NEW TRENDS IN FRACTIONAL OPTIMAL CONTROL PROBLEMS

GADRIAH JAMAAH ALI MADI

APRIL 2018

NEW TRENDS IN FRACTIONAL OPTIMAL CONTROL PROBLEMS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND
APPLIED SCIENCES OF
ÇANKAYA UNIVERSITY

BY
GADRIAH JAMAAH ALI MADI

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF

MASTER OF SCIENCE

IN

THE DEPARTMENT OF

MATHEMATICS

20 APRIL 2018

Title of the Thesis: **New Trends in Fractional Optimal Control Problems**

Submitted by **GADRIAH JAMAAH ALI MADI**

Approval of the Graduate School of Natural and Applied Sciences,Cankaya University

Prof. Dr. Can ÇOĞUN
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Fahd JARAD
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Baleanu

Dr. Instructor Dumitru BALEANU
Supervisor

Examination Date: 20-4-2018

Examining Committee Members

Dr. Instructor Dumitru BALEANU (Cankaya Univ.) Baleanu

Assoc. Prof. Dr. Fahd JARAD (Cankaya Univ.)

Dr. Instructor Tugba AKMAN YILDIZ (UTAA)

STATEMENT OF NON-PLAGARISM PAGE

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name : Gadriah MADI

Signature :

Date : 20-4-2018

ABSTRACT

New Trends in Fractional Optimal Control Problems

Gadriah Jamaah Ali MADI
M.Sc: Department of Mathematics
Supervisor: Dr. Instructor Dumitru BALEANU
Co-Supervisor: Dr. Instructor Özlem DEFTERLI

April 2018, 45 pages

In this thesis, I study the basics of some fractional derivatives (e. g. Riemann-Liouville, Caputo) with the corresponding approximation based on Grünwald-Letnikov definitions. Later, the fundamentals of fractional optimal control problem are presented via mentioned fractional derivatives which are used in the definition of constraints and optimality conditions given through the formulation. Some new aspects are studied for the numerical solutions of fractional optimal control problems in the sense of integrating new orthogonal polynomials to approximate the considered fractional derivatives. In this respect, Bernstein polynomials, shifted Chebyshev polynomials and shifted Legendre orthonormal polynomials are newly used within the Legendre-Gauss quadrature method in order to approximate and solve numerically the Caputo based fractional partial differential equations coming from the formulation of fractional optimal control problem. Two dynamical systems are considered as illustrative examples based on the given control functions and the corresponding responses of the systems are presented under fractional derivatives. Then the comparison with the classical derivative is discussed. It is observed from numerical results and presented simulations that the system response increases as the fractional order of the derivative decreases for the same point of the variable t.

Keywords: Riemann-Liouville fractional derivative, Caputo fractional derivative, Grünwald-Letnikov definition, Bernstein polynomials, Chebyshev polynomials, Legendre polynomials, Legendre-Gauss quadrature.

ÖZ

Kesirli optimal control problemerinde yeni eğilimler

Gadriah Jamaah Ail MADI

Yüksek Lisans,, Matematik

Tez Yöneticisi: Dr. Öğr. Üyesi Dumitru BALEANU

Ortak Tez Yöneticisi: Dr. Öğr. Üyesi Özlem DEFTERLI

Nisan 2018, 45 sayfa

Bu tez çalışmasında, bazı kesirli türevlerin (örn. Riemann-Liouville, Caputo) temelleri Grünwald–Letnikov tanımlarına dayanan ilgili yaklaşığıyla beraber çalışılmıştır. Daha sonra, kesirli optimal control probleminin temelleri, formülasyon da verilen kısıtlamaların ve optimalite koşullarının tanımlanmasında kullanılan bu kesirli türevler aracılığıyla sunulmaktadır. Göz önüne alınan kesirli türevlerin yaklaşığı için yeni orthogonal polinomların integrasyonu anlamında kesirli optimal kontrol problemlerinin sayısal çözümleri için bazı yeni yönelimler üzerinde çalışılmıştır. Bu bağlamda, Bernstein polinomları, kaydırılmış Chebyshev polinomları ve kaydırılmış Legendre ortonormal polinomları Legendre-Gauss kareleme yöntemin icerisinde kesirli optimal control probleminin formülasyonundan gelen Caputo tabanlı kesirli kısmi diferansiyel denklemleri sayısal olarak çözmek için kullanılmaktadır.Verilen control fonksiyonlarına dayanan iki dinamik sistem,açıklayıcı örnekler olarak ele alınmış ve sistemlerin ilişkili gelen karşılığı, kesirli türevler altında sunulmuş, daha sonra klasik türev ile karşılaştırılmıştır. Sayısal sonuçlardan ve verilen simulasyonlardan system cevabının, kesirli türevin derecesi azaldıkça t değişkeninin bazı noktalarında arttığı gözlemlenmiştir.

Anahtar Kelimeler: Riemann-Liouville kesirli türevi, Caputo tanımı, Grünwald – Letnikov kesirli türevi , dik polinomlar , Legendre-Gauss kareleme yontemi.

ACKNOWLEDGEMENTS

First of all, I am thankful to Allah for all gifts given to me. I shall also thank my government because for the scholarship provided for my graduate studies. I would like to thank my supervisor Dr. Instructor Dumitru Baleanu for his guidance, encouragement and endless patience during this thesis. I also would like to express appreciation to Co-supervisor Dr. Instructor Özlem Defterli. This thesis is dedicated to my mother and father. Iwould like to extend my deepest thanks to them because of every thing they haved one for me through out my whole life. Finally, I would also like to say thanks for all my family.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

Fractional calculus deals with the study of so-called fractional integrals and differential operators over real or complex domains and their application [1-20]. Over the past few decades, fractional differentiation have attracted more and more attention in the study of so-called anomalous social and physical behaviors, where the fractional scale power law seems to be commonly used as an empirical description of this complex phenomenon [21-43]. It is worth noting that the standard mathematical model with the integer-order derivatives, including the nonlinear model, does not work well in many cases where the power law is obviously observed [1-30]. In order to accurately reflect the non-local, frequency, and historical correlation properties of the power law phenomenon, several alternative modeling tools must be introduced into this fractional calculus. The study of fractional differentiation is essentially multidisciplinary, and its application is dispersed in various disciplines. As one would expect, since the fractional derivative is a generalization of ordinary derivatives, it loses many of its basic properties. For example, it loses its geometric or physical interpretation. The indexing method works only when working in a very specific function space. The derivative of the product of the two functions is difficult to obtain, and the chain rules are not suitable for direct application. Fractional –order differential equations (i.e., equations involving real or complex derivatives) play an important role in simulating the anomalous dynamic processes of many processes associated with complex systems in the most diverse areas of science and engineering. The thesis consists of six chapters. Chapter 1 is the introduction part. Chapter 2 include some basic definitions identified with fractional derivatives. In chapter 3, the principles of fractional optimal control problems are given. In chapter 4, the descriptions of some use full orthonormal polynomials are presented. Chapter 5, is about the useful Chebyshev polynomials to find the numerical solutions of fractional optimal control problems. Chapter 6 is about Legendre polynomials to be used for the same purpose.

CHAPTER 2
BASIC TOOLS

2.1 The Definitions of Riemann-Liouville and Caputo Fractional Derivatives

In this section we present some essential definitions identified with fractional derivatives. The left Riemann-Liouville fractional integral and the right Riemann-Liouville fractional-order integral are characterized separately by $[1-5]$

$$ {}_aI^{\beta} f(s) = \frac{1}{\Gamma(\beta)} \int_a^s (s-\tau)^{\beta-1} f(\tau) d\tau, \tag{2.1} $$

$$ I_b^{\beta} f(s) = \frac{1}{\Gamma(\beta)} \int_s^b (\tau-s)^{\beta-1} f(\tau) d\tau, \tag{2.2} $$

where $\beta > 0$,$m-1 < \beta < m$ and $\Gamma(\beta)$ represents the Gamma function.

The left Riemann-Liouville fractional derivative is defined as $[1-5]$

$$ {}_aD^{\beta} f(s) = \frac{1}{\Gamma(m-\beta)} \left(\frac{d}{ds}\right)^m \int_a^s (s-\tau)^{m-\beta-1} f(\tau)\, d\tau. \tag{2.3} $$

The right Riemann-Liouville fractional derivative is defined by $[1-5]$

$$ D_b^{\beta} f(s) = \frac{1}{\Gamma(m-\beta)} \left(-\frac{d}{ds}\right)^m \int_s^b (\tau-s)^{m-\beta-1} f(\tau) d\tau. \tag{2.4} $$

The Riemann-Liouville fractional derivatives of a constant can be calculated as $[1-5]$

$$ {}_aD^{\beta} C = C \frac{(s-a)^{-\beta}}{\Gamma(1-\beta)}\,, \tag{2.5} $$

where as the fractional derivative of a power function takes the form $[1-5]$

$$ {}_aD^{\beta} (s-a)^{\alpha} = \frac{\Gamma(\beta+1)(s-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \tag{2.6} $$

for $\alpha > -1, \beta \geq 0$.

The left Caputo fractional derivative is defined as $[1-5]$

$$ {}_{a}^{C}D^{\beta}f(s)=\frac{1}{\Gamma(m-\beta)}\int_{a}^{s}(s-\tau)^{m-\beta-1}\left(\frac{d}{d\tau}\right)^{m}f(\tau)d\tau, \tag{2.7}$$

and the Right Caputo fractional derivative has the form $[1-5]$

$$ {}^{c}D_{b}^{\beta}f(s)=\frac{1}{\Gamma(m-\beta)}\int_{a}^{b}(\tau-s)^{m-\beta-1}\left(-\frac{d}{d\tau}\right)^{m}f(\tau)d\tau, \tag{2.8}$$

where β represents the order of the derivative such that $m-1<\beta<m$. Note that the Caputo partial derivative of a constant function is zero $[1-5]$.

The Riemann-Liouville and Caputo fractional derivatives are related each other by $[1-5]$

$$ {}_{a}^{C}D^{\beta}f(s)={}_{a}D^{\beta}f(s)-\sum_{j=0}^{m-1}\frac{f^{(j)}(a)}{\Gamma(j-\beta+1)}(s-a)^{j-\beta}, \tag{2.9}$$

$$ {}^{c}D_{b}^{\beta}f(s)=D_{b}^{\beta}f(s)-\sum_{j=0}^{m-1}\frac{(-1)^{j}f^{(j)}(b)}{\Gamma(j-\beta+1)}(b-s)^{j-\beta}. \tag{2.10}$$

The formulation in below gives a formula [1] for the fractional integration by parts in [a, b].

Lemma 2.1.1 [1-5]

Let $\beta>0$, w, z≥ 1, and $\frac{1}{w}+\frac{1}{z}\leq 1+\beta$($w\neq 1$ and $z\neq 1$ in this case when $\frac{1}{w}+\frac{1}{z}=1+\beta$).

if $\eta\in L_{w}(a,b)$ and $\chi\in L_{z}(a,b)$, then

$$\int_{a}^{b}\eta(s)({}_{a}I^{\beta}\chi)(s)ds=\int_{a}^{b}\chi(s)(I_{b}^{\alpha}\eta)(s)ds \tag{2.11}$$

if $e\in I_{b}^{\beta}(L_{w})$ and $f\in{}_{a}I^{\beta}(L_{z})$, then

$$\int_{a}^{b}e(s)({}_{a}D^{\beta}f)(s)ds=\int_{a}^{b}f(s)\left(D_{b}^{\beta}e\right)(s)ds, \tag{2.12}$$

where ${}_{a}I^{\beta}(L_{z}):=\{f: f={}_{a}I^{\beta}e, e\in L_{w}(a,b)\}$ and

$I_b^\beta(L_w) \coloneqq \left\{f : f = I_b^\beta e, e \in L_w(a,b)\right\}.$

In [6] and [7] the fractional integration by parts on the subintervals [a, r] and [r, b] are given by the next lemmas.

Lemma 2.1.2 [1-7]

Let $\beta > 0, w,\ z \geq 1, r \in (a,b)$ and $\frac{1}{w} + \frac{1}{z} \leq 1 + \beta$,$w \neq 1$ and $z \neq 1$ in the case when $\frac{1}{w} + \frac{1}{z} = 1 + \beta$.

(a) If $\eta \in L_w(a,b)$ and $\chi \in L_w(a,b)$, then

$$\int_a^r \eta(s)({}_aD^\beta \chi)(s)\, ds = \int_a^r \chi(s)\left(I_r^\beta \eta\right)(s)\, ds. \tag{2.13}$$

So if e $\in I_r^\beta(L_w)$ and $f \in_a I^\beta(L_z)$, then

$$\int_a^r e(s)({}_aD^\beta f)(s)\, ds = \int_a^r f(s)\left(D_r^\beta e\right)(s)\, ds. \tag{2.14}$$

(b) If $\eta \in L_w(a,b)$ and $\chi \in L_z(a,b)$, then

$$\int_r^b \eta(s)({}_aI^\beta \chi)(s)\, ds = \int_r^b \chi(s)\left(I_b^\beta \eta\right)(s)\, ds +$$

$$\frac{1}{\Gamma(\beta)}\int_a^r \chi(s)\left(\int_r^b \eta(t)(t-s)^{\beta-1} dt\right) ds\ . \tag{2.15}$$

If e $\in I_b^\beta(L_w)$ and $f \in aI^\beta(L_z)$, then

$$\int_r^b e(s)({}_aD^\beta f)(s)\, ds = \int_r^b f(s)\left(D_b^\beta e\right)(s) ds -$$

$$\frac{1}{\Gamma(\beta)}\int_a^r \left(aD^\beta f\right)(s)\left(\int_r^b \left(D_b^\beta e\right)(t)(t-s)^{\beta-1} dt\right) ds. \tag{2.16}$$

so

$$\int_r^b e(s)({}_aD^\beta f)(s)\, ds = \int_r^b f(s)\left(D_b^\beta e\right)(s) ds -$$

$$\frac{1}{\Gamma(\beta)}\int_a^r f(s) D_r^\beta \left(\int_r^b \left(D_b^\beta e\right)(t)(t-s)^{\beta-1}\, dt\right) ds. \tag{2.17}$$

Lemma 2.1.3 [1-7]

Let $\beta > 0, w, z \geq 1, r \in (a,b)$ and $\frac{1}{w} + \frac{1}{z} \leq 1 + \beta, w \neq 1\ \ z \neq 1$ and $\frac{1}{w} + \frac{1}{z} = 1 + \beta$ If $\eta \in L_w(a,b)$ and $\chi \in L_z(a,b)$, then

$$\int_r^b \eta(s)\left(I_b^\beta \chi\right)(s)\,ds = \int_r^b \chi(s)({}_rI^\beta \chi)(s)\,ds (2.18)$$

and thus if $e \in {}_rI^\beta(L_w)$ and $f \in I_b^\beta(L_z)$, then

$$\int_r^b e(s)\left(D_b^\beta f\right)(s)\,ds = \int_r^b f(s)({}_rD^\beta e)(s)\,ds \tag{2.19}$$

(a) If $\eta \in L_w(a,b)$ and $\chi \in L_z(a,b)$, then

$$\int_a^r \eta(s)({}_bI^\beta \chi)(s)\,ds = \int_a^r \chi(s)\left(I_a^\beta \eta\right)(s)\,ds +$$

$$\frac{1}{\Gamma(\beta)}\int_r^b \chi(s)\left(\int_a^r \eta(t)(s-t)^{\beta-1}\,dt\right)ds. \tag{2.20}$$

and hence if $e \in {}_aI^\beta(L_w)$ and $f \in I_b^\beta(L_z)$, then

$$\int_a^r e(s)\left(D_b^\beta f\right)(s)\,ds = \int_a^r f(s)({}_aD^\beta e)(s)ds -$$

$$\frac{1}{\Gamma(\beta)}\int_r^b \left(D_b^\beta f\right)(s)\left(\int_a^r ({}_aD^\beta e)(t)(s-t)^{\beta-1}\,dt\right)ds. \tag{2.21}$$

So, we have

$$\int_a^r e(s)\left(D_b^\beta f\right)(s)\,ds = \int_a^r f(s)\left({}_aD^\beta e\right)(s)ds -$$

$$\frac{1}{\Gamma(\alpha)}\int_r^b f(s)\ {}_rD^\beta\left(\int_a^r \left(aD^\beta e\right)(s)(s-t)^{\beta-1}\,dt\right)ds. \tag{2.22}$$

2.2 The Grünwald-Letnikov Approximation

The Grünwald-Letnikov approximation of fractional derivatives defined as $[1-14]$

$${}_0D_s^\alpha y\left(s_{l-1/2}\right) \cong \frac{1}{h^\alpha}\sum_{j=0}^{l} w_j^{(\alpha)} y_{l-j}\,, \quad l = 1,\dots,m, \tag{2.23}$$

$${}_s\mathrm{D}_1^\alpha r\left(s_{l+1/2}\right) \cong \frac{1}{h^\alpha}\sum_{j=0}^{m-l} w_j^{(\alpha)} r_{l+j}\,, \qquad l = m-1, m-2, \dots\dots, 0, \tag{2.24}$$

where $w_j^{(\alpha)}$, j= 0, 1,..,m are the transactions. A recursive method for computing $w_j^{(\alpha)}$ is given by

$$w_0^{(\alpha)} = 1, \quad w_j^{(\alpha)} = \left(1 - \frac{\alpha+1}{j}\right) w_{j-1}^{(\alpha)}, \qquad j = 1,\dots,m.$$

It could be shown that for α = 1, Eqs. (2.23) and (2.24) became [1-14].

$$\frac{dy(s_{l-1/2})}{ds} = \frac{y_l - y_{l-1}}{h}, \qquad \frac{-dy(s_{l+1/2})}{ds} = \frac{y_l - y_{l+1}}{h}.$$

To build up a numerical technique the time space [0, 1] is divided into n equal parts and the fractional derivatives ${}_0D_s^{\alpha}y$ and ${}_sD_1^{\alpha}\lambda$ are approximated at the midpoint from every subinterval through these approximations.

Additionally take y ($s_{l-1/2}$) as an average of the two end values of the subinterval. Therefore, $y(s_{l-1/2}) = (y_{l-1} + y_i)/2$. It makes the same approximations for $\lambda(s_{l-1/2})$, y ($s_{l-1/2}$), and $\lambda(s_{l+1/2})$. Substituting these approximations into Euler – Lagrange equations (${}_0D_s^{\alpha}y = a(s)y - r^{-1}(s)b^2(s)\lambda, {}_sD_1^{\alpha}\lambda = q(s)y + a(s)\lambda$) it gets [1-14]

$$\frac{1}{h^{\alpha}}\sum_{j=0}^{l} w_j^{(\alpha)} y_{l-j} = \frac{1}{2}a(l_1h)(y_{l-1} + y_l) - \frac{1}{2}r^{-1}(l_1h)b^2(l_1h)(\lambda_{l-1} + \lambda_l)$$

(l=1,…,m), (2.25)

$$\frac{1}{h^{\alpha}}\sum_{j=0}^{m-l} w_j^{(\alpha)} \lambda_{l+j} = \frac{1}{2}q(l_2h)(y_{l+1} + y_l) + \frac{1}{2}a(l_2h)(\lambda_{l-1} + \lambda_l),$$

(l= m-1,…, 0). (2.26)

Here $l_1 = l - \frac{1}{2}, l_2 = l + \frac{1}{2}$ equations (2.25) and (2.26) produce linear equations in 2m unknowns. One can also develop an iterative plan in that one can march forward to compute y_i's and in backward to compute λ_i's to save storage space and perhaps computational time $[1 - 14]$.

CHAPTER 3

THE FUNDAMENTALS OF FRACTIONAL OPTIMAL CONTROL

3.1 The Fractional Euler-Lagrange Equations

In this section, we briefly study the fractional Euler–Lagrange equations (FELE) introduced in [10]. Among all functions x(s), the function $x^*(s)$ minimizes the functional

$$J[x] = \int_a^b F\left(s, x, {}_a D_s^\beta x\right) ds \tag{3.1}$$

and satisfies the boundary conditions [10]

$$x(a) = x_a \text{ and } x(b) = x_b. \tag{3.2}$$

At this point ${}_a D_s^\beta x$ be the β order left Riemann-Liouville fractional derivative which can be defined as [10]

$${}_a D_s^\beta x(s) = \frac{1}{\Gamma(m-\beta)} \left(\frac{d}{ds}\right)^m \int_a^s (s-\vartheta)^{m-\beta-1} x(\vartheta)\, d\vartheta, \tag{3.3}$$

where m-1$< \beta < m$ .The fractional derivative reaches to an ordinary derivative in case β be an integer. It is exhibited in [10] that the solution of the problem which was stated above must satisfy

$$\frac{\partial F}{\partial x} + {}_s D_b^\beta \frac{\partial F}{\partial_a D_s^\beta x} = 0, \tag{3.4}$$

where ${}_s D_b^\beta x$ is the right Riemann –Liouville fractional derivative (RRLFD) of order β define as [14]

$${}_s D_b^\beta x(s) = \frac{1}{\Gamma(m-\beta)} \left(-\frac{d}{ds}\right)^m \int_a^b (\vartheta - s)^{m-\beta-1} x(\vartheta)\, d\vartheta. \tag{3.5}$$

Equation (3.5) is going to played an impotant role in the formulation for fractional control optimal problems [14].

3.2 The Formulation of Fractional Optimal Control Problems

The problem here is to find the optimal control r(s) for a fractional derivative schemes that minimizes the performance index [14]

$$f(r) = \int_0^1 j(y, r, s)ds \tag{3.6}$$

and also, fulfills the dynamical constraints

$${}_0D_s^\alpha y = g(y, r, s) \tag{3.7}$$

with the initial condition

$$y(0) = y_0 \,. \tag{3.8}$$

Here y(s) is the state variable, s is for the time, j and g stand for arbitrary functions. Not that order of the fractional derivative $0 < \alpha < 1$ and the upper limit of integration is 1. Additionally; it is consider that y(s), r(s) and g(y, r, s) are all scalar functions. The similar technique could be considered if the upper limit from integration and α is bigger than 1, y(s), r(s) and g(y, r, s) are vector functions.

The performance index is modified as follows to get the optimal control [14]

$$\bar{f}(r) = \int_0^1 [H(y, r, s) - \lambda_0 D_s^\alpha y]\, ds, \tag{3.9}$$

where H (y, r, λ, s) is the Hamiltonian of the system defined via [14]

$$H(y, r, \lambda, s) = j(y, r, s) + \lambda g(y, r, s). \tag{3.10}$$

Also, λ represents the Lagrange multiplier. Applying the necessary conditions in terms of Hamiltonian Eq. (3.4) result that [14]

$${}_sD_1^\alpha \lambda = \frac{\partial H}{\partial y}, \tag{3.11}$$

$$\frac{\partial H}{\partial r} = 0, \tag{3.12}$$

$${}_0D_s^\alpha y = \frac{\partial H}{\partial \lambda}. \tag{3.13}$$

According to the approach given in [11], the following condition is imposed

$$\lambda(1) = 0. \tag{3.14}$$

Here, the time derivative of the Hamiltonian does not zero along the optimal trajectory even when j and g are not directly depend on s, which is a derivative of the integer order optimal control theory. The following quadratic performance index can be considered [14] as an example

$$f(r)=\frac{1}{2}\int_0^1 [q(s)y^2(s) + u(s)r^2]\, ds, \tag{3.15}$$

where q(s) ≥ 0 and u(s) > 0, and the system whose dynamics is described by the linear FDE

$$_0D_s^{\alpha} y = a(s)y + b(s)r. \tag{3.16}$$

Utilizing Eqs. (3.11) to (3.13), the essential Euler-Lagrange equations for the same system obtained as [11]

$$_0D_s^{\alpha} y = a(s)y - u^{-1}(s)b^2(s)\lambda, \tag{3.17}$$

$$_sD_1^{\alpha} \lambda = q(s)y + a(s)\lambda, \tag{3.18}$$

and

$$r = -u^{-1}(s)b(s)\lambda, \tag{3.19}$$

where they are all going to be used to build up a direct numerical method for a fractional optimal control problem.

3.2.1 An Example in 2D

Considering the minimization problem of the performance index [14]

$$J = \frac{1}{2}\int_0^2 [{}_0D_s^{\alpha}\, {}_0D_s^{\alpha}\theta]^2 ds, \tag{3.20}$$

subjected to the next dynamic condition, ${}_0\, D_s^{\alpha}\, {}_0D_s^{\alpha}\theta(s)$= r(s). Taking a= 0 and b= 2, and ${}_0D_s^{\alpha}\, {}_0D_s^{\alpha}\theta(s)$ is the successive derivative of θ. Using $\theta(s)$ =$y_1(s)$,

${}_0D_s^{\alpha}\theta(s) = y_2(s)$.

The modified performance index in (3.20) becomes [14]

$$J = \int_0^2 [H(y, r, \lambda) - \lambda^T\, {}_0D_s^{\alpha} u(s)]\, ds, \tag{3.21}$$

where

$$H(y,r,\lambda) = \frac{1}{2} r^2(s) + \lambda^T \big(Ay(s) + br(s)\big) \tag{3.22}$$

is the Hamiltonian of the system and [14]

$$y(s) = \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix}, \qquad \lambda(s) = \begin{pmatrix} \lambda_1(s) \\ \lambda_2(s) \end{pmatrix},$$

$$b(s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.23}$$

Applying (3.7)-(3.9) the following system of equations are obtained [14]

$$_sD_2^\alpha \lambda_1 = 0 \,,_s D_2^\alpha \lambda_2 - \lambda_1 = 0, r + \lambda_2 = 0 \,,_0 D_s^\alpha u_1 - u_2 = 0,_0 D_s^\alpha u_2 - r = 0. \tag{3.24}$$

Take note of that variable r from the above equations could be eliminated by the third equality introduction in (3.24). In addition to the above equations,

$\theta(0) =_0 D_s^\alpha \theta(0) = 0$ and $\theta(2) =_0 D_s^\alpha \theta(2) = 1$

which translate into $y_1(0) = y_2(0) = 0$ and $y_1(2) = y_2(2) = 1$.

The presentation of the numerical strategy which is utilized to solve the cores pending equations in (3.24) is shown below. This technique utilizes Gr$\ddot{u}$nwald-Letnikov approximation, quickly, the schemes is given as [14]:

(1) Split the time area into N sub-spaces, where N is an integer.
(2) Estimate the fractional derivatives in (3.24) at every grid point the Grünwald- Letnikov definitions given in (3.11).
(3) Impose the terminal conditions.
(4) Solve the concluding equations.

CHAPTER 4

BERNSTEIN AND SHIFTED LEGENDRE ORTHONORMAL POLYNOMIALS

4.1 Bernstein Polynomials and Their Properties

The Bernstein polynomials (BPs) of z^th-degree are characterized on the interval $[0,1]$ as $[24-25]$

$$A_{i,z}(u) = \binom{z}{i} u^i (1-u)^{z-i}, i = 0,1, \dots, z. \quad (4.1)$$

Corollary 4.1.1 [24-25]

The set $\{A_{0,z}(u), A_{1,z}(u), \dots, A_{z,z}(u)\}$ is a full basis in the Hilbert space $L^2[0, 1]$ and polynomials from degree z; they are expanded by the linear combination from $A_{i,z}(u)(i = 0,1, \dots, z)$ as:

$$p(u) = \sum_{i=0}^{z} c_i A_{i,z}(u). \quad (4.2)$$

Lemma 4.1.2 [24-25]

We can write $\phi_z(u) = BT_z(u)$ such that B is an upper triangular matrix $T_z(u) = [1, u, \dots, u^z]^T and\ \phi_z(u) = [A_0(u), A_2(u), \dots, A_z(u)]^T$.

Proof $[24-25]$ Using binomial increase of $(1-u)^{z-i}$ we

$$A_{i,z}(u) = \binom{z}{i} u^i (1-u)^{z-i}$$

$$= \binom{z}{i} u^i \left(\sum_{j=0}^{z-i} (-1)^j \binom{z}{i} \binom{z-i}{j} u^j \right) = \sum_{j=0}^{z-i} (-1)^j \binom{z}{i} \binom{z-i}{j} u^{i+j}, i = 0, \dots, z$$

In this way can be writing $[24-25]$

$$\phi_z(u) = BT_z(u), \quad (4.3)$$

where $[24-25]$

$B = \left(a_{i,p}\right)_{i,p=1}^{z+1}$ and $a_{i+1,p+1}$

$$= \begin{cases} (-1)^{p-i}\binom{z}{i}\binom{z-i}{p-i} & i \le p \\ 0 & i > p, \end{cases} \quad i,p = 0,1,..,z.$$

Lemma 4.1.3 [26]

Let $L^2[0,1]$ be a Hilbert space with the inside product $< k, r >=$ $\int_0^1 k(u)r(u)\,dy,$ and $x \in L^2[0,1]$. After that, the single vector c=$[c_1, c_2, \dots, c_z]^T$can be taken such that

$$x(u) \approx \sum_{i=0}^{z} c_i A_{i,z}(u) = c^T \phi_z(u). \tag{4.4}$$

For Lemma 4.1.3 gets $c^T =< k, \phi_z > \varrho^{-1}$, such that

$$< k, \phi_z > \int_0^1 k(u)\phi_z(u)^T\,du = [< k, A_{0,z} >, < k, A_{1,z} >, \dots, < k, A_{z,z} >]$$

and

$\varrho = (\varrho_{i,p})_{i,p=1}^{z+1}$

as follows

$$\varrho_{i+1,p+1} = \int_0^1 A_{i,z}(u)A_{p,z}(u)\,du = \frac{\binom{z}{i}\binom{z}{p}}{(2z+1)\binom{2z}{i+p}}, i,p = \ \ 0,1,\dots,z. \tag{4.5}$$

Lemma 4.1.5 [26]

Assume $C_{(z+1)\times 1}$ is a random vector. The operational matrix of the product $C^{\wedge}{}_{(z+1)\times(z+1)}$using BPs can give the following:

$$c^T\phi_z(u)\phi_z(u)^T \approx \phi_z(u)^T c. \tag{4.6}$$

Proof [26]

By equation (4.6) we have

$c^T\phi_z(u)\phi_z(u)^T = c^T\phi_z(u)(T_z(u)^T B^T)$

$= [c^T\phi_z(u), u(c^T\phi_z(u)), \dots, u^z(c^T\phi_z(u))]B^T$

$$= \left[\sum_{i=0}^{z} c_i A_{i,z}(u), \sum_{i=0}^{z} c_i u A_{i,z}(u), \dots, \sum_{i=0}^{z} c_i u^z A_{i,z}(u)\right] B.^T$$

This time, $u^j A_{i,z}(u)$ is used to approximate all functions $\phi_z(u)$. So we characterize $e_{j,i} = [e_{j,i}^0, e_{j,i}^1, \dots, e_{j,i}^z]^T$, and by equation (4.46) it is written as:

$u^j A_{i,Z}(u) \approx e_{j,i}\phi_z(u), i, j = 0,1, \dots\dots z$ [26].

So,

$$e_{j,i} = \varrho^{-1}\left(\int_0^1 u^j A_{i,z}(u)\phi_z(u)du\right)$$

$$= \varrho^{-1}\left[\int_0^1 u^j A_{i,z}(u)A_{0,z}(u)du, \times \int_0^1 u^j A_{i,z}(u)A_{1,z}(u)du, \dots, \right.$$

$$\left. \times \int_0^1 u^j A_{i,z}(u)A_{z,z}(u)du\right]^T$$

$$= \frac{\varrho^{-1}\binom{z}{i}}{2z+j+1}\left[\frac{\binom{z}{0}}{\binom{2z+j}{i+j}}, \frac{\binom{z}{1}}{\binom{2z+j}{i+j+1}}, \dots, \frac{\binom{z}{z}}{\binom{2z+j}{i+j+z}}\right]^T i, j = 0,1, \dots z.$$

We have [26]

$$\sum_{i=0}^{z} c_i u^j A_{i,z}(u) \approx \sum_{i=0}^{z} c_i\left(\sum_{p=0}^{z} e_{j,i}^{p} A_{p,z}(u)\right)$$

$$= \sum_{i=0}^{z} A_{p,z}(u)\left(\sum_{i=0}^{z} c_i e_{j,i}^{p}\right)$$

$$= \phi_z(u)^T\left[\sum_{i=0}^{z} c_i e_{j,i}^{0}, \sum_{i=0}^{z} c_i e_{j,i}^{1}, \dots., \sum_{i=0}^{z} c_i e_{j,i}^{z}\right]^T$$

$$= \phi_z(u)^T\left[e_{j,0}, e_{j,1}, \dots, e_{j,z}\right]c = \phi_z(u)^T V_{j+1}c \quad ,$$

where $V_{j+1}(j = 0,1, \dots, z)$ is matrix of (z+1) × (z+1) and each column has vector $e_{j,i}(i = 0,1, \dots, z)$. If it defines $\bar{C} = [V_1 c \quad , V_2 c \quad , \dots, V_z zc \quad]$, then get

$$C^T\phi_z(u)\phi_z(u)^T \approx \phi_z(u)^T\bar{C}B^T. \tag{4.7}$$

Then it obtains the operational matrix of product $\hat{C} = \bar{C}B^T$.

Corollary 4.1.6 [26]

Let $y(s) \approx c^T\phi_z(s)$, $u(s) \approx d^T\phi_z(s)$,$y(s) \approx d^T\phi_z(s)$ and $\hat{C}_{(z+1)\times(z+1)}$ be the operational matrix of the product using BPs for vector c. it is obtained that the approximate function for u(s) y(s) using BPs as :

$$y(s)u(s) \approx \phi_z(s)^T\hat{C}d. \tag{4.8}$$

Proof [26] it can be proved by using Lemma 4.1.5.

Corollary 4.1.7

Assume that $y(s) \approx c^T\phi_z(s)$ and $\hat{C}_{(z+1)\times(z+1)}$ are the operational matrix of a vector c using the product of BPs. Then can get an approximate function of $y^j(s)(j \in N)$, using the BPs as [26]

$$y^j(s) \approx \phi_z(s)^T\tilde{C}_j, \tag{4.9}$$

where $\tilde{C}_j = \hat{C}^{j-1}c$.

Proof [26]

By using induction, get an approximate value of $y^j(s), (j \in N)$ as:

for j=1 by (4.4) $y(s) \approx c^T\phi_z(s)$ and, for j=2 by Lemma 4.1.5 it gets

$y^2(s) \approx c^T\phi_z(s)\phi_z(s)^T \approx \phi_z(s)^T\hat{C}c$.

For j=3 it is obtained $y^3(s) \approx c^T\phi_z(s)\phi_z(s)^T C^{\wedge} c \approx \phi_z(s)^T\hat{C}^2 c$.

So, by induction write it as [26]

$y^j(s) \approx c^T\phi_z(s)\phi_z(s)^T\hat{C}^{j-2} \approx \phi_z(s)^T\tilde{C}_j$, where $\tilde{C}_j = \hat{C}^{j-1}c$.

4.2 Shifted Legendre Orthonormal Polynomials

The Legendre polynomial of degree j is noted by $P_j(w)$ and it is characterized on the interval (–1, 1). $P_j(w)$ might be created by the repeating the formula $[27-30]$

$$P_{j+1}(w) = \frac{2j+1}{j+1} \, {}_wP_j(w) - \frac{j}{j+1}P_{j-1}(w) \qquad 1 \le j,$$

$$P_0(w) = 1, \qquad P_1(w) = w.$$

Presenting w= 2s −1, the Legendre polynomials are characterized on the interval$(0,1)$ which might be called shifted Legendre polynomials $P_j^*(s)$ and are produced by utilizing the iterative formulae $[27-30]$

$$P_{j+1}^*(s) = \frac{2j+1}{j+1}(2s-1)P_j^*(s) - \frac{j}{j+1}P_{j-1}^*(s), \quad 1 \le j,$$

$$P_0^*(s) = 1, \qquad P_1^*(s) = 2s-1.$$

The orthogonality relation is $[27-30]$

$$\int_0^1 P_p^*(s)ds = \begin{cases} \dfrac{1}{2j+1} \text{for} & \text{p = j,} \\ 0 \quad \text{for} & \text{p} \neq \text{j.} \end{cases} \tag{4.10}$$

The analytical type of shifted Legendre polynomial $P_j^*(s)$ of degree j might be composed as $[27-30]$ for

$$\int_0^1 P_p^*(s) P_j^*(s) ds = \begin{cases} \dfrac{1}{2j+1} & \text{for } \ p = j, \\ 0 & \text{for } \ \mathrm{p} \neq \mathrm{j}\,. \end{cases} \tag{4.11}$$

Presenting the shifted Legendre orthonormal polynomials for $P_j^{\dot{a}}(s); P_j^{\dot{a}}(s) \equiv \sqrt{2j+1} P_j^*(s)$ we have $[27-30]$

$$\int_0^1 P_p^{\dot{a}}(s) P_p^{\dot{a}}(s) ds = \begin{cases} 1 & \text{for} \quad \mathrm{p} = \mathrm{j}, \\ 0 & \text{for} \quad \mathrm{p} \neq \mathrm{j}, \end{cases} \tag{4.12}$$

and

$$P_j^{\dot{a}}(s) = \sqrt{2j+1} \sum_{i=0}^{j} (-1)^{j+i} \frac{(j+i)!}{(j-i)!\,(i!)^2} s^i. \tag{4.13}$$

Any squared integrable function x_j characterized on the interival (0, 1), can be represented by a shifted Legendre orthonormal polynomials $P_j^{\dot{a}}(s)$ as$[27-30]$

$$x(s) = \sum_{j=0}^{\infty} x_j P_j^{\dot{a}}(s).$$

From that the coefficients, x_j are given by

$$x_j = \int_0^1 x(s) P_j^{\dot{a}}(s)\, ds, \quad 0 \leq j, \tag{4.14}$$

if $x(s)$ approximates by firstly (H+1) –terms ,it can write $[27-30]$

$$x_H(s) = \sum_{j=0}^{H} x_j P_j^{\dot{a}}(s). \tag{4.15}$$

which alternatively can be written in the matrix form$[27-30]$:

$$x_H(s) = X^T \Delta_H(s) \tag{4.16}$$

with

$$X = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \Delta_H(s) \begin{bmatrix} P_0^{\dot{a}}(s) \\ P_1^{\dot{a}}(s) \\ \vdots \\ P_N^{\dot{a}}(s) \end{bmatrix}. \tag{4.17}$$

4.3 Shifted Jacobi Orthonormal Polynomials

The Jacobi polynomials of degree j, indicated by $P_j^{(\zeta,\rho)}(w); \zeta \geq -1, \rho \geq -1$ characterized on the interval $[-1,1]$, constitutes an orthogonal system regarding the weight function $[31-38]$

$z^{(\zeta,\rho)}(w) = (1-w)^{\zeta}(1+w)^{\rho}, i, e,$

$$\int_{-1}^{1} P_j^{(\zeta,\rho)}(w) P_h^{(\zeta,\rho)}(w) z^{(\zeta,\rho)}(w)\, dw = \delta_{ph}\varphi_h^{(\zeta,\rho)}, \tag{4.18}$$

where δ_{jh} is the Kronecker function and $[31-38]$

$$\varphi_h^{(\zeta,\rho)} = \frac{2^{\zeta+\rho+1}\Gamma(h+\zeta+1)\Omega(h+\rho+1)}{(2h+\zeta+\rho+1)h!\,\Gamma(h+\zeta+\rho+1)}.$$

A shifted Jacobi polynomial with degree j, that is represented by $P_{T,j}^{(\zeta,\rho)}(s); \zeta \geq -1, \rho \geq -1$ and characterized at interval [0, T]. It is created by the change of variable w= $\frac{2s}{T} - 1, i.e., P_j^{(\zeta,\rho)}\left(\frac{2s}{T} - 1\right) \equiv P_{T,j}^{(\zeta,\rho)}(s)$. At that point the shifted Jacobi polynomials form an orthogonal system with respect to weight function $w_T^{(\zeta,\rho)}(s) = s^{\rho}(T-s)^{\zeta}$ with the orthogonality property $[31-37]$

$$\int_{0}^{T} P_{T,p}^{(\zeta,\rho)}(s) P_{T,h}^{(\zeta,\rho)}(s) w_T^{(\zeta,\rho)}(s)\, ds = L_{T,h}^{(\zeta,\rho)}\delta_{jh,} \tag{4.19}$$

where $[31-37]$

$$L_{T,h}^{(\zeta,\rho)} = \left(\frac{T}{2}\right)^{\zeta+\rho+1} \varphi_j^{(\zeta,\rho)} = \frac{T^{\zeta+\rho+1}\Gamma(h+\zeta+1)\Gamma(h+\rho+1)}{(2h+\zeta+\rho+1)h!\,\Gamma(h+\zeta+\rho+1)}.$$

Presenting the shifted Jacobi orthonormal polynomials $P_{T,h}^{(\zeta,\rho)}(s)$, where $[31-37]$

$$\grave{P}_{T,h}^{(\zeta,\rho)}(s) = \frac{1}{\sqrt{L_{T,h}^{(\zeta,\rho)}}} P_{T,h}^{(\zeta,\rho)}(s). \tag{4.20}$$

Thus, it has

$$\int_{0}^{T} \grave{P}_{T,j}^{(\zeta,\rho)}(s) \grave{P}_{T,h}^{(\zeta,\rho)}(s) z_T^{(\zeta,\rho)}(s)\, ds = \delta_{jh}. \tag{4.21}$$

The shifted Jacobi orthonormal polynomials are built up from the three-term iterative formula $[31-37]$

$$\grave{P}_{T,j+1}^{(\zeta,\rho)}(s) = \left(\mu_j s - \xi_j\right)\grave{P}_{T,j}^{(\zeta,\rho)}(s) - \xi_j \grave{P}_{T,j-1}^{(\zeta,\rho)}(s), \;\; j \geq 1, \tag{4.22}$$

with

$$\grave{P}_{T,0}^{(\zeta,\rho)}(s) = \frac{\sqrt{\Gamma(\zeta+\rho+2)}}{\sqrt{T^{\zeta+\rho+1}\Gamma(\zeta+1)\Gamma(\rho+1)}},$$

$$\grave{P}_{T,1}^{(\zeta,\rho)}(s) = \frac{\sqrt{(\zeta+\rho+3)\Gamma(\zeta+\rho+2)}}{\sqrt{T^{\zeta+\rho+3}\Gamma(\zeta+2)\Gamma(\rho+2)}}\big((\zeta+\rho+2)s - T(\rho+1)\big), \tag{4.23}$$

where $[31-37]$

$$\mu_j = \frac{(2j+\zeta+\rho+2)\sqrt{(2j+\zeta+\rho+3)(2j+\zeta+\rho+1)}}{T\sqrt{(j+\zeta+1)(j+\rho+1)(j+1)(j+\zeta+\rho+1)}},$$

$$\xi_j = \frac{\left(2j^2+(1+\rho)(\zeta+\rho)+2j(\zeta+\rho+1)\right)\sqrt{(2j+\zeta+\rho+3)(2j+\zeta+\rho+1)}}{(2j+\zeta+\rho)\sqrt{(j+\zeta+1)(j+\rho+1)(j+1)(j+\zeta+\rho+1)}}.$$

$$\xi_j = \frac{(2j+\zeta+\rho+2)\sqrt{(2j+\zeta+\rho+3)j(j+\zeta+\rho)(j+\zeta)(j+\rho)}}{(2j+\zeta+\rho)\sqrt{(2j+\zeta+\rho-1)(j+\zeta+1)(j+\rho+1)(j+1)(j+\zeta+\rho+1)}}.$$

From the shifted orthonormal Jacobi polynomials, $\grave{P}_{T,j}^{(\zeta,\rho)}(s)$ of degree j are given by $[31-37]$

$$\grave{P}_{T,j}^{(\zeta,\rho)}(s) = \sum_{h=0}^{j} \frac{(-1)^{j-h}\sqrt{(2j+\zeta+\rho+1)j!}\,\Gamma(j+h+\zeta+\rho+1)}{\sqrt{T^{2h+\zeta+\rho+1}\Gamma(j+\zeta+1)}\Gamma(h+\rho+1)(j-h)h!} s^j. \qquad (4.24)$$

Furthermore, this turn means $[31-37]$

$$\grave{P}_{T,j}^{(\zeta,\rho)}(0) = \frac{(-1)^j\sqrt{(2j+\xi+\rho+1)\Gamma(j+\rho+1)\Gamma(j+\zeta+\rho+1)}}{\sqrt{T^{\zeta+\rho+1}\Gamma(j+\zeta+1)j!\,\Gamma(\rho+1)}}, \qquad (4.25)$$

$$D^e\grave{P}_{T,j}^{(\zeta,\rho)}(0)$$
$$= \frac{(-1)^{j-e}\sqrt{(2j+\zeta+\rho+1)p!\,\Gamma(j+\rho+1)}\Omega(j+e+\zeta+\rho+1)}{\sqrt{T^{2e+\zeta+\rho+1}\Gamma(j+\zeta+1)\Gamma(j+\zeta+\rho+1)}\Gamma(e+\rho+1)(j-e)!}, \qquad (4.26)$$

which is going to be of important later $[31-37]$.

Assuming that x(s) is a square integralable function of the Jacobi weight function $z_T^{(\zeta,\rho)}(s)$ in (0, T), then it can be expressed as shifted Jacobi orthonormal polynomials as $[31-37]$

$$x(s) = \sum_{j=0}^{\infty} x_j \grave{P}_{T,j}^{(\zeta,\rho)}(s).$$

From that the coefficient x_J presented by $[31-37]$

$$x_j = \int_0^T z_T^{(\zeta,\rho)}(s)x(s)\grave{P}_{T,j}^{(\zeta,\rho)}(s)\,ds, \qquad j = 0,1,\dots,. \qquad (4.27)$$

If it approximates x_J by the first (H+1) terms, then it can be composed

$$x_H(s) \simeq \sum_{p=0}^{H} x_p \grave{P}_{T,J}^{(\zeta,\rho)}(s). \qquad (4.28)$$

This on the other hand might be composed in a grid frame

$$x_H(s) \approx X^T \psi_{T,H}(s) \tag{4.29}$$

with

$$X=\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_H \end{pmatrix}, \quad \psi_{T,H}(s) = \begin{pmatrix} \grave{P}_{T,0}^{(\zeta,\rho)}(s) \\ \grave{P}_{T,1}^{(\zeta,\rho)}(s) \\ \vdots \\ \grave{P}_{T,H}^{(\zeta,\rho)}(s) \end{pmatrix}. \tag{4.30}$$

CHAPTER 5
THE USE OF CHEBYSHEV POLYNOMIALS FOR NUMERICAL SOLUTIONS OF FRACTIONAL OPTIMAL CONTROL PROBLEMS

5.1 Shifted Orthonormal Chebyshev Polynomial

The classical orthogonal Chebyshev polynomial denoted by $\{T_j(z); j = 0,1, \dots\}$, and whose degree is j over the interval [-1, 1], constitutes an orthogonal weighted system [39]

$$\int_{-1}^{1} T_j(w)T_k(w)\omega(w)dw = \delta_{jk}h_{k,}$$

where $\omega(w) = \frac{1}{\sqrt{1-w^2}}, \delta_{jk}$ be the Kronecker function and

$$h_k = \frac{\epsilon_k}{2}\pi, \quad \varepsilon_0 = 2, \qquad \varepsilon_k = 1, \qquad k \geq 1.$$

In order to use the orthogonal Chebyshev polynomials to solve the problem defined in [0,1], we assume the change of variable $w = 2s - 1$ to propose the shifted Chebyshev polynomials, i.e. $T_j(2s-1) \equiv T_j^{`}(s)$ .Then the Chebyshev polynomials form an orthogonal system with the Wight function $\omega^*(s) = \frac{1}{\sqrt{s-s^2}}$ which satisfy the orthogonality feature [39]

$$\int_{0}^{1} \grave{T}_k(s)\grave{T}_j(s)\omega^*(s)ds = \delta_{jk}h_k. \tag{5.1}$$

The transformation Chebyshev polynomials get the shape [37]

$$\grave{T}_{j+1}(s) = 2(2s-1)\grave{T}_j - \grave{T}_{j-1}(s), \qquad j \geq 1$$

with [39]

$$\grave{T}_0(s) = 1, \qquad \grave{T}_1(s) = 2s - 1.$$

The clear format for analysis of $\grave{T}_j(s)$ with degree j is represented as [39]

$$\grave{T}_j(s) = \sum_{k=0}^{j} (-1)^{j-k} \frac{j(j+k-1)!\, 2^{2k}}{(j-k)!\,(2k)!} s^k. \tag{5.2}$$

Introducing the shifted orthonormal Chebyshev polynomials

$$T_j^*(s); T_j^*(s) \equiv \frac{1}{\sqrt{h_j}} \grave{T}_j(s),$$

$$\int_0^1 T_j^*(s) T_k^*(s) \omega^*(s) ds = \delta_{jk} \tag{5.3}$$

and

$$T_j^*(s) = \frac{1}{\sqrt{h_j}} \sum_{k=0}^{j} (-1)^{j-k} \frac{j(j+k-1)!\, 2^{2k}}{(j-k)!\,(2k)!} s^k. \tag{5.4}$$

Assuming that u(x) is a square-integrable function with regard to the shifted Chebyshev weight function $\omega^*(s)$ in [0,1], then it can be shifted orthonormal Chebyshev polynomials $T_k^*(s)$ as

$$u(s) = \sum_{k=0}^{\infty} u_k T_k^*(s).$$

From that the transactions u_k are given by [39]

$$u_k = \int_0^1 u(s) T_k^*(s) \omega^*(s) ds\,, \quad 0 \le k. \tag{5.5}$$

Approximate u(x) by means of the shifted orthonormal Chebyshev polynomials yields [39]

$$u_N(s) = \sum_{k=0}^{N} u_k T_k^*(s). \tag{5.6}$$

That can be expressed in terms of the matrix shape [39]

$$u_N(s) \simeq U^T ב_N(s), \tag{5.7}$$

with

$$U=\begin{pmatrix}u_0\\u_1\\u_2\\\vdots\\u_N\end{pmatrix},\qquad ב_N(s)=\begin{pmatrix}T_0^*(s)\\T_1^*(s)\\T_2^*(s)\\\vdots\\T_N^*(s)\end{pmatrix}. \tag{5.8}$$

5.2 The Numerical Technique

The shifted orthonormal Chebyshev polynomials are basis functions of the operational matrix of fractional derivatives for approximating the solution of the fractional optimal control problem [39]:

$$\text{Minimize. } J=\int_{s_0}^{s_1} p(x(s),u(s),s)ds, \tag{5.9}$$

such that

$$a\dot{x}(s)+bD^{\gamma}x(s)=e(s)x(s)+f(s)u(s)+g(s). \tag{5.10}$$

According to boundary conditions [39]

$$x(s_0)=c,\qquad x(s_1)=d, \tag{5.11}$$

where

$a,b\neq 0, s_0\leq s\leq s_1, 0\leq\gamma\leq 1.$

5.2.1 Shifted Orthonormal Chebyshev Approximation

The shifted –based orthonormal Chebyshev polynomials $T_k^*(x)$ approximate x(s) and u(s) as [39]

$$x(s)\simeq X^T ב_N(s),\quad u(s)\simeq U^T ב_N(s), \tag{5.12}$$

where series X and series U given [39]

$$X=\begin{pmatrix}x_0\\\vdots\\x_N\end{pmatrix},\qquad U=\begin{pmatrix}u_0\\\vdots\\u_N\end{pmatrix}.$$

Also, we expand e(s), f(s) and g(s) as [39]

$$e(s)\simeq E^T ב_N(s),\qquad f(s)\simeq F^T ב_N(s),\quad g(s)\simeq G^T ב_N(s), \tag{5.13}$$

where E, F and G are be written as [39]

$$E = \begin{pmatrix} e_0 \\ \vdots \\ e_N \end{pmatrix}, \qquad F = \begin{pmatrix} f_0 \\ \vdots \\ f_N \end{pmatrix}, \;\; G = \begin{pmatrix} g_0 \\ \vdots \\ g_N \end{pmatrix},$$

and

$$e_i = \int_0^1 e(s) T_i^*(s) \omega^*(s) ds\,, \qquad\qquad i = 0,1, \dots, N,$$

$$f_{\mathrm{i}} = \int_0^1 \mathrm{f(s)T_i^*(s)\omega^*(s)ds}, \qquad\qquad \mathrm{i} = 0,1, \dots, \mathrm{N},$$

$$g_{\mathrm{i}} = \int_0^1 \mathrm{g(s)T_i^*(s)\omega^*(s)ds}, \qquad\qquad \mathrm{i} = 0,1, \dots, \mathrm{N}. \tag{5.14}$$

Using the relations (5.12) and (5.13), the dynamic constraint (5.10) is able to be approximated as [39]

$$aD\big(X^T \beth_N(s)\big) + bD^\gamma\big(U^T \beth_N(s)\big) \\ = \big(E^T \beth_N(s)\big)\big(X^T \beth_N(s)\big) + \big(F^T \beth_N(s)\big)\big(U^T \beth_N(s)\big) + \big(G^T \beth_N(s)\big).$$

That can be reduced using relation the Caputo fractional derivative of order the shifted orthonormal Chebyshev polynomials vector writes as

with the matrix operations inside $D^\gamma \beth_N(s) = D^{(\gamma)} \beth_N(s)$

$$aX^T D^{(1)} \beth_N(s) + bX^T D^{(\gamma)} \beth_N(s) \\ = E^T \beth_N(s) \beth_N^T(s) X + F^T \beth_N(s) \beth_N^T(s) U + G^T \beth_N(s). \tag{5.15}$$

Let that [39]

$$E^T \beth_N(s) \beth_N^T(S) \simeq \beth_N^T(s) R^T, F^T \beth_N(s) \Delta_N^T(s) \simeq \beth_N^T(s) L^T, \tag{5.16}$$

where R and L are N× N matrices. For illustrating R and L, it might rewrite equation (5.16) as [39]

$$\sum_{k=0}^{N} e_k T_k^*(s) T_j^*(s) = \sum_{k=0}^{N} R_{kj} T_k^*(s),$$

$$\sum_{k=0}^{N} f_k T_k^*(s) T_j^*(s) = \sum_{k=0}^{N} L_{kj} T_k^*(s), j = 1,2,\dots,N. \quad (5.17)$$

Multiplying both sides of equation (5.17) by $T_i^*(s)\omega^*(s)$ and integrating from 0 to 1 yield

$$\sum_{k=0}^{N} e_k \int_0^1 T_k^*(s) T_j^*(s) T_i^*(s) \omega^*(s) ds = \sum_{k=0}^{N} R_{kj} \int_0^1 T_k^*(s) T_i^*(s) \omega^*(s)\, ds,$$

$$\sum_{k=0}^{N} f_k \int_0^1 T_k^*(s) T_j^*(s) T_i^*(s) \omega^*(s)\, ds =$$

$$\sum_{k=0}^{N} \mathrm{L_{kj}} \int_0^1 \mathrm{T_k^*(s) T_i^*(s) \omega^*(s) ds}, \qquad \mathrm{i, j} = 1,\dots,\mathrm{N}. \quad (5.18)$$

So, using (5.3) are given [39]

$$R_{kj} = \sum_{k=0}^{N} e_k \int_0^1 T_k^*(s) T_j^*(s) T_i^*(s) ds,$$

$$L_{\mathrm{kj}} = \sum_{\mathrm{k}=0}^{\mathrm{N}} \mathrm{f_k} \int_0^1 \mathrm{T_k^*(s) T_j^*(s) T_i^*(s) ds}, \quad \mathrm{i, j} = 1,\dots,\mathrm{N}. \quad (5.19)$$

By equation (5.16), rewrite equation (5.15) as [39]

$$aX^T D^{(1)} ℶ_N(s) + bX^T D^{(\gamma)} ℶ_N(s) - ℶ_N^T(s) R^T X - ℶ_N^T(s) L^T U - G^T ℶ_N(s) = 0,$$

$$\left(aX^T D^{(1)} + bX^T D^{(\gamma)} - R^T X - L^T U - G^T\right) ℶ_N(s) = 0. \quad (5.20)$$

The Caputo fractional derivative for the function f(s) is written as [39]

$$D^\gamma\left(\lambda f(s) + \mu g(s)\right) = \lambda D^\gamma f(s) + \mu D^\gamma g(s). \quad (5.21)$$

Using equation (5.20), the dynamical system (5.21) is reduced into a linear system of algebraic equations [39]

$$aX^T D^{(1)} + bX^T D^{(\gamma)} - R^T X - L^T U - G^T = 0. \quad (5.22)$$

In addition, it can be approximated the boundary conditions [39]

$x(s_0) = c, \;\; x(s_1) = d, \quad where a, b \neq 0, s_0 \leq s \leq s_1, 0 \leq \gamma \leq 1,$

as

$$x(s_0) \simeq X^T \beth_N(s_0), \qquad x(s_1) \simeq X^T \beth_N(s_1) \tag{5.23}$$

or

$$c - X^T \beth_N(s_0) \simeq 0, \qquad d - X^T \beth_N(s_1) \simeq 0. \tag{5.24}$$

5.2.2 Legendre -Gauss Quadrature Method

This time, using Equation (5.12) for approximating the performance index [39]

as $\mathrm{J} \equiv \mathrm{J}[\mathrm{C}, \mathrm{U}]$

$$J[X, U] \simeq \int_{s_0}^{s_1} p\left(\left(X^T \beth_N(s)\right), \left(U^T \beth_N(s)\right), s\right) ds. \tag{5.25}$$

In general, the previous integral cannot be computed exactly, in this case, it can use the Legendre-Gauss quadrature formula [39]

first, we suppose the change of variable

$$s = \frac{s_1 - s_0}{2} \grave{s} + \frac{s_1 + s_0}{2}. \tag{5.26}$$

That will be used for transforming the integration in the performance index (5.25) into another one in the $[-1,1]$ interval .Then, the equation (5.25) is equivalent to

$$J[X, U] \simeq \left(\frac{s_1 - s_0}{2}\right) \int_{-1}^{1} \grave{p}(\grave{s}) d\grave{s}, \tag{5.27}$$

where

$$\grave{p}(\grave{s}) = p\left(\left(X^T \grave{\beth}_N(\grave{s})\right), \left(U^T \grave{\beth}_N(\grave{s})\right), \grave{s}\right),$$

with

$$\grave{\beth}_N(\grave{s}) = \beth_N\left(\frac{s_1 - s_0}{2} s^{\grave{}} + \frac{s_0 + s_1}{2}\right).$$

Also, one can compute the integration in equation (5.27) by employing the Legendre-Gauss quadrature rule as [39]

$$J[X,U] \simeq \left(\frac{s_1 - s_0}{2}\right) \sum_{r=0}^{N} \grave{p}\left(\grave{s}_{N,r}\right)_{\varpi_{N,r}}, \tag{5.28}$$

where $\varpi_{N,r} 0 \leq r \leq N$, and $s_{N,r}, 0 \leq r \leq N$, are the Christoffel numbers and zeros of Legendre-Gauss quadrature respectively.

5.2.3 Lagrange Multiplier Technique

Here, there is a need to merge the linear algebraic equations derived as of the dynamical system (5.22) with the boundary conditions (5.24) inside the performance index at Legendre-Gauss quadrature rule as

$$J[X,U] \cong \left(\frac{s_1 - s_2}{2}\right) \sum_{r=0}^{N} \grave{p}(\grave{s}) \varpi_{N,r}$$

therefore, the Lagrange multiplier method can be applied. Let [39]

$$\begin{aligned} J^*[X,U,\mu_1,\mu_2,\mu_3] \\ &= J[X,U] + \left(aX^T D^{(1)} + bX^T D^{(\gamma)} - R^T X - L^T U - G^T\right)\mu_1 + Y^T \mu_2 \\ &+ Z^T \mu_3. \end{aligned} \tag{5.29}$$

Here

$$Y^T = [x_0 - X^T \beth_N(s_0), 0, \ldots, 0],$$

$$\mathrm{Z}^{\mathrm{T}} = [\mathrm{x}_1 - \mathrm{X}^{\mathrm{T}} \beth_{\mathrm{N}}(\mathrm{s}_1), 0, \ldots 0], \tag{5.30}$$

and μ_1, μ_2, μ_3 denote the unknown Lagrange multipliers that are able to be expressed as [39]

$$\mu_1 = \begin{pmatrix} \mu_{10} \\ \mu_{11} \\ \mu_{12} \\ \vdots \\ \mu_{1N} \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} \mu_{20} \\ \mu_{21} \\ \mu_{22} \\ \vdots \\ \mu_{2N} \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} \mu_{30} \\ \mu_{31} \\ \mu_{32} \\ \vdots \\ \mu_{3N} \end{pmatrix}. \tag{5.31}$$

The important conditions for the optimality of the performance index (5.9) subject to dynamic constraints (5.10) and (5.11) are

$$\frac{\partial J^*}{\partial X} = 0, \qquad \frac{\partial J^*}{\partial U} = 0, \qquad \frac{\partial J^*}{\partial \mu_1} = 0, \qquad \frac{\partial J^*}{\partial \mu_2} = 0, \qquad \frac{\partial J^*}{\partial \mu_3} = 0. \tag{5.32}$$

Here, $\frac{\partial J^*}{\partial X} = 0$ represents the system $\frac{\partial J^*}{\partial x_i} = 0, i = 0,1, \dots, N$. The above-mentioned system is able to be solved for X, U, $\mu_1,\ \mu_2,\ \mu_3$ using every standard iterative technique .Thus, X, U, $\mu_1,\ \mu_2$ and μ_3 given in (5.12) and (5.31) can be calculated.

CHAPTER 6

THE USE OF LEGENDRE POLYNOMIALS FOR NUMERICAL SOLUTIONS OF FRACTIONAL OPTIMAL CONTROL PROBLEMS

A fractional differential equation can be solved numerically for example, by using the finite difference method, and discretization technique [41]. The first order derivative of a continuous function *f(t)* can be approximated by using the forward finite difference method as following [41]

$$\dot{f}(\mathrm{t}) = \frac{f(\mathrm{t}+\mathrm{h}) - f(\mathrm{t})}{\mathrm{h}} + \mathrm{O}(\mathrm{h}),$$

where *h* is a small increment of the variable *t.* The left side Caputo fractional derivative of *f(t)* is defined by [41]

$${}_{a}^{C}D_t^{\gamma} f(t) = \frac{1}{\Gamma(m-\gamma)} \int_a^t (t-z)^{m-\gamma-1} \frac{d^m}{dz^m} f(z) dz,$$

where the fractional derivative order is $\gamma \in R$ and given within the range

$m-1 < \gamma \leq m, \qquad m \in N.$

For *a=0* and *m=1*, the left side Caputo fractional derivative is [41]

$${}_{0}^{C}D_t^{\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-z)^{-\gamma} \frac{d}{dz} f(z) dz.$$

The variable *t* is discrtized time such that the time increment $h = \frac{t_N}{N}$, where t_N is the upper limit of the variable *t* and *N* is the number of discrete points of vector t_n. (for more details refer to [41]) .

6.1 Operational Matrix for Fractional Derivatives

Theorem 6.1.1 $[27-30]$ The fractional derivative of order ν from shifted Legendre orthonormal polynomial vector $\Delta_H(s)$ given by $[27-30]$

$$D^v \Delta_H(s) = D_{(v)} \Delta_H(s), \tag{6.1}$$

where

$$D^v f(y) = \frac{1}{\Gamma(h-v)} \int_0^y (y-s)^{h-v-1} f^{(h)}(s)\, ds, h-1 < v \leq h. \tag{6.2}$$

The function $f(y)$ in the Caputo logic, while $D^{(v)}$ is the $(H+1)\times(H+1)$ operational matrix of fractional derivative of order v and is defined by

$$D_{(v)}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ \psi_v(h,0) & \psi_v(h,1) & \psi_v(h,2) & \psi_v(h,H) \\ \vdots & \vdots & \vdots & \vdots \\ \psi_v(i,0) & \psi_v(i,1) & \psi_v(i,2) & \psi_v(i,H) \\ \vdots & \vdots & \vdots & \vdots \\ \psi_v(H,0) & \psi_v(H,1) & \psi_v(H,2) & \psi_v(H,H) \end{bmatrix},$$

where

$\psi_v(i,p,j)$

$$=\sqrt{(2p+1)(2i+1)}\sum_{j=n}^{i}\sum_{i=0}^{p}\frac{(-1)^{i+p+j+1}(i+j)!\,(l+p)!}{(i-j)!\,j!\,\Gamma(\mathrm{j-v+1})(\mathrm{p-l})!\,(\mathrm{l!})^2(\mathrm{j+l-v+1})}\,. \quad (6.3)$$

Proof [27-30] Utilizing (4.13) and (6.2), the fractional derivative of order ν is shifted Legendre orthonormal polynomials $P_i^{\dot{a}}(s)$ is given by $[27-30]$

$$D^vP_i^{\dot{a}}(s)=\sqrt{2i+1}\sum_{j=0}^{i}(-1)^{i+j}\frac{(i+j)!}{(i-j)!\,(j!)^2}D^vs^j$$

$$=\sqrt{2i+1}\sum_{j=h}^{i}(-1)^{i+j}\frac{(i+j)!}{(i-j)!\,j!\,\Gamma(j-v+1)}s^{j-v} \quad (6.4)$$

approximated s^{j-v} by H+1 terms of shifted Legendre orthonormal polynomials $P_p^{\dot{a}}(s)$ as $[27-30]$:

$$s^{j-v}=\sum_{p=0}^{H}\mu_{jp}P_p^{\dot{a}}(s), \quad (6.5)$$

where μ_{jp} is given like in Eq. (6.3) with $y(s)=s^{j-v}$, then $[27-30]$

$$\mu_{jp}=\int_0^1 s^{j-v}P_p^{\dot{a}}(s)ds=$$

$$=\sqrt{2p+1}\sum_{l=0}^{p}(-1)^{p+1}\frac{(p+l)!}{(p-l)!\,(l!)^2}\int_0^1 s^{l+j-v}\,ds$$

$$=\sqrt{2p+1}\sum_{l=0}^{p}(-1)^{p+1}\frac{(p+l)!}{(p-l)!(l!)^2(j-v+l+1)}. \quad (6.6)$$

Employing Eqs.(6.5)-(6.6)

$$D^{v}P_i^{\dot{a}}(s) = \sqrt{2i+1}\sum_{j=n}^{i}\sum_{p=0}^{H}(-1)^{i+j}\frac{(i+j)!}{(i-j)!\,j!\,\Gamma(j-v+1)}\mu_{jp}P_p^{\dot{a}}(s)$$

$$= \sum_{p=0}^{H}\psi_v\,(i,p)P_p^{\dot{a}}(s), \qquad (6.7)$$

where $\psi(i,p)$is given by Eq.(6.3). In the end, one can rewrite Eq.(6.7) in a vector form as

$$D^{v}P_i^{\dot{a}}(s) = \left[\psi_v(i,0),\psi_v(i,1),\dots,\psi_v(i,p),\dots,\psi_v(i,H)\right]\Delta_H(s). \qquad (6.8)$$

Equation (6.8) completes the proof.

6.2 The Studied Model

Consider the dynamic system that is analytically studied in [40]

$$\dot{x}(\mathrm{t}) + \mathrm{D}^{\gamma}\mathrm{x}(\mathrm{t}) = \mathrm{u}(\mathrm{t}) + \mathrm{t}^2. \qquad (6.9)$$

The response x*(t)* of the fractional order system in Eq. (6.9) can be obtained numerically for a given input *u(t).* This can be accomplished by applying the discretization technique that is described in the previous section. The first term of the left hand side of Eq. (6.9) can be expressed by means of the forward finite difference method as follows [39]

$$\dot{x}(\mathrm{t}) = \frac{x(\mathrm{t}+\mathrm{h}) - x(\mathrm{t})}{\mathrm{h}} + \mathrm{O}(\mathrm{h}). \qquad (6.10)$$

Applying the aforementioned discretization of the variable t, expressed as follows [41]

$${}_{0}^{C}D_t^{\gamma}x(t) = \frac{1}{\Gamma(1-\gamma)}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t-z)^{-\gamma}\frac{d}{dz}x(z)dz. \qquad (6.11)$$

The forward finite difference is utilized to express the first derivative in Eq. (6.11) generate the following [41]

$${}_{0}^{C}D_t^{\gamma}x(t) = \frac{1}{\Gamma(1-\gamma)}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t-z)^{-\gamma}\left[\frac{x(t_{j+1}) - x(t_j)}{h}\right]dz. \qquad (6.12)$$

Solving the integral in Eq. (6.12) gives

$$ {}_{0}^{C}D_t^{\gamma}x(t) = \frac{1}{\Gamma(1-\gamma)}\sum_{j=0}^{n-1}\frac{x(t_{j+1})-x(t_j)}{h}\left\{-\left[\left(t_n-t_{j+1}\right)^{1-\gamma}\right.\right.$$

$$\left.\left. -\left(t_n-t_j\right)^{1-\gamma}\right]\right\}. \quad (6.13)$$

Equation (6.13) can be rewritten as [41]

$$ {}_{0}^{C}D_t^{\gamma}x(t) = \frac{h^{-\gamma}}{\Gamma(2-\gamma)}\sum_{j=0}^{n-1}[(n-j)^{1-\gamma}-(n-j-1)^{1-\gamma}]\left[x(t_{j+1})-x(t_j)\right]. \quad (6.14)$$

Substitute Eq. (6.13) and Eq. (6.14) into Eq. (6.9) to obtain

$$\frac{x(t_{n+1})-x(t_n)}{h}+\frac{h^{-\gamma}}{\Gamma(2-\gamma)}\sum_{j=0}^{n-1}[(n-j)^{1-\gamma}-(n-j-1)^{1-\gamma}]\left[x(t_{j+1})-x(t_j)\right]$$

$$= u(t_n)+t_n^2. \quad (6.15)$$

By applying the initial condition $x(t_0)=x_0$ and for given input function *u(t)* the response of the system can be numerically obtained.

6.2.1 Numerical Result 1

The response of the dynamic system that given by Eq. (6.9) is obtained for different values of the fractional order γ and the given input $u(t)=t^2e^{bt};$ where $b\in\mathbb{R}$. Figure 1 shows the system responses for $\gamma=0.35\ and\ \gamma=0.85$. The solution verification is also illustrated in Figure 1 by comparing the response of classic integer case with the fractional system response for $\gamma=1$.

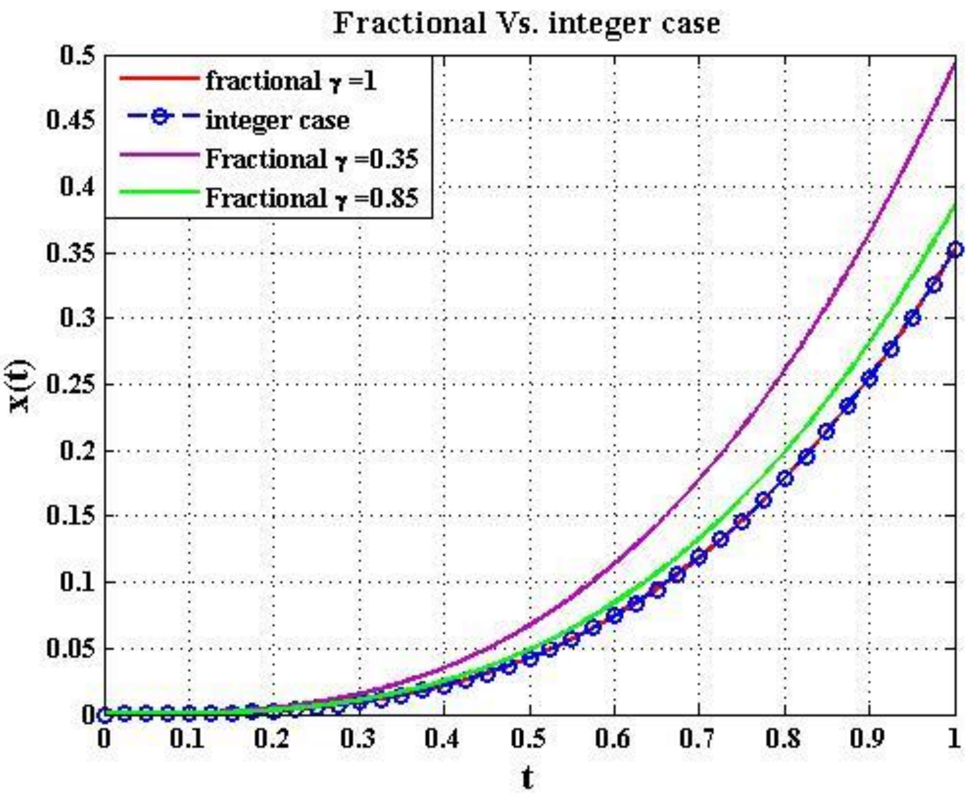

Figure 1: The solution verification and the responses of the system in Eq.(6.9) for γ= 0.35 and γ= 0.85.

6.2.2 Numerical Result 2

Consider the dynamic system that was studied in [40]

$$\frac{1}{2}\dot{x}(\mathrm{t}) + \frac{1}{2}\mathrm{D}^{\gamma}x(\mathrm{t}) = -x(\mathrm{t}) + \mathrm{u}(\mathrm{t}). \qquad (6.16)$$

The procedure that is used to obtain the response of case study 1 can be used to obtain the response x*(t)* of the fractional order system in Eq. (6.16) for a given input *u (t).*

The response of the dynamic system that given by Eq. (6.16) is obtained for different values of the fractional order γ, the given input $u(t)$ as [40] parabola, and initial value $x(t_0) = x_0$.Figure 2 shows the system responses for $\gamma = 0.15$ through $\gamma = 0.75$.

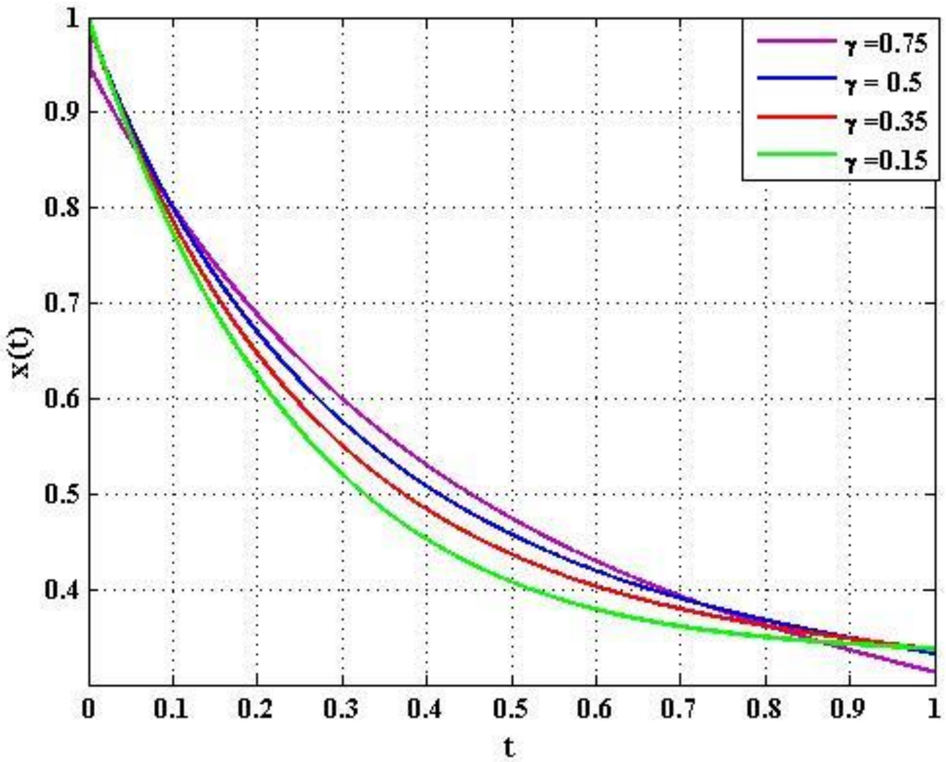

Figure 2*: The system in Eq.*(6.16) *responses for γ= 0.15 through γ= 0.75***.**

6.3 Shifted Legendre Polynomials

Assume that the Legendre polynomial of degree r is denoted by $L_r(w)$ (interval definition [-1, 1]).Then $L_r(w)$ can be generated by the recurrence formulae [40]

$$L_{r+1}(w) = \frac{2r+1}{r+1} wL_r(w) - \frac{r}{r+1} L_r(w), \quad 1 \leq r,$$
$$L_0(w) = 1 \ , L_1(w) = w.$$

The shifted Legendre polynomials $L_r^*(s)$ is defined in the interval [0, 1] and is generated using the following recurrence formula [40]

$$L_{r+1}^*(s) = \frac{2r+1}{r+1}(2s-1)L_r^*(s) - \frac{r}{r+1} L_{r-1}^*(s), \qquad 1 \leq r,$$
$$L_0^*(s) = 1, \qquad\qquad L_1^*(s) = 2s-1.$$

The orthogonality relation is [40]

$$\int_0^1 L_j^*(s)L_j^*(s)ds = \begin{cases} \frac{1}{2r+1}, & \text{for} j = r, \\ 0, & \text{for} j \neq r. \end{cases} \tag{6.17}$$

We can write the explicit analytical form of the k degree shifted Legendre polynomial $L_r^*(s)$ as [40]

$$L_r^*(s) = \sum_{i=0}^{r} (-1)^{r+i} \frac{(r+i)!}{(r-i)!\,(i!)^2} s^i. \tag{6.18}$$

Introducing the shifted Legendre orthonormal polynomials $L_r^*(s), L_r^*(s) \equiv \sqrt{2r+1} L_r^*(s)$, we have [40]

$$\int_0^1 L_j^*(s) L_j^*(s) ds = \begin{cases} 1, & \text{for} j = r, \\ 0, & \text{for } j \neq r. \end{cases} \tag{6.19}$$

and

$$L_r^*(s) = \sqrt{2r+1} \sum_{i=0}^{r} (-1)^{r+i} \frac{(r+i)!}{(r-i)!\,(i!)^2} s^i. \tag{6.20}$$

Let

$\mathrm{P}_N = Span\,\{L_0^*(s), L_1^*(s), \ldots, L_N^*(s)\}$ and q is an arbitrary element at L^2 [0, 1]. Because P_N is a finite-dimensional vector space, q has the single best approximation out of $\mathrm{P}_N$like $q_N \in \mathrm{P}_N$ such that [40]

$\forall_g \in \mathrm{P}_N, \|q - q_N\|_2 \leq \|q - g\|_2,$ where $\|q\|_2 = \sqrt{\langle q, q\rangle}$.

Every square integral function q(s) definite on the interval [0, 1] might be expressed in terms of shifted Legendre polynomials $L_r^*(s)$ as [40]

$$q(s) = \sum_{r=0}^{\infty} q_r L_r^*(s),$$

and the coefficients q_r give via [40]

$$q_r = \int_0^1 q(s) L_r^*(s) ds \qquad 0 \leq r. \tag{6.21}$$

If we approximate q (s) by the firstly (N+1) terms, we can write

$$q_N(s) = \sum_{r=0}^{M} q_r L_r^*(s), \tag{6.22}$$

that alternatively might be written in the matrix shape [40]

$$q_N(s) \simeq Q^T \Omega_N(s) \tag{6.23}$$

by means of

$$Q = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{pmatrix}, \Omega_N(s) = \begin{pmatrix} L_0^*(s) \\ L_1^*(s) \\ \vdots \\ L_N^*(s) \end{pmatrix}. \tag{6.24}$$

the fractional derivative of $\Omega_N(s)$of order υas can be written as

$$D^{\upsilon}\Omega_N(s) \simeq D^{(\upsilon)}\Omega_N(s), \tag{6.25}$$

where $D^{(\upsilon)}$ is the$(N+1)\times(N+1)$ operational matrix of fractional derivatives of order υ and is defined by Lotfi et al [40]

$$D^{(\upsilon)} = \begin{pmatrix} D_{11} & D_{12} & D_{13} & \cdots & D_{1(N+1)} \\ D_{21} & D_{22} & D_{23} & \cdots & D_{2(N+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{(N+1)1} & D_{(N+1)2} & D_{(N+1)3} & \dots & D_{(N+1)(N+1)} \end{pmatrix}, \tag{6.26}$$

where

$$D_{ij} = \hat{B}_{i-1j-1}, \qquad 1 \le i,j \le N+1.$$

and

$$\hat{B}_{ij} = \sqrt{(2i+1)(2j+1)},$$

$$\sum_{r=1}^{i}\sum_{l=0}^{j}\frac{(-1)^{i+j+r+1}(i+r)!\,(l+j)!}{(i-r)!\,r!\,\Gamma(r-\upsilon+1)(j-l)!\,(l!)^2(r+l-\upsilon+1)}.$$

6.4 The Numerical Technique

In this section, we use the operational matrix of the fractional derivatives to solve the following problems by means of the properties of the shifted Legendre orthogonal polynomials [40].

$$\min J = \frac{1}{2}\int_{s_0}^{s_1}\big(q(s)x^2(s) + r(s)u^2(s)\big)ds, \tag{6.27}$$

Limited by the dynamical system [40]

$$n_1\dot{x}(s) + n_2 D^{\upsilon}x(s) = a(s)x(s) + b(s)u(s). \tag{6.28}$$

$$x(s_0) = x_0, \qquad x(s_1) = x_1, \tag{6.29}$$

where $n_1, n \neq 0, s_0 \le s \le s_1, 0 \le \nu \le 1$.

6.4.1 Shifted Orthonormal Legendre Approximation

Now, we approximate x(s) and u(s) by the shifted Legendre orthonormal polynomials $L_k^*(s)$ as [39]

$$x(s) \simeq P^T\Omega_N(s), \qquad u(s) \simeq Y^T\Omega_N(s), \tag{6.30}$$

where P and Y are indefinite coefficient matrices that are able to be written as [40]

$$P=\begin{pmatrix}p_0\\p_1\\\vdots\\p_N\end{pmatrix},\quad Y=\begin{pmatrix}y_0\\y_1\\\vdots\\y_N\end{pmatrix},$$

due to (6.30), approximate the performance index W≡W[P,Y] as

$$W[P,Y]\simeq\frac{1}{2}\int_{s_0}^{s_1}\left(q(s)\left(P^T\Omega_N(s)\right)^2+r(s)\left(Y^T\Omega_N(s)\right)^2\right)ds. \tag{6.31}$$

Consequentially,

$$a(s)\simeq A^T\Omega_N(s),\qquad b(s)\simeq B^T\Omega_N(s), \tag{6.32}$$

where A and B are famous coefficients matrices that might be written as [39]

$$A=\begin{pmatrix}a_0\\a_1\\\vdots\\a_N\end{pmatrix},\qquad B=\begin{pmatrix}b_0\\b_1\\\vdots\\b_N\end{pmatrix},$$

and

$$a_i=\int_0^1 a(s)L_i^*(s)ds,\qquad i=0,1,\dots,N,$$

$$b_i=\int_0^1 b(s)L_i^*(s)ds,\qquad i=0,1,\dots,N. \tag{6.33}$$

For general functions a(s) and b(s), it is hard to calculate the earlier integrals precisely. Using the Legendre–Gauss quadrature formula, approximate the coefficients $a_i$and b_i as [40]

$$a_i=\sum_{\epsilon=0}^{N}a_i\left(s_{N,\epsilon}\right)L_k^*\left(s_{N,\epsilon}\right)\varpi_{N,\epsilon},\qquad i=0,1,\dots,N,$$

$$b_i=\sum_{\epsilon=0}^{N}b_i\left(s_{N,\epsilon}\right)L_k^*\left(s_{N,\epsilon}\right)\varpi_{N,\epsilon}, i=0,1,\dots,N,$$

where $s_{N,\epsilon}, 0\le\epsilon\le N$, are the zeros of the Legendre Gauss quadrature in the interval (0,1), with $s_{N,\epsilon}, 0\le\epsilon\le N$ being corresponding Christoffel numbers. Due to (6.25), (6.30) and (6.31), the dynamic constraint (6.28) be able to be approximated like [40]

$$n_1P^TD^{(1)}\Omega_n(s)+n_2P^TD^{(v)}\Omega_N(s)$$

$$=\left(A^T\Omega_N(s)\left(P^T\Omega_N(s)\right)\right)+\left(B^T\Omega_N(s)\left(Y^T\Omega_N(s)\right)\right)$$

$$nP^TD^{(1)}\Omega_N(s)+n_2P^TD^{(v)}\Omega_N(s)-A^T\Omega_N(s)\Omega_N^T(s)P-B^T\Omega_N(s)\Omega_N^T(s)Y$$
$$=0 \tag{6.34}$$

Assume $A^T\Omega_N(s)\Omega_N^T(s)$ and $B^T\Omega_N(s)\Omega_N^T(s)$are written in vector forms like [40]

$$A^T\Omega_N(s)\Omega_N^T(s) = [f_0(s), f_1(s), \dots, f_N(s)]$$

$$B^T\Omega_N(s)\Omega_N^T(s) = [n_0(s), n_1(s), \dots, n_N(s)], \qquad (6.35)$$

where

$$f_i(s) = \bar{f}_{i0}L_0^*(s) + \bar{f}_{i1}L_{i1}^*(s) + \cdots + \bar{f}_{iN}L_N^*(s)$$

$$h_i(s) = \bar{h}_{i0}L_0^*(s) + hL_{i1}^*(s) + \cdots + \bar{h}_{iN}L_N^*(s), \qquad 0 \le i \le N, \qquad (6.36)$$

and [40]

$$\bar{f}_{ij} = \int_0^1 f_i(s)L_j^*(s)ds,$$

$$\bar{h}_{ij} = \int_0^1 h_i(s)L_j^*(s)ds, \qquad 0 \le i,j \le N. \qquad (6.37)$$

So, we are able to write

$$A^T\Omega_N(s)\Omega_N^T(s) \simeq \Omega_N^T(s)\bar{F}^T,$$

$$B^T\Omega_N(s)\Omega_N^T(s) \simeq \Omega_N^T(s)\bar{H}^T, \qquad (6.38)$$

where [40]

$$F = [\bar{f}_{ij}]_{0\le i,j\le N}, \quad H = [\bar{h}_{ij}]_{0\le i,j\le N} \;. \qquad (6.39)$$

Employing (6.38) in (6.34), one can

$$n_1P^TD^{(1)}\Omega_N(s) + n_2P^TD^{(v)}\Omega_N(s) -$$

$$\Omega_N^T(s)\bar{F}^TP - \Omega_N^T(s)\bar{H}^TY = 0\,. \qquad (6.40)$$

or

$$\left(n_1P^TD^{(1)} + n_2P^TD^{(v)} - P^T\bar{F} - Y^T\bar{H}\right)\Omega_N(s) = 0. \qquad (6.41)$$

By virtue of (6.38), the dynamical system (6.28) is changed to the following linear system of algebraic equations:

$$n_1P^TD^{(1)} + n_2P^TD^{(v)} - P^T\bar{F} - Y^T\bar{H} = 0. \qquad (6.42)$$

Using (5.54), one can write

$$x(s_0) \simeq P^T\Omega_N(s_0) \;, x(s_1) \simeq P^T\Omega_N(s_1) \qquad (6.43)$$

or

$$x_0 - P^T\Omega_N(s_0) = 0 \quad , \; x_1 - P^T\Omega_N(s_1) = 0. \qquad (6.44)$$

6.4.2 The Lagrange Multiplier Technique

Let [40]

$$H^*[P,Y,\lambda_1,\lambda_2,\lambda_3] = H[P,Y] + \left(n_1 P^S D^{(1)} + n_2 P^S D^{(v)}\right) \times \lambda_1 + V^S\lambda_2 + Z^S\lambda_3, \quad (6.45)$$

where

$$V^S = [x_0 - P^S\Omega_N(s_0), 0 \dots ,0],$$
$$Z^S = [x_1 - P\Omega_N(s_1), 0 \dots ,0], \quad (6.46)$$

and $\lambda_1, \lambda_2, \lambda_3$ are unknown Lagrange multipliers that can be expressed as [41]

$$\lambda_1^S = [\lambda_{10}, \lambda_{11}, \dots, \lambda_{1N}],$$
$$\lambda_2^S = [\lambda_{20}, \lambda_{21}, \dots, \lambda_{2N}],$$
$$\lambda_3^S = [\lambda_{30}, \lambda_{31}, \dots, \lambda_{3N}]. \quad (6.47)$$

The current extreme conditions are

$$\frac{\partial H^*}{\partial P} = 0, \frac{\partial H^*}{\partial Y} = 0, \frac{\partial H^*}{\partial \lambda_1} = 0, \frac{\partial H^*}{\partial \lambda_2} = 0, \frac{\partial H^*}{\partial \lambda_3} = 0. \quad (6.48)$$

where $\frac{\partial H^*}{\partial p_i} = 0$, is the system $\frac{\partial H^*}{\partial p_i} = 0$, $i = 0,1, \dots N$. All equations in this part can be solved for $P, Y, \lambda_1, \lambda_2, \lambda_3$ using the Newton iterative method.Consequently, $P, Y, \lambda_1, \lambda_2$ and λ_3 given in (6.30) and (6.47) can be calculated.

CONCLUSION

The fractional calculus techniques were given intensively to discuss the optimal control problems due to their huge applications in science and engineering.

In this thesis we presented the basic definitions and theorems of fractional calculus. After that we discuss the fundamentals of fractional optimal control problems formulation. Also we discus the description of some orthonormal polynomials and more on properties of Chebyshev polynomials. Then, the numerical schemes based on the use of these special polynomials are presented for obtaining the approximate solutions of fractional optimal control problems.

Finally, two original examples were discussed in details and results are simulated Figure 1 and Figure 2.

We hope that this thesis will be utilized by researchers who would like to do research in the area of fractional optimal control with applications.

REFERENCES

[1] **Kilbas, A., Srivastava, H., and Trujillo, J. (2006).** *Theory and applications of fractional differential equations*, North-Holland Math. *Studies*, *204*, 135-209.

[2] Samko, S. G., Kilbas, A. A ., and Marichev, O. I. (1993). *Fractional integrals and derivatives: theory and applications, Gordon and Breach Science Publishers, Amsterdam*.

[3] Podlubny, I. (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications* (Vol. 198). Academic Press.

[4] Magin, R. L. (2006). *Fractional calculus in bioengineering*. Redding: Begell House.

[5] West, B., Bologna, M., and Grigolini, P. (2012). *Physics of fractal operators*. Springer' New-York.

[6] Baleanu, D., Maaraba, T., and Jarad, F. (2008). *Fractional variational principles with delay*. Journal of Physics A: Mathematical and Theoretical, 41(31), 315403.

[7] Jarad, F., Abdeljawad, T., and Baleanu, D. (2010). *Fractional variational principles with delay within Caputo derivatives*. Reports on Mathematical Physics, *65*(1), 17-28.

[8] Agrawal, O. P. (1989). *General formulation for the numerical solution of optimal control problems.* International Journal of Control, 50(2), 627-638.

[9] Agrawal, O. P. (2001). *A new Lagrangian and a new Lagrange equation of motion for fractionally damped systems*. Journal of Applied Mechanics, 68(2), 339-341.

[10] Agrawal, O. P. (2002). *Formulation of Euler–Lagrange equations for fractionalvariational problems.* Journal of Mathematical Analysis and Applications, 272(1), 368-379.

[11] Agrawal, O. P. (2004). *A general formulation and solution scheme for fractional optimal control problems*. Nonlinear Dynamics, 38(1-4), 323-337.

[12] Baleanu, D., Defterli, O., and Agrawal, O. P. (2009). *A central difference numerical scheme for fractional optimal control problems*. Journal of Vibration and Control, 15(4), 583-597.

[13] Defterli, O. (2010). *Anumerical scheme for two-dimensional optimal control problems with memory effect Computers and Mathematics with Applications,* 59(5), 1630-1636.

[14] Agrawal, O. P., Defterli, O., and Baleanu, D. (2010). *Fractional optimal control problems with several state and control variables Journal of Vibration and Control*, 16(13), 1967-1976.

[15] Bryson, A. E. (1975). *Applied optimal control: optimization, estimation and control*. CRC Press.

[16] Muslih, S. I., and Baleanu, D. (2005). *Hamiltonian formulation of systems with linear velocities within Riemann–Liouville fractional derivatives*. Journal of Mathematical Analysis and Applications, *304*(2), 599-606.

[17] Muslih, S. I., and Baleanu, D. (2005). *Formulation of Hamiltonian equations for fractional variational problems*. Czechoslovak Journal of Physics, *55*(6), 633-642.

[18] Muslih, S, I., Baleanu ,D., and Rabei ,E., (2006). *Hamiltonian formulation of classical fields within Riemann- liouville fractional derivatives,* PhysicaScripta 73,436-438.

[19] El-Nabulsi, R. A., and Torres, D. F. (2008). *Fractional actionlike variational problems*. Journal of Mathematical Physics, *49*(5), 053521.

[20] Frederico, G. S. F. and Torres, D. F. M., (2007). *Non-conservation Noethers theorem for fractional action-like variaational problems with intrinsic and observer times,* International Journal of Ecological Economics and Statistics 9(F07), 74-82.

[21] Frederico, G. S., and Torres, D. F. (2008a). *Fractional conservation laws in optimal control theory*, Nonlinear Dynamics, *53*(3), 215-222.

[22] Frederico, G. S. F. and Torres, D. F. M., (2008b). *Fractional Optimal control in the sense of Caputo and the fractional Noethers theorem*, International Mathematical Forum 3(9-12), 479-493.

[23] Frederico, G. S. F. and Torres, D. F. M., (2008 c). *Necessary Optimalality conditions for fractional action-like problems with intrinsic and observer times,* WSEAS Transactions on Mathematics 7(1).6-11.

[24] Alipour, M., Rostamy, D., and Baleanu, D. (2013). *Solving multi-dimensional fractional optimal control problems with inequality constraint by Bernstein polynomials operational matrices*. Journal of Vibration and Control, *19*(16), 2523-2540.

[25] Marzban, H. R., and Razzaghi, M. (2003). *Hybrid functions approach for linearly constrained quadratic optimal control problems*. Applied Mathematical Modelling, 27(6), 471-485.

[26] Kreyszig, E. (1978). *Introduction to Functional Analysis with Applications*, New York John Wiley.

[27] Doha, E. H., Bhrawy, A. H., and Ezz-Eldien, S. S. (2011). *A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order*. Computers and Mathematics with Applications, 62(5), 2364-2373.

[28] Doha, E. H., Bhrawy, A. H., and Ezz-Eldien, S. S. (2012). *A new Jacobi operational matrix: an application for solving fractional differential equations*. Applied Mathematical Modelling, *36*(10), 4931-4943.

[29] Saadatmandi, A. (2014). *Bernstein operational matrix of fractional derivatives and its applications*. Applied Mathematical Modelling, *38*(4), 1365-1372.

[30] Doha, E., Bhrawy, A., and Ezz-Eldien, S. (2013). *Numerical approximations for fractional diffusion equations via a Chebyshev spectral-tau method*. Open Physics, 11(10), 1494-1503.

[31] Bhrawy, A. H., and Zaky, M. A. (2015). *A method based on the Jacobi tau approximation for solving multi-term time–space fractional partial differential equations*. Journal of Computational Physics, 281, 876-895.

[32] Bhrawy, A. H., Zaky, M. A., and Baleanu, D. (2015). *New numerical approximations for space-time fractional Burgers' equations via a Legendre spectral-collocation method*. Rom. Rep. Phys, *67*(2), 1-13.

[33] Bhrawy, A. H. (2016). *A Jacobi spectral collocation method for solving multi-dimensional nonlinear fractional sub-diffusion equations*. Numerical Algorithms, 73(1), 91-113.

[34] Erjaee, G. H., Akrami, M. H., and Atabakzadeh, M. H. (2013). *The operational matrix of fractional integration for shifted Legendre polynomials*. Iranian Journal of Science and Technology (Sciences), 37(4), 439-444.

[35] Doha, E. H., Bhrawy, A. H., and Ezz-Eldien, S. S. (2015). *An efficient Legendre spectral tau matrix formulation for solving fractional subdiffusion and reaction subdiffusion equations*. Journal of Computational and Nonlinear Dynamics, 10(2), 021019.

[36] Bhrawy, A. H., Doha, E. H., Baleanu, D., and Ezz-Eldien, S. S. (2015). *A spectral tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations*. Journal of Computational Physics, 293, 142-156.

[37] Bhrawy, A. H., Baleanu, D., and Assas, L. M. (2014).*Efficient generalized Laguerre-spectral methods for solving multi-term fractional differential equations on the half line.* Journal of Vibration and Control, 20(7), 973-985.

[38] Jarad, F., Abdeljawad, T., and Baleanu, D. (2010). *Fractional variational optimal control problems with delayed arguments.* Nonlinear Dynamics, 62(3), 609-614.

[39] Bhrawy, A. H., Ezz-Eldien, S. S., Doha, E. H., Abdelkawy, M. A., and Baleanu, D. (2017). *Solving fractional optimal control problems within a Chebyshev–Legendre operational technique.* International Journal of Control, 90(6), 1230-1244.

[40] Ezz-Eldien, S. S., Doha, E. H., Baleanu, D., and Bhrawy, A. H. (2017). *A numerical approach based on Legendre orthonormal polynomials for numerical solutions of fractional optimal control problems*. Journal of Vibration and Control, 23(1), 16-30.

[41] Agila, A., Baleanu, D., Eid, R., and Irfanoglu, B. (2017). *A freely damped oscillating fractional dynamic system modeled by fractional Euler–Lagrange equations*. Journal of Vibration and Control, 1077546316685228.

[42] Smale, S., Hirsch, M. W., and Devaney, R. L. (2003). *Differential equations, dynamical systems, and an introduction to chaos* (Vol. 60). Academic Press.

[43] Iooss, G., and Langford, W. F. (1980). *Conjectures on the routes to turbulence via bifurcations.* Annals of the New York Academy of Sciences, *357*(1), 489-505.

[44] Kelley, W. G., and Peterson, A. C. (2001). *Difference equations: an introduction with applications*. Academic Press.

[45] Heymans, N., and Podlubny, I. (2006). *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives*. Rheologica Acta, 45(5), 765-771.

[46] Mainardi, F., Pagnini, G., and Luchko, Y. (2007). *The fundamental solution of the space-time fractional diffusion equation.* Fractional Calculs and Applied Analysis. 153-192.

[47] Momani, S. (2006). *A numerical scheme for the solution of multi-order fractional differential equations.* Applied Mathematics and Computation, *182*(1), 761-770.

[48] Mainardi, F., Luchko, Y., and Pagnini, G. (2001). *The fundamental solution of the space-time fractional the space-time fractional diffusion* equation. Freactional. Calculs and Applied Analysis.4(2),153-192.

[49] Scalas, E., Gorenflo, R., and Mainardi, F. (2004). *Uncoupled continuous-time random walks: Solution and limiting behavior of the master equation*. Physical Review E, 69(1), 011107.

[50] Riewe, F. (1996). *Non conservative lagrangian and Hamiltonian mechanics.* Physical Review E, 53(2), 1890.

[51] Riewe, F. (1997). *Mechanics with fractional derivatives*. Physical Review E, 55(3), 3581.

[52] Klimek, M. (2001). *Fractional sequential mechanics—models with symmetric fractional derivative.* Czechoslovak Journal of Physics, 51(12), 1348-1354.

[53] Klimek, M. (2002*).* *Lagrangean and Hamiltonian fractional sequential mechanics.* Czechoslovak Journal of Physics, *52*(11), 1247-1253.

[54] Baleanu, D., and Agrawal, O. P. (2006). *Fractional Hamilton formalism within Caputo's derivative*. Czechoslovak Journal of Physics, *56*(10), 1087-1092.

[55] Agrawal, O. P. (2002). *Formulation of Euler–Lagrange equations for fractional variational problems*. Journal of Mathematical Analysis and Applications, 272(1), 368-379.

[56] Agrawal, O. P. (2006). *Fractional variational calculus and the transversality conditions.* Journal of Physics A: Mathematical and General, 39(33), 10375.

[57] Agrawal, O. P. (2007). *Generalized Euler—Lagrange equations and transversality conditions for FVPs in terms of the Caputo derivative*. Journal of Vibration and Control, 13(9-10), 1217-1237.

[58] Agrawal, O. P., and Baleanu, D. (2007). *A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems*. Journal of Vibration and Control, 13(9-10), 1269-1281.

[59] Chen, Y., Vinagre, B. M., and Podlubny, I. (2004). *Continued fraction expansion approaches to discretizing fractional order derivatives—an expository review*. Nonlinear Dynamics, 38(1), 155-170.

[60] Tarasov, V. E., and Zaslavsky, G. M. (2006). *Non holonomic constraints with fractional derivatives*. Journal of Physics A: Mathematical and General, 39(31), 9797.

[61] Agrawal, O. P. (2004). *Ageneral formulation and solution scheme for fractional optimal control problems*. Nonlinear Dynamics, 38(1-4), 323-337.

[62] Rabei, E. M., Nawafleh, K. I., Hijjawi, R. S., Muslih, S. I., and Baleanu, D. **(2007)**. *The Hamilton formalism with fractional derivatives*. Journal of Mathematical Analysis and Applications, 327(2), 891-897.

[63]Baleanu, D., and Muslih, **S. I. (2005).** *About Lagrangian formulation of classical fields within Riemann-Liouville fractional derivatives*. In ASME 2005 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference (pp. 1457-1464). American Society of Mechanical Engineers.

[64] Muslih, S. I., and Baleanu, D. (2005). *Hamiltonian formulation of systems with linear velocities within Riemann–Liouville fractional derivatives*. Journal of Mathematical Analysis and Applications, 304(2), 599-606.

[65] Baleanu, D., and Avkar, T. (2004). *Lagrangians with linear velocities within Riemann-Liouville fractional derivatives*. Nuovo Cimento. B, 119(1), 73-79.