ALMOST CUBIC NONLINEAR SCHRÖDINGER EQUATION: EXISTENCE, UNIQUENESS AND SCATTERING

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ABSTRACT

ALMOST CUBIC NONLINEAR SCHRÖDINGER EQUATION: EXISTENCE, UNIQUENESS AND SCATTERING

In this thesis, a unified treatment is given for a class of nonlinear non-local 2D elliptic and hyperbolic Schrödinger equation which includes the 2D nonlinear Schrödinger (NLS) equation with a purely cubic nonlinearity, Davey-Stewartson (DS) system in the hyperbolic-elliptic (HE) and elliptic-elliptic (EE) cases and the generalized Davey-Stewartson (GDS) system in the hyperbolic-elliptic-elliptic (HEE) and elliptic-elliptic-elliptic (EEE) cases. Local in time existence and uniqueness of solutions are established for the Cauchy problem when initial data is in $L^2(\mathbb{R}^2)$, $H^1(\mathbb{R}^2)$, $H^2(\mathbb{R}^2)$ and in $\Sigma = H^1(\mathbb{R}^2) \cap L^2(|\boldsymbol{x}|^2 \, \mathrm{d}\boldsymbol{x})$ and the maximal time of existence for the solutions all agree. Conserved quantities corresponding to mass, momentum, energy are derived, as well as scale and pseudo-conformal invariance of solutions. Virial identity is also established and its relation to pseudo-conformal invariance is discussed. Various routes to global existence of solutions are also explored in the elliptic case, namely, for small mass solutions in $L^2(\mathbb{R}^2)$; in the defocusing case for solutions in $H^1(\mathbb{R}^2)$ and finally in the focusing case for $H^1(\mathbb{R}^2)$-solutions with subminimal mass. In all such cases the scattering of such solutions in $L^2(\mathbb{R}^2)$ and Σ topologies are discussed. Moreover, in the focusing case when initial energy is negative, it is shown that solutions in Σ blow-up. The existence and uniqueness results are also considered for more general nonlinearities.

ÖZET

NERDEYSE KÜBİK DOĞRUSAL OLMAYAN SCHRÖDINGER DENKLEMİ: VARLIK, TEKLİK VE SAÇILMA

Bu tezde yerel olmayan terim içeren doğrusal olmayan iki boyutlu bir sınıf Schrödinger denklemi incelenmektedir. Bu denklem sınıfı kübik doğrusal olmayan Schrödinger denklemini içerdiği gibi indirgenmiş haliyle Davey-Stewartson (DS) sisteminin hiperbolik-eliptik (HE) ve eliptik-eliptik (EE) durumlarıyla genelleştirilmiş Davey-Stewartson (GDS) sisteminin hiperbolik-eliptik-eliptik (HEE) ve eliptik-eliptik-eliptik (EEE) durumlarını da içerdiğinden sunulan neticeler bu denklemler için de geçerlidir. $L^2(\mathbb{R}^2)$, $H^1(\mathbb{R}^2)$, $H^2(\mathbb{R}^2)$ ve $\Sigma = H^1(\mathbb{R}^2) \cap L^2(|\boldsymbol{x}|^2\, \mathrm{d}\boldsymbol{x})$ gibi fonksiyon uzaylarinda zamana göre başlangıç sınır değer probleminin varlık ve teklik neticeleri verilmekle birlikte sözü edilen çözümlerin maksimal varlık zamanlarının aynı olduğu gözlenmektedir. Kütle , momentum, enerji gibi büyüklüklerin korunduğu, viryal özdeşliğin, sözde-konformal ve ölçek dönüşümlerinin denklem için geçerli olduğu gösterilmektedir. $L^2(\mathbb{R}^2)$ uzayındaki küçük başlangıç değerli çözümün, yoğunlaşmama durumundaki $H^1(\mathbb{R}^2)$ çözümünün ve yoğunlaşma durumunda kritik kütle altında bir kütleye sahip başlangıç değerli çözümün global varlık neticeleri gösterilmekte, bu gibi durumlarda $L^2(\mathbb{R}^2)$ ve Σ topolojilerinde çözümlerin saçılması ve son olarak benzer tipte daha genel doğrusal olmayan terimler için ifade edilen neticelerin geçerliliği incelenmektedir.

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LIST OF SYMBOLS

$C(E;F)$	space of continuous functions from the topological space E to the topological space F
$C^k(\Omega)$	space of continuous functions $u:\Omega\to\mathbb{C}$ such that $D^\alpha u\in C(\Omega)$ for all $\alpha\in\mathbb{N}^n$ with $\lvert\alpha\rvert\le k$
D^α	$\dfrac{\partial^{\lvert\alpha\rvert}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$ for $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$ and $\lvert\alpha\rvert=\alpha_1+\dots+\alpha_n$
$\mathcal{D}$	denotes $\mathcal{D}(\mathbb{R}^n)$, space of infinitely differentiable compactly supported functions
$\mathcal{D}(I;B)$	space of C^∞ functions $I\to B$ compactly supported in I, where $I\subset\mathbb{R}$ is open and B is any Banach space
$\mathcal{D}'(I;B)$	space of linear continuous mappings $\mathcal{D}(I)\to B$
$E\hookrightarrow F$	E is continuously imbedded in F
$\hat{f}$	Fourier transform of f, defined to be $\hat{f}(\boldsymbol{\xi})=\int_{\mathbb{R}^n}f(\boldsymbol{x})e^{2\pi i\boldsymbol{x}\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{x}$ for $f\in L^1(\mathbb{R}^n)$
H^k	$W^{k,2}$ for $k\in\mathbb{Z}$
$H^s(\mathbb{R}^n)$	Banach space of elements $u\in\mathcal{S}'(\mathbb{R}^n)$ such that $(1+\lvert\xi\rvert^2)^{s/2}\hat{u}\in L^2(\mathbb{R}^n)$ for $s\in\mathbb{R}$. $H^s(\mathbb{R}^n)$ is equipped with the norm $$\lVert u\rVert_{H^s}=\lVert(1+\lvert\xi\rvert^2)^{s/2}\hat{u}\rVert_2$$
$\mathrm{Im}\ z$	imaginary part of z
$J\Subset I\subset\mathbb{R}$	$\bar{J}$ is a compact subset of I
$L^p(\Omega)$	Banach space of classes of measurable functions $u:\Omega\to\mathbb{C}$ such that $\lVert u\rVert_p<\infty$, where $$\lVert u\rVert_p=\begin{cases}\left(\int_\Omega\lvert u(\boldsymbol{x})\rvert\mathrm{d}\boldsymbol{x}\right)^{1/p}, & \text{if } p<\infty\\ \text{ess sup}_\Omega\lvert u\rvert, & \text{if } p=\infty\end{cases}$$
L^p	$L^p(\mathbb{R}^2)$
$L^p(I;B)$	Banach space of classes of measurable functions $u:I\to B$ such that $\lVert u(t)\rVert_B\in L^p(I)$ where $I\subset\mathbb{R}$

p'	conjugate of $p \geq 1$ given by $\frac{1}{p} + \frac{1}{p'} = 1$
Re z	real part of z
$\mathcal{S}$	denotes $\mathcal{S}(\mathbb{R}^n)$, Schwartz space i.e. functions $u : \mathbb{R}^n \to \mathbb{C}$ such that $\sup_{\boldsymbol{x}\in\mathbb{R}^n} \lvert \boldsymbol{x}^\beta D^\alpha u(\boldsymbol{x})\rvert < \infty$ for all $\alpha, \beta \in \mathbb{N}^n$
$\mathcal{S}'$	denotes tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, topological dual of $\mathcal{S}(\mathbb{R}^n)$
u_r	$\frac{\partial u}{\partial r} = \frac{1}{r}\boldsymbol{x} \cdot \nabla u$
u_t	partial derivative of $u(t, \boldsymbol{x})$ with respect to t
u_{x_k}	$\frac{\partial u}{\partial x_k}$
$\boldsymbol{u} \in L^p(I; L^q)$	$u_k \in L^p(I; L^q)$ for each component u_k of $\boldsymbol{u}$
$W^{m,p}(\mathbb{R}^n)$	Banach space of elements $u \in \mathcal{S}'(\mathbb{R}^n)$ such that for $\lvert\alpha\rvert \leq m \in \mathbb{N}$, $D^\alpha u$, as an element of $\mathcal{S}'(\mathbb{R}^n)$, is generated by a function in $L^p(\mathbb{R}^n)$. $W^{m,p}(\mathbb{R}^n)$ is considered with one of the equivalent norms (giving the same topology) $$\lVert u\rVert_{W^{m,p}} = \left(\sum_{\lvert\alpha\rvert\leq m} \lVert D^\alpha u\rVert_p^p\right)^{1/p}$$
$W^{-m,p'}(\mathbb{R}^n)$	dual of $W^{m,p}(\mathbb{R}^n)$
$W^{k,p}$	$W^{k,p}(\mathbb{R}^2)$ for $k \in \mathbb{Z}$
$W^{m,p}(I; B)$	Banach space of classes of measurable functions $u : I \to B$ such that $\frac{d^k u}{dt^k} \in L^p(I; B)$ where B is any Banach space
$x \vee y$	$\max\{x, y\}$
$x \wedge y$	$\min\{x, y\}$
$\boldsymbol{x}$	denotes $(x, y) \in \mathbb{R}^2$
$\lVert \cdot \rVert_{p,q}$	norm in $L^p(I; L^q)$ when $I = [0, T]$, $T < \infty$
$\langle \cdot, \cdot \rangle_{-1,1}$	duality pairing between H^{-1} and H^1
∇u	gradient of u, $(u_{x_1}, \ldots, u_{x_n})$ in $\mathbb{R}^n$
Δu	$\sum_{k=1}^n u_{x_k x_k}$ in $\mathbb{R}^n$

1. INTRODUCTION

In one of the two basic approaches related to the existence and uniqueness theory for the initial value problem

$$\begin{aligned} iu_t + \Delta u &= g(u) \\ u(0) &= \varphi, \end{aligned} \tag{1.0.1}$$

where g is of either local or nonlocal nature with certain assumptions such as

$$\operatorname{Im}(g(u)\bar{u}) = 0 \text{ a.e. on } \mathbb{R}^n,$$
$$g = G' \text{ for some } G \in C^1(H^1(\mathbb{R}^n); \mathbb{R}),$$

the Cauchy problem (1.0.1) is considered in terms of energy techniques in the spaces where the energy

$$E(u) = \|\nabla u\|_2^2 + 2G(u)$$

is defined. From the formal computations similar to done in Chapter 2 for a model nonlinearity, it can be seen that the first assumption is in general to obtain mass conservation and the second is fundamental for the arguments where energy conservation is used. Instead of dealing with (1.0.1) one considers approximate equations in which nonlinearity is expected to satisfy finer local Lipchitz properties than g. The approximate equations are obtained from the original via different regularization techniques and conservation laws are derived for the approximate solutions. By utilizing these, we try to obtain a solution as a limit of approximate solutions. Then it is possible to show that the limit function also satisfies the conservation laws corresponding to mass and energy. In this method conserved quantities are essential for the arguments leading to the existence of solutions (see e.g. [1, Theorem 3.3.5] for a class of nonlinearities to be considered below).

Another approach, mainly due to Kato, [2], uses fixed point arguments utilizing Strichartz's estimates, Theorem A.2.1. In this method one may relax the above conditions on the nonlinearity, which is necessary to prove conservations and still obtain existence and uniqueness results since arguments do not require the conserved quantities.

We will consider some typical nonlinearities, that can be seen in the literature, to which the above mentioned techniques are applicable.

1.1. Typical Nonlinearities

The local nonlinearity. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be such that $f(\boldsymbol{x}, u)$ is measurable in $\boldsymbol{x}$, continuous in u and $f(\boldsymbol{x}, 0) = 0$ for a.a. $\boldsymbol{x} \in \mathbb{R}^n$. Assume also that for every $M > 0$ there exists $L(M) < \infty$ such that

$$|f(\boldsymbol{x}, u) - f(\boldsymbol{x}, v)| \leq L(M)|u - v|$$

for a.a. $\boldsymbol{x} \in \mathbb{R}^n$ and all u, v with $|u|, |v| \leq M$ where $L \in C([0, \infty))$ if $n = 1$ and $L(t) \leq Ct^\alpha$ with $0 \leq \alpha < 4/(n-2)$ if $n \geq 2$. Define F by

$$F(\boldsymbol{x}, u) = \int_0^u f(\boldsymbol{x}, s)\mathrm{d}s$$

for all $u \geq 0$. One can extend f as $\dfrac{u}{|u|} f(\boldsymbol{x}, |u|)$ for all $u \in \mathbb{C}, u \neq 0$. The nonlinearity is given by

$$g(u)(\boldsymbol{x}) = f(\boldsymbol{x}, u(\boldsymbol{x}))$$

a.e. on $\mathbb{R}^n$ for all measurable $u : \mathbb{R}^n \to \mathbb{C}$. We also need

$$G(u) = \int_{\mathbb{R}^n} F(\boldsymbol{x}, |u(\boldsymbol{x})|)\mathrm{d}\boldsymbol{x}$$

which is defined for all $u : \mathbb{R}^n \to \mathbb{C}$ such that $F(\cdot, |u(\cdot)|) \in L^1(\mathbb{R}^n)$.

An immediate example of such kind of local nonlinearities is the pure power type nonlinearity where $f(\boldsymbol{x}, u) = \lambda |u|^\alpha u$ for all $\boldsymbol{x} \in \mathbb{R}^n$ with $0 \le \alpha < 4/(n-2)$ ($0 \le \alpha < \infty$ if $n = 1$) and λ is a constant.

The Hartree nonlinearity. Let W be an even, real valued function in $L^p(\mathbb{R}^n)$ for some $p \ge 1$ and $p > n/4$. We consider the nonlocal nonlinearity

$$g(u) = (W * |u|^2)u$$

for all measurable $u : \mathbb{R}^n \to \mathbb{C}$ such that $W * |u|^2$ is measurable and define G as

$$G(u) = \frac{1}{4} \int_{\mathbb{R}^n} (W * |u|^2)(\boldsymbol{x}) |u(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x}$$

for all measurable u such that $(W * |u|^2)|u|^2$ is integrable.

The key properties for these nonlinearities which allow a common treatment as described above can be given in the following proposition ([1, Propositions 3.2.5, 3.2.9]):

Proposition 1.1 *Let $g(u)$ be the local or the Hartree nonlinearity as given above. Let $r = \alpha + 2$ if $n \ge 2$ ($r = 2$ if n=1) in the case of local nonlinearity and $r = 4p/(2p-1)$ in the case of Hartree nonlinearity. Then for g and the corresponding potential G, the following holds:*

(i) $G \in C(H^1(\mathbb{R}^n; \mathbb{R}))$, $g \in C(H^1(\mathbb{R}^n); H^{-1}(\mathbb{R}^n)))$ *and* $G' = g$.

(ii) $g \in C(L^r(\mathbb{R}^n); L^{r'}(\mathbb{R}^n))$.

(iii) For all $M > 0$ there exists $C(M) < \infty$ such that $\|g(u) - g(v)\|_{r'} \le C(M)\|u - v\|_r$ for all $u, v \in H^1(\mathbb{R}^n)$ with $H^1(\mathbb{R}^n)$-norms $\le M$.

(iv) Im$(g(u)\bar{u}) = 0$ a.e in $\mathbb{R}^n$ for all $u \in H^1(\mathbb{R}^n)$.

In order to obtain the continuity properties above one mainly uses the estimate

$$\|g(u) - g(v)\|_{r'} \leq C(\|u\|_r^\alpha + \|v\|_r^\alpha)\|u - v\|_r$$

in the case of local nonlinearity with C is independent of u and v. We use the same type of estimate for the Hartree nonlinearity where $\alpha = 2$, C depends on $\|W\|_p$ and it can be derived by using

$$\|(W * (uv))w\|_{r'} \leq \|W\|_p\|u\|_r\|v\|_r\|w\|_r$$
$$\int_{\mathbb{R}^n} (W * (uv))wz\mathrm{d}\boldsymbol{x} \leq \|W\|_p\|u\|_r\|v\|_r\|w\|_r\|z\|_r$$

which are obtained by Hölder and Young inequalities. In the next section we will consider a new nonlocal nonlinearity satisfying similar properties mentioned above.

1.2. A Nonlocal Nonlinearity

In this thesis, the Cauchy problem

$$\begin{aligned} &iu_t + \delta u_{xx} + u_{yy} = K(|u|^2)u, \quad \delta = \pm 1, \\ &u(0) = \varphi, \end{aligned} \tag{1.2.1}$$

where $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ and the non-local term K is defined on L^2 by

$$\widehat{K(f)}(\boldsymbol{\xi}) = \alpha(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi}), \tag{1.2.2}$$

for $\boldsymbol{\xi} \in \mathbb{R}^2$ and the symbol α is assumed to satisfy

(H1) α is even and homogenous of degree zero,
(H2) $\alpha \in C^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$,

is considered in terms of the fixed point arguments and their extensions which is also applicable to $\delta = -1$. This is partly because we want to consider the case $\delta = -1$. We also want to avoid complications in the energy techniques pointed out above related to the nonlocal nature of the nonlinearity. We call $\delta = 1$ the elliptic and $\delta = -1$ the hyperbolic case.

We have $\|K(f)\|_2 \leq \|\alpha\|_\infty \|f\|_2$ which implies that K is a bounded linear operator on L^2. Although there exists a corresponding kernel $W \in C^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$ which is the Fourier inverse transform of α in the sense of distributions and W is homogeneous of degree -2 ([3, Proposition 5.2]), we may not have sufficient information about the integrability properties of W which is not the case in the Hartree nonlinearity. But L^2-boundedness of K will suffice to obtain the estimates similar to what we have stated for the Hartree nonlinearity. We get

$$\|K(uv)w\|_{4/3} \leq \|K(uv)\|_2 \|w\|_4 \leq \|\alpha\|_\infty \|u\|_4 \|v\|_4 \|w\|_4$$

using Hölder pairs $(3/2, 3)$ and $(2, 2)$ in order. Similarly we obtain

$$\int_{\mathbb{R}^2} (K(uv)) wz \, \mathrm{d}\boldsymbol{x} \leq \|\alpha\|_\infty \|u\|_4 \|v\|_4 \|w\|_4 \|z\|_4.$$

By using these estimates and $K(|u|^2)u - K(|v|^2)v = K(|u|^2)(u - v) + v(K(|u|^2 - |v|^2))$ we deduce that $\|K(|u|^2)u - K(|v|^2)v\|_{4/3} \leq C(\|u\|_4^2 + \|v\|_4^2)\|u - v\|_4$ where C depends on $\|\alpha\|_\infty$. This in turn implies $H \in C(L^4; L^{4/3})$ and so it is in $C(H^1; H^{-1})$ where $H(u) = K(|u|^2)u$. Also since α is even $K(f)$ is real valued if f is real valued and this gives the fact that $\mathrm{Im}\,(H(u)\bar{u})$=0. Finally if we define $G(u) = \dfrac{1}{4}\displaystyle\int_{\mathbb{R}^2} K(|u|^2)|u|^2 \, \mathrm{d}\boldsymbol{x}$ then again by the estimates obtained above we get

$$\begin{aligned} &\left| G(u+v) - G(u) - \mathrm{Re} \int_{\mathbb{R}^2} K(|u|^2) u\bar{v} \right| \\ &\qquad = \left| \int_{\mathbb{R}^2} K(|v|^2) \left(\frac{|u|^2}{2} + \frac{|v|^2}{4} + \mathrm{Re}(u\bar{v}) \right) + \int_{\mathbb{R}^2} K(\mathrm{Re}(u\bar{v}))\mathrm{Re}(u\bar{v}) \right| \\ &\qquad \leq C(\|u\|_4^2 + \|v\|_4^2)\|v\|_4^2 \end{aligned}$$

C depending on $\|\alpha\|_\infty$. With these, Proposition 1.1 is true for $g = H$ with $r = 4$, $n = 2$. We will consider a generalized form of (1.2.1) in Chapter 7.

1.3. Motivation

The motivation for this work comes from the Generalized Davey-Stewartson (GDS) system that was derived in Babaoğlu and Erbay [4] and is given by

$$\begin{aligned} iu_t + \delta u_{xx} + u_{yy} &= \chi |u|^2 u + b(\phi_{1,x} + \phi_{2,y})u, \\ \phi_{1,xx} + m_2\phi_{1,yy} + n\phi_{2,xy} &= \left(|u|^2\right)_x, \\ \lambda\phi_{2,xx} + m_1\phi_{2,yy} + n\phi_{1,xy} &= \left(|u|^2\right)_y, \end{aligned} \tag{1.3.1}$$

where the corresponding symbol α is

$$\alpha(\boldsymbol{\xi}) = \chi + b\frac{\lambda\xi_1^4 + (1 + m_1 - 2n)\xi_1^2\xi_2^2 + m_2\xi_2^4}{\lambda\xi_1^4 + (m_1 + \lambda m_2 - n^2)\xi_1^2\xi_2^2 + m_1 m_2\xi_2^4}. \tag{1.3.2}$$

The system is classified as elliptic-elliptic-elliptic (EEE) and hyperbolic-elliptic-elliptic (HEE) when the corresponding signs of the parameters $(\delta, m_1, m_2, \lambda)$ are $(+, +, +, +)$ and $(-, +, +, +)$ respectively. In [5] the conserved quantities corresponding to mass, momentum and energy were derived for smooth solutions that decay to zero at infinity. A global existence result was given for the Cauchy problem in the defocusing (the case in which all H^1-solutions are global and bounded in H^1) EEE case. An argument was outlined for the local in time existence result for H^1-solutions for the Cauchy problem. Moreover, a virial identity was derived and was utilized to show the non-existence of global solutions. However, the existence and uniqueness of solutions in $\Sigma = H^1 \cap L^2(|\boldsymbol{x}|^2\, \mathrm{d}\boldsymbol{x})$ where the virial identity is meaningful was not addressed. Another conserved quantity related to the pseudo-conformal invariance of solutions of the GDS system was derived in Eden, Erbay and Muslu, [6] and was utilized for studying the time asymptotics of solutions. There were two types of results there, in the HEE case, a specific blow-up profile was found in the spirit of Ozawa's work [7] for the DS system. In the EEE case, using an argument of Weinstein [8] it was shown that the L^p-norms of the solutions decay to zero in time for $p > 2$. The blow-up profile that is considered in [6] only belongs to L^2 but not in H^1 so an existence and uniqueness of solutions in L^2 for the HEE problem was also needed.

A form for the non-local nonlinearity given in (1.2.1) was suggested in Eden and Erbay, [9] where the existence of standing wave solutions was established in the focusing (in which case there exists Σ-solutions blowing-up in finite time) EEE case. The situation is different for the HEE case, the existence of traveling wave solutions was considered in [10], necessary conditions were derived using Pohozaev type identities. These identities are valid for H^1-solutions in the HEE case. In [9] another type of global existence result was established for solution with small initial mass. These results were improved in Eden and Topaloğlu, [11] under stronger assumptions on the structure of the non-local term. The results on the global existence and non-existence given in [5] was improved in [12] by a more careful analysis of the non-local term. All of the above mentioned works have implicitly or explicitly assumed the existence and uniqueness results in various function spaces for the Cauchy problem (1.2.1) with α as in (1.3.2) as well as the validity of the conserved quantities in these spaces.

The analysis of the cubic nonlinear Scrödinger (NLS) equation in [2] and of the Davey-Stewartson (DS) system in [13] and the GDS system in [5] shares striking similarities which leads us to treat all these cases under a unified framework as described by (1.2.1) and (1.2.2). The class of equations that can be considered under the present framework also include some cases of Zakharov-Schulman equations (see (4.9) and (4.13) in [14]). In [15], p. 138, Figure 3, an outline of an argument is given for 2D cubic NLS equation. The main observation is that the nonlinearity considered there and in [2] need not be local and only the operator theoretic properties between various mixed space-time function spaces play a role. The nature of this nonlinearity was already implicit in [16, 17, 13] and is made explicit here by considering (1.2.2).

The aim of this work is three fold: the first is to lay the proper mathematical foundations for various results on global existence and blow-up of solutions for the GDS system in the EEE case; the second is to obtain scattering results for solutions and the third is to extend the range of applicability of the existence and uniqueness theorems on nonlinear Schrödiger (NLS) equations that are given by Kato in [2] to NLS equations with non-local terms as given above as well as the results of Ghidaglia and Saut on DS system [13]. Throughout the text the results are stated for the general Cauchy problem (1.2.1) and implications are considered in the conclusion. In Chapter 2 the validity of the conservation laws, virial identity, scale and pseudo-conformal invariance on a formal level, i.e. assuming the existence of solutions for the Cauchy problem on the appropriate function spaces, are discussed. In the third chapter, the Cauchy problem in L^2 are considered. As in the standard NLS equation in two space dimensions with cubic nonlinearity this is the critical scaling space for existence. The main theorem in this chapter, Theorem 3.4, establishes the existence of a unique maximal solution that satisfies conservation of mass. In Chapter 4, we follow the steps described in [5] in conjunction with the proof of Theorem 2.2 in [13] to establish a local in time existence theorem in H^1 with maximal interval of existence $[0, T^*)$ (Theorem 4.5). On this space, one also has the conservation of energy for the solutions and as a corollary a generalized version (Corollary 4.6) of [5, Theorem 5.1], on global existence of H^1-solutions in the defocusing case, is obtained.

In contrast to the global existence results, global-nonexistence results usually require the solutions to be in Σ. In the fifth chapter, the argument given in [13] for DS system is followed to deduce an existence and uniqueness result for the Cauchy problem in Σ (Theorem 5.2). This leads to a generalized version (Corollary 5.3) of [5, Theorem 6.1] on the blow-up of solutions for the Cauchy problem for data with negative energy. The other known route to blow-up passes from the pseudo-conformal invariance of solutions. Combining the existence of a ground state ([9, 11]) with the pseudo-conformal transformation a simple blow-up solution with minimal mass can be obtained. Moreover, the existence of the pseudo-conformal invariance of solutions allows us to deduce time-asymptotics for the L^p-norms of solutions for $p > 2$. In the sixth chapter as a prelude to the scattering result we verify this time asymptotics, extending the result given in [6] to the present setting. This asymptotic behaviour is valid in the EEE defocusing case where the global existence of solutions has been established. This paves the way for the scattering of solutions very much in the sprit of Tsutsumi and Yajima [18] (see also Cazenave [19, 1]). Chapter 7 deals with H^2-regularity of H^1-solutions. When $\delta = 1$, it is proved in Theorem 7.7 that the maximal time for existence of H^2-solutions agree with L^2 and H^1-solutions. We also consider a pure power nonlinearity of the form $|u|^{p-1}u$ with $p > 2$ and the corresponding non-local term and discuss to what extent the results obtained in the previous chapters remain valid. In conclusion, we summarize the impact of our results as they pertain to the previous work. All these results are to appear in [20].

2. CONSERVATION LAWS AND OTHER INVARIANTS

The quantities corresponding to mass, momentum and energy for $(1.2.1)_1$ are given by

$$M(u) = \int_{\mathbb{R}^2} |u|^2 \, \mathrm{d}x \, \mathrm{d}y, \tag{2.1}$$

$$J_x(u) = \int_{\mathbb{R}^2} (u\bar{u}_x - \bar{u}u_x) \, \mathrm{d}x \, \mathrm{d}y, \quad J_y(u) = \int_{\mathbb{R}^2} (u\bar{u}_y - \bar{u}u_y) \, \mathrm{d}x \, \mathrm{d}y, \tag{2.2}$$

$$E(u) = \int_{\mathbb{R}^2} \left[\delta|u_x|^2 + |u_y|^2 + \frac{1}{2} K(|u|^2)|u|^2 \right] \, \mathrm{d}x \, \mathrm{d}y. \tag{2.3}$$

Throughout this exposition the dependence of functions u on x, y and t will be assumed but will be suppressed for ease of notation. Let us show that these quantities are actually conserved for sufficiently smooth solutions which vanish at infinity. Multiplying $(1.2.1)_1$ by $2\bar{u}$ and considering imaginary parts we obtain

$$(|u|^2)_t + 2\mathrm{Im}[\delta(u_x\bar{u})_x + (u_y\bar{u})_y] = 0, \tag{2.4}$$

which implies the conservation of mass (2.1). Multiplying $(1.2.1)_1$ by $\bar{u}_x$ gives

$$iu_t\bar{u}_x + \bar{u}_x(\delta u_{xx} + u_{yy}) = K(|u|^2)u\bar{u}_x. \tag{2.5}$$

Next, adding (2.5) and its complex conjugate, we obtain

$$i(u_t\bar{u}_x - \bar{u}_t u_x) = K(|u|^2)(|u|^2)_x - 2\mathrm{Re}\,\bar{u}_x(\delta u_{xx} + u_{yy}). \tag{2.6}$$

Let $f := |u|^2$, then

$$\int_{\mathbb{R}^2} K(f)f_x \, \mathrm{d}x \, \mathrm{d}y = \mathrm{Re} \int_{\mathbb{R}^2} \alpha(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi})\overline{\widehat{(f_x)}}(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi} = \frac{1}{2} \int_{\mathbb{R}^2} (K(f)f)_x \, \mathrm{d}x \, \mathrm{d}y = 0,$$

by using Parseval identity and the fact that α is even. For the second term on the righthand side of (2.6),

$$\int_{\mathbb{R}^2} 2\mathrm{Re}\, \bar{u}_x(\delta u_{xx} + u_{yy})\, \mathrm{d}x\, \mathrm{d}y = \int_{\mathbb{R}^2} \{\delta(|u_x|^2)_x + (|u_y|^2)_x\}\, \mathrm{d}x\, \mathrm{d}y = 0.$$

As a result we obtain the conservation of momentum J_x and similarly for J_y. For the conservation of energy, multiplying $(1.2.1)_1$ by $2\bar{u}_t$ and then taking real parts give $2\mathrm{Re}\, \bar{u}_t(\delta u_{xx} + u_{yy}) = K(|u|^2)(|u|^2)_t$ from which it follows that

$$\begin{aligned} 0 &= \frac{d}{dt} \int_{\mathbb{R}^2} (\delta |u_x|^2 + |u_y|^2)\, \mathrm{d}x\, \mathrm{d}y + \mathrm{Re} \int_{\mathbb{R}^2} \alpha(\boldsymbol{\xi}) \hat{f}(\boldsymbol{\xi}) \overline{\widehat{(f_t)}}(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} \\ &= \frac{d}{dt} \left[\int_{\mathbb{R}^2} (\delta |u_x|^2 + |u_y|^2)\, \mathrm{d}x\, \mathrm{d}y + \frac{1}{2} \int_{\mathbb{R}^2} K(|u|^2)|u|^2\, \mathrm{d}x\, \mathrm{d}y \right], \end{aligned}$$

again by Parseval identity and since α is even. As long as solutions remain in H^1 this quantity makes sense. The following identity, known as the virial identity, will play a crucial role in the blow-up result, see Corollary 5.3,

$$\frac{d^2 I}{dt^2} = 8E(u), \tag{2.7}$$

where I is defined as

$$I = \int_{\mathbb{R}^2} (\delta x^2 + y^2)|u|^2\, \mathrm{d}x\, \mathrm{d}y. \tag{2.8}$$

In order to show that this identity holds for $\delta = \pm 1$, let us write (2.4) as

$$(|u|^2)_t = -i\{\delta(u\bar{u}_x - \bar{u}u_x)_x + (u\bar{u}_y - \bar{u}u_y)_y\},$$

and using

$$\begin{aligned} \frac{dI}{dt} &= 2i \int_{\mathbb{R}^2} \{\delta^2 x(u\bar{u}_x - \bar{u}u_x) + y(u\bar{u}_y - \bar{u}u_y)\}\, \mathrm{d}x\, \mathrm{d}y \\ &= 4\mathrm{Im} \int_{\mathbb{R}^2} (x\bar{u}u_x + y\bar{u}u_y)\, \mathrm{d}x\, \mathrm{d}y, \end{aligned} \tag{2.9}$$

we obtain

$$\frac{d^2 I}{dt^2} = 4\mathrm{Im} \int_{\mathbb{R}^2} \{x(u_{xt}\bar{u} + u_x \bar{u}_t) + y(u_{yt}\bar{u} + u_y\bar{u}_t)\}\, \mathrm{d}x\, \mathrm{d}y.$$

The identity $\int_{\mathbb{R}^2} x u_{xt}\bar{u}\, \mathrm{d}x\, \mathrm{d}y = -\int_{\mathbb{R}^2} u_t(\bar{u} + x\bar{u}_x)\, \mathrm{d}x\, \mathrm{d}y$ and the corresponding integral relation in y results in

$$\frac{d^2 I}{dt^2} = -8\mathrm{Im} \int_{\mathbb{R}^2} (x\bar{u}_x u_t + y\bar{u}_y u_t + \bar{u}u_t)\, \mathrm{d}x\, \mathrm{d}y. \tag{2.10}$$

Next, using the equation $(1.2.1)_1$, we have

$$\begin{aligned}
\frac{d^2 I}{dt^2} &= 8\mathrm{Re} \int_{\mathbb{R}^2} (K(|u|^2)u - \delta u_{xx} - u_{yy})(x\bar{u}_x + y\bar{u}_y + \bar{u})\, \mathrm{d}\boldsymbol{x} \\
&= 8\left\{ \int_{\mathbb{R}^2} \left[\frac{1}{2}K(|u|^2)(\boldsymbol{x}\cdot\nabla)|u|^2 + K(|u|^2)|u|^2\right] \mathrm{d}\boldsymbol{x} \right. \\
&\qquad \left. -\mathrm{Re} \int_{\mathbb{R}^2} (\delta u_{xx} + u_{yy})(\boldsymbol{x}\cdot\nabla)\bar{u}\, \mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^2} (\delta|u_x|^2 + |u_y|^2)\, \mathrm{d}\boldsymbol{x} \right\} \\
&= 8\int_{\mathbb{R}^2} (\delta|u_x|^2 + |u_y|^2)\, \mathrm{d}\boldsymbol{x} + 4\int_{\mathbb{R}^2} [2K(|u|^2)|u|^2 + K(|u|^2)(\boldsymbol{x}\cdot\nabla)|u|^2]\, \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

So in order to prove (2.7), we need to show that $\int_{\mathbb{R}^2}[K(|u|^2)|u|^2 + K(|u|^2)(\boldsymbol{x}\cdot\nabla)|u|^2]\, \mathrm{d}\boldsymbol{x} = 0$. Let f represent $|u|^2$ as before and also let $g = |\hat{f}|^2$. Then, with $\boldsymbol{\xi} \in \mathbb{R}^2$,

$$\begin{aligned}
\mathcal{J} &:= \int_{\mathbb{R}^2} K(f)(\boldsymbol{x}\cdot\nabla)f\, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^2} \bar{\hat{f}}(\boldsymbol{\xi}\cdot\nabla)\widehat{K(f)}\, \mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^2} \bar{\hat{f}}(\boldsymbol{\xi}\cdot\nabla)(\alpha\hat{f})\, \mathrm{d}\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^2} g(\boldsymbol{\xi}\cdot\nabla)\alpha\, \mathrm{d}\boldsymbol{\xi} + \int_{\mathbb{R}^2} \alpha\bar{\hat{f}}(\boldsymbol{\xi}\cdot\nabla)\hat{f}\, \mathrm{d}\boldsymbol{\xi}.
\end{aligned} \tag{2.11}$$

First integral in the line just above vanishes since α, being homogeneous of degree 0, satisfies $(\boldsymbol{\xi}\cdot\nabla)\,\alpha = 0$. The second integral is real since $\mathcal{J}$ is so and we have,

$$\mathcal{J} = \int_{\mathbb{R}^2} \frac{\alpha}{2}(\boldsymbol{\xi}\cdot\nabla)g\, \mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^2} \frac{\alpha}{2}\nabla\cdot(\boldsymbol{\xi}g)\, \mathrm{d}\boldsymbol{\xi} - \int_{\mathbb{R}^2} \alpha g\, \mathrm{d}\boldsymbol{\xi} = -\frac{1}{2}\int_{\mathbb{R}^2} g\,(\boldsymbol{\xi}\cdot\nabla)\alpha\, \mathrm{d}\boldsymbol{\xi} - \int_{\mathbb{R}^2} \alpha g\, \mathrm{d}\boldsymbol{\xi}$$

similar to above. By using Parseval identity once more in the last integral we achieve our aim and this establishes the virial identity on a formal level.

We also consider the invariance of $(1.2.1)_1$ under the pseudo-conformal and scale transformations. To define the pseudo-conformal transformation, given $(t, \boldsymbol{x}) \in \mathbb{R}\times\mathbb{R}^2$, define the conjugate time and space variables $(T, \boldsymbol{X}) \in \mathbb{R} \times \mathbb{R}^2$ by

$$\boldsymbol{X} = \frac{\boldsymbol{x}}{a+bt}, \quad T = \frac{c+dt}{a+bt}, \quad \text{for} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Given u defined on $(-T_1, T_2) \times \mathbb{R}^2$ with $0 \le T_1, T_2 \le \infty$, we set

$$t_1 = \begin{cases} \infty & \frac{c+aT_1}{d+bT_1} < 0 \text{ or } bT_1 \le -d \\ \frac{c+aT_1}{d+bT_1} & aT_1 \ge -c \text{ and } bT_1 > -d \end{cases}, \quad t_2 = \begin{cases} \infty & \frac{-c+aT_2}{d-bT_2} < 0 \text{ or } bT_2 \ge d \\ \frac{-c+aT_2}{d-bT_2} & aT_2 \ge c \text{ and } bT_2 < d. \end{cases}$$

We define U on $(-t_1, t_2) \times \mathbb{R}^2$ by

$$U(t, \boldsymbol{x}) = \frac{1}{a+bt} \exp\left\{ ib\frac{\delta x^2 + y^2}{a+bt} \right\} u(T, \boldsymbol{X}) \tag{2.12}$$

where $\boldsymbol{X}$ stands for $(X, Y) \in \mathbb{R}^2$. Now let u be a sufficiently smooth solution of $(1.2.1)_1$ on $(-T_1, T_2) \times \mathbb{R}^2$. Since

$$\begin{aligned} K(|U(t,\boldsymbol{x})|^2) &= \int_{\mathbb{R}^2} \alpha(\boldsymbol{\xi})[|U(t,\cdot)|^2]\hat{}(\boldsymbol{\xi})\, e^{2\pi i\boldsymbol{x}\cdot\boldsymbol{\xi}}\, \mathrm{d}\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^2} \alpha(\boldsymbol{\xi})[|u(T,\cdot)|^2]\hat{}((a+bt)\boldsymbol{\xi})\, e^{2\pi i\boldsymbol{x}\cdot\boldsymbol{\xi}}\, \mathrm{d}\boldsymbol{\xi} \\ &= \frac{1}{(a+bt)^2} \int_{\mathbb{R}^2} [K(|u(T,\cdot)|^2)]\hat{}(\boldsymbol{\xi})\, e^{2\pi i\frac{\boldsymbol{x}}{a+bt}\cdot\boldsymbol{\xi}} \mathrm{d}\boldsymbol{\xi} \\ &= \frac{1}{(a+bt)^2} K(|u(T, \boldsymbol{x}/(a+bt))|^2) \end{aligned}$$

by the fact that α is homogeneous of order 0, U is a solution of $(1.2.1)_1$ on $(-t_1, t_2)\times\mathbb{R}^2$, which in turn implies the pseudo-conformal invariance of the equation $(1.2.1)_1$. Similarly solutions are invariant under the scale transformation $u \mapsto U$ given by $U(t, \boldsymbol{x}) = \mu u(\mu^2 t, \mu\boldsymbol{x})$, $\mu > 0$.

The conserved quantity corresponding to scale invariance is given by $E_{\mathrm{sc}}(u) := (1/2)I' - 4tE(u)$ whose conservation follows from the virial identity (2.7), see (2.3) and (2.9). Finally we consider the pseudo-conformal conservation law with the corresponding conserved quantity

$$E_{\mathrm{pc}}(u) := \int_{\mathbb{R}^2} [\delta|xu + 2i\delta t u_x|^2 + |yu + 2itu_y|^2 + 2t^2 K(|u|^2)|u|^2]\, \mathrm{d}x\, \mathrm{d}y, \tag{2.13}$$

which can also be written as $E_{\mathrm{pc}}(u) = I - 4t\mathrm{Im} \int_{\mathbb{R}^2} \bar{u}(\boldsymbol{x} \cdot \nabla)u\, \mathrm{d}\boldsymbol{x} + 4t^2 E(\varphi)$, with I as in (2.8) and $\varphi = u(0)$. So

$$\frac{dE_{\mathrm{pc}}(u)}{dt} = I' - 4\mathrm{Im} \int_{\mathbb{R}^2} \bar{u}(\boldsymbol{x} \cdot \nabla)u\, \mathrm{d}\boldsymbol{x} - t\frac{d}{dt} 4\mathrm{Im} \int_{\mathbb{R}^2} \bar{u}(\boldsymbol{x} \cdot \nabla)u\, \mathrm{d}\boldsymbol{x} + 8tE(\varphi) = 0$$

from (2.7) and (2.9). This quantity, like the virial identity, makes sense for Σ-solutions to be considered in the fifth section where Σ is the Hilbert space $H^1 \cap L^2(|\boldsymbol{x}|^2 \mathrm{d}\boldsymbol{x})$ equipped with the norm $\|\cdot\|_\Sigma^2 = \|\cdot\|_{H^1}^2 + \||\boldsymbol{x}|\cdot\|_2^2$.

3. THE CAUCHY PROBLEM IN L^2

Lemma 2.1 in [9] generalizes a result of Cipolatti [16] and summarizes the key properties of the linear operator K. In particular $K : L^p \to L^p$ is bounded for all $p \in (1, \infty)$ and $\|K(f)\|_2 \leq \|\alpha\|_\infty \|f\|_2$ which follows by Calderon-Zygmund theorem (see e.g. [3, Theorem 5.16]). The following trilinear estimates follow directly by a series of applications of Hölder's inequality

$$\begin{aligned} \|K(fg)h\|_{4/3} &\leq \|\alpha\|_\infty \|f\|_4 \|g\|_4 \|h\|_4, \\ \|K(fg)h\|_{L^{4/3}(I;L^{4/3})} &\leq \|\alpha\|_\infty \|f\|_{L^4(I;L^4)} \|g\|_{L^4(I;L^4)} \|h\|_{L^4(I;L^4)}, \\ \|K(fg)h\|_2 &\leq C_{8/3} \|f\|_8 \|g\|_4 \|h\|_8, \end{aligned} \tag{3.1}$$

where $C_{8/3}$ denotes the operator norm of K on $L^{8/3}$. Set $H(u) = K(|u|^2)u$. We give some of the operator theoretic properties of H which immediately follows from $(3.1)_1$ and $(3.1)_2$:

Corollary 3.1 *Let I be an interval of $\mathbb{R}$. Then there exists a constant $C_\alpha > 0$ depending only on $\|\alpha\|_\infty$ but not on I such that for every $u, v \in L^4(I; L^4)$*

$$\|H(u)\|_{L^{4/3}(I;L^{4/3})} \leq C_\alpha \|u\|^3_{L^4(I;L^4)}, \tag{3.2}$$

$$\|H(u) - H(v)\|_{L^{4/3}(I;L^{4/3})} \leq C_\alpha (\|u\|^2_{L^4(I;L^4)} + \|v\|^2_{L^4(I;L^4)}) \|u - v\|_{L^4(I;L^4)}. \tag{3.3}$$

To fix some notation let $S(t)$ represents the solution semigroup for the linear problem $iu_t + \delta u_{xx} + u_{yy} = 0$ (see also A.1) and given $\varphi \in L^2$ define $\Lambda H(u)$ and $\mathcal{T}u$ by

$$\Lambda H(u)(t) = \int_0^t S(t - s) H(u(s))\, \mathrm{d}s, \tag{3.4}$$

$$(\mathcal{T}u)(t) = S(t)\varphi - i\Lambda H(u)(t). \tag{3.5}$$

We consider a fixed point problem for the map $\mathcal{T}$ (in the appropriate function space) equivalent to (1.2.1) with $\varphi \in L^2$. Strichartz's estimates for both $\delta = \pm 1$ will be essential for the existence and uniqueness results (see A.2). As a reduction to the case $n = 2$, a pair (q, r) is called admissible if $2/q = 1 - 2/r$ and $2 \leq r < \infty$. From this point on all pairs denoted by (q, r) will be admissible. We will be using Theorem A.2.1 many times especially for $q = r = \gamma = \rho = 4$ with $n = 2$.

Corollary 3.2 *Let I be an interval of $\mathbb{R}$ containing zero. Then for every (q, r) there exists a constant $C_{q,r} = C_{q,r}(\|\alpha\|_\infty)$, independent of I, such that*

$$\|\Lambda H(u)\|_{L^q(I;L^r)} \leq C_{q,r}\|u\|^3_{L^4(I;L^4)}, \tag{3.6}$$

$$\|\Lambda(H(u) - H(v))\|_{L^q(I;L^r)} \leq C_{q,r}(\|u\|^2_{L^4(I;L^4)} + \|v\|^2_{L^4(I;L^4)})\|u - v\|_{L^4(I;L^4)}, \tag{3.7}$$

for every $u, v \in L^4(I; L^4)$. Moreover $\Lambda H(u) \in C(\bar{I}; L^2)$.

Proof. The result follows by Theorem A.2.1 and Corollary 3.1. □

The next proposition is needed for the unique continuation of local solutions.

Proposition 3.3 *Let I be an interval of $\mathbb{R}$ such that $0 \in \bar{I}$ and let $\varphi \in L^2$. Then there exists at most one $u \in C(\bar{I}; L^2) \cap L^4(I; L^4)$ which satisfies (1.2.1) in $\mathcal{D}'(I; H^{-2})$.*

Proof. Assume v also satisfies (1.2.1) in $\mathcal{D}'(I; H^{-2})$ and let $I = (0, T)$, $T < \infty$ without loss of generality. $u, v \in L^1(I; L^2)$, $H(u), H(v) \in L^{4/3}(I; L^{4/3}) \hookrightarrow L^1(I; H^{-2})$ imply $u, v \in W^{1,1}((0, T); H^{-2})$ satisfy $w = \mathcal{T}w$ in H^{-2} a.e. on $\bar{I}$. Let $0 \in J \subset I$ with $|J|$ sufficiently small. We have $\|u - v\|_{L^4(J;L^4)} \leq \|\Lambda(H(u) - H(v))\|_{L^4(J;L^4)} \leq C_{4,4}\left(\|u\|^2_{L^4(J;L^4)} + \|v\|^2_{L^4(J;L^4)}\right)\|u - v\|_{L^4(J;L^4)}$ from (3.7). For $|J|$ small, $C_{4,4}\left(\|u\|^2_{L^4(J;L^4)} + \|v\|^2_{L^4(J;L^4)}\right) < 1$. As a result $\|u - v\|_{L^4(J;L^4)} = 0$ and $u = v$ on J. J can be taken $J = (0, \tau)$ for $0 < \tau \leq T$, τ sufficiently small. Define $\theta = \sup\{0 < \tau < T : u = v \text{ on } (0, \tau)\}$. Since $u, v \in C([0, T]; L^2)$, we have $u = v$ on $[0, \theta]$. If $\theta = T$ then $u = v$ on I. Assume

$\theta < T$. Let $u_1 := u(\theta + \cdot)$, $v_1 := v(\theta + \cdot)$. $u_1, v_1 \in C([0, T - \theta]; L^2) \cap L^4(0, T - \theta; L^4)$ solves $(1.2.1)_1$ a.e. on $[0, T - \theta]$. So as above, where $(0, T)$ is replaced by $(0, T - \theta)$, by a similar definition to θ, there exists $\epsilon > 0$ such that $u_1 = v_1$ on $[0, \epsilon]$. This in turn implies $u = v$ on $[\theta, \theta + \epsilon]$ and so on $[0, \theta + \epsilon]$ which contradicts with the definition of θ. So $\theta = T$. By continuation we obtain the result for more general I. □

The following theorem is the main result of this chapter.

Theorem 3.4 *Given $\varphi \in L^2$, there exists a unique maximal solution $u \in C([0, T^*); L^2) \cap L^4([0, t]; L^4)$, for every $t < T^*$ solving (1.2.1) in H^{-2} a.e. on $[0, T^*)$ with the following properties:*

- *(i) (Further regularity) $u \in L^q([0, t]; L^r)$ for every (q, r) and for every $t < T^*$,*
- *(ii) (Blow-up) $T^* < \infty$ implies that $\|u\|_{L^q([0,T^*);L^r)} = \infty$ for every (q, r) with $r \geq 4$,*
- *(iii) (H^1-regularity) $\varphi \in H^1$ implies $u \in C([0, T^*); H^1)$ where also $[0, T^*)$ coincides with the maximal interval of existence for the corresponding H^1-solution to be considered in the next section,*
- *(iv) (Mass Conservation) $\|u(t)\|_2 = \|\varphi\|_2$ for $0 \leq t < T^*$,*
- *(v) (Continuous Dependence) If $\varphi_n \to \varphi$ in L^2 and u_n's are the corresponding solutions, then for any $I \Subset [0, T^*)$ and for any n sufficiently large u_n's are defined on I and $u_n \to u$ in $L^q(I, L^r)$ for every (q, r).*

Proof. By Corollary 3.2, the proof is exactly the one for [1, Theorem 4.7.1], which is true for the hyperbolic case as well.

Step 1: Local existence. Let $\mu > 0$ be such that $8C_{4,4}\,\mu^2$, $8C_{\infty,2}\,\mu^2 < 1$. By Theorem A.2.1, $S(\cdot)\varphi \in C(\mathbb{R}; L^2) \cap L^q(\mathbb{R}; L^r)$ for every (q, r). So $S(\cdot)\varphi \in L^4(\mathbb{R}; L^4)$ and there exists I such that $0 \in \bar{I} \subset [0, \infty)$ with $|I|$ sufficiently small to satisfy

$$\|S(\cdot)\varphi\|_{L^4(I;L^4)} < \mu \tag{3.8}$$

by DCT. (Hence I depends on μ and φ).

Claim 1. There exists unique $u \in C(\bar{I}; L^2) \cap L^q(I; L^r)$ for every (q, r), solving (1.2.1) in H^{-2} a.e. on I.

For Claim 1, define

$$E = \{u \in L^4(I; L^4) : \|u\|_{L^4(I;L^4)} \leq 2\mu\}.$$

E is complete with the metric induced by $L^4(I; L^4)$-norm. For $u, v \in E$, by using the estimates (3.6) and (3.7), $\|\mathcal{T}u\|_{L^4(I;L^4)} \leq \|S(\cdot)\varphi\|_{L^4(I;L^4)} + C_{4,4}\|u\|^3_{L^4(I;L^4)} \leq \mu + 8C_{4,4}\,\mu^3$, and $\|\mathcal{T}u - \mathcal{T}v\|_{L^4(I;L^4)} \leq C_{4,4}(\|u\|^2_{L^4(I;L^4)} + \|v\|^2_{L^4(I;L^4)})\|u - v\|_{L^4(I;L^4)} \leq 8C_{4,4}\,\mu^2\|u - v\|_{L^4(I;L^4)}$. Then $8C_{4,4}\,\mu^2 < 1$ and the last inequalities imply $\mathcal{T} : E \to E$ is a strict contraction on $E \subset L^1(I; H^{-2})$. So there exists a unique $u \in E$ such that $u = \mathcal{T}u$. By Strichartz's estimates $u \in C(\bar{I}; L^2) \cap L^q(I; L^r)$ for every (q, r). $u \in C(\bar{I}; L^2) \cap L^4(I; L^4)$ implies $H(u) \in L^{4/3}(I; L^{4/3}) \hookrightarrow L^1(I; H^{-2})$. Hence $u \in W^{1,1}(I; H^{-2})$ solves (1.2.1) in H^{-2} a.e. on I. By Proposition 3.3, u is the unique such solution.

Claim 2. Let $\varphi, \psi \in L^2$ and μ as before. Assume $|I|$ is sufficiently small to satisfy (3.8) also with φ, ψ and let u, v be the corresponding solutions as in Claim 1. Then there exists a constant C, independent of I, u, v, such that

$$\|u - v\|_{L^\infty(I;L^2)} + \|u - v\|_{L^4(I;L^4)} \leq C\|\varphi - \psi\|_2. \tag{3.9}$$

For Claim 2, let u, v be solutions on I and hence they satisfy the corresponding integral equations. Let C_s denotes maximum of the constants for $(q, r) = (\infty, 2)$ and $(4, 4)$ coming from the Strichartz's estimates. So

$$\begin{aligned}\|u - v\|_{L^\infty(I;L^2)} &\leq \|S(\cdot)(\varphi - \psi)\|_{L^\infty(I;L^2)} + \|\Lambda(H(u) - H(v))\|_{L^\infty(I;L^2)} \\ &\leq C_s\|\varphi - \psi\|_2 + 4C_{\infty,2}\,\mu^2\|u - v\|_{L^4(I;L^4)}\end{aligned}$$

by using Theorem A.2.1 for the first norm on the righthand side and (3.7) for the latter together with the fact $u, v \in E$. Similarly, $\|u-v\|_{L^4(I;L^4)} \leq C_s\|\varphi-\psi\|_2 + 4C_{4,4}\,\mu^2\|u-v\|_{L^4(I;L^4)}$. Adding these two inequalities,

$$\|u-v\|_{L^\infty(I;L^2)} + \|u-v\|_{L^4(I;L^4)} \leq C\|\varphi-\psi\|_2 + 8\left(C_{4,4} \vee C_{\infty,2}\right)\mu^2\,\|u-v\|_{L^4(I;L^4)}.$$

So by the assumption on μ, namely $8\left(C_{4,4} \vee C_{\infty,2}\right)\mu^2 < 1$, at the beginning of Step 1, there exists a constant C as claimed.

Step 2: H^1-regularity on I. Let $\varphi \in H^1$ and I, u be as in Claim 1. We show that $u \in C(I; H^1)$. For this we assume the existence of a unique maximal solution $v \in C([0,T_1^*); H^1) \cap C^1([0,T_1^*); H^{-1})$ such that $v \in L^4([0,t]; W^{1,4})$ for $t < T_1^*$ which is central to the next chapter. If $I \subset [0, T_1^*)$ then by Proposition 3.3 $u = v$ on I and this gives the necessary regularity. So we will show that $I \subset [0, T_1^*)$.

It suffices to consider $I = (0, b)$, $b < \infty$ where I is still satisfying (3.8). Assume on the contrary $b > T_1^*$. For $h \in \mathbb{R}^2$ sufficiently small, $\|\varphi(\cdot+h)-\varphi\|_2$ is small so that we also have $\|S(\cdot)\varphi(\cdot+h)\|_{L^4(I;L^4)} < \mu$, since $\|S(\cdot)\varphi(\cdot+h)\|_{L^4(I;L^4)} \leq C_s\|\varphi(\cdot+h)-\varphi\|_2 + \|S(\cdot)\varphi\|_{L^4(I;L^4)}$. By Claim 2,

$$\begin{aligned}\|\tau_h v - v\|_{L^\infty((0,T_1^*);L^2)} &\leq \|\tau_h u - u\|_{L^\infty((0,b);L^2)} + \|\tau_h u - u\|_{L^4((0,b);L^4)} \\ &\leq C\,\|\varphi(\cdot+h)-\varphi\|_2\end{aligned}$$

where $t \mapsto (\tau_h v)(t) = v(t, \cdot+h)$ is the corresponding solution for the initial data $\varphi(\cdot+h)$. Dividing both sides of the above inequality by $|h|$ and letting $|h| \to 0$, we obtain $\|\nabla v\|_{L^\infty((0,T_1^*);L^2)} \leq C\,\|\nabla\varphi\|_2$. Since $\|v\|_{L^\infty([0,T_1^*);L^2)} < \infty$, last inequality implies $\|v\|_{L^\infty((0,T_1^*);H^1)} < \infty$ which contradicts with the blow-up results for H^1-solutions ($b > T_1^*$ implies $T_1^* < \infty$).

Step 3: Mass conservation on I. For $\varphi \in L^2$, let u be the corresponding solution as in Claim 1. Consider $\varphi_n \in H^1$ such that $\varphi_n \to \varphi$ in L^2. For n large, $\|\varphi_n - \varphi\|_2$ will be small so that (3.8) holds for φ_n by using $\|S(\cdot)\varphi_n\|_{L^4(I;L^4)} \leq C_s\|\varphi_n-\varphi\|_2 + \|S(\cdot)\varphi\|_{L^4(I;L^4)}$

as above. Let u_n's be the corresponding solutions for those φ_n's. Claim 2 implies $\|u_n - u\|_{L^\infty(I;L^2)} \leq C\,\|\varphi_n - \varphi\|_2$, which in turn gives $u_n(t) \to u(t)$ in L^2 for $t \in I$ ($u_n, u \in C(\bar{I}; L^2)$). But we have $\|u_n(t)\|_2 = \|\varphi_n\|_2$ since we have mass conservation for H^1-solutions (see Chapter 4). So $\|\varphi_n\|_2 \to \|u(t)\|_2$ for $t \in I$ from which the result follows.

Step 4: Maximal solution and Blow-up. Choose $I = [0, T]$ where T is sufficiently small to satisfy (3.8). By Claim 1, there exists unique u such that $u \in C([0,T]; L^2) \cap L^q([0,T]; L^r)$ for every (q, r) and u solves (1.2.1) in $\mathcal{D}'((0,T); H^{-2})$. We have also $\|u(t)\|_2 = \|\varphi\|_2$ on I by Step 3. Define T^* be the supremum of such T's. By Proposition 3.3, u can be extended uniquely to satisfy $u \in C([0,T^*); L^2) \cap L^q([0,t]; L^r)$ for every (q, r), $t < T^*$ and that u solves (1.2.1) in $\mathcal{D}'((0,T^*); H^{-2})$ with mass conserved on $[0, T^*)$.

Assume $T^* < \infty$ and $u \in L^4([0,T^*); L^4)$. Let $0 \leq t < T^*$. Since $u(t + \cdot)$ satisfies (1.2.1) in $\mathcal{D}'((0, T^* - t); H^{-2})$, $u(t+s) = S(s)u(t) - i\int_0^s S(t-\sigma)H(u(t+\sigma))\,\mathrm{d}\sigma$ in H^{-2} a.e. on $[0, T^* - t)$ and so by (3.7)

$$\|S(\cdot)u(t)\|_{L^4([0,T^*-t);L^4)} \leq \|u\|_{L^4([t,T^*);L^4)} + C_{4,4}\|u\|^3_{L^4([t,T^*);L^4)}. \tag{3.10}$$

Both terms on the righthand side of (3.10) are finite by assumption. By (3.10) for t close enough to T^*, $\|S(\cdot)u(t)\|_{L^4([0,T^*-t);L^4)} < \mu/2$. Since $u(t) \in L^2$ by Strichartz's estimates $S(\cdot)u(t) \in L^4(\mathbb{R}; L^4)$. So for $0 \leq T < t$ sufficiently close to t, $\|S(\cdot)u(t)\|_{L^4([0,T^*-T);L^4)} < \mu$. Now, take $I = [0, T^* - T)$ and $u(t)$ as the initial data to use Step 1 and we obtain a unique $\tilde{u} \in C(\bar{I}; L^2) \cap L^q(I; L^r)$ solving (1.2.1) in $\mathcal{D}'((0, T^* - T); H^{-2})$ with φ replaced by $u(t)$. Extend u beyond t on $[0, T^* - T + t)$ as

$$u(s) = \begin{cases} u(s) & \text{if } 0 \leq s \leq t \\ \tilde{u}(s-t) & \text{if } t \leq s < T^* - T + t. \end{cases}$$

By Proposition 3.3, u is the unique solution of (1.2.1) in $\mathcal{D}'((0, T^* - T + t); H^{-2})$ which contradicts with the definition of T^* since $T < t$. So for $T^* < \infty$,

$$\|u\|_{L^4((0,T^*);L^4)} = \infty. \tag{3.11}$$

This establishes *(ii)* for $r = 4$. For the blow-up results with $r > 4$, let $T < T^* < \infty$. By interpolation in space ($r > 4 > 2$) and the Hölder inequality for the time integral part of the norm, we get $\|u\|_{L^4((0,T);L^4)} \leq \|u\|^{\nu}_{L^\infty((0,T);L^2)} \|u\|^{1-\nu}_{L^q((0,T);L^r)}$, for $\nu = (r-4)/2(r-2)$. By mass conservation $\|u\|_{L^4((0,T);L^4)} \leq \|\varphi\|_2^{\nu} \|u\|^{1-\nu}_{L^q((0,T);L^r)}$, from which the necessary blow-up results follow by (3.11).

Step 5: H^1-regularity on the maximal interval of existence. Let $\varphi \in H^1$ and T_1^* denotes the endpoint of the maximal interval of existence for the H^1-solution to be considered in the next chapter. By Proposition 3.3 and by definition of T^*, $T_1^* \leq T^*$. Assume $T_1^* < T^*$. So we have $u \in C([0, T_1^*]; H^1)$ by definition of T^* and by Step 2. But this contradicts with the blow-up result for the H^1-solutions since $T_1^* < T^*$ implies $T_1^* < \infty$. Thus, $T_1^* = T^*$.

Step 6: Continuous dependence. Let $T < T^* = T^*(\varphi)$ and μ as before. We have

$$\sup_{t \in [0,T]} \|S(\cdot)u(t)\|_{L^4(\mathbb{R};L^4)} \leq C_s \sup_{t \in [0,T]} \|u(t)\|_2 < \infty$$

by using Theorem A.2.1 and the fact that $u \in C([0, T]; L^2)$. So there exists $\tau > 0$ such that

$$\|S(\cdot)u(t)\|_{L^4((0,\tau);L^4)} < \frac{\mu}{2}, \quad \forall t \in [0, T]. \tag{3.12}$$

Let n be a fixed integer satisfying $T \leq n\tau$. Increase C in Claim 2 to satisfy $C \geq 1$ if necessary. We want to show that for $\|\varphi - \psi\|_2$ small corresponding maximal solution v is defined on $[0, T]$ and $\|u - v\|_{\infty,2} + \|u - v\|_{4,4}$ is also small. Let ε be sufficiently small such that $C_s C^{n-1} \varepsilon < \mu/2$. ($C, n, \tau$ depend on T fixed.)

Claim. $\|\varphi - \psi\|_2 \le \varepsilon$ implies $T^*(\psi) > T$ and

$$\|u - v\|_{L^\infty([0,T];L^2)} + \|u - v\|_{L^4([0,T];L^4)} \le nC^{n-1}\|\varphi - \psi\|_2 \tag{3.13}$$

where v stands for the corresponding maximal solution on $[0, T^*(\psi))$.

For the proof, since $\|\varphi - \psi\|_2 \le \varepsilon$,

$$\|S(\cdot)\psi\|_{L^4([0,T/n];L^4)} \le \|S(\cdot)\varphi\|_{L^4([0,T/n];L^4)} + \|S(\cdot)(\varphi - \psi)\|_{L^4([0,T/n];L^4)} \le \mu/2 + C_s\,\varepsilon < \mu$$

by using (3.12) and Strichartz's estimates. Let $I = [0, T/n]$ in step 1, so by Claims 1, 2 and the definition of $T^*(\psi)$,

$$\|u - v\|_{C([0,T/n];L^2)} + \|u - v\|_{L^4([0,T/n];L^4)} \le C\|\varphi - \psi\|_2 \tag{3.14}$$

and we have $T^*, T^*(\psi) > T/n$. In particular this implies $\|u(T/n) - v(T/n)\|_2 \le C\,\|\varphi - \psi\|_2 \le C\varepsilon$. Now take $\varphi_{new} = u(T/n)$ and $\psi_{new} = v(T/n)$, we have

$$\begin{aligned}&\|S(\cdot)v(T/n)\|_{L^4([0,T/n];L^4)}\\ &\le \|S(\cdot)u(T/n)\|_{L^4([0,T/n];L^4)} + \|S(\cdot)(u(T/n) - v(T/n))\|_{L^4([0,T/n];L^4)} \le \mu/2 + C_sC\varepsilon < \mu\end{aligned}$$

similar to above by (3.12) and Strichartz's estimates. Again by using Claims 1, 2 with $I = [0, T/n]$ and with $\varphi_{new}, \psi_{new}$, we obtain

$$\begin{aligned}\|\tilde{u} - \tilde{v}\|_{C([0,T/n];L^2)} + \|\tilde{u} - \tilde{v}\|_{L^4(0,T/n;L^4)} &\le C\|u(T/n) - v(T/n)\|_2\\ &\le C^2\|\varphi - \psi\|_2 \ (\le C^2\varepsilon).\end{aligned} \tag{3.15}$$

The maximal solution u defined on $[0, T^*)$ equals to

$$u(t) = \begin{cases} u(t) & \text{if } 0 \le t \le T/n \\ \tilde{u}(t - T/n) & \text{if } T/n \le t \le 2T/n. \end{cases}$$

on $[0, 2T/n]$ by Proposition 3.3 and $2T/n < T^*$ and similar result holds for v which is the maximal solution with the initial data ψ. (3.14), (3.15) and the last remarks imply

$$\|u - v\|_{C([0,2T/n];L^2)} + \|u - v\|_{L^4(0,2T/n;L^4)} \le C^2 \|\varphi - \psi\|_2$$

since $C \ge 1$. Continuing in this way after finitely many steps, we obtain (3.13) and the claim is proved. Also since u and v are solutions on $(0, T)$, they satisfy the corresponding integral equations which implies that $\|u - v\|_{L^q((0,T);L^r)}$ is controlled by a constant multiple, independent of T, of $\|\varphi - \psi\|_2 + \|u - v\|_{L^4((0,T);L^4)}$ again using Strichartz's estimates and (3.7). We obtain *(v)* for $I = (0, T)$ and the result follows for more general $I \Subset [0, T^*)$ by using the intervals of the form $(0, T)$ with $T < T^*$. □

Remark 3.5 (See also [1, Remarks 4.7.4-5]) With certain assumptions on the initial data and the sign of α it is possible to obtain global L^2-solutions in $u \in L^4([0, \infty); L^4)$ which will be important for the scattering results to be considered in Chapter 6.

(i) For small initial data we obtain global solutions: The claim is that there exists $\eta > 0$ such that if $\varphi \in L^2$ satisfies

$$\|S(\cdot)\varphi\|_{L^4(\mathbb{R};L^4)} < \eta \tag{3.16}$$

then $T^* = \infty$. Indeed, let φ satisfies $\|S(\cdot)\varphi\|_{L^4(\mathbb{R};L^4)} < \mu$ where μ as in step 1 of the above proof. Then using Claim 1 in that step with $I = [0, T^*)$, we have the solution $u \in C(I; L^2) \cap L^q(I; L^r)$ for any (q, r) admissible pair. This implies that $\|u\|_{L^q([0,T^*);L^r)} < \infty$ and by the blow-up alternative $T^* = \infty$. So one can take $\eta = \mu$. (3.16) can be satisfied if $\|\varphi\|_2$ is sufficiently small considering Theorem A.2.1.

(ii) The other case is when $\alpha > \alpha_G > 0$ and $\varphi \in L^2((1 + |\boldsymbol{x}|^2)\mathrm{d}\boldsymbol{x})$ in the elliptic case. Here α_G being a positive constant is a lower bound for the symbol α. If $\varphi_n \in \Sigma$ are such that $\varphi_n \to \varphi$ in L^2, then $|\boldsymbol{x}|\varphi_n$ is bounded in L^2. Let u_n's be the corresponding solutions, as given in Theorem 5.2, which are global by Corollary 4.6.

Considering pseudo-conformal conservation for u_n's (see (2.13)) which will be justified after establishing virial identity for Σ-solutions in Chapter 5,

$$\|\boldsymbol{x}\varphi_n\|_2^2 = \|\boldsymbol{x}u_n + 2it\nabla u_n\|_2^2 + 2t^2 \int_{\mathbb{R}^2} K(|u_n|^2)|u_n|^2 \geq 2\alpha_G t^2 \|u_n\|_4^4$$

implies that $\|u_n(t)\|_4^4 \leq Ct^{-2}$ globally for all n. By Theorem 3.4, *(v)*, $\|u_n(t)\|_4^4 \to \|u(t)\|_4^4$ for a.a. $t \in [0, T^*)$ which implies $\|u(t)\|_4^4 \leq Ct^{-2}$ a.e. on $[0, T^*)$. This gives $T^* = \infty$ by the same theorem, *(ii)*.

We have performed the formal computations to establish the pseudo-conformal invariance in the previous section. With L^2-solutions, this invariance property holds for $\delta = \pm 1$ in the sense of [1, Theorem 6.7.1].

Theorem 3.6 *Suppose* $u \in C((-T_1, T_2); L^2) \cap L^4_{loc}((-T_1, T_2); L^4)$ *is a solution of* $(1.2.1)_1$ *in the sense of Theorem 3.4. Let* a, b, c, d, t_1, t_2 *and* U *be as given in the previous section (see* (2.12) *and the definitions above it). It follows that*

$$U \in C((-t_1, t_2); L^2) \cap L^4_{loc}((-t_1, t_2); L^4)$$

is also a solution of $(1.2.1)_1$ *in the same sense.*

With the existence and uniqueness result obtained above the proof is as in [1] and mainly uses Theorem 3.4, *(v)* with more regular solutions so an existence and uniqueness result for H^2-solutions to be given in Theorem 7.6 and Remark 7.8 is needed. A similar result holds with the scale transformation defined in the previous section.

4. THE CAUCHY PROBLEM IN H^1

In this chapter (1.2.1) is considered with $\varphi \in H^1$. Let us recall the spaces

$$X = L^\infty(I; L^2) \cap L^4(I; L^4), \quad \| \cdot \|_X = \| \cdot \|_{\infty,2} \vee \| \cdot \|_{4,4}$$
$$X_0 = L^\infty(I; L^2) \cap L^\infty(I; L^4), \quad \| \cdot \|_{X_0} = \| \cdot \|_{\infty,2} \vee \| \cdot \|_{\infty,4}$$

for $I = [0, T]$, $T < \infty$ (see [2]). We will also need $\bar{X} = C(I; L^2) \cap L^4(I; L^4)$ which is a closed subspace of X. For $u \in X$, u solves $(1.2.1)_1$ in $\mathcal{D}'((0,T); H^{-2})$ if and only if $u = \mathcal{T}u$ in H^{-2} a.e. on I as in the argument for Proposition 3.3. We consider the maps H and ΛH to obtain the necessary contraction property for the map $\mathcal{T}$ defined by (3.5).

Lemma 4.1 *Let X, X_0 be given by (4) then for any $T < \infty$*

(i) H maps X (so X_0) boundedly and continuously into $L^{4/3}(I; L^{4/3})$.

(ii) ΛH maps X_0 boundedly and continuously into X. Also for any $R > 0$, ΛH is a contraction on $B_{X_0}(0, R) = \{v \in X_0 : \|v\|_{X_0} \leq R\}$, when considered with the metric induced by the norm $\| \cdot \|_X$, provided T is sufficiently small.

Proof. *(i)* Using (3.2), (3.3) and $\| \cdot \|_X \leq T^{1/4} \| \cdot \|_{X_0}$

$$\|H(u)\|_{4/3,4/3} \leq C_\alpha \|u\|_{4,4}^3 \leq C_\alpha T^{3/4} \|u\|_{\infty,4}^3, \tag{4.1}$$
$$\|H(u) - H(v)\|_{4/3,4/3} \leq 2C_\alpha T^{1/2} (\|u\|_{X_0}^2 \vee \|v\|_{X_0}^2) \|u - v\|_{4,4}, \tag{4.2}$$

where C_α is independent of T. The claim follows from these estimates.

(ii) Let $u, v \in X_0$, then

$$\|\Lambda H(u)\|_{q,r} \leq C_{q,r}\|u\|_{4,4}^3 \leq C_{q,r}\, T^{3/4}\|u\|_{\infty,4}^3 \tag{4.3}$$

$$\|\Lambda(H(u) - H(v))\|_{q,r} \leq C_{q,r}\, T^{3/4}(\|u\|_{\infty,4}^2 + \|v\|_{\infty,4}^2)\|u - v\|_{\infty,4}, \tag{4.4}$$

for $(q, r) = (\infty, 2)$ and $(4, 4)$ using (3.6), (3.7) and $X_0 \subset X$. The continuity property now follows. We have also $\|\Lambda(H(u) - H(v))\|_{q,r} \leq C_{q,r}\, T^{1/2}(\|u\|_{\infty,4}^2 + \|v\|_{\infty,4}^2)\|u - v\|_{4,4}$. So if in addition $u, v \in B_{X_0}(0, R)$, the last inequality implies that there exists a constant C_γ independent of T such that

$$\|\Lambda(H(u) - H(v))\|_X \leq C_\gamma T^{1/2} R^2 \|u - v\|_X \tag{4.5}$$

from which we obtain the contraction property by choosing T sufficiently small. □

We need some additional spaces for dealing with derivatives (see also [2]):

$$\begin{aligned} Y &= L^\infty(I; H^1) \cap L^4(I; W^{1,4}),\ \|\cdot\|_Y = \|\cdot\|_X \vee \|\nabla\cdot\|_X, \\ Y' &= \{f \in L^{4/3}(I; L^{4/3}) : \nabla f \in L^{4/3}(I; L^{4/3})\},\ \|\cdot\|_{Y'} = \|\cdot\|_{4/3,4/3} \vee \|\nabla\cdot\|_{4/3,4/3} \\ \bar{Y} &= C(I; H^1) \cap L^4(I; W^{1,4}). \end{aligned}$$

Since $H^1 \hookrightarrow L^p$ for $p \in [2, \infty)$, $L^\infty(I; H^1) \hookrightarrow X_0$. This implies $Y \subset X_0$ and $\|\cdot\|_Y \leq C_e\|\cdot\|_{X_0}$ where C_e is independent of T. By Theorem A.2.1 and the fact that $S(t)\nabla = \nabla S(t)$, $S(\cdot) \in \mathcal{L}(H^1; \bar{Y})$ and $\Lambda \in \mathcal{L}(Y'; \bar{Y})$. A property of H in Y-spaces is the following.

Lemma 4.2 *H maps Y boundedly into Y' and in fact, there exists a constant M independent of T such that*

$$\|H(u)\|_{Y'} \leq M\, T^{1/2}\|u\|_Y^3. \tag{4.6}$$

Proof. Being different from the argument in [13] for Lemma 2.3, we consider an alternative way for obtaining the estimate

$$\|\nabla H(u)\|_{4/3,4/3} \leq CT^{1/2}\|u\|_Y^3 \tag{4.7}$$

for some C independent of T by utilizing the properties of K. For $u \in Y$, we have $u \in H^1 \cap W^{1,4}$. Therefore $|u|^2 \in H^1$. By [9, Lemma 2.1, *(iii)*], $K(|u|^2) \in H^1$ and $\nabla K(|u|^2) = K(\nabla|u|^2) := (K((|u|^2)_x), K((|u|^2)_y))$. So $K(|u|^2)v \in W^{1,1}$ and $\nabla(K(|u|^2)v) = K(u\nabla\bar{u} + \bar{u}\nabla u)v + K(|u|^2)\nabla v$. For some C depending only on $\|\alpha\|_\infty$,

$$\|\nabla(K(|u|^2)v)\|_{4/3,4/3} \leq C(\|u\|_{4,4}\|\nabla u\|_{4,4}\|v\|_{4,4} + \|u\|_{4,4}^2\|\nabla v\|_{4,4}) \tag{4.8}$$

by $(3.1)_2$. As a result $\nabla H(u) \in L^{4/3}(I; L^{4/3})$ with $\nabla H(u) = K(\nabla|u|^2)u + K(|u|^2)\nabla u$ and $\|\nabla H(u)\|_{4/3,4/3} \leq C\|u\|_{4,4}^2\|\nabla u\|_{4,4}$. This implies (4.7) since $\|\nabla u\|_{4,4} \leq \|u\|_Y$ and $\|u\|_{4,4}^2 \leq T^{1/2}\|u\|_{\infty,4}^2$. Also by using (3.2) and $Y \hookrightarrow X_0 \hookrightarrow X$

$$\|H(u)\|_{4/3,4/3} \leq C_\alpha\|u\|_{4,4}^3 \leq C_\alpha T^{1/2}\|u\|_{4,4}\|u\|_{\infty,4}^2. \tag{4.9}$$

We have $\|\cdot\|_{4,4} \leq \|\cdot\|_Y$ and $\|\cdot\|_{\infty,4} \leq \|\cdot\|_{X_0} \leq C_e\|\cdot\|_Y$. So (4.9) implies

$$\|H(u)\|_{4/3,4/3} \leq CT^{1/2}\|u\|_Y^3 \tag{4.10}$$

for some constant C depending on $\|\alpha_N\|_\infty$, C_e but not on T. (4.7) and (4.10) prove the claim. □

Let us note that unlike in [2] we have only worked with reflexive spaces in the above argument and this eased our task. Now we can state the contraction property for $\mathcal{T}$, fundamental to the existence of H^1-solutions.

Lemma 4.3 *Let $\varphi \in H^1$, $R > 0$ be given and $\mathcal{T}$ be defined by (3.4) and (3.5). Also let $B_Y(S(\cdot)\varphi, R) = \{v \in Y : \|v - S(\cdot)\varphi\|_Y \leq R\}$ be considered as a metric space with* *the metric induced by the norm* $\|\cdot\|_X$*. Then* $\mathcal{T}$ *is a strict contraction on* $B_Y(S(\cdot)\varphi, R)$ *provided* T *is sufficiently small.*

Proof. $\varphi \in H^1$ implies $S(\cdot)\varphi \in \bar{Y}$ by Strichartz's estimates as noted before Lemma 4.2. If $u \in B_Y(S(\cdot)\varphi, R)$, then $H(u) \in Y'$ by Lemma 4.2 and

$$\begin{aligned} \|\mathcal{T}u - S(\cdot)\varphi\|_Y &= \|\Lambda H(u)\|_Y \le \|\Lambda\|_{op} M T^{1/2} \|u\|_Y^3 \\ &\le \|\Lambda\|_{op} M T^{1/2} (R + \|S(\cdot)\varphi\|_Y)^3 \le \|\Lambda\|_{op} M' T^{1/2} (R + \|\varphi\|_{H^1})^3 \end{aligned} \tag{4.11}$$

by (4.6) and Theorem A.2.1. Let $v \in B_Y(S(\cdot)\varphi, R)$ also. $B_Y(S(\cdot)\varphi, R) \subset B_{X_0}(0, C_e(R + \|S(\cdot)\varphi\|_Y)$ since $Y \hookrightarrow X_0$. Then

$$\begin{aligned} \|\mathcal{T}u - \mathcal{T}v\|_X = \|\Lambda(H(u) - H(v))\|_X &\le C_\gamma C_e^2 T^{1/2} (R + \|S(\cdot)\varphi\|_Y)^2 \|u - v\|_X \\ &\le C_f T^{1/2} (R + \|\varphi\|_{H^1})^2 \|u - v\|_X \end{aligned} \tag{4.12}$$

by (4.5) and Stricahrtz's estimates. (C_f is independent of T). (4.11), (4.12) holds for arbitrary $T < \infty$. Choosing T small will prove the claim. □

Remark 4.4 $B_Y(S(\cdot)\varphi, R)$ is a complete metric space when considered with $\|\cdot\|_X$. Indeed, let $\{v_n\}_{n\ge 0} \subset B_Y(S(\cdot)\varphi, R)$ such that $v_n \to v$ in $L^4(I; L^4)$. Now let J be an open interval in $\mathbb{R}$ with $I \subset J$ and extend $v, v_n, S(\cdot)\varphi$ to J being 0 on $J \setminus I$. There exists a subsequence, which we still denote by v_n, such that $v_n(t) \to v(t)$ for a.a. $t \in J$. $W^{1,4} \hookrightarrow L^4$ and $\|v_n - S(\cdot)\varphi\|_{L^4(J;W^{1,4})} \le R$. So by [1, Theorem 1.2.5], $v \in L^4(J; W^{1,4})$ and $\|v - S(\cdot\varphi)\|_{L^4(J;W^{1,4})} \le \liminf_{n\to\infty} \|v_n - S(\cdot)\varphi\|_{L^4(J;W^{1,4})} \le R$. Also $H^1 \hookrightarrow L^4$ and v_n bounded in $L^\infty(I; H^1)$, so similarly $v \in L^\infty(I; H^1)$ and $\|v - S(\cdot)\varphi\|_{L^\infty(J;H^1)} \le \liminf_{n\to\infty} \|v_n - S(\cdot)\varphi\|_{L^\infty(J;H^1)} \le R$. As a result we have $v \in B_Y(S(\cdot)\varphi, R)$ which implies that $B_Y(S(\cdot)\varphi, R)$ is closed in $L^4(I; L^4)$ and so in X and this establishes the claim since X is a Banach space.

As a result of the lemmas above we have the main theorem of this chapter:

Theorem 4.5 *Given $\varphi \in H^1$, there exists a unique maximal solution $u \in C([0,T^*); H^1) \cap C^1([0,T^*); H^{-1})$ solving (1.2.1) on $[0,T^*)$ with the following properties:*

(i) (Further regularity) $\nabla u \in L^4([0,t]; L^4)$ for every $t < T^*$,

(ii) (Blow-up) $T^* < \infty$ implies that $\|u\|_{L^\infty([0,T^*);H^1)} = \infty$,

(iii) (Continuous Dependence) $\varphi_n \to \varphi$ in H^1 and u_n's are the corresponding solutions. Then for any $I \Subset [0,T^*)$ and for any n sufficiently large u_n's are defined on I and $u_n \to u$ in $C(I; H^1)$.

(iv) (Mass and Energy Conservation) $\|u(t)\|_2 = \|\varphi\|_2$ and $E(u(t)) = E(\varphi)$ for $0 \leq t < T^*$.

Proof. By Lemma 4.3, for T sufficiently small there exists unique $u \in B_Y(S(\cdot)\varphi, R)$ such that $u = \mathcal{T}u$ in Y since $B_Y(S(\cdot)\varphi, R)$ is a complete metric space when furnished with $\|\cdot\|_X$ as shown in Remark 4.4. This gives local existence and the local uniqueness is by considering Proposition 3.3.

Maximal solution and Blow-up. Let $T^* = \sup\{T > 0 : \exists u \in C([0,T]; H^1) \cap C^1([0,T]; H^{-1})$ solving (1.2.1) on $[0,T]$ and $\nabla u \in L^4([0,T]; L^4)\}$. By Proposition 3.3, we can extend the solution uniquely to $u \in C([0,T^*); H^1) \cap C^1([0,T^*); H^{-1})$ solving (1.2.1) on $[0,T^*)$ and also $\nabla u \in L^4((0,t); L^4)$ for every $t < T^*$.

Assume $T^* < \infty$ and $\|u\|_{L^\infty((0,T^*);H^1)} < \infty$. Let $\{t_j\}$ be a sequence such that $t_j \uparrow T^*$ and $\|u(t_j)\|_{H^1} \leq A$ for some $A > 0$. Let T satisfy $C_f T^{1/2}(R + (\|\varphi\|_{H^1} \vee A))^2 < 1$ and $\|\Lambda\|_{op} M' T^{1/2}(R + (\|\varphi\|_{H^1} \vee A))^3 \leq R$. Let k be such that $t_k + T > T^*$. So there exists unique $\tilde{u} \in C([0,T]; H^1) \cap C^1([0,T]; H^{-1})$ solving $(1.2.1)_1$ on $[0,T]$ and $\tilde{u}(0) = u(t_k)$. Then

$$u(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq t_k \\ \tilde{u}(t - t_k) & \text{if } t_k \leq t \leq T + t_k \end{cases}$$

is the unique solution of (1.2.1) on $[0, T + t_k]$ which contradicts with the definition of T^* since $t_k + T > T^*$. This shows *(ii)*.

Continuous dependence. Consider any $T' < T^*(\varphi)$. Set $A = \sup_{t\in[0,T']} \|u(t)\|_{H^1}$. Let $\varphi_n \to \varphi$ in H^1 and $R = A$. Let T satisfy $\|\Lambda\|_{op} M' T^{1/2}(3A)^3 \leq A$ and $C_f T^{1/2}(3A)^2 < 1$. There exists $k \in \mathbb{N}$ such that $T'/k \leq T$. For some n_0, $n \geq n_0$ implies $\|\varphi_n\| \leq 2\|\varphi\|_{H^1}$. We have $\|\Lambda\|_{op} M'(T'/k)^{1/2}(A + \|\varphi_n\|_{H^1})^3 \leq A$, $C_f(T'/k)^{1/2}(A + \|\varphi_n\|_{H^1})^2 < 1$ for such n which imply that there exist unique solutions $u_n \in B_Y(S(\cdot)\varphi_n, A)$ on $[0, T'/k]$ with initial conditions φ_n (note that $Y = Y([0, T'/k])$). Also $\|\Lambda\|_{op} M'(T'/k)^{1/2}(A + \|\varphi\|_{H^1})^3 \leq A$, $C_f(T'/k)^{1/2}(A + \|\varphi\|_{H^1})^2 < 1$ which imply $u|_{[0,T'/k]} \in B_Y(S(\cdot)\varphi, A)$. Since $u_n \in B_Y(S(\cdot)\varphi_n, A)$ and $\|\varphi_n\|_{H^1} \leq 2\|\varphi\|_{H^1} \leq 2A$, we have

$$\|u_n\|_Y \leq \|S(\cdot)\varphi_n\|_Y + A \leq C\|\varphi_n\|_{H^1} + A \leq 2(C+1)A$$

by using Lemma 4.2. Similar result holds for u since $u \in B_Y(S(\cdot)\varphi, A)$. So there exists a constant $\bar{C}$ such that

$$\|u_n\|_Y, \|u\|_Y \leq \bar{C}A \tag{4.13}$$

(we consider u_n's with $n \geq n_0$). Since

$$u_n - u = S(\cdot)(\varphi_n - \varphi) - i\Lambda(H(u_n) - H(u)), \tag{4.14}$$

we have the estimates

$$\begin{aligned}
\|u_n - u\|_{\infty,2} &\leq C_s\|\varphi_n - \varphi\|_2 + C_{\infty,2}(\|u_n\|_{4,4}^2 + \|u\|_{4,4}^2)\|u_n - u\|_{4,4} \\
&\leq C_s\|\varphi_n - \varphi\|_{H^1} + C_{\infty,2}(T'/k)^{1/2}(\|u_n\|_{X_0}^2 + \|u\|_{X_0}^2)\|u_n - u\|_{4,4} \\
&\leq C_s\|\varphi_n - \varphi\|_{H^1} + 2C_{\infty,2}C_e^2(\bar{C})^2(T'/k)^{1/2}A^2\|u_n - u\|_{4,4}
\end{aligned}$$

by using Theorem A.2.1, Corollary 3.2, $X \hookrightarrow X_0$, (4.13) and $Y \hookrightarrow X_0$. Similarly

$$\|u_n - u\|_{4,4} \leq C_s\|\varphi_n - \varphi\|_{H^1} + 2C_{4,4}C_e^2(\bar{C})^2(T'/k)^{1/2}A^2\|u_n - u\|_{4,4}.$$

As a result of the last two inequalities there exist constants C_1, C_2 where C_1 is independent of T'/k and C_2 depends only on A such that

$$\|u_n - u\|_{\infty,2} + \|u_n - u\|_{4,4} \leq C_1\|\varphi_n - \varphi\|_{H^1} + C_2(T'/k)^{1/2}\|u_n - u\|_{4,4}. \tag{4.15}$$

We can increase k further to satisfy $C_2(T'/k)^{1/2} < 1$ and still have that u_n, u are defined on $[0, T'/k]$. So at last we have

$$\|u_n - u\|_{\infty,2} + \|u_n - u\|_{4,4} \leq C_3\|\varphi_n - \varphi\|_{H^1}, \tag{4.16}$$

for some constant C_3 which implies $u_n \to u$ both in $L^\infty([0, T'/k]; L^2)$ and in $L^4([0, T'/k]; L^4)$ as $n \to \infty$.

Taking gradients of both sides of (4.14) implies that

$$\nabla(u_n - u) = S(\cdot)\nabla(\varphi_n - \varphi) - i\Lambda\nabla(H(u_n) - H(u)). \tag{4.17}$$

From (4.17) we obtain

$$\begin{aligned}\|\nabla(u_n - u)\|_{\infty,2} &\leq C_s(\|\nabla(\varphi_n - \varphi)\|_2 + \|\nabla(H(u_n) - H(u))\|_{4/3,4/3}),\\ \|\nabla(u_n - u)\|_{4,4} &\leq C_s(\|\nabla(\varphi_n - \varphi)\|_2 + \|\nabla(H(u_n) - H(u))\|_{4/3,4/3}),\end{aligned} \tag{4.18}$$

by Theorem A.2.1 where C_s represents the maximum of the constants for $(q, r) = (\infty, 2)$ and $(4, 4)$ as before. We have to deal with $\|\nabla(H(u_n) - H(u))\|_{4/3,4/3}$. By the proof of Lemma 4.2, for $u, v \in Y$, $\nabla(K(|u|^2)v) \in L^{4/3}(I; L^{4/3})$ and $\nabla(K(|u|^2)v) = K(\nabla|u|^2)v + K(|u|^2)\nabla v$. We have

$$\begin{aligned}\|\nabla(H(u_n) - H(u))\|_{4/3,4/3} \leq C(&2\|u_n\|_{4,4}\|\nabla u_n\|_{4,4}\|u_n - u\|_{4,4} + \|u_n\|_{4,4}^2\|\nabla(u_n - u)\|_{4,4}\\ &+ (\|u_n\|_{4,4} + \|u\|_{4,4})\|\nabla u\|_{4,4}\|u_n - u\|_{4,4} + 2\|u_n\|_{4,4}\|u\|_{4,4}\|\nabla(u_n - u)\|_{4,4}\\ &+ 2\|u\|_{4,4}\|\nabla u\|_{4,4}\|u_n - u\|_{4,4}),\end{aligned}$$

where C depends only on $\|\alpha\|_\infty$, by first rearranging and then by using $(3.1)_2$. Again by a series of the application of the facts $Y \hookrightarrow X_0 \hookrightarrow X$ and (4.13), we obtain that for some constants D_1, D_2 depending only on A,

$$\begin{aligned}\|\nabla(H(u_n) - H(u))\|_{4/3,4/3} \\ \leq D_1(T'/k)^{1/4}\|u_n - u\|_{4,4} + D_2(T'/k)^{1/4}\|\nabla(u_n - u)\|_{4,4}.\end{aligned} \tag{4.19}$$

By (4.18) and (4.19), there exist constants C_4, C_5, C_6 where C_4 is independent of T'/k and C_5, C_6 depends only on A such that

$$\begin{aligned}&\|\nabla(u_n - u)\|_{\infty,2} + \|\nabla(u_n - u)\|_{4,4} \\ &\quad \leq C_4\|\varphi_n - \varphi\|_{H^1} + C_5(T'/k)^{1/4}\|u_n - u\|_{4,4} + C_6(T'/k)^{1/4}\|\nabla(u_n - u)\|_{4,4}.\end{aligned}$$

We can increase k further as before so that $C_6(T'/k)^{1/4} < 1$. As a result, there exist C_7, C_8 such that

$$\|\nabla(u_n - u)\|_{\infty,2} + \|\nabla(u_n - u)\|_{4,4} \leq C_7\|\varphi_n - \varphi\|_{H^1} + C_8(T'/k)^{1/4}\|u_n - u\|_{4,4}.$$

Since $u_n \to u$ in $L^4([0, T'/k]; L^4)$, we have $\nabla u_n \to \nabla u$ in $L^\infty([0, T'/k]; L^2)$ which together with the first part implies $u_n \to u$ in $L^\infty([0, T'/k]; H^1)$. Now taking $u(T'/k)$, $\{u_n(T'/k)\}_{n \geq n_0}$ instead of $\varphi, \{\varphi_n\}$ as initial conditions and modifying the above argument give $u_n \to u$ (after some $n_1 \geq n_0$) in $L^\infty([0, 2T'/k]; H^1)$ where we continue the previous solutions u_n beyond T'/k with this second application of the argument (k will not increase further in this second and later applications since all the constants, seen above, to determine k will appear in the same way) . Iterating this k times gives $u_n \to u$ in $L^\infty([0, T']; H^1)$. And the result for any compact subinterval of $[0, T^*)$ will follow from here (recall that we have chosen $T' < T^*$ arbitrary in the above argument).

Conservations. Taking the $H^{-1} - H^1$ duality product of $(1.2.1)_1$ with $2u$ gives

$$2i\langle u_t, u\rangle_{-1,1} = 2(\delta\|u_x\|_2^2 + \|u_y\|_2^2) + 2\int_{\mathbb{R}^2} K(|u|^2)|u|^2 \,\mathrm{d}\boldsymbol{x}.$$

on $[0, T^*)$. Since the right hand side is real, we obtain the mass conservation on $[0, T^*)$.

The computations done in Chapter 2 to establish energy conservation is meaningful with H^2-solutions which will be shown to exist in Chapter 7 . So let $\varphi_n \in H^2$ such that $\varphi_n \to \varphi$ in H^1. Given any compact subinterval I of $[0, T^*)$, for sufficiently large n, the corresponding solutions $u_n \in C([0, T^*(\varphi_n)); H^2) \cap C^1([0, T^*(\varphi_n)); L^2)$ are defined on I and $u_n \to u$ in $C(I; H^1)$ by *(iii)*. So we have $E(u_n(t)) \to E(u(t))$ for any $t \in I$. Since $E(u_n(t)) = E(\varphi_n)$ and $E(\varphi_n) \to E(\varphi)$, we obtain $E(u(t)) = E(\varphi)$ for $t \in I$, I being any compact subinterval of $[0, T^*)$. $\square$

We have global existence for the case $\delta = 1$ in the defocusing case (i.e. $\alpha \geq 0$):

Corollary 4.6 *Suppose that $\alpha(\xi) \geq 0$ for all $\xi \in \mathbb{R}^2$. Then H^1-solutions of (1.2.1), $\delta = 1$ are global.*

Proof. Let $f = |u|^2$ as before. We have

$$E(u(t)) = \|\nabla u(t)\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^2} K(|u(t)|^2)|u(t)|^2 \,\mathrm{d}\boldsymbol{x} = \|\nabla u(t)\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^2} \alpha|\hat{f}|^2(t)\,\mathrm{d}\boldsymbol{\xi}. \quad (4.20)$$

(4.20) and conservation of energy implies $\|\nabla u(t)\|_2^2 \leq E(u(t)) = E(\varphi)$ for $0 \leq t < T^*$ since we have the assumption on α. By mass conservation, $\|u(t)\|_{H^1}^2 \leq E(\varphi) + \|\varphi\|_2^2$ on $[0, T^*)$ which gives $T^* = \infty$ by Theorem 4.5, *(ii)*. $\square$

5. THE CAUCHY PROBLEM IN Σ

In order to justify rigorously what was done in [5, Section 6] for the global nonexistence results for GDS system and to generalize the argument to the problem (1.2.1), we need to establish local existence and uniqueness results for (1.2.1) in Σ.

For the weight considered in Σ we will define

$$Z = \{v \in Y : |\boldsymbol{x}|v \in X\} \tag{5.1}$$

with the norm $\|\cdot\|_Z = \|\cdot\|_Y + \|J\cdot\|_X$ where $(Jv)(t) = ((J_1 v)(t), (J_2 v)(t)) := (J_1(t)v(t), J_2(t)v(t))$ and

$$J_1(t)\psi := 2it\frac{\partial\psi}{\partial x} + \delta x\psi, \quad J_2(t)\psi := 2it\frac{\partial\psi}{\partial y} + y\psi, \tag{5.2}$$

for any sufficiently smooth space function ψ. So we consider $Z = L^\infty(I;\Sigma) \cap L^4(I;W^{1,4}) \cap L^4(I;L^4(|\boldsymbol{x}|^4\,\mathrm{d}\boldsymbol{x}))$ with $\|\cdot\|_Z$ where $I = [0,T]$, $T < \infty$ as before. We will investigate the contraction properties of $\mathcal{T}$ on Z for $H(u) = K(|u|^2)u$ where K is defined by (1.2.2).

Lemma 5.1 *Let Z, J be defined by (5.1), (5.2).*

(i) If $v \in Z$ then for every $t \in I$, $J(t)H(v(t)) \in L^{4/3} \cap L^2$ and there exists a constant κ depending on $\|\alpha\|_\infty$ and the operator norm of K on $L^{8/3}$ such that for $(p,q) = (4/3, 4)$ and $(2, 8)$

$$\|J(t)H(v(t))\|_p \le \kappa \|v(t)\|_q^2 \|J(t)v(t)\|_4. \tag{5.3}$$

(ii) Given $\varphi \in \Sigma$ and $R > 0$, $\mathcal{T}$ maps $B_Z(S(\cdot)\varphi, R)$ into itself and is a contraction on it with respect to the metric induced by $\|\cdot\|_X + \|J\cdot\|_X$ provided that T is sufficiently small.

Proof. *(i)* Let $v \in Z$ and define $\phi(x,y) = \delta x^2 + y^2$. Note that for $t \neq 0$,

$$J_k(t)v(t) = 2it\, e^{i\phi/4t} \frac{\partial}{\partial x_k}\{v(t)e^{-i\phi/4t}\},$$

for $k = 1, 2$ where $x_1 = x, x_2 = y$. Set $w(t) = v(t)e^{-i\phi/4t}$ for $t \neq 0$. Then $|w(t)| = |v(t)|$ and $w(t) \in H^1 \cap L^4$ (so $H(w(t)) \in W^{1,4/3}$) since $v(t) \in \Sigma$. We have

$$J_k(t)H(v(t)) = 2it\, e^{i\phi/4t} \frac{\partial}{\partial x_k}\{H(v(t))e^{-i\phi/4t}\} = 2ite^{i\phi/4t} \frac{\partial}{\partial x_k} H(w(t))$$

($v(t) \in H^1 \cap L^4 \cap L^4(|\boldsymbol{x}|^4 \,\mathrm{d}\boldsymbol{x})$ gives $J_k(t)H(v(t)) \in L^{4/3}$) and this implies $\|J_k(t)H(v(t))\|_{4/3} = 2|t|\|\partial_{x_k} H(w(t))\|_{4/3}$ for any $t \neq 0$. As in Lemma 4.2, $\nabla H(w) = K(\nabla|w|^2)w + K(|w|^2)\nabla w$ for $t \neq 0$. We proceed as

$$\begin{aligned} \|J(t)H(v(t))\|_{4/3} &\leq C_1(\|J_1(t)H(v(t))\|_{4/3} + \|J_2(t)H(v(t))\|_{4/3}) \\ &= 2C_1|t|(\|\partial_x H(w(t))\|_{4/3} + \|\partial_y H(w(t))\|_{4/3}) \\ &\leq C_2|t|\|w(t)\|_4^2\|\nabla w(t)\|_4 = C_2\|v(t)\|_4^2\|J(t)v(t)\|_4, \end{aligned} \tag{5.4}$$

where, to obtain the second inequality, $(3.1)_1$ is used so $C_2 = C_2(\|\alpha\|_\infty)$. Similarly,

$$\int_{\mathbb{R}^2} |J(t)H(v(t))|^2 \,\mathrm{d}\boldsymbol{x} = 4|t|^2 \int_{\mathbb{R}^2} |\nabla H(w(t))|^2 \,\mathrm{d}\boldsymbol{x} \leq C_3\|v(t)\|_8^4\|J(t)v(t)\|_4^2,$$

by using $(3.1)_3$. So we have

$$\|J(t)H(v(t))\|_2 \leq C_4\|v(t)\|_8^2\|J(t)v(t)\|_4, \tag{5.5}$$

with C_4 depending on the operator norm of K on $L^{8/3}$. (5.4) and (5.5) imply (5.3) for $t \neq 0$. For $t = 0$, $J(0)H(v(0)) = (\delta x H(v(0)), yH(v(0)))$, $J(0)v(0) = (\delta x v(0), yv(0))$ and estimates (3.1) give the result as above by using $v(0) \in \Sigma \cap L^4(|\boldsymbol{x}|^4 \,\mathrm{d}\boldsymbol{x})$.

(ii) A formal computation shows that J_k commutes with $i\partial_t + \delta\partial_{xx} + \partial_{yy}$. By density we have $J(t)S(t)\varphi = S(t)J(0)\varphi$ on $\mathbb{R}$ for $\varphi \in \Sigma$ which gives $S(\cdot)\varphi \in C(\mathbb{R}; \Sigma)$

(this property is known as the conformal invariance property of the linear Schrödinger operator [21] and an extension to non-elliptic case is by [13, Lemma 3.1]). By (5.3), $JH(v) \in L^{4/3}(I; L^{4/3})$ with $\|JH(v)\|_{4/3,4/3} \leq \kappa \|v\|_{4,4}^2 \|Jv\|_{4,4}$ and $J\Lambda H(v) = \Lambda JH(v)$ for $v \in Z$ as a result of the above commutation, where Λ is defined by (3.4) on I.

Let $\varphi \in \Sigma$. $S(\cdot)\varphi \in Z$ by Strichartz's estimates. Let $v \in B_Z(S(\cdot)\varphi, R) \subset B_Y(S(\cdot)\varphi, R)$. As in (4.11),

$$\|\mathcal{T}v - S(\cdot)\varphi\|_Y \leq \|\Lambda\|_{op} M T^{1/2} (R + \|S(\cdot)\varphi\|_Z)^3. \tag{5.6}$$

$J(\mathcal{T}v - S(\cdot)\varphi) = \Lambda JH(v)$ by the commutation and $\Lambda JH(v) \in X$ since $JH(v) \in L^{4/3}(I; L^{4/3})$, by Theorem A.2.1. We get

$$\|J(\mathcal{T}v - S(\cdot)\varphi)\|_X = \|\Lambda JH(v)\|_X \leq C_1 \|JH(v)\|_{4/3,4/3} \leq C_2 \|v\|_{4,4}^2 \|Jv\|_{4,4}$$

for some C_1, C_2 which are multiples of the constants in Strichartz's estimates and κ using Theorem A.2.1, *(ii)* and (5.3). $\|v\|_{4,4}^2 \leq T^{1/2} \|v\|_{\infty,4}^2 \leq C_e^2 T^{1/2} \|v\|_Y^2$, $\|Jv\|_{4,4} \leq \|Jv\|_X \leq \|v\|_Z$ imply

$$\|J(\mathcal{T}v - S(\cdot)\varphi)\|_X \leq C_2 C_e^2 T^{1/2} \|v\|_Z^3 \leq C_2 C_e^2 T^{1/2} (R + \|S(\cdot)\varphi\|_Z)^3, \tag{5.7}$$

where C_e, C_2 are independent of T. By (5.6) and (5.7), for some C_3 independent of T,

$$\|\mathcal{T}v - S(\cdot)\varphi\|_Z \leq C_3 T^{1/2} (R + \|S(\cdot)\varphi\|_Z)^3, \tag{5.8}$$

for any $v \in B_Z(S(\cdot)\varphi, R)$.

Let $v_1, v_2 \in B_Z(S(\cdot)\varphi, R) \subset B_Y(S(\cdot)\varphi, R) \subset B_{X_0}(0, C_e(R + \|S(\cdot)\varphi\|_Y))$, by (4.5)

$$\|\mathcal{T}v_1 - \mathcal{T}v_2\|_X \leq C_\gamma C_e^2 T^{1/2} (R + \|S(\cdot)\varphi\|_Z)^2 \|v_1 - v_2\|_X. \tag{5.9}$$

Next we need to estimate $\|J(\mathcal{T}v_1 - \mathcal{T}v_2)\|_X$. $J(\mathcal{T}v_1 - \mathcal{T}v_2) = J\Lambda(H(v_1) - H(v_2)) = \Lambda J(H(v_1) - H(v_2))$. So

$$\begin{aligned} \|J(\mathcal{T}v_1 - \mathcal{T}v_2)\|_X &\leq \|\Lambda J(H(v_1) - H(v_2))\|_{\infty,2} \vee \|\Lambda J(H(v_1) - H(v_2))\|_{4,4} \\ &\leq C_4\|J(H(v_1) - H(v_2))\|_{4/3,4/3} \end{aligned} \tag{5.10}$$

where C_4 is a constant depending on the constants coming from Strichartz's estimates. Let $w_j(s) = v_j(s)e^{-i\phi/4s}$, $j = 1, 2$ for $s \neq 0$ then

$$|J(s)(H(v_1(s)) - H(v_2(s)))| = 2|s||\nabla(H(w_1(s)) - H(w_2(s)))| \tag{5.11}$$

for $s \neq 0$. By rearranging first and then by using $(3.1)_1$

$$\begin{aligned} \|\nabla(H(w_1) - H(w_2))\|_{4/3} \leq C_5\{&\|w_1 - w_2\|_4(\|w_1\|_4 + \|w_2\|_4)(\|\nabla w_1\|_4 + \|\nabla w_2\|_4) \\ &+ \|\nabla(w_1 - w_2)\|_4(\|w_1\|_4^2 + \|w_2\|_4^2)\}. \end{aligned}$$

where C_5 depends on C_α and is independent of $s \in I$ and of T. (5.11) and the last inequality imply

$$\begin{aligned} \|J(H(v_1) - H(v_2))\|_{4/3} \leq C_5\{&\|v_1 - v_2\|_4(\|v_1\|_4 + \|v_2\|_4)(\|Jv_1\|_4 + \|Jv_2\|_4) \\ &+ \|J(v_1 - v_2)\|_4(\|v_1\|_4^2 + \|v_2\|_4^2)\} \end{aligned}$$

a.e. on I. So we obtain

$$\begin{aligned} \|J(H(v_1) - H(v_2))\|_{4/3,4/3} &\leq C_6\{\|v_1 - v_2\|_{4,4}(\|v_1\|_{4,4} + \|v_2\|_{4,4})(\|Jv_1\|_{4,4} \\ &\quad + \|Jv_2\|_{4,4}) + \|J(v_1 - v_2)\|_{4,4}(\|v_1\|_{4,4}^2 + \|v_2\|_{4,4}^2)\} \\ &\leq C_7T^{1/4}(R + \|S(\cdot)\varphi\|_Z)^2\{\|v_1 - v_2\|_X + \|J(v_1 - v_2)\|_X\} \end{aligned} \tag{5.12}$$

where we apply the Hölder inequality to the time integral for the first inequality and we obtain the second one from the facts $B_Z(S(\cdot)\varphi, R) \subset B_Y(S(\cdot)\varphi, R)) \subset B_{X_0}(0, C_e(R + \|S(\cdot)\varphi\|_Z))$, $\|v_i\|_{4,4} \leq T^{1/4}\|v_i\|_{\infty,4} \leq T^{1/4}\|v_i\|_{X_0}$, $\|\cdot\|_X \leq \|\cdot\|_Y \leq \|\cdot\|_Z$, and $Y \subset X_0$. C_{11} is independent of T as in the the previous estimates. Finally by (5.9), (5.10) and

(5.12) for some constant C_8 independent of T,

$$\|\mathcal{T}v_1 - \mathcal{T}v_2)\|_X + \|J(\mathcal{T}v_1 - \mathcal{T}v_2))\|_X \\ \leq C_8 T^{1/4}(R + \|S(\cdot)\varphi\|_Z)^2(\|v_1 - v_2\|_X + \|J(v_1 - v_2)\|_X) \quad (5.13)$$

for $v_1, v_2 \in B_Z(S(\cdot)\varphi, R)$ and T sufficiently small. The proof is complete when we consider (5.8), (5.13) and reduce T further if necessary. □

With this contraction property, the following existence and uniqueness result follows:

Theorem 5.2 *Given $\varphi \in \Sigma$, there exists a unique maximal solution $u \in C([0,T^*);\Sigma) \cap C^1([0,T^*);H^{-1})$ solving (1.2.1) on $[0,T^*)$ with the following properties:*

(i) (Further regularity) $|\boldsymbol{x}|u$, $\nabla u \in L^4([0,t];L^4)$ for every $t < T^*$.

(ii) (Blow-up) $T^* < \infty$ implies that $\|u\|_{L^\infty([0,T^*);\Sigma)} = \infty$.

(iii) (Σ-solutions are H^1-solutions) $[0,T^*)$ coincides with the maximal interval of existence for the H^1-solution in Theorem 4.5 with initial data φ.

(iv) (Virial Identity, Pseudo-conformal Conservation) For $\delta = 1$, $t \mapsto I(t) = \int_{\mathbb{R}^2} |\boldsymbol{x}|^2 |u(t,\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} \in C^2([0,T^*))$ and for every $t \in [0,T^*)$

$$I'(t) = 4Im \int_{\mathbb{R}^2} (x\bar{u}u_x + y\bar{u}u_y) \, \mathrm{d}x \, \mathrm{d}y,$$
$$I''(t) = 8E(u(t)).$$

Moreover $E_{pc}(u)$ as defined in (2.13) is in $C^1([0,T^*))$ and is conserved.

(v) (Continuous Dependence) $\varphi_n \to \varphi$ in Σ and u_n's are the corresponding solutions. Then for any $I \Subset [0,T^*)$ and for any n sufficiently large u_n's are defined on I and $u_n \to u$ in $C(I;\Sigma)$.

Proof. $B_Z(S(\cdot)\varphi, R)$ is complete when considered with the norm $\|\cdot\|_X + \|J\cdot\|_X$ with an argument similar to given in Remark 4.4. So we have a unique local solution

$u = \mathcal{T}u$ in $C(I;H^1) \cap C^1(I;H^{-1})$ $(B_Z(S(\cdot)\varphi), R) \subset B_Y(S(\cdot)\varphi, R))$ solving (1.2.1) on I (uniqueness follows from Proposition 3.3). $Ju = J\mathcal{T}u = S(\cdot)J(0)\varphi - i\Lambda J(H(u)) \in C(I;L^2)$ by Strichartz's estimates since $\varphi \in \Sigma$ and $JH(u) \in L^{4/3}(I;L^{4/3})$. As a result $u \in C(I;\Sigma)$. Moreover $u \in Z$ implies $\boldsymbol{x}u, \nabla u \in L^4(I;L^4)$. Similar to the previous definitions of maximal interval of existence, let $T^* = \sup\{T > 0 : \exists u \in C(I;\Sigma) \cap C^1(I;H^{-1})$ solving (1.2.1) on I with $|\boldsymbol{x}|u, \nabla u \in L^4([0,T];L^4)\}$. By Proposition 3.3, we obtain the unique maximal solution which also satisfies *(i)*. We also deduce *(ii)* in a similar way leading to H^1-blow-up in Theorem 4.5. (by replacing H^1-norm with Σ-norm and using the fact that $\|S(\cdot)\varphi\|_Z$ is a constant (independent of I) multiple of $\|\varphi\|_\Sigma$ which follows from Theorem A.2.1).

(iii) Let $[0, T_1^*)$ be the maximal interval of existence for the solution in Theorem 4.5 with initial data $\varphi \in \Sigma$ and let v denote this solution. $T_1^* \geq T^*$ (since otherwise $u \in C([0,T^*);H^1) \cap C^1([0,T^*);H^{-1})$ satisfies (1.2.1) and also $\nabla u \in L^4([0,t];L^4)$ for every $t < T^*$ which contradicts with the definition of T_1^*). We have $u = v$ on $[0,T^*)$ by Proposition 3.3. We claim that $T^* = T_1^*$. Assume on the contrary that $T^* < T_1^*$. This implies $T^* < \infty$, $\|u\|_{L^\infty([0,T^*);\Sigma)} = \infty$ by *(ii)* and $\|u\|_{L^\infty([0,T^*);H^1)} < \infty$. So $\lim_{t\to T^*-} \|u\|_{Y([0,t])} + \|Ju\|_{X([0,t])} = \infty$ ($u \in C([0,T^*);\Sigma)$). But $T^* < T_1^*$ implies $\|u\|_{Y([0,T^*])} < \infty$ and this, in turn, yields $\lim_{t\to T^*-} \|Ju\|_{X([0,t])} = \infty$. Since $u = \mathcal{T}u$ in Z on $[0,T^*)$, $J(t)u(t) = S(t)J(0)\varphi - (\Lambda JH(u))(t)$ on $[0,T^*)$ by the commutation property of J. We deduce that, for $t \in [0,T^*)$

$$\begin{aligned}
\|J(t)u(t)\|_4 &\leq \|S(t)J(0)\varphi\|_4 + C\int_0^t (t-s)^{-1/2}\|J(s)H(u(s))\|_{4/3}\,\mathrm{d}s \\
&\leq \|S(t)J(0)\varphi\|_4 + C\int_0^t (t-s)^{-1/2}\|u(s)\|_4^2\|J(s)u(s)\|_4\,\mathrm{d}s \\
&\leq \|S(t)J(0)\varphi\|_4 + Ct^{1/4}\left(\int_0^t \|J(s)u(s)\|_4^4\,\mathrm{d}s\right)^{1/4}
\end{aligned}$$

by using the property $\|S(t)\psi\|_p \leq (4\pi|t|)^{-(1-2/p)}\|\psi\|_{p'}$, $p \in [2,\infty]$, $t \neq 0$ (see e.g. A.9, [13]), (5.4), the fact that $\|u\|_{L^\infty([0,T^*];L^4)}$ is finite and the Hölder inequality in the order they were mentioned. C denotes changing constants and is independent of the particular time $t < T^*$. So for $t \in [0, T^*)$,

$$\|J(t)u(t)\|_4^4 \le C\left(\|S(t)J(0)\varphi\|_4^4 + \int_0^t \|J(s)u(s)\|_4^4 \,\mathrm{d}s\right). \tag{5.14}$$

$\|S(\cdot)J(0)\varphi\|_{L^4(\mathbb{R};L^4)} \le C_s\|J(0)\varphi\|_2$, we obtain $\|Ju\|_{L^4((0,T^*);L^4)} < \infty$ using Gronwall's lemma with (5.14). Similarly,

$$\begin{aligned}\|J(t)u(t)\|_2 &\le \|S(t)J(0)\varphi\|_2 + C_4\int_0^t \|u(s)\|_8^2\|J(s)u(s)\|_4 \,\mathrm{d}s\\ &\le C\left(\|S(t)J(0)\varphi\|_2 + \int_0^t \|J(s)u(s)\|_4 \,\mathrm{d}s\right)\end{aligned}$$

since $S(t)$ is unitary on L^2 and (5.5) and $\|u\|_{L^\infty([0,T^*];L^8)} < \infty$ hold. We get $\|Ju\|_{L^\infty([0,T^*);L^2)} \le C(\|\varphi\|_\Sigma + \|Ju\|_{L^1([0,T^*);L^4)}) < \infty$ after taking supremum of each side of the above inequality on $[0, T^*)$ and using $\|Ju\|_{L^4((0,T^*);L^4)} < \infty$ obtained in the previous step. Finiteness of the last two norms contradicts with $\lim_{t\to T^*-}\|Ju\|_{X([0,t])} = \infty$. So we have $T^* = T_1^*$.

(iv), (v) (We modify the arguments in [1, Proposition 6.5.1]) Since $(1.2.1)_1$ holds in H^{-1} on $[0, T^*)$, we can not just multiply by $|\boldsymbol{x}|^2\bar{u} \notin H^1$, take the imaginary part and then integrate as we did in Chapter 2. Instead we need a regularization. Let $J = [0, T]$, $T < T^*$. For $\varepsilon > 0$, $\boldsymbol{x} \mapsto e^{-2\varepsilon|\boldsymbol{x}|^2}|\boldsymbol{x}|^2 u(t,\boldsymbol{x}) \in H^1$ for every $t \in [0, T^*)$ since $\boldsymbol{x} \mapsto e^{-2\varepsilon|\boldsymbol{x}|^2} \in \mathcal{S}$. We have

$$\begin{aligned}\operatorname{Re}\langle u_t, e^{-2\varepsilon|\boldsymbol{x}|^2}|\boldsymbol{x}|^2 u\rangle_{-1,1} &= \operatorname{Im}\int_{\mathbb{R}^2}\{\delta u_x[e^{-2\varepsilon|\boldsymbol{x}|^2}|\boldsymbol{x}|^2\bar{u}]_x + u_y[e^{-2\varepsilon|\boldsymbol{x}|^2}|\boldsymbol{x}|^2\bar{u}]_y\}\,\mathrm{d}\boldsymbol{x}\\ &= \operatorname{Im}\int_{\mathbb{R}^2}\bar{u}\{\delta u_x[e^{-2\varepsilon|\boldsymbol{x}|^2}|\boldsymbol{x}|^2]_x + u_y[e^{-2\varepsilon|\boldsymbol{x}|^2}|\boldsymbol{x}|^2]_y\}\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{5.15}$$

by considering the $H^{-1}-H^1$ duality product of $(1.2.1)_1$ with $e^{-2\varepsilon|\boldsymbol{x}|^2}|\boldsymbol{x}|^2 u$ and then taking the imaginary part. Since $u \in C(J; H^1) \cap C^1(J; H^{-1})$, $I'_\varepsilon(t) = 2\operatorname{Re}\langle u_t, e^{-2\varepsilon|\boldsymbol{x}|^2}|\boldsymbol{x}|^2 u\rangle_{-1,1}$ in $\mathcal{D}'((0,T))$ where $I_\varepsilon(t) = \int_{\mathbb{R}^2} e^{-2\varepsilon|\boldsymbol{x}|^2}|\boldsymbol{x}|^2|u(t)|^2\,\mathrm{d}\boldsymbol{x}$. So we obtain

$$I'_\varepsilon(t) = 4\operatorname{Im}\int_{\mathbb{R}^2} e^{-2\varepsilon|\boldsymbol{x}|^2}\{\delta(1-2\varepsilon|\boldsymbol{x}|^2)x\bar{u}u_x + (1-2\varepsilon|\boldsymbol{x}|^2)y\bar{u}u_y\}\,\mathrm{d}\boldsymbol{x},$$

by (5.15) and this in turn implies

$$I_\varepsilon(t) = I_\varepsilon(0) + 4\int_0^t \operatorname{Im}\int_{\mathbb{R}^2} e^{-2\varepsilon|\boldsymbol{x}|^2}\{\delta(1-2\varepsilon|\boldsymbol{x}|^2)x\bar{u}u_x + (1-2\varepsilon|\boldsymbol{x}|^2)y\bar{u}u_y\}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \quad (5.16)$$

on J since both sides are real valued continuous functions of time. For $t \in J$ fixed, let $\varepsilon \to 0$ in (5.16). Using DCT for time and space integrals gives

$$\int_{\mathbb{R}^2} |\boldsymbol{x}|^2|u(t)|^2\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} |\boldsymbol{x}|^2|\varphi|^2\,\mathrm{d}x\,\mathrm{d}y + 4\int_0^t \operatorname{Im}\int_{\mathbb{R}^2}(\delta x\bar{u}u_x + y\bar{u}u_y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \quad (5.17)$$

on J so by FTC, $I \in C(J)$ and the first equality of *(iv)* (with $\delta = 1$) holds on J. T is any time less than T^*, so the result holds on $[0, T^*)$.

Next we show *(v)*. Let $J = [0, T]$, $T < T^*$ as above. $\varphi_n \to \varphi$ in Σ implies, for n sufficiently large, (we consider such n's hereafter) the corresponding solutions $u_n \in C([0, T^*(\varphi_n); \Sigma))$ are defined on J and $u_n \to u$ in $C(J; H^1)$ by Theorem 4.5, *(iii)*. We want to show that $|\boldsymbol{x}|u_n \to |\boldsymbol{x}|u$ in $C(J; L^2)$. Assume on the contrary that there exists $\varepsilon > 0$ such that for every n there exists t_n with $\||\boldsymbol{x}|u_n(t_n) - |\boldsymbol{x}|u(t_n)\|_2 \geq \varepsilon$. For some subsequence of t_n which we still denote by t_n, $t_n \to \tau$ for some $\tau \in J$. Since $u_n \to u$ in $C(J; H^1)$, $(u_n)_n$ is bounded in $L^\infty(J; H^1) \subset L^2(J; H^1)$ and $(H(u_n))_n$ is bounded in $L^\infty(I; H^{-1})$ by using $\|H(u_n(t))\|_{4/3} \leq C_\alpha\|u_n(t)\|_4^3$. $(1.2.1)_1$ implies that $(u_{nt})_n$ is bounded in $L^\infty(J; H^{-1}) \subset L^2(J; H^{-1})$. We obtain $u_n \in C(J; L^2)$ and for $t, s \in J$

$$\begin{aligned}\|u_n(t) - u_n(s)\|_2^2 &= 2\int_s^t \operatorname{Re}\langle u_{nt}(\tau), u_n(\tau) - u_n(s)\rangle_{-1,1}\,\mathrm{d}\tau \\ &\leq 4\|u_n\|_{L^\infty(J;H^1)}\|u_{nt}\|_{L^\infty(J;H^{-1})}|t-s|,\end{aligned}$$

which implies that $(u_n)_n$'s are bounded in $C^{0,1/2}(J; L^2)$. So $u_n(t_n) \to u(\tau)$ in L^2 since $\|u_n(t_n) - u_n(\tau)\|_2 \leq C|t_n - \tau|^{1/2}$. We also have

$$\||\boldsymbol{x}|u_n(t)\|_2^2 = \||\boldsymbol{x}|\varphi_n\|_2^2 + 4\int_0^t \operatorname{Im}\int_{\mathbb{R}^2}(\delta x\bar{u}_n u_{nx} + y\bar{u}_n u_{ny})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \quad (5.18)$$

on J. (5.18) gives that

$$\||\boldsymbol{x}|u_n(t)\|_2^2 \leq \||\boldsymbol{x}|\varphi_n\|_2^2 + C\int_0^t \||\boldsymbol{x}|u_n(s)\|_2 \|\nabla u_n(s)\|_2 \,\mathrm{d}s$$

from which the uniform boundedness of $\||\boldsymbol{x}|u_n(t)\|_2$ in $t \in J$ and n is deduced by using Gronwall's lemma and the fact that $\|\nabla u_n\|_{L^\infty(J;L^2)}$'s are bounded. Therefore there exists a weakly convergent subsequence of $|\boldsymbol{x}|u_n(t_n)$ which we still denote as $|\boldsymbol{x}|u_n(t_n)$. We have $|\cdot|u_n(t_n) \rightharpoonup |\cdot|u(\tau)$ in L^2 since $u_n(t_n) \to u(\tau)$ in L^2. By using (5.17) and (5.18) we also obtain $\||\boldsymbol{x}|u_n(t_n)\|_2 \to \||\boldsymbol{x}|u(\tau)\|_2$. Norm and weak convergence in L^2 imply $|\boldsymbol{x}|u_n(t_n) \to |\boldsymbol{x}|u(\tau)$ in L^2 and this gives $|\boldsymbol{x}|u_n(t_n) \to |\boldsymbol{x}|u(t_n)$ in L^2 which contradicts with the assumption. So we have $|\boldsymbol{x}|u_n \to |\boldsymbol{x}|u$ in $C(J; L^2)$. The result on compact sets follows since $J = [0, T]$ and T is arbitrary satisfying $T < T^*$.

For the virial identity in *(iv)*, we first consider a regular initial data. We will give the details for the computations related to the nonlocal nonlinearity and refer to [1] for the standard computations.

Case 1: $\varphi \in H^2 \cap \Sigma$. By the regularity results, Corollary 7.7, the corresponding solution $u \in C([0, T^*); \Sigma)$ is also in $C([0, T^*); H^2) \cap C^1([0, T^*); L^2)$. Let $\theta_\varepsilon(\boldsymbol{x}) = e^{-\varepsilon|\boldsymbol{x}|^2}$ for $\varepsilon > 0$. Define

$$h_\varepsilon(t) = \operatorname{Im}\int_{\mathbb{R}^2} \theta_\varepsilon \bar{u}\,(\boldsymbol{x}\cdot\nabla)u \,\mathrm{d}\boldsymbol{x} \quad \text{for } t \in [0, T^*).$$

As obtained in [1, Step 1, p. 181], $h_\varepsilon \in C^1([0, T^*))$ with $h_\varepsilon'(t) = -\operatorname{Im}\int_{\mathbb{R}^2} u_t\{2\theta_\varepsilon r\partial_r\bar{u} + (2\theta_\varepsilon + r\partial_r\theta_\varepsilon)\bar{u}\}\,\mathrm{d}\boldsymbol{x}$ so by the equation $(1.2.1)_1$ we obtain

$$h_\varepsilon'(t) = \operatorname{Re}\int_{\mathbb{R}^2} (H(u) - \Delta u)\{2\theta_\varepsilon r\partial_r\bar{u} + (2\theta_\varepsilon + r\partial_r\theta_\varepsilon)\bar{u}\}\,\mathrm{d}\boldsymbol{x}. \tag{5.19}$$

We proceed by assuming $u \in \mathcal{D}$. We have

$$\begin{aligned} &-\mathrm{Re}\int_{\mathbb{R}^2} \Delta u\{2\theta_\varepsilon r\partial_r\bar{u} + (2\theta_\varepsilon + r\partial_r\theta_\varepsilon)\bar{u}\}\,\mathrm{d}\boldsymbol{x} \\ &\qquad = 2\int_{\mathbb{R}^2} \theta_\varepsilon|\nabla u|^2\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^2} \{2r\partial_r\theta_\varepsilon|\partial_r u|^2 + (3\partial_r\theta_\varepsilon + r\partial_r^2\theta_\varepsilon)\mathrm{Re}\,(\bar{u}\partial_r u)\}\,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{5.20}$$

which also follows from [1, (6.5.15)]. For the part containing the nonlinearity $H(u)$ in (5.19), using $\mathrm{Re}\{u[2\theta_\varepsilon r\partial_r\bar{u} + (2\theta_\varepsilon + r\partial_r\theta_\varepsilon)\bar{u}]\} = \nabla\cdot(\boldsymbol{x}\theta_\varepsilon|u|^2)$ we get

$$\begin{aligned} \mathrm{Re}\int_{\mathbb{R}^2} H(u)[2\theta_\varepsilon r\partial_r\bar{u} + (2\theta_\varepsilon + r\partial_r\theta_\varepsilon)\bar{u}]\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^2} K(|u|^2)\,\nabla\cdot(\boldsymbol{x}\theta_\varepsilon|u|^2)\,\mathrm{d}\boldsymbol{x} \\ &= -\int_{\mathbb{R}^2} \theta_\varepsilon|u|^2\boldsymbol{x}\cdot K(\nabla|u|^2)\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{5.21}$$

On the other hand, using the identity $\nabla\cdot(H(u)\bar{u}\boldsymbol{x}\theta_\varepsilon) = 2\theta_\varepsilon H(u)\bar{u} + H(u)\bar{u}\nabla\theta_\varepsilon\cdot\boldsymbol{x} + \theta_\varepsilon|u|^2\boldsymbol{x}\cdot K(\nabla|u|^2) + K(|u|^2)\theta_\varepsilon(\boldsymbol{x}\cdot\nabla)|u|^2$,

$$\begin{aligned} &-\int_{\mathbb{R}^2} \theta_\varepsilon|u|^2\boldsymbol{x}\cdot K(\nabla|u|^2)\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} 2\theta_\varepsilon H(u)\bar{u}\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^2} K(|u|^2)\theta_\varepsilon\,(\boldsymbol{x}\cdot\nabla)|u|^2\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^2} H(u)\bar{u}\,(\boldsymbol{x}\cdot\nabla)\theta_\varepsilon\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{5.22}$$

By the last equality, (5.21) and (5.20), (5.19) can be written as

$$\begin{aligned} h_\varepsilon'(t) =& 2\int_{\mathbb{R}^2} \theta_\varepsilon|\nabla u|^2\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^2} [2r\partial_r\theta_\varepsilon|\partial_r u|^2 + (3\partial_r\theta_\varepsilon + r\partial_r^2\theta_\varepsilon)\mathrm{Re}\,(\bar{u}\partial_r u)]\,\mathrm{d}\boldsymbol{x} \\ &+ 2\int_{\mathbb{R}^2} \theta_\varepsilon H(u)\bar{u}\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^2} K(|u|^2)\theta_\varepsilon(\boldsymbol{x}\cdot\nabla)|u|^2\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^2} H(u)\bar{u}\,(\boldsymbol{x}\cdot\nabla)\theta_\varepsilon\,\mathrm{d}\boldsymbol{x} \end{aligned}$$

for every $u \in H^2$ by density. By using DCT

$$\lim_{\varepsilon\to 0} h_\varepsilon'(t) = 2\|\nabla u\|_2^2 + 2\int_{\mathbb{R}^2} H(u)\bar{u}\,\mathrm{d}\boldsymbol{x} + \lim_{\varepsilon\to 0}\int_{\mathbb{R}^2} K(|u|^2)\theta_\varepsilon(\boldsymbol{x}\cdot\nabla)|u|^2\,\mathrm{d}\boldsymbol{x}.$$

Now we claim that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2} K(|u|^2)\theta_\varepsilon(\boldsymbol{x}\cdot\nabla)|u|^2\,\mathrm{d}\boldsymbol{x} = -\int_{\mathbb{R}^2} H(u)\bar{u}\,\mathrm{d}\boldsymbol{x}. \tag{5.23}$$

For the result, we exploit the properties of α as in Chapter 2. Let $f = |u|^2$ as before.

$$\begin{aligned}\int_{\mathbb{R}^2} K(f)\theta_\varepsilon(\boldsymbol{x}\cdot\nabla)f\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^2} [\widehat{K(f)}]_{\xi_1}[(\xi_1\bar{\hat{f}}) * \hat{\theta}_\varepsilon] + [\widehat{K(f)}]_{\xi_2}[(\xi_2\bar{\hat{f}}) * \hat{\theta}_\varepsilon]\,\mathrm{d}\boldsymbol{\xi}\\ &= \int_{\mathbb{R}^2} (\alpha_{\xi_1}\hat{f} + \alpha\hat{f}_{\xi_1})[(\xi_1\bar{\hat{f}}) * \hat{\theta}_\varepsilon] + (\alpha_{\xi_2}\hat{f} + \alpha\hat{f}_{\xi_2})[(\xi_2\bar{\hat{f}}) * \hat{\theta}_\varepsilon]\,\mathrm{d}\boldsymbol{\xi}.\end{aligned}$$

Since $\iint_{\mathbb{R}^2}\hat{\theta}_\varepsilon\,\mathrm{d}\boldsymbol{\xi} = 1$ for any $\varepsilon > 0$, by approximations to the identity, the last integral converges to $\mathcal{J}$, see (2.11). Handling $\mathcal{J}$ as we did there gives (5.23).

Also $\lim_{\varepsilon\to 0} h_\varepsilon(t) = \operatorname{Im}\int_{\mathbb{R}^2}\bar{u}\boldsymbol{x}\cdot\nabla u\,\mathrm{d}\boldsymbol{x}$, which is equal to $h(t)$ by definition. So we obtain $h \in C^1([0,T^*))$ and $h'(t) = 2E(u(t))$ and by definitions of h and I, $4h = I'$.

Case 2: $\varphi \in \Sigma$. This follows as in , [1, Step 2, p. 182] where the necessary continuous dependence in our case follows from Theorem 5.2, *(v)*. □

According to Theorem 5.2, *(iv)*, $I(t) = I(0) + I'(0)t + 4E(\varphi)t^2$ for $t \in [0,T^*)$ using also the conservation of energy. Any of the three conditions on the initial data to be given below forces I to attain negative values after a finite time. So we can state a generalized version of [5, Theorem 6.1] on the blow-up of the solutions with initial data in Σ:

Corollary 5.3 *Let u be the solution of the Cauchy problem* (1.2.1), $\delta = 1$ *with initial value $\varphi \in \Sigma$. If one of the conditions $E(\varphi) < 0$ or $E(\varphi) = 0$ and Im $\int_{\mathbb{R}^2}\bar{\varphi}\boldsymbol{x}\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x} < 0$ or $E(\varphi) > 0$ and $-$Im $\int_{\mathbb{R}^2}\bar{\varphi}\boldsymbol{x}\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x} > \sqrt{E(\varphi)I(0)}$ holds then $T^* < \infty$ and so, as a result of Theorem 4.5, (ii), u blows up in finite time.*

The existence of an initial data φ satisfying one of the conditions above requires more specific information on α, for DS system see [13] and for GDS system see [5, 12].

It is possible to obtain explicit blow-up solutions in the elliptic case. Generalizing the results in [9, 11] as described in Conclusion we obtain global solutions of the form $u(T,\boldsymbol{X}) = e^{iT}R(\boldsymbol{X})$ where R is a ground state solution satisfying $\Delta R - R = K(R^2)R$.

We have $\|u\|_{L^4(\mathbb{R};L^4)} = \infty$. By using the pseudo-conformal transformation (2.12) with $a = d = 0$, $b = -1$, $c = 1$ we obtain the solution $U(t, \boldsymbol{x}) = \frac{1}{t} e^{-\frac{i}{t} + i\frac{|\boldsymbol{x}|^2}{4t}} R(\boldsymbol{x}/t)$ (see Theorem 3.6). After translating U in the positive direction in time, we obtain a solution having subminimal mass $\|R\|_2$ and blowing-up in finite positive time since (6.5) implies that$\|U\|_{L^4((-\infty,0);L^4)} = \infty$. We did not consider in this work the characterization of the minimal blow-up solutions which was established in [22] with initial data in H^1 in the case of NLS with pure power nonlinearity with the critical exponent.

6. ASYMPTOTIC BEHAVIOR AND SCATTERING OF SOLUTIONS

In this section we consider the asymptotic behavior of the global solutions. We mainly deal with the case $\delta = 1$. For the results we will utilize the pseudo-conformal invariance of $(1.2.1)_1$. The pseudo-conformal transformation was defined in Chapter 2 (see (2.12) and the definitions preceding it). With these

$$\|U(t)\|_2 = \|u(T)\|_2. \tag{6.1}$$

where $T = (c + dt)/(a + bt)$. Also for $\beta \geq 0$ and $d - bT > 0$

$$\|U(t)\|_{\beta+2} = (d - bT)^{\frac{\beta}{\beta+2}} \|u(T)\|_{\beta+2}, \tag{6.2}$$

$$\|\boldsymbol{x}U(t)\|_2 = (d - bT)^{-1} \|\boldsymbol{X}u(T)\|_2, \tag{6.3}$$

$$\|\nabla U(t)\|_2 = \frac{1}{2} \|(-b\boldsymbol{X} + 2i(d - bT)\nabla))u(T)\|_2. \tag{6.4}$$

For $bT_1 \geq -d$, $bT_2 \leq d$ and t_1, t_2 as defined above (2.12) in Section 2,

$$\|U\|_{L^4((-t_1,t_2);L^4)} = \|u\|_{L^4((-T_1,T_2);L^4)}. \tag{6.5}$$

By (6.1), (6.3)-(6.4), $u \in C((-T_1, T_2); \Sigma)$ implies that $U \in C((-t_1, t_2); \Sigma)$. Note that the pseudo-conformal transformation preserves the spaces L^2 and Σ but not H^1.

Now we will generalize a previously obtained result on the asymptotic behavior of solutions in EEE case of GDS system in [6] to the problem $(1.2.1)_1$ when $\delta = 1$.

Proposition 6.1 *Consider the equation $(1.2.1)_1$, $\delta = 1$. Let $\varphi \in \Sigma$ be such that the corresponding maximal solution u is global and for some constant C, $\|\nabla|u|\|_2^2 \leq$*

$CE(|u|)$ holds globally. Then

$$\|u(s)\|_p^p \leq N(1+|s|)^{2-p} \tag{6.6}$$

for $p > 2$ and $s \in \mathbb{R}$ where N depends only on φ and p.

Proof. We will only argue for positive times. Let $u \in C([0,\infty);\Sigma)$ be as claimed. Let $T_1 = 0$ and $T_2 = \infty$ and U be defined as in (2.12) with $a = d = 1$, $b = -1$, $c = 0$. We have $t_1 = 0$, $t_2 = 1$, $t = T/(1+T)$ for $T \in [0,\infty)$ and $U \in C([0,1);\Sigma)$ solves $(1.2.1)_1$ on $[0,1)$ with $U(0) = e^{-i|\boldsymbol{x}|^2/4}\varphi =: \varphi_{-1}$ since we have the pseudo-conformal invariance property as given by Theorem 3.6. So we have $T^*(\varphi_{-1}) \geq 1$ and by Proposition 3.3 the maximal solution corresponding to φ_{-1} coincides with U on $[0,1)$. From $\|\nabla|U|\|_2 \leq \|\nabla U\|_2$ (by Stampacchia's inequality) and the definition of energy, we get

$$E(|U(t)|) \leq E(U(t)) = E(\varphi_{-1}), \tag{6.7}$$

for $t \in [0,1)$ where we have used the conservation of energy. $|U(t,\boldsymbol{x})| = \frac{1}{1-t}|u(T,\boldsymbol{X})|$ and $1/(1-t) = 1+T$ gives

$$E(|u(T)|) = (1-t)^2 E(|U(t)|) \leq (1-t)^2 E(\varphi_{-1}) = (1+T)^{-2} E(\varphi_{-1}),$$

for $T \in [0,\infty)$ by (6.7). The above inequality, Gagliardo-Nirenberg inequalities and the assumption $\|\nabla|u|\|_2^2 \leq CE(|u|)$ on $[0,\infty)$ imply that

$$\|u(T)\|_p^p \leq C_1 \|\nabla|u(T)|\|_2^{p-2} \|u(T)\|_2^2 \leq C_2 E(|u(T)|)^{(p-2)/2} \|\varphi\|_2^2 \leq N(1+T)^{2-p}$$

for $p > 2$ and $T \in [0,\infty)$ where N depends only on φ and p. This proves the claim for positive times. The result for negative times follows by a time reversal argument. □

Given $\varphi \in \Sigma$, for $\alpha \geq 0$ on $\mathbb{R}^2$, the global solution satisfies the assumption in the above proposition (see Corollary 4.6). On the other hand, for $\alpha(\xi) < 0$ for all $\xi \in \mathbb{R}^2$

the global solution as in Remark 6.4, *(iii)* has the same property.

We need another proposition before stating the scattering results, the proof follows the same line of argument as in [1, Proposition 7.5.1].

Proposition 6.2 *Let $u \in C([0,\infty); L^2)$ (respectively $\in C([0,\infty);\Sigma)$) be a solution of (1.2.1)$_1$ and $U \in C([0,1); L^2)$ (respectively $\in C([0,1);\Sigma)$) be as in the above proof. It follows that $S(-T)u(T)$ has a strong limit in L^2 (respectively in Σ) as $T \to \infty$ if and only if $U(t)$ has a strong limit in L^2 (respectively in Σ) as $t \uparrow 1$, in which case*

$$\lim_{T\to\infty} S(-T)u(T) = e^{i|\boldsymbol{x}|^2/4} S(-1)U(1) \quad \text{in } L^2 \ (respectively \ in \ \Sigma). \tag{6.8}$$

In terms of the scattering theory in L^2, $\mathcal{R}_+ := \{\varphi \in L^2 : T^* = \infty$ and $\lim_{t\to\infty} S(-t)u(t)$ exists in $L^2\}$. By (6.5) and the L^2 well-posedness of (1.2.1), a necessary and sufficient condition for $U(t)$ to have a limit in L^2 as $t \to 1$ is that $\|u\|_{L^4([0,\infty);L^4)} < \infty$, where u and U is as in Proposition 6.2 and so we obtain $\mathcal{R}_+ = \{\varphi \in L^2 : T^* = \infty$ and $\|u\|_{L^4([0,\infty);L^4)} < \infty\}$. Similarly, $\{\varphi \in \Sigma : T^* = \infty$ and $\lim_{t\to\infty} S(-t)u(t)$ exists in $\Sigma\} = \mathcal{R}_+ \cap \Sigma$ from Σ-regularity of the solutions for (1.2.1). If we define $\mathcal{R}_- := \{\varphi \in L^2 : \bar{\varphi} \in \mathcal{R}_+\}$, similar results hold for negative times by time reversal arguments (see [19, Theorem 4.13] for purely cubic power nonlinearity). The following theorem shows the existence of the scattering states in Σ for the global solutions with the assumptions given in Proposition 6.1. For a similar but stronger result in the case of purely power nonlinearity, see [18] and [1, Theorem 7.5.4].

Theorem 6.3 *Consider (1.2.1)$_1$ with $\delta = 1$. Let $\varphi \in \Sigma$ be such that the corresponding maximal solution u is global and for some constant C, $\|\nabla|u|\|_2^2 \leq CE(|u|)$ holds globally. Then there exist $u_\pm \in \Sigma$ such that $\lim_{s\to\pm\infty} S(-s)u(s) = u_\pm$ in Σ.*

Proof. Again we argue only for positive times. By Proposition 6.2 it is enough to show that $\{U(t)\}_{t\in[0,1)}$ has a strong limit in Σ as $t \uparrow 1$. Let U be as in the above proofs and φ as in Proposition 6.1. So by using (6.2) with $\beta = 2$, (6.6) with $p = 4$ and $1 + T = 1/(1-t)$, we have

$$\|U(t)\|_4^4 = (1+T)^2 \|u(T)\|_4^4 \leq \frac{1}{(1-t)^2} N(1+T)^{-2} = N$$

on $[0,1)$ which implies $\|U(t)\|_4 \leq C$ on $[0,1)$ for some constant C. We noted that $T^*(\varphi_{-1}) \geq 1$. If $T^*(\varphi_{-1}) = 1$ then by the blow-up alternative given by Theorem 3.4, *(ii)*, $\|U\|_{L^4((0,1);L^4)} = \infty$ but this contradicts with $\|U(t)\|_4 \leq C$ on $[0,1)$ (in the previous sections by regularity results it was seen that for an initial data in Σ the maximal solutions in L^2, H^1, Σ are the same and T^* is common so we can use Theorem 3.4, *(ii)*). As a result $T^*(\varphi_{-1}) > 1$ from which we deduce that $U \in C([0,1];\Sigma)$. This implies $\lim_{t\to 1} U(t) = U(1)$ in Σ. So by Proposition 6.2 the proof is complete. □

Remark 6.4 (i) Note that for any $\varphi \in \Sigma$ as in the above theorem there exist unique scattering states $u_\pm$ in Σ by uniqueness of solutions. One can see from the proof that $\varphi \in \mathcal{R}_+ \cap \mathcal{R}_- \cap \Sigma$. So for $\alpha \geq 0$ on $\mathbb{R}^2$, $\Sigma \subset \mathcal{R}_+ \cap \mathcal{R}_-$.

(ii) As a result of Remark 3.5, Proposition 6.2 and the observation following it, for $\alpha > \alpha_G > 0$ where α_G as described in Remark 3.5, $L^2((1+|\boldsymbol{x}|^2)\,\mathrm{d}\boldsymbol{x}) \subset \mathcal{R}_+ \cap \mathcal{R}_-$, and without any assumption on the signs of α, sufficiently small neighborhoods of 0 in L^2 are in $\mathcal{R}_+ \cap \mathcal{R}_-$. In both cases we have unique scattering states in L^2 for the corresponding initial data.

(iii) When $\alpha(\xi) < 0$ for all $\xi \in \mathbb{R}^2$ and $\varphi \in \Sigma$ satisfies $\|\varphi\|_2 < \|R\|_2$ for R being the ground state solution of $\Delta R - R = K(R^2)R$, the corresponding solution in Σ is global. Also we have the estimate $-(K(|u|^2),|u|^2) \leq \dfrac{2}{\|R\|_2^2}\|u\|_2^2 \|\nabla |u|\|_2^2$ which holds globally (see Conclusion for the generalization of the results in [9]). This implies $E(|u|) = \|\nabla |u|\|_2^2 + \dfrac{1}{2}(K(|u|^2),|u|^2)_2 \geq \|\nabla |u|\|_2^2 \left(1 - \|\varphi\|_2^2/\|R\|_2^2\right)$, for every $t \in [0,\infty)$ so we have the assertions of Theorem 6.3.

With another smallness assumption on the initial data we obtain a similar result.

Let $\varphi \in \Sigma$ such that the corresponding maximal solution u is global and $\|\varphi\|_2 < \|R\|_2/(\|\alpha\|_\infty)^{1/2}$ where R is the ground state solution of $\Delta R - R + R^3 = 0$. Similar to above we have $E(|u|) \geq \|\nabla |u|\|_2^2(1 - \|\alpha\|_\infty \|\varphi\|_2^2/\|R\|_2^2)$, globally from $(K(|u|^2), |u|^2)_2 \leq \|\alpha\|_\infty \|u\|_4^4 \leq \|\alpha\|_\infty (2/\|R\|_2^2)\|\nabla |u|\|_2^2 \|u\|_2^2$ where we utilize the inequality (I.2) in [23] to control $\|u\|_4^4$. So again we obtain scattering results in Σ by Theorem 6.3.

(iv) More generally similar to [19, Corollary 4.9] for purely cubic power nonlinearity, $\mathcal{R}_+$ and $\mathcal{R}_+ \cap \Sigma$ are open in L^2 and Σ respectively which implies low-energy scattering in those spaces i.e. with initial data having small norm in L^2 (respectively in Σ) and for which the corresponding maximal solution is global, we have unique scattering states in L^2 (respectively in Σ). This is by Proposition 6.2 and the remarks following it.

(v) The following claim related to asymptotic behavior holds for the case $\delta = -1$ as well. The problem (1.2.1) that we are considering here also falls under the class considered by Constantin in [17] hence Theorem 2.3 there applies. In particular, when $m = 2$ (since for $n = 2$ this is the smallest possible m we can take there) and $\|\varphi\|_\Sigma^2 \vee \|\Delta\varphi\|_2^2 \vee \||\boldsymbol{x}|^2\varphi\|_2^2$ is small enough we have $|u(t, \boldsymbol{x})| \leq C(1 + |t|)^{-1}$, i.e. the L^∞-norm of the solutions decay to 0.

7. THE CAUCHY PROBLEM FOR A GENERALIZED EQUATION AND FURTHER REGULARITY

We will be dealing with the Cauchy problem

$$\begin{aligned} &iu_t + \delta u_{xx} + u_{yy} = K(|u|^{p-1})u, \quad \delta = \pm 1,\ \infty > p > 2 \\ &u(0) = \varphi. \end{aligned} \tag{7.1}$$

in this section where K is defined by (1.2.2). We have considered the problem for $p = 3$ in L^2, H^1 and Σ in the previous sections. Here we will show that for $\infty > p > 2$ we have H^1-solutions and in addition we will consider (7.1) in H^2. Let $H(u)$ denote $K(|u|^{p-1})u$ throughout this section. The following result which is a corollary to [9, Lemma 2.1] gives some of the operator theoretic properties of H:

Corollary 7.1 *$H \in C(L^{p+1}; L^{(p+1)/p}) \cap C(L^{2p}; L^2)$ with*

$$\begin{aligned} \|H(u)\|_{(p+1)/p} &\le C_H \|u\|_{p+1}^p \quad \forall u \in L^{p+1}, \\ \|H(u) - H(v)\|_{(p+1)/p} &\le C_H(\|u\|_{p+1}^{p-1} + \|v\|_{p+1}^{p-1})\|u - v\|_{p+1} \quad \forall u, v \in L^{p+1}, \end{aligned} \tag{7.2}$$

$$\begin{aligned} \|H(u)\|_2 &\le C_H \|u\|_{2p}^p \quad \forall u \in L^{2p}, \\ \|H(u) - H(v)\|_2 &\le C_H(\|u\|_{2p}^{p-1} + \|v\|_{2p}^{p-1})\|u - v\|_{2p} \quad \forall u, v \in L^{2p}. \end{aligned} \tag{7.3}$$

Proof. Let $u, v \in L^{p+1}$ then by using the Hölder inequality we get $\|K(|u|^{p-1})v\|_{(p+1)/p} \le C_p \|u\|_{p+1}^{p-1} \|v\|_{p+1}$ where C_p denotes the operator norm of K on $L^{(p+1)/(p-1)}$ by using [9, Lemma 2.1, *(i)*]. This gives $(7.2)_1$. Similarly

$$\|H(u) - H(v)\|_{(p+1)/p} \le C_p \left\{ \|u\|_{p+1}^{p-1} \|u - v\|_{p+1} + C\|(|u|^{p-2} + |v|^{p-2})|u - v|\|_{\frac{p+1}{p-1}} \|v\|_{p+1} \right\}.$$

If we use $\|(|u|^{p-2}+|v|^{p-2})|u-v|\|_{\frac{p+1}{p-1}} \le (\|u\|_{p+1}^{p-2}+\|u\|_{p+1}^{p-2})\|u-v\|_{p+1}$ with $p>2$, we obtain $(7.2)_2$. (7.3) can also be obtained by using the same Hölder pairs. $\square$

Remark 7.2 (i) For $k=1-1/p$, $H^k \hookrightarrow L^{2p}$ by the Sobolev embedding. $0<k<1$ implies $H^1 \hookrightarrow H^k$. With the above inequalities $H \in C(H^k;L^2)\cap C(H^2;L^2)$.

(ii) For $(r,p+1)$ admissible and $u,v \in L^\infty(I;L^{p+1})$, I bounded,

$$
\begin{aligned}
&\|H(u)-H(v)\|_{L^{r'}(I;L^{\frac{p+1}{p}})} \\
&\qquad \le C_H|I|^{1/r'-1/r}(\|u\|^{p-1}_{L^\infty(I;L^{p+1})}+\|v\|^{p-1}_{L^\infty(I;L^{p+1})})\|u-v\|_{L^r(I;L^{p+1})}. \quad (7.4)
\end{aligned}
$$

(iii) We generalize the definitions of X, X_0, $\bar{X}$, Y, Y', $\bar{Y}$ (see Chapter 4 for the previous definitions) by replacing L^4 with L^{p+1}, $L^4(I;L^4)$ with $L^r(I;L^{p+1})$ and $L^{4/3}(I;L^{4/3})$ with $L^{r'}(I;L^{\frac{p+1}{p}})$ where $(r,p+1)$ is admissible (in terms of Theorem A.2.1, *(ii)* we have considered the case $(\gamma,\rho)=(4,4)$ previously when $p=3$ and here we consider the more general case $(\gamma,\rho)=(r,p+1)$, $p>2$. Note that the constants appearing in Strichartz's estimates also depend on the particular choice of (γ,ρ)). By a modification of the argument given in Chapter 4 due to the change in the definition of the above spaces and using (7.2), (7.4) to obtain general versions of the previous results in the same section, given $\varphi \in H^1$, there exists unique maximal solution $u \in C([0,T^*);H^1)\cap C^1([0,T^*);H^{-1})$ solving (7.1) such that $\nabla u \in L^r([0,t];L^{p+1})$ for every $t<T^*$. We also have $\|u\|_{L^\infty([0,T^*);H^1)}=\infty$ provided $T^*<\infty$. Another existence and uniqueness result can be obtained in L^2 for $2<p<3$ in the elliptic case as in [1, Theorem 4.6.1] since we have (7.2) and $p+1<r$.

We define another group of spaces in order to deal with (7.1) in H^2 ($I=[0,T]$, with any $T<\infty$ as before). Let

$$
\begin{aligned}
\mathcal{Z} &= \{v \in X : v_t \in X,\ \Delta v \in L^\infty(I;L^2)\}, \quad \|\cdot\|_{\mathcal{Z}} = \|\cdot\|_X \vee \|\partial_t \cdot\|_X \vee \|\Delta\cdot\|_{\infty,2} \\
\mathcal{Z}' &= \{f \in L^\infty(I;L^2) : f_t \in L^{r'}(I;L^{\frac{p+1}{p}})\}, \quad \|\cdot\|_{\mathcal{Z}'} = \|\cdot\|_{\infty,2} \vee \|\partial_t\cdot\|_{r',\frac{p+1}{p}}
\end{aligned}
\quad (7.5)
$$

and $\bar{\mathcal{Z}} = \{v \in \bar{X} : v_t \in \bar{X},\, \Delta v \in C(I; L^2)\} \subset \bar{Y}$, $\bar{\mathcal{Z}}$ is a closed subspace of $\mathcal{Z}$. We have

$$\|v\|_Y \leq CT^{1/r}\|v\|_{\mathcal{Z}} \tag{7.6}$$

with C independent of T. In order to obtain further contraction property for $\mathcal{T}$ on $\mathcal{Z}$ we need some preliminary lemmas. We prove the existence and uniqueness of H^2-solutions in the elliptic case by a modification of the argument in [2].

Lemma 7.3 *Let* Λ *be defined by* (3.4) *and* $\delta = 1$ *then*

i) there exists some constant C_s *being independent of* T *such that for every* $f \in \mathcal{Z}'$

$$\begin{aligned} \|\Lambda f\|_X &\leq C_s\|f\|_{1,2}, \\ \|(\Lambda f)_t\|_X &\leq C_s(\|f(0)\|_2 + \|f_t\|_{r',(p+1)/p}), \end{aligned} \tag{7.7}$$

ii) $S(\cdot) \in \mathcal{L}(H^2; \bar{\mathcal{Z}})$, $\Lambda \in \mathcal{L}(\mathcal{Z}'; \bar{\mathcal{Z}})$.

Proof. *(i)* $(7.7)_1$ follows by Theorem A.2.1, *(ii)* with $(\gamma, \rho) = (\infty, 2)$. Let $v := \Lambda f$. $\mathcal{Z}' \subset L^1(I; L^2)$ so $f \in W^{1,1}((0,T); H^{-1}) \hookrightarrow C(I; H^{-1})$. If $f \in C^1(I; H^{-1})$ then

$$v_t(t) = S(t)f(0) + \int_0^t S(t-s)f_t(s)\,\mathrm{d}s \quad \text{in } H^{-1} \text{ for } t \in I. \tag{7.8}$$

We have $v_t \in L^1((0,T); H^{-1})$. For f_n sufficiently smooth and $f_n \to f$ in $W^{1,1}((0,T); H^{-1})$, $v_n \to v$ and $v_{nt} \to S(\cdot)f(0) + \int_0^{\cdot} S(\cdot - s)f_t(s)\,\mathrm{d}s$ in $L^1((0,T); H^{-1})$. So (7.8) holds for $f \in W^{1,1}((0,T); H^{-1})$ and by Strichartz's estimates (taking $(\gamma, \rho) = (r, p+1)$) we obtain $(7.7)_2$.

(ii) $\|S(\cdot)\varphi\|_X \leq C\|\varphi\|_2$ by Strichartz's estimates. $\varphi \in H^2$ implies that $S(\cdot)\varphi \in C(\mathbb{R}; H^2) \cap C^1(\mathbb{R}; L^2)$ solves $iu_t + \Delta u = 0$. So we obtain $\|\Delta S(\cdot)\varphi\|_X = \|S(\cdot)\Delta\varphi\|_X = \|(S(\cdot)\varphi)_t\|_X \leq C\|\Delta\varphi\|_2$, by using the linear equation and Strichartz's estimates. We

obtain the first claim with

$$\|S(\cdot)\varphi\|_{\mathcal{Z}} \leq C_1\|\varphi\|_{H^2} \tag{7.9}$$

with C_1 independent of T.

By using (7.7), $\|\Lambda f\|_X \leq C_s T\|f\|_{\mathcal{Z}'}$ and $\|(\Lambda f)_t\|_X \leq 2C_s\|f\|_{\mathcal{Z}'}$. $\Lambda f \in C(I; L^2)$ solves $iv_t + \Delta v = f$ such that $v(0) = 0$ and $v \in C(I; L^2) \cap C^1(I; H^{-2})$ since $f \in C(I; H^{-1})$. By the equation $\Delta\Lambda f \in L^\infty(I; L^2)$ and we have $\|\Delta\Lambda f\|_{\infty,2} \leq \|(\Lambda f)_t\|_X + \|f\|_{\infty,2} \leq (2C_s + 1)\|f\|_{\mathcal{Z}'}$. Also $\Delta\Lambda f$ and $(\Lambda f)_t \in C(I; L^2)$ by approximation since Λ maps smooth functions into $\bar{\mathcal{Z}}$. So the second assertion is obtained. Note that the operator norm of the map $\Lambda : \mathcal{Z}' \to \bar{\mathcal{Z}}$ depends on T and for $T \leq 1$

$$\|\Lambda f\|_{\mathcal{Z}} \leq (2C_s + 1)\|f\|_{\mathcal{Z}'} = C_2\|f\|_{\mathcal{Z}'}. \tag{7.10}$$

See [2, Lemma 3.2] and [1, Lemmas 4.8.2, 4.8.5] for these results. □

Next lemma is in parallel with [2, Lemmas 3.3, 3.4], we get similar results for H:

Lemma 7.4 *Let $\theta = 1 - k/2$ with $k = 1 - 1/p$.*

(i) For every $v \in \mathcal{Z}$, $H(v) \in C^\theta(I; L^2)$ and

$$\|H(v(t)) - H(v(s))\|_2 \leq \bar{C}\|v\|_{\mathcal{Z}}^p |t - s|^\theta, \quad \forall t, s \in I$$

with $\bar{C}$ being independent of T.

(ii) H maps $\mathcal{Z}$ into $\mathcal{Z}'$ boundedly and if $T \leq 1$

$$\|H(v) - H(v(0))\|_{\mathcal{Z}'} \leq MT^{1/r' - 1/r}\|v\|_{\mathcal{Z}}^p, \quad \forall v \in \mathcal{Z}$$

where M is independent of T and $H(v(0))$ represents $t \mapsto H(v(0))$ for $t \in I$.

Proof. *(i)* $0 < k < 2$ and $v \in \mathcal{Z} \subset W^{1,\infty}((0,T);L^2) \hookrightarrow C^{0,1}(I;L^2)$ so

$$\begin{aligned}\|v(t)-v(s)\|_{H^k} &\leq \|v(t)-v(s)\|_{H^2}^{1-\theta}\|v(t)-v(s)\|_2^{\theta}\\ &\leq C\|v\|_{\mathcal{Z}}^{1-\theta}|t-s|^{\theta}\|v_t\|_{\infty,2}^{\theta} \leq C\|v\|_{\mathcal{Z}}|t-s|^{\theta}\end{aligned}$$

for every $t, s \in I$ where C is independent of I, v. Since $H^k \hookrightarrow L^{2p}$ we obtain

$$\|v(t)-v(s)\|_{2p} \leq C\|v\|_{\mathcal{Z}}|t-s|^{\theta} \quad \forall t,s \in I,\, v \in \mathcal{Z}, \tag{7.11}$$

with C independent of I and changing accordingly. By using $(7.3)_2$, (7.11) and $H^2 \hookrightarrow L^{2p}$ we obtain *(i)*.

(ii) We need to show the estimates:

$$\begin{aligned}\|H(v)-H(v(0))\|_{\infty,2} &\leq CT^{1/r'-1/r}\|v\|_Z^p\\ \|(H(v))_t\|_{r',(p+1)/p} &\leq CT^{1/r'-1/r}\|v\|_{X_0}^{p-1}\|v_t\|_X\end{aligned}$$

for every $v \in \mathcal{Z}$ where C is independent of T. By *(i)* for any $t \in I$, $\|H(v((t)) - H(v(0))\| \leq \bar{C}t^{\theta}\|v\|_{\mathcal{Z}}^p$. Taking the supremum of each side on I gives the first inequality when $T \leq 1$ $(\theta > 1/r' - 1/r)$.

Consider $H(v(t+h)) - H(v(t))$ for $t \in I$ and h small. By using $(7.2)_2$ we have

$$\|H(v(t+h))-H(v(t))\|_{\frac{p+1}{p}} \leq 2C_H\|v\|_{X_0}^{p-1}\int_t^{t+h}\|v_t(s)\|_{p+1}\,\mathrm{d}s \tag{7.12}$$

for which $\mathcal{Z} \hookrightarrow Y \hookrightarrow X_0$ is used. It was seen that $\|\cdot\|_{X_0} \leq C_e\|\cdot\|_Y$ and C_e is independent of T, same is true for $\mathcal{Z} \subset Y$ for $T \leq 1$ from (7.6). Also for $u \in \mathcal{Z}$, $\|H(u)\|_{r',\frac{p+1}{p}} \leq C\|u\|_{X_0}^{p-1}T^{1/r'-1/r}\|u\|_{r,p+1}$ for $T \leq 1$ where we used the above embeddings and $(7.2)_1$. So by (7.12), [1, Proposition 1.3.12] implies that $H(v) \in W^{1,r'}((0,T);L^{(p+1)/p})$ and

$$\|(H(v))_t\|_{r',\frac{p+1}{p}} \leq C_H\|v\|_{X_0}^{p-1}\|v_t\|_{r',p+1} \leq 2C_HT^{1/r'-1/r}\|v\|_{X_0}^{p-1}\|v_t\|_{r,p+1}, \tag{7.13}$$

which establishes the second inequality and the claim is proved by these inequalities.□

With these results we can state the contraction property needed to establish the existence and uniqueness of H^2-solutions:

Lemma 7.5 *Let $\varphi \in H^2$, $\delta = 1$ and $R > C_1\|\varphi\|_{H^2} + C_2\|H(\varphi)\|_2$ where C_1, C_2 are as in* (7.9) *and* (7.10)*. Set $E = \{v \in B_{\mathcal{Z}}(0, R) : v(0) = \varphi\}$. Then for sufficiently small T, $\mathcal{T} : E \mapsto E$ and is a strict contraction on E when considered with the metric induced by the X-norm. (Given $\varphi \in H^2$, $\mathcal{T}$ and Λ are defined by* (3.4)*,* (3.5)*.)*

Proof. $S(\cdot)\varphi \in E \neq \emptyset$. Let $v \in E$ then $(\mathcal{T}v)(0) = \varphi$ and by (7.9), (7.10) and Lemma 7.4, *(ii)* for $T \leq 1$

$$\begin{aligned}\|\mathcal{T}v\|_{\mathcal{Z}} &\leq C_1\|\varphi\|_{H^2} + C_2(\|H(v) - H(\varphi)\|_{\mathcal{Z}'} + \|H(\varphi)\|_2)\\ &\leq C_1\|\varphi\|_{H^2} + C_2(MT^{1/r'-1/r}\|v\|^p_{\mathcal{Z}} + \|H(\varphi)\|_2)\\ &\leq C_1\|\varphi\|_{H^2} + C_2(MT^{1/r'-1/r}R^p + \|H(\varphi)\|_2)\end{aligned}$$

where $H(\varphi)$ on I represents $t \mapsto H(\varphi)$. Now let $w \in E$ also. We have $\|\mathcal{T}v - \mathcal{T}w\|_X = \|\Lambda(H(v) - H(w))\|_X \leq C_3T^{1/2}R^2\|v - w\|_X$ by using $B_{\mathcal{Z}}(0, R) \subset B_{X_0}(0, \bar{R})$ and the estimate similar to (4.5) where $(4, 4)$ in the X-norm is replaced by $(r, p + 1)$ (see Remark 7.2, *(iii)*). So for T small we obtain the result by the estimates above. □

Theorem 7.6 *Given $\varphi \in H^2$, there exists a unique maximal solution $u \in C([0, T^*); H^2) \cap C^1([0, T^*); L^2)$ solving* (7.1)*, $\delta = 1$ with the following properties:*

(i) $u_t \in L^r([0, t]; L^{p+1})$ *for every* $t \in [0, T^*)$*,*

(ii) $T^* < \infty$ *implies that* $\|u\|_{L^\infty([0,T^*);H^2)} = \infty$.

Proof. For T sufficiently small and R as in Lemma 7.5, there exists a unique $u \in E$ such that $u = \mathcal{T}u$ by the fact that E is complete when considered with the metric induced by the X-norm. Since $\mathcal{T} : \mathcal{Z} \to \bar{\mathcal{Z}}$ by Lemma 7.3, *(ii)* and Lemma 7.4, *(ii)*, $u \in C(I; H^2)$, $u_t \in C(I; L^2) \cap L^r(I; L^{p+1})$ with $I = [0, T]$. $H(u) \in L^1(I; L^2)$ so u satisfies $(7.1)_1$ with $\delta = 1$ in $\mathcal{D}'((0, T); L^2)$ and $u(0) = \varphi$. Since $u \in \mathcal{Z}$, $u \in C(I; H^2) \cap C^1(I; L^2)$ with $u_t \in L^r(I; L^{p+1})$ solves (7.1) on I. Let $v \in C(I; H^2) \cap C^1(I; L^2)$ be another solution on I. Let $0 \in J \subset I$, we have

$$\begin{aligned}\|u - v\|_{L^r(J;L^{p+1})} = \|\mathcal{T}u - \mathcal{T}v\|_{L^r(J;L^{p+1})} &\le C\|H(u) - H(v)\|_{L^{r'}(J;L^{\frac{p+1}{p}})} \\ &\le C|J|^{1/r'-1/r}(\|u\|^{p-1}_{L^\infty(J;L^{p+1})} + \|v\|^{p-1}_{L^\infty(J;L^{p+1})})\|u - v\|_{L^r(J;L^{p+1})}\end{aligned}$$

which implies $u = v$ on J for $|J|$ sufficiently small and by a continuation argument the same is true on I. As a result we have a unique u with the properties given above and so u can be uniquely extended to $[0, T^*)$, where $T^* = \sup\{T > 0 : \exists u \in C([0, T]; H^2) \cap C^1([0, T]; L^2)$ solving (7.1) on $[0, T]$ such that $u_t \in L^r([0, T]; L^{p+1})\}$. This gives *(i)*.

Assume $T^* < \infty$ and let $t_j \uparrow T^*$ such that there exists $A > 0$ satisfying $\|\varphi\|_{H^2}, \|u(t_j)\|_{H^2} \le A$. Let $R > C_1A + C_2\tilde{C}A^p$ where we use $\|H(\cdot)\|_2 \le \tilde{C}\| \cdot \|^p_{H^2}$ by $(7.3)_1$, $\tilde{C}$ appears because of $H^2 \hookrightarrow L^{2p}$. Choose T sufficiently small such that $C_1A + C_2(MT^{1/r'-1/r}R^p + \tilde{C}A^p) \le R$ and $C_3T^{1/2}R^2 < 1$ where C_3 is as in the proof of Lemma 7.5. Let t_k be such that $t_k + T > T^*$. Proceeding as in the proof of Theorem 4.5, Step 2 (after the choice of t_k there) yields a contradiction which implies *(ii)*. □

After considering the Cauchy problem in H^2 in the sense of the above theorem we can state the H^2-regularity of H^1-solutions.

Corollary 7.7 *Let $\varphi \in H^2$. Then the H^1-solution $u \in C([0, T^*); H^1)$ noted in Remark 7.2, (iv) is in $C([0, T^*); H^2) \cap C^1([0, T^*); L^2)$ and $[0, T^*)$ coincides with the maximal interval of existence of the corresponding H^2-solutions.*

Proof. Let T_2^* be the end point of the maximal interval of existence for the H^2-solution v. By the uniqueness obtained in the above proof, $T_2^* \le T^*$ and $u = v$ on $[0, T_2^*)$.

Assume on the contrary that $T_2^* < T^*$. So $\|u(t)\|_{H^2} \to \infty$ as $t \uparrow T_2^*$ since $T_2^* < \infty$. $u \in C([0,T^*); H^1)$ implies

$$\|u\|_{L^\infty([0,T_2^*];H^1)} < \infty. \tag{7.14}$$

By the above theorem

$$\begin{aligned} &u \in L^\infty([0,\tau]; H^2) \quad \text{and} \\ &u_t \in L^\infty([0,\tau]; L^2) \cap L^r([0,\tau]; L^{p+1}) \subset L^{r'}([0,\tau]; L^{p+1}) \end{aligned} \tag{7.15}$$

for every $\tau < T_2^*$. We want to show that $u_t \in L^\infty([0,T_2^*); L^2)$. u satisfies $u(t) = S(t)\varphi - i\Lambda H(u)(t)$ in L^2 on $[0,T_2^*)$. Let $\tau < T_2^*$,

$$\begin{aligned} \|u_t\|_{L^r([0,\tau];L^{p+1})} &\le C_1\|\varphi\|_{H^2} + C_s(\|H(\varphi)\|_2 + \|(H(u))_t\|_{L^{r'}([0,\tau];L^{\frac{p+1}{p}})}) \\ &\le C_1\|\varphi\|_{H^2} + C_s(\|H(\varphi)\|_2 + C\|u\|^{p-1}_{X_0([0,\tau])}\|u_t\|_{L^{r'}([0,\tau];L^{p+1})}) \\ &\le C_1\|\varphi\|_{H^2} + C_s\|H(\varphi)\|_2 + C_5\|u_t\|_{L^{r'}([0,\tau];L^{p+1})} \end{aligned} \tag{7.16}$$

with C_5 being independent of τ by using Lemma 7.3 with $f = H(u)$ and (7.13). Let $\varepsilon < \tau$, we have

$$\begin{aligned} \|u_t\|_{L^{r'}([0,\tau];L^{p+1})} &\le \|u_t\|_{L^{r'}([0,\tau-\varepsilon];L^{p+1})} + \|u_t\|_{L^{r'}([\tau-\varepsilon,\tau];L^{p+1})} \\ &\le \|u_t\|_{L^{r'}([0,T_2^*-\varepsilon];L^{p+1})} + \varepsilon^{1/r'-1/r}\|u_t\|_{L^r([\tau-\varepsilon,\tau];L^{p+1})} \\ &\le C_\varepsilon + \varepsilon^{1/r'-1/r}\|u_t\|_{L^r([0,\tau];L^{p+1})}. \end{aligned}$$

So by (7.16)

$$\|u_t\|_{L^r([0,\tau];L^{p+1})} \le C_\varphi + C_5C_\varepsilon + C_5\varepsilon^{1/r'-1/r}\|u_t\|_{L^r([0,\tau];L^{p+1})}.$$

By fixing $\varepsilon < \tau$ sufficiently small $\|u_t\|_{L^r([0,\tau];L^{p+1})} \le C$ as $\tau \to T_2^*$. Similar to (7.16) we obtain

$$\|u_t\|_{L^\infty([0,\tau];L^2)} \le C_1\|\varphi\|_{H^2} + C_s\|H(\varphi)\|_2 + C_6\|u_t\|_{L^{r'}([0,\tau];L^{p+1})}$$

which together with the last result imply that $u_t \in L^\infty([0,T_2^*);L^2)$ since $T_2^* < \infty$. We also have

$$\|H(u(t))\|_2 \le C_H \|u(t)\|_{2p}^{p-1}\|u(t)\|_{2p} \le C_7\|u(t)\|_{H^1}^{p-1}\|u(t)\|_{H^k} \le C_8\|u(t)\|_{H_k}$$

for $t < T_2^*$ by $(7.3)_1$, $H^1 \hookrightarrow H^k \hookrightarrow L^{2p}$ for $k = 1-1/p$ and (7.14). So by interpolation and Young's inequality

$$\|H(u(t))\|_2 \le C_8\|u(t)\|_2^{k/2}\|u(t)\|_{H^2}^{(2-k)/2} \le C_9 + \frac{1}{2}\|u(t)\|_{H^2} \tag{7.17}$$

for $t < T_2^*$ since $u \in L^\infty([0,T_2^*];L^2)$. Using (7.17) and the equation, we get $\|\Delta u(t)\|_2 \le \|u_t\|_{L^\infty([0,T_2^*);L^2)} + C_9 + \frac{1}{2}\|u(t)\|_{H^2}$ which implies $\|u(t)\|_{H^2} \le C_{10} + 1/2\|u(t)\|_{H^2}$ for $t < T_2^*$ by (7.14) and the fact $u_t \in L^\infty([0,T_2^*);L^2)$ obtained above. Hence we get $\|u\|_{L^\infty([0,T_2^*);H^2)} < \infty$ which contradicts with Theorem 7.6, *(ii)* since $T_2^* < \infty$. □

Remark 7.8 When $\delta = -1$, we cannot extend Kato's framework to the H^2-regularity of solutions and to their maximal interval of existence. However, for $p = 3$, an alternative argument, which works for the hyperbolic case as well, gives the result. Indeed, given $\varphi \in H^2$ and $R > 0$, for $I = [0,T]$, T sufficiently small, $\mathcal{T}$ is from $B_{C(I;H^2)}(S(\cdot)\varphi, R)$ into itself and is a strict contraction on it, by the estimates

$$\begin{aligned}\|\mathcal{T}u - S(\cdot)\varphi\|_{L^\infty(I;H^2)} &\le CT^{3/4}\|u\|_{L^\infty(I;H^2)},\\ \|\mathcal{T}u - \mathcal{T}v\|_{L^\infty(I;H^2)} &\le CT^{3/4}(\|u\|_{L^\infty(I;H^2)}^2 + \|u\|_{L^\infty(I;H^2)}^2)\|u-v\|_{L^\infty(I;H^2)},\end{aligned}$$

which hold for every $u, v \in C(I;H^2)$. These are obtained by Theorem A.2.1 and the control of ΔH in $L^{4/3}(I;L^{4/3})$. The existence and uniqueness of a maximal solution on $[0,T_2^*)$ and the blow-up alternative in H^2 follow as in the case of H^1-solutions. For the regularity result, $\|\Delta H(u)\|_{4/3,4/3}$ is estimated more carefully in order to make use of Y-norm of the solution. For this, let T^* be as in Theorem 4.5 and v be the corresponding H^1-solution. Then by definitions of T_2^*, T^* and by Proposition 3.3, $T_2^* \le T^*$ and $u = v$ on $[0,T_2^*)$. We want to show that $T_2^* = T^*$.

Assume on the contrary that $T_2^* < T^*$. Then by H^2 blow-up, $\|\Delta u\|_{L^\infty([0,T_2^*);L^2)} = \infty$ since $\|u\|_{L^\infty([0,T_2^*];L^2)} < \infty$. Let $\tau < T_2^*$ and $J = [0,\tau)$. Since $\Delta u = S(\cdot)\Delta\varphi - i\Lambda\Delta H(u)$ on J, by Theorem A.2.1,

$$\begin{aligned}\|\Delta u\|_{L^\infty(J;L^2)} &\le C_1(\|\varphi\|_{H^2} + \|\Delta H(u)\|_{L^{4/3}(J;L^{4/3})}) \\ &\le C_2(\|\varphi\|_{H^2} + \|u\|^2_{L^4(J;L^8)}\|\Delta u\|_{L^4(J;L^2)} + \|\nabla u\|^2_{L^4(J;L^4)}\|u\|_{L^4(J;L^4)} \\ &\quad + \|u\|_{L^4(J;L^{4p/(3-2p)})}\|\Delta u\|_{L^4(J;L^2)}\|u\|_{L^4(J;L^{4p'/3})}),\end{aligned}$$

for some $1 < p < 3/2$. By the last inequality we obtain

$$\|\Delta u\|_{L^\infty(J;L^2)} \le C_3(\|\varphi\|_{H^2} + T^{3/4}M^2\|\Delta u\|_{L^\infty(J;L^2)} + T^{1/4}M^3),$$

where $M = \|u\|_{Y([0,T_2^*])}$. Specify τ by choosing $\tau = T_2^*/n$ for fixed n sufficiently large. This gives $\frac{1}{2}\|\Delta u\|_{L^\infty(J;L^2)} \le 1 + C_3\|\varphi\|_{H^2}$. Iterating n times we get $\frac{1}{2}\|\Delta u\|_{L^\infty([0,T_2^*);L^2)} \le 1 + C_3 \max_{0\le k\le n-1} \|u(k\tau)\|_{H^2}$ which contradicts with $\|\Delta u\|_{L^\infty([0,T_2^*);L^2)} = \infty$.

8. CONCLUSION

In the elliptic case, we claim that all of the previous results on the purely elliptic GDS system that are mentioned in the introduction [5, 9, 6, 12, 11] are now fully justified.

When $\alpha(\boldsymbol{\xi}) \equiv \chi$ we recover the cubic NLS. To recover the DS system in the HE and EE cases just take $\alpha(\boldsymbol{\xi}) = \chi + b\frac{\xi_1^2}{\xi_1^2+m\xi_2^2}$. We have in this case $\chi + b \geq \alpha(\xi) \geq \chi$ when $b > 0$ and $\chi \geq \alpha(\xi) \geq \chi + b$ when $b < 0$ for all $\xi \in \mathbb{R}^2$. For the HEE and EEE cases of the GDS system, the corresponding symbol is given by (1.3.2). All of these three symbols satisfy (H1) and (H2). The case $\alpha \geq 0$ where the global existence of H^1-solutions as well as Σ-solutions are proven in Corollary 4.6 and Theorem 5.2, *(iii)* was already considered for the GDS system in [5] as well as in [12]. In fact, the conditions that are assumed on the parameters were in order to ensure $\alpha(\xi) \geq 0$ for all $\xi \in \mathbb{R}^2 \setminus \{(0,0)\}$. On the other hand, when $\alpha < 0$ in the special case of the GDS system with $b \leq 0$, it was shown that there can be data φ with negative energy resulting in the blowing-up of solutions. When $b > 0$, in the case where α takes both positive and negative values, neither a global existence nor a blow-up of solutions was obtained, this is still the case in this work. (see [12])

The framework considered here for the nonlinearity was inspired by [9]. If we let $K(|u|^2)u = \chi|u|^2u + b\mathcal{K}(|u|^2)u$ then we obtain the framework presented in [9]. In fact replacing the nonlinear functional J there (see (35) in [9]) by

$$J(v) = \frac{-2\|v\|_2^2\|\nabla v\|_2^2}{(K(|v|^2), |v|^2)_{L^2}}$$

we can obtain the result of Theorem 2.2. Hence we generalize the work of Weinstein [23] on the critical NLS. The assumptions on the parameters in [9] imply that $\alpha < 0$. In [11] an alternative route was taken leading to the existence of standing waves. To rephrase the main result there (Theorem 1): If either $\lim_{s\to\infty} \alpha(s\xi_1, \xi_2) = \bar{\alpha}_1 < 0$ or $\lim_{s\to 0^+} \alpha(s\xi_1, \xi_2) = \bar{\alpha}_2 < 0$ then the constrained minimization has a positive solution.

Note that $\bar{\alpha}_1 = \chi + \alpha_1 b$ and $\bar{\alpha}_2 = \chi + \alpha_2 b$ in the terminology of that paper. In the DS case, $\bar{\alpha}_1 \wedge \bar{\alpha}_2 < 0$ is equivalent to $\chi < (-b) \vee 0$. Moreover, in the GDS case for $b < 0$ it is observed in [12] that $\bar{\alpha}_1 \wedge \bar{\alpha}_2 < 0$ suffices to guarantee solutions with negative energy (Lemma 2, *(ii)*). Both of these results also implied a global existence result for initial data with subminimal mass. These observations indicate that the focusing and defocusing cases for (1.2.1) are not as sharply demarcated yet as in the cases of NLS and DS system. For solutions in Σ these two cases can be seperated as follows: (i) either there exists $u \in \Sigma$ such that $(K(|u|^2), |u|^2) < 0$ (focusing) then there exists initial value with negative energy. By Corollary 5.3 the corresponding solution in Σ blows up in finite time (ii) or for every $u \in \Sigma$, $(K(|u|^2), |u|^2) \geq 0$ (defocusing) hence conservation of energy and mass gives boundedness of H^1-norm of the H^1-solutions. From the proof of Corollary 4.6 it follows that H^1-solutions are global so by Theorem 5.2, *(iii)*, the Σ-solutions exist globally. In [6], asymptotic behavior of L^p-norms were also considered, we have rephrased this in Section 6, Proposition 6.1. All of the papers mentioned so far were on the purely elliptic case of the GDS system. We also presented new results on the global existence of solutions for the GDS system. Firstly, in the HEE and EEE cases, when the initial mass is small enough L^2-solutions are global (Remark 3.5). Secondly , in the elliptic case when $\alpha \geq 0$ and $\varphi \in \Sigma$ the Cauchy problem has a global solution in Σ. Moreover, the same is true under the conditions described above i.e. for small solutions with subminimal mass when $\bar{\alpha}_1 < 0$ or $\bar{\alpha}_2 < 0$ or when $\alpha < 0$ when $\varphi \in \Sigma$.

It is clear that the key observation that makes all of the arguments go smoothly in the present work is that the nonlinear term that we are considering acts in the same way as the purely cubic power nonlinearity acts as an operator on the mixed space-time norm spaces (see e.g. Corollaries 3.1, 3.2, 7.1, Lemmas 4.1, 4.2, 5.1, *(i)*, 7.4 and Remark 7.2). Since the pure power case is a special case of the term that we have considered this may have been a natural consequence of the general framework considered in [2]. Note however that we have refrained from decomposing the nonlinear term into two parts as done there by a cut-off function since we do not know how to deal with such an operation in the presence of a real non-local nonlinearity. Consequently we have also avoided some of the technical details that had to be considered in [2].

In deriving the conservation laws we always had to recourse to same type of regularity result, hence H^1-L^2 and H^2-H^1 regularity results were needed. We deduce mass conservation and the pseudo-conformal invariance for $\delta = \pm 1$ with L^2-solutions, this justifies the argument on the blow-up profile given in [6]. Energy conservation is also valid for $\delta = \pm 1$ with H^1-solutions .

As a final comment we would like to mention the situation in the $2D$ case when the nonlinearity is like a pth power nonlinearity, i.e. when $H(u) = K(|u|^{p-1})u$ for $\infty > p > 2$. We have the operator estimates for H given in Corollary 7.1 which leads to the existence and uniqueness of H^1-solutions (for both $\delta = \pm 1$) for $p > 2$ and of L^2-solutions (in the case $\delta = 1$) for $3 > p > 2$. We also obtain H^2-solutions for $p > 2$ in the elliptic case which is important in establishing the pseudo-conformal invariance property, the energy conservation in H^1 and the virial identity. And finally with the $H^1 - H^2$ regularity result we obtain that for an initial data in H^2 the maximal interval of existence of L^2, H^1, Σ and H^2-solutions coincides and those solutions are the same.

APPENDIX A: LINEAR SCHRÖDINGER EQUATION

Although the following results are considered for $n = 2$ throughout the text, they hold in general as stated here. We refer to [1] for the elliptic case and [13] for the hyperbolic case.

A.1. Fundamental Properties

Let $(S(t))_{t\in\mathbb{R}}$ represents the solution semigroup for the linear problem $iu_t+\delta u_{xx}+u_{yy} = 0$, $\delta = \pm 1$. We have the following result:

Proposition A.1.1 *Let I be a bounded, open interval of $\mathbb{R}$ with $0 \in \mathbb{R}$. Let $s \in \mathbb{R}$, $\varphi \in H^s(\mathbb{R}^n)$, $f \in L^1(I; H^{s-2}(\mathbb{R}^n))$ and $u \in L^1(I; H^s(\mathbb{R}^n))$. Then u satisfies*

$$u(t) = S(t)\varphi - i\int_0^t S(t-s)f(s)\mathrm{d}s$$

for a.a. $t \in I$ if and only if $u \in W^{1,1}(I; H^{s-2}(\mathbb{R}^n))$ and

$$\begin{aligned} &iu_t + \delta u_{xx} + u_{yy} = f \quad \textit{for a.a. } t \in I \\ &u(0) = \varphi \end{aligned}$$

If, in addition, $f \in C(I; H^{s-2}(\mathbb{R}^n))$ and $u \in C(I; H^s(\mathbb{R}^n))$, then $u \in C^1(I; H^{s-2}(\mathbb{R}^n))$ and the equation holds on I.

A.2. Strichartz's Estimates for Schrödinger

Let $S(t)$ be as above and $\Lambda f(t) = \int_0^t S(t-s)f(s)\,\mathrm{d}s$ and $(\mathcal{T}u)(t) = S(t)\varphi - i\Lambda f(t)$. We say that a pair (q,r) is admissible if

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{r}\right) \text{ and } 2 \le r \le \frac{2n}{n-2}$$

($2 \le r \le \infty$ if $n = 1$, $2 \le r < \infty$ if $n = 2$). We have :

Theorem A.2.1 *The following properties hold:*

(i) *For every* $\varphi \in L^2(\mathbb{R}^n)$, *the function* $t \mapsto S(t)\varphi \in L^q(\mathbb{R}; L^r(\mathbb{R}^n)) \cap C(\mathbb{R}; L^2(\mathbb{R}^n))$ *for every* (q, r) *admissible pair and there exists a constant* C *depending on* q *and* r *such that*

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \le C\|\varphi\|_2.$$

(ii) *Let* I *be an interval of* $\mathbb{R}$, $J = \bar{I}$, *and* $0 \in J$. *If* (γ, ρ) *is an admissible pair and* $f \in L^{\gamma'}(I; L^{\rho'}(\mathbb{R}^n))$, *then for every* (q, r) *admissible pair, the function* $t \mapsto \Lambda f(t) := \int_0^t S(t-s)f(s)\mathrm{d}s \quad$ *for* $t \in I$, *belongs to* $L^q(I; L^r(\mathbb{R}^n)) \cap C(J; L^2(\mathbb{R}^n))$ *and there exists a constant* C *depending on* q, r, γ *and* ρ *and is independent of* I *such that*

$$\|\Lambda f\|_{L^q(I;L^r(\mathbb{R}^n))} \le C\|f\|_{L^{\gamma'}(I;L^{\rho'}(\mathbb{R}^n))}.$$

Regarding Theorem A.2.1, see [1, Theorem 2.2.3] and [13, Lemma A.1] for elliptic and hyperbolic cases respectively.

APPENDIX B: SOME INEQUALITIES

Proposition B.1 (Young's inequality) [24, Appendix B.2, c.] *Let* $1 < p < \infty$ *then for* $a, b > 0$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proposition B.2 (Young's inequality for convolutions) [25, Theorem 1.5.2] *Let* $1 \leq p \leq \infty$ *and* $f \in L^1(\mathbb{R}^n)$*,* $g \in L^p(\mathbb{R}^n)$*. Then* $f * g$ *is well-defined for almost every* $\boldsymbol{x}$ *and is in* $L^p(\mathbb{R}^n)$ *with*

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Proposition B.3 (Gronwall's lemma) [26, Lemma 4.2.1] *Let* $T > 0$*,* $\lambda \in L^1((0,T))$*,* $\lambda \geq 0$ *a.e. and* $C_1, C_2 \geq 0$*. Let* $\varphi \in L^1((0,T))$*,* $\varphi \geq 0$ *a.e., be such that* $\lambda\varphi \in L^1((0,T))$ *and*

$$\varphi(t) \leq C_1 + C_2 \int_0^t \lambda(s)\varphi(s)\,\mathrm{d}s,$$

for a.e. $t \in (0,T)$*. Then*

$$\varphi(t) \leq C_1 \exp\left(C_2 \int_0^t \lambda(s)\,\mathrm{d}s,\right)$$

for a.e. $t \in (0,T)$*.*

Proposition B.4 (Gagliardo-Nirenberg's inequality) [19, Theorem 1.3.7] *Let* $1 \leq p, q, r \leq \infty$ *and let* j, m *be two integers,* $0 \leq j < m$*. If*

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{(1-a)}{q}$$

for some $a \in [j/m, 1]$ *(*$a < 1$ *if* $r > 1$ *and* $m - j - \frac{n}{r} = 0$*), then there exists* $C(n, m, j, a, q, r)$ *such that*

$$\sum_{|\alpha|=j} \|D^\alpha u\|_p \leq C \left(\sum_{|\alpha|=m} \|D^\alpha u\|_r \right)^a \|u\|_q^{1-a}$$

for every $u \in \mathcal{D}(\mathbb{R}^n)$.

APPENDIX C: SOBOLEV EMBEDDING RESULTS

Theorem C.1 [25, Theorem 2.4.5] *Let $m \geq 1$ be an integer and $1 \leq p < \infty$. Then*

i) if $\frac{1}{p} - \frac{m}{n} > 0$, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n))$ *with* $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$,

ii) if $\frac{1}{p} - \frac{m}{n} = 0$, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n))$, *for* $p \leq q < \infty$,

iii) if $\frac{1}{p} - \frac{m}{n} < 0$, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n))$.

We have also an embedding result for partial fractions. It can be deduced from [27, Theorem 6.5.1] that $H^s(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ when $s = \frac{n}{2} - \frac{n}{q}$ with $s \in \mathbb{R}$.

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