum in $P = \{y \in C^2[0, c] : y(0) = A, y(c) = B\}$ at some $f \in P$, where $L:[0, c] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. It follows that

$$\left[\mathrm{L}_1(\mathrm{s}) + {}^{\mathrm{ABR}}_{\ \ \ 0}\mathrm{D}^{\alpha}_{\mathrm{c}}\mathrm{L}_2(\mathrm{s})\right] = 0 \text{ for all } \mathrm{s} \in [0, \mathrm{c}],$$

where $\mathrm{L}_1(\mathrm{s}) = \frac{\partial \mathrm{L}}{\partial \mathrm{f}}(\mathrm{s})$ and $\mathrm{L}_2(\mathrm{s}) = \frac{\partial \mathrm{L}}{\partial\, {}^{\mathrm{ABC}}_{\ \ \ 0}\mathrm{D}^{\alpha}\, \mathrm{f}}(\mathrm{s})$.

Proof [2]: Without losing the generality, it is possible to suppose that *J* possesses a local maximum in *P* located at *f*. As a result, we have the existence of $\epsilon < 0$ such that $J(\hat{f}) - J(f) \leq 0$ for all cases of $\hat{f} \in P$ with $\|\hat{f} - f\| = sup_{k\in\mathbb{N}_{a\cap c\mathbb{N}}}|\hat{f}(k) - f(k)| < \epsilon$. For any case of $\hat{f} \in P$ we also have $\eta \in H = \{y \in C^2[0, c],\ y(0) = y(c) = 0\}$ such that $\hat{f} = f + \epsilon\eta$. Following from this, the ϵ-Taylor theorem implies this equality:

$$\begin{aligned} L(k, f, \hat{f}) &= L(k, f + \epsilon\eta, {}^{ABC}_{\ \ \ 0}D^{\alpha} f + \epsilon\, {}^{ABC}_{\ \ \ 0}D^{\alpha}\eta) \\ &= L(k, f, {}^{ABC}_{\ \ \ 0}D^{\alpha} f) + \epsilon[\eta L_1 + {}^{ABC}_{\ \ \ 0}D^{\alpha}\eta L_2] + O(\epsilon^2). \end{aligned}$$

Furthermore,

$$\begin{aligned} J(\hat{f}) - J(f) &= \int_0^{c} L\left(k, \hat{f}(k), {}^{ABC}_{\ \ \ 0}D^{\alpha} \hat{f}(t)\right) - \int_0^{c} L\left(k, f(k), {}^{ABC}_{\ \ \ 0}D^{\alpha} f(k)\right) \\ &= \epsilon \int_0^{c} [\eta(k) L_1(k)({}^{ABC}_{\ \ \ 0}D^{\alpha}\eta)(k) L_2(k)] + O(\epsilon^2). \end{aligned}$$

Now we take $\delta J(\eta, y) = \int_0^c [\eta(k) L_1(k) + ({}^{ABC}_{\ 0}D^{\alpha}\eta)(k) L_2(k)]dt$ as the first variation of J.

Evidently, if η ∈ H it follows that −η ∈ H, and δJ(η, y) = −δJ(−η, y). For ϙ small, the sign of $J(\hat{f}) - J(f)$ is determined by the first variation's sign, excepting the case in which δJ(η, y) = 0 for every η ∈ H. For the parameter to be η free, the integration by part formula of Proposition 4.2 can be used in order to obtain

$$\delta J(\eta, y) = \int_0^c \eta\,(s)[L_1(s) + ({}^{ABC}_{\ 0}D^{\alpha}_{c} L_2(s)] + \eta(k)\frac{B(\alpha)}{1-\alpha}\left(\boldsymbol{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha},c^-} L_2\right)(t)|_0^c = 0,$$

for all η ∈ H. Now the conclusion follows due to the calculus of variation and its fundamental lemma.

The term $\left(\boldsymbol{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha},c^-} L_2\right)(k)|_0^c = 0$ as shown above is referred to as the natural boundary condition.

Employing a similar approach, if we assume that the Lagrangian is dependent upon the right Caputo fractional derivative, the following can be stated:

Theorem 5.2[2]: Take 0 < α ≤ 1 as a value that is a noninteger, c ∈ R, 0 < c, and let us suppose that the functional J:
$C^2[0, c] \to \mathbb{R}$ taking the form

$$\mathrm{J(f)} = \int_0^c \mathrm{L}\left(\mathrm{k}, \mathrm{f(k)}, \quad {}^{\mathrm{ABC}}\mathrm{D}^{\alpha}_{\mathrm{c}}\mathrm{f(k)}\right)\mathrm{dk}, \tag{5.2}$$

possesses a local extremum in S = {y ∈ C^2[0, c] : y(0) = A, y(c) = B} at some f ∈ S, where L:
[0, b] × ℝ × ℝ → ℝ. It follows now that

$$[\mathrm{L}_1(\mathrm{s}) + {}^{\mathrm{ABR}}_{\ \ 0}\mathrm{D}^{\alpha}\ \mathrm{L}_2(\mathrm{s})] = 0 \ \text{ for all } \ \mathrm{s} \in [0, \mathrm{c}].$$

Proof [2]: The proof is the same as in Theorem 5.1. from Proposition 4.2, the second integration by parts can be applied with the aim of obtaining a natural boundary condition that can be written as follows: $\left(\boldsymbol{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha},c^+} L_2\right)(k)|_0^c = 0.$

Theorem 5.3[2]: If we assume σ, λ , ρ, μ, γ, ν ∈ C (Re(μ), Re(ν), Re(ρ) > 0), it follows that

$$\int_0^x (x - t)^{\pi-1} E^{\gamma}_{\rho,\mu}(\lambda[x-t]^{\rho}) t^{\nu-1} E^{\sigma}_{\rho,\nu}(\lambda t^{\rho}) dt = x^{\mu+\nu-1} E^{\gamma+\sigma}_{\rho,\mu+\nu}(\lambda t^{\rho}). \tag{5.3}$$

Furthermore, if $\mu = 1$, $\rho = \alpha$ and $\gamma = 1$, the following is yielded:

$$\int_0^x E_{\alpha}(\lambda[x-t]^{\alpha}) t^{\nu-1} E^{\sigma}_{\alpha,\nu}(\lambda t^{\alpha}) dt = x^{\nu} E^{1+\sigma}_{\alpha,1+\nu}(\lambda t^{\alpha}). \tag{5.4}$$

From [10] we have the following helpful differentiation formula. For $\gamma, \lambda , \alpha, \mu \in C$ ($Re(\alpha > 0)$ and $n \in N$ the following can be written:

$$\left(\frac{d}{dt}\right)^n [z^{\mu-1} E^{\gamma}_{\rho,\mu}(\lambda z^{\alpha})] = z^{\mu-n-1} E^{\gamma}_{\alpha,\mu-n}(\lambda z^{\alpha}). \tag{5.5}$$

Now, by the help of (5.4) and (5.5), we have

$${}^{ABR}_{0}D^{\alpha} [x^{\nu-1} E^{\sigma}_{\alpha,\nu}(\lambda x^{\alpha})] = \frac{B(\alpha)}{1-\alpha} \frac{d}{dx} [x^{\nu} E^{1+\sigma}_{\alpha,1+\nu}(\lambda x^{\alpha})] = \frac{B(\alpha)}{1-\alpha} x^{\nu-1} E^{1+\sigma}_{\alpha,\nu}(\lambda x^{\alpha}). \tag{5.6}$$

By the help of (5.4) and (5.5), we also have

$$\begin{aligned} {}^{ABR}_{0}D^{\alpha} [x^{\nu-1} E^{\sigma}_{\alpha,\nu}(\lambda x^{\alpha})] &= \frac{B(\alpha)}{1-\alpha} \int_0^x x^{\nu} E_{\alpha}(\lambda(x-1)^{\alpha}) \frac{d}{dx} [t^{\nu-1} E^{\sigma}_{\alpha,\nu}(\lambda x^{\alpha})] dt \\ &= \frac{B(\alpha)}{1-\alpha} x^{\nu-1} E^{1+\sigma}_{\alpha,\nu}(\lambda x^{\alpha}). \end{aligned} \tag{5.7}$$

Remark 5.1[2]: Due to (5.6) and (5.7), the function

$$\begin{aligned} g(y) &= \lim_{\nu \to 0} \frac{1-\alpha}{B(\alpha)} y^{\nu-1} E^{-1}_{\alpha,\nu}(\lambda y^{\alpha}) \\ &= \frac{\alpha x^{\alpha-1}}{B(\alpha)\Gamma(\alpha)}, \end{aligned} \tag{5.8}$$

is a nonzero function; furthermore, its fractional ABC and ABR derivative is also found to be zero. We can see that this is the case because $(-1)_0 = 1$, $(-1)_1 = -1$ and $(-1)_k = 0$ for $k = 2, 3, 4, \ldots$ and also because

$$E^{0}_{\alpha,\nu}(\lambda, y) = \frac{y^{\nu-1}}{\Gamma(\nu)} \to 0, \qquad \nu \to 0^{+}. \tag{5.9}$$

Here we can also see that when α tends to 1 the function $g(y)$ will tend to the constant function 1.

If we apply relation [53]

$$\left({}^{ABC}_{0}D^{\alpha} f\right)(t) = \left({}^{ABC}_{0}D^{\alpha} f\right)(t) - \frac{B(\alpha)}{1-\alpha} f(0) E_{\alpha}(\lambda t^{\alpha}) , \qquad \lambda = \frac{-\alpha}{1-\alpha}, \tag{5.10}$$

along with the identity [54]

$$\left({}_{0}I^{\alpha}\, t^{\beta-1} E_{\mu,\beta}[\lambda t^{\mu}](y) \right) = y^{\alpha+\beta-1} E_{\mu,\alpha+\beta}[\lambda t^{\mu}], \tag{5.11}$$

where the M-L function possessing the two parameters of α and β can be given by

$$E_{\mu,\beta}(m)\sum_{k=0}^{\infty}\frac{m^k}{\Gamma(\alpha k+\beta)}\ , \qquad (m,\beta\in\mathbb{C};\ \ Re(\alpha)>0), \tag{5.12}$$

for which $E_{\alpha,\beta}(m) = E_{\alpha}(m)$, then the following conclusion can be stated. It is helpful for solving fractional dynamical systems within Caputo fractional derivatives having M-L kernels.

5.2 Discrete Fractional Euler-Langrange Equations

In this section, Euler-Langrange equations for the case of a Lagrangian possessing the left new discrete Caputo derivative will be demonstrated[4].

Theorem 5.4[4]: Take $0 < \alpha \leq 1$ to be a noninteger, $a, c \in \mathbb{R}, a < c, a \equiv c$ (mod1). Furthermore, we can suppose that the functional able to be written as

$$J(f) = \sum_{s=a}^{b-1} L\left(s, f^{\rho}(s), {}_{a-1}^{ABC}\nabla^{\alpha} f(s)\right), \tag{5.13}$$

has its local extremum in $S = \{y: (\mathbb{N}_{a-1} \cap_{c-1} \mathbb{N}) \to \mathbb{R}: y(a-1) = A, y(c-1) = B\}$ at some $f \in S$, for which $L: (\mathbb{N}_{a-1} \cap_{c-1} \mathbb{N}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Subsequently,

$$[L_1(t) + {}^{ABR}\nabla_{b-1}^{\alpha} L_2(t)] = 0 \quad \text{for all} \quad t \in (\mathbb{N}_{a-1} \cap_{c-1} \mathbb{N}), \tag{5.14}$$

where $L_1(t) = \frac{\partial L}{\partial f^{\rho}}(t)$ and $L_2(t) = \frac{\partial L}{\partial_{a-1}^{ABC}\nabla^{\alpha} f}(t)$.

Proof [4]: Without losing generality here, we can say that J possesses a local maximum in S at f. Therefore, we have the existence of $\varepsilon > 0$ such that $J(\hat{f}) - J(f) \leq 0$ for all cases of $f \in S$ with $\parallel \hat{f} - f \parallel = \sup_{t \in \mathbb{N}_a \cap_b \mathbb{N}} | \hat{f}(s) - f(s) < \varphi$. For any $\hat{f} \in S$ we have $\eta \in H = \{y: (\mathbb{N}_{a-1} \cap_{c-1} \mathbb{N}) \to \mathbb{R}: y(a-1) = y(c-1) = 0\}$ such that we also have $\hat{f} = f + \varphi\eta$. Subsequently, the φ-Taylor theorem together with the assumption imply the first variation quantity $\delta J(\eta, y) = \sum_{t=a}^{c-1}[\eta^{\rho}(s) L_1(s) + ({}_{a-1}^{ABC}\nabla^{\alpha}\eta)(s) L_2(s)] ds = 0$, for all cases of $\eta \in H$. For us to guarantee that the parameter is η free, the integration by parts equation (4.69) can be utilized for obtaining the following:

$$\delta J(\eta, f) = \sum_{t=a}^{c-1} \eta^{\rho}\,(t)[L_1(t) + {}^{ABR}\nabla_{c-1}^{\alpha} L_2(t)] \tag{5.15}$$

$$+\eta^{\rho}(s)\frac{B(\alpha)}{1-\alpha}(\boldsymbol{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha},c^{-}}L_2)(s)|_a^c = 0,$$

for all $\eta \in H$. The result then follows due to calculus of variation and its discrete fundamental lemma.

$(\boldsymbol{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha},c^{-}}L_2)(s)|_a^c = 0$ above is referred to as the natural boundary condition.

Likewise, if the Lagrangian here depends on the discrete right Caputo fractional derivative, it is possible for the following result to be written.

Theorem 5.5[4]: Take $0 < \alpha \leq 1$ to be a noninteger, $a, c \in \mathbb{R}, a < c, a \equiv c$ (mod1). Furthermore, it is assumed that the functional J possessing the following form of

$$J(f) = \sum_{a+1}^{c} L\left(s, f^{\sigma}(s), {}^{ABC}\nabla^{\alpha}_{c+1}f(s)\right), \tag{5.16}$$

has its local extremum in $S = \{y: (\mathbb{N}_{a+1} \cap_{c+1} \mathbb{N}) \to \mathbb{R}: y(a+1) = A, y(c+1) = B\}$ at some $f \in S$, for which $L: (\mathbb{N}_{a+1} \cap_{c+1} \mathbb{N}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Subsequently,

$$[L_1(t) + {}^{ABR}_{a=1}\nabla^{\alpha}L_2(t)] = 0 \quad \text{for all} \quad t \in (\mathbb{N}_{a+1} \cap_{c+1} \mathbb{N}), \tag{5.17}$$

where $L_1(t) = \frac{\partial L}{\partial f^{\sigma}}(t)$ and $L_2(t) = \frac{\partial L}{\partial {}^{ABC}\nabla^{\alpha}_{c+1}f}(t)$.

Proof [4]: This proof is found to be similar to that of Theorem 5.11 with application of equation (4.70) for the second integration by parts in order to obtain the natural boundary condition able to be written as $(\boldsymbol{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha},c^{+}}L_2)(s)|_a^c = 0$.

Example 5.1[4]: Now an interesting physical action will be presented as a means of support for Theorem 5.4. The fractional discrete action can be considered as follows: $(y) = \sum_{t=a}^{b-1}[\frac{1}{2}({}^{ABC}_{a-1}\nabla^{\alpha}y(t))^2 - V(y^{\rho}(t))]$, where we have $0 < \alpha < 1$ and additionally with $y(b-1), y(a-1)$ being assigned or having the natural boundary condition

$$(\boldsymbol{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha},b^{-}} \quad {}^{ABC}_{a-1}\nabla^{\alpha}(y))(t)|_a^b = 0. \tag{5.18}$$

With application of Theorem 5.4, the E-L equation is

$$({}^{ABR}\nabla^{\alpha}_{b-1} \circ^{ABC}_{a-1} \nabla^{\alpha}y)(s) - \frac{dV}{dy}(s) = 0 \quad \text{for all} \quad s \in (\mathbb{N}_{a-1} \cap_{b-1} \mathbb{N}). \tag{5.19}$$

The E-L equation above is interesting in cases of a composition comprising the discrete right and discrete left types of fractional derivatives. Please consult [50] to find valuable comparisons with cases of classical discrete fractional E-L equations within ∇. We likewise suggest for the reader the references for the classical fractional dynamical systems comprising left and right fractional operators in such a case that delay exists.

5.3 Fractional Bateman-Feshbach-Tikochinsky Oscillator

The classical Lagrangian of the Bateman-Feshbach-Tikochinsky oscillator can be given as follows[5]:

$$L = m\dot{q}_1\dot{q}_2 + \rho(q_1\dot{q}_2 - \dot{q}_1 q_2) - Kq_1q_2. \tag{5.20}$$

In this equation, q_1 is the damped harmonic oscillator coordinate and q_2 represents the time-reversed counterpart, while parameters $m, \rho,$ and K are time-independent. The fractional Lagrangian (5.20) can be written as [4]

$$L^F = m_a D_t^\alpha q_1 \, {}_a D_t^\alpha q_2 + \rho(q_1 \, {}_a D_t^\alpha q_2 - {}_a D_t^\alpha q_1 q_2,) - Kq_1q_2 \tag{5.21}$$

while the Lagrange model of fractional order is

$$\begin{gathered} m_a D_t^\alpha \, {}_a D_t^\alpha q_1 + \rho_a D_t^\alpha q_1 + Kq_1 = 0, \\ m_a D_t^\alpha \, {}_a D_t^\alpha q_2 - \rho_a D_t^\alpha q_2 + Kq_2 = 0. \end{gathered} \tag{5.22}$$

The generalized momentum can subsequently be written as follows [4]:

$$p_i = \frac{\partial L^F}{\partial_a D_t^\alpha q_i}, \tag{5.23}$$

where L^F is the Lagrangian of fractional order and i = 1, 2. The two generalized momentums can be written as

$$\begin{gathered} p_1 = \frac{\partial L^F}{\partial_a D_t^\alpha q_1} = m_a D_t^\alpha q2 - \frac{\rho}{2} q_2, \\ p_2 = \frac{\partial L^F}{\partial_a D_t^\alpha q_2} = m_a D_t^\alpha q1 - \frac{\rho}{2} q_1. \end{gathered} \tag{5.24}$$

When the Legendre transformation is applied, the Hamiltonian of fractional order can be obtained [4]:

$$H^F(t, q_i, p_i) = \sum_i p_i \, {}_a D_t^\alpha q_i(q_i, p_i) - L\big(t, q_{i,a} D_t^\alpha q_i(q_i, p_i)\big). \tag{5.25}$$

Utilizing equation (5.25), we have the following:

$$H^F = \left(K - \frac{\rho^2}{4m}\right) q_1 q_2 + \frac{\rho}{2m}(q_2 p_2 - q_1 p_1) + \frac{p_1 p_1}{m}. \tag{5.26}$$

We define $\omega = \sqrt{K \frac{\rho_2}{4m}}$ and the Hamiltonian takes the following form:

$$H^1 = \omega^2 q_1 q_2 + \frac{\rho}{2m}(q_2 p_2 - q_1 p_1) + \frac{p_1 p_2}{m}. \tag{5.27}$$

The fractional Hamilton model of the B-F-T oscillator can be given as follows:

$$
\begin{aligned}
{}_aD_t^{\alpha}q_1 &= -\frac{\rho q_1}{2m}+\frac{p_2}{m},\\
{}_aD_t^{\alpha}q_2 &= \frac{\rho q_2}{2m}+\frac{p_1}{m},\\
{}_aD_t^{\alpha}p_1 &= \frac{\rho^2 q_2}{4m}+\frac{\rho p_1}{2m}-Kq_2,\\
{}_aD_t^{\alpha}p_2 &= \frac{\rho^2 q_1}{4m}-\frac{\rho p_2}{2m}-Kq_1.
\end{aligned}
\tag{5.28}
$$

Next the fractional operators of Liouville-Caputo (L-C) and Caputo-Fabrizio-Caputo (C-F-C) and the fractional derivative based in the M-L kernel will be considered.

Case 5.1: We have the following in the L-C sense [5]:

$$
\begin{aligned}
q_1(t) &= \sum_{i=0}^{n-1} q_1\,(0)^{(i)}\frac{t^i}{i!}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\kappa)^{\alpha-1}\left(-\frac{\chi q_1(\kappa)}{2m}+\frac{p_2(\kappa)}{m}\right)d\kappa,\\
q_2(t) &= \sum_{i=0}^{n-1} q_2\,(0)^{(i)}\frac{t^i}{i!}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\kappa)^{\alpha-1}\left(\frac{\chi q_2(\kappa)}{2m}+\frac{p_1(\kappa)}{m}\right)d\kappa, \quad t<T,\\
p_1(t) &= \sum_{i=0}^{n-1} p_1\,(0)^{(i)}\frac{t^i}{i!}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\kappa)^{\alpha-1}\left(\frac{\chi^2 q_2(\kappa)}{4m}+\frac{\chi p_1(\kappa)}{2m}-Kq_2(\kappa)\right)d\kappa,\\
p_2(t) &= \sum_{i=0}^{n-1} p_2\,(0)^{(i)}\frac{t^i}{i!}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\kappa)^{\alpha-1}\left(\frac{\chi^2 q_1(\kappa)}{4m}-\frac{\chi p_2(\kappa)}{2m}-Kq_1(\kappa)\right)d\kappa.
\end{aligned}
\tag{5.29}
$$

Numerical approximation of (5.29) can be achieved with the assistance of algorithm 4.

Case 5.2: In the C-F-C sense [5]:

$$
\begin{aligned}
q_{1(l+1)}(t) &= q_{(1)}(t)+\left\{\frac{1-\alpha}{B(\alpha)}\left[-\left(\frac{\rho}{2m}\right)q_{1(l+1)}(t)+\left(\frac{1}{m}\right)p_{2(l+1)}(t)\right]\right\}\\
&\quad+\frac{\alpha}{B(\alpha)}\sum_{z=0}^{\alpha}\varepsilon_{1,z,l}\left[-\left(\frac{\rho}{2m}\right)q_{1(l)}(t)+\left(\frac{1}{m}\right)p_{2(l)}(t)\right],\\
q_{2(l+1)}(t) &= q_{(2)}(t)+\left\{\frac{1-\alpha}{B(\alpha)}\left[\left(\frac{\rho}{2m}\right)q_{2(l+1)}(t)+\left(\frac{1}{m}\right)p_{1(l+1)}(t)\right]\right\}\\
&\quad+\frac{\alpha}{B(\alpha)}\sum_{z=0}^{\alpha}\varepsilon_{2,z,l}\left[\left(\frac{\rho}{2m}\right)q_{2(l)}(t)+\left(\frac{1}{m}\right)p_{2(l)}(t)\right],\\
p_{1(l+1)}(t) &= p_{(1)}(t)+\left\{\frac{1-\alpha}{B(\alpha)}\left[\left(\frac{\rho^2}{4m}\right)q_{2(l+1)}(t)+\left(\frac{\rho}{2m}\right)p_{1(l+1)}(t)-Zq_{1(l+1)}(t)\right]\right\}\\
&\quad+\frac{\alpha}{B(\alpha)}\sum_{z=0}^{\alpha}\varepsilon_{3,z,l}\left[\left(\frac{\rho^2}{4m}\right)q_{2(l)}(t)+\left(\frac{\rho}{2m}\right)p_{1(l)}(t)-Zq_{2(l)}(t)\right],\\
p_{2(l+1)}(t) &= p_{(2)}(t)+\left\{\frac{1-\alpha}{B(\alpha)}\left[\left(\frac{\rho^2}{4m}\right)q_{1(l+1)}(t)-\left(\frac{\rho}{2m}\right)p_{2(l+1)}(t)-Zq_{1(l+1)}(t)\right]\right\}\\
&\quad+\frac{\alpha}{B(\alpha)}\sum_{z=0}^{\alpha}\varepsilon_{4,z,l}\left[\left(\frac{\rho^2}{4m}\right)q_{1(l)}(t)+\left(\frac{\rho}{2m}\right)p_{2(l)}(t)-Zq_{1(l)}(t)\right],
\end{aligned}
\tag{5.30}
$$

where

$$\varepsilon_{(1,2,3,4),z,l+1}\begin{cases} l^{\alpha}-(1-(1-\alpha)(l-\alpha), & z=0, \\ (l-z+2)^{\alpha+1}+(l-z)^{\alpha+1}-2(l-z+1)^{\alpha+1}, & 0\leq z\leq 1. \end{cases} \tag{5.31}$$

Case 5.3: It is possible to use the numerical approximation scheme developed in [59] for the fractional derivative based on the M-L kernel [5]:

$$\begin{aligned} {}_{0}^{AB}I_{t}^{\alpha}[f(t_{l+1})] = & \frac{1-\alpha}{B(\alpha)}\left[\frac{f(t_{l+1})-f(t_{l})}{2}\right] \\ & +\frac{\alpha}{\Gamma(\alpha)}\sum_{z=0}^{\infty}\left[\frac{f(t_{z+1})-f(t_{z})}{2}\right]b_{z}^{\alpha}, \end{aligned} \tag{5.32}$$

where

$$b_{z}^{\alpha}=(z+1)^{1-\alpha}-(z)^{1-\alpha}, \tag{5.33}$$

and the system of (5.28) is further represented as follows:

$$\begin{aligned} q_{1(l+1)}(t)-q_{1(l)}(t) = & \, q_{(1)}^{l}(t)+\left\{\frac{1-\alpha}{B(\alpha)}\left[-\left(\frac{\chi}{2m}\right)\left(\frac{q_{1(l+1)}(t)-q_{1(l)}(t)}{m}\right)\right.\right. \\ & \left.\left.+\left(\frac{1}{m}\right)\left(\frac{p_{2(l+1)}(t)-p_{2(l)}(t)}{2}\right)\right]\right\}+\frac{\alpha}{B(\alpha)}\sum_{z=0}^{\infty}b_{z}^{\alpha}\left[-\left(\frac{\rho}{2m}\right)\left(\frac{q_{1(z+1)}(t)-q_{1(z)}(t)}{2}\right)\right. \\ & \left.+\left(\frac{1}{2}\right)\left(\frac{p_{2(z+1)}(t)-p_{2(z)}(t)}{2}\right)\right], \end{aligned} \tag{5.34}$$

$$\begin{aligned} q_{2(l+1)}(t)-q_{2(l)}(t) = & \, q_{(2)}^{l}(t)+\left\{\frac{1-\alpha}{B(\alpha)}\left[\left(\frac{\rho}{2m}\right)\left(\frac{q_{2(l+1)}(t)-q_{2(l)}(t)}{2}\right)\right.\right. \\ & \left.\left.+\left(\frac{1}{m}\right)\left(\frac{p_{1(l+1)}(t)-p_{1(l)}(t)}{2}\right)\right]\right\}+\frac{\alpha}{B(\alpha)}\sum_{z=0}^{\infty}b_{z}^{\alpha}\left[\left(\frac{\rho}{2m}\right)\left(\frac{q_{2(z+1)}(t)-q_{2(z)}(t)}{2}\right)\right. \\ & \left.+\left(\frac{1}{2}\right)\left(\frac{p_{1(z+1)}(t)-p_{1(z)}(t)}{2}\right)\right], \end{aligned} \tag{5.35}$$

$$\begin{aligned} q_{1(l+1)}(t)-q_{1(l)}(t) = & \, q_{(1)}^{l}(t) \\ & +\left\{\frac{1-\alpha}{B(\alpha)}\left[\left(\frac{\rho}{2m}\right)\left(\frac{q_{2(l+1)}(t)-q_{2(l)}(t)}{m}\right)+\left(\frac{1}{m}\right)\left(\frac{p_{1(l+1)}(t)-p_{1(l)}(t)}{2}\right)\right.\right. \\ & \left.\left.-Z\left(\frac{q_{2(z+1)}(t)-q_{2(z)}(t)}{2}\right)\right\}+\frac{\alpha}{B(\alpha)}\sum_{z=0}^{\infty}b_{z}^{\alpha}\left(\frac{\rho}{2m}\right)\left(\frac{q_{2(z+1)}(t)-q_{2(z)}(t)}{2}\right)\right. \\ & \left.+\left(\frac{1}{m}\right)\left(\frac{p_{1(l+1)}(t)-p_{1(l)}(t)}{2}\right)-Z\left(\frac{q_{2(z+1)}(t)-q_{2(z)}(t)}{2}\right)\right], \end{aligned} \tag{5.36}$$

$$p_{2(l+1)}(t)-p_{2(l)}(t)=p_{(2)}^{l}(t)$$

$$+\left\{\frac{1-\alpha}{B(\alpha)}\left[\left(\frac{\rho}{2m}\right)\left(\frac{p_{2(l+1)}(t)-p_{2(l)}(t)}{m}\right)+\left(\frac{\rho}{m}\right)\left(\frac{p_{2(l+1)}(t)-p_{2(l)}(t)}{2}\right)\right.\right.$$

$$\left.\left.-Z\left(\frac{q_{1(z+1)}(t)-q_{1(z)}(t)}{2}\right)\right]\right\}+\frac{\alpha}{B(\alpha)}\sum_{z=0}^{\infty} b_z^{\alpha}\left(\frac{\rho}{2m}\right)\left(\frac{q_{1(z+1)}(t)-q_{1(z)}(t)}{2}\right)$$

$$-\left(\frac{\rho}{m}\right)\left(\frac{p_{2(l+1)}(t)-p_{2(l)}(t)}{2}\right)-Z\left(\frac{q_{1(z+1)}(t)-q_{1(z)}(t)}{2}\right). \tag{5.37}$$

5.4 Fractional RLC Electrical Circuit

With the work in [60], auxiliary parameter σ was presented together with the finality to preserve the temporal operator's dimensionality:

$$\frac{d}{dt} \to \frac{1}{\sigma^{1-\alpha}} \cdot D_t^{\alpha},\ \ \nu-1<\alpha\leq\nu,\ \ \nu=1,2,3,\ldots, \tag{5.38}$$

and

$$\frac{d^2}{dt^2} \to \frac{1}{\sigma^{2-\alpha}} \cdot D_t^{2\alpha},\ \ \nu-1<\alpha\leq\nu,\ \ \nu=1,2,3,\ldots, \tag{5.39}$$

Here, s is in the dimension of seconds. It was stated that this parameter is related to temporal elements of the system [60]; in the case that a = 1, we see that (5.38) and (5.39) are recovered in the traditional sense. With the application of Kirchhoff's laws, the RLC circuit has the following equation:

$$D_t^2 I(t)+\frac{R}{L}D_t I(t)+\frac{1}{LC}I(t)=\frac{1}{L}E(t). \tag{5.40}$$

Here, L is inductance, R is resistance, and $E(t)$ is source voltage.

5.4.1. RLC Electrical Circuit with Fractional Operator with Mittag-Leffler Kernel

In light of (5.38) and (5.39), the fractional equation that corresponds to (5.40) via the fractional operator with M-L kernel can be written as follows:

$${}_{0}^{ABC}D_k^{2\alpha} I(k)+A\,{}_{0}^{ABC}D_k^{\alpha} I(k)=B\,C\,E(k)-BI(k),\ \ \ 0<\alpha\leq 1 \tag{5.41}$$

and the analytical solutions of equations (5.41) can be obtained with the consideration of different source terms[6].

Case 5.4[6]: The unit step source, $E(k)=u(k), I(0)=I_0, (I_0>0), \dot{I}(0)=0$, (5.41) has the following definition:

$$ {}_{0}^{ABC}D_k^{2\alpha}I(k) + A\,{}_{0}^{ABC}D_k^{\alpha}I(k) = B\,C\,u(k) - BI(k). \tag{5.42}$$

When we apply LT to (5.42), the current's expression is:

$$\tilde{I}(k) = B\left[\frac{(1-\alpha)s^{2\alpha-1}}{s^{2\alpha}K + s^{\alpha}L + M} + \frac{2\alpha(1-\alpha)s^{\alpha-1}}{s^{2\alpha}K + s^{\alpha}L + M} + \frac{\alpha^2}{s^{2\alpha}K + s^{\alpha}L + M} + \frac{1}{s}\right]$$

$$B(\alpha)^2\frac{s^{2\alpha-1}I_0}{s^{2\alpha}K + s^{\alpha}L + M} + AB(\alpha)I_0\frac{s^{\alpha-1}(s^{\alpha}(1-\alpha)+\alpha)}{s^{2\alpha}K + s^{\alpha}L + M}. \tag{5.43}$$

With the inverse LT of (5.43) it is possible to obtain the analytical solution:

$$I(k) = \left[B(1-\alpha) + \frac{B(\alpha)^2 I_0}{K} + \frac{AB(\alpha)^2 I_0(1-\alpha)}{K}\right]$$

$$\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-C)^m(-H)^t\binom{m+t}{t}}{\Gamma[t\alpha + (m+1)2\alpha - 2\alpha + 1]}k^{\alpha(t+2m)}\left[2\alpha(1-\alpha) + \frac{AB(\alpha)I_0\alpha}{K}\right]$$

$$\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-C)^m(-H)^t\binom{m+t}{t}}{\Gamma[t\alpha + (m+1)2\alpha - 2\alpha + 1]}k^{\alpha(t+2m)} \tag{5.44}$$

$$+\frac{\alpha^2}{K}\int_0^k \tau^{2\alpha(m+1)-1}\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-C)^m(-H)^t\binom{m+t}{t}}{\Gamma[k\alpha + (n+1)2\alpha]}k^{t\alpha}d\tau,$$

where

$$K = B(\alpha)^2 + AB(\alpha(1-\alpha) + D(1-\alpha)^2,$$
$$L = AB(\alpha) + 2D(\alpha)(1-\alpha),$$
$$M = D(\alpha)^2, \tag{5.45}$$
$$C = \frac{M}{K},$$
$$H = \frac{L}{K}.$$

Case 5.5[6]: The exponential source, $E(k) = e_{-ak}, I(0) = I_0, (I_0 > 0), \dot{I}(0) = 0$, (5.41) has the following definition:

$$ {}_{0}^{ABC}D_k^{2\alpha}I(k) + A\,{}_{0}^{ABC}D_k^{\alpha}I(k) = B\,C\,e^{-ak} - BI(k). \tag{5.46}$$

Upon application of LT to (5.46), the current's expression is obtained as follows:

$$I(k) = B\left[\frac{1}{s+a} + \left(\frac{(1-\alpha)s^{2\alpha-1}}{s^{2\alpha}K + S^{\alpha}L + M} + \frac{2\alpha(1-\alpha)s^{\alpha}}{s^{2\alpha}K + S^{\alpha}L + M} + \frac{2\alpha}{s^{2\alpha}K + S^{\alpha}L + M}\right)\right]$$

$$+B(\alpha)^2\frac{s^{2\alpha-1}I_0}{s^{2\alpha}K + S^{\alpha}L + M} + AB(\alpha)I_0\frac{s^{\alpha-1}(s^{\alpha}(1-\alpha)+\alpha)}{s^{2\alpha}K + S^{\alpha}L + M}. \tag{5.47}$$

After we take the inverse LT to (5.47), we are able to obtain the following solution:

$$I(k)=\left[\frac{B(\alpha)^2 I_0}{K}+\frac{AB(\alpha)^2 I_0(1-\alpha)}{K}\right]$$

$$\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-C)^m(-H)^t\binom{m+t}{t}}{\Gamma[t\alpha+(m+1)2\alpha-2\alpha+1]}k^{\alpha(t+2m)}$$

$$+\frac{AB(\alpha)I_0(\alpha)}{K}k^{\alpha}\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-C)^m(-H)^t\binom{m+t}{t}}{\Gamma[t\alpha+(m+1)2\alpha-2\alpha+1]}k^{\alpha(t+2m)}$$

$$+\frac{2B\alpha(1-\alpha)}{K}\int_0^k E_{\alpha,\alpha}\,(a-(k-\tau))\tau^{\alpha-1}$$

$$\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-C)^m(-H)^t\binom{m+t}{t}}{\Gamma[t\alpha+(m+1)2\alpha]}\tau^{\alpha(t+2m)}d\tau \tag{5.48}$$

$$+\frac{B(1-\alpha)}{K}\int_0^k E_{\alpha,\alpha}\,(a-(k-\tau))\tau^{-1}\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-C)^m(-H)^t\binom{m+t}{t}}{\Gamma[t\alpha+(m+1)2\alpha]}\tau^{\alpha(t+2m)}\tau$$

$$+\frac{B(\alpha)^2}{K}\int_0^k E_{\alpha,\alpha}\,(a-(k-\tau))\tau^{2\alpha(m+1)-1}\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-C)^m(-H)^t\binom{m+t}{t}}{\Gamma[t\alpha+(\ddot{o}+1)2\alpha]}\tau^{t\alpha}d\tau$$

for which K, L, M, C, and H are defined by (5.45).

Case 5.6[6]: The periodic source, $E(k)=\sin(jk), I(0)=I_0, (I_0>0), \dot{I}(0)=0$, (5.41) has the following definition:

$$ {}_{0}^{ABC}D_k^{2\alpha}I(k)+A\,{}_{0}^{ABC}D_k^{\alpha}I(k)=B\,C\,\sin(\varphi k)-BI(k) \tag{5.49}$$

Upon application of LT to (5.49), the current's expression is obtained as follows:

$$\tilde{I}(s)=\left[\frac{s^2\alpha(1-\alpha)^2+2\alpha(1-\alpha)s^{\alpha}+a^2}{s^2\alpha K+s^{\alpha}L+M}\right]\frac{\varphi}{s^2+\varphi^2}$$

$$+B(\alpha)^2\frac{s^{2\alpha-1}I_0}{s^2\alpha K+s^{\alpha}L+M}+AB(\alpha)\frac{s^{\alpha-1}I_0(s^{\alpha}(1-\alpha)+\alpha}{s^2\alpha K+s^{\alpha}L+M}. \tag{5.50}$$

With the inverse LT to (5.50), we have the following solution:

$$I(k)=\frac{(1-\alpha)^2}{t}\int_0^k \sin(\varphi(k-\tau))\tau^{-1}$$

$$\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-C)^m(-H)^t\binom{m+t}{t}}{\Gamma[t\alpha+(m+1)2\alpha-2\alpha+1]}k^{\alpha(t+2m)}d\tau$$

$$+\frac{2\alpha(1-\alpha)}{K}\int_0^k \sin(\varphi(k-\tau))\tau^{-1}$$

$$\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-C)^m(-H)^t\binom{m+t}{t}}{\Gamma[t\alpha+(m+1)2\alpha-2\alpha+1]}k^{\alpha(t+2m)}d\tau \tag{5.51}$$

$$+\frac{\alpha^2}{K}\int_0^k \sin\,(\varphi(k-\tau))\tau^{2\alpha(m+1)-1}\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-C)^m(-H)^t\binom{m+t}{t}}{\Gamma[t\alpha+(m+1)2\alpha-2\alpha+1]}k^{t\alpha}d\tau$$

$$+\left[\frac{B(\alpha)^2I_0}{K}+\frac{AB(\alpha)I_0(1-\alpha)}{K}+\frac{AB(\alpha)I_0\alpha}{K}k^{\alpha}\right]$$

$$\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-C)^m(-H)^t\binom{m+t}{t}}{\Gamma[t\alpha+(m+1)2\alpha-2\alpha+1]}k^{\alpha(t+2m)}d\tau,$$

for which $K, L, M, C,$ and H are defined by (5.45).

In order to compare our results with same other produced by different fractional derivatives we present below two different approches.

5.5 Fractional Schrödinger Equation with Atangana-Baleanu Fractional Derivative

Here the one-dimensional time-dependent Schrödinger equation involving the C-F fractional operator and the new operator with M-L kernel with arbitrary order will be analyzed [7].

Case 5.7[7]: Considering ${}^{ABC}_{\ a}D_t^{\alpha}f(t)=\frac{B(\alpha)}{1-\alpha}\int_a^t f(\theta)E_{\alpha}\left[-\alpha\frac{(t-\theta)^{\alpha}}{1-\alpha}\right]d\theta$, the fractional Schrödinger equation via the C-F fractional operator is given by

$$i\hbar{}_0^{CF}\mathcal{D}_t^{\alpha}\Psi(x,t)=\frac{\hbar^2}{2m}\nabla^2\Psi(x,t)+V(x)\Psi(x,t), \tag{5.52}$$

in which $\hbar$ is the Planck constant, m is mass, and $\psi(x,t)$ is the particle's wave function. We consider that $V(x)=0$ for $|x|<1, V(x)=\infty$ for $|x|\geq 1$ (infinite square well). Now, considering $V(x)=0$, it is possible for us to write

$${}_0^{CF}\mathcal{D}_t^{\alpha}\Psi(x,t)=a\frac{\partial^2\Psi}{\partial x^2}\psi(x,t), \tag{5.53}$$

for which $\Psi(x,0)=\Psi_0(x)$ and $a=i\frac{\hbar}{2m}$.

Upon application of LT to (5.53), the following is obtained:

$$\frac{s\Psi(x,s)-\Psi(x,0)}{s+\alpha(1-s)}=a\frac{\partial^2\Psi}{\partial x^2}\Psi(x,s) \tag{5.54}$$

The relationship between the Fourier transform (FT) operator and the C-F fractional operator was given in [61]. Applying the FT to (5.54), we have

$$\frac{s\Psi(k,s)-\Psi_0(k)}{s+\alpha(1-s)} = ak^2\Psi(x,t), \tag{5.55}$$

where [7]

$$\Psi(k,s) = \frac{\Psi_0(k)}{s+ak^2\big(s+\alpha(1-s)\big)}. \tag{5.56}$$

Now, applying the inverse FT and inverse LT to (5.56), we have

$$\Psi(x,t) = \frac{1}{2\pi i}\overset{st}{\exp}\, d\, s\, \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\overset{-ikx}{exp}\,\Psi_0\,(k)}{s+ak^2\big(s+\alpha(1-s)\big)}\, dk. \tag{5.57}$$

Considering the initial condition to be equal to $\Psi(x,0)=\delta(x)$, equation (5.56) yields [7]

$$\Psi(k,s) = \frac{1}{s\big(1+ak^2(1-\alpha)\big)+a\alpha k^2}, \tag{5.58}$$

applying the inverse Laplace and inverse FT to equation (5.58), we have

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\overset{-ikx}{exp}}{1+ak^2(1-\alpha)}\left(\frac{a\alpha k^2}{1+ak^2(1-\alpha)}\right) t\, dk. \tag{5.59}$$

In the case when $\alpha \rightarrow 1$, we have

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overset{-ikx}{\exp} \quad \exp\,(-ak^2)t\, dk. \tag{5.60}$$

Equation (5.59) describes the Schrödinger equation using the C-F fractional operator.

Case 5.8[7]: Considering ${}^{ABC}_{\ a}D^{a}_{t} f(t) = \frac{B(a)}{1-a}\int_a^t f(\theta)E_a\left[-a\frac{(t-\theta)^a}{1-a}\right]d\theta$, the fractional Schrödinger equation via the fractional operator with M-L kernel understood in the L-C sense is given by [7]

$$i\hbar\, {}^{ABC}_{\ 0}\mathcal{D}^{\alpha}_{t}\Psi(x,t) = -\frac{\hbar^2}{2m}\nabla^2\Psi(x,t) + V(x)\Psi(x,t), \tag{5.61}$$

where we consider that $V(x)=0$ for $|x|<1$, $V(x)=\infty$ for $|x|\geq 1$ (infinite square well). Now, considering V(x) = 0, we have [7]

$${}^{ABC}_{\ 0}\mathcal{D}^{\alpha}_{t}\Psi(x,t) = a\frac{\partial^2\Psi}{\partial x^2}\Psi(x,t), \tag{5.62}$$

where $\Psi(x,0)=\Psi_0(x)$ and $a = i\frac{\hbar}{2m}$.

Upon application of LT (5.62), we have the following [7]:

$$\frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}\Psi(x,s)-s^{\alpha-1}\Psi(x,0)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}=\frac{\partial^{2}\Psi}{\partial x^{2}}\Psi(x,s), \tag{5.63}$$

The relationship between the FT operator and the fractional operator with the M-L kernel understood in the sense of L-C was given in [62]. Upon application of the FT to (5.55), we have the following equation [7]:

$$\frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}\Psi(x,s)-s^{\alpha-1}\Psi(x,0)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}=-ak^{2}\Psi(x,s), \tag{5.64}$$

where

$$\Psi(k,s)=\frac{B(\alpha)}{1-\alpha}\frac{1-\alpha}{B(\alpha)+ak^{2}(1-\alpha)}\frac{s^{\alpha-1}\Psi_{0}(k)}{s^{\alpha}+\frac{a\alpha k^{2}}{B(\alpha)+ak^{2}(1-\alpha)}}. \tag{5.65}$$

Finally, applying the inverse Laplace and inverse FT(5.65) , we have [7]

$$\Psi(x,t)=\frac{B(\alpha)}{2\pi}\int_{-\infty}^{\infty}\frac{\overset{ikx}{exp}}{B(\alpha)+ak^{2}(1-\alpha)} E_{\alpha,1}\left[\left(-\frac{a\alpha k^{2}}{B(\alpha)+ak^{2}(1-\alpha)}\right)t^{\alpha}\right]\Psi_{0}(k)\,dk. \tag{5.66}$$

In the case when $\Psi(x0)=\delta 1$, we have

$$\Psi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overset{-ikx}{\exp}\quad \exp\left(-ak^{2}\right)t\Psi_{0}(k)\,dk, \tag{5.67}$$

where $\Psi(x,0)=\delta(x)$. Equation (5.66) describes the Schrödinger equation using the fractional operator with the M-L kernel understood in the sense of L-C. It should be noted now that this representation is the general case and the representation given in Equation (5.59) is a particular case of this representation.

6. CONCLUSION

During the last few years in the area of fractional calculus some new fractional derivatives were introduced. Among those, there is one-so called-Atangana-Baleanu derivative which is a non-singular fractional operator with a Mittag-Leffler kernel. The main aim of this thesis is to understand the fundamentals of this operator as well as its discrete version.

In this work, I reviewed, understood and applied the properties of the Mittag-Leffler operator in several real-world problems appearing in engineering sciences. I realized through the presented applications that this operator is a good potential candidate to describe better the properties of non-locality which is different than the one described by Riemann-Liouville and Caputo fractional derivatives.

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