CAN HYPERBOLIC PHASE OF BRANS-DICKE FIELD ACCOUNT FOR DARK MATTER AND DARK ENERGY?

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Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Physics
Boğaziçi University
2008

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DATE OF APPROVAL: 13.06.2008

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to Prof. Metin Arık for providing me the opportunity to work with him, for his guidance, kindness and patience. It has been a honor to work with him.

It has also been a wonderful chance to work and study with Assist. Prof. Mehmet Çalık, without whom this thesis would have never come to existence. Your passion on science gave me the greatest inspiration to study on the Universe. Thank you for your unending support and patience, not only for your guidance but also for your advices to encourage me to enlighten the "DARK" both in my life and in the universe.

My special thanks belong to Assoc. Prof. Serkant Ali Çetin for his sincerity and understanding in everything and his trust in me and giving me the chance to work at CERN.

I am grateful to Burcu Özdemir and Serkan Sütlü for their presence and support during this work. Burcu, you are a "real" friend for me and you were always with me no matter what life encounters. Serkan, I appreciate for your friendship, you are such a reliable and good person. Thank you both for making life easier and making me remember the times we spent with Mustafa. I also thank my old friend and present housemate Ayşe Özge Kuruoğlu for always being one of the life sources in my life and for her understanding and help during this work.

The most special thanks go to Yücel Altundal. It makes me happy to know that you are always "there". I would not think of being here without you. I also thank to my parents for believing in me and for standing by me through the many trials and decisions of my life. I would like to thank very much to a very good physicist of the future, my little sunshine, my sister Gizem Çifter. You made me believe that our dreams will come true.

I would like to dedicate this thesis to my professors and my friends, Prof. Engin Arık, Prof Şenel Boydağ, Assoc. Prof. İskender Hikmet, Res. Assist. Mustafa Fidan, Res. Assist. Özgen Berkol Doğan and Res. Assist. Engin Abat, who we lost in 30 November, 2007. Each of them had various contribution in this thesis which is very valuable to me. We will never forget what you've done for the love of science.

ABSTRACT

CAN HYPERBOLIC PHASE OF BRANS-DICKE FIELD ACCOUNT FOR DARK MATTER AND DARK ENERGY?

The main purpose of this thesis is to explain the dark matter and dark energy contribution to standard Friedmann Equation which relates the expansion rate of the universe, H, to the various fractions of the present energy densities of the universe by using Brans-Dicke theory of gravity in late-time regime.

Firstly, vacuum solutions of the field equations will be analyzed and it will be shown that there is some anomaly in explaining the dark matter issue if Brans-Dicke field ϕ is solely assumed to be complex. Then, for the next step, a hyperbolic phase of Brans-Dicke field ϕ is worked out in the field equations and hence the anomaly seen under the assumption of complex Brans-Dicke field ϕ has been disappeared automatically. Therefore, we showed that Brans-Dicke scalar tensor theory well accounts for dark matter and dark energy provided that the Brans-Dicke field ϕ is modified suitably.

ÖZET

BRANS-DICKE ALANININ HİPERBOLİK FAZI KARA MADDE VE KARA ENERJİYİ AÇIKLAYABİR Mİ?

Bu tezin ana amacı, geç zaman rejiminde Brans-Dicke skaler-tensörel kütle çekim teorisini kullanarak, bugünkü enerji yoğunluğunu, evrenin genişleme hızına bağlayan, Friedmann denklemine kara madde ve kara enerjinin katkısını açıklamaktır.

İlk olarak, kompleks bir Brans-Dicke alanının, vakumdaki çözümleri analiz edilecek ve gösterilecektir ki; Brans-Dicke alanını kara madde ve kara enerjiyi açıklamak için kompleks bir alan olarak ele almak bir anomali içermektedir.

Bir sonraki adımda, Brans-Dicke alanı için hiperbolik faz tanımlanacak ve hiperbolik faz kullanılarak alan denklemlerinin çözülmesi ile, Brans-Dicke alanındaki anomalinin kaldırdığı gösterilecektir. Bu sebeple, hiperbolik fazlı Brans-Dicke alanı için vakum geç zaman çözümünün kullanılması ile gösterilecektir ki; Brans-Dicke alanı uygun olarak modifiye edildiğinde, Brans-Dicke skaler-tensörel kütle çekim teorisi, karanlık enerji ve karanlık maddeye açıklık getirebilmiştir.

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LIST OF SYMBOLS/ABBREVIATIONS

a	Scale size of the universe
$\dot{a}$	Time rate of the scale size of the universe
a_{obs}	Scale size of the universe measured by observer
a_{em}	Scale size of the universe when the light is emitted
a_0	Present value of the scale size of the universe
c	Speed of light
ds	Line element
d^4x	Four dimensional volume element
$F(a)$	Rate of change of Brans Dicke field
$F_{1\infty}, F_{11}, F_{12}$	Perturbative constants of $F_1(a)$
$F_{2\infty}, F_{21}, F_{22}$	Perturbative constants of $F_2(a)$
$g_{\mu\nu}$	Metric tensor
$\sqrt{g}$	Metric tensor density
G_N	Newtonian gravitational constant
G	Dynamical Newtonian gravitational constant
G^{μ}_{ν}	Einstein tensor
$H(a)$	Rate of change of the scale size of the universe
H_∞, H_1, H_2	Perturbative constants of $H(a)$
H_0	Present Hubble parameter value
$\hbar$	Reduced Plank's constant
i^*	imaginary number i
i, j	Spatial indices
k	Curvature of space-time
l	Astrophysical length scale
L	Length dimension
L_M	Matter field Lagrangian
m	Mass of the BD scalar field
p	Pressure
p_M	Pressure of everything except BD scalar field

R^{μ}_{ν}	Ricci tensor
R	Ricci scalar
$\vec{r}$	Physical distance vector
r	Radial parameter
t	Cosmological time
t_0	Present cosmological time
T^{μ}_{ν}	Energy momentum tensor
u^{μ}	4-velocity vector for fluid
v	Velocity of nearby galaxies
$\vec{x}$	Co-moving distance vector
z	red-shift
α	Dimensionless constant
$\phi(t)$	Canonical formed Brans-Dicke scalar field
∂	Ordinary differential operator
γ	Equation of state parameter
μ, ν	Both spatial and time indices
Λ	Cosmological constant
Ω	Density parameter
Ω_{Λ}	Density parameter due to cosmological constant
Ω_{DE}	Density parameter due to dark energy
Ω_M	Density parameter due to matter
Ω_{DM}	Density parameter due to dark matter
ρ	Energy density
ρ_{rad}	Energy density due to radiation
ρ_M	Energy density of everything except BD scalar field
ρ_c	Critical energy density
ρ_0	Present energy density
ω	Brans-Dicke coupling constant

BD	Brans Dicke
CMBR	Cosmic microwave background radiation

DM	Dark Matter
FLRW	Freedmann Lemaitre Robertson Walker
Mpc	Mega parsec

1. INTRODUCTION

Discoveries of the secrets of universe have always carried forward the questions of human being including the term "Why?". Starting in 1960s, spectacular observational breakthroughs like the discovery of the cosmic microwave background [1] radiation and large number of galaxies and other discrete objects, often those with high redshifts renewed interest in cosmology and made it a 'hot' research topic for many theoretical physicists and experimental particle physicists. With this expansion has come a great deal of new information and a model for the Universe. The foundation stone of modern cosmology and the starting point of all those interests was the discovery of the expansion of the universe by famous astronomer Edwin Hubble in 1929 [2]. He demonstrated that all galaxies are moving away from us and from each other and this was a direct observational evidence for an expanding universe. This observation was in agreement with the expanding solution of Einstein's equations. After that, the inflation theories, firstly predicted by Alan Guth [3], had been able to explain not only the accelerated expansion of the universe but also many other problems of cosmology like homogenity, isotropy and flatness problem. Undoubtedly the most impressive solutions have been the most recent analysis of the CMB data based on the WMAP [4] observations which shows that the Universe is close to spatially flat with a large cosmological constant Λ, and these WMAP data can be fit very well with the value of Hubble constant. The expansion of universe results from an adequate negative pressure of dark energy. Recent observations show that the dark energy behaves like Einstein's cosmological constant [5] that arises from the vacuum energy leading to the inflation of the universe. In respect of recent WMAP data [4], dark energy constitutes nearly 72 % of our universe together with the remaining energy density composed of dark matter which can not be observed directly although its gravitational effects on visible matter validate its presence. Unlike normal matter (whose pressure is always positive) scalar fields can have an effective negative pressure, therefore providing an inflationary theory. The first attempt to construct a connection between the inflation field and a scalar field was introduced with the new inflation theory [6] suggesting that inflation would be explained by a scalar tensor theory of gravity. The aim of this thesis is to show that the contribution of modified BD field to Friedmann Equation yields to the explanation of dark energy and dark matter issues in BD cosmology. The reason to choose the underlying theory as BD theory is that, it is one of the alternative theories to general relativity.

In the first chapter, the general review of standard cosmology together with scalar tensor theories and dark matter and dark energy will briefly be discussed. The second chapter gives the methodology for the contribution of the BD field in respect of a complex BD field and of a hyperbolic phase of BD field. Lastly, in the third chapter, it will be concluded that BD theory well accounts for dark matter and dark energy.

1.1. Friedmann-Lemaitre-Robertson-Walker Cosmology-The Metric

In the early years of the 20th century, scientists had little knowledge about the structure and the distribution of matter in the Universe. In order to make further steps, they constructed a model to widen the aspects of the Universe which was based on an idea called the Cosmological Principle. The Cosmological Principle states that, the universe appears to be isotropic and homogeneous on large scales. When we say that the universe is isotropic, we mean that an observer will see the same characteristics in the universe whichever direction he observes. When we say that it is homogeneous, we mean that the universe will appear the same to any observer, independently of his position. These do not automatically imply each other however if we require that a distribution is isotropic about every point, then that does enforce homogeneity as well according to a basic theorem of geometry. Astronomical observations of the Cosmic Microwave Background show that the Universe appears to be isotropic on large scales to 1 part in 10^5 [1]. Despite the existence of inhomogeneous structures such as stars and galaxies, the observable universe is remarkably homogeneous and isotropic at scales larger than about $150h^{-1}$ Mpc [7], where 1 Mpc$\approx 3 \times 10^{24}$ cm is a convenient unit for extragalactic astronomy and $h = 0.72 \pm 0.07$ characterizes the current rate of expansion of the universe in dimensionless form. The mean distance between galaxies is about 1 Mpc while the size of the visible universe is about $3000h^{-1}$ Mpc.

The Friedmann-Lemaitre-Robertson-Walker metric, which gives quite consistent results with the observations in cosmological studies, is the most conventional metric used as in solving Einstein's equations for an isotropic and homogeneous universe which arises the question why the universe has chosen such a special state.

Assuming flat space-time with no gravity, the Minkowski metric with a signature (+, -, -, -) for distance ds in four dimensions (x, y, z, and t) is given by

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2. \tag{1.1}$$

In a static universe, this form of the metric would adequately describe the distance between stars. However, the universe is expanding. As in a balloon, the coordinates x, y, z in the universe remain constant; however, as more air is added to the balloon, the distance increases proportional to the expansion factor a. The metric becomes

$$ds^2 = c^2 dt^2 - a^2(t)[dx^2 + dy^2 + dz^2]. \tag{1.2}$$

In spherical coordinates, this flat metric becomes

$$ds^2 = c^2 dt^2 - a^2(t)\left[d\xi^2 + r^2(\xi)(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{1.3}$$

The definition of $r(\xi)$ varies with the value of the curvature of space k. The three possibilities for $r(\xi)$ to get homogeneous and isotropic spatial sections are

$$r(\xi) = \{\sin\xi,\ \xi,\ \sinh\xi\}. \tag{1.4}$$

These alternatives of $r(\xi)$ are due to a purely geometric fact, independent of the details of general relativity. In this thesis we will be working with the natural units in which $\hbar = c = 1$. The metric we use is the spatially conformal flat form of FLRW metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right]. \tag{1.5}$$

In the FLRW metric the curvature k, governed by the amount of matter and energy inside the universe, does not change with the expansion of the universe. It can take three different values:

- $k = 1$ if $r(\xi) = \sin\xi$, corresponds to positively curved spatial sections, geometrically spherical universe,
- $k = 0$ if $r(\xi) = \xi$, corresponds to local flatness namely zero spatial curvature, a geometrically flat, Euclidean universe,
- $k = -1$ if $r(\xi) = \sinh\xi$, corresponds to negatively curved spatial sections, geometrically hyperbolic universe.

1.2. Expansion and Red-Shift

Expansion of the universe means that early in its history the distance between us and distant galaxies was smaller than it is today. The basis of this expansion begins with redshift. The expansion of the universe is detected through the extension of the light emitted by the receding galaxies. Due to Doppler shifts, wavelenghts received from galaxies moving away from the milky way are elongated and contain less energy. The observed redshift indicates that most of the galaxies are moving away from Milky Way [8].

Redshift itself is the stretching of light emitted from galaxies due to the emitting object's recession. Cosmic expansion, denoted by a, cause the distances to increase, so if the ratio of the emitted wavelength λ to the observed wavelength is directly related to the ratio of cosmic expansion at the time when the light was emitted (t_{em}) to when it was observed (t_{obs}), then

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \frac{a_{\text{obs}}}{a_{\text{em}}}. \tag{1.6}$$

where z represents redshift. If an object is receding at a speed v, then its red-shift is

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}}, \tag{1.7}$$

for $v \ll c$

$$z = \frac{v}{c}, \tag{1.8}$$

In 1929, Hubble used these relationships to discover that a galaxy's recession velocity increase linearly with its distance from Earth, thus establishing the expansion of the universe. Because the expansion is uniform, the relationship between real distance $\vec{dr}$ and the co-moving distance $\vec{dx}$ for nearby objects can be written

$$\vec{dr} = a(t)\vec{dx}, \tag{1.9}$$

where the homogeneity property ensures that a is a function of time only. The recession velocity of the nearby galaxies at fixed co-moving coordinates is $\vec{v} = d\vec{r}/dt$, and is proportional to their physical distance vector, $\vec{dr}$,

$$\vec{v} = H\vec{dr}. \tag{1.10}$$

This relation is known as Hubble's Law, where H is the Hubble parameter, the rate of change of the scale factor, a. Hubble parameter, defined as $H = \dot{a}/a$, having the dimension $[LENGTH]^{-1}$, is constant over space but not over time; it determines the expansion rate of the universe, where $\dot{a} = da(t)/dt$.

1.3. Dynamics: The Friedmann Equations

In general relativity, the expansion rate of the universe follows from the Friedmann Equation which is found from the G_0^0 component of Einstein's Field Equations [9, 10];

$$G^{\mu}_{\ \nu} \equiv R^{\mu}_{\ \nu} - \frac{1}{2} R \delta^{\mu}_{\ \nu} = 8\pi G_N T^{\mu}_{\ \nu}, \tag{1.11}$$

where R^{μ}_{ν} is the Ricci tensor, $R = R^{\mu}_{\mu}$ is the Ricci scalar, T^{μ}_{ν} is the stress-energy tensor and G_N is the gravitational constant. Here we consider that the matter is in the perfect fluid form whose energy-momentum tensor can be written as

$$T^{\mu}_{\ \nu} = (p + \rho) u^{\mu} u_{\nu} - p \delta^{\mu}_{\ \nu}. \tag{1.12}$$

From homogenity and isotropy, the fluid should be at rest in comoving coordinates which implies $u^{\mu} = u_{\nu} = (1, 0, 0, 0)$. The individual elements are thus:

$$T^{0}_{\ 0} = \rho, \tag{1.13}$$

$$T^{1}_{\ 1} = T^{2}_{\ 2} = T^{3}_{\ 3} = -p, \tag{1.14}$$

$$T^{0}_{\ i} = T^{i}_{\ 0} = T^{i}_{\ j} = 0 \ (i \neq j). \tag{1.15}$$

Here ρ is the energy density and p is the pressure of the fluid. Solving Einstein's equations gives us two equations for the scale factor: $G^{0}_{\ 0}$ of Equation (1.11) yields

$$G^{0}_{\ 0} = 3(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}) = 8\pi G_N \rho,$$

so that the ‘Friedmann equation’ is

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{1.16}$$

which relates expansion rate of the universe to the energy density ρ. The $G^i_{\ i}$ component of Equation (1.11) on the other hand yields

$$G^i_{\ i} = 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G_N p. \tag{1.17}$$

Another equation which is called ‘contunity equation’ is as follows,

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0 \tag{1.18}$$

Let us consider the evolution of the universe with an equation of state which is defined by

$$\gamma = p/\rho \tag{1.19}$$

where γ is assumed to be constant. Using (1.18) and (1.19) gives

$$\rho \sim a^{-3(1+\gamma)} \tag{1.20}$$

For a radiation dominated universe $\gamma = 1/3$ and for a matter dominated universe $\gamma = 0$. Most of the time during history of the universe, a single component dominates a certain period of evolution. An example is the radiation dominated era where $\rho \sim a^{-4}$, $a \sim t^{\frac{1}{2}}$. At later times, when matter came to dominate, this dependence switched to, $\rho \sim a^{-3}$, $a \sim t^{\frac{2}{3}}$. It is now believed that, very recently, a has stopped growing as $t^{\frac{2}{3}}$, a signal that a new form of energy, namely dark energy, has come to dominate the cosmological landscape, where $\gamma = -1$.

1.4. The Density Parameter Ω

The total density of the universe is $\rho = \rho_M + \rho_\Lambda$ where M is denoted by matter (Baryons+Dark Matter) and the symbol Λ has been introduced to represent dark energy; it is also known as the cosmological constant or vacuum energy. As the volume of space in the universe increases, the mean density of matter decreases according to the relation $\rho = \rho_{M,0}(\frac{a_0}{a})^3$. However, the density of the dark energy, though, is a constant. For any given energy density component i, it is conventional to define the ratio of its today's energy density (where the 0 index represent the value of today) and today's critical density by a dimensionless density parameter Ω, as

$$\Omega_{i,0} = \frac{\rho_{i,0}}{\rho_{c,0}}, \tag{1.21}$$

where ρ_c is the critical density required in order to make the geometry of the universe flat, $(k = 0)$ with

$$\rho_c(t) = \frac{3H^2}{8\pi G_N}, \qquad \rho_{c,0} = \frac{3H_0^2}{8\pi G_N} \tag{1.22}$$

With this new notation, one can rewrite the Friedmann Equation (1.16) as

$$\Omega - 1 = \frac{k}{a^2 H^2}, \tag{1.23}$$

and can state the possibilities for the geometries of the universe depending on the density parameter where $\Omega = \Omega_\mathrm{M} + \Omega_\Lambda$.

- Open Universe: $0 < \Omega < 1 : k < 0 : \rho < \rho_c$
- Flat Universe: $\Omega = 1 : k = 0 : \rho = \rho_c$
- Closed Universe: $\Omega > 1 : k > 0 : \rho > \rho_c$.

The ratios of matter and dark energy in the universe to the critical density are respectively given by

$$\Omega_{M,0} = \frac{8\pi G_N \rho_{M,0}}{3H_0^2}, \qquad \Omega_{\Lambda,0} = \frac{8\pi G_N \rho_{\Lambda,0}}{3H_0^2}. \tag{1.24}$$

Although theoretically it can easily be seen from equation (1.23) that, for a flat universe, $k = 0$ and $\Omega_{tot} = \Omega_{\mathrm{M}} + \Omega_{\Lambda} = 1$, determining Ω_{tot} has been one of the key challenges in cosmology for decades. There has been a sudden breakthrough using measurements of the cosmic microwave background [4] combined with other measurements like galaxy clustering, and the latest constraint is $\Omega_{tot} = 1.0002 \pm 0.0295$, which is very well consistent with perfectly flat space, $\Omega_{tot} = 1$.

1.5. Dark Matter and Dark Energy

1.5.1. Dark Matter

In 1933, Swiss astronomer Fritz Zwicky suggested that gravity must keep the galaxies in the cluster together, since otherwise they would move apart from each other due to their own motion [11]. By determining the speed of motion of many galaxies within a cluster from the measurement of the doppler shift of the spectral lines, he could infer the required gravitational pull and thereby the total mass in the cluster. Suprisingly, the required mass by far exceeded the visible mass in the cluster. However, his results was not accepted by most astronomers at that time and it took sixty years until he was proven to be right.

In the 1970's Vera Rubin and her team found out that the visible stars are not the only objects making up the mass of the galaxies [12]. They measured the orbital speeds of stars around the center of spiral galaxies and that they move with a constant velocity independent of their radial distance from the center. This is in apparent disagreement with Kepler's law, which describes the orbital motion of the planets in our solar system very correctly. If Kepler's law is valid everywhere in the universe then, the rotational velocities of the stars can only be explained if the mass of the galaxy is increasing with the radial distance from its center. Numerical calculations show that there must be at least an order of magnitude more matter in the galaxies than is visible. From their measurements, which they repeated on hundreds of different galaxies, they concluded that each galaxy must be embedded in an enormous halo of dark matter, which reaches out even beyond the visible diameter of the galaxy.

It seems that the gravitational pull of huge amounts of dark matter is preventing individual galaxies from moving away from each other and is keeping them bound together in large clusters. By adding the total matter in galaxies and clusters of galaxies one ends up with a total mass which corresponds to about 28% of the critical mass of the universe.

1.5.2. Dark Energy

After the observations of type1a supernovae by the Supernova Cosmology Project and the High-z Supernova Search Team in 1998 [13], astronomers learned that the universe has been expanding with acceleration. A negative pressure is required in order to accelerate the expansion of the universe and that may be provided by some unidentified form of dark energy. This mysterious dark energy amounts to 72% of the critical mass of the universe and has the strange feature that its gravitational force does not attract, on the contrary it repels. This is hard to imagine since Newton's law of gravity tell us that matter is gravitationally attractive. In Einstein's law of gravity, however, the strength of gravity depends not only on mass and other forms of energy, but also on pressure. From the Einstein's equation (1.11), which describes the state of the universe, it follows that gravitation is repulsive if the pressure is sufficiently negative and it is attractive if the pressure is positive. In order to provide enough negative pressure to counterbalance the attractive force of gravity, Einstein originally introduced the so-called cosmological constant to keep the universe in a steady state. At that time all observations seemed to favour a steady state universe with no evolution and no knowledge about its beginning and its end. When Einstein learned about the Hubble expansion of the universe he discarded the cosmological constant by admitting that it was his biggest blunder. For a long time cosmologists assumed the cosmological constant to be negligibly small and set its value to zero. However this has changed very recently since we know about the accelerated expansion of the universe.

In the light of this introductory information about standard Einstein cosmology and observational results, one can see that how strange the present universe behaves cosmologically. Although the only component of the universe which we can understand theoretically well is the radiation (Cosmic Microwave Radiation), understanding the baryonic and dark matter density components is not trivial. Moreover, the issue of dark energy and acceleration in present universe under the influence of this mysterious energy today is somehow perplexing and beyond the expectations of standard Einstein cosmology. Hence, to solve this puzzle, many alternative theories to Einstein's gravitational theory are widely being used in the literature. Scalar tensor theories are the most favorite ones of these theories. So, in this thesis, we have chosen BD Dicke scalar tensor theory of gravity to explain dark matter and dark energy.

1.6. Scalar-Tensor Theories and Brans-Dicke Scalar Tensor Theory

In spite of the widely recognized success of general relativity, it is now called the standard theory of gravitation, the theory has also feeded by many "alternative" theories. Among them, we particularly focus on the Jordan and Brans-Dicke scalar tensor theories of gravitation. The reason why these theories are also labeled with "tensor" is that this type of theories do not merely combine the two kinds of fields (scalar field and gravitational field), but also they are built on the solid foundation of general relativity and besides, since Einstein's theory of relativity is a geometrical theory of space-time and the fundamental building block is a metric tensor field, $g_{\mu\nu}$, these kind of theories are named as "scalar-tensor" theories.

BD theory effectively replaces the Newtonian gravitational constant G_N in the Einstein-Hilbert action [16] by a power of the BD scalar field in such a way that

$$G^{-1} = \frac{2\pi}{\omega}\phi^2, \tag{1.25}$$

where G is the effective gravitational constant as long as the dynamical scalar field ϕ varies slowly. In units where $c = \hbar = 1$, the dimension of the scalar field is chosen to be L^{-1} so that G_{eff} has a dimension L^2. In the PhD thesis of M. Çalık, it is shown that a cosmological non-vacuum solution with flat space-like section is capable of explaining how the Hubble parameter H evolves with the scale size of the universe $a(t)$ and how the solution of fractional rate of change of BD scalar field, F contributes to the evolution of H in the late era in which the universe is expanding at a slow rate [16]. Besides, in the context of (BD) [17] theory, the action for a real BD scalar field is given by

$$S = \int d^4x \sqrt{g} \left[-\frac{1}{8\omega} \phi^2 R + \frac{1}{2} g^{\mu v} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) + L_M \right], \tag{1.26}$$

where ϕ represents the BD scalar field and ω denotes the dimensionless BD parameter taken to be much larger than 1, $\omega > 10^4 \gg 1$ [18]. L_M, on the other hand, is the matter Lagrangian such that the scalar field ϕ does not couple with it. The nonminimal coupling term is $\phi^2 R$ and R is the Ricci scalar. The kinetic and potential terms of the scalar field behave effectively as time dependent cosmological constants. At this point, we have to point out three simple assumptions made in this work:

- The BD field ϕ does not couple to any other field except gravity.
- The Lagrangian of the field, in addition to the kinetic term of ϕ, contains the simplest potential energy density $V(\phi) = \frac{1}{2} m^2 \phi^2$ which is composed only of the scalar field mass term [16].
- In particular we may expect that ϕ is spatially uniform, but varies slowly with time. For simplicity we also restrict our analysis to the Robertson Walker metric 1.5 to emphasize that ϕ is necessarily spatially homogeneous.

After applying the variational procedure [16] to the action (1.26) and assuming $\phi = \phi(t)$ and energy momentum tensor of matter and radiation excluding ϕ is in the perfect fluid form of $T^\mu_\nu = diag(\rho, -p, -p, -p)$ where ρ is the energy density and p is the pressure, and also noting that the right hand side of the ϕ equation below is set to be zero in accordance with our first assumption on L_M being independent of ϕ, the field equations considering the scalar ϕ field only are

$$\frac{3}{4\omega}\,\phi^2\left(\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)-\frac{1}{2}\,\dot{\phi}^2-\frac{1}{2}\,m^2\,\phi^2+\frac{3}{2\omega}\,\frac{\dot{a}}{a}\,\dot{\phi}\,\phi=\rho_M, \tag{1.27}$$

$$\frac{-1}{4\omega}\phi^2\left(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)-\frac{1}{\omega}\,\frac{\dot{a}}{a}\,\dot{\phi}\,\phi-\frac{1}{2\omega}\,\ddot{\phi}\,\phi-\left(\frac{1}{2}+\frac{1}{2\omega}\right)\,\dot{\phi}^2+\frac{1}{2}\,m^2\,\phi^2=p_M, \tag{1.28}$$

$$\ddot{\phi}+3\,\frac{\dot{a}}{a}\,\dot{\phi}+\left[m^2-\frac{3}{2\omega}\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)\right]\,\phi=0, \tag{1.29}$$

where k is the curvature parameter with $k=-1,\ 0,\ 1$ corresponding to open, flat, closed universes respectively and $a\,(t)$ is the scale factor of the universe (dot denotes $\frac{d}{dt}$). Since in the standard theory of gravitation, the total energy density ρ is assumed to be composed of $\rho=\rho_\Lambda+\rho_M$ where ρ_Λ is the energy density of the universe due to the cosmological constant which in modern terminology is called as "*dark energy*", the right hand sides of (1.27) and (1.28) are adopted to the matter energy density term ρ_M instead of ρ and p_M instead of p where M denotes everything except the ϕ field. The main reason behind doing such an organization is that whether if the ϕ terms on the left-hand side of (1.27) can accommodate a contribution to due to what is called dark matter and dark energy. In addition, the right hand side of the ϕ equation (1.29) is set to be zero according to the assumption imposed on the matter Lagrangian L_M being independent of the scalar field ϕ.

2. Methodology

2.1. Complex ϕ field in Vacuum Late Time Regime

Since in the previous section of this thesis, the field equations (1.27)-(1.29) are obtained on an assumption that BD field is solely scalar, at this point of this thesis we will start to work in BD cosmology by redefining scalar BD field ϕ as a complex BD field like $\phi = \phi_1 + i\phi_2$. Hence, the action (1.26) is modified to

$$S = \int \sqrt{g}\left[\frac{-1}{8\omega}\phi\phi^* R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^* - \frac{1}{2}m^2\phi\phi^* + L_M\right] d^4x, \tag{2.1}$$

where $\phi^2 = \phi\phi^*$.

2.1.1. Field Equations

Since we search for vacuum solutions ($\rho_M = p_M = 0$) of BD field equations (1.27)-(1.29) in late time regime where space-like sections of today's universe is approximately flat ($k = 0$), the modified field equations are

$$\frac{3}{4\omega}\phi\phi^*\left(\frac{\dot{a}^2}{a^2}\right) - \frac{1}{2}\dot{\phi}\dot{\phi}^* + \frac{3}{2\omega}\frac{\dot{a}}{a}\dot{\phi}\,\phi - \frac{1}{2}m^2\phi\phi^* = 0, \tag{2.2}$$

$$\begin{aligned} -\frac{1}{4\omega}\phi\phi^*\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) - \left(\frac{1}{2} + \frac{1}{2\omega}\right)\dot{\phi}\dot{\phi}^* \\ -\frac{1}{\omega}\frac{\dot{a}}{a}\dot{\phi}\,\phi - \frac{1}{2\omega}\ddot{\phi}\,\phi + \frac{1}{2}m^2\phi\phi^* = 0, \end{aligned} \tag{2.3}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \left[m^2 - \frac{3}{2\omega}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\right]\phi = 0. \tag{2.4}$$

where

$$\dot{\phi}\,\phi = \frac{\dot{\phi}\phi^* + \dot{\phi}^*\,\phi}{2}, \ddot{\phi}\,\phi = \frac{\ddot{\phi}\phi^* + \ddot{\phi}^*\phi}{2}. \tag{2.5}$$

Whenever we substitute equation (2.5) into (2.2) and (2.3) we get the following field equations for complex BD field ϕ

$$\frac{3}{4\omega}\left(\frac{\dot{a}^2}{a^2}\right) - \frac{1}{2}\frac{\dot{\phi}\dot{\phi}^*}{\phi\phi^*} + \frac{3}{4\omega}\left(\frac{\dot{a}}{a}\right)\left[\frac{\dot{\phi}\phi^* + \dot{\phi}^*\,\phi}{\phi\phi^*}\right] - \frac{1}{2}m^2 = 0, \tag{2.6}$$

$$-\frac{1}{4\omega}\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) - \left(\frac{1}{2} + \frac{1}{2\omega}\right)\frac{\dot{\phi}\dot{\phi}^*}{\phi\phi^*} \tag{2.7}$$
$$-\frac{1}{2\omega}\left(\frac{\dot{a}}{a}\right)\left[\frac{\dot{\phi}\phi^* + \dot{\phi}^*\,\phi}{\phi\phi^*}\right] - \frac{1}{4\omega}\left[\frac{\ddot{\phi}\phi^* + \ddot{\phi}^*\phi}{\phi\phi^*}\right] + \frac{1}{2}m^2 = 0,$$

$$\frac{\ddot{\phi}}{\phi} + 3\left(\frac{\dot{a}}{a}\right)\frac{\dot{\phi}}{\phi} + \left[m^2 - \frac{3}{2\omega}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\right] = 0. \tag{2.8}$$

Since solving the field equations (2.80)-(2.82) for $a(t)$ and $\phi(t)$ is hard enough, we first define the Hubble parameter as $H = \frac{\dot{a}}{a}$ and the fractional rate of change of ϕ as $F = \frac{\dot{\phi}}{\phi} = F_1 + iF_2$, where

$$\frac{\ddot{a}}{a} = H^2 + aHH' \tag{2.9}$$

$$F_1 = \frac{\dot{\phi}_1\phi_1 + \dot{\phi}_2\phi_2}{\phi_1^2 + \phi_2^2}, \qquad F_2 = \frac{\dot{\phi}_2\phi_1 - \dot{\phi}_1\phi_2}{\phi_1^2 + \phi_2^2} \tag{2.10}$$

such that

$$\frac{\dot{\phi}\dot{\phi}^*}{\phi\phi^*} = \frac{\dot{\phi}_1^2 + \dot{\phi}_2^2}{\phi_1^2 + \phi_2^2} = F_1^2 + F_2^2 \tag{2.11}$$

$$\left[\frac{\dot{\phi}\phi^* + \dot{\phi}^*\phi}{\phi\phi^*}\right] = \frac{2(\dot{\phi}_1\phi_1 + \dot{\phi}_2\phi_2)}{\phi_1^2 + \phi_2^2} = 2F_1 \tag{2.12}$$

$$\left[\frac{\ddot{\phi}\phi^* + \ddot{\phi}^*\phi}{\phi\phi^*}\right] = \frac{2(\ddot{\phi}_1\phi_1 + \ddot{\phi}_2\phi_2)}{\phi_1^2 + \phi_2^2} = 2(aHF_1' + F_1^2 - F_2^2) \tag{2.13}$$

$$\frac{\ddot{\phi}}{\phi} = \dot{F} + F^2 = F'aH + F^2 = (F_1' + iF_2')aH + (F_1 + iF_2)^2. \tag{2.14}$$

Hence, we rewrite (2.80)-(2.82) in terms of H, F_1 and F_2 and their derivatives with respect to a:

$$3H^2 - 2\omega F_1^2 - 2\omega F_2^2 + 6HF_1 - 2\omega m^2 = 0 \tag{2.15}$$

$$3H^2 + (2\omega + 4)\,F_1^2 + 2\omega F_2^2 + 4HF_1 + 2aHF_1' + 2aHH' - 2\omega m^2 = 0 \tag{2.16}$$

$$-6H^2 + 2\omega F_1^2 - 2\omega F_2^2 + 6\omega HF_1 + 2\omega aHF_1' - 3aHH' + 2\omega m^2 = 0 \tag{2.17}$$

$$2\omega aHF_2' + 4\omega F_1F_2 + 6\omega HF_2 = 0. \tag{2.18}$$

We have to note that equation (2.17) is the real part of ϕ equation and (2.18) is the imaginary part of ϕ equation and prime denotes the derivative with respect to a. In order to solve (2.15)-(2.18) we propose the transformation

$$u = \left(\frac{a_0}{a}\right)^{\alpha} \tag{2.19}$$

such that

$$aH^{'} = a\frac{dH}{da} = -\alpha u\left(\frac{dH}{du}\right) \tag{2.20}$$

$$aF_1^{'} = a\frac{dF_1}{da} = -\alpha u\left(\frac{dF_1}{du}\right) \tag{2.21}$$

$$aF_2^{'} = a\frac{dF_2}{da} = -\alpha u\left(\frac{dF_2}{du}\right). \tag{2.22}$$

Now, let's rewrite (2.15)-(2.18) in terms of $H(u)$, $F_1(u)$, $F_2(u)$, α and their derivatives with respect to u.

$$3H^2 - 2\omega F_1^2 - 2\omega F_2^2 + 6HF_1 - 2\omega m^2 = 0, \tag{2.23}$$

$$3H^2 + (2\omega + 4)F_1^2 + 2\omega F_2^2 + 4HF_1 - 2\alpha uH\left(\frac{dF_1}{du}\right) - 2\alpha uH\left(\frac{dH}{du}\right) - 2\omega m^2 = 0, \tag{2.24}$$

$$-6H^2 + 2\omega F_1^2 - 2\omega F_2^2 + 6\omega HF_1 - 2\omega\alpha uH\left(\frac{dF_1}{du}\right) + 3\alpha uH\left(\frac{dH}{du}\right) + 2\omega m^2 = 0, \tag{2.25}$$

$$-2\omega\alpha uH\left(\frac{dF_2}{du}\right) + 4\omega F_1F_2 + 6\omega HF_2 = 0. \tag{2.26}$$

Since these coupled equations are still too hard to be solved analytically for H, F_1 and F_2, our approach is to determine a perturbative solution in which H, F_1 and F_2 vary about some perturbation constants. Hence the solution to the equation system

(2.23)-(2.26) is in the form of

$$H = H_\infty + H_1 \left(\frac{a_0}{a}\right)^\alpha + H_2 \left(\frac{a_0}{a}\right)^{2\alpha} = H_\infty + H_1 u + H_2 u^2 \tag{2.27}$$

$$F_1 = F_{1\infty} + F_{11} \left(\frac{a_0}{a}\right)^\alpha + F_{12} \left(\frac{a_0}{a}\right)^{2\alpha} = F_{1\infty} + F_{11} u + F_{12} u^2 \tag{2.28}$$

$$F_2 = F_{2\infty} + F_{21} \left(\frac{a_0}{a}\right)^\alpha + F_{22} \left(\frac{a_0}{a}\right)^{2\alpha} = F_{2\infty} + F_{21} u + F_{22} u^2 \tag{2.29}$$

where H_∞, H_1, H_2, $F_{1\infty}$, F_{11}, F_{12}, $F_{2\infty}$, F_{21}, F_{22} are perturbation constants and α is an exponential factor to be determined from the theory. Once we proposed the solutions (2.108)-(2.110) we are able to write

$$\left(\frac{dH}{du}\right) = H_1 + 2H_2 u \tag{2.30}$$

$$\left(\frac{dF_1}{du}\right) = F_{11} + 2F_{12} u \tag{2.31}$$

$$\left(\frac{dF_2}{du}\right) = F_{21} + 2F_{22} u \tag{2.32}$$

$$H^2 = H_\infty^2 + (2H_1 H_\infty)u + \left(H_1^2 + 2H_\infty H_2\right) u^2 \tag{2.33}$$

$$F_1^2 = F_{1\infty}^2 + (2F_{11} F_{1\infty})u + \left(F_{11}^2 + 2F_{1\infty} F_{12}\right) u^2 \tag{2.34}$$

$$F_2^2 = F_{2\infty}^2 + (2F_{21} F_{2\infty})u + \left(F_{21}^2 + 2F_{2\infty} F_{22}\right) u^2 \tag{2.35}$$

$$HF_1 = F_{1\infty}H_\infty + (F_{1\infty}H_1 + F_{11}H_\infty)u + (F_{1\infty}H_2 + F_{11}H_1 + F_{12}H_\infty)u^2 \tag{2.36}$$

$$HF_2 = F_{2\infty}H_\infty + (F_{2\infty}H_1 + F_{21}H_\infty)u + (F_{2\infty}H_2 + F_{21}H_1 + F_{22}H_\infty)u^2 \tag{2.37}$$

$$F_2F_1 = F_{1\infty}F_{2\infty} + (F_{1\infty}F_{21} + F_{11}F_{2\infty})u + (F_{1\infty}F_{22} + F_{11}F_{21} + F_{12}F_{2\infty})u^2 \tag{2.38}$$

where the equations are written to the second power of u. Therefore, substituting these equations into (2.23)-(2.26) and keeping only the zeroth, first and second order terms of u and neglect higher terms, we get the following equations to be solved:

$$\begin{gathered} 3[H_\infty^2 + (2H_1H_\infty)u + \left(H_1^2 + 2H_\infty H_2\right)u^2]- \\ 2\omega[F_{1\infty}^2 + (2F_{11}F_{1\infty})u + \left(F_{11}^2 + 2F_{1\infty}F_{12}\right)u^2]- \\ 2\omega[F_{2\infty}^2 + (2F_{21}F_{2\infty})u + \left(F_{21}^2 + 2F_{2\infty}F_{22}\right)u^2]+ \\ 6[F_{1\infty}H_\infty + (F_{1\infty}H_1 + F_{11}H_\infty)u + (F_{1\infty}H_2 + F_{11}H_1 + F_{12}H_\infty)u^2] - 2\omega m^2 = 0, \end{gathered} \tag{2.39}$$

$$\begin{gathered} 3[H_\infty^2 + (2H_1H_\infty)u + \left(H_1^2 + 2H_\infty H_2\right)u^2]+ \\ (2\omega+4)[F_{1\infty}^2 + (2F_{11}F_{1\infty})u + \left(F_{11}^2 + 2F_{1\infty}F_{12}\right)] \\ +2\omega[F_{2\infty}^2 + (2F_{21}F_{2\infty})u + \left(F_{21}^2 + 2F_{2\infty}F_{22}\right)u^2]+ \\ 4[F_{1\infty}H_\infty + (F_{1\infty}H_1 + F_{11}H_\infty)u + (F_{1\infty}H_2 + F_{11}H_1 + F_{12}H_\infty)u^2]- \\ 2\alpha u(H_\infty + H_1u + H_2u^2)(F_{11} + 2F_{12}u) \\ -2\alpha u(H_\infty + H_1u + H_2u^2)(H_1 + 2H_2u) - 2\omega m^2 = 0, \end{gathered} \tag{2.40}$$

$$-6[H_\infty^2 + (2H_1H_\infty)u + \left(H_1^2 + 2H_\infty H_2\right)u^2] \tag{2.41}$$
$$+2\omega[F_{1\infty}^2 + (2F_{11}F_{1\infty})u + \left(F_{11}^2 + 2F_{1\infty}F_{12}\right)]$$
$$-2\omega[F_{2\infty}^2 + (2F_{21}F_{2\infty})u + \left(F_{21}^2 + 2F_{2\infty}F_{22}\right)u^2]+$$
$$6\omega[F_{1\infty}H_\infty + (F_{1\infty}H_1 + F_{11}H_\infty)u + (F_{1\infty}H_2 + F_{11}H_1 + F_{12}H_\infty)u^2]$$
$$-2\omega\alpha u(H_\infty + H_1u + H_2u^2)(F_{11} + 2F_{12}u)$$
$$+3\alpha u(H_\infty + H_1u + H_2u^2)(H_1 + 2H_2u) + 2\omega m^2 = 0,$$

$$-2\omega\alpha u(H_\infty + H_1u + H_2u^2)(F_{21} + 2F_{22}u)+ \tag{2.42}$$
$$4\omega(F_{1\infty}F_{2\infty} + (F_{1\infty}F_{21} + F_{11}F_{2\infty})u + (F_{1\infty}F_{22} + F_{11}F_{21} + F_{12}F_{2\infty})u^2)+$$
$$6\omega(F_{2\infty}H_\infty + (F_{2\infty}H_1 + F_{21}H_\infty)u + (F_{2\infty}H_2 + F_{21}H_1 + F_{22}H_\infty)u^2) = 0.$$

From these equations (2.39)-(2.42) we get the following equations according to their order of u;
In the zeroth order:

$$3H_\infty^2 - 2\omega F_{1\infty}^2 - 2\omega F_{2\infty}^2 + 6F_{1\infty}H_\infty - 2\omega m^2 = 0, \tag{2.43}$$

$$3H_\infty^2 + (2\omega + 4)F_{1\infty}^2 + 2\omega F_{2\infty}^2 + 4F_{1\infty}H_\infty - 2\omega m^2 = 0, \tag{2.44}$$

$$-6H_\infty^2 + 2\omega F_{1\infty}^2 - 2\omega F_{2\infty}^2 + 6\omega F_{1\infty}H_\infty + 2\omega m^2 = 0, \tag{2.45}$$

$$(4\omega F_{1\infty} + 6\omega H_\infty)F_{2\infty} = 0. \tag{2.46}$$

In the first order of u;

$$(6F_{1\infty} + 6H_\infty)H_1 + (6H_\infty - 4\omega F_{1\infty})F_{11} - 4\omega F_{21}F_{2\infty} = 0, \tag{2.47}$$

$$((6{-}2\alpha)H_\infty + 4F_{1\infty})H_1 + ((4{-}2\alpha)H_\infty + (4\omega+8)\,F_{1\infty})F_{11} + 4\omega F_{21}F_{2\infty} = 0, \quad (2.48)$$

$$((3\alpha-12)H_\infty + 6\omega F_{1\infty})H_1 + ((6\omega{-}2\omega\alpha)H_\infty + 4\omega F_{1\infty})F_{11} - 4\omega F_{21}F_{2\infty} = 0, \quad (2.49)$$

$$[(-2\omega\alpha+6\omega)H_\infty + 4\omega F_{1\infty}]F_{21} + 4\omega F_{11}F_{2\infty} + 6\omega H_1 F_{2\infty} = 0, \quad (2.50)$$

In the second order of u;

$$3H_1^2 + 6H_\infty H_2 - 2\omega F_{11}^2 - 4\omega F_{1\infty}F_{12} \quad (2.51)$$
$$-4\omega F_{2\infty}F_{22} - 2\omega F_{21}^2 + 6F_{1\infty}H_2 + 6F_{11}H_1 + 6F_{12}H_\infty = 0$$

$$(3-2\alpha)H_1^2 + (2\omega{+}4)\,F_{11}^2 + (4-2\alpha)F_{11}H_1 + (4F_{1\infty} + 6H_\infty - 4\alpha H_\infty)H_2 \quad (2.52)$$
$$+[4H_\infty + (4\omega{+}8)\,F_{1\infty} - 4\alpha H_\infty]F_{12} + 2\omega F_{21}^2 + 4\omega F_{2\infty}F_{22} = 0$$

$$(3\alpha-6)H_1^2 + 2\omega F_{11}^2 + (-2\omega\alpha+6\omega)F_{11}H_1 + (-12H_\infty + 6\alpha H_\infty + 6\omega F_{1\infty})H_2 \quad (2.53)$$
$$+(-4\omega\alpha H_\infty + 4\omega F_{1\infty} + 6\omega H_\infty)F_{12} - 2\omega F_{21}^2 - 4\omega F_{2\infty}F_{22} = 0$$

$$(4\omega F_{1\infty} - 4\omega\alpha H_\infty + 6\omega H_\infty)F_{22} + (-2\omega\alpha H_1 + 4\omega F_{11} + 6\omega H_1)F_{21} \quad (2.54)$$
$$+4\omega F_{2\infty}F_{12} + 6\omega F_{2\infty}H_2 = 0.$$

2.1.2. Solutions

Solving the equation set (2.43)-(2.46) and (2.47)-(2.50) respectively, provide

$$F_{2\infty} = 0 \qquad H_\infty = \frac{2\,(\omega+1)\,\sqrt{\omega}m}{\sqrt{(6\omega^2+17\omega+12)}} \qquad F_{1\infty} = \frac{H_\infty}{2\omega+2}, \quad (2.55)$$

$$\alpha = 3 + \frac{1}{\omega + 1} \qquad F_{11} = -\frac{3}{2}H_1 \qquad F_{21} = \text{free} - \text{parameter}. \tag{2.56}$$

and afterwards, substituting (2.55) and (2.56) into the equation set (2.51)-(2.54) yields the following equation set to be solved for H_2, F_{12}, F_{21}, F_{22} as

$$(12\omega + 18)H_\infty H_2 + (8\omega + 12)H_\infty F_{12} - 4\omega(\omega + 1)F_{21}^2 = (9\omega^2 + 21\omega + 12)H_1^2, \tag{2.57}$$

$$-(12\omega + 16)H_\infty H_2 - (12\omega + 16)H_\infty F_{12} + 4\omega(\omega + 1)F_{21}^2 = -(9\omega^2 + 27\omega + 20)H_1^2, \tag{2.58}$$

$$(18\omega + 24)H_\infty H_2 - (12\omega^2 + 16\omega)H_\infty F_{12} - 4\omega(\omega + 1)F_{21}^2 = -(9\omega^2 + 21\omega + 12)H_1^2, \tag{2.59}$$

$$F_{22} = -\frac{H_1}{H_\infty}F_{21}. \tag{2.60}$$

To proceed one step further, we write the standard Friedmann equation

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{\Lambda,0} + \Omega_{M,0}\left(\frac{a_0}{a}\right)^3 \tag{2.61}$$

and we fit all theory parameters to the observational density parameters;

$$\Omega_\Lambda = \frac{H_\infty^2}{H_\Sigma^2}, \tag{2.62}$$

$$\Omega_M = \frac{2H_\infty H_1}{H_\Sigma^2}, \tag{2.63}$$

where

$$H_\Sigma^2 = H_\infty^2 + 2H_\infty(H_1 + H_2) + H_1^2. \tag{2.64}$$

With these relations above and the constraint $\Omega_\Lambda + \Omega_M = 1$, where $\Omega_M = \Omega_{\rm VM} + \Omega_{\rm DM}$, we can express H_1 in terms of the observational density parameters Ω_Λ, Ω_M and H_∞ as

$$H_1 = \frac{\Omega_M}{2\Omega_\Lambda} H_\infty, \tag{2.65}$$

Using recent observational results [4] on density parameters $\Omega_{DM} \simeq 0.28$, $\Omega_\Lambda \simeq 0.72$ and $\Omega_{\rm VM} = 0$ (since the universe we study in this theory is vacuum) together with (2.65) we determine;

$$H_1 = \frac{0.28}{1.44} H_\infty \simeq 0.19 H_\infty. \tag{2.66}$$

Similarly, using (2.66) in solving the equations (2.57)-(2.60); the solutions are;

$$H_2 H_\infty = \frac{9\omega^3 H_1^2 + 4F_{21}^2\omega^3 + 18\omega^2 H_1^2 + 8F_{21}^2\omega^2 - \omega H_1^2 + 4F_{21}^2\omega - 12H_1^2}{12\omega^2 + 34\omega + 24}, \tag{2.67}$$

$$F_{12} H_\infty = \frac{45\omega^2 H_1^2 + 4\omega^2 F_{21}^2 + 123\omega H_1^2 + 4F_{21}^2\omega + 84H_1^2}{24\omega^2 + 68\omega + 48}, \tag{2.68}$$

As $\omega \to \infty$;

$$H_2 \simeq \left(\frac{1}{3}\omega \frac{F_{21}^2}{H_\infty} + \frac{3}{4}\omega \frac{H_1^2}{H_\infty} \right), \tag{2.69}$$

$$F_{12} \simeq \left(\frac{4}{24H_\infty} F_{21}^2 + \frac{45}{24H_\infty} H_1^2 \right), \tag{2.70}$$

$$F_{22} = -\frac{H_1}{H_\infty} F_{21}. \tag{2.71}$$

2.1.3. Results

At this point, we emphasize that H_2 must be zero in order to make sense with recent observational data on density parameters of the universe and to find the exact value for F_{21}. Therefore, when we insert $H_2 = 0$ we see that

$$F_{21}^2 \simeq -\frac{9}{4}H_1^2 \tag{2.72}$$

Namely, F_{21} is non-physical. In the previous work of Arik and Calik [19], ϕ was accepted as solely scalar and this assumption showed that, while BD scalar tensor theory well accounts for dark energy, it does not contribute to dark matter. On the other hand, in our work, we expect to find a contribution to dark matter in the presence of F_2 term. Since F_{21} is the coupling term with $(\frac{a_0}{a})^3$, we see that there is some anomaly in considering BD field ϕ as a complex field to explain the dark matter in BD cosmology. Therefore, to get rid of this problem we will shift to the hyperbolic phase of ϕ field.

2.2. Hyperbolic ϕ Field in Vacuum Late Time Regime

In the first part of the thesis we have used a complex ϕ field such that

$$\phi = \phi_1 + i\phi_2 = \phi_{\mathrm{R}} e^{i\beta} \tag{2.73}$$

where ϕ_{R} is real scalar field amplitude. Such complex BD field can also be represented as in matrix form

$$\phi = \begin{pmatrix} \phi_1 & \phi_2 \\ -\phi_2 & \phi_1 \end{pmatrix} = \phi_1 + i\sigma_2\phi_2 \tag{2.74}$$

where σ_2 is a Pauli spin matrix.

However, we will take the phase of ϕ to be hyperbolic by replacing the term

$i\beta = \Psi$ in (2.73) such that ϕ becomes

$$\phi = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_1 \end{pmatrix} \tag{2.75}$$

and its conjugate matrix becomes

$$\phi^* = \begin{pmatrix} \phi_1 & -\phi_2 \\ -\phi_2 & \phi_1 \end{pmatrix} \tag{2.76}$$

together with

$$\phi_1 = \phi_R \cosh \Psi \tag{2.77}$$

$$\phi_2 = \phi_R \sinh \Psi \tag{2.78}$$

where Ψ is real. With this modification, we note here that Ψ gains a "ghost" character since its kinetic contribution ($\phi\phi^* = \phi_1^2 - \phi_2^2$) brings a minus sign. With this new modification the action equation (2.1) becomes

$$S = \frac{1}{2} tr \int d^4x\sqrt{g}\left[\frac{-1}{8\omega}\phi\phi^* R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^* - \frac{1}{2}m^2\phi\phi^* + IL_M\right]. \tag{2.79}$$

where I is the unit matrix.

2.2.1. Field Equations

Remembering our modified field equations from the first part;

$$\frac{3}{4\omega}\left(\frac{\dot{a}^2}{a^2}\right) - \frac{1}{2}\frac{\dot{\phi}\dot{\phi}^*}{\phi\phi^*} + \frac{3}{4\omega}\left(\frac{\dot{a}}{a}\right)\left[\frac{\dot{\phi}\phi^* + \dot{\phi}^*\phi}{\phi\phi^*}\right] - \frac{1}{2}m^2 = 0, \tag{2.80}$$

$$-\frac{1}{4\omega}\left(\frac{2\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)-\left(\frac{1}{2}+\frac{1}{2\omega}\right)\frac{\dot{\phi}\dot{\phi}^*}{\phi\phi^*} \tag{2.81}$$
$$-\frac{1}{2\omega}\left(\frac{\dot{a}}{a}\right)\left[\frac{\dot{\phi}\phi^*+\dot{\phi}^*\phi}{\phi\phi^*}\right]-\frac{1}{4\omega}\left[\frac{\ddot{\phi}\phi^*+\ddot{\phi}^*\phi}{\phi\phi^*}\right]+\frac{1}{2}m^2=0,$$

$$\frac{\ddot{\phi}}{\phi}+3\left(\frac{\dot{a}}{a}\right)\frac{\dot{\phi}}{\phi}+\left[m^2-\frac{3}{2\omega}\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)\right]=0. \tag{2.82}$$

We also recall that, we define the Hubble parameter as $H=\frac{\dot{a}}{a}$ and the fractional rate of change of ϕ as $F=\frac{\dot{\phi}}{\phi}$, where

$$\frac{\ddot{a}}{a}=H^2+aHH' \tag{2.83}$$

$$F=\frac{\dot{\phi}\phi^*}{\phi\phi^*}=\begin{pmatrix}\frac{\phi_1\dot{\phi}_1-\phi_2\dot{\phi}_2}{\phi_1^2-\phi_2^2} & \frac{\phi_1\dot{\phi}_2-\dot{\phi}_1\phi_2}{\phi_1^2-\phi_2^2}\\ \frac{\phi_1\dot{\phi}_2-\dot{\phi}_1\phi_2}{\phi_1^2-\phi_2^2} & \frac{\phi_1\dot{\phi}_1-\phi_2\dot{\phi}_2}{\phi_1^2-\phi_2^2}\end{pmatrix}=\begin{pmatrix}F_1 & F_2\\ F_2 & F_1\end{pmatrix}$$

$$F_1=\frac{\phi_1\dot{\phi}_1-\phi_2\dot{\phi}_2}{\phi_1^2-\phi_2^2}=\frac{\dot{\phi}_R}{\phi_R}, \qquad F_2=\frac{\phi_1\dot{\phi}_2-\dot{\phi}_1\phi_2}{\phi_1^2-\phi_2^2}=\dot{\Psi}$$

such that

$$\phi\phi^*=\begin{pmatrix}\phi_1^2-\phi_2^2 & 0\\ 0 & \phi_1^2-\phi_2^2\end{pmatrix}, \tag{2.84}$$

$$\dot{\phi}\dot{\phi}^*=\begin{pmatrix}\dot{\phi}_1^2-\dot{\phi}_2^2 & 0\\ 0 & \dot{\phi}_1^2-\dot{\phi}_2^2\end{pmatrix} \tag{2.85}$$

$$\frac{\dot{\phi}\dot{\phi}^*}{\phi\phi^*} = \begin{pmatrix} \frac{\dot{\phi}_1^2 - \dot{\phi}_2^2}{\phi_1^2 - \phi_2^2} & 0 \\ 0 & \frac{\dot{\phi}_1^2 - \dot{\phi}_2^2}{\phi_1^2 - \phi_2^2} \end{pmatrix} = \begin{pmatrix} F_1^2 - F_2^2 & 0 \\ 0 & F_1^2 - F_2^2 \end{pmatrix} \tag{2.86}$$

$$\frac{\dot{\phi}\phi^* + \dot{\phi}^*\phi}{\phi\phi^*} = \begin{pmatrix} \frac{2(\phi_1\dot{\phi}_1 - \phi_2\dot{\phi}_2)}{\phi_1^2 - \phi_2^2} & 0 \\ 0 & \frac{2(\phi_1\dot{\phi}_1 - \phi_2\dot{\phi}_2)}{\phi_1^2 - \phi_2^2} \end{pmatrix} = \begin{pmatrix} 2F_1 & 0 \\ 0 & 2F_1 \end{pmatrix} \tag{2.87}$$

$$\ddot{\phi}\phi^* = \begin{pmatrix} \phi_1\ddot{\phi}_1 - \phi_2\ddot{\phi}_2 & -\ddot{\phi}_1\phi_2 + \phi_1\ddot{\phi}_2 \\ -\ddot{\phi}_1\phi_2 + \phi_1\ddot{\phi}_2 & \phi_1\ddot{\phi}_1 - \phi_2\ddot{\phi}_2 \end{pmatrix} \tag{2.88}$$

$$\phi\ddot{\phi}^* = \begin{pmatrix} \phi_1\ddot{\phi}_1 - \phi_2\ddot{\phi}_2 & -\phi_1\ddot{\phi}_2 + \ddot{\phi}_1\phi_2 \\ -\phi_1\ddot{\phi}_2 + \ddot{\phi}_1\phi_2 & \phi_1\ddot{\phi}_1 - \phi_2\ddot{\phi}_2 \end{pmatrix} \tag{2.89}$$

$$\frac{\ddot{\phi}\phi^* + \ddot{\phi}^*\phi}{\phi\phi^*} = \begin{pmatrix} \frac{2(\phi_1\ddot{\phi}_1 - \phi_2\ddot{\phi}_2)}{\phi_1^2 - \phi_2^2} & 0 \\ 0 & \frac{2(\phi_1\ddot{\phi}_1 - \phi_2\ddot{\phi}_2)}{\phi_1^2 - \phi_2^2} \end{pmatrix} = \begin{pmatrix} F_1'aH + F_1^2 + F_2^2 & 0 \\ 0 & F_1'aH + F_1^2 + F_2^2 \end{pmatrix} \tag{2.90}$$

$$\frac{\dot{\phi}}{\phi} = \begin{pmatrix} \frac{\phi_1\dot{\phi}_1 - \phi_2\dot{\phi}_2}{\phi_1^2 - \phi_2^2} & \frac{\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2}{\phi_1^2 - \phi_2^2} \\ \frac{\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2}{\phi_1^2 - \phi_2^2} & \frac{\phi_1\dot{\phi}_1 - \phi_2\dot{\phi}_2}{\phi_1^2 - \phi_2^2} \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_2 & F_1 \end{pmatrix} \tag{2.91}$$

$$\frac{\ddot{\phi}}{\phi} = \begin{pmatrix} \frac{\phi_1\ddot{\phi}_1 - \phi_2\ddot{\phi}_2}{\phi_1^2 - \phi_2^2} & \frac{-\ddot{\phi}_1\phi_2 + \phi_1\ddot{\phi}_2}{\phi_1^2 - \phi_2^2} \\ \frac{-\ddot{\phi}_1\phi_2 + \phi_1\ddot{\phi}_2}{\phi_1^2 - \phi_2^2} & \frac{\phi_1\ddot{\phi}_1 - \phi_2\ddot{\phi}_2}{\phi_1^2 - \phi_2^2} \end{pmatrix} = \begin{pmatrix} F_1'aH + F_1^2 + F_2^2 & F_2'aH + 2F_1F_2 \\ F_2'aH + 2F_1F_2 & F_1'aH + F_1^2 + F_2^2 \end{pmatrix}, \tag{2.92}$$

Having specified all terms (2.84)-(2.92) in matrix form, provides us to write the matrix equations of the field equations (2.80)-(2.82):

$$\frac{3}{4\omega}H^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}F_1^2-F_2^2 & 0\\ 0 & F_1^2-F_2^2\end{pmatrix}+$$
$$\frac{3}{4\omega}H\begin{pmatrix}2F_1 & 0\\ 0 & 2F_1\end{pmatrix}-\frac{1}{2}m^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \tag{2.93}$$

$$-\frac{1}{4\omega}(3H^2+2aHH')\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\left(\frac{1}{2}+\frac{1}{2\omega}\right)\begin{pmatrix}F_1^2-F_2^2 & 0\\ 0 & F_1^2-F_2^2\end{pmatrix}-$$
$$\frac{1}{2\omega}H\begin{pmatrix}2F_1 & 0\\ 0 & 2F_1\end{pmatrix}+$$
$$\frac{1}{4\omega}\begin{pmatrix}F_1'aH+F_1^2+F_2^2 & 0\\ 0 & F_1'aH+F_1^2+F_2^2\end{pmatrix}+$$
$$\frac{1}{2}m^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \tag{2.94}$$

$$\begin{pmatrix}F_1'aH+F_1^2+F_2^2 & F_2'aH+2F_1F_2\\ F_2'aH+2F_1F_2 & F_1'aH+F_1^2+F_2^2\end{pmatrix}+3H\begin{pmatrix}F_1 & F_2\\ F_2 & F_1\end{pmatrix}+$$
$$\left[m^2-\frac{3}{2\omega}(2H^2+aHH')\right]\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{2.95}$$

At this point, writing ρ, p and ϕ equations just in one matrix form can be much more useful and they are;

$$\begin{pmatrix}\begin{matrix}\frac{3}{4\omega}H^2-\frac{1}{2}(F_1^2-F_1^2)+\\ \frac{3}{2\omega}HF_1-\frac{1}{2}m^2\end{matrix} & 0\\ 0 & \begin{matrix}\frac{3}{4\omega}H^2-\frac{1}{2}(F_1^2-F_1^2)+\\ \frac{3}{2\omega}HF_1-\frac{1}{2}m^2\end{matrix}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \tag{2.96}$$

$$\begin{pmatrix} \begin{array}{c} -\frac{1}{4\omega}(3H^2+2aHH') \\ -(\frac{1}{2}+\frac{1}{2\omega})(F_1^2-F_2^2) \\ -\frac{1}{\omega}HF_1 \\ +\frac{1}{4\omega}(F_1'aH+F_1^2+F_2^2)+\frac{1}{2}m^2 \end{array} & 0 \\ 0 & \begin{array}{c} -\frac{1}{4\omega}(3H^2+2aHH') \\ -(\frac{1}{2}+\frac{1}{2\omega})(F_1^2-F_2^2) \\ -\frac{1}{\omega}HF_1 \\ +\frac{1}{4\omega}(F_1'aH+F_1^2+F_2^2)+\frac{1}{2}m^2 \end{array} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.97}$$

$$\begin{pmatrix} \begin{array}{c} F_1'aH+F_1^2+F_2^2+3HF_1 \\ +\left[m^2-\frac{3}{2\omega}(2H^2+aHH')\right] \end{array} & F_2'aH+2F_1F_2+3HF_1 \\ F_2'aH+2F_1F_2+3HF_1 & \begin{array}{c} F_1'aH+F_1^2+F_2^2+3HF_1 \\ +\left[m^2-\frac{3}{2\omega}(2H^2+aHH')\right] \end{array} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.98}$$

where, while the diagonal terms of the matrix form of the ϕ equation represents the real part and the non-diagonal terms of the matrix form of the ϕ equation represents the imaginary part. Now, it is time to write our field equations in terms of H, F_1, F_2 and their derivatives with respect to a:

$$3H^2-2\omega F_1^2+2\omega F_2^2+6F_1H-2\omega m^2=0 \tag{2.99}$$

$$3H^2+(2\omega+4)\,F_1^2-2\omega F_2^2+4F_1H+2aHF_1'+2aHH'-2\omega m^2=0 \tag{2.100}$$

$$-6H^2+2\omega F_1^2+2\omega F_2^2+6\omega F_1H+2\omega aHF_1'-3aHH'+2\omega m^2=0 \tag{2.101}$$

$$(4\omega F_1+6\omega H)\,F_2+2\omega aHF_2'=0 \tag{2.102}$$

Again with the same transformation

$$u = \left(\frac{a_0}{a}\right)^{\alpha} \tag{2.103}$$

we rewrite (2.99)-(2.102) in terms of $H(u), F_1(u)$ and $F_2(u)$;

$$3H^2 - 2\omega F_1^2 + 2\omega F_2^2 + 6HF_1 - 2\omega m^2 = 0, \tag{2.104}$$

$$3H^2 + (2\omega + 4) F_1^2 - 2\omega F_2^2 + 4HF_1 - 2\alpha u H \left(\frac{dF_1}{du}\right) - 2\alpha u H \left(\frac{dH}{du}\right) - 2\omega m^2 = 0, \tag{2.105}$$

$$-6H^2 + 2\omega F_1^2 + 2\omega F_2^2 + 6\omega HF_1 - 2\omega\alpha u H \left(\frac{dF_1}{du}\right) + 3\alpha u H \left(\frac{dH}{du}\right) + 2\omega m^2 = 0, \tag{2.106}$$

$$-2\omega\alpha u H \left(\frac{dF_2}{du}\right) + 4\omega F_1 F_2 + 6\omega H F_2 = 0. \tag{2.107}$$

Remembering our perturbative solution to the equation system (2.104)-(2.107);

$$H = H_\infty + H_1 \left(\frac{a_0}{a}\right)^{\alpha} + H_2 \left(\frac{a_0}{a}\right)^{2\alpha} = H_\infty + H_1 u + H_2 u^2 \tag{2.108}$$

$$F_1 = F_{1\infty} + F_{11} \left(\frac{a_0}{a}\right)^{\alpha} + F_{12} \left(\frac{a_0}{a}\right)^{2\alpha} = F_{1\infty} + F_{11} u + F_{12} u^2 \tag{2.109}$$

$$F_2 = F_{2\infty} + F_{21} \left(\frac{a_0}{a}\right)^{\alpha} + F_{22} \left(\frac{a_0}{a}\right)^{2\alpha} = F_{2\infty} + F_{21} u + F_{22} u^2 \tag{2.110}$$

If we substitute (2.108)-(2.110) into (2.104)-(2.107) and keep only the zeroth, first and second order terms of u and neglect higher terms, we get the following equations to be solved:

$$3[H_\infty^2 + (2H_1H_\infty)u + \left(H_1^2 + 2H_\infty H_2\right)u^2] -$$
$$2\omega[F_{1\infty}^2 + (2F_{11}F_{1\infty})u + \left(F_{11}^2 + 2F_{1\infty}F_{12}\right)u^2] +$$
$$2\omega[F_{2\infty}^2 + (2F_{21}F_{2\infty})u + \left(F_{21}^2 + 2F_{2\infty}F_{22}\right)u^2] +$$
$$6[F_{1\infty}H_\infty + (F_{1\infty}H_1 + F_{11}H_\infty)u + (F_{1\infty}H_2 + F_{11}H_1 + F_{12}H_\infty)u^2] - 2\omega m^2 = 0, \tag{2.111}$$

$$3[H_\infty^2 + (2H_1H_\infty)u + \left(H_1^2 + 2H_\infty H_2\right)u^2] +$$
$$(2\omega + 4)[F_{1\infty}^2 + (2F_{11}F_{1\infty})u + \left(F_{11}^2 + 2F_{1\infty}F_{12}\right)u^2]$$
$$-2\omega[F_{2\infty}^2 + (2F_{21}F_{2\infty})u + \left(F_{21}^2 + 2F_{2\infty}F_{22}\right)u^2] +$$
$$4[F_{1\infty}H_\infty + (F_{1\infty}H_1 + F_{11}H_\infty)u + (F_{1\infty}H_2 + F_{11}H_1 + F_{12}H_\infty)u^2] -$$
$$2\alpha u(H_\infty + H_1u + H_2u^2)(F_{11} + 2F_{12}u)$$
$$-2\alpha u(H_\infty + H_1u + H_2u^2)(H_1 + 2H_2u) - 2\omega m^2 = 0, \tag{2.112}$$

$$-6[H_\infty^2 + (2H_1H_\infty)u + \left(H_1^2 + 2H_\infty H_2\right)u^2]$$
$$+2\omega[F_{1\infty}^2 + (2F_{11}F_{1\infty})u + \left(F_{11}^2 + 2F_{1\infty}F_{12}\right)u^2]$$
$$+2\omega[F_{2\infty}^2 + (2F_{21}F_{2\infty})u + \left(F_{21}^2 + 2F_{2\infty}F_{22}\right)u^2] +$$
$$6\omega[F_{1\infty}H_\infty + (F_{1\infty}H_1 + F_{11}H_\infty)u + (F_{1\infty}H_2 + F_{11}H_1 + F_{12}H_\infty)u^2]$$
$$-2\omega\alpha u(H_\infty + H_1u + H_2u^2)(F_{11} + 2F_{12}u)$$
$$+3\alpha u(H_\infty + H_1u + H_2u^2)(H_1 + 2H_2u) + 2\omega m^2 = 0, \tag{2.113}$$

$$-2\omega\alpha u(H_\infty + H_1u + H_2u^2)(F_{21} + 2F_{22}u) +$$
$$4\omega(F_{1\infty}F_{2\infty} + (F_{1\infty}F_{21} + F_{11}F_{2\infty})u + (F_{1\infty}F_{22} + F_{11}F_{21} + F_{12}F_{2\infty})u^2) +$$
$$6\omega(F_{2\infty}H_\infty + (F_{2\infty}H_1 + F_{21}H_\infty)u + (F_{2\infty}H_2 + F_{21}H_1 + F_{22}H_\infty)u^2) = 0. \tag{2.114}$$

Writing (2.111)-(2.114) according to their order of u, we obtain the following equations firstly according to zeroth order of u:

$$3H_{\infty}^{2}-2\omega F_{1\infty}^{2}+2\omega F_{2\infty}^{2}+6F_{1\infty}H_{\infty}-2\omega m^{2}=0, \tag{2.115}$$

$$3H_{\infty}^{2}+(2\omega+4)\,F_{1\infty}^{2}-2\omega F_{2\infty}^{2}+4F_{1\infty}H_{\infty}-2\omega m^{2}=0, \tag{2.116}$$

$$-6H_{\infty}^{2}+2\omega F_{1\infty}^{2}+2\omega F_{2\infty}^{2}+6\omega F_{1\infty}H_{\infty}+2\omega m^{2}=0, \tag{2.117}$$

$$(4\omega F_{1\infty}+6\omega H_{\infty})F_{2\infty}=0. \tag{2.118}$$

secondly according to first order of u:

$$(6F_{1\infty}+6H_{\infty})H_{1}+(6H_{\infty}-4\omega F_{1\infty})F_{11}+4\omega F_{21}F_{2\infty}=0, \tag{2.119}$$

$$[(6-2\alpha)H_{\infty}+4F_{1\infty}]H_{1}+[(4-2\alpha)H_{\infty}+(4\omega+8)\,F_{1\infty}]F_{11}-4\omega F_{21}F_{2\infty}=0, \tag{2.120}$$

$$[(3\alpha-12)H_{\infty}+6\omega F_{1\infty}]H_{1}+((6\omega-2\omega\alpha)H_{\infty}+4\omega F_{1\infty})F_{11}+4\omega F_{21}F_{2\infty}=0, \tag{2.121}$$

$$[(-2\omega\alpha+6\omega)H_{\infty}+4\omega F_{1\infty}]F_{21}+4\omega F_{11}F_{2\infty}+6\omega H_{1}F_{2\infty}=0, \tag{2.122}$$

and lastly according to second order of u:

$$\begin{gathered}3H_{1}^{2}+6H_{\infty}H_{2}-2\omega F_{11}^{2}-4\omega F_{1\infty}F_{12}+4\omega F_{2\infty}F_{22}\\ +2\omega F_{21}^{2}+6F_{1\infty}H_{2}+6F_{11}H_{1}+6F_{12}H_{\infty}=0\end{gathered} \tag{2.123}$$

$$(3-2\alpha)H_1^2+(2\omega+4)\,F_{11}^2+(4-2\alpha)F_{11}H_1+(4F_{1\infty}+6H_\infty-4\alpha H_\infty)H_2 \tag{2.124}$$
$$+[4H_\infty+(4\omega+8)\,F_{1\infty}-4\alpha H_\infty]F_{12}-$$
$$2\omega F_{21}^2-4\omega F_{2\infty}F_{22}=0$$

$$(3\alpha-6)H_1^2+2\omega F_{11}^2+(-2\omega\alpha+6\omega)F_{11}H_1+ \tag{2.125}$$
$$(-12H_\infty+6\alpha H_\infty+6\omega F_{1\infty})H_2+(-4\omega\alpha H_\infty+4\omega F_{1\infty}+6\omega H_\infty)F_{12}+$$
$$2\omega F_{21}^2+4\omega F_{2\infty}F_{22}=0$$

$$(4\omega F_{1\infty}-4\omega\alpha H_\infty+6\omega H_\infty)F_{22}+(-2\omega\alpha H_1+4\omega F_{11}+6\omega H_1)F_{21} \tag{2.126}$$
$$+4\omega F_{2\infty}F_{12}+6\omega F_{2\infty}H_2=0.$$

2.2.2. Solutions

Solving the equation set (2.115)-(2.118) and (2.119-(2.122) respectively, will give us the same solutions as we found in the first part of the thesis;

$$H_\infty=\frac{2\,(\omega+1)\,\sqrt{\omega}m}{\sqrt{(6\omega^2+17\omega+12)}} \tag{2.127}$$

$$F_{1\infty}=\frac{H_\infty}{2\omega+2} \tag{2.128}$$

$$F_{2\infty}=0, \tag{2.129}$$

$$F_{11}=-\frac{3}{2}H_1, \tag{2.130}$$

$$F_{21} = \text{free} - \text{parameter} \tag{2.131}$$

$$\alpha = 3 + \frac{1}{\omega + 1} \simeq 3, \qquad (as \quad \omega \to \infty). \tag{2.132}$$

Continuing with the same procedure applied in the first part, we substitute (2.127)-(2.132) into the equation set (2.123)-(2.126) and this yields to the following equation set to be solved for H_2, F_{12}, F_{21}, and F_{22} as

$$(12\omega + 18)H_\infty H_2 + (8\omega + 12)H_\infty F_{12} + 4\omega(\omega + 1)F_{21}^2 = (9\omega^2 + 21\omega + 12)H_1^2 \tag{2.133}$$

$$-(12\omega + 16)H_\infty H_2 - (12\omega + 16)H_\infty F_{12} - 4\omega(\omega + 1)F_{21}^2 = -(9\omega^2 + 27\omega + 20)H_1^2 \tag{2.134}$$

$$(18\omega + 24)H_\infty H_2 - (12\omega^2 + 16\omega)H_\infty F_{12} + 4\omega(\omega + 1)F_{21}^2 = -(9\omega^2 + 21\omega + 12)H_1^2 \tag{2.135}$$

$$F_{22} = -\frac{H_1}{H_\infty} F_{21}. \tag{2.136}$$

Recall that

$$H_1 = \frac{0.28}{1.44} H_\infty \simeq 0.19 H_\infty. \tag{2.137}$$

Using (2.137), in the solutions to the equation system (2.133)-(2.136) provides;

$$H_2 H_\infty = \frac{9\omega^3 H_1^2 - 4F_{21}^2\omega^3 + 18\omega^2 H_1^2 - 8F_{21}^2\omega^2 - \omega H_1^2 - 4F_{21}^2\omega - 12H_1^2}{12\omega^2 + 34\omega + 24}, \tag{2.138}$$

$$F_{12} H_\infty = \frac{45\omega^2 H_1^2 - 4\omega^2 F_{21}^2 + 123\omega H_1^2 - 4F_{21}^2\omega + 84H_1^2}{24\omega^2 + 68\omega + 48}, \tag{2.139}$$

As $\omega \to \infty$;

$$H_2 \simeq \left(-\frac{1}{3}\omega\frac{F_{21}^2}{H_\infty} + \frac{3}{4}\omega\frac{H_1^2}{H_\infty}\right) \tag{2.140}$$

$$F_{12} \simeq \left(-\frac{4}{24H_\infty}F_{21}^2 + \frac{45}{24H_\infty}H_1^2\right), \tag{2.141}$$

$$F_{22} = -\frac{H_1}{H_\infty}F_{21}, \tag{2.142}$$

2.2.3. Results

As soon as we set $H_2 = 0$, we find that

$$F_{21} \simeq \frac{3}{2}H_1, \tag{2.143}$$

which physically means the anomaly related with the contribution to dark matter via F_{21} is totally eliminated. Substituting F_{21} into (2.141) and (2.142) and using (2.128)-(2.130) and (2.137) simultaneously, enables us to write all perturbation constants in terms of H_∞:

$$H_1 \simeq 0.19H_\infty, \qquad H_2 = 0, \tag{2.144}$$

$$F_{1\infty} = \frac{H_\infty}{2\omega+2}, \qquad F_{11} = -\frac{3}{2}H_1 \simeq -0.28H_\infty, \qquad F_{12} \simeq \frac{3}{2}\frac{H_1^2}{H_\infty} \simeq 0.05H_\infty \tag{2.145}$$

$$F_{2\infty} = 0, \qquad F_{21} \simeq \frac{3}{2}H_1 \simeq 0.28H_\infty, \qquad F_{22} \simeq -\frac{3}{2}\frac{H_1^2}{H_\infty} \simeq -0.05H_\infty \tag{2.146}$$

where

$$H_\infty = \frac{2(\omega+1)\sqrt{\omega}m}{\sqrt{(6\omega^2+17\omega+12)}}. \tag{2.147}$$

Hence, with these perturbation constants we can express (2.108)-(2.110);

$$H = H_\infty + 0.19 H_\infty \left(\frac{a_0}{a}\right)^3 \tag{2.148}$$

$$F_1 = \frac{H_\infty}{2\omega + 2} - 0.28 H_\infty \left(\frac{a_0}{a}\right)^3 + 0.05 H_\infty \left(\frac{a_0}{a}\right)^6 \tag{2.149}$$

$$F_2 = 0.28 H_\infty \left(\frac{a_0}{a}\right)^3 - 0.05 H_\infty \left(\frac{a_0}{a}\right)^6 \tag{2.150}$$

where

$$H_\infty \simeq 0.84 H_0 \tag{2.151}$$

if (2.148) is satisfied for $H = H_0$, and H_0 is the present value of the Hubble parameter [4].

3. DISCUSSION AND CONCLUSION

In this thesis, we tried to analyze the dark matter (Ω_{DM}) and the dark energy contribution (Ω_Λ) to Friedmann Equation (2.61) solely by BD theory of gravitation with no other input. As far as we know, the scalar field ϕ was always examined individually however we brought forward a new idea such that it can have different components and each of these components can account for different energy densities. First, we have attempted to use a complex scalar field ϕ, such as $\phi = \phi_1 + i\phi_2$. After we obtain field equations in terms of H, F_1 and F_2, we put up the argument of perturbative solutions with the constant terms H_∞, H_1, H_2, $F_{1\infty}$, F_{11}, F_{12}, $F_{2\infty}$, F_{21} and F_{22}. All of these constants have made it possible to originate new predictions on dark matter and dark energy contribution of BD theory. Similarly, the term H_∞ which has no scale factor term, just like the energy density term due to the cosmological constant Ω_Λ in the Friedman Equation, was found purely from theory;

$$H_\infty = \frac{2(\omega+1)\sqrt{\omega}m}{\sqrt{(6\omega^2+17\omega+12)}}. \tag{3.1}$$

For this reason, using a complex ϕ field has revealed meaningful solutions for H_∞, H_1, $F_{1\infty}$, F_{11} and $F_{2\infty}$. More importantly, in the asymptotic limit $\omega \to \infty$ (where BD approaches Einstein theory), α was found 3, as it appears in the Friedman Equation in the form $\Omega_M \left(\frac{a_0}{a}\right)^3$. However, the solution of F_{21} was found non-physical.Therefore, although this approach has brought brand new considerations and aspects to Friedman Equation in the concept of dark matter and dark energy, a more suitable solution was found with the modification of scalar field by using a hyperbolic phase $i\beta = \Psi$. This method has canceled out the anomaly in F_{21} and made us able to find all perturbation constants. If we rewrite our solutions with obtained values;

$$H = H_\infty + 0.19H_\infty \left(\frac{a_0}{a}\right)^3 \tag{3.2}$$

$$F_1 = \frac{H_\infty}{2\omega + 2} - 0.28H_\infty \left(\frac{a_0}{a}\right)^3 + 0.05H_\infty \left(\frac{a_0}{a}\right)^6 \tag{3.3}$$

$$F_2 = 0.28H_\infty \left(\frac{a_0}{a}\right)^3 - 0.05H_\infty \left(\frac{a_0}{a}\right)^6 \tag{3.4}$$

We see that in equation (3.2), the second term is smaller than the first one and the third term is smaller than the second one. Namely, the dominating term is the first one which can be predicted to be the contribution to dark energy. The second term can be interpreted as the contribution of dark matter. However, the situation is different for F_1 and F_2 equations. Looking for different contributions from different components, it is agreeable to predict that while $F_{1\infty}$ is contributing to dark energy, F_{21} is contributing to dark matter.

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