UNITARY MATRIX HOPF ALGEBRAS AND THETA DEFORMED FERMION ALGEBRA

by

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ABSTRACT

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The Hopf Algebras and their properties are reviewed. It is shown that unitary matrices with non-commutative elements can represent Hopf Algebras, and new matrix algebras were examined to see if they are Hopf Algebras. Fermion algebra is deformed with a central element c and properties and representations of this algebra are studied. Finally tensor product representation of this algebra is defined and it is shown that this deformed fermion algebra describes the orbifold S1/Z2.

ÖZET

UNİTER MATRİS HOPF CEBİRLERİ VE TETA DEFORME FERMİYON CEBİRİ

Hopf Cebirleri ve özellikleri gözden geçirildi. Elemanları komütatif olmayan bir üniter matrisin Hopf Cebirlerini temsil edebildiği gösterildi ve Hopf Cebiri olup olmadıklarını görmek için yeni uniter matris cebirleri araştırıldı. Fermion cebiri merkezi bir c öğesiyle deforme edildi, yeni cebirin özellikleri ve temsilleri çıkarıldı. Son olarak, bu cebirin tansör çarpımı temsili tanımlandı ve bu deforme edilmis fermion cebirinin S1/Z2 orbifoldunu tanımladığı gösterildi.

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LIST OF SYMBOLS

m	Product map
Δ	Coproduct map
ϵ	Counit map
η	Unit map
id	Identity map
S	Antipode map
c	Deformation coefficient
θ	Deformation angle
s	Sine operator of deformed fermion algebra
g	Hermitian Unitary operator of deformed fermion algebra
a	Annihilation operator of deformed fermion algebra
a^*	Creation operator of deformed fermion algebra
N	Number operator
$U_G(2)$	Unitary group of 2×2 matrices
S^1 / Z_2	1 dimensional Orbifold

1. INTRODUCTION

Group theory has various applications in physics, especially in quantum mechanics. Hopf Algebras are generalizations of ordinary groups, and have useful applications in situations where ordinary groups may be insufficient. When a classical system is quantized, the invariance group of the classical system may also need quantization and thus be promoted into a Hopf Algebra. First, Milnor and Moore used the term Hopf Algebra. It is named after Heinz Hopf, a German mathematician who had important works on topology.

In a Hopf Algebras there are 3 characteristic operations on elements: Coproduct(Δ), counit(ϵ) and the antipode(S). Understanding Hopf Algebras and these operations can be achieved by use of commutative diagrams, which are used in category theory. It is a diagram shown by arrows, one can go in both ways in the diagram to reach the same result. Commutative diagrams are analogies to equations in an algebra. These diagrams are easier ways of writing complicated equations. In the second chapter, it will help us a lot to understand basic concepts about Hopf Algebras.

One can write matrices, whose elements form Hopf Algebras. The properties of these algebras can be obtained from matrix properties like matrix inverse, matrix multiplication and unit matrix. In this thesis we will be using 2×2 matrices. Furthermore, these matrices can be set to be unitary and may again obey the Hopf Algebra axioms. In the third chapter we will be seeking for unitary matrices, whose elements form a Hopf Algebras. If the elements of these matrices does not form Hopf Algebras, we will seek for deformations to these matrices to set them as Hopf Algebras.

The algebra that explains the behavior of fermions can again be expressed as a matrix, and it is not a Hopf Algebra. Thus, one can make a deformation on that to make it Hopf Algebra by adding an element c. The fourth chapter will deal with the deformed fermion algebra.

There are previous works by Arikan and Arik[5] that analyzed deformed fermion algebra, however in this thesis we use the deformation in a physically more consistent way and discuss it in a more complete manner. We will seek for representations of the deformed fermion algebra. As the non-deformed fermion algebra, deformed algebra has 2 dimensional representations, in which eigenstates are common eigenstates of deformation c and number element N.

The coproduct of the Hopf Algebra enables us to relate the representations of the tensor product algebra $A \otimes A$ to the representations of A. Many problems in physics use Hopf Algebra's, although in most texts it is not mentioned. For instance; addition of two angular momenta $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$, and operating $\mathbf{J}_z^{(1)} + \mathbf{J}_z^{(2)}$ on a state of the tensor product algebra $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$ is an important example in understanding quantum mechanics. One mostly says that $\mathbf{J}_z^{(1)}$ applies on the first state vector,$|j_{(1)} m_1\rangle$ and $\mathbf{J}_z^{(2)}$ applies on the second,$|j_2 m_2\rangle$. It is more proper to formulate this as follows. The coproduct of $\mathbf{J}_z$ is $\Delta(\mathbf{J}_z) = \mathbf{J}_z^{(1)} \otimes 1 + 1 \otimes \mathbf{J}_z^{(2)}$, where application of 1 does not make any change in the state vector. So the addition of angular momentum is explained in a more consistent way by using Hopf Algebras.

Similarly, coproduct of c in the deformed fermion algebra will be our interest in the last part of the thesis. We will check the tensor product algebra and construct the tensor product states. The eigenvalues of c will be obtained from properties of the two algebras in the tensor product.

In the representations deformed fermion algebra we will also see that, we can use angle θ instead of coefficient c. Eigenvalues of the elements of the algebra will be in terms of sine and cosine of θ.

This choice of coefficient exposes an important result. Replacing θ with $-\theta$ does not create a different algebra. Algebras with θ and $-\theta$ are related by a similarity transformation, thus they are equivalent. This is a 1-dimensional orbifold S^1 / Z_2. Orbifolds may lead to important discussion in physics, however in the thesis, we do not investigate further than the relation of 1-dimensional orbifold with deformed fermion algebra. From this point one may relate some Hopf Algebras with higher dimensional orbifolds.

2. PRELIMINARIES

2.1. Algebra, Coalgebra and Hopf Algebra

2.1.1. Algebra

An algebra consists of a k-vector space A, a ring k, unit map η and multiplication map m. This maps has the following properties:

$$\eta \;:\; k \longrightarrow A \quad \text{such that} \quad \eta(\alpha) = \alpha I_A \;\; (I_A \text{ is the unit element of } A) \tag{2.1}$$

$$m \;:\; A \otimes A \longrightarrow A \tag{2.2}$$

These operators satisfy a set of relations and one can express the relations in terms of commutative diagrams:

$$\begin{tikzcd}
& A \otimes A \otimes A \arrow[dl, "\mathrm{id} \otimes m"'] \arrow[dr, "m \otimes \mathrm{id}"] & \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[drr, "m"] & & \\
k \otimes A \arrow[u, "\eta \otimes \mathrm{id}"] & = & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[drr, "m"] & & \\
A \otimes k \arrow[u, "\mathrm{id} \otimes \eta"] & = & A
\end{tikzcd}$$

Figure 2.1. Algebra

This diagrams are equivalent to the equation:

$$m(id \otimes m) = m(m \otimes id)$$

$$m(\eta \otimes id) = m(id \otimes \eta)$$

2.1.2. Coalgebra

A coalgebra consists of a k-vector space C, a ring k, counit map ϵ and coproduct map Δ, with the following properties:

$$\epsilon \;:\; C \longrightarrow k \tag{2.3}$$

$$\Delta \;:\; C \longrightarrow C \otimes C \tag{2.4}$$

In a coalgebra, the diagrams in Figure 2.2 commute,

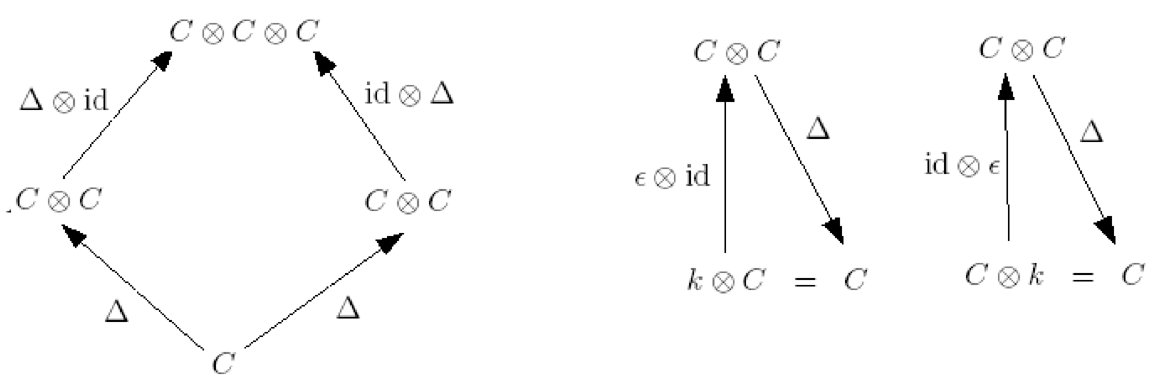

Figure 2.2. Coalgebra

One can also express these as follows:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

$$(\epsilon \otimes id)\Delta = (id \otimes \epsilon)\Delta$$

2.1.3. Hopf Algebra

Hopf Algebra is a bialgebra, which has algebra and coalgebra structures with an additional map called antipode. So Hopf Algebra has a k-vector space A, ring k and the following maps:

$$\text{Multiplication} : m \quad : \quad A \otimes A \longrightarrow A \tag{2.5}$$

$$\text{Coproduct} : \Delta \quad : \quad A \longrightarrow A \otimes A \tag{2.6}$$

$$\text{Unit} : \eta \quad : \quad k \longrightarrow A \qquad \text{such that} \qquad \eta(\alpha) = \alpha I_A \tag{2.7}$$

$$\text{Antipode} : S \quad : \quad A \longrightarrow A \tag{2.8}$$

$$\text{Counit} : \epsilon \quad : \quad A \longrightarrow k \tag{2.9}$$

The antipode is somehow like an inverse, it is also called co-inverse. However to have a Hopf Algebra it is not necessary for each element to have an inverse. Antipode is a bit more complex structure than an inverse. Thus, actions of this operators are characterized by Figure 2.3, and it is equivalent to the equations below:

$$\begin{tikzcd}
A \arrow[r, "\epsilon"] \arrow[d, "\Delta"'] & k \arrow[r, "\eta"] & A \\
A \otimes A \arrow[rr, "\mathrm{id} \otimes S\ ,\quad S \otimes \mathrm{id}"] & & A \otimes A \arrow[u, "m"']
\end{tikzcd}$$

Figure 2.3. Hopf Algebra

$$\begin{aligned} m(S \otimes id)\Delta &= \eta\epsilon && (2.10) \\ (\Delta \otimes id)\Delta &= (id \otimes \Delta)\Delta && (2.11) \\ (\epsilon \otimes id)\Delta &= (id \otimes \epsilon)\Delta && (2.12) \end{aligned}$$

2.2. Matrix Algebra

Matrix Algebra is an example of Hopf Algebra. If we take a 2×2 matrix, each elements coproduct, counit and antipode is determined by properties of the matrix. Coproduct is determined by matrix multiplication, counit by unit matrix and antipode by matrix inverse. With all these, matrix algebra satisfies all Hopf Algebra rules.

Consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose elements may be noncommutative. Coproduct of this matrix is defined as follows:

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix}$$

The coproduct of each matrix element can be written according to the previous tensor product of matrices:

$$\Delta(a) = a \otimes a + b \otimes c \tag{2.13}$$

$$\Delta(b) = a \otimes b + b \otimes d \tag{2.14}$$

$$\Delta(c) = c \otimes a + d \otimes c \tag{2.15}$$

$$\Delta(d) = c \otimes b + d \otimes d \tag{2.16}$$

For simplicity say $ad - bc = 1$. We see how antipode is like inverse, because action of antipode transforms the matrix A into the inverse A of A. The action of ϵ transforms A into the unit matrix. However, it can not be said for any element α that $S(\alpha) = \alpha^{-1}$.

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad , \quad \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Again, action of antipode and ϵ on each elements is deduced from this, as follows:

$$S(a) = d \quad , \quad \epsilon(a) = 1 \tag{2.17}$$

$$S(b) = -b \quad , \quad \epsilon(b) = 0 \tag{2.18}$$

$$S(c) = -c \quad , \quad \epsilon(c) = 0 \tag{2.19}$$

$$S(d) = a \quad , \quad \epsilon(d) = 1 \tag{2.20}$$

When one applies (2.10) to this algebra, one sees that it is consistent.

$$\cdot(S \otimes \mathrm{id})\Delta(a) = \cdot(S \otimes \mathrm{id})(a \otimes a + b \otimes c) \quad (2.21)$$

$$= \cdot(S(a) \otimes a + S(b) \otimes c) \quad (2.22)$$

$$= da - bc \quad (2.23)$$

$$= 1 = \eta\epsilon(a) \quad (2.24)$$

Other elements also obey the Hopf Algebra rules:

$$\cdot(S \otimes \mathrm{id})\Delta(a) = 1 = \eta\epsilon(a) \quad (2.25)$$

$$\cdot(S \otimes \mathrm{id})\Delta(b) = 0 = \eta\epsilon(b) \quad (2.26)$$

$$\cdot(S \otimes \mathrm{id})\Delta(c) = 0 = \eta\epsilon(c) \quad (2.27)$$

$$\cdot(S \otimes \mathrm{id})\Delta(d) = 1 = \eta\epsilon(d) \quad (2.28)$$

Then, as stated before, one can say this matrix algebra is a ”Hopf Algebra”.

2.2.1. Unitary Matrix Algebra

Now we will add another constraint to matrix algebra by making matrices with noncommutative elements to be unitary. We call this the Unitary group $U_G(2)$. In this group, $a, b, c, d \in A$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_G(2)$ such that;

$$MM^{\dagger} = I \qquad \text{i.e.} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.29}$$

By using this one can prove that it is also valid in reverse order: $M^{\dagger}M = I$.

Proof: From (2.29) we have the following relations:

$$aa^* + bb^* = 1 \tag{2.30}$$

$$ac^* + bd^* = 0 \tag{2.31}$$

$$cc^* + dd^* = 1 \tag{2.32}$$

We assume that each element has an inverse. Without this assumption, no solutions seem possible. So $d = -ca^*b^{*-1}$ and $d^* = -b^{-1}ac^*$. From (2.32),

$$cc^* + ca^*b^{*-1}b^{-1}ac^* = cc^* + ca^*(bb^*)^{-1}ac^* = cc^* + ca^*(1 - aa^*)^{-1}ac^* = 1$$

Since for a function f, one has: $a^*f(aa^*) = f(a^*a)a^*$,[1] thus,

[1]Say $f(aa^*) = \Sigma c_k(aa^*)^k \Rightarrow a^*f(aa^*) = \Sigma c_k a^*(aa^*)^k = \Sigma c_k(a^*a)^k a^* = f(a^*a)a^*$

$$
\begin{aligned}
cc^* + c(1-a^*a)^{-1}a^*ac^* &= cc^* + c\left[\frac{a^*a}{1-a^*a}\right]c^* = 1 && (2.33)\\
&\Rightarrow c\left[\frac{1}{1-a^*a}\right]c^* = 1 && (2.34)\\
&\Rightarrow (c^*c)^{-1} = (1-a^*a)^{-1} && (2.35)\\
&\Rightarrow c^*c + a^*a = 1 && (2.36)
\end{aligned}
$$

By the same way one obtains other relations :

$$
\begin{aligned}
a^*b + c^*d &= 0 && (2.37)\\
b^*b + d^*d &= 1 && (2.38)
\end{aligned}
$$

These equations are equivalent to:

$$
M^\dagger M = I \qquad \text{i.e.} \quad \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

Finally, one can write the relations clearly as follows;

$$aa^* + bb^* = I \tag{2.39}$$

$$ac^* + bd^* = 0 \tag{2.40}$$

$$cc^* + dd^* = I \tag{2.41}$$

$$a^*a + c^*c = I \tag{2.42}$$

$$a^*b + c^*d = 0 \tag{2.43}$$

$$b^*b + d^*d = I \tag{2.44}$$

From these relations one can prove $U_G(2)$ is also a Hopf Algebra. The actions of Δ, ϵ and S on each element is:

$$S(a) \ = a^*, \quad \Delta(a) = a \otimes a + b \otimes c, \quad \epsilon(a) = 1 \tag{2.45}$$

$$S(b) \ = c^*, \quad \Delta(b) = a \otimes b + b \otimes d, \quad \epsilon(b) = 0 \tag{2.46}$$

$$S(c) \ = b^*, \quad \Delta(c) = c \otimes a + d \otimes c, \quad \epsilon(c) = 0 \tag{2.47}$$

$$S(d) \ = d^*, \quad \Delta(d) = c \otimes b + d \otimes d, \quad \epsilon(d) = 1 \tag{2.48}$$

With the antipode defined in that way, $U_G(2)$ obeys the Hopf Algebra rules from (2.10) to (2.12).

Now, we have the unitary groups, which can be treated as Hopf Algebras. To the unitary group, one can add more constraints, and see if it still satisfies Hopf Algebra rules.

3. SEARCH FOR NEW UNITARY MATRIX HOPF ALGEBRAS

Elements of matrices in the unitary matrix algebra $U_G(2)$ may form a Hopf Algebra as stated. Now we will seek some special unitary matrices and see if they still have Hopf Algebra structure. The general form of a unitary matrix has the following form:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

One can change it by adding new elements or implying some relation between the current elements a, b, c, d. For instance one may set the elements to be equal (*i.e.* $d = a$) or use hermitian conjugates of elements in the relations (*i.e.* $b = a^*$). Elements of the matrix above form a Hopf Algebra, however these special cases may lack the Hopf Algebra structure; beyond that, characteristic functions of the Hopf Algebra like antipode or coproduct may impose some changes on the relations between the elements. Some of these may have a physical significance and lead us to some problems in physics.

Now, we will start with simple matrix algebras and see if Hopf Algebra structure is satisfied. In each of them we will have a general way of investigating structure of the algebra. This can be summarized as following steps:

1. Write the relations between the elements of the algebra by using equations we deduced from the fact that $M^\dagger M = MM^\dagger = 1$:

$$aa^* + bb^* = I \tag{3.1}$$

$$ac^* + bd^* = 0 \tag{3.2}$$

$$cc^* + dd^* = I \tag{3.3}$$

$$a^*a + c^*c = I \tag{3.4}$$

$$a^*b + c^*d = 0 \tag{3.5}$$

$$b^*b + d^*d = I \tag{3.6}$$

2. See if any of these contradict with each other, if not; see if these imply more relations between the algebra elements
3. Write the counit ϵ, antipode S and coproduct Δ of each element according to equations:

$$S(a) \;\; = a^*, \qquad \Delta(a) = a \otimes a + b \otimes c, \qquad \epsilon(a) = 1 \tag{3.7}$$

$$S(b) \;\; = c^*, \qquad \Delta(b) = a \otimes b + b \otimes d, \qquad \epsilon(b) = 0 \tag{3.8}$$

$$S(c) \;\; = b^*, \qquad \Delta(c) = c \otimes a + d \otimes c, \qquad \epsilon(c) = 0 \tag{3.9}$$

$$S(d) \;\; = d^*, \qquad \Delta(d) = c \otimes b + d \otimes d, \qquad \epsilon(d) = 1 \tag{3.10}$$

4. See if these basic functions contradict with the equations found in the first and second steps. Here, one needs to consider the elements which are same and hermitian conjugates of elements, if they exist in the algebra. Since Δ and ϵ are morphism and S is an anti-morphism, the following should be satisfied for a general element g. If not, one does not have a Hopf Algebra[9]:

$$\Delta(g^*) = [\Delta(g)]^*, \qquad \epsilon(g^*) = [\epsilon(g)]^* \qquad S\left(S(g)^*\right) = g^*$$

Finally this process as a whole may either show us that the algebra in consideration is not a Hopf Algebra, or it needs more specific properties. This properties will lead to some physical cases.

Now we will start from the simplest one and try to add more elements to have different algebras.

3.1. XU(2)

The first algebra we will consider is the simplest one. We set $c = b$ and $d = a$. The matrix becomes:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

With the help of relations (3.1) to (3.6), one can see:

$$a^*a + b^*b = 1 \tag{3.11}$$

$$aa^* + bb^* = 1 \tag{3.12}$$

$$a^*b = -ba^* \tag{3.13}$$

$$b^*a = -ab^* \tag{3.14}$$

From these relations there seems no new relation. Next one should check the coproducts of a and b. Here two elements of the matrix is a, and other two is b. So one needs to check if coproduct, antipode and counit of both a's and both b's are the same. It turns out that, they are same as follows:

$$S(a) = a^*, \qquad \Delta(a) = a \otimes a + b \otimes b \quad , \quad \epsilon(a) = 1 \tag{3.15}$$

$$S(b) = b^*, \qquad \Delta(b) = a \otimes b + b \otimes a \quad , \quad \epsilon(b) = 0 \tag{3.16}$$

$$S\left(S(a)^*\right) = a^* \qquad , \qquad S\left(S(b)^*\right) = b^* \tag{3.17}$$

These tell that $XU(2)$ is a Hopf Algebra. We will now study some algebras in which hermitian conjugates of elements exist.

3.2. SU(2)

As stated, we will add elements which are hermitian conjugates of other elements. We set $c = -b^*$ and $d = a^*$. This algebra is represented by the following matrix:

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

Relations (3.1) to (3.6) are used and one obtains the new relations for $SU(2)$.

$$aa^* + bb^* = 1 \tag{3.18}$$

$$aa^* + b^*b = 1 \tag{3.19}$$

$$a^*a + bb^* = 1 \tag{3.20}$$

$$a^*a + b^*b = 1 \tag{3.21}$$

$$ab = ba \tag{3.22}$$

$$b^*a^* = a^*b^* \tag{3.23}$$

$$a^*b = ba^* \tag{3.24}$$

$$ab^* = b^*a \tag{3.25}$$

The first four equalities imply that:

$$aa^* = a^*a \tag{3.26}$$

$$bb^* = b^*b \tag{3.27}$$

Combining all these relations, one can easily see that algebra is summarized as follows:

$$[a, a^*] = [a, b^*] = [a, b] = [a^*, b] = [a^*, b^*] = [b, b^*] = 0$$

Since each element commute with each other, one may call $SU(2)$ commutative algebra. Next, one need to check characteristic functions of Hopf Algebra. The antipode, coproduct and counit are as follows:

$$S(a) = a^*, \qquad \Delta(a) = a \otimes a - b \otimes b^*, \qquad \epsilon(a) = 1 \tag{3.28}$$

$$S(b) = -b, \qquad \Delta(b) = a \otimes b + b \otimes a^*, \qquad \epsilon(b) = 0 \tag{3.29}$$

$$S(a^*) = a, \qquad \Delta(a^*) = a^* \otimes a^* - b^* \otimes b, \qquad \epsilon(a^*) = 1 \tag{3.30}$$

$$S(-b^*) = b^*, \qquad \Delta(-b^*) = -a^* \otimes b^* - b^* \otimes a, \qquad \epsilon(b^*) = 0 \tag{3.31}$$

One has to check the rules stated in step 4.

$$[\Delta(a)]^* = a^* \otimes a^* - b^* \otimes b = \Delta(a^*)$$

$$[\Delta(b)]^* = a^* \otimes b^* + b^* \otimes a = \Delta(b^*)$$

Conjugate commutation of the coproduct is satisfied as above, The rule for the antipode ($S(S(g)^*) = g^*$) is also satisfied, since:

$$S(S(a)^*) = S((a^*)^*) = S(a) = a^* \quad \text{and} \quad S(S(b)^*) = S(b^*) = b^*$$

Thus, $SU(2)$ is a Hopf Algebra.

An important result emerges from this particular algebra. In the beginning, elements of the algebra were not necessarily commutative, however Hopf Algebra rules imposed commutativity among each pair of elements. In other words if any two elements do not commute, $SU(2)$ cannot be a Hopf Algebra.

3.3. $\mathbf{SU}_q\mathbf{(2)}$

This algebra is kind of a generalization of $SU(2)$. Instead of using $c = -b^*$, we set $c = -q^{-1}b^*$ where q is a real number. As in $SU(2)$, $d = a^*$. So the matrix representation is as follows:

$$\begin{pmatrix} a & b \\ -q^{-1}b^* & a^* \end{pmatrix}$$

Relations (3.1) to (3.6) imply the following:

$$aa^* + b^*b = 1 \tag{3.32}$$

$$aa^* + bb^* = 1 \tag{3.33}$$

$$a^*a + q^{-2}bb^* = 1 \tag{3.34}$$

$$a^*a + q^{-2}b^*b = 1 \tag{3.35}$$

$$q^{-1}b^*a^* = a^*b^* \tag{3.36}$$

$$q^{-1}ab = ba \tag{3.37}$$

$$q^{-1}ba^* = a^*b \tag{3.38}$$

$$q^{-1}ab^* = b^*a \tag{3.39}$$

From first two relations, one can see that $b^*b = bb^*$.

If one checks the coproducts and antipodes of elements according to equations

(3.7) to (3.10), the following are found:

$$\Delta(a) = a \otimes a - b \otimes q^{-1}b^* \tag{3.40}$$

$$\Delta(a^*) = q^{-1}b^* \otimes b + a^* \otimes a^* \tag{3.41}$$

$$\Delta(b) = a \otimes b + b \otimes a^* \tag{3.42}$$

$$\Delta(-q^{-1}b^*) = -q^{-1}(b^* \otimes a + a^* \otimes b^*) \tag{3.43}$$

These equations obviously satisfy $[\Delta(a)]^* = \Delta(a^*)$ and $[\Delta(b)]^* = \Delta(b^*)$. Next, antipodes should be checked. One finds out that:

$$S(a) = a^* \tag{3.44}$$

$$S(a^*) = a \tag{3.45}$$

$$S(b) = -q^{-1}b \tag{3.46}$$

$$S(-q^{-1}b^*) = b^* \tag{3.47}$$

When one checks the rule for antipode, one sees:

$$S(S(a)^*) = S((a^*)^*) = S(a) = a^* \quad \textit{and} \quad S(S(b)^*) = S((-q^{-1}b)^*) = -q^{-1}S(b^*) = (-q^{-1})(-q)b^* =$$

Which shows that rule for antipode is satisfied too. Finally, $SU_q(2)$ is a Hopf Algebra, and it is more general than $SU(2)$

3.4. $\mathbf{SU}_q\mathbf{(2)}$ for $q \in \mathbb{C}$

Another generalization of $SU(2)$ is $SU_q(2)$ for $q \in \mathbb{C}$. Its only difference from $SU_q(2)$ is q being a complex number. The algebra can be represented as follows:

$$\begin{pmatrix} a & b \\ -q^{-1}b^* & a^* \end{pmatrix}$$

Although the matrix is the same as $SU_q(2)$, the relations that one deduce from equations (3.1) to (3.6) are not the same. By using $Q \equiv |q|^2$ one obtains the following equations.

$$aa^* + b^*b = 1 \tag{3.48}$$

$$aa^* + bb^* = 1 \tag{3.49}$$

$$a^*a + Q^{-1}bb^* = 1 \tag{3.50}$$

$$a^*a + Qb^*b = 1 \tag{3.51}$$

$$q^{-1}b^*a^* = a^*b^* \tag{3.52}$$

$$\bar{q}^{-1}ab = ba \tag{3.53}$$

$$q^{-1}ba^* = a^*b \tag{3.54}$$

$$\bar{q}^{-1}ab^* = b^*a \tag{3.55}$$

One can see that also following equations are satisfied:

$$bb^* = b^*b \tag{3.56}$$

$$(1 - a^*a)Q = 1 - aa^* \tag{3.57}$$

Without checking coproduct and antipodes, one can use the previous equations to prove $SU_q(2)$ for $q \in \mathbb{C}$ is not a Hopf Algebra. By using equations (3.48), one sees:

$$b(aa^* + bb^*) = b \tag{3.58}$$

$$\Rightarrow \quad baa^* + b^2b^* = b \tag{3.59}$$

$$\Rightarrow \quad \bar{q}^{-1}aba^* + b^2b^* = b \tag{3.60}$$

$$\Rightarrow \quad \frac{q}{\bar{q}}aa^*b + b^*b^2 = b \tag{3.61}$$

$$\Rightarrow \quad \left(\frac{q}{\bar{q}}aa^* + b^*b\right) b = b \tag{3.62}$$

The last line tells that, one must have $\frac{q}{\bar{q}}aa^* + b^*b = 1$. However, we already know that $aa^* + b^*b = 1$. Thus:

$$\frac{q}{\bar{q}} = 1 \Rightarrow q = \bar{q} \Rightarrow q \in \mathbb{R}$$

This result tells us that there is no subgroup $SU_q(2)$ of $U_G(2)$ such that q is a complex number. The characteristic relations of unitary matrix algebra $U_G(2)$ impose that q has to be a real number. We do not need to check further aspect such as antipode and coproduct.

3.5. $\mathbf{SU}_p\mathbf{(2)}$

Now consider another algebra where $c = -b^*$ and $d = pa^*$ with p is a complex number. $SU_p(2)$ is expressed as:

$$\begin{pmatrix} a & b \\ -b^* & pa^* \end{pmatrix}$$

The relations become;$(P \equiv |p|^2)$

$$aa^* + bb^* = 1 \tag{3.63}$$

$$b^*b + Pa^*a = 1 \tag{3.64}$$

$$b^*b + Paa^* = 1 \tag{3.65}$$

$$a^*a + bb^* = 1 \tag{3.66}$$

$$pa^*b^* = b^*a^* \tag{3.67}$$

$$\bar{p}ba = ab \tag{3.68}$$

$$pba^* = a^*b \tag{3.69}$$

$$\bar{p}ab^* = b^*a \tag{3.70}$$

From these one obtains:

$$aa^* = a^*a$$

Thus one may check if there is any constraint on p. From the equations above, there seems no restriction p. However, one can check the coproduct to see if the rule $[\Delta(a)]^* = \Delta(a^*)$ impose any change on p. The coproduct of a and pa^* are as follows:

$$\begin{aligned} \Delta(a) &= a \otimes a - b \otimes b^* && (3.71) \\ \Delta(a^*) &= p^2 a^* \otimes a^* - b^* \otimes b && (3.72) \\ [\Delta(a)]^* &= a^* \otimes a^* - b^* \otimes b && (3.73) \end{aligned}$$

The rule $[\Delta(a)]^* = \Delta(a^*)$ implies:

$$a^* \otimes a^* - b^* \otimes b = pa^* \otimes a^* - \frac{1}{p} b^* \otimes b$$

Thus, one must have $p = 1$. The equality $[\Delta(b)]^* = \Delta(b^*)$ also implies that $p = 1$. We see that the algebra $SU_p(2)$ turns out to be nothing different from $SU(2)$.

3.6. $\mathbf{SU}_{q,p}\mathbf{(2)}$

The most general case that can exist for the last 4 algebras is the $SU_{q,p}(2)$, in which $c = -qb^*$ and $d = pa^*$. It is represented as follows:

$$\begin{pmatrix} a & b \\ -qb^* & pa^* \end{pmatrix}$$

By creating new variables as $P \equiv |p|^2$ and $Q \equiv |q|^2$, one obtains the following relations:

$$aa^* + bb^* = 1 \tag{3.74}$$

$$Qb^*b + Pa^*a = 1 \tag{3.75}$$

$$b^*b + Paa^* = 1 \tag{3.76}$$

$$a^*a + Qbb^* = 1 \tag{3.77}$$

$$pa^*b^* = qb^*a^* \tag{3.78}$$

$$\bar{p}ba = \bar{q}ab \tag{3.79}$$

$$\bar{q}pba^* = a^*b \tag{3.80}$$

$$q\bar{p}ab^* = b^*a \tag{3.81}$$

Now let us create new variables for the sake of simplicity as follows:

$$A \equiv aa^*, \qquad B \equiv bb^*, \qquad A' \equiv a^*a, \qquad B' \equiv b^*b,$$

With these variables one has new relations:

$$B = (1 - A')/Q, \quad B = 1 - A \tag{3.82}$$

$$\Rightarrow (1 - A') = Q(1 - A) \tag{3.83}$$

$$\Rightarrow A' = 1 - Q(1 - A) \tag{3.84}$$

One also has the following relations:

$$B' = (1 - PA')/Q, \qquad B' = 1 - PA \tag{3.85}$$

$$\Rightarrow (1 - PA')/Q = (1 - PA) \tag{3.86}$$

By combining (3.84) and (3.86), one obtains:

$$1 - P(1 - Q(1 - A)) = Q(1 - PA) \tag{3.87}$$

$$\Rightarrow 1 - P + QP - QPA = Q - QPA \tag{3.88}$$

$$\Rightarrow 1 - P = Q(1 - P) \tag{3.89}$$

The final equation has two solutions. Either $Q = 1(q = e^{i\alpha})$ or $P = 1(p = e^{i\beta})$.

With these results, the following is true.

$$If \quad Q = 1 \Rightarrow A = A' \quad \text{and} \quad B' = 1 - P + PB$$

$$If \quad P = 1 \Rightarrow B = B' \quad \text{and} \quad A' = 1 - Q + QA$$

Next, one may check coproducts by choosing $P = 1$

$$\Delta(a) = pa \otimes a - qb \otimes b^* \tag{3.90}$$

$$p\Delta(a^*) = p^2 a^* \otimes a^* - qb^* \otimes b \tag{3.91}$$

$$p[\Delta(a)]^* = p\bar{p}a^* \otimes a^* - \bar{q}pb^* \otimes b \tag{3.92}$$

We already know that $p\bar{p} = |p|^2 = 1$. To have the equality $[\Delta(a)]^* = \Delta(a^*)$, $p^2 = 1$ and $\bar{q}p = q$ must be satisfied.

There are again two options, $p = 1$ and $p = -1$. By checking the coproducts of b and b^* one may have chance to eliminate one of these.

$$\Delta(b) = a \otimes b + pb \otimes a^* \tag{3.93}$$

$$-q\Delta(b^*) = -q(b^* \otimes a + pa^* \otimes b^*) \tag{3.94}$$

$$[\Delta(b)]^* = \bar{p}b^* \otimes a + a^* \otimes b^* \tag{3.95}$$

To satisfy $[\Delta(a)]^* = \Delta(a^*)$, condition $p = \bar{p} = 1$ is required. If $p = 1$, from $q = p\bar{q}$, we see that $q = \bar{q}$, which means that q is a real number.

Thus, if we choose $P = 1$, $SU_{q,p}(2)$ reduces to $SU_q(2)$.

Instead of $P = 1$, condition $Q = 1$ can be chosen. If so, the relations $[\Delta(a)]^* = \Delta(a^*)$ and $[\Delta(a)]^* = \Delta(a^*)$ impose that $p = 1$ and $q = \pm 1$. However, this is again a subgroup of $SU_q(2)(q \in \mathbb{R})$.

As a result, the algebra $SU_{q,p}(2)$ turns out to be not different from $SU_q(2)$.

We have followed the regular steps that are introduced in the beginning of the chapter. However, one can prove that $SU_{q,p}(2)$ is not more general than $SU_q(2)$, in a more elegant way without using the coproducts.

Again, one can check for two different cases $P = 1$ and $Q = 1$

1. $P = 1 \Rightarrow p = e^{2i\alpha}$

$$\begin{pmatrix} a & b \\ -qb^* & e^{2i\alpha}a^* \end{pmatrix} = e^{i\alpha} \begin{pmatrix} e^{-i\alpha}a & e^{-i\alpha}b \\ -qe^{-i\alpha}b^* & e^{i\alpha}a^* \end{pmatrix}$$

Now, one can set new variables:

$$a' = e^{-i\alpha}a \tag{3.96}$$

$$b' = e^{-i\alpha}b \tag{3.97}$$

$$q' = qe^{-2i\alpha} \tag{3.98}$$

With these variables, algebra turns into following algebra:

$$e^{i\alpha} \begin{pmatrix} a' & b' \\ -q'b'^* & a'^* \end{pmatrix}$$

$e^{i\alpha}$ is an element of 1-dimensional unitary group $U_G(1)$, while the two dimensional matrix belongs to $SU_q(2)$. Thus the multiplication is again in $SU_q(2)$. The factor $e^{i\alpha}$ is eliminated in relations (3.1) to (3.6), since each has a multiplication with a complex conjugated element.

2. $Q = 1 \Rightarrow q = e^{2i\beta}$

The matrix is as follows

$$\begin{pmatrix} a & b \\ -e^{2i\beta}b^* & pa^* \end{pmatrix} = e^{i\beta}\begin{pmatrix} e^{-i\beta}a & e^{-i\beta}b \\ -e^{i\beta}b^* & pe^{-i\beta}a^* \end{pmatrix}$$

Now, one can set new variables:

$$a' = e^{-i\beta}a \tag{3.99}$$

$$b' = e^{-i\beta}b \tag{3.100}$$

$$p' = pe^{-2i\beta} \tag{3.101}$$

With these variables, the matrix becomes:

$$e^{i\beta}\begin{pmatrix} a' & b' \\ -b'^* & pa'^* \end{pmatrix}$$

One can treat this result as the previous case, however this time the algebra is same as $SU_p(2)$. And it is already proven that $SU_p(2)$ is not more general than $SU(2)$.

To sum up, $SU_{q,p}(2)$ is proven in two different ways to be equivalent to $SU_q(2)$, which means $p = 1$ and $q \in \mathbb{R}$

3.7. Fermion Algebra

Next we may look for Fermion Algebra which we know is not a Hopf Algebra. However, it is important to mention it in this chapter. Because it will lead us to a similar algebra, which is a Hopf Algebra.

Fermion algebra is represented as:

$$\begin{pmatrix} a & a^* \\ a^* & a \end{pmatrix}$$

From equations (3.1) to (3.6) only two relations emerge:

$$\begin{aligned} a^2 + a^{*2} &= 0 \qquad (3.102) \\ aa^* + a^*a &= 1 \qquad (3.103) \end{aligned}$$

One can calculate the coproduct as follows:

$$\begin{aligned} \Delta(a) &= a \otimes a + a^* \otimes a^* \qquad (3.104) \\ \Delta(a^*) &= a \otimes a^* + a^* \otimes a \qquad (3.105) \end{aligned}$$

From the coproduct above, it is easily seen that $[\Delta(a)]^* \neq \Delta(a^*)$. This i a violation of Hopf Algebra structure, so Fermion Algebra is not a Hopf Algebra.

So, we seek some little deformations to the algebra, which will make it a Hopf Algebra. In the last unitary matrix algebra of this chapter, we will look for this deformation.

3.8. θ-deformed Fermion Algebra

Since fermion algebra is not a Hopf Algebra, we may try to create a similar algebra which is a Hopf Algebra. Let us add an element as follows:

$$\begin{pmatrix} a+c & a^* \\ a^* & a+c \end{pmatrix}$$

This algebra is a Hopf Algebra. Now, we are quitting searching new algebras and focus on this specific algebra. We've worked on different algebras so far, and " θ-deformed Fermion Algebra" will be the last one we will be studying. In the next chapter we will seek useful properties about this algebra which may have some physical meaning.

4. θ DEFORMED FERMION ALGEBRA

Non deformed fermion algebra: The regular fermion algebra is characterized by the following matrix:

$$\begin{pmatrix} a & a^* \\ a^* & a \end{pmatrix}$$

Unitarity of this matrix is equivalent to the equations below:

$$aa^* + a^*a = 1 \tag{4.1}$$

$$a^2 = 0 \tag{4.2}$$

These are the basic relations of a fermion algebra, which obey the Pauli exclusion principle. And these satisfy the fact that $MM^\dagger = M^\dagger M = 1$. From these relations there only exists a 2-dimensional representation.

$$a|0\rangle = 0 \tag{4.3}$$

$$a|1\rangle = |0\rangle \tag{4.4}$$

$$a^*|0\rangle = |1\rangle \tag{4.5}$$

$$a^*|1\rangle = 0 \tag{4.6}$$

In the representation $|0\rangle$ and $|1\rangle$ are eigenstates of the number operator $N = a^*a$. $|0\rangle$ is the vacuum state and $|1\rangle$ is the 1-particle state. The lacking of Hopf Algebra structure involves the coproduct. In a Hopf Algebra coproduct is a homomorphism, so it must satisfy $\Delta(a^*) = [\Delta(a)]^*$. However the fermion algebra does not satisfy this. One can make the deformation by adding a hermitian element c.

4.1. Deformed fermion algebra

One can add a hermitian element c to fermion algebra as follows:

$$M = \begin{pmatrix} a+c & a^* \\ a^* & a+c \end{pmatrix}$$

In previous works of Arik and Arikan[5] on this subject the matrix were chosen as

$$M = \begin{pmatrix} a & a^*-c \\ a^*-c & a \end{pmatrix}$$

With this matrix every calculation is similar with our matrix, however when building up a representations, there occurs a physically irrelevant result in the tensor product of the number operator. For now, we leave it just like that. While building the representation, we will explain in detail, why the matrix we chose is physically more relevant.

With the deformed fermion algebra defined as above, one can use the fact $M^\dagger M = MM^\dagger = 1$ to deduce following relations:

$$aa^* + a^*a + a^*c + ca + c^2 = 1 \tag{4.7}$$

$$a^{*2} + a^2 = -ca^* - ac \tag{4.8}$$

$$aa^* + a^*a + ca^* + ac + c^2 = 1 \tag{4.9}$$

$$a^{*2} + a^2 = -ca - a^*c \tag{4.10}$$

From these relations, one obtains:

$$c(a - a^*) = (a - a^*)c \tag{4.11}$$

$$c^2 - (a - a^*)^2 = 1 \tag{4.12}$$

$$(a + a^* + c)^2 = 1 \tag{4.13}$$

Since we intend to find new relations, one can write the algebra in terms of self-adjoint set of variables s, c and g; instead of a, a^* and c:

$$s = i(a^* - a) \tag{4.14}$$

$$g = a + a^* + c \tag{4.15}$$

This transformation is invertible, therefore one can say:

$$a = \frac{1}{2}(g - c + is) \tag{4.16}$$
$$a^* = \frac{1}{2}(g - c - is) \tag{4.17}$$

In the previous steps of our research, instead of g we used an element $h = a^* + a$. Later we realized that, choosing g as the 3^{rd} variable is much more convenient for its properties that we will stress later. These new variables will lead us to new set of relations:

$$c^2 + s^2 = 1 \tag{4.18}$$
$$cs = sc \tag{4.19}$$
$$g^2 = 1 \tag{4.20}$$

By using the previous knowledge of $U_G(2)$, one can find the action of ϵ, Δ and antipode S on each three element, by using relations (2.13) to (2.16), (2.17) to (2.20) and the facts that Δ and ϵ are morphism, and S is an anti-morphism. (i.e. $S\,(S(g)^*) = g^*$, $\Delta(g^*) = [\Delta(g)]^*$, $\epsilon(g^*) = [\epsilon(g)]^*$, $\Delta(c^2+s^2) = 1\otimes 1$, $\Delta(cs) = \Delta(sc)$, $\Delta(g^2) = 1\otimes 1$).

$$\Delta(a+c) = a\otimes a + a^*\otimes a^* + a\otimes c + c\otimes a + c\otimes c \tag{4.21}$$

$$\Delta(a^*) = a\otimes a^* + a^*\otimes a + a^*\otimes c + c\otimes a^* \tag{4.22}$$

$$\Delta(a) = [\Delta(a^*)]^* = a\otimes a^* + a^*\otimes a + a\otimes c + c\otimes a \tag{4.23}$$

$$\begin{aligned}
\Delta(c) &= \Delta(a+c) - \Delta(a) && (4.24)\\
&= a\otimes a + a^*\otimes a^* + c\otimes c - a\otimes a^* - a^*\otimes a && (4.25)\\
&= (a-a^*)\otimes(a-a^*) + c\otimes c && (4.26)\\
&= -s\otimes s + c\otimes c && (4.27)
\end{aligned}$$

$$\begin{aligned}
\Delta(s) &= i\Delta(a^*-a) && (4.28)\\
&= i(\Delta(a^*) - \Delta(a)) && (4.29)\\
&= i(a^*-a)\otimes c + c\otimes i(a^*-a) && (4.30)\\
&= s\otimes c + c\otimes s && (4.31)
\end{aligned}$$

$$\begin{aligned}
\Delta(g) &= \Delta(a) + \Delta(a^*) + \Delta(c) && (4.32)\\
&= 2a^*\otimes a + 2a\otimes a^* + (a^*+a)\otimes c + c\otimes(a^*+a) + \Delta(c) && (4.33)\\
&= (a+a^*+c)\otimes(a+a^*+c) && (4.34)\\
&= g\otimes g && (4.35)
\end{aligned}$$

$$S(a^*) = a \tag{4.36}$$

$$S(a+c) = a^* + c \tag{4.37}$$

$$\text{One may choose } S(a) = a^* \quad \text{so it obeys } S\left(S(a)^*\right) = a^* \tag{4.38}$$

$$\Rightarrow S(c) = c \tag{4.39}$$

$$S(s) = -s \tag{4.40}$$

$$S(g) = g \tag{4.41}$$

$$\epsilon(a^*) = 0 \tag{4.42}$$

$$\epsilon(a+c) = 1 \tag{4.43}$$

$$\epsilon(a) = [\epsilon(a)]^* = 0 \tag{4.44}$$

$$\Rightarrow \epsilon(c) = 1 \tag{4.45}$$

$$\epsilon(s) = 0 \tag{4.46}$$

$$\epsilon(g) = 1 \tag{4.47}$$

Now it is time to mention how the coproduct of c is important. We have $\Delta(c) = (a - a^*) \otimes (a - a^*) + c \otimes c$. In the fermion algebra $c = 0$. But when one replaces c with 0 in the last equation, one gets: $\Delta(0) = (a - a^*) \otimes (a - a^*) + 0 \otimes 0$. However $\Delta(0) = 0 \otimes 0$ is what we must have normally. So, fermion algebra is a Hopf Algebra only if $a - a^* = 0$, but with this, it looses the characteristic of fermion algebra. Thus one must use the deformation coefficient c to have a Hopf Algebra.

To sum up operators have the following actions:

$$\Delta(c) = -s \otimes s + c \otimes c \ , \quad S(c) = c \ , \quad \epsilon(c) = 1 \tag{4.48}$$

$$\Delta(s) = s \otimes c + c \otimes s \ , \quad S(s) = -s \ , \quad \epsilon(s) = 0 \tag{4.49}$$

$$\Delta(g) = g \otimes g \ , \quad S(g) = g \ , \quad \epsilon(g) = 1 \tag{4.50}$$

With these relations, Hopf Algebra rules from (2.10) to (2.12) are satisfied for s, c, g. Furthermore coproduct acts like a morphism, as we wish to have a Hopf Algebra.(i.e. $\Delta(c^2 + s^2) = \Delta(1) = 1 \otimes 1$, $\Delta(cs) = \Delta(sc)$, $\Delta(g^2) = \Delta(1) = 1 \otimes 1$)

Also because of properties of $\Delta(c)$ and $\Delta(s)$, one sees that c and s is somehow acts like a cosine-sine algebra, where $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$ and $\sin(\theta_1 + \theta_2) = \cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1$.

With what we have now, it is not possible to find representations. Thus, we will consider less general algebras by adding more constraints. These constraints will be on commutation relations of s,c and g.

One can take pairs from the variables s,c and g and make them commute, anti-commute, or impose other commutation relations. However, we are limited by the characteristic rules of our Hopf Algebra. We will look for algebras satisfying relations:

$$cs = k_1 sc \tag{4.51}$$

$$cg = k_2 gc \tag{4.52}$$

$$sg = k_3 gs \tag{4.53}$$

From these relations we will proceed as follows to find the possible values of these constants:

$$
\begin{aligned}
cs = k_1 sc \quad &\Rightarrow \quad (cs)^* = (k_1 sc)^* && (4.54)\\
&\Rightarrow \quad s^* c^* = k_1^* c^* s^* && (4.55)\\
&\Rightarrow \quad sc = k_1^* cs && (4.56)\\
&\Rightarrow \quad k = \frac{1}{k_1^*} && (4.57)\\
&\Rightarrow \quad |k_1|^2 = 1 && (4.58)\\
&\Rightarrow \quad k_1 = e^{i\alpha} && (4.59)
\end{aligned}
$$

In a similar way, one can see that $k_2 = e^{i\beta}$ and $k_3 = e^{i\gamma}$. Now we will use the relations $s^2 + c^2 = 1$ and $g^2 = 1$ to put extra constraints on constants:

$$
\begin{aligned}
cs = e^{i\alpha} sc \quad &\Rightarrow \quad cs^2 = e^{i\alpha} scs && (4.60)\\
&\Rightarrow \quad cs^2 = e^{i\alpha} scs && (4.61)\\
&\Rightarrow \quad c(1 - c^2) = e^{2i\alpha}(1 - c^2)c && (4.62)\\
&\Rightarrow \quad e^{2i\alpha} = 1 && (4.63)\\
&\Rightarrow \quad \alpha = 0, \pi && (4.64)
\end{aligned}
$$

$$\begin{aligned} cg = e^{i\beta} gc \quad &\Rightarrow \quad cg^2 = e^{i\beta} gcg && (4.65) \\ &\Rightarrow \quad c = e^{2i\beta} c && (4.66) \\ &\Rightarrow \quad \beta = 0, \pi && (4.67) \end{aligned}$$

$$\begin{aligned} sg = e^{i\gamma} gs \quad &\Rightarrow \quad sg^2 = e^{i\gamma} gsg && (4.68) \\ &\Rightarrow \quad s = e^{2i\gamma} s && (4.69) \\ &\Rightarrow \quad \gamma = 0, \pi && (4.70) \end{aligned}$$

There remains one more constraint on value of the constants, Δ commutation relations;

$$\Delta(c^2 + s^2) = 1 \otimes 1 \tag{4.71}$$

$$\Delta(cg - e^{i\beta} gc) = 0 \otimes 0 \tag{4.72}$$

$$\Delta(sg - e^{i\gamma} gs) = 0 \otimes 0 \tag{4.73}$$

$$\begin{aligned}
\Delta(c^2+s^2) &= c^2\otimes c^2+s^2\otimes s^2-cs\otimes sc-sc\otimes sc+ \\
&\quad +s^2\otimes c^2+c^2\otimes s^2+sc\otimes cs+cs\otimes sc && (4.74)\\
&= 1\otimes 1-cs\otimes cs-e^{-i\alpha}cs\otimes e^{-i\alpha}cs+ \\
&\quad +e^{-i\alpha}cs\otimes cs+e^{-i\alpha}cs\otimes cs && (4.75)\\
&= 1\otimes 1+cs\otimes cs(-1-e^{-2i\alpha}+2e^{-i\alpha}) && (4.76)\\
&= 1\otimes 1+cs\otimes cs(-2+2e^{-i\alpha}) && (4.77)\\
&\Rightarrow e^{-i\alpha}=1 && (4.78)\\
&\Rightarrow \alpha=0 && (4.79)
\end{aligned}$$

$$\begin{aligned}
\Delta(cg-e^{i\beta}gc) &= cg\otimes cg-sg\otimes sg-e^{i\beta}(gc\otimes gc-gs\otimes gs) && (4.80)\\
&= cg\otimes cg(1-e^{-i\beta})+sg\otimes sg(e^{-2i\gamma}e^{i\beta}-1) && (4.81)\\
&= cg\otimes cg(1-e^{-i\beta})+sg\otimes sg(e^{i\beta}-1) && (4.82)\\
&\Rightarrow e^{i\beta}=1 && (4.83)\\
&\Rightarrow \beta=0 && (4.84)
\end{aligned}$$

$$\begin{aligned}
\Delta(sg-e^{i\gamma}gs) &= sg\otimes cg+cg\otimes sg-e^{i\gamma}(gs\otimes gc+gc\otimes gs) && (4.85)\\
&= sg\otimes cg(1-e^{-i\beta})+cg\otimes sg(1-e^{-i\beta}) && (4.86)\\
&= 0\otimes 0 && (4.87)\\
&\Rightarrow \text{No more restriction on } \gamma && (4.88)
\end{aligned}$$

With this results, one finds out we can have following two algebras:

1. $cs = sc$

   $cg = gc$

   $sg = gs$

2. $cs = sc$

   $cg = gc$

   $sg = -gs$

So, we have made up some relations and searched for constants, that can satisfy these relations. We used the following facts:

1. s, c and g are hermitian
2. $s^2 + c^2 = 1$ and $g^2 = 1$
3. Δ is a morphism.

From these we saw that there are two possible algebras. First one is the case we call commutative algebra.

In previous results one sees g has an important place in this algebra. This is why we used $g = a + a^* + c$ instead of $(a + a^*)$. g itself acts like an operator from a different space. It's coproduct and antipode does not depend on other elements. With using this information and new commutation relations of g one can transform algebra into two distinct algebras. First is s, c algebra, second is g algebra. To separate this algebras we will consider a tensor product $CSC \otimes HU$ where CSC is the "Commutative Sine Cosine Algebra", that includes s, c; and HU, "Hermitian Unitary Algebra" includes g, which is both hermitian and unitary.

1. CSC: $s = s^*$, $c = c^*$, $c^2 + s^2 = 1$, $cs = sc$; $s = s \otimes 1$, $c = c \otimes 1$.
2. HU: $g = g^*$, $g^2 = 1$, $g = 1 \otimes g$

The tensor product algebra $CSC \otimes HU$ is the commutative algebra. Because, c and s already commutes and since g is in a different space, it commutes both with c and s. This approach fits well to the commutative algebra we have found. The algebra can be summarized as follows:

$$s = s^*, \quad c = c^*, \quad g = g^*, \qquad [s, c] = [s, g] = [c, g] = 0, \qquad s^2 + c^2 = g^2 = 1$$

However, this first case is out of our interest. There are no nontrivial representations that we can use, since a, a^*, c commute. With these elements one can not proceed in finding representations as done to Fermion Algebra. However, a non-commutative algebra is usable for finding representations. Thus we prefer to have a non-commutative algebra, which let us have extra conditions to find representations.

In the second algebra s and g anticommute and c is the central element. In both possible algebras c is central. One realizes that, in both cases the algebra can only be deformed by a central element(i.e. an element which commutes with the whole algebra).

4.2. Representations of the θ-Deformed Fermion Algebra

The second algebra has the following relations:

$$cs = cs, \qquad cg = gc, \qquad sg = -gs \tag{4.89}$$

$$c^2 + s^2 = 1, \qquad g^2 = 1 \tag{4.90}$$

If one returns to a,a^*,c algebra, one sees that previous relations correspond to the following ones:

$$[c, a] = [c, a^*] = 0 \tag{4.91}$$

$$ca = -a^2, \quad ca^* = -a^{*2} \tag{4.92}$$

$$a^*a + aa^* = -c(a + a^*) + 1 - c^2 \tag{4.93}$$

As in the Fermion algebra case, one may define the number operator $N = a^*a$. $|c', n\rangle$ is the common eigenstate of N and c, with eigenvalues n and c' respectively. Then the following relations follow:

$$aN = a \tag{4.94}$$

$$Na^* = a^* \tag{4.95}$$

$$Na = -cN \tag{4.96}$$

$$a^*N = -cN \tag{4.97}$$

With using these relations, one can find possible representations of the θ-Deformed Fermion Algebra. Let us take a state $|c', n\rangle$ and apply aN on it.

$$aN|c', n\rangle = na|c', n\rangle = a|c', n\rangle$$

This can imply two possibilities. First $n = 1$, so there is a state $|c', 1\rangle$ and secondly $a|c', n\rangle = 0$, so there is a state on which action of a gives 0. First, let us say there is a state $|c', 1\rangle$.

$$\begin{aligned}
a^* N|c', 1\rangle = -cN|c', 1\rangle &\;\Rightarrow\; a^*|c', 1\rangle = -c'|c', 1\rangle & (4.98)\\
Na|c', 1\rangle = -cN|c', 1\rangle = -c'|c', 1\rangle &\;\Rightarrow\; a|c', 1\rangle = -c'|c', 1\rangle & (4.99)\\
(a^*a + aa^*)|c', 1\rangle = (-c(a + a^*) + 1 - c^2)|c', 1\rangle &\;\Rightarrow\; 2c'^2 = c'^2 + 1 & (4.100)\\
&\;\Rightarrow\; c' = \pm 1 & (4.101)
\end{aligned}$$

From the existence of $|c', 1\rangle$ we found a 1-dimensional representation, which can be characterized as:

$$a = a^* = -c = \pm 1$$

Next, we will find other representations by using the fact that there exist a state $|c', n\rangle$ which vanishes with the action of a. It is obvious that this state is $|c', 0\rangle$ which has eigenvalue 0. The proof is as follows.

$$\begin{aligned}
Na|c', n\rangle = -cN|c', n\rangle = 0 &\;\Rightarrow\; -nc'|c', n\rangle = 0 & (4.102)\\
&\;\Rightarrow\; n = 0 & (4.103)\\
&\;\Rightarrow\; a|c', 0\rangle = 0 & (4.104)
\end{aligned}$$

We proved that we have a state with eigenvalue 0. Now we can check the action of a^* on $|c', 0\rangle$.

$$Na^*|c', 0\rangle = a^*|c', 0\rangle$$

So $a^*|c', 0\rangle$ is a state with eigenvalue 1(i.e. $a^*|c', 0\rangle = \alpha|c', 1\rangle$). To find the normalization constant, consider the following:

$$\|a^*|c', 0\rangle\|^2 = \langle c', 0|aa^*|c', 0\rangle = \langle c', 0|1 - c^2|c', 0\rangle = 1 - c'^2$$

$$\Rightarrow a^*|c', 0\rangle = \pm\sqrt{1 - c'^2}|c', 1\rangle$$

Next, the action of a^* on $|c', 1\rangle$ is trivially found. As in 1-dimensional case, $a^*|c', 1\rangle = -c'|c', 1\rangle$. Lastly we shall check $a|c', 1\rangle$:

$$a|c', 1\rangle = \frac{aa^*|c', 0\rangle}{\pm\sqrt{1 - c'^2}} = \frac{-ca^*|c', 0\rangle}{\pm\sqrt{1 - c'^2}} + \frac{(1 - c'^2)|c', 0\rangle}{\pm\sqrt{1 - c'^2}} = -c'|c', 1\rangle \pm \sqrt{1 - c'^2}|c', 0\rangle$$

So far we have build up our representation. One can see that representation is 2-dimensional. Action of a and a^* to $|c', 0\rangle$ and $|c', 1\rangle$ again gives linear combination of these two states. Let us summarize our 2-dimensional representation as follows:

$$
\begin{aligned}
a|c', 0\rangle &= 0 && (4.105)\\
a^*|c', 0\rangle &= \pm\sqrt{1 - c'^2}|c', 1\rangle && (4.106)\\
a|c', 1\rangle &= -c'|c', 1\rangle \pm \sqrt{1 - c'^2}|c', 0\rangle && (4.107)\\
a^*|c', 1\rangle &= -c'|c', 1\rangle && (4.108)\\
c|c', 0\rangle &= c'|c', 0\rangle && (4.109)\\
c|c', 1\rangle &= c'|c', 1\rangle && (4.110)
\end{aligned}
$$

Here seems to be two 2-dimensional representations, because of $\pm$ signs. In fact these are equivalent. Now we only consider the representation with $+$ sign and explain later the reason of equivalence.

The cosine-sine algebra structure of operators c and s allows us to set $c' = \cos\theta$ as the eigenvalue of c, which will make $\sqrt{1 - c'^2} = \sqrt{1 - \cos^2\theta} = \sin\theta$. With this new eigenvalue representation we can denote states $|c', 0\rangle$ and $|c', 1\rangle$ by $|\theta, 0\rangle$ and $|\theta, 1\rangle$.

One can now go back to our $s - c - g$ variables and write their actions on state vectors.

$$
\begin{aligned}
s|\theta, 0\rangle &= i\sin\theta|\theta, 1\rangle && (4.111)\\
s|\theta, 1\rangle &= -i\sin\theta|\theta, 0\rangle && (4.112)\\
g|\theta, 0\rangle &= \sin\theta|\theta, 1\rangle + \cos\theta|\theta, 0\rangle && (4.113)\\
g|\theta, 1\rangle &= -\cos\theta|\theta, 1\rangle + \sin\theta|\theta, 0\rangle && (4.114)
\end{aligned}
$$

With these two sets of relations and choice of a simple basis one can write matrix representations of the operators as follows.

$$a = \begin{pmatrix} 0 & \sin\theta \\ 0 & -\cos\theta \end{pmatrix} \quad a^* = \begin{pmatrix} 0 & 0 \\ \sin\theta & -\cos\theta \end{pmatrix} \quad |c', 0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |c', 1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.115}$$

$$c = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} \quad s = i \begin{pmatrix} 0 & -\sin\theta \\ \sin\theta & 0 \end{pmatrix} \quad g = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \tag{4.116}$$

As stated, another representation seems possible, which has:

$$a^*|c', 0\rangle = c'|c', 0\rangle - \sqrt{1 - c'^2}|c', 1\rangle \quad \text{i.e.} \quad a^*|\theta, 0\rangle = \cos\theta|\theta, 0\rangle - \sin\theta|\theta, 1\rangle$$

This representation has the following matrices:

$$a = \begin{pmatrix} 0 & -\sin\theta \\ 0 & -\cos\theta \end{pmatrix} \quad a^* = \begin{pmatrix} 0 & 0 \\ -\sin\theta & -\cos\theta \end{pmatrix} \quad |c', 0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |c', 1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.117}$$

$$c = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} \quad s = i \begin{pmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{pmatrix} \quad g = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \tag{4.118}$$

In fact this is the case where θ becomes $-\theta$. These two representations are equivalent. It can be shown by finding a similarity transformation between two representations[2] .

There is no similarity transformation between $a(c,s) = a(\theta)$ and $a(c,-s) = a(-\theta)$[3] but there is a transformation between $a(\theta)$ and $a^*(-\theta)$[4] . It also shows two representation are equivalent, because a and a^* can be obtained from each other by hermitian conjugation.

What is just found is an orbifold structure since ($\theta \to -\theta$) does not change the representation. In next chapter, importance of the orbifold structure will be explained in further detail.

[2]If there is a similarity transformation between the representation $\mathcal{G}$ and $\mathcal{G}'$, they are equivalent. i.e. if there exists a transformation T such that $\mathcal{G}' = T\mathcal{G}T^{-1} \Rightarrow \mathcal{G}' \simeq \mathcal{G}$

[3]$a(c,s) = a(\theta) = \begin{pmatrix} 0 & \sin\theta \\ 0 & -\cos\theta \end{pmatrix}$ and $a(c,-s) = a(-\theta) = \begin{pmatrix} 0 & -\sin\theta \\ 0 & -\cos\theta \end{pmatrix}$

[4]The matrix representation of transformation T is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, indeed one really obtains $Ta(\theta)T^{-1} = a^*(-\theta)$

4.3. The Addition of c and the Orbifold $S^1 \diagup Z_2$

As addition of angular momentum, one can have addition of c. In this case, we will be seeking representations for $\Delta(c)$ and $\Delta(a^*a)$, which is the tensor product representation. Let us denote the common eigenvector of $\Delta(c)$ and $\Delta(a^*a)$ by $|\theta, n\rangle$. One can write:

$$|\theta, n\rangle = \sum C^{\theta,n}_{\theta_1,n_1;\theta_2,n_2} |\theta_1, n_1\rangle \otimes |\theta_2, n_2\rangle \tag{4.119}$$

where $C^{\theta,n}_{\theta_1,n_1;\theta_2,n_2}$ is a Clebsch-Gordon coefficient. Remember that:

$$\Delta(c) = -s \otimes s + c \otimes c \tag{4.120}$$

$$\Delta(s) = s \otimes c + c \otimes s \tag{4.121}$$

$$\Delta(g) = g \otimes g \tag{4.122}$$

From these one calculate coproduct of the number operator as:

$$\Delta(N) = \Delta(a^*)\Delta(a) = 1 \otimes N + N \otimes 1 - 2N \otimes N \tag{4.123}$$

With all these calculated, one can write actions of each coproduct on tensor product eigenvectors.

$$\Delta(c)|\theta_1,0\rangle\otimes|\theta_2,0\rangle = c_1c_2|\theta_1,0\rangle\otimes|\theta_2,0\rangle + s_1s_2|\theta_1,1\rangle\otimes|\theta_2,1\rangle \tag{4.124}$$

$$\Delta(c)|\theta_1,0\rangle\otimes|\theta_2,1\rangle = c_1c_2|\theta_1,0\rangle\otimes|\theta_2,1\rangle - s_1s_2|\theta_1,1\rangle\otimes|\theta_2,0\rangle \tag{4.125}$$

$$\Delta(c)|\theta_1,1\rangle\otimes|\theta_2,0\rangle = c_1c_2|\theta_1,1\rangle\otimes|\theta_2,0\rangle - s_1s_2|\theta_1,0\rangle\otimes|\theta_2,1\rangle \tag{4.126}$$

$$\Delta(c)|\theta_1,1\rangle\otimes|\theta_2,1\rangle = c_1c_2|\theta_1,1\rangle\otimes|\theta_2,1\rangle + s_1s_2|\theta_1,0\rangle\otimes|\theta_2,0\rangle \tag{4.127}$$

$$\Delta(N)|\theta_1,0\rangle\otimes|\theta_2,1\rangle = |\theta_1,0\rangle\otimes|\theta_2,1\rangle \tag{4.128}$$

$$\Delta(N)|\theta_1,0\rangle\otimes|\theta_2,0\rangle = \Delta(N)|\theta_1,1\rangle\otimes|\theta_2,1\rangle = 0 \tag{4.129}$$

$$\Delta(N)|\theta_1,1\rangle\otimes|\theta_2,0\rangle = |\theta_1,1\rangle\otimes|\theta_2,0\rangle \tag{4.130}$$

$$\Delta(s)|\theta_1,0\rangle\otimes|\theta_2,0\rangle = ic_1s_2|\theta_1,0\rangle\otimes|\theta_2,1\rangle + is_1c_2|\theta_1,1\rangle\otimes|\theta_2,0\rangle \tag{4.131}$$

$$\Delta(s)|\theta_1,0\rangle\otimes|\theta_2,1\rangle = -ic_1s_2|\theta_1,0\rangle\otimes|\theta_2,0\rangle + is_1c_2|\theta_1,1\rangle\otimes|\theta_2,1\rangle \tag{4.132}$$

$$\Delta(s)|\theta_1,1\rangle\otimes|\theta_2,0\rangle = ic_1s_2|\theta_1,1\rangle\otimes|\theta_2,1\rangle - is_1c_2|\theta_1,0\rangle\otimes|\theta_2,0\rangle \tag{4.133}$$

$$\Delta(s)|\theta_1,1\rangle\otimes|\theta_2,1\rangle = -ic_1s_2|\theta_1,1\rangle\otimes|\theta_2,0\rangle - is_1c_2|\theta_1,0\rangle\otimes|\theta_2,1\rangle \tag{4.134}$$

$$\Delta(g)|\theta_1,0\rangle \otimes |\theta_2,0\rangle = s_1 s_2|\theta_1,1\rangle \otimes |\theta_2,1\rangle + s_1 c_2|\theta_1,1\rangle \otimes |\theta_2,0\rangle \tag{4.135}$$

$$+c_1 s_2|\theta_1,0\rangle \otimes |\theta_2,1\rangle + c_1 c_2|\theta_1,0\rangle \otimes |\theta_2,0\rangle \tag{4.136}$$

$$\Delta(g)|\theta_1,0\rangle \otimes |\theta_2,1\rangle = s_1 s_2|\theta_1,1\rangle \otimes |\theta_2,0\rangle - s_1 c_2|\theta_1,1\rangle \otimes |\theta_2,1\rangle \tag{4.137}$$

$$+c_1 s_2|\theta_1,0\rangle \otimes |\theta_2,1\rangle - c_1 c_2|\theta_1,0\rangle \otimes |\theta_2,1\rangle \tag{4.138}$$

$$\Delta(g)|\theta_1,1\rangle \otimes |\theta_2,0\rangle = s_1 s_2|\theta_1,0\rangle \otimes |\theta_2,1\rangle + s_1 c_2|\theta_1,0\rangle \otimes |\theta_2,0\rangle \tag{4.139}$$

$$-c_1 s_2|\theta_1,1\rangle \otimes |\theta_2,1\rangle - c_1 c_2|\theta_1,1\rangle \otimes |\theta_2,0\rangle \tag{4.140}$$

$$\Delta(g)|\theta_1,1\rangle \otimes |\theta_2,1\rangle = s_1 s_2|\theta_1,0\rangle \otimes |\theta_2,0\rangle - s_1 c_2|\theta_1,0\rangle \otimes |\theta_2,1\rangle \tag{4.141}$$

$$-c_1 s_2|\theta_1,1\rangle \otimes |\theta_2,0\rangle + c_1 c_2|\theta_1,1\rangle \otimes |\theta_2,1\rangle \tag{4.142}$$

Here it is necessary to have a break and explain why we have not chosen deformed algebra as done in Arikan's paper[5]. If the matrix elements were chosen as $(a+c)$ and a^* instead of a and (a^*-c), most results would have been the same. However, action of the coproduct of number operator on tensor product states(equations 4.128 to 4.130) would be as follows:

$$\Delta(N)|\theta_1,0\rangle \otimes |\theta_2,0\rangle = |\theta_1,0\rangle \otimes |\theta_2,0\rangle \tag{4.143}$$

$$\Delta(N)|\theta_1,0\rangle \otimes |\theta_2,1\rangle = \Delta(N)|\theta_1,1\rangle \otimes |\theta_2,0\rangle = 0 \tag{4.144}$$

$$\Delta(N)|\theta_1,1\rangle \otimes |\theta_2,1\rangle = |\theta_1,1\rangle \otimes |\theta_2,1\rangle \tag{4.145}$$

This result is physically not desired, because tensor product of two states with eigenvalue 0 (which may be considered as vacuum states), has eigenvalue 1 in the tensor product space. One expects to have eigenvalue of this tensor product to be 0, and in our choice of matrix elements, it is so.

Since $|c, n\rangle$ is a common eigenvector of $\Delta(N)$ and $\Delta(c)$ it should satisfy the following equations.

$$\Delta(N)|\theta, n\rangle = n|\theta, n\rangle \tag{4.146}$$

$$\Delta(c)|\theta, n\rangle = \cos\theta|\theta, n\rangle \tag{4.147}$$

One can write two eigenvectors $|\theta, 0\rangle$ and $|\theta, 1\rangle$ as in (4.119).

$$|\theta, 1\rangle = \alpha_1|\theta_1, 0\rangle \otimes |\theta_1, 0\rangle + \alpha_2|\theta_1, 0\rangle \otimes |\theta_2, 1\rangle + \alpha_3|\theta_1, 1\rangle \otimes |\theta_2, 0\rangle + \alpha_4|\theta_1, 1\rangle \otimes |\theta_2, 1\rangle \tag{4.148}$$

$$|\theta, 0\rangle = \beta_1|\theta_1, 0\rangle \otimes |\theta_1, 0\rangle + \beta_2|\theta_1, 0\rangle \otimes |\theta_2, 1\rangle + \beta_3|\theta_1, 1\rangle \otimes |\theta_2, 0\rangle + \beta_4|\theta_1, 1\rangle \otimes |\theta_2, 1\rangle \tag{4.149}$$

Next, one uses (4.146) and (4.147) to find the coefficients.

$$\Delta(N)|\theta, 1\rangle = \alpha_2|\theta_1, 0\rangle \otimes |\theta_2, 1\rangle + \alpha_3|\theta_1, 1\rangle \otimes |\theta_2, 0\rangle = |\theta, 1\rangle \Rightarrow \alpha_1 = \alpha_4 = 0 \tag{4.150}$$

$$\Delta(N)|\theta, 0\rangle = \beta_2|\theta_1, 0\rangle \otimes |\theta_2, 1\rangle + \beta_3|\theta_1, 1\rangle \otimes |\theta_2, 0\rangle = 0 \Rightarrow \beta_2 = \beta_3 = 0 \tag{4.151}$$

So one can write our tensor product state vectors as follows:

$$|\theta, 1\rangle = \alpha_2|\theta_1, 0\rangle \otimes |\theta_2, 1\rangle + \alpha_3|\theta_1, 1\rangle \otimes |\theta_2, 0\rangle \tag{4.152}$$

$$|\theta, 0\rangle = \beta_1|\theta_1, 0\rangle \otimes |\theta_1, 0\rangle + \beta_4|\theta_1, 1\rangle \otimes |\theta_2, 1\rangle \tag{4.153}$$

With using (4.147) on $|\theta, 0\rangle$ and $|\theta, 1\rangle$ one finds the coefficients.[5]

$$\Delta(c)|\theta,1\rangle = (\alpha_2 c_1 c_2 - \alpha_3 s_1 s_2)|\theta_1, 0\rangle \otimes |\theta_2, 1\rangle + (-\alpha_2 s_1 s_2 + \alpha_3 c_1 c_2)|\theta_1, 1\rangle \otimes |\theta_2, 0\rangle \tag{4.154}$$

$$Also \quad \Delta(c)|\theta,1\rangle = c|\theta,1\rangle = c(\alpha_2|\theta_1,0\rangle \otimes |\theta_2,1\rangle + \alpha_3|\theta_1,1\rangle \otimes |\theta_2,0\rangle) \tag{4.155}$$

With these two, one can write:

$$\alpha_2 c_1 c_2 - \alpha_3 s_1 s_2 = c\alpha_2 \tag{4.156}$$

$$\alpha_3 c_1 c_2 - \alpha_2 s_1 s_2 = c\alpha_3 \tag{4.157}$$

With acting $\Delta(c)$ on $|\theta, 0\rangle$ one finds:

$$\beta_1 c_1 c_2 + \beta_4 s_1 s_2 = c\beta_1 \tag{4.158}$$

$$\beta_4 c_1 c_2 + \beta_1 s_1 s_2 = c\beta_4 \tag{4.159}$$

One can also use (4.111) and (4.112) by using $\Delta(s)$:

$$\Delta(s)|\theta,0\rangle = is|\theta,1\rangle \tag{4.160}$$

$$\Delta(s)|\theta,1\rangle = -is|\theta,0\rangle \tag{4.161}$$

[5]For simplicity, $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$

These two can be used to obtain the following equations:

$$-\beta_1 c_1 s_2 + \beta_4 s_1 c_2 = s\alpha_2 \tag{4.162}$$

$$-\beta_1 s_1 c_2 + \beta_4 c_1 s_2 = s\alpha_3 \tag{4.163}$$

$$\alpha_2 c_1 s_2 + \alpha_3 s_1 c_2 = -s\beta_1 \tag{4.164}$$

$$\alpha_2 s_1 c_2 + \alpha_3 c_1 s_2 = s\beta_4 \tag{4.165}$$

With the equations (4.156) to (4.159) and (4.162) to (4.165) it can be written that in general:

$$\alpha_2 = \pm\alpha_3 = 1/\sqrt{2} \tag{4.166}$$

$$\beta_1 = \mp\beta_4 = 1/\sqrt{2} \tag{4.167}$$

$$s = \pm c_1 s_2 \pm s_1 c_2 \tag{4.168}$$

$$c = c_1 c_2 \pm s_1 s_2 \tag{4.169}$$

However, there are some constraints in this equations, like when $\alpha_2 = -\alpha_3$, one can only have $\beta_1 = \beta_4$.

The cosine-sine algebra structure of c and s is more obvious here. For instance, instead of writing $s = c_1 s_2 + s_1 c_2$ and $c = c_1 c_2 - s_1 s_2$ we can write $c_1 = \cos\theta_1$, $c_2 = \cos\theta_2$, $s_1 = \sin\theta_1$, $s_2 = \sin\theta_2$, $s = \sin\theta$, $c = \cos\theta$ where $\theta = \theta_1 + \theta_2$. With this approach, we can make the table in Table 4.1.

Table 4.1. Relation of the resulting angle and coefficients of the tensor product states

α_2	α_3	β_1	β_4	θ
$+$	$-$	$+$	$+$	$\theta_1 - \theta_2$
$-$	$+$	$+$	$+$	$-(\theta_1 - \theta_2)$
$+$	$+$	$+$	$-$	$-(\theta_1 + \theta_2)$
$+$	$+$	$-$	$+$	$\theta_1 + \theta_2$

In the tensor product representation of the c-deformed (θ-deformed) fermion algebra, resulting eigenvalues of $\Delta(c)$ and $\Delta(s)$ obey the same rules which occur in addition of angles. However θ and $-\theta$ gives the equivalent representations. Thus the tensor product representation gives 2 representations in total, which are $(\theta_1 - \theta_2)$ and $(\theta_1 + \theta_2)$. Each are again 2-dimensional like the anti-commutative algebra itself.

When one takes tensor product of 2, 2-dimensional algebras, obtains again 2, 2-dimensional algebras. In popular notation it is expressed as,

$$2 \otimes 2 = 2 \oplus 2$$

The resulting algebra has the following eigenvalues for c:

$$c = c_1 c_2 - s_1 s_2$$

$$c = c_1 c_2 + s_1 s_2$$

Since θ and $-\theta$ gives the same algebra we have an S^1 whose points symmetrical with respect to the diameter $\theta = 0$ are identified. This defines an orbifold structure S^1/Z_2. The structure of the algebra, and especially its coproduct respect this orbifold structure. This orbifold structure is a requirement imposed by the Hopf Algebra structure which implies that c is cosine of an angle. The orbifold structure is shown by the diagram in Figure 4.1.

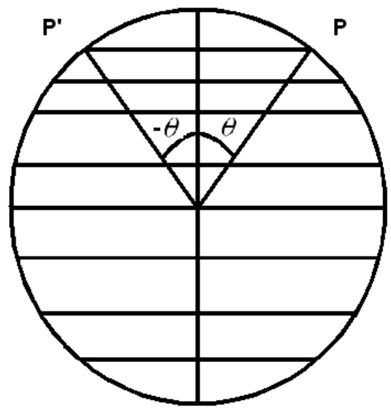

Figure 4.1. The Orbifold S^1/Z_2

In this diagram, the angles are equivalent, which corresponds to two sides of each horizontal line. For any θ, $-\theta$ gives the exact same structure. Whereas P and P' are different points of S^1, they are identical on S^1/Z_2.

5. CONCLUSION

Hopf Algebras are structures that consist of an algebra and a coalgebra, and an additional map antipode. We gave the basic definitions of Hopf algebra and showed that some 2×2 matrices can represent Hopf Algebras.

Unitarity of matrices whose elements may be noncommutative still imply Hopf Algebra structures, with the elements of the matrices as algebra elements. Some unitary matrices may not be representing a Hopf Algebra. However, with adding some constraints on the relations between the elements, we can turn them into Hopf Algebras. One may do it by operating the coproduct, antipode and counit on elements of these unitary matrices. Some unitary matrices directly contradict with the Hopf Algebra structure, and can not be turned into a Hopf Algebra by adding constraints. However, the action of these operators may impose additional relations between elements of matrices to turn them into Hopf Algebras. In the second chapter we looked over 8 different unitary matrices, and searched if it they represented Hopf Algebras.

The last one was the fermion algebra, which cannot be turned into a Hopf Algebra without breaking the former relations of the algebra. We changed fermion algebra by adding an element c to deform it into a Hopf Algebra. We started from a unitary matrix, and by using properties of Hopf Algebras, we constructed two possible algebras. First is a commutative algebra and has no non-trivial representation. In the second one, two elements anti-commute which made this algebra our main interest. Having this deformed fermion algebra may lead to new solutions to some problems, with using properties of Hopf Algebras.

One sees that the element c can be represented by an angle θ and eigenvalues of the algebra elements are in terms of sine and cosines of θ. Later we constructed 2-dimensional representations of this algebra, which led to the tensor product representation.

In the deformed fermion algebra, there is a hermitian conjugate algebra consisting of one element and a commutative sin-cosine Hopf Subalgebra, which is geometrically equivalent to S^1. However, in the deformed algebra, θ can be replaced by $-\theta$ and it describes the orbifold $S^1 \diagup Z_2$. Perhaps such structures may be used to construct Hopf Algebras related to other and higher dimensional orbifolds.

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