VARIATIONAL METHODS FOR NONLINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH NONLOCAL TERMS

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ABSTRACT

VARIATIONAL METHODS FOR NONLINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH NONLOCAL TERMS

In this thesis, existence of standing waves for the Davey–Stewartson (DS) and generalized Davey–Stewartson (GDS) systems are established using variational methods. Since both the DS system and the GDS system reduce to a non-linear Schrödinger (NLS) equation with the only difference in their non-local term, arguments used in this thesis apply to a larger class of equations which include the DS and GDS systems as special cases. Existence of standing waves for an NLS equation is investigated in two ways: by considering an unconstrained minimization problem and a constrained minimization problem. These two variational methods apply to the GDS system as well and here the sufficient conditions on the existence of standing wave solutions for the GDS system which are imposed by these methods and the minimizers obtained are investigated in comparison.

ÖZET

YEREL OLMAYAN TERİMLER İÇEREN DOĞRUSAL OLMAYAN ELİPTİK KISMİ TÜREVLİ DENKLEMLER İÇİN VARYASYONEL METOTLAR

Bu tezde Davey–Stewartson (DS) ve genelleştirilmiş Davey–Stewartson (GDS) sistemlerinin durağan dalga çözümlerinin varlığı varyasyonel metotlar kullanılarak incelenmektedir. DS ve GDS sistemleri bir doğrusal olmayan Schrödinger (NLS) denklemine indirgenebildiğinden bu tezde kullanılan argümanlar DS ve GDS sistemlerini özel durum olarak içeren daha genel denklem sınıfları için de geçerlidir. Burada NLS denkleminin durağan dalga çözümlerinin varlığını göstermek için koşullu ve koşulsuz minimizasyon problemleri ele alınarak iki farklı varyasyonel metot kullanılmaktadır. Ayrıca GDS sistemi için de geçerli olan bu iki metodun durağan dalga çözümlerinin varlığı için getirdiği yeter koşullar ve iki metodun ürettiği çözümler karşılaştırmalı olarak incelenmektedir.

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LIST OF SYMBOLS

$\square$	End of proof
$B_\rho(x)$	The closed disk in $\mathbb{R}^2$ with center x and radius ρ
$C(\Omega)$	The space of continuous functions from Ω to $\mathbb{R}$ (or $\mathbb{C}$)
$C_0(\Omega)$	The space of continuous functions from Ω to $\mathbb{R}$ (or $\mathbb{C}$) compactly supported in Ω
$C^k(\Omega)$	The space of continuous functions u such that $D^\alpha u \in C(\Omega)$ for all $\alpha \in \mathbb{N}^n$ with $\lvert\alpha\rvert \leqslant k$
$C^\infty(\Omega)$	The space of infinitely differentiable functions
$C_0^\infty(\Omega)$	The space of infinitely differentiable functions compactly supported in Ω
D^α	$\dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\lvert\alpha\rvert = \alpha_1 + \cdots + \alpha_n$
$\hat{f}$	Fourier transform of f
$H^m(\Omega)$	$W^{m,2}(\Omega)$
$H^{-m}(\Omega)$	$W^{-m,2}(\Omega)$
$H^s(\mathbb{R}^2)$	The Banach space of classes of measurable functions $u : \mathbb{R}^2 \to \mathbb{R}$ (or $\mathbb{C}$) such that $(1 + \lvert y\rvert^s)\widehat{u} \in L^2(\mathbb{R}^2)$ for $0 < s < \infty$. For noninteger s, $$\|u\|_{H^s(\mathbb{R}^2)} = \|(1 + \lvert y\rvert^s)\widehat{u}\|_2$$
$L^p(\Omega)$	The Banach space of classes of measurable functions $u : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) such that $\int_\Omega \lvert u(x)\rvert^p < \infty$ if $1 \leqslant p < \infty$, or $\operatorname{ess\,sup}_\Omega \lvert u\rvert < \infty$ if $p = \infty$. $L^p(\Omega)$ is equipped with the norm $$\|u\|_p = \begin{cases} \left(\int_\Omega \lvert u(x)\rvert^p dx\right)^{1/p}, & \text{if } p < \infty, \\ \operatorname{ess\,sup}_\Omega \lvert u\rvert, & \text{if } p = \infty. \end{cases}$$
$\operatorname{meas}(\Omega)$	Lebesgue measure of the set Ω
p'	The conjugate of p given by $\dfrac{1}{p} + \dfrac{1}{p'} = 1$

$\Re(z)$	Real part of z
u_t	Partial derivative of $u(t,x)$ with respect to t
u_{x_i}	Partial derivative of $u(t,x)$ or $u(x)$ with respect to the ith space variable x_i
$W^{m,p}(\Omega)$	The Banach space of classes of measurable functions $u : \Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$) such that $D^\alpha u \in L^p(\Omega)$ in the sense of distributions, for every multi-index α with $\lvert\alpha\rvert \leqslant m$. $W^{m,p}$ is equipped with the norm $$\|u\|_{W^{m,p}} = \left(\sum_{\lvert\alpha\rvert \leqslant m} \|D^\alpha u\|_{L^p}^p \right)^{1/p}$$
$W_0^{m,p}(\Omega)$	The closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$
$W^{-m,p'}(\Omega)$	The dual of $W_0^{m,p}(\Omega)$
$\lvert x\rvert$	Used interchangably to denote the absolute value if $x \in \mathbb{R}$, the modulus of a complex number if $x \in \mathbb{C}$ and the Euclidean norm if $x \in \mathbb{R}^2$
$\lfloor x \rfloor$	Integer part of x
$x_n \rightharpoonup x$	Denotes that x_n converges to x weakly
$X \hookrightarrow Y$	Denotes that $X \subset Y$ with continuous injection
$X \subset\subset Y$	Denotes that $\overline{X} \subset Y$ and $\overline{X}$ is compact
$\mathbf{1}_\Omega$	Characteristic function of the set Ω, i.e., $$\mathbf{1}_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega. \end{cases}$$
Δu	Laplacian of u, i.e., $\sum_{i=1}^{n} u_{x_i x_i}$ in $\mathbb{R}^n$

1. INTRODUCTION

Variational methods play an important role in the analysis of nonlinear partial differential equations (PDEs). There is no general theory for establishing solutions for nonlinear PDEs however using variational methods we can recover solutions of some nonlinear PDE as being critical points of an appropriate "energy" functional J, exactly when the nonlinear differential operator A is the "derivative" of J in the variational sense. Symbolically we can write $A = J'$, hence the problem of finding a u such that $A(u) = 0$ becomes to find a u such that $J'(u) = 0$. The point is that although it might be difficult to show directly that the PDE has a solution, we may easily find a critical point to the functional. Since the critical points of the functional correspond to the solutions of the PDE in that case, the problem of existence of solutions may be addressed following a variational route. To begin with we can consider minimizers of the functional J which are critical points as well.

Assuming that J has the explicit form $J(u) := \int_{\mathbb{R}^2} L(\nabla u(x), u(x), x)dx$, where $L : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, we can say that any smooth minimizer of J is a solution of the Euler–Lagrange PDE and conversely we can try to find a solution to the Euler–Lagrange equation by searching for minimizers of J (see [1, Section 8.1.2]). Since our main aim is the guarantee the existence of a solution to a given nonlinear PDE we define J not only for smooth functions, but also for functions in some Sobolev space, $W^{1,q}$ for $1 < q < \infty$. After all, the wider the class of functions for which J is defined, the more candidates we will have for a minimizer.

The process of finding minimizers for a functional J runs as follows: We take a minimizing sequence $\{u_n\}$, that is, $J(u_n) \to m := \inf J$. We would now like to show that some subsequence of $\{u_n\}$ converges to an actual minimizer. For this to happen, one possibility is the presence of some kind of compactness. However, even if we utilize a coercivity condition on J, it turns out that we can only conclude that the minimizing sequence lies in a bounded set. On bounded domains we can overcome this problem by considering the weak topologies. Since we are assuming $1 < q < \infty$, so that L^q is reflexive, we can extract a weakly convergent subsequence and then utilizing strong convergence by compact imbeddings of Sobolev spaces in appropriate L^p-spaces. Yet in this thesis our interest is lies in nonlinear PDEs defined on $\mathbb{R}^2$ and we definitely have a problem in unbounded domains since there, Sobolev spaces cannot be imbedded compactly into an L^p-space. Indeed, as $\mathbb{R}^2$ is the "most" unbounded domain, using its radial symmetry and translation invariance we can extract some sort of compactness and this is the idea in Strauss' Compactness Lemma [2] and Lions' Concentration Compactness Principle [3]. These compactness results are given in the second chapter.

As we minimize J over a Sobolev space, failure of smoothness results in the need of some extra growth conditions on L and its derivatives so that any minimizer of J solves the Euler–Lagrange equation in the weak sense (see Theorem A.3.1).

In this thesis variational methods are applied to the nonlinear Schrödinger (NLS) equation on $\mathbb{R}^2$ given by

$$iv_t + \Delta v + g(v) = 0$$

with a nonlinear term g. For the NLS equation above, we take $g(v) = |v|^\sigma v$ for the sake of simplicity, where $0 < \sigma < \infty$. We construct the so-called *standing wave* solutions of the NLS equation which are of the form

$$v(t, x) = e^{i\omega t}u(x),$$

where $\omega \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^2)$, $u \neq 0$. In the third chapter we show existence of standing waves in two ways. First we use Weinstein's approach [4] and minimize the functional $J_\sigma(f) = \dfrac{\sigma\|\nabla f\|_2^\sigma \|f\|_2^2}{\|f\|_{\sigma+2}^{\sigma+2}}$ over $H^1(\mathbb{R}^2)$. In contrast, in the alternative approach, we introduce the kinetic and potential energies and minimize the kinetic energy over a space where the potential energy is zero [5]. The first approach is called an unconstrained minimization whereas the second approach is called a constrained minimization.

In dimensionless form the Davey-Stewartson (DS) system given in [6] reads as the following system for the complex amplitude $v(t, x, y)$ and the real mean velocity potential $\phi(t, x, y)$:

$$\begin{aligned} iv_t + \delta v_{xx} + \mu v_{yy} &= \chi |v|^\sigma v + b_1 v \phi_x \\ \nu \phi_{xx} + \phi_{yy} &= -b_2(|v|^2)_x, \end{aligned} \tag{1.0.1}$$

where δ, μ, ν, χ, b_1, b_2 are real constants, δ being positive. In [7], b_1 and b_2 are assumed to be positive. However during the flow of arguments positivity of these constants is not needed. Here b_1 and b_2 are of arbitrary sign. According to the signs of μ and ν as positive–positive, positive–negative, negative–positive and negative–negative, these systems may be classified as elliptic–elliptic, elliptic–hyperbolic, hyperbolic–elliptic and hyperbolic–hyperbolic, respectively.

In the fourth chapter we consider the elliptic–elliptic case in two dimensions and assume that $\delta = \mu = \nu = 1$, so that we have

$$\begin{aligned} iv_t + \Delta v &= \chi |v|^\sigma v + b_1 v \phi_x \\ -\Delta \phi &= b_2(|v|^2)_x, \end{aligned} \tag{1.0.2}$$

where $\chi \in \mathbb{R}$, $\sigma > 0$. The system (1.0.2) may be reduced to a single equation in v by applying the Fourier transform. Let E_1 be the nonlocal linear operator defined by

$$[\widehat{E_1(\psi)}](\xi) = \gamma_1(\xi)\widehat{\psi}(\xi),$$

where $\gamma_1(\xi) = \dfrac{\xi_1^2}{|\xi|^2}$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. To write this system as a single equation for v, we begin by expressing ϕ in terms of v by solving the Poisson equation $(1.0.2)_2$. Thus (1.0.2) can be reduced to the following nonlinear Schrödinger equation

$$iv_t + \Delta v = \chi |v|^\sigma v + b E_1(|v|^2) v, \tag{1.0.3}$$

where $b = -b_1 b_2$. In [8] the existence of standing waves for the DS system is guaranteed by considering an unconstrained minimization problem like in [4] for the NLS equation. However, in the forth chapter we mainly follow [7] and establish existence of standing waves by considering a constrained minimization problem.

Finally, the fifth chapter is devoted to the cubic NLS equation with an additional nonlocal term in two space dimensions:

$$iv_t + \Delta v = \chi |v|^2 v + bK(|v|^2)v, \tag{1.0.4}$$

where the nonlocal term is given in terms of Fourier transform variables $\xi = (\xi_1, \xi_2)$ as $\widehat{K(f)}(\xi) = \alpha(\xi)\widehat{f}(\xi)$. The symbol $\alpha(\xi)$ is assumed to satisfy:

(A1) $\alpha(\xi)$ is even and homogenous of degree zero,

(A2) $0 \leqslant \alpha(\xi) \leqslant \alpha_M$ for all $\xi \in \mathbb{R}^2$,

(A3) $\alpha_1 := \lim_{s\to\infty} \alpha(s\xi_1, \xi_2)$ and $\alpha_2 := \lim_{s\to 0^+} \alpha(s\xi_1, \xi_2)$ exist.

The generalized Davey–Stewartson (GDS) system

$$\begin{aligned} iv_t + \delta v_{xx} + v_{yy} &= \chi |v|^2 v + b(\phi_{1,x} + \phi_{2,y})v, \\ \phi_{1,xx} + m_2\phi_{1,yy} + n\phi_{2,xy} &= \left(|v|^2\right)_x, \\ \lambda\phi_{2,xx} + m_1\phi_{2,yy} + n\phi_{1,xy} &= \left(|v|^2\right)_y, \end{aligned} \tag{1.0.5}$$

which is derived by Babaoglu and Erbay [9] to model the propagation of waves in a bulk medium composed of an elastic medium with couple stresses, is classified in [10] as elliptic–elliptic–elliptic (EEE), elliptic–hyperbolic–hyperbolic and elliptic–elliptic–hyperbolic according to the signs of the physical parameters (m_1, m_2, λ): $(+,+,+)$, $(+,-,-)$ and $(+,+,-)$, respectively. The GDS system can be written in the EEE case as in (1.0.4) with

$$\alpha(\xi) = \frac{\lambda\xi_1^4 + (1 + m_1 - 2n)\xi_1^2\xi_2^2 + m_2\xi_2^4}{\lambda\xi_1^4 + (m_1 + \lambda m_2 - n^2)\xi_1^2\xi_2^2 + m_1 m_2 \xi_2^4}, \tag{1.0.6}$$

given explicitely. The symbol $\alpha(\xi)$ then satisfies (A1)–(A3) where $\alpha_M = \max\{1, 1/m_1\}$ (see [10]) and $\alpha_1 = 1$, $\alpha_2 = \dfrac{1}{m_1}$. Here we do not assume that the symbol $\alpha(\xi)$ is given by the explicit form in (1.0.6), instead we only assume that it satisfies (A1)–(A3). Hence the results apply to the GDS system a special case.

In [11] the problem of existence of travelling waves for GDS system is considered for the cases EEE and HEE. The necessary conditions for existence are Pohozaev type identities. In [12] Pohozaev type identities play an important role in restricting the parameters ω, χ and b in order to establish the existence of standing waves. The existence of standing waves for a GDS system is established in [12] by extending the analysis done by Weinstein for the NLS equation [4] and by Papanicolaou et. al. for the DS system [8]. However here we choose a different route and obtain the existence of standing waves for a GDS system under less stringent conditions on the parameters. The arguments in [5, 7] can be modified so that they apply to a larger class of equations that include the GDS system as a special case. Here, however, due to assumption (A3) the more general case considered in [12] is not treated.

2. MATHEMATICAL PRELIMINARIES

In fact the compactness results given in this chapter apply to $\mathbb{R}^n$ with the same arguments. However to be in harmony with the following chapters we consider them in $\mathbb{R}^2$.

2.1. Strauss' Compactness Lemma

To be compact in L^p on an unbounded domain, a class of functions has to be uniformly small at infinity. To achieve this we consider *radial* functions, i.e., functions f of a single variable $|x|$. Before stating the main result of this section we will need the following technical lemma whose proof follows as in [13, 2]. In this section we assume that the given functions are real valued.

Lemma 2.1.1. [13, Radial Lemma A.II] *Every radial function $u \in H^1(\mathbb{R}^2)$ is almost everywhere equal to a function $U(x)$, continuous for $x \neq 0$ and such that*

$$|U(x)| \leqslant C|x|^{-1/2}\|u\|_{H^1} \quad \textit{for} \ \ |x| \geqslant 1.$$

Proof. Let $u \in H^1$ be a radial function. Define $r(x) = |x|$. Since u is radial $u(x) = F(r(x))$ for some F. Clearly r is continuous. Now, we need to show that F is continuous. Since $u \in H^1$ and since for any $x \in S^1$ $F(\rho) = u(\rho x)$ implies $|F'(\rho)| = |\nabla u(\rho x)|$, we have

$$\begin{aligned}\|F\|_{H^1(0,\infty)} &= \int_0^\infty |u(\rho x)|^2 + |\nabla u(\rho x)|^2 d\rho \\ &\qquad < \int_0^\infty \left(\int_{\rho S^1} |u(y)|^2 + |\nabla u(y)|^2 dA \right) d\rho = \|u\|_{H^1(\mathbb{R}^2)} < \infty.\end{aligned}$$

This gives that F is absolutely continuous on $(0, \infty)$. Now define $U(x) = (F \circ r)(x)$ for $x \neq 0$ and $U(x) = 0$ for $x = 0$. Then U is continuous except $x = 0$ and almost everywhere equal to u as claimed.

Note that $\|U\|_{H^1(\mathbb{R}^2)} = \|u\|_{H^1(\mathbb{R}^2)} < \infty$. Since $C^\infty \cap H^1$ is dense H^1 it suffices to show the estimate for functions in $C^\infty \cap H^1$. Then the result follows from density.

Let $v \in C^\infty \cap H^1$. We have

$$\begin{aligned} -(rv^2)_r &= -[(r^{1/2}v)^2]_r \\ &= -2(r^{1/2}v)_r(r^{1/2}v) \\ &\leqslant \left|2(r^{1/2}v)_r(r^{1/2}v)\right| \\ &\leqslant [(r^{1/2}v)_r]^2 + [r^{1/2}v]^2 \\ &= r(v_r^2 + v^2) + \left(\frac{1}{2}v^2\right)_r + \frac{1}{4r}v^2. \end{aligned}$$

Now integrating over $[r, \infty]$ we get

$$\begin{aligned} rv^2 &\leqslant \int_r^\infty \rho(v_r^2 + v^2)d\rho - \frac{1}{2}v^2 + \int_r^\infty \frac{1}{4\rho}v^2 d\rho \\ &\leqslant \int_r^\infty \rho(v_r^2 + v^2)d\rho - \frac{1}{2}v^2 + \int_r^\infty \frac{1}{4}\rho v^2 d\rho, \end{aligned}$$

for $r \geqslant 1$. Thus

$$rv^2(r) \leqslant \frac{5}{8\pi}\|v\|_{H^1}^2,$$

and the estimate is established. □

Let us denote by $H_r^1(\mathbb{R}^2)$ the subspace of $H^1(\mathbb{R}^2)$ formed by the radial functions.

Lemma 2.1.2 (Strauss Compactness Lemma)**.** [13, Theorem A.I'] *The injection* $H_r^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ *is compact for* $2 < p < \infty$.

Proof. Sobolev's imbedding theorem (Section C.1) guarantees the continuous imbedding $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ for $2 < p < \infty$. Let $(u_n) \subset H_r^1(\mathbb{R}^2)$ be a sequence such that

$\|u_n\|_{H^1} \leqslant M$ for some M. Then from the lemma above we have

$$\lim_{|x|\to\infty} |u_n(x)| = 0, \text{ uniformly with respect to } n. \tag{2.1.1}$$

Since (u_n) is bounded in H^1 it has a subsequence, say (u_{n_k}), converging weakly to $u \in H^1$. Extracting a subsequence of (u_{n_k}), denoted by (u_{n_k}), such that it converges almost everywhere in $\mathbb{R}^2$ to u we have that u is radial.

Since $||u_{n_k}|^p - |u|^p| \leqslant |u_{n_k} - u|^p$, $|u_{n_k}|^p \to |u|^p$ almost everywhere. We want to show that (u_{n_k}) converges strongly to u in $L^p(\mathbb{R}^2)$.

Now, let $\epsilon > 0$ and q be such that $p < q < \infty$. Since $\dfrac{|s|^p}{|s|^2 + |s|^q} \to 0$ as $s \to 0$, and (2.1.1) holds true for (u_{n_k}), there exists $R_0 > 0$ such that

$$|x| \geqslant R_0 \text{ implies } |u_{n_k}(x)|^p \leqslant \epsilon \left(|u_{n_k}(x)|^2 + |u_{n_k}(x)|^q\right) \text{ for all } n_k \in \mathbb{N}.$$

Also we have $\|u_{n_k}\|_q \leqslant C_1\|u_{n_k}\|_{H^1}$ by the continuous injection and $\|u_{n_k}\|_2 \leqslant \|u_{n_k}\|_{H^1}$ by definition. Therefore, by Fatou's lemma, $u \in L^p(\mathbb{R}^2)$, and

$$\int_{\{|x|\geqslant R_0\}} |u(x)|^p dx \leqslant \epsilon C,$$

where $C = M(1 + C_2)$. We know that for bounded domains the injection $H^1 \hookrightarrow L^p$ is compact, hence there exists $N_0 \in \mathbb{N}$ such that for any $n_k \geqslant N_0$,

$$\int_{\{|x|<R_0\}} ||u_{n_k}(x)|^p - |u(x)|^p|\, dx \leqslant \epsilon.$$

Hence we have for $n_k \geqslant N_0$,

$$\int_{\mathbb{R}^2} ||u_{n_k}(x)|^p - |u(x)|^p|\, dx \leqslant 2\epsilon C + \epsilon.$$

Also since we have almost everywhere convergence we conclude that

$$u_{n_k} \to u \quad \text{strongly in } L^p(\mathbb{R}^2) \text{ as } n_k \to \infty.$$

This proves the lemma. $\square$

2.2. Concentration Compactness Principle

The invariance of $\mathbb{R}^2$ under the actions of noncompact groups of translations and dilations might cause loss of compactness, e.g., Rellich–Kondrashov theorem is no more valid in $\mathbb{R}^2$. However using this invariance of $\mathbb{R}^2$, we state the following theorem which indicates that the only possible loss of compactness for minimizing sequences stems from splitting of the functions at least in two parts which are going infinitely away from each other [3]. This method enables us to solve problems with some form of "local compactness". In this section we take the given functions complex valued.

Theorem 2.2.1 (Concentration Compactness Lemma)**.** [14, Lemma 8.3.8] *If $\mu > 0$ and if (u_n) is a bounded sequence of $H^1(\mathbb{R}^2)$ such that*

$$\int_{\mathbb{R}^2} |u_n(x)|^2 dx = \mu,$$

then there exists a subsequence, which we still denote by (u_n), for which one of the following properties holds.

- *(i)* (Concentration) *There exists a sequence $(y_n) \subset \mathbb{R}^2$ such that for every $\epsilon > 0$, there exists $R < \infty$ so that $\displaystyle\int_{\{|x-y_n| \leqslant R\}} |u_n(x)|^2 dx \geqslant \mu - \epsilon$.*
- *(ii)* (Vanishing) $\displaystyle\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{\{|x-y|\leqslant R\}} |u_n(x)|^2 dx = 0$ *for all $R > 0$.*

(iii) (Dichotomy) *There exists* $\gamma \in (0,\mu)$ *so that for every* $\epsilon > 0$, *there exist* $N_0 \geqslant 0$ *and two sequences* $(v_n), (w_n) \subset H^1(\mathbb{R}^2)$, *with disjoint supports, such that for* $n \geqslant N_0$

$$\|v_n\|_{H^1} + \|w_n\|_{H^1} \leqslant 4 \sup_{n\in\mathbb{N}} \|u_n\|_{H^1}; \tag{2.2.1}$$

$$\|u_n - v_n - w_n\|_2 \leqslant \epsilon; \tag{2.2.2}$$

$$\left| \int_{\mathbb{R}^2} |v_n(x)|^2 dx - \gamma \right| \leqslant \epsilon; \tag{2.2.3}$$

$$\left| \int_{\mathbb{R}^2} |w_n(x)|^2 dx + \gamma - \mu \right| \leqslant \epsilon; \tag{2.2.4}$$

$$\int_{\mathbb{R}^2} |\nabla u_n(x)|^2 - |\nabla v_n(x)|^2 - |\nabla w_n(x)|^2 dx \geqslant -\epsilon. \tag{2.2.5}$$

For this proof we follow along the lines of [14].

Proof. Consider the functions

$$Q_n(t) = \sup_{y\in\mathbb{R}^2} \int_{\{x-y\leqslant t\}} |u_n(x)|^2 dx.$$

Clearly, $0 \leqslant Q_n(t) \leqslant \mu$ for all $t \geqslant 0$ and for all $n \in \mathbb{N}$. Also since the integrand is positive, $Q_n(t)$ is an increasing function of t. Define

$$F_n(t) = \int_0^t Q_n(s) ds.$$

It can be easily shown that F_n is a sequence of C^1, convex, increasing, nonnegative functions. Also, as

$$\begin{aligned} |F_n(t) - F_n(\tau)| &= \left| \int_0^t Q_n(s)ds - \int_0^\tau Q_n(s)ds \right| \\ &= \left| \int_\tau^t Q_n(s)ds \right| \leqslant \mu|t-\tau| \qquad \text{for all } t,\tau \geqslant 0, \end{aligned}$$

F_n are uniformly Lipschitz continuous. Hence by the Arzela-Ascoli theorem, there exists a subsequence of F_n, still denoted by F_n, such that $F_n \to F$ uniformly on compact subsets of $\mathbb{R}_+$. It follows that F is also convex, increasing, nonnegative, and Lipschitz continuous. Moreoever, Rademacher theorem [1, Theorem 5.8.6] implies that F is differentiable almost everywhere. Then $F_n'(t) \to F'(t)$ for almost every t as $n \to \infty$. Let $Q(t) = F'(t)$. $Q'(t) \geqslant 0$ by almost everywhere convergence, so Q is increasing. Also $Q(t) = F'(t) \geqslant 0$ since F increasing. Clearly, $0 \leqslant Q(t) \leqslant \mu$ for almost every $t \in \mathbb{R}_+$. Let $\gamma = \lim_{t\to\infty} Q(t)$. We will consider three cases separately:

(a) *The case $\gamma = \mu$.* We claim that in this case *(i)* occurs. Consider first $0 < \lambda < \mu$. Since Q is increasing, there exists $R' > 0$ such that $Q(R') > \lambda$. Thus $Q_n(R') > \lambda$ for $n \geqslant N_0 \geqslant 0$. For every $n \leqslant N_0$, there exists R_n such that $Q_n(R_n) > \lambda$. Taking $R(\lambda) = \max\{R', R_0, \dots, R_{N_0}\}$, it follows that $Q_n(R(\lambda)) > \lambda$, for all $n \in \mathbb{N}$. Therefore, as Q_n is a supremum, there exists $x_n(\lambda)$ such that

$$\int_{\{|x-x_n(\lambda)|\leqslant R(\lambda)\}} |u_n(x)|^2 dx > \lambda.$$

Let $y_n = x_n(\mu/2)$. Given $\lambda > \mu/2$, let $R = R(\mu/2) + 2R(\lambda)$. We claim that

$$\int_{\{|x-y_n|\leqslant R\}} |u_n(x)|^2 dx > \lambda \quad \text{for all } n \in \mathbb{N}.$$

Indeed, $|x_n(\lambda) - y_n| \leqslant R(\mu/2) + R(\lambda)$ since otherwise

$$\{x : |x - x_n(\lambda)| \leqslant R(\lambda)\} \cap \{x : |x - y_n| \leqslant R(\mu/2)\} = \emptyset$$

which then would imply

$$\int_{\mathbb{R}^2} |u_n|^2 dx \geqslant \int_{\{|x-y_n|\leqslant R(\mu/2)\}} |u_n|^2 dx + \int_{\{|x-x_n(\lambda)|\leqslant R(\lambda)\}} |u_n|^2 dx \geqslant \frac{\mu}{2} + \lambda > \mu,$$

which contradicts the assumption. Hence

$$\{x : |x - x_n(\lambda)| \leqslant R(\lambda)\} \subset \{x : |x - y_n| \leqslant R\}.$$

To see this, let $x \in \{x : |x - x_n(\lambda)| \leqslant R(\lambda)\}$, i.e., $|x - x_n(\lambda)| \leqslant R(\lambda)$. Then

$$\begin{aligned}|x - y_n| &= |x - x_n(\lambda) + x_n(\lambda) - y_n| \\ &\leqslant |x - x_n(\lambda)| + |x_n(\lambda) - y_n| \\ &\leqslant R(\lambda) + (R(\mu/2) + R(\lambda)) = R,\end{aligned}$$

hence $x \in \{x : |x - y_n| \leqslant R\}$. Therefore we have

$$\int_{\{x:|x-y_n|\leqslant R\}} |u_n(x)|^2 dx \geqslant \int_{\{x:|x-x_n(\lambda)|\leqslant R(\lambda)\}} |u_n(x)|^2 dx \geqslant \lambda,$$

and so *(i)* follows.

(b) *The case* $\gamma = 0$. Here we want to show that *(ii)* occurs. This is easy to see as follows: Since Q is increasing,

$$0 = \gamma = \lim_{t\to\infty} Q(t) \geqslant Q(R) = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{\{|x-y|\leqslant R\}} |u_n(x)|^2 dx \geqslant 0.$$

(c) *The case* $\gamma \in (0, \mu)$. We claim that in this case *(iii)* occurs. Let $\epsilon > 0$. There exits R_0 such that $\gamma - \epsilon < Q(R) < \gamma + \epsilon$ for all $R \geqslant R_0$. In particular, there exist $R > 0$, $N_0 \geqslant 0$ and $\overline{R} > \max\{2R, 1/\epsilon\}$ such that

$$\gamma - \epsilon < Q_n(R) \leqslant Q_n(\overline{R}) < \gamma + \epsilon, \qquad \text{for all } n \geqslant N_0,$$

hence, there exists (y_n) such that

$$\gamma - \epsilon < \int_{\{|x-y_n|\leqslant R\}} |u_n(x)|^2 dx < \gamma + \epsilon.$$

Let $\rho \in C_0^\infty(\mathbb{R}^2)$ be such that $\rho \equiv 1$ on $\{|x| \leqslant R\}$, $\rho \equiv 0$ on $\{|x| \geqslant \overline{R}/2\}$, $0 \leqslant \rho \leqslant 1$, and $|\nabla\rho| \leqslant 2\epsilon$. Also, let $\theta \in C_0^\infty(\mathbb{R}^2)$ be such that $\theta \equiv 0$ on $\{|x| \leqslant \overline{R}/2\}$, $\theta \equiv 1$ on $\{|x| \geqslant \overline{R}\}$, $0 \leqslant \theta \leqslant 1$, and $|\nabla\theta| \leqslant 2\epsilon$.

Define $\rho_n(x) = \rho(x - y_n)$ and $\theta_n(x) = \theta(x - y_n)$ and finally let $v_n = \rho_n u_n$ and $w_n = \theta_n u_n$. Clearly, the supports of v_n and w_n are disjoint as ρ and θ are chosen to be so. Also, as Q_n is increasing,

$$
\begin{aligned}
Q_n(R) \leqslant \int_{\mathbb{R}^2} |v_n(x)|^2 dx = \int_{\mathbb{R}^2} |\rho_n(x) u_n(x)|^2 dx = \int_{\mathbb{R}^2} |\rho(x-y_n) u_n(x)|^2 dx \\
= \int_{\{|x-y_n| \leqslant \overline{R}/2\}} |u_n(x)|^2 dx \leqslant Q_n(\overline{R}/2) \leqslant Q_n(\overline{R}),
\end{aligned}
$$

hence equation (2.2.3) follows easily. For the equation (2.2.1), note that $\nabla v_n = \rho_n \nabla u_n + u_n \nabla \rho_n$, and so, $|\nabla v_n|^2 \leqslant \rho_n^2 |\nabla u_n|^2 + 2\epsilon |u_n||\nabla u_n| + 4\epsilon^2 |u_n|^2$. Hence we have,

$$
\int_{\mathbb{R}^2} |\nabla v_n(x)|^2 dx \leqslant \int_{\mathbb{R}^2} \rho_n^2(x) |\nabla u_n(x)|^2 dx + C\epsilon,
$$

similarly,

$$
\int_{\mathbb{R}^2} |\nabla w_n(x)|^2 dx \leqslant \int_{\mathbb{R}^2} \theta_n^2(x) |\nabla u_n(x)|^2 dx + C\epsilon.
$$

Thus equation (2.2.1) can be established. From the two integrals above it also follows that

$$
\int_{\mathbb{R}^2} |\nabla u_n|^2 - |\nabla v_n|^2 - |\nabla w_n|^2 dx \geqslant \int_{\mathbb{R}^2} \left(1 - \rho_n^2 - \theta_n^2\right) |\nabla u_n|^2 dx \ - 2C\epsilon \geqslant -2C\epsilon,
$$

where $1 - \rho_n^2 - \theta_n^2 \geqslant 0$, as the supports of ρ_n and θ_n are disjoint. Hence we get (2.2.5). Now it remains to prove (2.2.2) and (2.2.4). To see these, consider

$$
\begin{aligned}
\int_{\mathbb{R}^2} |u_n - v_n - w_n|^2 dx &\leqslant \int_{\{R \leqslant |x-y_n| \leqslant \overline{R}\}} |u_n|^2 dx \\
&= \int_{\{|x-y_n| \leqslant \overline{R}\}} |u_n|^2 dx - \int_{\{|x-y_n| \leqslant \ R\}} |u_n|^2 dx \\
&\leqslant Q_n(\overline{R}) - \int_{\{|x-y_n| \leqslant R\}} |u_n|^2 dx \\
&\leqslant (\gamma + \epsilon) - (\gamma - \epsilon) = 2\epsilon.
\end{aligned}
$$

Hence (2.2.2) and (2.2.4) follow, and this completes the proof. $\square$

Hence we see that given any minimizing sequence (u_n), we can consider a new minimizing sequence $(u_n^{y_k,s_k})$ for which by appropriate choices of translations (y_k) and dilations (s_k) we can compansate the failure of compactness [15]. An alternative approach to this principle is given also in [15].

2.3. Positive Solutions of Elliptic PDE's

This technical theorem is used in the following chapters to show positivity of standing waves.

Theorem 2.3.1. [14, Lemma 8.1.12] *Let $a : \mathbb{R}^2 \to \mathbb{R}$ be continuous, and assume that $a(x) \to 0$ as $|x| \to \infty$. If there exists $v \in H^1(\mathbb{R}^2)$ such that*

$$\int_{\mathbb{R}^2} |\nabla v|^2 - a|v|^2 dx < 0$$

then there exist $\lambda > 0$ and a positive solution $u \in H^1(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ of the equation

$$-\Delta u + \lambda u = au.$$

In addition, if $w \in H^1(\mathbb{R}^2)$ is nonnegative, $w \neq 0$, and if there exists $\nu \in \mathbb{R}$ such that $-\Delta w + \nu w = aw$, then there exists $c > 0$ such that $w = cu$. In particular, $\nu = \lambda$.

Sketch of the proof. First show that the following minimization problem has a nonnegative solution:

$$\begin{aligned} &\|u\|_2 = 1, \\ &J(u) = \min\{J(v) : v \in H^1, \|v\|_2 = 1\}, \end{aligned}$$

where $J(u) = \displaystyle\int_{\mathbb{R}^2} |\nabla u|^2 - a|u|^2 dx$.

Taking a minimizing sequence (v_n) yields that (u_n), where $u_n = |v_n|$, is also a minimizing sequence which is bounded in $H^1(\mathbb{R}^2)$ (Section B.1). Passing to a subsequence one can show that there exists $u \in H^1(\mathbb{R}^2)$ such that $u_n \rightharpoonup u$ and u satisfies the minimization problem above.

Therefore, there exists a Lagrange multiplier λ such that

$$-\Delta u + \lambda u = au.$$

By standard arguments, we get

$$\lambda = -\inf\{J(v) : \|v\|_2 = 1\} > 0,$$

also easily we see that $u \in H^2(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ and from the strong maximum principle, $u > 0$ on $\mathbb{R}^2$ [16, Corollary 8.21]. So the first part of the statement follows.

The claim $\nu = \lambda$ follows easily when one assumes the existence of such a ν. Second claim of the second part, i.e., $w = cu$ for some $c > 0$ follows by the way of contradiction. $\square$

3. NONLINEAR SCHRÖDINGER EQUATION

The existence of solutions to equations of the form $-\Delta u = f(u)$ has been extensively studied [1]. The first results investigate the case $f(u) = |u|^\sigma u$, which is associated to standing wave solutions of the NLS equation. These results were later generalized to larger classes of functions f in [2, 13] in dimensions $n \geqslant 3$ and in [5] in dimension two. The results below apply in dimensions $n \geqslant 3$ with some modifications however to be consistent with the following chapters we consider the NLS equation in $\mathbb{R}^2$ only.

3.1. Unconstrained Minimization Problem

Considering solutions of

$$iv_t + \Delta v + |v|^\sigma v = 0$$

of the form $v(t,x) = e^{i\omega t}u(x)$ we see that u then satisfies

$$\Delta u - \omega u + |u|^\sigma u = 0. \tag{3.1.1}$$

Throughout the arguments in this section we take $\omega = 1$ which can be achieved after a normalization: $u(x) = \omega^{1/\sigma}\psi(\sqrt{\omega}x)$. We also assume that the mentioned functions in this section are real valued.

To show existence of solutions of (3.1.1) it suffices to minimize the functional

$$J_\sigma(f) := \frac{\sigma\|\nabla f\|_2^\sigma \|f\|_2^2}{\|f\|_{\sigma+2}^{\sigma+2}}.$$

We show that the minimum is attained at some H^1-function u^*. By scaling we can take $\|\nabla u^*\|_2 = 1$ and $\|u^*\|_2 = 1$. Then computing the Euler-Lagrange equation leads to (3.1.1) [4].

Theorem 3.1.1. [4, Theorem B] *For* $0 < \sigma < \infty$*,*

$$\alpha := \inf_{f \in H^1(\mathbb{R}^2)} J_\sigma(f)$$

is attained at a function u *with the following properties:*

(i) u *is positive and a function of* $|x|$ *only.*
(ii) $u \in C^2(\mathbb{R}^2)$*.*
(iii) u *is a solution of* (3.1.1) *of minimal* L^2 *norm (the ground state).*

In addition,

$$\alpha = \frac{2\|u\|_2^\sigma}{\sigma + 2}.$$

Thus we see that the equation (3.1.1) has a positive, radial solution of class $H^1(\mathbb{R}^2)$. The proof of this theorem runs as in [4] by using one parameter scalings and Strauss' Compactness Lemma.

Proof. If we set $u^{a,b}(x) = bu(ax)$, then we have

$$\begin{aligned} J_\sigma(u^{a,b}) &= J_\sigma(u), \\ \|u^{a,b}\|_2^2 &= a^{-2} b^2 \|u\|_2^2, \\ \|\nabla(u^{a,b})\|_2^2 &= b^2 \|\nabla u\|_2^2. \end{aligned} \tag{3.1.2}$$

Since $J_\sigma \geqslant 0$, there exists a minimizing sequence $f_n \in H^1(\mathbb{R}^2) \cap L^{\sigma+2}(\mathbb{R}^2)$, i.e.,

$$\alpha = \inf J_\sigma(f) = \lim_{n\to\infty} J_\sigma(f_n) < \infty.$$

We can assume $f_n > 0$, and moreover by Schwarz symmetrization (Section B.1) we can take $f_n = f_n(|x|)$.

Choosing $a_n = \dfrac{\|f_n\|_2}{\|\nabla f_n\|_2}$ and $b_n = \dfrac{1}{\|\nabla f_n\|_2}$, we obtain a sequence $u_n(x) = f^{a_n,b_n}(x)$. (u_n) clearly satisfies the followings:

(a) $u_n \geqslant 0$, $u_n = u_n(|x|)$,

(b) $u_n \in H^1(\mathbb{R}^2)$,

(c) $\|u_n\|_2 = 1$ and $\|\nabla u_n\|_2 = 1$,

(d) $J_\sigma(u_n) \to \alpha$ as $n \to \infty$.

Note that by (c), (u_n) is bounded in $H^1(\mathbb{R}^2)$, hence, has a subsequence, denoted by (u_n) again, converging to $u^* \in H^1(\mathbb{R}^2)$ weakly, i.e., $u_n \rightharpoonup u^*$ in $H^1(\mathbb{R}^2)$.

Since (u_n) is uniformly bounded in H^1 and u_n are radial, it follows from the Strauss Compactness Lemma that $u_n \to u^*$ strongly in $L^{\sigma+2}(\mathbb{R}^2)$ for $0 < \sigma < \infty$.

By weak convergence, we have $\|u^*\|_2 \leqslant 1$, $\|\nabla u^*\|_2 \leqslant 1$, hence,

$$\alpha \leqslant J_\sigma(u^*) \leqslant \frac{\sigma}{\int |u^*(x)|^{\sigma+2}dx} = \lim_{n\to\infty} J_\sigma(u_n) = \alpha.$$

This implies that $\|\nabla u^*\|_2^\sigma \|u^*\|_2^2 = 1$ and since $\|u^*\|_2 \leqslant 1$, $\|\nabla u^*\|_2 \leqslant 1$, we get $\|u^*\|_2 = 1$ and $\|\nabla u^*\|_2 = 1$. Therefore

$$u_n \to u^* \quad \text{strongly in } H^1(\mathbb{R}^2),$$

and this proves *(i)*.

For *(ii)*, we refer to [5, Theorem 1].

Now, the minimizing function u^* is in H^1 and satisfies the Euler-Lagrange equation:

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} J_\sigma(u^* + \epsilon\eta) = 0 \quad \text{for all } \eta \in C_0^\infty(\mathbb{R}^2).$$

Since $\|u^*\|_2 = 1$ and $\|\nabla u^*\|_2 = 1$ the Euler-Lagrange equation becomes

$$\Delta u^* - u^* + \alpha \left(\frac{\sigma+2}{2}\right) |u^*|\sigma u^* = 0.$$

Let $u^* = \left[\alpha\left(\frac{\sigma+2}{2}\right)\right]^{-1/\sigma} u$. Then

$$\begin{aligned} 0 = \left[\alpha\left(\frac{\sigma+2}{2}\right)\right]^{-1/\sigma} \Delta u - \left[\alpha\left(\frac{\sigma+2}{2}\right)\right]^{-1/\sigma} u \\ + \alpha\left(\frac{\sigma+2}{2}\right)\left[\alpha\left(\frac{\sigma+2}{2}\right)\right]^{-(\sigma+1)/\sigma} |u|^\sigma u \end{aligned}$$

and dividing both sides by $\left[\alpha\left(\frac{\sigma+2}{2}\right)\right]^{-1/\sigma}$, which is nonzero, we see that u satisfies (3.1.1).

Also, $1 = \|u^*\|_2^\sigma = \left(\int |u^*(x)|^2 dx\right)^{\sigma/2} = \frac{2}{\alpha(\sigma+2)}\left(\int |u(x)|^2 dx\right)^{\sigma/2}$ gives

$$\alpha = \frac{2\|u\|_2^\sigma}{\sigma+2}.$$

Thus *(iii)* follows. □

3.2. Constrained Minimization Problem

Now, we show the existence of a non-zero $u \in H^1(\mathbb{R}^2)$ satisfying

$$-\Delta u + \omega u - |u|^\sigma u = 0 \tag{3.2.1}$$

using a constrained minimization problem as in [5]. First let us fix some notation. Given $0 < \sigma < \infty$ and $\omega > 0$ define

$$T(u) := \|\nabla u\|_2^2, \qquad V(u) := \frac{1}{\sigma+2}\|u\|_{\sigma+2}^{\sigma+2} - \frac{\omega}{2}\|u\|_2^2.$$

Note that u is a solution of (3.2.1) if and only if u is a critical point of $S(u) := \frac{1}{2}T(u) - V(u)$. Also to be consistent with the notation in [14] define

$$\mathcal{V} := \{u \in H^1(\mathbb{R}^2) : u \neq 0 \text{ and } u \text{ satisfies } (3.2.1)\},$$
$$\mathcal{W} := \{u \in \mathcal{V} : S(u) \leqslant S(\phi), \forall \phi \in \mathcal{V}\}.$$

It can be easily shown that if u solves (3.2.1) then $V(u) = 0$ and hence $S(u) = \frac{1}{2}T(u)$. This result follows from [14, Lemma 8.1.3].

Now we give the theorem stating the existence of standing waves. Proof of this theorem can be found in [14], where arguments of [5] are used.

Theorem 3.2.1. [14, Theorem 8.1.6] *For $0 < \sigma < \infty$ and $\omega > 0$ the following hold:*

(i) $\mathcal{V}$ *and* $\mathcal{W}$ *are non-empty.*

(ii) $u \in \mathcal{W}$ *if and only if* $u \in N$ *with* $\|u\|_2^2 = c$ *solves*

$$S(u) = \min\{S(\phi) : \phi \in N\},$$

where $N := \{u \in H^1(\mathbb{R}^2) : u \neq 0, V(u) = 0\}$ *and* $c = \dfrac{4}{\omega\sigma}\min\{S(\phi) : \phi \in N\}$.

(iii) There exists a real valued, positive, spherically symmetric and decreasing function $\varphi \in \mathcal{W}$ *such that* $\mathcal{W} = \bigcup\{e^{i\theta}\varphi(\cdot - y) : \theta \in \mathbb{R}, y \in \mathbb{R}^2\}$.

4. DAVEY–STEWARTSON SYSTEM

In this chapter we study the existence and regularity of standing waves for (1.0.3), i.e., periodic solutions of the form

$$v(t,x,y) = e^{i\omega t}u(x,y),$$
$$\phi(t,x,y) = \varphi(x,y),$$

where $\omega > 0$, $u, \varphi \in H^1(\mathbb{R}^2)$ and $u, \varphi \neq 0$. Then v is a standing wave solution of (1.0.3) implies that u must solve the following problem:

$$-\Delta u + \omega u = -bE_1(|u|^2)u - \chi|u|^\sigma u. \tag{4.0.1}$$

Clearly, u is a solution of (4.0.1) if and only if u is a critical point of the functional

$$S(u) = \frac{1}{2}\int |\nabla u|^2 dx + \frac{b}{4}\int |u|^2 E_1(|u|^2)dx + \frac{\chi}{\sigma+2}\int |u|^{\alpha+2}dx + \frac{\omega}{2}\int |u|^2 dx.$$

This can be seen by taking the Gâteaux derivative of S at u in any direction and setting it equal to zero.

Before proceeding further let us introduce some notation to be used from now on. We define the function sets

$$\mathcal{X} := \{u \in H^1(\mathbb{R}^2) : u \neq 0,\ u \text{ solves } (4.0.1)\},$$
$$\mathcal{G} := \{u \in \mathcal{X} : S(u) \leqslant S(\psi)\ \ \forall \psi \in \mathcal{X}\},$$

where $\mathcal{G}$ is called the set of *ground states*, and we introduce the set of admissible parameters

$$\mathcal{R}_{\omega,b} = \{(\sigma,\chi) : 0 < \sigma < \infty \text{ and } \chi < \chi^*_\sigma\},$$

where

$$\chi_\sigma^* = \begin{cases} +\infty & \text{if } \sigma < 2, \\ -b & \text{if } \sigma = 2, \\ -b^{\sigma/2}\omega^{(2-\sigma)/2}\left(\frac{2}{\sigma}\right)\left(\frac{\sigma-2}{\sigma}\right) & \text{if } \sigma > 2. \end{cases}$$

Note that the factor $\gamma_1(\xi)$ in the definition of E_1 prevents the existence of radial solutions for problem (4.0.1). In fact, $H_r^1(\mathbb{R}^2)$ is not invariant under E_1. However to indemnify the lack of compactness in the imbedding $H^1 \hookrightarrow L^p$, we apply the concentration compactness principle due to P. L. Lions [3].

4.1. Regularity

First we investigate the regularity of solutions of (4.0.1). To do so let us mention some properties of E_1. In general, let E_j be the singular integral operator defined in Fourier variables by

$$\widehat{[E_j(\psi)]}(\xi) = \gamma_j(\xi)\widehat{\psi}(\xi),$$

where $\gamma_j(\xi) = \dfrac{\xi_j^2}{|\xi|^2}$, $j = 1, 2$. Then E_j satisfies the following.

Lemma 4.1.1. [7, Lemma 2.1] *For $j = 1, 2$ and $1 < p < \infty$ we have:*

(i) *E_j is a linear operator from L^p into L^p,*

(ii) *$E_1 + E_2 = I$,*

(iii) *If $\psi \in H^s$ then $E_j(\psi) \in H^s$, $s \in \mathbb{R}$,*

(iv) *If $\psi \in W^{m,p}$ then $E_j(\psi) \in W^{m,p}$ and $\partial_k E_j(\psi) = E_j(\partial_k \psi)$, $k = 1, 2$,*

(v) *E_j preserves the following operations:*

- (translation) $E_j(\psi(\cdot + y))(x) = E_j(\psi)(x + y)$, *for all* $y \in \mathbb{R}^2$,
- (dilatation) $E_j(\psi(\lambda\cdot))(x) = E_j(\psi)(\lambda x)$, *for* $\lambda > 0$,
- (conjugation) $\overline{E_j(\psi)} = E_j(\overline{\psi})$.

Proof. Since γ_j is homogeneous of order zero and $|\gamma_j(\xi)| = \left|\frac{\xi_j^2}{|\xi|^2}\right| \leqslant 1$, *(i)* follows from the Calderon-Zygmund theorem [17].

Now,

$$(\widehat{E_1 + E_2})(\psi)(\xi) = \widehat{E_1(\psi)}(\xi) + \widehat{E_2(\psi)}(\xi) = (\gamma_1(\xi) + \gamma_2(\xi))\,\widehat{\psi}(\xi) = \widehat{\psi}(\xi),$$

hence, $E_1(\psi) + E_2(\psi) = \psi$ implies that $E_1 + E_2 = I$. This proves *(ii)*.

To prove *(iii)*, recall that

$$\psi(x) \in H^s \quad \text{if and only if} \quad \left(1 + |\xi|^2\right)^{s/2} \widehat{\psi}(\xi) \in L^2.$$

Let $\psi \in H^s$. Then $(1 + |\xi|^2)^{s/2}\,\widehat{\psi}(\xi) \in L^2$. To show that $E_j(\psi) \in H^s$, it suffices to show that $\left(1 + |\xi|^2\right)^{s/2} \widehat{E_j(\psi)} \in L^2$. Since $\gamma_j(\xi) \leqslant 1$ almost everywhere in $\mathbb{R}^2$,

$$\left\|\left(1 + |\xi|^2\right)^{s/2} \widehat{E_j(\psi)}\right\|_2 = \left\|\left(1 + |\xi|^2\right)^{s/2} \gamma_j \widehat{\psi}\right\|_2 \leqslant \left\|\left(1 + |\xi|^2\right)^{s/2} \widehat{\psi}\right\|_2 < +\infty,$$

so *(iii)* follows.

The remaining claims follow easily when one considers them in the Schwartz space $\mathcal{S}$ of rapidly decreasing functions. The result then is established by a density argument. □

Let B_j be the quadratic functional on L^2 defined by

$$B_j(\psi) = \int \gamma_j(\xi)|\widehat{\psi}(\xi)|^2 d\xi.$$

It follows from Parseval identity that

$$B_j(\psi) = \int E_j(\psi)\overline{\psi} d\xi,$$

and in particular,

$$B_j(\psi) \leqslant \|\psi\|_2^2,$$

since $\|\gamma_j\|_\infty \leqslant 1$. Also $B_j \in C^\infty(L^2;\mathbb{R})$, with $B_j' = 2E_j$. To see this, note that E_j is linear and by Parseval's identity $\int E_j(\psi)\overline{u} = \int E_j(u)\overline{\psi}$ and hence we have

$$\begin{aligned}&\left|B(\psi+u) - B(\psi) - 2\int E_j(\psi)\overline{u}dx\right| \\ &\quad = \left|\int E_j(\psi+u)(\overline{\psi+u})dx - \int E_j(\psi)\overline{\psi}dx - \int 2E_j(\psi)udx\right| = |B_j(u)| \leqslant \|u\|_2^2.\end{aligned}$$

Lemma 4.1.2. [7, Lemma 2.3] *Let $u \in L^4$, $\lambda \geqslant 0$ and $f(\lambda) := B_j(|u_\lambda|^2)$, where*

$$u_\lambda = \lambda^{1/4} u(\Lambda_\lambda x),$$

with $\Lambda_\lambda = \operatorname{diag}(\lambda, 1)$ or $\operatorname{diag}(1, \lambda)$ for $j = 1, 2$, respectively, where $\operatorname{diag}(a, b)$ denotes the 2×2 matrix with diagonal a and b and zero elsewhere. Then f is an increasing function satisfying $f(0) = 0$ and

$$\lim_{\lambda\to\infty} f(\lambda) = \|u\|_4^4.$$

Proof. It suffices to consider only $j = 1$. By a change of variables, $\xi \leftrightarrow \Lambda_\lambda \xi$, we get

$$B_1(|u_\lambda|^2) = \int \gamma_1(\xi) \left|\widehat{(|u_\lambda|^2)}\right|^2 d\xi = \int \gamma_1(\Lambda_\lambda \xi) \left|\widehat{(|u|^2)}\right|^2 d\xi,$$

where $\gamma_1(\Lambda_\lambda \xi) = \dfrac{\lambda^2 \xi_1^2}{\lambda^2 \xi_1^2 + \xi_2^2}$. Since

$$\gamma_1(\Lambda_\lambda \xi) \to 1, \quad \text{as } \lambda \to \infty,$$

by the Dominated Convergence Theorem we get

$$f(\lambda) \to \left\|\widehat{(|u|^2)}\right\|_2^2 = \left\||u|^2\right\|_2^2 = \|u\|_4^4.$$

Also, $\dfrac{d}{d\lambda}\gamma_1(\Lambda_\lambda \xi) = \dfrac{2\lambda\xi_1^2\xi_2^2}{(\lambda^2\xi_1^2+\xi_2^2)^2} > 0$, hence, f is increasing. $\square$

Now we can state the regularity result.

Theorem 4.1.3 (Regularity)**.** [7, Theorem 2.4] *If* $0 < \sigma < \infty$ *and* $u \in H^1(\mathbb{R}^2)$ *is a weak solution of* (4.0.1)*, then the following hold.*

(i) $u \in W^{2,p}$ *for all* $2 \leqslant p < \infty$*,*

(ii) $\lim\limits_{|x|\to\infty} \left\{|\nabla u(x)| + |u(x)| + |E_1(|u|^2)(x)|\right\} = 0$*,*

(iii) $u \in C^2$*,*

(iv) There exist positive constants C *and* ν *such that*

$$e^{\nu|x|}\left\{|u(x)| + |\nabla u(x)|\right\} \leqslant C \quad \forall x \in \mathbb{R}^2.$$

Proof. Theorem will be proved in several steps.

Step 1: We want to show that $u \in L^2 \cap L^\infty$. Since $H^1 \hookrightarrow L^p$ for all $2 \leqslant p < \infty$, we can find $r > 2$ such that $E_1(|u|^2), |u|^\sigma u \in L^r$. As u solves (4.0.1), $u \in W^{2,r}$ and then $u \in W^{1,\infty}$ from the Sobolev imbedding.

Step 2: It suffices to show that the right hand side of (4.0.1) is in L^p for all $p \geqslant 2$. Let $u \in L^2 \cap L^\infty$ and $p \geqslant 2$ be fixed. Then, for $\mu = \dfrac{2}{p(\sigma+1)}$

$$\left\||u|^\sigma u\right\|_p = \|u\|_{p(\sigma+1)}^{\sigma+1} \leqslant \|u\|_2^{\mu(\sigma+1)} \|u\|_\infty^{(1-\mu)(\sigma+1)} < \infty,$$

hence, $|u|^\sigma u \in L^p$. Also, since $|u|^2 \in L^{p/2}$ for all $2 \leqslant p < \infty$ we get

$$E_1(|u|^2)u \in L^q, \quad \text{for all } 1 < q < \infty.$$

Thus *(i)* follows from the regularity of elliptic equations.

Step 3: From *Step 1*, we know that $u \in W^{1,\infty}$, i.e., u is globally Lipschitz continuous. Since $u \in L^2$, we have,

$$\lim_{|x|\to\infty} |u(x)| = 0.$$

Step 4: As $u \in W^{1,\infty}$ and $u \in W^{2,p}$, for all $2 \leqslant p < \infty$ we get $u \in W^{1,p}$, for all $2 \leqslant p \leqslant \infty$. Now, let $2 \leqslant p \leqslant \infty$ be fixed. Then

$$\int (|u|^2)^p dx = \int |u|^{2p} dx < \infty, \quad \text{since } 2 \leqslant 2p \leqslant \infty.$$

Also $\nabla(|u|^2) = u\nabla\overline{u} + \overline{u}\nabla u$ implies

$$\left\|\nabla(|u|^2)\right\|_p = \|u\nabla\overline{u} + \overline{u}\nabla u\|_p \leqslant \|u\|_\infty \|\nabla\overline{u}\|_p + \|\overline{u}\|_\infty \|\nabla u\|_p < \infty.$$

Thus $|u|^2 \in W^{1,p}$. For $2 \leqslant p < \infty$, $|u|^2 \in W^{1,p}$ implies that $E_1(|u|^2) \in W^{1,p}$. Since $W^{1,p}$ is a Banach algebra [18, Theorem 5.23], $E_1(|u|^2)u \in W^{1,p}$. Similarly, $|u|^\sigma u \in L^p$ for all $2 \leqslant p < \infty$. Thus we get, $(-\Delta + \omega)\partial_k u \in L^p$ and this yields

$$u \in W^{3,p}, \quad \text{for all } 2 \leqslant p < \infty.$$

Since $W^{3,p} \hookrightarrow C^2$ if $p \geqslant 3$, *(iii)* follows.

This also implies that $|\nabla u| \in W^{1,\infty}$ and like in *Step 3*, we get $\lim_{|x|\to\infty} |\nabla u(x)| = 0$.

Step 5: From Lemma 4.1.1, $|u|^2 \in W^{2,p}$ implies $E_1(|u|^2) \in W^{2,p}$, and $E_1(|u|^2) \in W^{1,\infty}$, as before. We therefore have $\lim_{|x|\to\infty} |E_1(|u|^2)(x)| = 0$. Thus *(ii)* is established.

Step 6: Now it remains to prove *(iv)*. It suffices to consider the case $\omega = 1$, as u is a solution of (4.0.1) if and only if ψ defined by $u(x) = \omega^{1/\sigma}\psi(\sqrt{\omega}x)$ is a solution of

$$-\Delta\psi + \psi = -\omega^{(2-\sigma)/\sigma} bE_1(|\psi|^2)\psi - \chi|\psi|^\sigma\psi,$$

like in the previous chapter. Let $\epsilon > 0$, $\theta_\epsilon(x) := \exp\left(\frac{|x|}{1+\epsilon|x|}\right)$. Then θ_ϵ is bounded, since $\exp\left(\frac{|x|}{1+\epsilon|x|}\right) \leqslant \exp\left(\frac{1}{\epsilon}\right) = M$. Also

$$|\nabla\theta_\epsilon(x)|^2 = \theta_\epsilon^2(x)\frac{x_1^2}{|x|^2(1+\epsilon|x|)^4} + \theta_\epsilon^2(x)\frac{x_2^2}{|x|^2(1+\epsilon|x|)^4} = \theta_\epsilon^2(x)\frac{1}{(1+\epsilon|x|)^4} \leqslant \theta_\epsilon^2(x),$$

gives that $|\nabla\theta_\epsilon| \leqslant \theta_\epsilon$ almost everywhere in $\mathbb{R}^2$, and since $\theta_\epsilon \leqslant M$, θ_ϵ is Lipschitz continuous.

Multiplying the equation (4.0.1) by $\theta_\epsilon\overline{u} \in H^1$ and integrating we get,

$$\int \Re(\nabla u \cdot \nabla(\theta_\epsilon\overline{u}))dx + \int \theta_\epsilon|u|^2 dx = -b\int \theta_\epsilon|u|^2 E_1(|u|^2)dx - \chi\int \theta_\epsilon|u|^{\sigma+2}dx.$$

Note that $\nabla\left(\theta_\epsilon\overline{u}\right) = \overline{u}\nabla\theta_\epsilon + \theta_\epsilon\nabla\overline{u}$ yields $\Re(\nabla u \cdot \nabla(\theta_\epsilon\overline{u})) \geqslant \theta_\epsilon|\nabla u|^2 - \theta_\epsilon|u||\nabla u|$, which in turn gives us

$$\int \theta_\epsilon|\nabla u|^2 dx - \int \theta_\epsilon|u||\nabla u|dx + \int \theta_\epsilon|u|^2 dx \leqslant |b|\int \theta_\epsilon|u|^2 E_1(|u|^2)dx + |\chi|\int \theta_\epsilon|u|^{\sigma+2}dx.$$

Let $\delta < \frac{1}{4(|b|+|\sigma|)}$. From *(ii)*, we know that, for some R_1,

$$|E_1(|u|^2)(x)| < \delta, \quad |u(x)|^\sigma < \delta,$$

for $|x| \geqslant R_1$.

By Cauchy inequality,

$$\frac{1}{2}\int \theta_\epsilon |\nabla u|^2 dx + \frac{1}{2}\int \theta_\epsilon |u|^2 dx \leqslant |b| \int_{\{|x|\leqslant R_1\}} e^{|x|}|u|^2 E_1(|u|^2) dx + |b|\delta \int_{\{|x|\geqslant R_1\}} \theta_\epsilon |u|^2 dx$$
$$+ |\chi| \int_{\{|x|\leqslant R_1\}} e^{|x|}|u|^{\sigma+2} dx + |a|\delta \int_{\{|x|\geqslant R_1\}} \theta_\epsilon |u|^2 dx,$$

and this yields

$$\frac{1}{2}\int \theta_\epsilon |\nabla u|^2 dx + \frac{1}{2}\int \theta_\epsilon |u|^2 dx \leqslant C_1 + \delta(|b| + |\chi|) \int_{\{|x|\geqslant R_1\}} \theta_\epsilon |u|^2 dx \leqslant C_1 + \frac{1}{4}\int \theta_\epsilon |u|^2 dx.$$

Thus, we get

$$\frac{1}{2}\int \theta_\epsilon |\nabla u|^2 dx + \frac{1}{4}\int \theta_\epsilon |u|^2 dx \leqslant C_1,$$

where C_1 is a positive constant independent of ϵ. Here letting $\epsilon \longrightarrow 0$ yields

$$\frac{1}{4}\int e^{|x|}(|\nabla u|^2 + |u|^2) dx \leqslant C_1, \tag{4.1.1}$$

by the Monotone Convergence Theorem.

Now, again *(ii)* implies that for some $R_2 > 0$, we have $|u(x)| + |\nabla u(x)| < 1$ provided $|x| \geqslant R_2$. On the other hand, for $|x| \leqslant R_2$, we have,

$$e^{\frac{|x|}{2}}(|u(x)| + |\nabla u(x)|) \leqslant e^{\frac{R_2}{2}} \|u\|_{W^{1,\infty}}. \tag{4.1.2}$$

Let $x \in \mathbb{R}^2$ be such that $|x| \geqslant R_2$. Since u and ∇u are globally Lipschitz continuous, there is $L > 0$ such that, for all $y \in \mathbb{R}^2$,

$$\begin{aligned} |\nabla u(y)| &\geqslant |\nabla u(x)| - \frac{L}{\sqrt{2}}|x-y| \\ |u(y)| &\geqslant |u(x)| - \frac{L}{\sqrt{2}}|x-y|, \end{aligned} \tag{4.1.3}$$

which then implies $|u(x)|^2 + |\nabla u(x)|^2 \leqslant 2(|u(y)|^2 + |\nabla u(y)|^2 + L^2|x-y|^2)$. Taking $\rho := \dfrac{1}{2L}(|u(x)|^2 + |\nabla u(x)|^2)^{1/2}$, gives

$$|u(x)|^2 + |\nabla u(x)|^2 \leqslant 4(|u(y)|^2 + |\nabla u(y)|^2), \quad \forall y \in B_\rho(x).$$

Integrating this inequality over $B_\rho(x)$ we obtain

$$C_2\rho^2(|u(x)|^2 + |\nabla u(x)|^2) \leqslant 4\int_{B_\rho(x)} (|u(y)|^2 + |\nabla u(y)|^2)dy,$$

and plugging ρ we have,

$$C_3(|u(x)|^2 + |\nabla u(x)|^2)^2 \leqslant 4\int_{B_\rho(x)} (|u(y)|^2 + |\nabla u(y)|^2)dy \tag{4.1.4}$$

For $|x| \geqslant R_2$ we have $\rho \leqslant \dfrac{1}{2L}$, so it follows that

$$|y| - |x| + \frac{1}{2L} \geqslant 0, \quad \forall y \in B_\rho(x),$$

and from (4.1.4)

$$\begin{aligned} C_3e^{|x|}(|u(x)|^2 + |\nabla u(x)|^2)^2 &\leqslant 4\int_{B_\rho(x)} e^{|x|}(|u(y)|^2 + |\nabla u(y)|^2)dy \\ &\leqslant 4\int_{B_\rho(x)} e^{\frac{1}{2L}}e^{|y|}(|u(y)|^2 + |\nabla u(y)|^2)dy \leqslant C, \end{aligned}$$

using (4.1.1). Thus for $|x| \geqslant R_2$,

$$e^{|x|}(|u(x)|^2 + |\nabla u(x)|^2)^2 \leqslant C_4, \tag{4.1.5}$$

and therefore *(iv)* follows combining (4.1.2) and (4.1.5). $\square$

To simplify the notation, we introduce the following functionals on H^1:

$$\begin{aligned} T(u) &= \|\nabla u\|_2^2, \\ V(u) &= -\frac{b}{4}B_1(|u|^2) - \frac{\chi}{\sigma+2}\|u\|_{\sigma+2}^{\sigma+2} - \frac{\omega}{2}\|u\|_2^2, \\ S(u) &= \frac{1}{2}T(u) - V(u). \end{aligned}$$

Here we have a proposition with useful identities which can be obtained from the Theorem 4.1.3 as in [7].

Proposition 4.1.4. [7, Proposition 2.5] *If $u \in H^1$ is a solution of (4.0.1), then*

(i) $T(u) + \omega\|u\|_2^2 = -bB_1(|u|^2) - \chi\|u\|_{\sigma+2}^{\sigma+2}$,
(ii) $2\omega\|u\|_2^2 = -bB_1(|u|^2) - \dfrac{4\chi}{\sigma+2}\|u\|_{\sigma+2}^{\sigma+2}$.

The following corollary follows easily from the above proposition.

Corollary 4.1.5. [7, Corollary 2.6] *If u is a solution of* (4.0.1)*, then*

$$S(u) = \frac{1}{2}T(u), \qquad V(u) = 0.$$

4.2. Existence of Standing Waves

Now we prove the existence of ground states for the problem (1.0.3), that means, solutions of (4.0.1) that minimizes the Lagrangian S over the set of solutions of (4.0.1).

Theorem 4.2.1 (Existence)**.** [7, Theorem 3.1] *Let $(\sigma, \chi) \in \mathcal{R}_{\omega,b}$. Then the following hold.*

(i) $\mathcal{X}$ and $\mathcal{G}$ contain a real valued positive function,
(ii) $u \in \mathcal{G}$ if and only if u solves the minimisation problem

$$\begin{aligned} u &\in \Sigma_0, \\ T(u) &= \min\{T(\psi) : \psi \in \Sigma_0\}, \end{aligned} \tag{4.2.1}$$

where $\Sigma_0 = \{u \in H^1 : u \neq 0, V(u) = 0\}$.

Before proving this theorem we need the following lemmas:

Lemma 4.2.2. [14, Lemma 8.3.7] *Let* $0 < q < \infty$. *Then there exists a constant* $C > 0$ *such that, for all* $\psi \in H^1$,

$$\|\psi\|_{q+2}^{q+2} \leqslant C \left(\sup_y \int_{B_1(y)} (|\nabla \psi|^2 + |\psi|^2) dx \right)^{q/2} \|\psi\|_{H^1}^2.$$

Proof. We can cover $\mathbb{R}^2$ by unit squares $\{S_k\}$ such that $S_k \cap S_j = \emptyset$ for $k \neq j$. Then

$$\|\psi\|_{q+2}^{q+2} = \sum_{k=1}^{\infty} \int_{S_k} |\psi|^{q+2} dx,$$

and,

$$\|\psi\|_{H^1}^2 = \sum_{k=1}^{\infty} \int_{S_k} (|\nabla \psi|^2 + |\psi|^2) dx.$$

Since $H^1(S_k) \hookrightarrow L^{q+2}(S_k)$, we get

$$\begin{aligned} \int_{S_k} |\psi|^{q+2} dx &\leqslant C \left(\int_{S_k} |\nabla \psi|^2 + |\psi|^2 dx \right)^{(q+2)/2} \\ &\leqslant C \left(\sup_{k \in \mathbb{N}} \int_{S_k} |\nabla \psi|^2 + |\psi|^2 dx \right)^{q/2} \int_{S_k} (|\nabla \psi|^2 + |\psi|^2) dx. \end{aligned}$$

Summing up in k yields,

$$\begin{aligned} \|\psi\|_{q+2}^{q+2} &\leqslant C \left(\sup_{k \in \mathbb{N}} \int_{S_k} |\nabla \psi|^2 + |\psi|^2 dx \right)^{q/2} \|\psi\|_{H^1}^2 \\ &\leqslant C \left(\sup_y \int_{B_1(y)} |\nabla \psi|^2 + |\psi|^2 dx \right)^{q/2} \|\psi\|_{H^1}^2. \end{aligned}$$

$\square$

For each $\mu \in \mathbb{R}$ we define

$$\Sigma_\mu := \{u \in H^1 : u \neq 0, V(u) = \mu\}, \quad j(\mu) = \inf\left\{\frac{1}{2}T(u) : u \in \Sigma_\mu\right\}.$$

Then we have

Lemma 4.2.3. [7, Lemma 3.4] *Let $(\sigma, \chi) \in \mathcal{R}_{\omega,b}$. Then the following hold:*

(i) $\Sigma_\mu \neq \emptyset$ *for all* $\mu \in \mathbb{R}$,

(ii) There exists a constant $I > 0$ *such that* $j(\mu) = I$, *for all* $\mu \in \mathbb{R}$.

Proof. Let $u \in H^1$, $u \neq 0$ and define for $\lambda > 0$, $u_\lambda(x) := u\left(\frac{x}{\sqrt{\lambda}}\right)$. Then by a change of variables, $x \leftrightarrow \frac{x}{\sqrt{\lambda}}$, we have $V(u_\lambda) = \lambda V(u)$. Also note that $V(\epsilon u) = \epsilon^2 P(\epsilon)$, where $P(\epsilon) := -\epsilon^2 \frac{b}{4} B_1(|u|^2) - \epsilon^\sigma \frac{\chi}{\sigma+2}\int |u|^{\sigma+2}dx - \frac{\omega}{2}\int |u|^2 dx$. Since $P(0) = -\frac{\omega}{2}\|u\|_2^2 < 0$ and P is a continuous function of ϵ, there exists $\tilde{\epsilon} > 0$, small enough, such that $P(\tilde{\epsilon}) < 0$, hence, so is $V(\tilde{\epsilon}u)$. As $V((\tilde{\epsilon}u)_\lambda) = \lambda V(\tilde{\epsilon}u)$ for all $\lambda > 0$, $\Sigma_\mu \neq \emptyset$ for all $\mu < 0$.

To prove $\Sigma_\mu \neq \emptyset$ for all $\mu \geqslant 0$, it suffices to show

$$\exists u_0 \in H^1 \quad \text{such that} \quad V(u_0) > 0. \tag{4.2.2}$$

Suppose, for the moment, that (4.2.2) holds true. Then $V(\epsilon u_0)|_{\epsilon=0} < 0$ and $V(\epsilon u_0)|_{\epsilon=1} > 0$ imply that there exists $\tau_0 < 1$ such that $V(\tau_0 u_0) = 0$, so $\Sigma_0 \neq \emptyset$. Also since $V(u_{0_\lambda}) = \lambda V(u_0)$ for all $\lambda > 0$, $\Sigma_\mu \neq \emptyset$ for all $\mu > 0$.

Now it remains to show (4.2.2). Let $(\sigma, \chi) \in \mathcal{R}_{\omega,b}$, and $u \in H^1$, $u \neq 0$.

First, if $\sigma < 2$, ϵ^4 is the dominant term in $V(\epsilon u)$ with positive coefficient. Similarly, if $\chi < 0$, the dominant term in $V(\epsilon u)$, ϵ^4 or $\epsilon^{\sigma+2}$, has positive coefficient. Thus taking $u_0 = \tau u$ with τ large enough gives $V(u_0) = V(\tau u) > 0$.

Second, assume $\sigma = 2$ and $\chi < -b$. For any $u \in L^4$, consider $u_\lambda(x_1, x_2) = \lambda^{1/4}u(\lambda x_1, x_2)$. Then using Lemma 4.1.2,

$$\lim_{\lambda\to\infty}\left(-\frac{b}{4}B_1(|u_\lambda|^2) - \frac{\chi}{4}\|u_\lambda\|_4^4\right) = \left(-\frac{b}{4} - \frac{\chi}{4}\right)\|u\|_4^4 > 0.$$

Thus there exists λ_0 such that $-\frac{b}{4}B_1(|u_{\lambda_0}|^2) - \frac{\chi}{4}\|u_{\lambda_0}\|_4^4 > 0$, which is exactly the coefficient of the ϵ^4 term in $V(\epsilon u_{\lambda_0})$. Taking $u = u_{\lambda_0}$ shows that (4.2.2) holds with $u_0 = \tau u$, for sufficiently large τ.

Third, for $\sigma > 2$ and $\chi < -b^{\sigma/2}\omega^{(2-\sigma)/2}\left(\frac{2}{\sigma}\right)\left(\frac{\sigma-2}{\sigma}\right)^{(\sigma-2)/2}$, define

$$G_\lambda(s) := -\frac{b}{4}s^4 - \frac{\chi}{\sigma+2}\lambda^{(\sigma-2)/4}s^{\sigma+2} - \frac{\omega}{2}\lambda^{-1/2}s^2.$$

Easily, for all $\lambda > 0$, there exists $s_0 > 0$ such that $G_\lambda(s_0) > 0$ if and only if $\chi < -b^{\sigma/2}\omega^{(2-\sigma)/2}\left(\frac{2}{\sigma}\right)\left(\frac{\sigma-2}{\sigma}\right)^{(\sigma-2)/2}$. Let $\epsilon > 0$ and $u = s\mathbf{1}_{B_R}$. From Lemma 4.1.2, there exists λ such that

$$-\frac{b}{4}B_1(|u_\lambda|^2) > -\frac{b}{4}\|u\|_4^4 - \epsilon.$$

Therefore,

$$\begin{aligned}
V(u_\lambda) &= -\frac{b}{4}B_1(|u_\lambda|^2) - \frac{\chi}{\sigma+2}\|u_\lambda\|_{\sigma+2}^{\sigma+2} - \frac{\omega}{2}\|u_\lambda\|_2^2 \\
&> -\frac{b}{4}\|u\|_4^4 - \epsilon - \frac{\chi\lambda^{(\sigma-2)/4}}{\sigma+2}\|u\|_{\sigma+2}^{\sigma+2} - \frac{\omega}{2}\lambda^{-1/2}\|u\|_2^2 \\
&= \operatorname{meas}(B_R)G_\lambda(s) - \epsilon.
\end{aligned}$$

Hence there exists $s_0 > 0$ such that $G_\lambda(s_0) > 0$. Thus choosing R large enough we get $V(u_\lambda) > 0$, so (4.2.2) follows and *(i)* is established.

To prove *(ii)*, let

$$I := j(0) = \inf\left\{\frac{1}{2}T(u) : u \in \Sigma_0\right\}.$$

We want to show that $I > 0$. Let $u \in \Sigma_0$ be arbitrary. Since $V(u) = 0$ and $B_1(u) \leqslant \|u\|_2^2$, we have

$$\frac{\omega}{2}\|u\|_2^2 \leqslant \frac{|b|}{4}\|u\|_4^4 + \frac{|\chi|}{\sigma+2}\|u\|_{\sigma+2}^{\sigma+2}. \tag{4.2.3}$$

Moreover, from Gagliardo-Nirenberg-Sobolev's inequality (Section B.2):

$$\begin{aligned} \|u\|_4^4 &\leqslant C_1\|\nabla u\|_2^2\|u\|_2^2 \\ \|u\|_{\sigma+2}^{\sigma+2} &\leqslant C_2\|\nabla u\|_2^\sigma\|u\|_2^2. \end{aligned} \tag{4.2.4}$$

Hence putting (4.2.3) and (4.2.4) together we get

$$\frac{\omega}{2} \leqslant C_3T(u) + C_4T(u)^{\sigma/2}.$$

Taking infimum over Σ_0 we receive that $I > 0$ as claimed.

Note that for $u_\lambda(x) = u\left(\frac{x}{\sqrt{\lambda}}\right)$ we have $V(u_\lambda) = \lambda V(u)$, hence,

$$u \in \Sigma_\mu \Leftrightarrow u_\lambda \in \Sigma_{\lambda\mu}.$$

As $T(u_\lambda) = T(u)$ for all $\lambda > 0$, $j(\mu)$ must be constant on $(-\infty, 0)$ and on $(0, +\infty)$. Let $\mu_n \searrow 0$ and $\epsilon > 0$. There exists $u \in \Sigma_0$ such that

$$I < \frac{1}{2}T(u) < I + \epsilon,$$

since I is the infimum.

Taking $\tau_n > 1$ such that $V(\tau_n u) = \mu_n$, we have:

$$j(\mu_n) < \frac{1}{2}T(\tau_n u) = \frac{1}{2}\tau_n^2 T(u), \text{ and}$$
$$-I < -\frac{1}{2}T(u) + \epsilon.$$

Adding up we get $j(\mu_n) - I < \frac{1}{2}(\tau_n^2 - 1)T(u) + \epsilon$, and so

$$\limsup_{n\to\infty}\,(j(\mu_n) - I) \leqslant 0. \tag{4.2.5}$$

On the other hand, let $\psi_n \in \Sigma_{\mu_n}$ be such that

$$\frac{1}{2}T(u_n) < j(\mu_n) + \epsilon.$$

Taking $\tau_n < 1$ such that $\tau_n u_n \in \Sigma_0$ yields

$$I \leqslant \frac{1}{2}T(\tau_n u_n) = \frac{1}{2}\tau_n^2 T(u_n) < j(\mu_n) + \epsilon,$$

hence,

$$\liminf_{n\to\infty}(j(\mu_n) - I) \geqslant 0. \tag{4.2.6}$$

From (4.2.5) and (4.2.6) we obtain $\lim_{n\to\infty} j(\mu_n) = I$, i.e., $j(0) = I$ and therefore

$$j(\mu) = I \quad \text{for all} \quad \mu \in [0, +\infty).$$

Same arguments with $\mu_n \nearrow 0$ give

$$j(\mu) = I \quad \text{for all} \quad \mu \in (-\infty, 0].$$

Therefore *(ii)* follows. $\square$

Here we state a lemma showing an equivalent formulation of the minimization problem in (4.2.1).

Lemma 4.2.4. [7, Lemma 3.6] *Let* $(\sigma, \chi) \in \mathcal{R}_{\omega,b}$. *Then the problem* (4.2.1) *is equivalent to*

$$\begin{gathered} V(u) = 0, \quad u \neq 0, \\ T(u) = \min\{T(\psi) : V(u) \geqslant 0\}. \end{gathered}$$

Proof. Let

$$\overline{I} := \inf\{T(u) : V(u) \geqslant 0\}, \quad I = \inf\{T(u) : V(u) = 0\}.$$

Clearly, $\overline{I} \leqslant I$. If $u \in H^1$, $u \neq 0$ is such that $V(u) \geqslant 0$, we can find $0 < \tau \leqslant 1$, as in the previous arguments, for which $V(\tau u) = 0$ and the assertion follows, since

$$I \leqslant T(\tau u) = \tau^2 T(u) \leqslant T(u),$$

and taking infimum yields $I \leqslant \overline{I}$. □

We are now ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. Using the arguments given in [7] we prove the theorem in several steps. Let $j(\mu)$ be as before and consider the minimization problem (4.2.1).

Step 1 [Existence of a solution of (4.2.1)]: Let (ψ_n) be a minimizing sequence and let us define $u_n(x) := \psi_n(\sqrt{\Lambda_n} x)$, where $\Lambda_n = \|\psi_n\|_2^2$. Since $T(u_n) = T(\psi_n)$, and $V(u_n) = \dfrac{1}{\Lambda_n} V(\psi_n) = 0$, (u_n) is also a minimizing sequence. Also $\|u_n\|_2^2 = \dfrac{1}{\Lambda_n}\|\psi_n\|_2^2 = 1$ implies that (u_n) is bounded in H^1, so there exists $u \in H^1$ and a subsequence of (u_n), still denoted by (u_n), such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1.$$

Now, we will apply the concentration compactness principle with

$$\rho_n = |\nabla u_n|^2 + |u_n|^2.$$

Observe that as $n \to \infty$

$$\int \rho_n = T(u_n) + 1 \to 2I + 1 > 0.$$

Assume "vanishing" occurs, then $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_1(y)} \rho_n dx = 0$. Using Lemma 4.2.2 we see that $\|u_n\|_{\sigma+2} \to 0$. Similarly, since $B_1(|u_n|^2) \leqslant \|u_n\|_4^4$, we get $B_1(|u_n|^2) \to 0$. But then, since $\omega > 0$, $V(u_n) = 0$ entails $\|u_n\|_2 \to 0$, which contradicts the fact that $\|u_n\|_2^2 = 1$. Thus "vanishing" does not occur.

Assume "dichotomy" occurs. Then for all $\epsilon > 0$, there exist $N_0 \in \mathbb{N}$ and two sequences $(u_n^{(1)}), (u_n^{(2)}) \subset H^1$ with disjoint supports such that for $n \geqslant N_0$,

$$\|\nabla u_n\|_2^2 - \|\nabla u_n^{(1)}\|_2^2 - \|\nabla u_n^{(2)}\|_2^2 \geqslant -C\epsilon,$$

for some $C > 0$ independent of ϵ. Since $u_n^{(1)}, u_n^{(2)} \neq 0$, for n large enough,

$$\begin{aligned} I + \epsilon > \frac{1}{2}T(u_n) &\geqslant \frac{1}{2}T(u_n^{(1)}) + \frac{1}{2}T(u_n^{(2)}) - \frac{C}{2}\epsilon \\ &\geqslant j(V(u_n^{(1)})) + j(V(u_n^{(2)})) - \frac{C}{2}\epsilon \quad = 2I - \frac{C}{2}\epsilon \end{aligned}$$

Hence, $I \leqslant \epsilon\left(1 + \frac{C}{2}\right)$, where I and C are independent of ϵ. Sending ϵ to zero yields $I \leqslant 0$, which is in contradiction with the fact that $I > 0$ and shows that "dichotomy" does not occur.

Therefore, "concentration" occurs, that means, there exists a sequence $(y_n) \subset \mathbb{R}^2$ such that for every $\epsilon > 0$ there is $R_\epsilon \geqslant \dfrac{1}{\epsilon}$ such that

$$\int_{\mathbb{R}^2 \setminus B_{R_\epsilon}(y_n)} \rho_n(x) dx \leqslant \epsilon. \tag{4.2.7}$$

Let $\widetilde{u_n}(\cdot) := u_n(\cdot - y_n)$. Then $\widetilde{u_n} \rightharpoonup \widetilde{u}$ weakly in H^1

As $H^1 \hookrightarrow L^p$ for all $2 \leqslant p < \infty$, (4.2.7) implies

$$\int_{\mathbb{R}^2 \setminus B_{R_\epsilon}(0)} |\widetilde{u_n}|^p \leqslant \epsilon^{p/2}, \quad \text{for all } 2 \leqslant p < \infty. \tag{4.2.8}$$

Defining

$$V_\Omega(\psi) := \int_\Omega |\psi|^2 \left\{ -\frac{b}{4} E_1(|\psi|^2) - \frac{\chi}{\sigma + 2} |\psi|^\sigma - \frac{\omega}{2} \right\} dx,$$

and using (4.2.8), we obtain that

$$|V_{\mathbb{R}^2 \setminus B_{R_\epsilon}}(\widetilde{u_n})| < \delta(\epsilon), \tag{4.2.9}$$

with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. Since the injection $H^1(B_{R_\epsilon}) \hookrightarrow L^p(B_{R_\epsilon})$ is compact, we have

$$V_{B_{R_\epsilon}}(\widetilde{u_n}) \to V_{B_{R_\epsilon}}(\widetilde{u}) \quad \text{as } n \to \infty. \tag{4.2.10}$$

Moreover, $0 = V(\widetilde{u_n}) = V_{B_{R_\epsilon}}(\widetilde{u_n}) + V_{\mathbb{R}^2 \setminus B_{R_\epsilon}}(\widetilde{u_n})$ and (4.2.9) entail

$$|V_{B_{R_\epsilon}}(\widetilde{u_n})| < \delta(\epsilon). \tag{4.2.11}$$

Letting $n \to \infty$ in (4.2.11) gives by (4.2.10),

$$|V_{B_{R_\epsilon}}(\widetilde{u})| < \delta(\epsilon).$$

As $\epsilon \searrow 0$, we get that $\widetilde{u} \in \Sigma_0$.

Since T is weakly lower semi-continuous, $T(\widetilde{u}) \leqslant \liminf_{n\to\infty} T(\widetilde{u_n}) = I$. Therefore $\widetilde{u}$ is the desired minimum.

Step 2 [$\mathcal{X}$ is nonempty]: Let u be a solution of (4.2.1). Then there exists a Lagrange multiplier λ (see Theorem A.4.2) such that

$$-\Delta u = \lambda(-bE_1(|u|^2)u - a|u|^\sigma u - \omega u).$$

Let $\phi \in H^1$ satisfy $\langle V'(u), \phi\rangle > 0$, where $\langle\cdot,\cdot\rangle$ denotes the H^{-1}-H^1 duality pairing. Since $T, V \in C^1(H^1, \mathbb{R})$, we have

$$\begin{aligned} V(u+t\phi) &= V(u) + \int_0^t \langle V'(u+s\phi), \phi\rangle ds, \\ T(u+t\phi) &= T(u) + t\lambda\langle V'(u), \phi\rangle + t^2\|\nabla\phi\|_2^2. \end{aligned} \tag{4.2.12}$$

If $\lambda < 0$, from (4.2.12) we have for $t \leqslant \dfrac{2}{\|\nabla\phi\|_2^2}(-\lambda\langle V'(u), \phi\rangle)$,

$$V(u+t\phi) > 0 \quad \text{and} \quad T(u+t\phi) < T(\varphi), \tag{4.2.13}$$

which contradict Lemma 4.2.4. Hence $\lambda > 0$. Let

$$u_\lambda(x) = u\left(\frac{x}{\sqrt{\lambda}}\right).$$

We claim that u_λ solves (4.0.1). This follows from (4.2.13) and by a simple change of variables, $x \leftrightarrow \dfrac{x}{\sqrt{\lambda}}$. Therefore $u_\lambda \in \mathcal{X}$, and so $\mathcal{X} \neq \emptyset$.

Step 3 [*(ii)* holds]: Let u be a solution of (4.2.1) and let $\psi \in \mathcal{X}$. Then ψ solves (4.0.1) and $V(\psi) = 0$ by Proposition 4.1.4. Hence $\psi \in \Sigma_0$. Since u is a solution of (4.2.1), $V(\varphi) = 0$ and $T(u) \leqslant T(\psi)$. This gives that $S(u) \leqslant S(\psi)$. Since ψ was chosen arbitrarily, $u \in \mathcal{G}$.

Conversely, assume $u \in \mathcal{G}$, then $S(u) \leqslant S(\psi)$ for all $\psi \in \mathcal{X}$. We know that the problem (4.2.1) has a solution, say $\tilde{u} \in \Sigma_0$, i.e., $T(\tilde{u}) = \min_{\psi\in\Sigma_0} T(\psi)$. However, $\tilde{u}$ need not be in $\mathcal{X}$, but as $\tilde{u}$ solves (4.2.1), $\tilde{u}_\lambda \in \mathcal{X}$ as in *Step 2.* Also note that $T(\tilde{u}) = T(\tilde{u}_\lambda)$. Since $\mathcal{X} \subset \Sigma_0$, we have

$$T(\tilde{u}_\lambda) = T(\tilde{u}) \leqslant T(\phi) \leqslant T(\psi) \qquad \forall\psi \in \mathcal{X}.$$

Taking $\psi = \tilde{u}_\lambda$ yields

$$T(\tilde{u}_\lambda) \leqslant T(u) \leqslant T(\tilde{u}_\lambda),$$

hence, $T(u) = T(\tilde{u}_\lambda) = \min\limits_{\psi\in\Sigma_0} T(\psi)$, i.e., u solves (4.2.1), so *(ii)* is established.

Step 4 [*(i)* holds]: From *Step 1* and *Step 3*, $\mathcal{G} \neq \emptyset$. Let $u \in \mathcal{G}$. Define $A(x) := -bE_1(|u(x)|^2) - \chi|\varphi(x)|^\sigma$. Then $A(x) \to 0$, as $|x| \to \infty$ by Theorem 4.1.3*(ii).* Also $J(u) = -\omega\|u\|_2^2 < 0$. Therefore, from the Theorem 2.3.1, there exists a function $\psi > 0$ and a constant $c > 0$ such that $u = c\psi > 0$, and so the claim follows. $\square$

5. GENERALIZED DAVEY–STEWARTSON SYSTEM

The existence of standing waves for a generalized Davey–Stewartson (GDS) system is shown by Eden and Erbay in [12] using an unconstrainted minimization problem. Here, we consider the same problem but relax the conditions on the parameters to $\chi + \alpha_1 b < 0$ or $\chi + \alpha_2 b < 0$ with α_1 and α_2 defined in (A3). Our approach, in the spirit of Berestycki, Gallouët and Kavian [5] and Cipolatti [7], is to use a constrained minimization problem and utilize Lions' concentration compactness theorem [3]. When both methods are applicable we show that they give the same minimizer and obtain a sharp bound for a Gagliardo–Nirenberg type inequality. As in [12], this leads to a global existence result for small-mass solutions. Moreover, following an argument in Eden, Erbay and Muslu [19] we show that when $p > 2$, the L^p-norms of solutions to the Cauchy problem for a GDS system converge to zero as $t \to \infty$. These results are to appear in an article by Eden and Topaloğlu [20].

5.1. Review of previous results

For $v_0 \in H^1(\mathbb{R}^2)$ the existence and uniqueness of solutions to the Cauchy problem for the GDS system is discussed in [10]. Moreover it is shown that the Hamiltonian

$$H(v) = \int_{\mathbb{R}^2} \left(|\xi|^2 |\widehat{v}|^2 + \frac{1}{2}(\chi + b\alpha(\xi)) \left| \widehat{|v|^2} \right|^2 \right) d\xi \tag{5.1.1}$$

for the GDS system is conserved in the EEE case. It can easily be checked that the same quantity $H(v)$ is conserved for solutions of (1.0.4) under (A1) and (A2), [21].

Looking for a solitary wave in (1.0.4) of standing wave type, that is, v is of the form $e^{i\omega t} u(x)$ with $u \in H^1(\mathbb{R}^2)$, one is led to the equation

$$-\Delta u + \omega u = -\chi |u|^2 u - bK(|u|^2)u. \tag{5.1.2}$$

One of the key properties of the map K is that $K : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)$ is bounded for all $1 < p < \infty$ and we have $\|K(f)\| \leqslant \alpha_M \|f\|_2^2$. This and further properties of K are given in [12, Lemma 2.1]. Also we know that if u is a solution of (5.1.2), then $u \in \bigcap_{m=1}^{\infty} W^{m,p}$ for all $2 \leqslant p < \infty$ and there exist positive constants C, ν such that $|u(x)| + |\nabla u(x)| \leqslant Ce^{-\nu|x|}$ for all $x \in \mathbb{R}^2$ and $\lim_{|x|\to\infty} K(|u|^2)(x) = 0$, [12, Lemma 2.2]. As in the previous chapters we can take $\omega = 1$ without loss of generality by defining ψ as $u(x) = \sqrt{\omega}\psi(\sqrt{\omega}x)$.

In [12, Theorem 2.1], the following necessary conditions are obtained for the existence of solutions of (5.1.2):

$$\int_{\mathbb{R}^2} (|\nabla R|^2 - \omega R^2)dx = 0, \qquad \int_{\mathbb{R}^2} (2\omega + \chi R^2 + bK(R^2))R^2 dx = 0. \tag{5.1.3}$$

From (5.1.3) the two inequalities $\omega > 0$ and $\chi\|R\|_4^4 + b(K(R^2), R^2) < 0$ follow as necessary conditions on the existence of solutions. To guarantee the latter it is assumed that $\chi < \min\{-b\alpha_M, 0\}$. In [12] under the assumption $\chi < \min\{-b\alpha_M, 0\}$, the functional

$$J(f) = \frac{-2\|f\|_2^2\|\nabla f\|_2^2}{\chi\|f\|_4^4 + b(K(|f|^2), |f|^2)}$$

is shown to have a minimum on $H^1(\mathbb{R}^2)$, say R, which satisfies (5.1.2) after a proper normalization. Hence the following Gagliardo–Nirenberg type inequality is obtained as a corollary to [12, Theorem 2.1]:

$$-\chi\|f\|_4^4 - b(K(|f|^2), |f|^2) \leqslant C_{\text{opt}}\|f\|_2^2\|\nabla f\|_2^2, \tag{5.1.4}$$

where $C_{\text{opt}} = \dfrac{2}{\|R\|_2^2}$.

Here we adapt the approach of Berestycki and Lions [13] and Berestycki, Gallouët and Kavian [5] for the NLS equation and consider a constrained minimization problem.

5.2. Existence of Standing Waves

We note that $u \neq 0$ solves (5.1.2) if and only if u is a critical point of the Lagrangian

$$L_\omega(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{b}{4}B(|u|^2) + \frac{\chi}{4}\|u\|_4^4 + \frac{\omega}{2}\|u\|_2^2,$$

where $B(f) := \int \alpha(\xi)|\widehat{f}(\xi)|^2 d\xi = \int K(f)(x)\overline{f(x)}dx.$

Various parts of this Lagrangian are invariant under different scalings, [12]: if

$$u_{a,b}(x) := s^a u(s^b x), \quad \text{for some } s > 0, \tag{5.2.1}$$

then we have

$$\begin{aligned} \|u_{a,b}\|_2^2 &= s^{2a-2b}\|u\|_2^2, & \|\nabla u_{a,b}\|_2^2 &= s^{2a}\|\nabla u\|_2^2, \\ \|u_{a,b}\|_4^4 &= s^{4a-2b}\|u\|_4^4, & B(|u_{a,b}|^2) &= s^{4a-2b}B(|u|^2). \end{aligned} \tag{5.2.2}$$

There is also a partial scaling that reveals the closer kinship between $B(|u|^2)$ and $\|u\|_4^4$. Letting

$$u_s(x) = u_s(x_1, x_2) = s^{1/4}u(sx_1, x_2), \tag{5.2.3}$$

we get $B(|u_s|^2) = \int \alpha(s\xi_1, \xi_2)\left|\widehat{(|u|^2)}(\xi_1, \xi_2)\right|^2 d\xi$. By (A3) and the Dominated Convergence Theorem it follows that $\lim_{s\to\infty} B(|u_s|^2) = \alpha_1\|u\|_4^4$ and $\lim_{s\to 0^+} B(|u_s|^2) = \alpha_2\|u\|_4^4$.

Using the standard terminology, as in [14, 7], we set

$$T(u) := \|\nabla u\|_2^2, \qquad V(u) := -\frac{b}{4}B(|u|^2) - \frac{\chi}{4}\|u\|_4^4 - \frac{\omega}{2}\|u\|_2^2$$

so that $L_\omega(u) = \dfrac{1}{2}T(u) - V(u)$ is to be minimized over $H^1(\mathbb{R}^2)$. Now define $\Sigma_0 := \{u \in H^1(\mathbb{R}^2) : u \neq 0, V(u) = 0\}$ and $I := \inf\left\{\frac{1}{2}T(u) : u \in \Sigma_0\right\}$ as in [7]. Then it can be easily shown that if $\Sigma_0 \neq \emptyset$ and $\omega > 0$ then $I > 0$.

Theorem 5.2.1. *For $\chi+\alpha_1 b < 0$ or $\chi+\alpha_2 b < 0$, and $\omega > 0$ the minimization problem*

$$\begin{aligned} &u \in \Sigma_0, \\ &T(u) = \min\{T(\psi) : \psi \in \Sigma_0\} = 2I, \end{aligned} \tag{5.2.4}$$

has a positive solution. This solution satisfies $0 < L_\omega(u) \leqslant L_\omega(\psi)$ among all $\psi \in H^1(\mathbb{R}^2)$ solving (5.1.2)*. Moreover, if u is properly scaled then it is a solution of* (5.1.2)*.*

Proof. First we will note that Σ_0 is not empty. To establish this we will use one parameter scalings given by (5.2.1) and (5.2.3). If $\chi+\alpha_1 b < 0$, for $u \in H^1(\mathbb{R}^2)$ defining u_s as in (5.2.3), $s \to \infty$ implies $(-bB(|u_s|^2) - \chi\|u_s\|_4^4) \longrightarrow -(\chi+\alpha_1 b)\|u\|_4^4 > 0$. Thus there exists s_0 large enough such that $-bB(|u_{s_0}|^2) - \chi\|u_{s_0}\|_4^4 > 0$. Since $V(su_{s_0})$ is a quintic polynomial in s with positive leading coefficient, there exists an s_1 so that $V(s_1 u_{s_0}) = 0$. Similarly if $\chi+\alpha_2 b < 0$ we let $s \to 0^+$ to have $(-bB(|u_s|^2) - \chi\|u_s\|_4^4) \longrightarrow -(\chi+\alpha_2 b)\|u\|_4^4 > 0$, hence we choose s_0 close to 0 such that $-bB(|u_{s_0}|^2) - \chi\|u_{s_0}\|_4^4 > 0$. Rest of the argument proceeds as above.

Now, let $(u_n) \subset \Sigma_0$ be a minimizing sequence such that $\|u_n\|_2 = 1$. Since $T(u_n)$ is bounded so is $\|u_n\|_{H^1}$, hence there exists $u \in H^1(\mathbb{R}^2)$ and a subsequence such that $u_n \rightharpoonup u$ weakly in H^1. In order to utilize the concentration compactness principle of Lions [3] we consider

$$\rho_n(x) = |\nabla u_n(x)|^2 + |u_n(x)|^2,$$

where $\displaystyle\int_{\mathbb{R}^2} \rho_n(x)dx = T(u_n) + \|u_n\|_2^2 \to 2I + 1$. There are three possibilities: vanishing, dichotomy or concentration. Since concentration is the only possibility, there exists $(y_n) \subset \mathbb{R}^2$ such that for every $\epsilon > 0$, there exists $R_\epsilon \geqslant \dfrac{1}{\epsilon}$ and

$$\int_{\mathbb{R}^2 \setminus B_{R_\epsilon}(y_n)} \rho_n(x)dx \leqslant \epsilon.$$

Replacing $u_n(x)$ by $\widetilde{u_n}(x) = u_n(x-y_n)$, $\widetilde{u_n} \rightharpoonup \widetilde{u}$ weakly in $H^1(\mathbb{R}^2)$ and by the imbedding $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ for $2 \leqslant p < \infty$, it follows that $\displaystyle\int_{\mathbb{R}^2\setminus B_{R_\epsilon}(0)} |\widetilde{\varphi_n}|^2 dx \leqslant \epsilon^{p/2}$ for $2 \leqslant p < \infty$. Over $B_{R_\epsilon}(0)$ the imbedding is compact and we can pass to the limit in V. Combining these two, from $V(\widetilde{u_n}) = 0$ it follows that $V(\widetilde{u}) = 0$, i.e., $\widetilde{u} \in \Sigma_0$ with $T(\widetilde{\varphi}) \leqslant \liminf_{n\to\infty} T(\widetilde{\varphi_n}) = 2I$. Hence $\widetilde{u}$ is the desired minimum. Positivity of this minimum is granted by [14, Lemma 8.1.12]. If u solves the minimization problem and ψ is any solution of (5.1.2) then by the Pohozaev like identities in [12] (see also Proposition 5.2.4), we get $V(\psi) = 0$, and hence $L_\omega(u) \leqslant L_\omega(\psi)$.

Let u be a solution of (5.2.4). Then there is a Lagrange multiplier $s > 0$ such that $-\Delta u = s(-bK(|u|^2)u - \chi|u|^2 u - \omega u)$ (see Theorem A.4.2). From that we have a solution of (5.1.2) under the scaling $u_{0,-\frac{1}{2}} = u\left(\dfrac{x}{\sqrt{s}}\right)$. □

Remark 5.2.2. The minimum of T does not change if we replace Σ_0 by $\{u \in H^1(\mathbb{R}^2) : u \neq 0, V(u) \geqslant 0\}$. This is easy to see using one parameter scalings defined in (5.2.1), i.e., the fact that if $V(u) \geqslant 0$ then there exists $0 < s \leqslant 1$ such that $V(su) = 0$.

Remark 5.2.3. Here we want to highlight that minimizers obtained from both methods coincide. By [12, Theorem 2.2], there exists R, which minimizes $J = \dfrac{-2\|f\|_2^2\|\nabla f\|_2^2}{\chi\|f\|_4^4 + bB(|f|^2)}$ over H^1. Furthermore R satisfies Pohozaev type identities, i.e., $T(R) = \omega\|R\|_2^2$ and $V(R) = 0$. Noting that for any u with $V(u) = 0$, $J(u) = \dfrac{1}{\omega}T(u)$ and hence $\dfrac{1}{\omega}T(R) \leqslant J(\psi)$ for all $\psi \in H^1$. Restricting this inequality to Σ_0 we see that R minimizes T over Σ_0. Conversely, let $u \in \Sigma_0$ be a minimizer of T and let $\psi \in H^1$. If $V(\psi) = 0$, clearly $J(u) \leqslant J(\psi)$. Otherwise consider $V(s\psi)$. Since $\chi < \min\{-b\alpha_M, 0\}$, there exists s_0 such that $V(s_0\psi) = 0$. Note that $J(\psi) = J(s_0\psi)$, hence we get that $J(u) \leqslant J(s_0\psi) = J(\psi)$ and so u is a minimizer of J over H^1.

Now we want to outline how to establish Pohozaev type identities given in [12] in an alternative way.

Proposition 5.2.4. *If $u \in H^1$ is a solution of (5.1.2) then*

$$T(u) + \omega\|u\|_2^2 = -bB(|u|^2) - \chi\|u\|_4^4, \qquad 2\omega\|u\|_2^2 = -bB(|u|^2) - \chi\|u\|_4^4.$$

Proof. Note that if u is a solution of (5.1.2) then it is a critical point of L_ω. To show the first identity, differentiate L_ω along the one parameter family defined by $s \longmapsto u_{1,0}$. Since $L_\omega(u_{1,0}) = s^2\frac{1}{2}T(u) + s^4\frac{b}{4}B(|u|^2) + s^4\frac{\chi}{4}\|u\|_4^4 + s^2\frac{\omega}{2}\|u\|_2^2$, the result follows from $\left.\frac{dL_\omega(u_{1,0})}{ds}\right|_{s=1} = 0$. For the second identity, differentiate L_ω along $s \longmapsto u_{0,-1}$. Using the scalings given in (5.2.2), $L_\omega(u_{0,-1}) = \frac{1}{2}T(u) - \lambda^2 V(u)$. Hence $\left.\frac{dL_\omega(u_{0,-1})}{ds}\right|_{s=1} = 0$ yields the second identity. □

5.3. A Gagliardo–Nirenberg Type Inequality and its Consequences

In this section we give an alternative derivation of the Gagliardo–Nirenberg type inequality using the constrained minimization problem described in the previous section. When $\chi + \alpha_1 b < 0$ or $\chi + \alpha_2 b < 0$, in the unconstrained minimization problem the denominator of the functional J can be zero for $u \in H^1(\mathbb{R}^2)$, hence this method does not seem to be applicable. On the other hand, in the constrained minimization problem the potential $V(u)$ can be arranged to change sign along a continuous one parameter family of functions passing through u. This fact plays an important role in the derivation of the main result of this section.

Theorem 5.3.1. *If $\chi + \alpha_1 b < 0$ or $\chi + \alpha_2 b < 0$ then for any $f \in H^1(\mathbb{R}^2)$ we have*

$$-\left(\chi\|f\|_4^4 + bB(|f|^2)\right) \leqslant \frac{\omega}{I}\|f\|_2^2\|\nabla f\|_2^2,$$

where $I = \frac{1}{2}T(u)$ and u is a solution of (5.1.2).

Proof. Let $f \in H^1(\mathbb{R}^2)$ be arbitrary. First, if $V(f) = 0$ then we know that $I \leqslant \frac{1}{2}\|\nabla f\|_2^2$. Hence we establish the result. Second, assume $V(f) > 0$. Since $\omega > 0$ we have $-\chi\|f\|_4^4 - bB(|f|^2) > 0$, and using scaling properties of V we can show the existence of an s such that $V(sf) = 0$. Since J is invariant under these type of scalings the assertion follows from the first case. Finally, if $V(f) < 0$ the claim is trivially true when $-\chi\|f\|_4^4 - bB(|f|^2) \leqslant 0$. If $V(f) < 0$ but $-\chi\|f\|_4^4 - bB(|f|^2) > 0$, considering

$V(sf)$ as a quintic polynomial as before we can find $s_0 > 1$ so that $V(s_0 f) = 0$ hence the first case applies. $\square$

Remark 5.3.2. The connection between I and C_{opt}, where C_{opt} is given in (5.1.4), is established as follows: For R obtained in [12, Theorem 2.2], we have $\frac{1}{\omega}T(R) \leqslant \frac{1}{\omega}T(u)$ for all $u \in \Sigma_0$. Hence $\frac{1}{\omega}T(R) \leqslant \frac{1}{\omega}\inf\{T(u) : u \in \Sigma_0\} = \frac{2I}{\omega}$. Since $R \in \Sigma_0$ via the Pohozaev type identities, $\inf T(u) \leqslant T(R)$. Noting that $T(R) = \omega\|R\|_2^2$ we have $\frac{\omega}{C_{\text{opt}}} = \frac{\omega}{2}\|R\|_2^2 = \frac{1}{2}T(R) = I$.

Using this estimate we can find an upper bound on the initial condition and hence state the following global existence result proof of which is as in [12].

Corollary 5.3.3. *For the Cauchy problem for the GDS system, if* $\chi + b < 0$ *or* $\chi + \frac{b}{m_1} < 0$, *and* $\|v_0\|_2 < \|u\|_2$, *where* $v_0 \in H^1(\mathbb{R}^2)$ *is the initial amplitude and* u *is a solution of* (5.1.2), *then the corresponding solution of the GDS system is global.*

The asymptotic behaviour of solutions is described in the corollary below.

Corollary 5.3.4. *Let* v *be a solution to the Cauchy problem for a GDS system and assume that* v *remains in* $\Sigma := \{v \in H^1(\mathbb{R}^2) : (x^2 + y^2)^{1/2}\, v \in L^2(\mathbb{R}^2)\}$. *If* $\chi + b < 0$ *or* $\chi + \frac{b}{m_1} < 0$, *and* $\|v_0\|_2 < \|u\|_2$, *where* u *is a solution of* (5.1.2), *then*

$$\|v(t)\|_p^p \leqslant C(1 + |t|)^{2-p},$$

for $t > 0$, $p > 2$ *where* C *depends only on* v_0 *and* p.

Proof. In fact, $\|v_0\|_2 < \|u\|_2$ implies that $\|\nabla v(t)\|_2^2 \leqslant M H(v_0)$ for every $t > 0$, with $M = \left(1 - \frac{\|v_0\|_2^2}{\|u\|_2^2}\right)^{-1}$. Proceeding as in [19, Section 4] the claim can be proved. $\square$

In order to adapt the argument in [19] to the present situation one needs the validity of the pseudoconformal invariance under (A1) and (A2). This is addressed in Eden and Kuz [21] as well as the existence and uniqueness of the Cauchy problem for (5.1.2) under (A1) and (A2).

6. CONCLUSION

The hypothesis (A3) given in the previous chapter is satisfied by the symbol of DS system with $\alpha_1 = \alpha_2 = 1$ and by the symbol of the GDS system with $\alpha_1 = 1$ and $\alpha_2 = \dfrac{1}{m_1}$. (A3) was not assumed in [12], hence in a certain sense the result in [12] on existence is more general. However, (A3) plays the key role in the scaling $u \leftrightarrow u_s$ defined in (5.2.3) and in the relation between $B(|u|^2)$ and $\|u\|_4^4$.

Note that the constraint $V(u) = 0$ in the minimization problem gives that $J(u) = \dfrac{1}{\omega}T(u)$, however for $V(u) \neq 0$ we have $J(u) = \dfrac{2\|u\|_2^2 T(u)}{4V(u) + 2\omega\|u\|_2^2}$. Nevertheless using the partial scaling $u \leftrightarrow u_s$ (5.2.1) and the scalings $u \leftrightarrow u_{a,b}$ with $a = 1$, $b = 0$ (5.2.3) one can make $V(u)$ equal to zero when $\chi + \alpha_1 b < 0$ or $\chi + \alpha_2 b < 0$ under the assumption (A3) on $\alpha(\xi)$. Here, although J is invariant under the scalings (5.2.3) it is no longer invariant under the scaling (5.2.1).

The stregth of this thesis lies in the fact that comparing the condition $\chi < \min\{-b\alpha_M, 0\}$ with $\chi + b < 0$ or $\chi + \dfrac{b}{m_1} < 0$ for the GDS system, we see that, when $b > 0$, the first condition reduces to $\chi + b\alpha_M < 0$. Since $\alpha_M \geqslant 1$ and $\alpha_M \geqslant \dfrac{1}{m_1}$ this is a stronger assumption than $\chi + b < 0$ or $\chi + \dfrac{b}{m_1} < 0$. When on the other hand $b < 0$, from the first condition we have $\chi < 0$, whereas $\chi < -b$ or $\chi < -\dfrac{b}{m_1}$ allows positive values for χ as well. When $m_1 = 1$, hence $\alpha_M = 1$, there is still improvement in $\chi + b < 0$ case.

APPENDIX A: THE CALCULUS OF VARIATIONS

A.1. Euler–Lagrange Equation

Let $L : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and consider

$$J(u) := \int_{\mathbb{R}^2} L(\nabla u(x), u(x), x)dx.$$

Computing the Gateâux derivative of J at u in any direction $v \in C_0^\infty(\mathbb{R}^2)$ gives the *Euler–Lagrange equation* associated with the energy functional J, which is the following second-order PDE:

$$-\sum_{i=1}^{n} \left(L_{p_i}(\nabla u, u, x)\right)_{x_i} + L_z(\nabla u, u, x) = 0,$$

where $p = (p_1, \ldots, p_n) = \nabla u(x)$ and $z = u(x)$.

A.2. Existence of Minimizer in Bounded Domains

Theorem A.2.1. [1, Theorem 8.2.2] *Assume that for some fixed $1 < q < \infty$, there exists constants $\alpha > 0$, $\beta \geqslant 0$ such that $L(p, z, x) \geqslant \alpha|p|^q - \beta$ for all $p \in \mathbb{R}^2$, $z \in \mathbb{R}$, $x \in \Omega$. Assume also that L is convex in the variable p. Suppose the space over which J is minimized is nonempty. Then there exists at least one function u such that u minimizes J.*

A.3. Weak Solutions of Euler–Lagrange Equation

Theorem A.3.1. [1, Theorem 8.2.4] *Assume L verifies the growth conditions:*

$$\begin{aligned} |L(p,z,x)| &\leqslant C\left(|p|^q + |z|^q + 1\right), \\ |D_pL(p,z,x)| &\leqslant C\left(|p|^{q-1} + |z|^{q-1} + 1\right), \\ |D_zL(p,z,x)| &\leqslant C\left(|p|^{q-1} + |z|^{q-1} + 1\right), \end{aligned}$$

for some constant C. Suppose u is a minimizer of J. Then u is a weak solution of the Euler–Lagrange equation associated with J.

A.4. Constraint Minimization

For Ω open and bounded consider the problem of minimizing $J(u) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx$ over all functions in $H_0^1(\Omega)$ but subject to the integral constraint $I(u) = \displaystyle\int_\Omega G(u)dx = 0$, where $G : \mathbb{R} \to \mathbb{R}$ is given a smooth function. Write g for G' and assume $|g(z)| \leqslant C(|z|+1)$. Then the following theorem holds true.

Theorem A.4.1. [1, Theorem 8.4.1] *Assume the admissible set $\mathcal{A} := \{w \in H_0^1(\Omega) : I(w) = 0\}$ is nonempty. Then there exists $u \in \mathcal{A}$ satisfying*

$$I(u) = \min_{w\in\mathcal{A}} I(w).$$

We can relax the bounded condition of the space by utilizing other compactness tools like Strauss Compactness Lemma or Concentration Compactness Lemma. Independent of the boundedness of the domain we have the following theorem.

Theorem A.4.2 (Lagrange multiplier)**.** [1, Theorem 8.4.2] *Let* $u \in \mathcal{A}$ *satisfy*

$$I(u) = \min_{w \in \mathcal{A}} I(w).$$

Then there exists a real number λ *such that*

$$\int_\Omega \nabla u \cdot \nabla v dx = \lambda \int_\Omega g(u) v dx$$

for all $v \in H_0^1(\Omega)$*.*

APPENDIX B: SOME BACKGROUND IN ANALYSIS

B.1. Schwarz Symmetrization

We introduce here the basic properties of Schwarz symmetrization without giving the proofs and refer them to [13, 2]. First, recall the definition of the spherical symmetrization of a function. Let $f \in L^1(\mathbb{R}^2)$, then f^*, the Schwarz symmetrized function of f, is a radial, nonincreasing, measurable function such that for any $\alpha > 0$,

$$\text{meas}\{f^* \geqslant \alpha\} = \text{meas}\{|f| \geqslant \alpha\}.$$

It is obvious that $\displaystyle\int_{\mathbb{R}^2} F(f)dx = \int_{\mathbb{R}^2} F(f^*)dx$ for every continuous function F such that $F(f)$ is integrable. A fundamental property of the mapping $f \mapsto f^*$ is the following:

Proposition B.1.1 (Riesz inequality)**.** *Let f, g be in $L^2(\mathbb{R}^2)$, then* $\displaystyle\int_{\mathbb{R}^2} f(x)g(x)dx \leqslant \int_{\mathbb{R}^2} f^*(x)g^*(x)dx.$

From this inequality we have $\|f^* - g^*\|_2 \leqslant \|f - g\|_2$ for all $f, g \in L^2(\mathbb{R}^2)$.

Another important consequence of the Riesz inequality is the following result.

Proposition B.1.2. *Let $u \in H^1(\mathbb{R}^2)$. Then $u^* \in H^1(\mathbb{R}^2)$ and we have*

$$\int_{\mathbb{R}^2} |\nabla u^*(x)|^2 dx \leqslant \int_{\mathbb{R}^2} |\nabla u(x)|^2 dx.$$

B.2. Gagliardo-Nirenberg-Sobolev Inequality

Theorem B.2.1. [14, Theorem 2.3.7] *Let $1 \leqslant p, q, r \leqslant \infty$ and let j, m be two integers, $0 \leqslant j < m$. If*

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{(1-a)}{q},$$

for some $a \in [j/m, 1]$ *(*$a < 1$ *if* $r > 1$ *and* $m - j - \frac{n}{r} = 0$*), then there exists* $C = C(n, m, j, a, q, r)$ *such that*

$$\sum_{|\alpha|=j} \|D^\alpha u\|_p \leqslant C \left(\sum_{|\alpha|=m} \|D^\alpha u\|_r \right)^a \|u\|_q^{1-a},$$

for every $u \in C_c^\infty(\mathbb{R}^n)$*.*

Indeed, taking $n = 2$, $p = 4$, $q = 2$, $r = 2$, $j = 0$, $m = 1$ and $a = \frac{1}{2}$ we get $\|u\|_4^4 \leqslant C_1 \|\nabla u\|_2^2 \|u\|_2^2$ for some C_1. Similarly for $n = 2$, $p = \sigma + 2$, $q = 2$, $r = 2$, $j = 0$, $m = 1$ and $a = \frac{\sigma}{\sigma + 2}$ there exists a constant C_2 such that $\|u\|_{\sigma+2}^{\sigma+2} \leqslant C_2 \|\nabla u\|_2^\sigma \|u\|_2^2$.

APPENDIX C: A FACT OF SOBOLEV SPACES

C.1. Sobolev Imbedding Theorem

Theorem C.1.1. [22, Theorem 2.4.5] *Let $m \geqslant 1$ be an integer and $1 \leqslant p < \infty$. Then*

(i) if $\dfrac{1}{p} - \dfrac{m}{n} > 0$, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, *with* $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$,
(ii) if $\dfrac{1}{p} - \dfrac{m}{n} = 0$, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, *for* $p \leqslant q < \infty$,
(iii) if $\dfrac{1}{p} - \dfrac{m}{n} < 0$, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$.

In particular $W^{m,p}(\mathbb{R}^n) \hookrightarrow C^k(\mathbb{R}^n)$ *for* $m > \dfrac{n}{p}$, *where* $k = \left\lfloor m - \dfrac{n}{p} \right\rfloor$.

From the theorem we have, for $n = 2$, $m = 1$ and $p = 2$, $H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for all $2 \leqslant q < \infty$.

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