INVENTORY POLICIES FOR AN ASSEMBLE-TO-ORDER SYSTEM WITH JOINT DISCOUNT INCENTIVES

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This work is dedicated to my family including Esin and (possible) other new members. This page is for them to fill.

ABSTRACT

INVENTORY POLICIES FOR AN ASSEMBLE-TO-ORDER SYSTEM WITH JOINT DISCOUNT INCENTIVES

We consider an assemble-to-order system to meet all of the stationary stochastic demand of a finished product in a periodic review setting. The finished product is assembled using two subassemblies (components). The demand must be met either by regular production or by using a faster but more expensive expedited mode. Components have independent setup, production, holding and expediting costs. However when both components fall short of demand they use the same expediting resource (same plane, same supplier channel, same overtime shift in a factory, etc.) causing a joint discount in unit expediting costs. This joint cost factor prevents solving of inventory control problem of each component independently and increases the time and space complexity of solving optimal inventory policy. We analyze models with and without setup costs. We prove that the optimal policy of the model without setup cost is a modified base stock policy, where target inventory for a component is a function of the other component's inventory level, both for a finite and an infinite horizon model. Similarly the optimal policy of the single and two period model with positive setup cost is a modified state dependent (s, S) policy, where (s, S) values of a component is a function of the other component's inventory level. Based on these results we develop an algorithm, which decreases time complexity, for solving finite and infinite horizon models in models without setup-costs optimally and in models with setup costs very close to optimal results.

ÖZET

ORTAK İNDİRİM TEŞVİĞİ OLAN SİPARİŞE GÖRE MONTAJ SİSTEMİNDE ENVANTER POLİTİKALARI

Siparişe göre montaj yapan ve bitmiş bir ürüne olan bütün stokastik talebi dönemsel gözden geçirme ortamında karşılayan bir sistem düşünülmektedir. Bitmiş ürün iki altparçadan (komponent) oluşturulmaktadır. Talep ya normal üretimle ya da hızlı tedarikle karşılanmaktadır . Komponentlerin bağımsız sabit, üretim, stok ve hızlı tedarik masrafları vardır. Ama eğer iki komponentin stoğu da talebi karşılayamazsa aynı hızlı tedarik kaynağı kullanılmaktadır (aynı uçak, aynı tedarikçi kanalı, aynı fabrikadaki gece mesaisi gibi). Aynı kaynağı paylaşarak kullanmak toplam hızlı tedarik masraflarına belli bir indirim getirmektedir. Bu ortak indirim stok kontrol probleminin her komponent için ayrı olarak en iyi şekilde çözülmesine engel olup en iyi çözümün zaman ve yer karmaşıklığını arttırmaktadır. Sabit üretim masraflı ve masrafsız modeller ayrı ayrı incelenmiştir. Sabit üretim masrafı olmayan, sonlu dönemli ve sonsuz dönemli modellerde sabit stok hedefi politikasının gelişmiş bir versiyonunun en iyi politika olduğu kanıtlanmıştır (bu gelişmiş versiyonda sabit stok hedefi diğer komponentin stoğunun bir fonksiyonu olmaktadır). Sabit üretim masrafı olan modelde ise tek ve iki dönemde en iyi politikanın gene duruma bağlı bir gelişmiş (s, S) politikası olduğu gösterilmiştir. Bu gelişmiş politikada her bir komponentin (s, S) değerleri diğer komponentin stoğunun fonksiyonu olarak ortaya çıkmıştır. Bu ispatlara dayanarak zaman karmaşıklığı az bir algoritma geliştirilmiştir. Bu algoritma çok ve sonsuz dönemli modellerde üretim masrafı yoksa en iyi, üretim masrafı varsa en iyiye yakın değerleri bulmaktadır.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS/ABBREVIATIONS

A	set of all possible actions in MDP
$A(h)$	set of all possible actions available to state h in MDP
c_i	regular ordering cost per unit i
d	random demand variable for product during the single period
d_{max}	maximum possible demand per period
d_n	random demand for the end-product during period n
$e(x_1, x_2)$	adjusted expediting (overtime) cost function for amount of x_i
$F(d)$	cumulative density function of demand d in continuous domain.
$f(d)$	probability density function of demand d in continuous domain.
H	set of all states in MDP
$h^r_{in}(x_i)$	adjusted holding cost function for excess inventory amount of x_i
$h_i(x_i)$	adjusted holding cost function for excess inventory amount of x_i
i	index for components $i = 1, 2$
K_i	regular fixed order cost of component i
$P(d)$	probability function of demand d in discrete domain.
R	reward matrix in MDP
T	transition matrix in MDP
X	number of possible inventory levels
x_i	inventory of unit i at the start of period
x_{in}	inventory of unit i at the start of period n
$\tilde{x}_i$	inventory of unit i before expediting
$\tilde{x}_{in}$	inventory of unit i before expediting in period n
y_i	target inventory position chosen for regular production of unit i
y_{in}	inventory position chosen for regular production of unit i at period n

Y	number of possible target inventory levels
$\tilde{y}_i$	target inventory position chosen after expediting unit i
$\tilde{y}_{in}$	inventory position chosen after expediting of unit i at period n
α	time discount for costs occurring in the next period.
α_d	unit expediting discount when two components are expedited

ATO	Assemble-to-order
IND	Independent Single Component Algorithm
MDP	Markov Decision Process
MOD	Modified (s, S) Algorithm
OPT	Optimal Policy Algorithm

1. INTRODUCTION

In sectors with high cost end products having uncertain demand and high variability of customer preferences, assemble to order (ATO) systems are both cost-efficient and responsive. Automotive and hardware are examples of sectors with high-cost high profit end products with high variability. PC manufacturer Dell is a popular example for ATO systems offering high variability products and delivering products quickly [1]. Procuring or producing estimated demand regularly and having fast procurement channels for expediting unexpected demand boosts the efficiency of ATO systems. Huggins and Olsen [2] give an example for a supplier of ATO system in automotive sector which uses expediting frequently. We consider an ATO system to meet all of the stationary stochastic demand of a finished product in a periodic review setting. The finished product is assembled using two subassemblies (components). Inventory holding and shortage costs of the components are incurred periodically in infinite horizon. All costs are different and independent for each component except a common expediting discount. This joint discount occurs when both components are out of stock and they are ordered in an expedited mode at the same time. The shortage of the components are handled by replenishing components in an expedited mode, which is a fast but expensive way of procurement. Inspired by an expediting model setting defined in Huggins and Olsen [2], each period has two phases:

1. In phase one, regular procurement decisions for components are made based on expected demand. Regular procurement is cheaper than expediting. However its relatively slow lead time prevents the decision maker waiting for the realization of the period demand.
2. Once the period demand is realized, in order to meet unexpected high demand, it is possible to use a form of expediting. Either using a faster and expensive transportation method or using overtime production the required components are procured until the end of the period. At the end of the period, all demand is met.

The only joint cost effect in our model is on the unit expediting costs of both components. When both components are expedited, they are using the same resource. Either they are put in the same plane or they are made in the same factory in a single overtime shift. This common resource usage causes a discount per both component units expedited. Had there been no such joint discount, both components would behave independently based on their independent holding and expediting costs. This joint discount gives an incentive for increasing the overall probability of both components falling short of demand in the long run, thereby closing the difference between optimal inventory levels of both components.

Using a dynamic programming technique of policy iteration, we investigate the nature of optimal inventory policies. The results of policy iteration algorithm show that, the optimal policies are between two extreme cases, based on the independent component holding and expediting costs and joint expediting discount:

- **Case 1:** The joint expedition discount is insignificant and both components behave independently as if there is no discount.
- **Case 2:** The expedition discount is very high and both components behave like a single component.

For different cost parameters optimal policies are like in case 1 or in case 2 or in a hybrid state somewhere between case 1 and 2.

During our computational tests on our most general model that involves setup times, we observed that optimal results exhibit a modified (s, S) form: the s and S parameters of a component do depend on the initial inventory level of the other component. $s_1(x_2)$,$s_2(x_1)$, $S_1(x_2)$, $S_2(x_1)$ functions can be defined; for component i, the optimal policy is to reach target inventory level $S_i(x_j)$ when initial inventory x_i of component i is less than $s_i(x_j)$, where x_j is the initial inventory of the other component j. If x_i is greater than $s_i(x_j)$ optimal policy is "not to order". We have analyzed the model in order to prove optimality of the modified (s, S) policy. In a single period model, we show that the model without setup costs is optimized by a modified base stock policy, which can be described only with $S_1(x_2), S_2(x_1)$ functions. (If there are no setup costs $s_1(x_2)$,$s_2(x_1)$ functions are set to zero.) We extend our results to the model with setup costs and prove that the single period model with setup cost has modified (s, S) policy as the optimal policy. Then we analyze the multiple period model and prove that no-setup cost model has optimal modified base stock policy for finite and infinite horizons and optimal cost function is always convex in the no-setup setting. In the model with setup costs we prove that the two period model has optimal modified (s, S) policy, based on the $[K_1K_2]$-convexity of the cost function. For more than two periods we fail to prove that $[K_1K_2]$-convexity of the cost function is preserved. Hence we continue our computational efforts to find counter examples.

Using policy iteration and exploiting modified form of (s, S) policies we develop a tailor-cut policy iteration algorithm which has less time and space complexity compared to the usual policy iteration algorithm, where we do not presume the optimal policy has the $s_1(x_2), s_2(x_1), S_1(x_2), S_2(x_1)$ form. For different test parameters optimal policy results of these algorithms are usually the same, although few counter-examples are found and reported.

To investigate whether the optimality of modified (s, S) policy depends on a lucky selection of the parameters or a general behaviour of our problem, we analyze the optimality conditions of the problem starting with the single period model. Single period cost function is convex where the optimal policy has $s_1(x_2), s_2(x_1), S_1(x_2), S_2(x_1)$ form. Applying the optimal policy yields a $[K_1K_2]$-convex optimal cost function. We prove that in two periods $[K_1K_2]$-convexity of the optimal cost function is preserved, given the demand distribution is *stationary and log-concave.* Preservation of $[K_1K_2]$-convexity of the cost function guarantees $s_1(x_2), s_2(x_1), S_1(x_2), S_2(x_1)$ form of the optimal policy in two periods. For n periods ($n > 2$) and infinite horizon we fail to prove or disprove that $[K_1K_2]$-convexity is preserved. Few of our computational tests have resulted in counter-examples to this conjecture. However even with counter examples, modified (s, S) policy (MOD) has reached the optimal solutions in most of the cases (536 out of 576), spending only one fourth policy iteration time of the exact optimal solution algorithm (OPT). X being number of possible inventory levels for each component and d_{max} being the maximum demand possible in a period, MOD has time complexity $O(log_2(X) \times X \times (X - d_{max})^2 \times d_{max})$, while OPT has time complexity $O(X^2 \times (X - d_{max})^2 \times d_{max})$, which makes MOD an efficient heuristic. It can be applied in larger problems where time complexity of OPT is prohibitively large.

1.1. LITERATURE SURVEY

In [2], Huggins and Olsen consider a model where a manager must make two inventory decisions during each period. At the beginning of the period the current inventory level is known and the manager must then decide the target inventory level for the regular production. After the regular production is determined, a stochastic demand is realized. At this point the manager knows the inventory level at the beginning of the expediting period, which is the difference between the inventory position chosen for regular production and the demand. The manager must now decide whether to run the overtime production and if it is run, the new level of inventory position. However, backlogged demand is not allowed for the following period such that if the inventory is negative then the decision becomes whether to just close the shortfall and start the next day with zero units or if not, up-to what positive level to produce for starting next period with some inventory. The inventory position chosen for overtime becomes the starting inventory level for the next period, a holding cost is charged for any positive inventory and the cycle continues. They show that optimal regular production policy is (s, S) type and they also investigate the case where there are two different expediting suppliers. Our problem setting is inspired by Huggins and Olsen, however we consider an assembly system with joint expediting cost rather than a single component as they do. Our results lead us to more general results in two dimensional assembly settings rather than handling only expediting. Expediting literature involves managing lead times in discrete [3, 4] or continuous time settings [4, 5]. Arslan and her colleagues [4] use a make-to-order system for a supplier who can choose overtime or subcontracting as a form of expediting. They have shown (s, S) policy is optimal in their continuous and discrete time settings. Gallego and his colleagues propose a solution where fixed leadtime can be converted when orders coming from an Erlang distribution reaches threshold limits. Lawson and Porteus [3] have shown that in a discrete time multi stage inventory management system without setup costs "top-down base stock" policies are optimal. The focus on these works is managing lead times for improving system performance, where we only deal with dual source supply problem in a multi-item environment.

We show that in the single period model a modified (s, S) policy for 2 dimensions is optimal. Scarf [6] has proved that (s, S) policies are optimal for periodic review inventory control problems with convex holding and shortage costs and non-negative setup costs. Our proof is based on Scarf's proof. In his proof he assumed convex costs per period and introduced the notion of *K-convexity.* Scarf's [6] definition of K-convexity is generalized to multi dimensions in Gallego and Sethi [7]. Our definition of $[K_1K_2]$-convexity is a special case of this generalization, where a joint setup cost component is missing from our definition of $[K_1K_2]$-convexity. Other references for (s, S) policies are Veinott [8] and Zheng [9]. Veinott relaxed convexity assumption to quasi-convexity. Zheng generalized the results of Scarf and Veinott to the infinite horizon. A recent modified (s, S) policy result is due to Chao and Zipkin [10]. They consider a model where regular production has zero setup cost if it is less than a specified capacity. They are able to partially characterize the optimal policy using K-convexity concept similar to our work. They propose a fast heuristic for finding optimal or near optimal solutions as we have proposed modified (s, S) policy for multi period model with setup costs, where we fail to characterize the optimal policy. However none of these works is related with a multiple component (i.e., two dimensional in our case) (s, S) policy in assemble-to-order system.

An in-depth review of inventory control problems in assemble-to-order systems is given in Song and Zipkin [11]. There are two relevant works to our model [12, 13]. Benjaafar and Elhafsi [12] propose a continuous review model with m components, which have exponential production processes with different rates. There are n different classes of customers. Demand for each class occurs according to a different Poisson process. Backorder costs or lost sales are also different for each class. Based on the priority of customer class the decision maker may meet demand from inventory or make the customers with less priority wait. They have shown that the optimal policy in their settings is a modified base stock policy where base stock levels depend on the inventory of the other components, similar to the results in our discrete time setting where there are no lead times between different production orders. The optimal policy for accepting different customer classes is a function of inventory levels, where at lower inventory levels one accepts only high priority classes (which have high lost sale or back order costs). Feng, Ou and Zang [13] use the same settings as [12]. However instead of different customer classes, they include to the model product pricing. The product price has two levels, where higher price level decreases the expected demand, which is a nonhomogeneous Poisson process. Assuming that it is always better to lever the product price before switching the production of components, they yield same optimal modified base stock policy where base stock levels depends on the inventory of the other components. In the optimal price policy a higher inventory level again (j_1, j_2), dominates a lower inventory level (i_1, i_2): the product price is always less than or equal to the product price of lower inventory level. Contrary to these works [12, 13], we have a periodic review model with zero component production lead times. In our model allocation of finished products to different customer classes or control of demand by levering the product price is not considered. No back orders or demand shortages are allowed. Demand is always met by expediting of components. Special disjoint expediting cost exists for expediting both components. However our model with no setup costs yield same modified base stock policy, where base stock levels depend on the inventory of the other components. Additionally we extend our work to model with setup costs, which are not present in these works.

Joint replenishment problems have also joint cost structure depending on multi items. Joint costs are based on fixed setup costs rather than unit costs. *Major* joint setup cost charged once if there is an at least one item to be ordered. Each item has also a specific additional *minor* setup cost which is charged when the related item is ordered. As in our problem optimal joint replenishment policy can be found by defining a Markov decision process [14], however if there are more than two items or many different levels of inventory, decision space grows exponentially as in our problem. In the joint setup cost literature the focus is on the simple heuristic policies rather than finding optimal policies. Policies can be continuous, where inventory levels of items are monitored continuously [15, 16, 17, 18], or periodic where inventory levels of items are checked periodically [19, 20, 21]. Can-order policy is one of the early proposed continuous policies [15]. In can-order policies each item has three parameters: s,c and S, where $s < c < S$. If inventory level of the item is below s than item must be ordered up to level base stock level S. If inventory level of item is between s and c than the item is ordered up to S, when there is another item that must be ordered. It is reported to perform better than periodic review policies when major setup costs is relatively high [16, 18]. One of the periodic review policies is $P(s, S)$ proposed by Viswanathan [19]. In $P(s, S)$ every t units of time all items are reviewed. Item i ordered upto S_i level, when its inventory level is below s_i. For calculation of (s, S) values only related minor setup costs is considered. Nielsen and Larsen [17] have developed $Q(s, S)$ policy which is an extension $P(s, S)$ policy in continuous setting, yielding better results. In joint replenishment literature although there are policies where modified (s, S) policies are used, there is no proposal about a policy where s, S levels depends on the inventory level of the other component(s). Our work may be extended as a joint replenishment problem for the future research.

1.2. PREVIEW

The thesis is organized as follows: Chapter 2 analyzes the optimal policy in a single period model. Starting from zero initial inventory, no setup cost model it shows that no-setup model has a convex cost function, building the form of optimal policy as functions of components $s_1(x_2), s_2(x_1),\ S_1(x_2), S_2(x_1)$, in the model with non-zero initial inventory and setup costs.

Chapter 3 extends the results to multiple periods and shows no setup-cost model preserves convexity of the optimal policy in finite and infinite horizons, proving the optimal policy has $S_1(x_2), S_2(x_1)$ form. In model with setup costs it shown optimal policy preserves $[K_1K_2]$-convexity of cost function preserved in two periods, hence two period model has modified (s, S) form of optimal policy. For more than two periods we fail to prove that $[K_1K_2]$-convexity and thereby modified (s, S) form of optimal policy is preserved.

However we propose a heuristic (MOD) which is based on the assumption that the optimal policy has modified (s, S) form in Chapter 4. Comparing the test results with the exact optimal algorithm (OPT) and a simple heuristic ignoring joint expediting costs (IND), MOD is shown to be an algorithm which is efficient both in accuracy and time complexity.

In chapter 5 we give our conclusions and point to further research directions.

2. SINGLE PERIOD MODEL

Inspired by an automotive industry problem, we consider an assembled component consisting two subassemblies (components) that are bought from external suppliers with unlimited capacity. The duration and the cost of the assembly process is negligible and because of vast customer configuration preferences it is undesirable to hold a finished product inventory. The situation is a typical assemble to order system where a random demand is faced by the manufacturer (assembler) and components are bought from different suppliers. The costs of the manufacturer are related with managing the inventories of subassemblies and shortages. The shortage cost of the finished product is so high that demand must be met at all times. The manufacturer makes two sets of decisions, replenishment of components before the demand is revealed and expedited replenishment at a higher unit cost after the demand is revealed. If both component inventories fall short of demand, expediting them together is cheaper than expediting each component separately. This might be because the components bought together are sharing a transportation modality, for example. Regular replenishment decisions are given at the beginning of the period, followed by demand realization. If production falls short of demand, expedited replenishment decisions are given next. Costs accrue and profits are collected at the end of the period.

The random demand is characterized by its probability density function, f and cumulative distribution, F. Inventory holding costs are charged to the manufacturer. Expediting cost includes only unit expedited cost for each component plus a discount if two different unit components are expedited together. However even with this discount, it is still cheaper to produce both components using regular replenishment (i.e., without using expedition).

In this chapter, we consider a single period model to gain insight about the characteristics of the problem.In Section 2.1 we introduce assumptions and the notation of the model. We derive expected costs and the optimality conditions for the model in Section 2.2. In this derivation, the setup costs are excluded from the cost function and it is assumed all components have zero starting inventory for the sake of simplicity. The special case where each component behaves independently is analyzed first, which results in the newsvendor problem. Having analyzed the single component single period model, optimality conditions of the two component single period model are derived.

The optimal policies derived from the cost function without setup costs and zero initial inventory (Section 2.3) are extended to the model with non-zero initial inventories in Section 2.4. Setup costs are included in the model of Section 2.5. The form of optimal policy for single period is fully analyzed.

2.1. Single Period Model: Notation and Assumptions

We consider a single period assembly model with two components. There is no inventory of the finished product. Components are assembled upon observing the demand. The lead time for assembly is assumed to be negligible. Related costs are holding costs h_1, h_2 and unit production costs c_1, c_2 of each component and joint expediting cost $e(x_1, x_2)$ for x_1,x_2 missing units of components, respectively.

Random demand d must be met either by regular production or expedition. Thus minimal cost $c_i d$ is a sunk cost for all situations, where inventory holding costs of regularly replenished components and the extra charge of expediting costs add to this sunk cost. Holding and expediting costs may be readjusted to include unit production costs, eliminating regular production costs in the model description. There is no salvage value for positive ending inventories of components.

It is assumed that holding costs $h_i(x_i)$ are non-negative, non-decreasing in x_i with $h_i(0) = 0$ and $lim_{x_i \to \infty} h_i(x_i) = \infty$. Furthermore $e(x_1, x_2)$ is non-negative and non-decreasing in x_1 and x_2 with $e(0, 0) = 0$ and $lim_{x_i \to \infty} e(x_i, x_{3-i}) = \infty$. For later reference, the assumptions are labelled as follows:

1. Demand d is a random variable with pdf f, cdf F and a finite mean.
2. $e(x_1, x_2) \geq \sum_{i=1}^{2} h_i x_i$

3. $e(x_1, x_2)$ is non-negative and non-decreasing in x_1, x_2 with $e(0,0) = 0$ and $lim_{x_i \to \infty} e(x_i, x_{3-i}) = \infty$.
4. $h_i(x_i)$ is non-negative, non-decreasing in x_i with $h_i(0) = 0$ and $lim_{x_i \to \infty} h_i(x_i) = \infty$, for $i = 1, 2$.
5. When both items are expedited (i.e., $x_1, x_2 > 0$) then $e(x_1, x_2) \leq e(x_1, 0) + e(0, x_2)$
6. The discount on joint expediting cost function $e(x_1, 0) + e(0, x_2) - e(x_1, x_2)$ is not greater than the expediting cost any component expedited single-handedly: $e(x_1, 0) + e(0, x_2) - e(x_1, x_2) < e(x_1, 0)$ and $e(x_1, 0) + e(0, x_2) - e(x_1, x_2) < e(0, x_2)$.

In this problem there are two state variables representing the inventory level at the beginning of regular time and at the end of the period just before expediting. Accordingly there are two decision variables presenting chosen inventory position for regular production and expedited production.

The notation is given in Table 2.1. Inventory of component i just before expediting is given by the balancing equation:

$$\tilde{x}_i = y_i - d$$

We assume that no backorders are allowed, hence :

$$y_i \geq x_i^+ \qquad \tilde{y}_i \geq \tilde{x}_i^+$$

where x^+ denotes $\max(0, x)$. The initial inventory is non-negative :

$$x_i \geq 0$$

Table 2.1. Notation

i	index for components $i = 1, 2$
c_i	regular ordering cost per unit i(embedded in adjusted holding and expediting costs)
K_i	regular fixed order cost of component i
y_i	target inventory position chosen for regular production of unit i
$\tilde{y}_i$	target inventory position chosen after expediting unit i
x_i	inventory of unit i at the start
$\tilde{x}_i$	inventory of unit i before expediting
d	random demand variable for product during the single period
$f(d)$	probability density function of demand in continuous domain.
$F(d)$	cumulative density function of demand in continuous domain.
$e(x_1, x_2)$	adjusted expediting (overtime) cost function for amount of x_i
$h_i(x_i)$	adjusted holding cost function for excess inventory amount of x_i
α_d	unit expediting discount when two components are expedited together.

2.2. Expected Cost Function for Single Period Model

For a two component problem let π be an admissible policy with variables

$$(x_1, x_2, y_1, y_2, \tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2)$$

where x_i are the starting inventory, y_i are regular replenishment quantities, $\tilde{x}_i$ are inventory levels just after demand is realized and $\tilde{y}_i$ are expedited quantity decisions. Let Π be the set of all admissible policies. The cost function is given as follows:

$$g(x_1, x_2, y_1, y_2, \tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2) = \sum_{i=1}^{2} [K_i \delta(y_i - x_i) + h_i(\tilde{y}_i)] + e(\tilde{y}_1 - \tilde{x}_1, \tilde{y}_2 - \tilde{x}_2)$$

Where $\delta(x)$ is defined as $\delta(x) = 1$ if $x > 0$ and $\delta(x) = 0$ otherwise. The related expediting cost function $e(a, b)$ is:

$$e(a, b) = e_1 a^+ + e_2 b^+ - \min(a^+, b^+)\alpha_d(e_1 + e_2)$$

where e_1, e_2 are unit expediting costs for individual components and α_d is the expediting discount if two units are expedited together. The related expected policy cost function is:

$$g_\pi(x_1, x_2) = E_d(g(x_1, y_{1\pi}, \tilde{x}_{1\pi}, \tilde{y}_{1\pi}, x_2, y_{2\pi}, \tilde{x}_{2\pi}, \tilde{y}_{2\pi}))$$

The optimal expected cost function is minimum of all admissible policies $\pi \in \Pi$.

There will be no salvage value for the end inventory. At the end of the period, stock components will be discarded. Since there is no need to expedite more than the demand in this case, *optimal expediting policy* is only covering shortages:

$$\tilde{y}_i^* = (\tilde{x}_i)^+$$

where $\tilde{x}_i = y_i - d$.

Using the optimal expediting policy and taking expectation, we may rewrite g which does not include $\tilde{y}_i$ and $\tilde{x}_i$:

$$g(x_1, x_2, y_1, y_2) = E_d\left[\sum_{i=1}^{2}[K_i\delta(y_i - x_i) + h_i(y_i - d)^+] + e[(d - y_1)^+, (d - y_2)^+]\right]$$

The optimal expected policy cost function becomes:

$$g^*(x_1, x_2) = \min_{y_1 \geq x_1, y_2 \geq x_2} g(x_1, x_2, y_1, y_2) \tag{2.1}$$

2.3. Optimal Policies in Single Period Model without Setup Costs and Initial Inventory

We start by assuming that setup costs $K_i = 0$, $i = 1, 2$ and beginning of period inventories as zero, $x_1 = x_2 = 0$. In this case the cost function reduces to:

$$g(0,0,y_1,y_2) = E_d\left[\sum_{i=1}^{2}[h_i(y_i-d)^+] + e[(d-y_1)^+,(d-y_2)^+]\right] \tag{2.2}$$

Taking the expectation in (2.2) and eliminating the $^+$ operator we have,
for $y_1 \leq y_2$:

$$\begin{aligned} g(0,0,y_1,y_2) \;=\; & \int_0^{y_1} [h_1(y_1-u)+h_2(y_2-u)]\, f(u)du + \\ & \int_{y_1}^{y_2} [e_1(u-y_1)+h_2(y_2-u)]\, f(u)du + \\ & \int_{y_2}^{\infty} [(e_1+e_2)(1-\alpha_d)(u-y_2)+e_2(y_2-y_1)]\, f(u)du \end{aligned} \tag{2.3}$$

for $y_1 > y_2$:

$$\begin{aligned} g(0,0,y_1,y_2) \;=\; & \int_0^{y_2} [h_1(y_1-u)+h_2(y_2-u)]\, f(u)du + \\ & \int_{y_2}^{y_1} [h_1(y_1-u)+e_2(u-y_2)]\, f(u)du + \\ & \int_{y_1}^{\infty} [(e_1+e_2)(1-\alpha_d)(u-y_1)+e_2(y_1-y_2)]\, f(u)du \end{aligned} \tag{2.4}$$

If the expediting discount $\alpha_d = 0$ then problem can be solved for each component separately (dropping the subscript i), which results in the newsvendor critical ratio solution:

$$F(y^*) = \frac{e}{(h+e)} \tag{2.5}$$

2.3.1. Two component Single Period Model

We start the evaluation of two component model by taking each component separately. Ignoring the combined expediting discount:

$$y_i^o = F^{-1}\left(\frac{e_i}{h_i + e_i}\right) \qquad i = 1, 2 \tag{2.6}$$

Using equation 2.6 we can find minimum and maximum of the optimal stock levels. Based on y_1^o, y_2^o values we may find the optimum joint stock levels y_1^*, y_2^*.

Proposition 2.1 *Let y_1^o, y_2^o be optimum stock levels of component 1 and 2, when $\alpha_d =$ 0. Then optimal inventory decisions (y_1^*, y_2^*) are characterized as follows:*

$$\begin{aligned}
y_1^* = y_1^o,\ y_2^* &= F^{-1}\left(\frac{e_2 - \alpha_d(e_1+e_2)}{h_2 + e_2 - \alpha_d(e_1+e_2)}\right) && \textit{for } \tfrac{e_1}{h_1+e_1} < \tfrac{e_2-\alpha_d(e_1+e_2)}{h_2+e_2-\alpha_d(e_1+e_2)} \\
y_2^* = y_2^o,\ y_1^* &= F^{-1}\left(\frac{e_1 - \alpha_d(e_1+e_2)}{h_1 + e_1 - \alpha_d(e_1+e_2)}\right) && \textit{for } \tfrac{e_2}{h_2+e_2} < \tfrac{e_1-\alpha_d(e_1+e_2)}{h_1+e_2-\alpha_d(e_1+e_2)} \\
y_1^* = y_2^* &= F^{-1}\left(\frac{(e_1+e_2)(1-\alpha_d)}{h_1 + h_2 + (e_1+e_2)(1-\alpha_d)}\right) && \textit{otherwise}
\end{aligned} \tag{2.7}$$

Proof: Without loss of generality assume $y_1^o \leq y_2^o$. There are two possibilities:

1. Optimal stock levels of components are different: $y_1^* < y_2^*$,
2. or they are equal: $y_1^* = y_2^*$

We will assume case 1 first and extend our results to case 2. If the first order conditions are satisfied, the convexity of cost function guarantees the optimality of (y_1^*, y_2^*). Taking partial derivatives of (2.3) we have:

$$\begin{aligned}
\frac{\partial g}{\partial y_1} &= h_1\left[F(y_1) - F(0)\right] - e_1\left[F(y_2) - F(y_1)\right] - e_1\left[1 - F(y_2)\right] \\
&= (h_1 + e_1)F(y_1) - e_1
\end{aligned}$$

and

$$\begin{aligned}\frac{\partial g}{\partial y_2} &= h_2\left[F(y_1)-F(0)\right]+h_2\left[F(y_2)-F(y_1)\right]-(e_2-\alpha_d(e_1+e_2))\left[1-F(y_2)\right]\\ &= (h_2+e_2-\alpha_d(e_1+e_2))F(y_2)-(e_2-\alpha_d(e_1+e_2))\end{aligned}$$

By setting the derivatives to zero we get:

$$F(y_1^*) = \frac{e_1}{h_1+e_1} \tag{2.8}$$

$$F(y_2^*) = \frac{e_2-\alpha_d(e_1+e_2)}{h_2+e_2-\alpha_d(e_1+e_2)} \tag{2.9}$$

The optimality conditions for the component with lower optimal stock level y_1^* is its independent solution found from (2.6). The expediting discount affects only component with greater optimal level. The optimal stock level decreases because the effective expediting cost decreases (2.9).

Equations (2.8) and (2.9) are only valid when $\frac{e_1}{h_1+e_1} < \frac{e_2-\alpha_d(e_1+e_2)}{h_2+e_2-\alpha_d(e_1+e_2)}$. If $\frac{e_2}{h_2+e_2} < \frac{e_1-\alpha_d(e_1+e_2)}{h_1+e_1-\alpha_d(e_1+e_2)}$ then

$$F(y_2^*) = \frac{e_2}{h_2+e_2} \tag{2.10}$$

$$F(y_1^*) = \frac{e_1-\alpha_d(e_1+e_2)}{h_1+e_1-\alpha_d(e_1+e_2)} \tag{2.11}$$

However there are cases, where neither $\frac{e_1}{h_1+e_1} < \frac{e_2-\alpha_d(e_1+e_2)}{h_2+e_2-\alpha_d(e_1+e_2)}$ nor $\frac{e_2}{h_2+e_2} < \frac{e_1-\alpha_d(e_1+e_2)}{h_1+e_1-\alpha_d(e_1+e_2)}$ can be satisfied. Our initial assumption is that optimum stock levels are different (either $y_1^* < y_2^*$ or $y_2^* < y_1^*$). W.L.O.G. assume $\frac{e_1}{h_1+e_1} < \frac{e_2}{h_2+e_2}$ and $\frac{e_1}{h_1+e_1} > \frac{e_2-\alpha_d(e_1+e_2)}{h_2+e_2-\alpha_d(e_1+e_2)}$, which is mathematically possible. In this case if $y_1^* < y_2^*$ is true, then y_1^* and y_2^* must be calculated according to the equations (2.8) and (2.9), yielding the result $y_1^* > y_2^*$. If $y_1^* > y_2^*$ is true, then y_1^* and y_2^* must be calculated according to the equations (2.10) and (2.11), yielding the result $y_2^* > y_1^*$. Hence our initial assumption that $y_1^* \neq y_2^*$ is false. Two components behave like a single component and their optimal inventory levels are equal: $y_1^* = y_2^*$. Then the cost function becomes:

$$\begin{aligned} g(0,0,y_1,y_1) &= \int_0^{y_1} [(h_1+h_2)(y_1-u)]\, f(u)du + \\ & \int_{y_1}^{\infty} [(e_1+e_2)(1-\alpha_d)(u-y_1)]\, f(u)du \end{aligned}$$

and y_1^* is calculated by the equation :

$$F(y_1^*) = \frac{(e_1+e_2)(1-\alpha_d)}{h_1+(e_1+e_2)(1-\alpha_d)} \tag{2.12}$$

To complete the proof we need to show convexity of (2.3) for which the Hessian is given as:

$$H = \begin{bmatrix} (e_1+h_1)f(y_1) & 0 \\ 0 & h_2+e_2-\alpha_d(e_1+e_2)f(y_2) \end{bmatrix}$$

which is positive definite as long as $h_2+e_2 > \alpha_d(e_1+e_2)$. Our initial Assumption 6 includes $e_2 > \alpha_d(e_1+e_2)$, satisfying this inequality. For a general proof which does not require this assumption please refer to Appendix A.

2.4. Optimal Policies in Single Period Model with Positive Initial Inventory and no Setup Costs

For the problem with zero initial inventories we have single unique optimum inventory level : (y_1^*, y_2^*) as characterized in Proposition 2.1. However this targeting may not be a possible action when there are positive initial inventories y_1^* or y_2^*.

The problem is similar to minimizing (2.2) except the constraints on y_1 and y_2 :

$$\min_{y_1\geq x_1, y_2\geq x_1} g(x_1,x_2,y_1,y_2) = E_d\left[\sum_{i=1}^{2}[h_i(y_i-d)^+] + e[(d-y_1)^+, (d-y_2)^+]\right] \tag{2.13}$$

With new initial inventory constraints the set of possible target inventory levels has been reduced from $(y_1, y_2) \in \Re^2$ to $(y_1, y_2) \in \{(y_1, y_2) | y_1 \geq x_1, y_1 \geq x_2\}$.

For investigating the optimal inventory level of this restricted solution set, it is helpful to determine possible actions. From initial inventory levels (x_1, x_2), possible target inventory levels (y_1, y_2) can be reached with four different actions :

1. **Stay at the current inventory position** $(y_1 = x_1, y_2 = x_2)$.
   This action is optimal when expected holding costs of ordering any component is more than the expected expediting costs of covering demand.
2. **Order only component 1** $(y_1 > x_1, y_2 = x_2)$**.** When x_2 is *sufficiently large* then ordering only component 1 is a reasonable action.
3. **Order only component 2** $(y_1 = x_1, y_2 > x_2)$**.** When x_1 is *sufficiently large* then ordering only component 2 is a reasonable action.
4. **Order both components** $(y_1 > x_1, y_2 > x_2)$**.** This action is reasonable when demand is *sufficiently large* with respect to x_1 and x_2. This action is optimal for zero initial inventory case $(x_1 = 0, x_2 = 0)$, unless marginal expediting costs are more than marginal holding costs.

Consider action 2, where only component 1 is ordered. Target inventory level of component 2 is fixed at the initial inventory level $(y_2 = x_2)$. The expected cost function is $g(x_1, x_2, y_1, y_2 = x_2)$, which is also a convex function and has a minimum. Let $y_1^*(x_2)$ be expected cost minimizer function given $x_1 = 0$ and $y_2 = x_2$.
$y_1^*(x_2) = \text{argmin}_{y_1}\{g(0, x_2, y_1, y_2 = x_2)\}$.
Depending on the value of x_1 the optimal inventory level of component 1 for action 2 is the maximum of x_1 and $y_1^*(x_2)$.

Similarly, the optimal target inventory level of component 2 in action 3 $(y_1 = x_1)$ is the related expected cost minimizer, which is the maximum of x_2 and $y_2^*(x_1) = \text{argmin}_{y_2}\{g(x_1, 0, y_1 = x_1, y_2)\}$. For graphical interpretation of $y_1^*(x_2)$ and $y_2^*(x_1)$ functions please see Figures 2.1 and 2.2 . Please note that $y_1^*(x_2)$ and $y_2^*(x_1)$ functions are

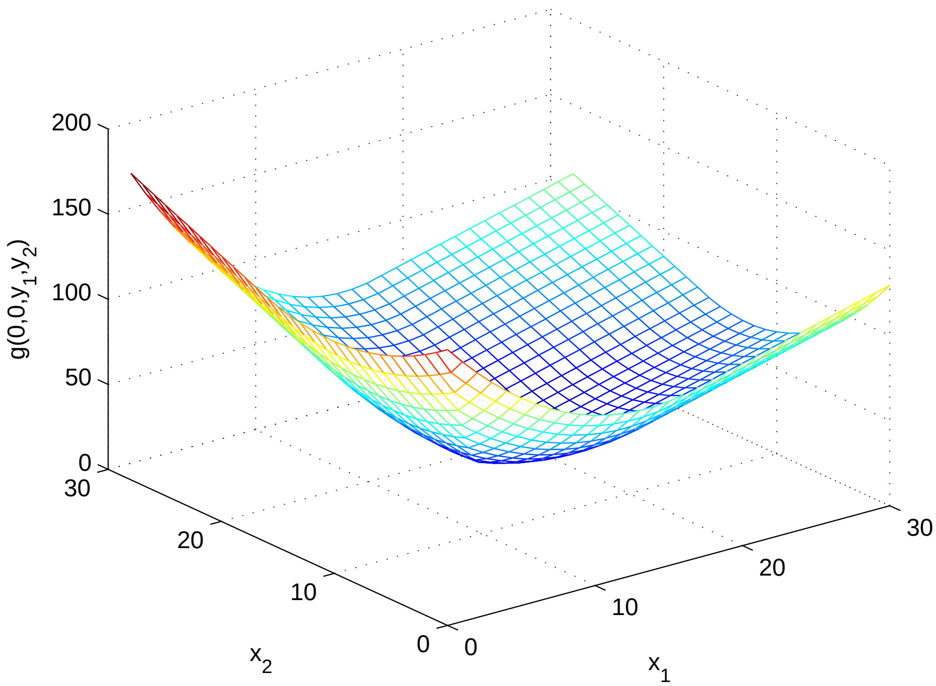


Figure 2.1. $g(0, 0, y_1, y_2)$ function

one dimensional functions which are shown in top view of Figure 2.2(b).

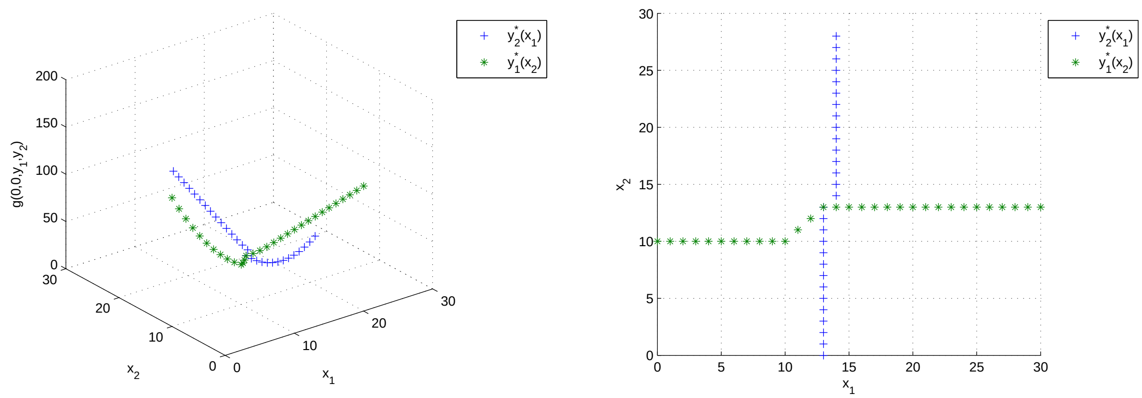
The optimal ordering decisions are characterized in Proposition 2.2. Let $y_1^*(x_1, x_2)$ and $y_2^*(x_1, x_2)$ be the optimal inventory levels satisfying initial inventory constraints.

Proposition 2.2 *Given the initial inventory levels* (x_1, x_2) *optimal target inventory levels* $(y_1^*(x_1, x_2), y_2^*(x_1, x_2))$ *are :*

(i) If $x_1 \leq y_1^*, x_2 \leq y_2^*$

$$\begin{aligned} y_1^*(x_1, x_2) &= y_1^* \\ y_2^*(x_1, x_2) &= y_2^* \end{aligned}$$

where y_1^* *and* y_2^* *are given as in Proposition 2.1.*



(a) (b)

Figure 2.2. $y_1^*(x_2)$ and $y_2^*(x_1)$ functions: minimum values of $g(0,0,y_1,y_2)$ at fixed x_1 and x_2 (a), their projection on $y_1 - y_2$ axis (top view of (a)) yield $y_1^*(x_2)$ and $y_2^*(x_1)$ functions (b)

(ii) If $x_1 > y_1^*, x_2 \leq y_2^*$

$$\begin{aligned} y_1^*(x_1,x_2) &= x_1 \\ y_2^*(x_1,x_2) &= \max(y_2^*(x_1),x_2) \end{aligned}$$

(iii) If $x_1 \leq y_1^*, x_2 > y_2^*$

$$\begin{aligned} y_1^*(x_1,x_2) &= \max(y_1^*(x_2),x_1) \\ y_2^*(x_1,x_2) &= x_2 \end{aligned}$$

(iv) If $x_1 > y_1^*, x_2 > y_2^*$*, then at least one of the following equations is true:*

$$\begin{aligned} y_1^*(x_1,x_2) &= \max(y_1^*(x_2),x_1) \\ y_2^*(x_1,x_2) &= \max(y_2^*(x_1),x_2) \end{aligned}$$

There is no possibility of $y_1^*(x_1,x_2) > x_1$ *and* $y_2^*(x_1,x_2) > x_2$ *at the same time.*

Proof:

(i) For $x_1 \leq y_1^*, x_2 \leq y_2^*$, the minimum (y_1^*, y_2^*) is accessible and this is the unique minimizer of the function.

(ii) For $x_1 \geq y_1^*, x_2 \leq y_2^*$, assume there exists $z = (z_1, z_2)$ such that $z_1 > x_1, z_2 \geq x_2$ and $g(x_1, x_2, z_1, z_2) < g(x_1, x_2, x_1, y_2^*(x_1, x_2))$. By the convexity of g:

$$g(0,0,\lambda y_1^*+(1-\lambda)z_1,\lambda y_2^*+(1-\lambda)z_2) \leq \lambda g(0,0,y_1^*,y_2^*)+(1-\lambda)g(0,0,z_1,z_2) \quad (2.14)$$

By the definition of (y_1^*, y_2^*), $g(0,0,y_1^*,y_2^*) \leq g(0,0,\lambda y_1^*+(1-\lambda)z_1, \lambda y_2^*+(1-\lambda)z_2)$. It is possible to multiply this equation by λ for any $\lambda \in (0,1)$ and add to equation 2.14 :

$$\begin{aligned} g(0,0,\lambda y_1^* + (1-\lambda)z_1, \lambda y_2^* + (1-\lambda)z_2) + \lambda g(0,0,y_1^*,y_2^*) \\ \leq \lambda g(0,0,y_1^*,y_2^*) + (1-\lambda)g(0,0,z_1,z_2) + \lambda g(0,0,x_1,\lambda y_2^* + (1-\lambda)z_2) \end{aligned}$$

After algebraic manipulation and dividing by $(1-\lambda)$ equation simplifies to :

$$g(0,0,\lambda y_1^* + (1-\lambda)z_1, \lambda y_2^* + (1-\lambda)z_2) \leq g(0,0,z_1,z_2)$$

Let λ_0 be the value satisfying:

$$x_1 = \lambda_0 y_1^* + (1-\lambda_0)z_1$$

Note that $g(0,0,x_1,\lambda y_2^*+(1-\lambda)z_2) = g(x_1,x_2,x_1,\lambda y_2^*+(1-\lambda)z_2)$ and $g(0,0,z_1,z_2) = g(x_1,x_2,z_1,z_2)$ for the model without setup costs. Resulting equation is :

$$g(x_1,x_2,x_1,\lambda_0 y_2^* + (1-\lambda_0)z_2) \leq g(x_1,x_2,z_1,z_2)$$

By definition $g(x_1,x_2,x_1,y_2^*(x_1,x_2)) \leq g(x_1,x_2,x_1,\lambda y_2^*+(1-\lambda)z_2)$, for $\lambda \in (0,1)$, by transitivity $g(x_1,x_2,x_1,y_2^*(x_1,x_2)) \leq g(0,0,z_1,z_2) = g(x_1,x_2,z_1,z_2)$ which contradicts the initial assumption about z.

Hence $y_1^*(x_1, x_2)$ can only be x_1, and $y_2^*(x_1, x_2)$ the related function minimizer $max(y_2^*(x_1), x_2)$.

(iii) For $x_1 \leq y_1^*, x_2 \geq y_2^*$, similar to the previous case assume the existence of $z = (z_1, z_2)$ such that $z_1 \geq x_1, z_2 > x_2$ and $g(x_1, x_2, z_1, z_2) < g(x_1, x_2, y_1^*(x_1, x_2), x_2)$. Take the linear combination of y^* and z where $x_2 = \lambda y_2^* + (1 - \lambda)$. It can be shown that $g(x_1, x_2, y_1^* + (1 - \lambda)z_1, x_2) \leq g(x_1, x_2, z_1, z_2)$ using convexity. By definition $g(x_1, x_2, y_1^*(x_1, x_2), x_2) \leq g(x_1, x_2, y_1^* + (1 - \lambda)z_1, x_2)$ which disproves the existence of z. Hence $y_2^*(x_1, x_2)$ can only be x_2, and $y_1^*(x_1, x_2)$ the related function minimizer $max(y_1^*(x_2), x_1)$.

(iv) For $x_1 \geq y_1^*, x_2 \geq y_2^*$, similar to the previous cases assume the existence of $z = (z_1, z_2)$ such that $z_1 > x_1, z_2 > x_2$ and further assume $g(x_1, x_2, z_1, z_2) < g(x_1, x_2, y_1^*(x_1, x_2), y_2^*(x_1, x_2))$. In order to show a contradiction it necessary to take a linear combination of z and y^*. Depending on the relative positions of the y^*, x and z different linear combinations are selected. There are three different cases, by comparing $(z_2 - y_2^*)/(z_1 - y_1^*)$ with $(x_2 - y_2^*)/(x_1 - y_1^*)$:

- If $(z_2 - y_2^*)/(z_1 - y_1^*) > (x_2 - y_2^*)/(x_1 - y_1^*)$, then there is a linear combination of (z_1, z_2) and (y_1^*, y_2^*), where $x_1 = \lambda y_1^* + (1 - \lambda)z_1$.
  $g(x_1, x_2, x_1, y_1^* + (1 - \lambda)z_1) \leq g(x_1, x_2, z_1, z_2)$ by the convexity. The function $g(x_1, x_2, x_1, y_2^*(x_1, x_2)) \leq g(x_1, x_2, x_1, y_2^* + (1 - \lambda)z_2)$ by definition, which leads to a contradiction.
- If $(z_2 - y_2^*)/(z_1 - y_1^*) < (x_2 - y_2^*)/(x_1 - y_1^*)$, then there is a linear combination of (z_1, z_2) and (y_1^*, y_2^*), where $x_2 = \lambda y_2^* + (1 - \lambda)z_2$.
  $g(x_1, x_2, y_1^* + (1 - \lambda)z_1, x_2) \leq g(x_1, x_2, z_1, z_2)$ by the convexity. The function $g(x_1, x_2, x_1, y_2^*(x_1, x_2)) \leq g(x_1, x_2, y_1^* + (1 - \lambda)z_1, x_2)$ by definition, which leads to a contradiction.
- If $(z_2 - y_2^*)/(z_1 - y_1^*) = (x_2 - y_2^*)/(x_1 - y_1^*)$, then there is a linear combination of (z_1, z_2) and (y_1^*, y_2^*), where $x = \lambda y^* + (1 - \lambda)z$.
  $g(x_1, x_2, x_1, x_2) \leq g(x_1, x_2, z_1, z_2)$ by the convexity. By the definition:
  $g(x_1, x_2, y_1^*(x_1, x_2), y_2^*(x_1, x_2)) \leq g(x_1, x_2, x_1, x_2)$, which leads to a contradiction.

For all possible different positions of z, its existence is contradicted. Hence

$y_1^*(x_1,x_2) > x_1$ and $y_2^*(x_1,x_2) > x_2$ at the same time is not possible. Either $y_1^*(x_1,x_2) = x_1$ or $y_2^*(x_1,x_2) = x_2$ or both.

2.4.1. Optimal Policy when Ordering only One Component Type

In the previous section we have defined $y_1^*(x_2)$ and $y_2^*(x_1)$ functions giving the optimal target inventory level of a component depending on the other component, where ordering only one component is possible. We now characterize these functions. Assume the component that can be ordered has zero initial inventory.

Proposition 2.3 *If it is not possible to order component i in the optimal solution ($y_i^*(x_i,x_j) = x_i$), then the optimal order policy y_j^* for the other component j is, given x_j=0 :*

$$\begin{aligned} y_j^*(x_i) &= y_j', \qquad && \text{for } x_i \leq y_j' \\ y_j^*(x_i) &= x_i, && \text{for } y_j' < x_i < y_j^o \\ y_j^*(x_i) &= y_j^o, && \text{for } x_i \geq y_j^o \end{aligned}$$

where y_j^o and y_j' are defined as :

$$\begin{aligned} y_j^o &= F^{-1}\left(\frac{e_j}{h_j+e_j}\right) \\ y_j' &= F^{-1}\left(\frac{e_j-\alpha_d(e_i+e_j)}{h_j+e_j-\alpha_d(e_i+e_j)}\right), \ \text{for} \ \left(\frac{e_j-\alpha_d(e_i+e_j)}{h_j+e_j-\alpha_d(e_i+e_j)}\right) > 0 \\ y_j' &= 0, \ \text{for} \ \left(\frac{e_j-\alpha_d(e_i+e_j)}{h_j+e_j-\alpha_d(e_i+e_j)}\right) \leq 0 \end{aligned}$$

Proof: W.L.O.G. say it is not possible to order component 1. Then the only component 1 policy available is to stay at the same level $y_1 = x_1$. Depending on the inventory level y_2 of the component 2, the cost function is:

$$g(x_1,0,x_1,y_2) = \int_0^{y_2} [h_1(x_1-u)+h_2(y_2-u)]\, f(u)du + \int_{y_2}^{x_1} [h_1(x_1-u)+e_2(u-y_2)]\, f(u)du + \int_{x_1}^{\infty} [(e_1+e_2)(1-\alpha_d)(u-x_1)+e_2(x_1-y_2)]\, f(u)du$$

for $y_2 \leq x_1$.

$$g(x_1,0,x_1,y_2) = \int_0^{x_1} [h_1(x_1-u)+h_2(y_2-u)]\, f(u)du + \int_{x_1}^{y_2} [e_1(u-x_1)+h_2(u-y_2)]\, f(u)du + \int_{x_1}^{\infty} [(e_1+e_2)(1-\alpha_d)(u-y_2)+e_2(y_2-x_1)]\, f(u)du$$

for $y_2 > x_1$. First derivatives with respect to y_2 are :

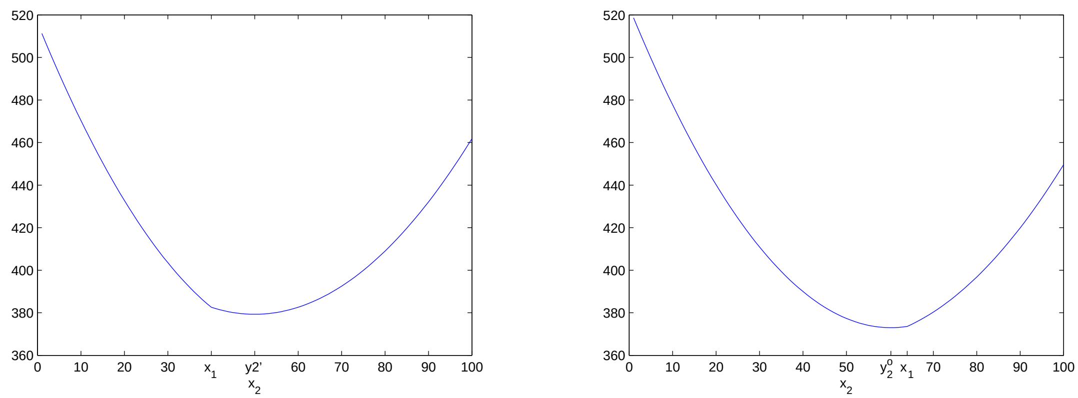
$$\begin{aligned} \frac{\partial g(x_1,0,x_1,y_2)}{\partial y_2} &= h_2\,[F(y_2)-F(0)] + e_2\,[F(x_1)-F(y_2)] - (e_2\,[1-F(y_2)] \\ &= (h_2+e_2)F(y_2) - e_2 \end{aligned}$$

for $y_2 \leq x_1$.

$$\begin{aligned} \frac{\partial g(x_1,x_2,x_1,y_2)}{\partial y_2} &= h_2\,[F(x_1)-F(0)] + h_2\,[F(y_2)-F(x_1)] - \\ &\quad (e_2-\alpha_d(e_1+e_2))\,[1-F(y_2)] \\ &= (h_2+e_2-\alpha_d(e_1+e_2))F(y_2) - (e_2-\alpha_d(e_1+e_2)) \end{aligned}$$

for $y_2 > x_1$.

There are three possibilities for the optimal inventory level y_2^* :
If $y_2^* < x_1$ then $y_2^* = y_2^o$, where $F(y_2^o) = \frac{e_2}{h_2+e_2}$.
If $y_2^* > x_1$ then $y_2^* = y_2'$, where $F(y_2') = \frac{e_2-\alpha_d(e_1+e_2)}{h_2+e_2-\alpha_d(e_1+e_2)}$.



(a) (b)

Figure 2.3. $g(x_1, 0, x_1, x_2)$ in two different cases

$\frac{e_2}{h_2+e_2} \geq \frac{e_2-\alpha_d(e_1+e_2)}{h_2+e_2-\alpha_d(e_1+e_2)}$ because subtracting same non-negative values both from numerator and denominator of a fraction, where numerator is less than the denominator decreases the value of the fraction. $F(y_2^o) \geq F(y_2')$, hence $y_2^o \geq y_2'$.

Depending on the value of x_1:

If $x_1 \leq y_2'$ then $y_2^* = y_2'$ (Figure 2.3a).

If $x_1 \geq y_2^o$ then $y_2^* = y_2^o$ (Figure 2.3b).

There is a gap between y_2' and y_2^o. If x_1 is in this gap, ($y_2' < x_1 < y_2^o$), then the third possibility occurs: $y_2^* = x_1$, to verify the optimality of this possibility, we have to take left and right sided derivatives.

$$\frac{\partial g(x_1, 0, x_1, x_1^-)}{\partial y_2} = (h_2 + e_2)F(x_1^-) - e_2 \tag{2.15}$$
$$x_1 < y_2^o \Longrightarrow (h_2 + e_2)F(x_1^-) - e_2 < 0$$

$$\frac{\partial g(x_1, 0, x_1, x_1^+)}{\partial y_2} = (h_2 + e_2 - \alpha_d(e_1 + e_2))F(x_1^-) - (e_2 - \alpha_d(e_1 + e_2)) - e_2 > 0 \tag{2.16}$$
$$y_2' < x_1 \Longrightarrow (h_2 + e_2)F(x_1^+) - e_2 > 0$$

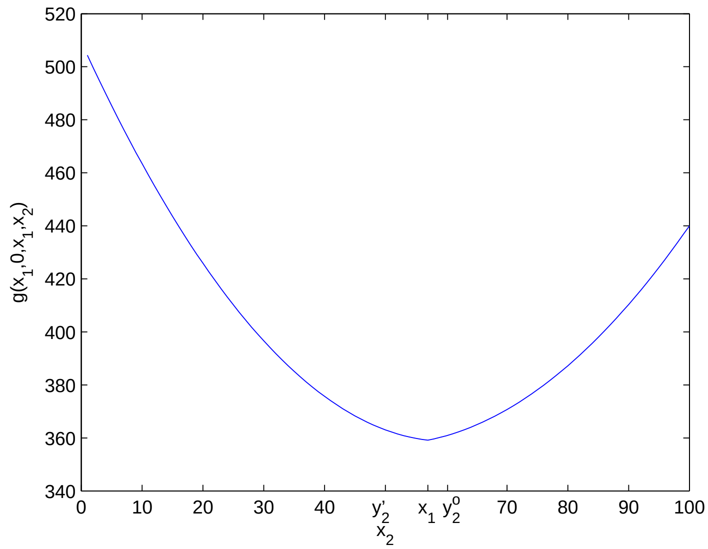


Figure 2.4. $g(x_1, 0, x_1, x_2)$ when $y_2' < x_1 < y_2^o$

Left and right side derivatives confirm the optimality of $y_2^* = x_1$, given $y_2' < x_1 < y_2^o$ (Figure 2.4).

Using Proposition 2.3 we may draw optimal "order only component 1" graph (Figure 2.5(a)) and "optimal order only component 2" graph (Figure 2.5(b)).

How Figure 2.5(a) and 2.5(b) come together depends on whether $y_1^* > y_2^*$, $y_2^* > y_1^*$ or $y_2^* = y_1^*$. W.L.O.G we will show the cases where $y_1^* \leq y_2^*$.

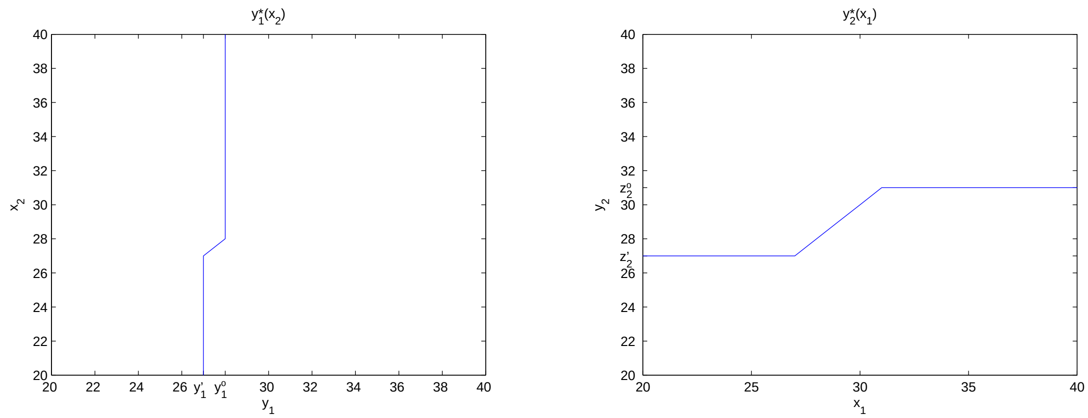
1. Case 1 : Optimal Stock Levels of components are different $y_1^* < y_2^*$

   Two graphs intersect at the optimal inventory policy point. (y_1^o, y_2')(Figure 2.6).
2. Case 2 : Optimal Stock Levels of components are equal $y_1^* = y_2^*$

   With $y_2' \leq y_1^o$, there is no guarantee that two graphs do connect at (y_1', y_2'). However they do connect at the y^o point, which is the inverse of the cumulative function F at the harmonic average of $F(y_2')$ and $F(y_1^o)$.

$$y^o = F^{-1}\left(\frac{(e_1 + e_2)(1 - \alpha_d)}{h_1 + h_2 + (e_1 + e_2)(1 - \alpha_d)}\right) \tag{2.17}$$

(y^o, y^o) is the optimal policy. Please note although there are many points where



(a) (b)

Figure 2.5. Optimal inventory policies when inventory level of other component is fixed

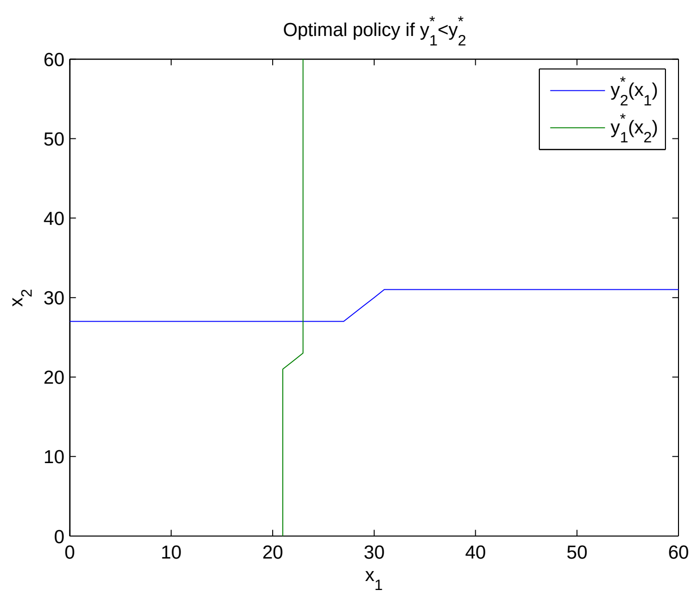


Figure 2.6. Optimal policy when stock levels are different

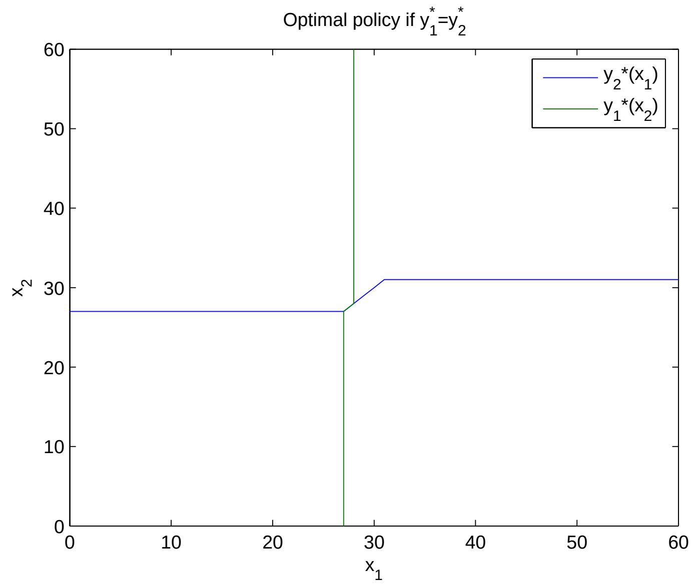


Figure 2.7. Optimal policy when stock levels are equal

$y_1^*(x_2)$ and $y_2^*(x_1)$ function intersects in Figure 2.7, (y^o, y^o) is a unique point, which gives minimum value of convex $g(0, 0, y^o, y^o)$ function.

For $y_1^* \geq y_2^*$, it is only necessary to replace x_1 and x_2 axes of the graphs.

With the help of order only one component graphs it is possible to fully characterize the optimal policy in Theorem 2.1.

Theorem 2.1 *For single period model with two components, non-negative initial inventory and no setup costs the optimal policy is generalized base stock policy* $S_1(x_2), S_2(x_1)$. *Functions* $S_1(x_2)$ *and* $S_2(x_1)$ *determine optimal target inventory levels* $y_1^*(x_1, x_2)$ *and* $y_2^*(x_1, x_2)$ *given as:*

$$y_1^*(x_1, x_2) = \max(x_1, S_1(x_2))$$
$$y_2^*(x_1, x_2) = \max(x_2, S_2(x_1)) \tag{2.18}$$

where

$$S_1(x_2) = \begin{cases} y_1^* & \text{for } x_2 \le y_2^* \\ y_1^*(x_2) & \text{for } x_2 > y_2^* \end{cases}, S_2(x_1) = \begin{cases} y_2^* & \text{for } x_1 \le y_1^* \\ y_2^*(x_1) & \text{for } x_1 > y_1^* \end{cases}$$

Proof: It is necessary to check that the functions cover all cases.

For $x_1 \le y_1^*, x_2 \le y_2^*$ the optimal policy is (y_1^*, y_2^*) by Proposition 2.1.

$$y_1^*(x_1, x_2) = \max(x_1, S_1(x_2)) = \max(x_1, y_1^*) = y_1^*$$
$$y_2^*(x_1, x_2) = \max(x_2, S_2(x_1)) = \max(x_2, y_2^*) = y_2^*$$

For $x_1 > y_1^*, x_2 \le y_2^*$ the optimal policy is $(x_1, y_2^*(x_1))$ by Proposition 2.3.

$$y_1^*(x_1, x_2) = \max(x_1, S_1(x_2)) = \max(x_1, y_1^*) = x_1$$
$$y_2^*(x_1, x_2) = \max(x_2, S_2(x_1)) = \max(x_2, y_2^*(x_1)) = y_2^*(x_1)$$

For $x_1 \le y_1^*, x_2 > y_2^*$ the optimal policy is $(y_1^*(x_2), x_2)$ by Proposition 2.3.

$$y_1^*(x_1, x_2) = \max(x_1, S_1(x_2)) = \max(x_1, y_1^*(x_2)) = y_1^*(x_2)$$
$$y_2^*(x_1, x_2) = \max(x_2, S_2(x_1)) = \max(x_2, y_2^*) = x_2$$

For $x_1 > y_1^*, x_2 > y_2^*$ there are further subcases :

If $x_1 > y_1^*(x_2), x_2 > y_2^*(x_1)$ the optimal policy is (x_1, x_2) by Proposition 2.3.

$$y_1^*(x_1, x_2) = \max(x_1, S_1(x_2)) = \max(x_1, y_1^*(x_2)) = x_1$$
$$y_2^*(x_1, x_2) = \max(x_2, S_2(x_1)) = \max(x_2, y_2^*(x_1)) = x_2$$

If $x_1 > y_1^*(x_2), x_2 \leq y_2^*(x_1)$ the optimal policy is $(x_1, y_2^*(x_1))$ by Proposition 2.3.

$$y_1^*(x_1, x_2) = \max(x_1, S_1(x_2)) = \max(x_1, y_1^*(x_2)) = x_1$$
$$y_2^*(x_1, x_2) = \max(x_2, S_2(x_1)) = \max(x_2, y_2^*(x_1)) = y_2^*(x_1)$$

If $x_1 \leq y_1^*(x_2), x_2 > y_2^*(x_1)$ the optimal policy is $(y_1^*(x_2), x_2)$ by Proposition 2.3.

$$y_1^*(x_1, x_2) = \max(x_1, S_1(x_2)) = \max(x_1, y_1^*(x_2)) = y_1^*(x_2)$$
$$y_2^*(x_1, x_2) = \max(x_2, S_2(x_1)) = \max(x_2, y_2^*(x_1)) = x_2$$

The convexity of cost function prevents the possibility $x_1 < y_1^*(x_2), x_2 < y_2^*(x_1)$ for the case $x_1 > y_1^*, x_2 > y_2^*$.

2.5. Optimal Policies with Setup Costs

In this section we introduce setup costs K_1 and K_2 of ordering components into the model. In addition to the assumptions of Section 2.1, it is assumed that $e(x_1, x_2) \geq \sum_{i=1}^{2} \alpha K_i \delta(x_i)$ for $x_1, x_2 \geq 0$; provided that the fixed costs associated with expediting are at least as those associated with performing regular production in the next period. The expected cost function is now given as :

$$\min_{y_1 \geq x_1, y_2 \geq x_1} g(x_1, x_2, y_1, y_2) = K_1\delta(y_1 - x_1)^+ + K_2\delta(y_2 - x_2)^+ + E_d\left[\sum_{i=1}^{2}[h_i(y_i - d)^+] + e[(d - y_1)^+, (d - y_2)^+]\right] \quad (2.19)$$

When setup costs are introduced, the cost function is not necessarily convex anymore. However, if the action is staying at the current inventory position (i.e., $y_1 = x_1$ and $y_2 = x_2$) then (2.19) reduces to Equation (2.13) of Section 2.4. Once again we have four different types of possible actions.

1. Stay at the current inventory position.

2. Order only component 1.
3. Order only component 2.
4. Order both components.

Assume it is not optimal to order component 1. Without setup costs, optimal inventory policy for the other component 2 would be $y_2^*(x_1)$ if $x_2 \leq y_2^*(x_1)$. For $x_2 > y_2^*(x_2)$ it would be also optimal not to order component 2 (Section 2.4.1). However with the introduction of the setup cost K_2, this base stock policy is not optimal for some $x_2 \leq y_2^*(x_1)$, which satisfies the following equation:

$$g(x_1, x_2, x_1, x_2) < K_2 + g(x_1, y_2^*(x_1), x_1, y_2^*(x_1)) \tag{2.20}$$

Proposition 2.4 *If it is impossible to order component i in the optimal solution, then the optimal inventory policy $y_j^*(x_i, x_j)$ for the other component j is:*

$$y_j^*(x_i, x_j) = \begin{cases} y_j^*(x_i) & \text{for } x_j < y_j^o(x_i) \\ x_j & \text{for } x_j \geq y_j^o(x_i) \end{cases}$$

where $y_j^o(x_i)$ is defined as the positive inventory level satisfying the following equation:

$$g(x_i, x_j, x_i, y_j^o(x_i)) = K_j + g(x_i, y_j^*(x_i), x_i, y_j^*(x_i)) \tag{2.21}$$

If there is no positive $y_j^o(x_i)$ satisfying equation (2.21), $y_j^o(x_i)$ is set to zero.

Proof: W.L.O.G. assume it is impossible to order component 1. For $x_2 < y_2^*(x_1)$ optimum cost of the type 3 action (order only component 2 policy) is $g(x_1, y_2^*(x_1), x_1, y_2^*(x_1)) + K_2$. If $g(x_1, x_2, x_1, x_2) < g(x_1, y_2^*(x_1), x_1, y_2^*(x_1)) + K_2$ this action is dominated by type 1 action (do not order any components policy). The convexity of the cost function guarantees a unique $y_2^o(x_1)$ value, where for $x_2 < y_2^o(x_1)$ type action 1 dominates type 3 action and for $x_2 \geq y_2^o(x_1)$ vice versa. If there exists no positive x_2 value which

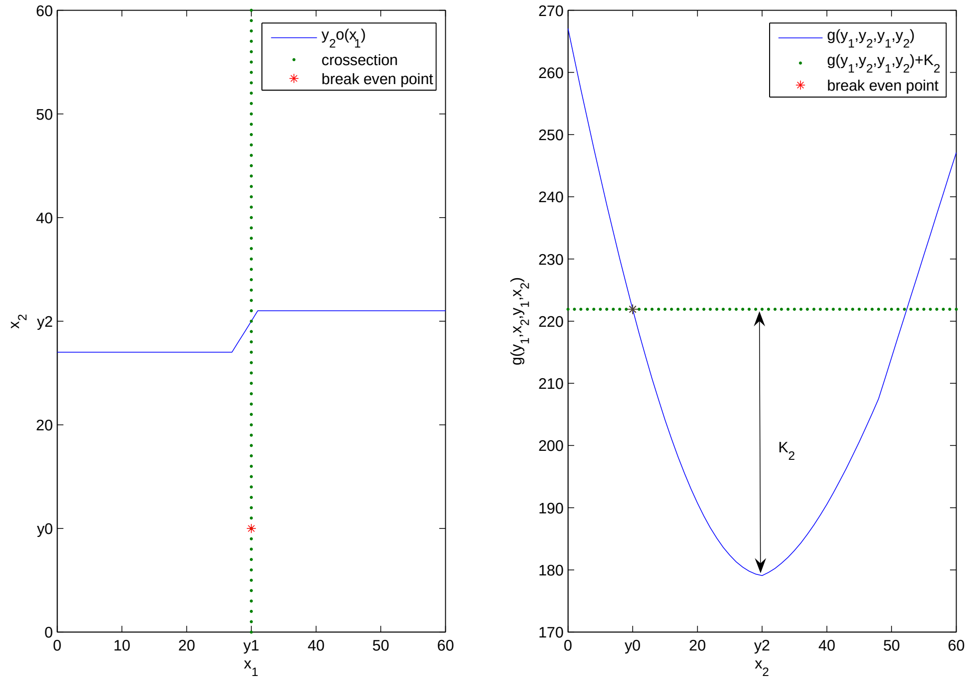


Figure 2.8. Break even point y_0 at crossection $x_1 = y_1$ (a) top view (b) side view

satisfies $g(x_1, x_2, x_1, x_2) \geq g(x_1, y_2^*(x_1), x_1, y_2^*(x_1)) + K_2$ then $y_2^o(x_1)$ is set 0, indicating best action for all levels of x_2 inventory is not to order component 2.

Figure 2.8 shows the *break-even point* $y_0 = y_2^o(x_1)$ at the crossection of $y_1 = x_1$ indicated by dots in both Figure 2.8 (a) and (b). $y_2 = y_2^*(x_1)$ is optimal target inventory level for component 2. As shown in the expected cost function graph at the right figure (Figure 2.8(b)), for inventory level y_0 ordering minimum cost of ordering new components and paying K_2 is equal to not ordering anything. The convexity of the cost function guarantees for any $x_2 < y_0$ it is economical to order component 2. Where for $x_2 > y_0$ not ordering policy is better.

Define the *break-even functions* with respect to $y_2^*(x_1)$, and $y_1^*(x_2)$:

$$y_2^o(x_1) = \{y_2 | y_2 < y_2^*(x_1), g(x_1, y_2, x_1, y_2) + K_2 = g(x_1, y_2^*(x_1), x_1, y_2^*(x_1))\}$$

$$y_1^o(x_2) = \{y_1 | y_1 < y_1^*(x_2), g(y_1, x_2, y_1, x_2) + K_1 = g(y_2^*(x_1), x_2, y_2^*(x_1), x_2)\}$$

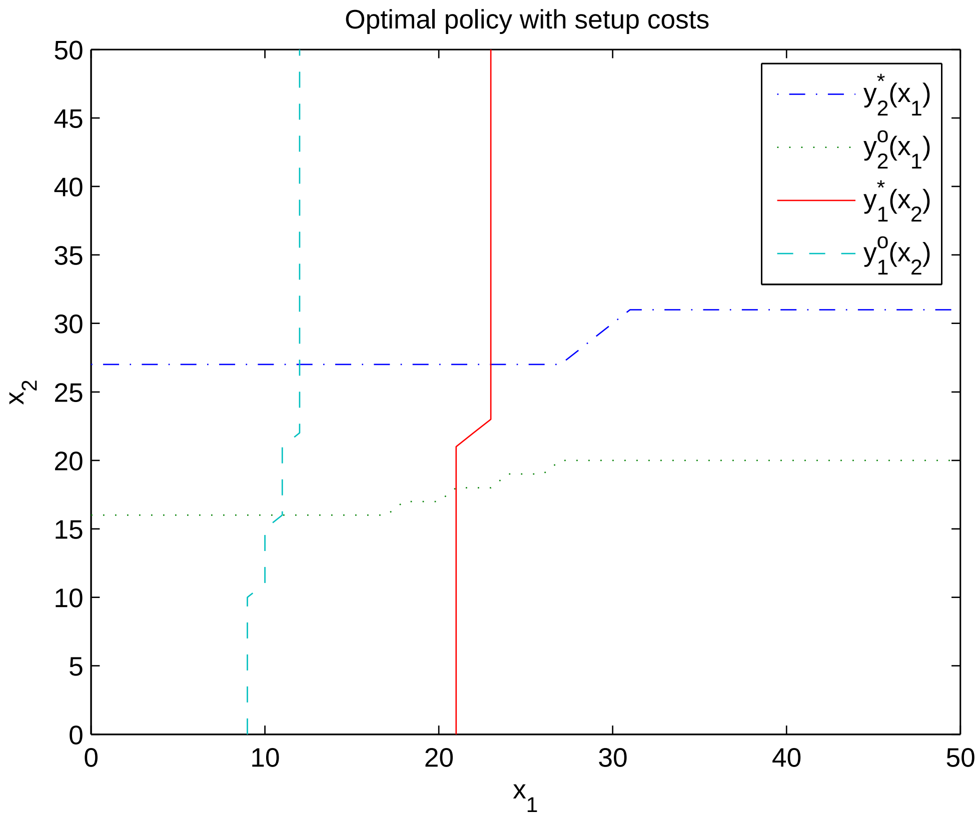


Figure 2.9. Break even lines for setup costs

With these functions also drawn in x_1,x_2 axis (Figure 2.9), we may identify the areas where action 2 (order only component 1) and 3 (order only component 2), dominates [1] action 4 (do not order anything) and vice versa. The action of optimal policy may not be dominated by any other available action. Using domination rules we may identify optimal actions for different areas depending on the initial inventory levels.

It is easier to detect optimal actions in these specific areas:

(i) For (x_1, x_2), where $x_1 < y_1^o(x_2), x_2 < y_2^o(x_1)$, it is better to order component 1 rather than not to order anything: $x_1 < y_1^o(x_2) \Rightarrow g(x_1, x_2, y_1^*(x_2), x_2) < g(x_1, x_2, x_1, x_2)$. We go to the better target inventory $(y_1^*(x_2), x_2)$. For this policy to be optimal, best action at $(y_1^*(x_2), x_2)$ must be of type 4 (do not order anything). However at this point it is better to order component 2 rather than not to order component 2: $x_2 < y_2^o(y_1^*(x_2)) \Rightarrow g(y_1^*(x_2), x_2, y_1^*(x_2), y_2^*(y_1^*(x_2))) <$

[1] An action dominates another action if it's expected cost is less than the other.

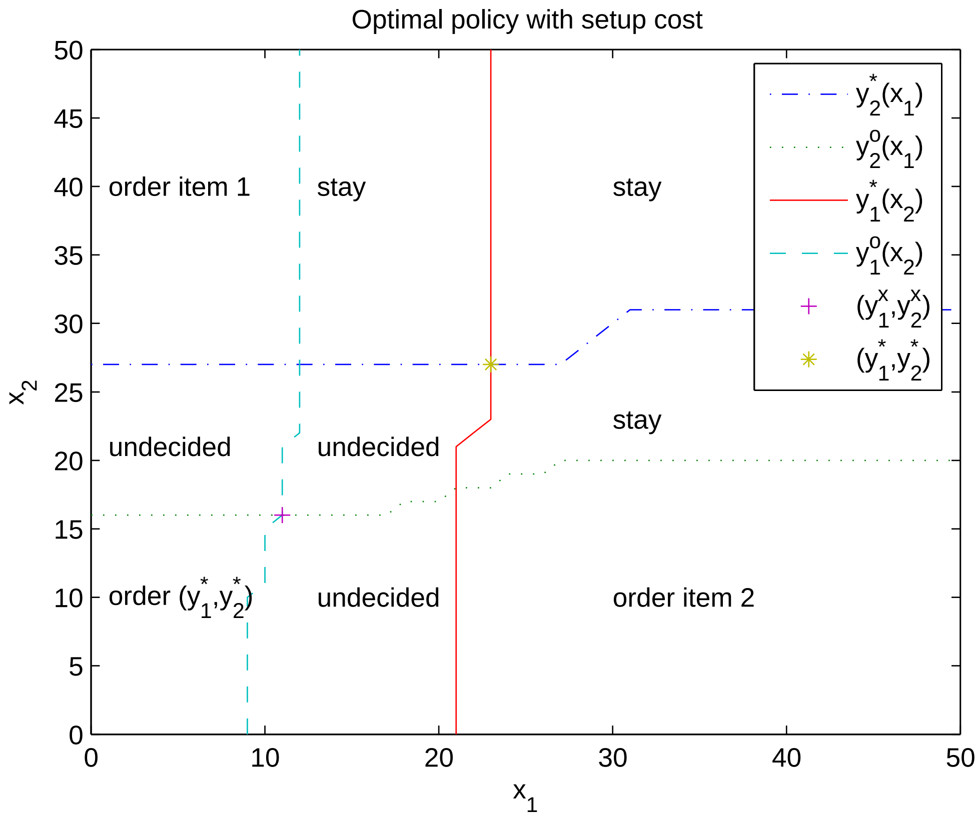


Figure 2.10. Optimal policies with undetermined areas

$g(y_1^*(x_2), x_2, y_1^*(x_2), x_2)$. Ordering both components to level (y_1^*, y_2^*) is the best policy.

(ii) For $x_1 < y_1^o(x_2), x_2 > y_2^*(x_1)$, ordering component 2 is not economical, because x_2 is above the related optimum level $y_2^*(x_1)$, however ordering component 1 is economical because x_1 is below the related break even level $y_1^o(x_2)$. Best policy is ordering component 1.

(iii) For $x_1 > y_1^*(x_2), x_2 < y_2^o(x_1)$, ordering component 1 is not economical, because x_1 is above the related optimum level $y_1^*(x_2)$, however ordering component 2 is economical because x_2 is below the related break even level $y_2^o(x_1)$. Best policy is ordering component 2.

(iv) For $x_1 > y_1^o(x_2), x_2 > y_2^*(x_1)$, ordering component 2 is not economical, because x_2 is above the related optimum level $y_2^*(x_1)$. Ordering component 1 is also not economical because x_1 is above the related break even level $y_1^o(x_1)$. Best policy is to stay at the level (x_1, x_2).

For $x_1 > y_1^*(x_2), x_2 > y_2^o(x_1)$, ordering component 1 is not economical, because

x_1 is above the related optimum level $y_1^*(x_2)$. Ordering component 2 is also not economical because x_2 is below the related break even level $y_2^o(x_1)$. Best policy is to stay at the level (x_1, x_2).

There are only three areas where the optimal policy is not defined yet (Figure 2.10).

These areas require further analysis and we take different points at each area (Figure 2.11) :

1. point A (a_1, a_2)
   $a_2 < y_2^o(a_1)$ suggests ordering component 2 is better than staying at a_2 level. Ordering only component 2 policy leads to point $A^*(a_1, y_2^*(a_1))$. However $a_1 < y_1^o(y_2^*(a_1))$ which shows it is better to order both components rather than ordering only component 2. If a_1 would be more than $y_1^o(y_2^*(a_1))$ which is the break even level when only component 2 is ordered, ordering only component 2 would be better. The area of point A is shared between order only component 2 and order both components policies.
2. point B (b_1, b_2)
   $b_1 < y_1^o(b_2)$ suggests ordering component 1 is better than staying at b_1 level. Ordering only component 2 policy leads to point $B^*(y_1^*(b_2), b_2)$. $b_2 > y_2^o(y_1^*(b_2))$ which shows it is better to order only component 2 rather than ordering both. If b_2 would be less than $y_2^o(y_1^*(b_2))$ which is the break even level when only component 2 is ordered, ordering only component 2 would be better. The area of point B is shared between order only component 1 and order both components policies.
3. point C (c_1, c_2)
   $c_1 > y_1^o(c_2)$ suggests ordering only component 1 is worse than staying at c_1 level. $c_2 > y_2^o(c_1)$ suggests ordering only component 2 is worse than staying at c_2 level. However it may be still better ordering both components. Because the facts that
   $g(c_1, c_2, c_1, c_2,) + K_1 > g(c_1, c_2, y_1^o(c_2), c_2,)$ and
   $g(c_1, c_2, c_1, c_2,) + K_2 > g(c_1, c_2, c_1, y_2^o(c_1))$ do not guarantee
   $g(c_1, c_2, c_1, c_2,) + K_1 + K_2 > g(c_1, c_2, y_1^*, y_2^*)$. By drawing the contour line for points in this area which satisfy $g(x_1, x_2, x_1, x_2) + K_1 + K_2 = g(x_1, x_2, y_1^*, y_2^*)$ we

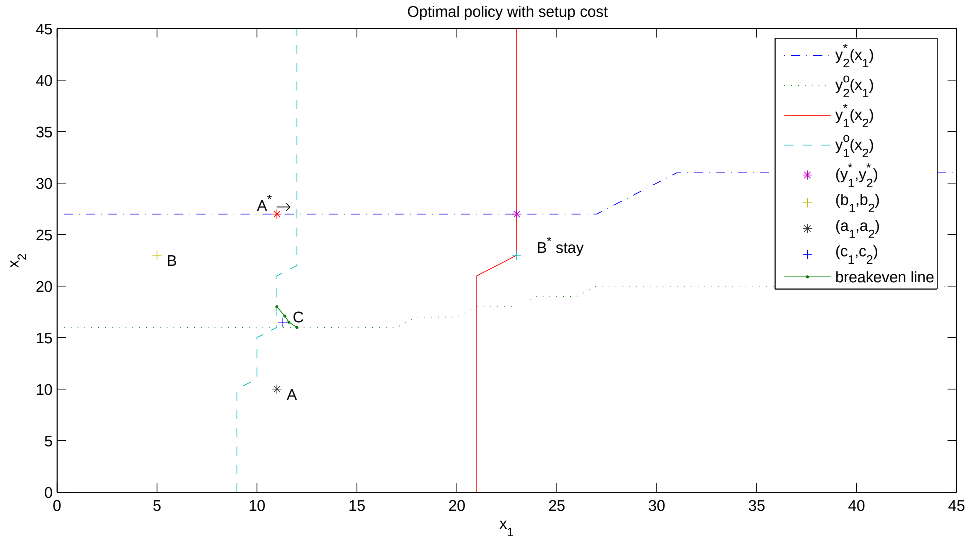


Figure 2.11. Optimal policies in sample areas

find that best policy for point C is ordering both components, by the convexity of the cost function. The area of point C is shared between order both components and stay at the level policies.

Let $y^{o*} = (y_1^{o*}, y_2^{o*})$ the point where $y_1^o(x_1)$ intersects $y_1^*(x_2)$, let $y^{*o} = (y_1^{*o}, y_2^{*o})$ the point where $y_1^*(x_2)$ intersects $y_2^o(x_1)$.

$$y_1^{o*} = y_1^o(y_2^{o*}),\ \ y_2^{o*} = y_2^*(y_1^{o*}) \tag{2.22}$$
$$y_1^{*o} = y_1^*(y_2^{*o}),\ \ y_2^{*o} = y_2^o(y_2^{*o})$$

These points are decisive for determining optimal policies in undetermined areas. Order both components area is extended from $x_2 < y_2^o(x_1)$ to $x_2 < y_2^{*o}$, for $x_1 < y_1^o(y_2^{*o})$, from $x_1 < y_1^o(x_2)$ to $x_1 < y_1^{o*}$, for $x_2 < y_2^o(y_1^{o*})$ and to a curve between $(y_1^o(y_2^{*o}), y_2^{*o})$, $(y_1^{o*}, y_2^o(y_1^{o*}))$ points (Figure 2.12).

Having determined the boundaries of the policy areas we may show them on the policy map (Figure 2.13).

The optimal single period policy can be characterized in following theorem.

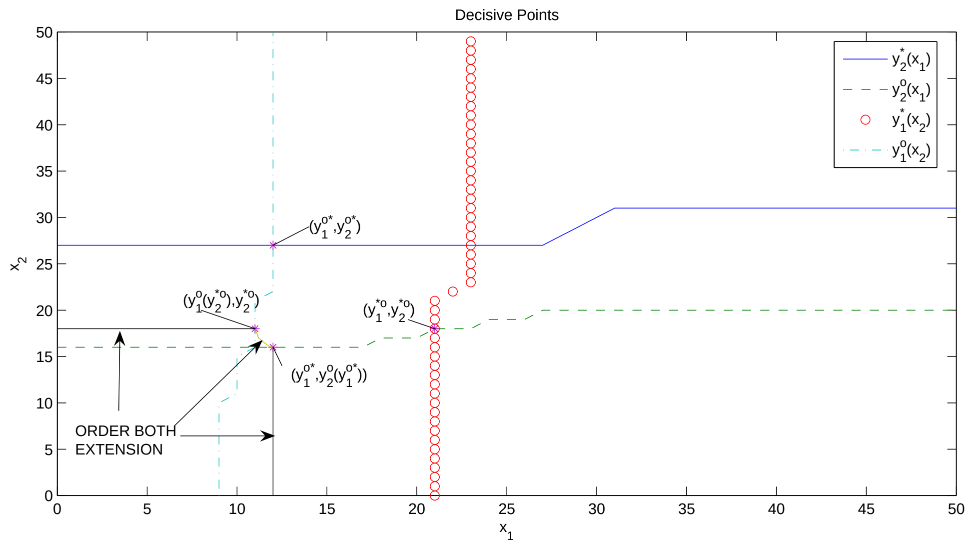


Figure 2.12. Order both policy extension

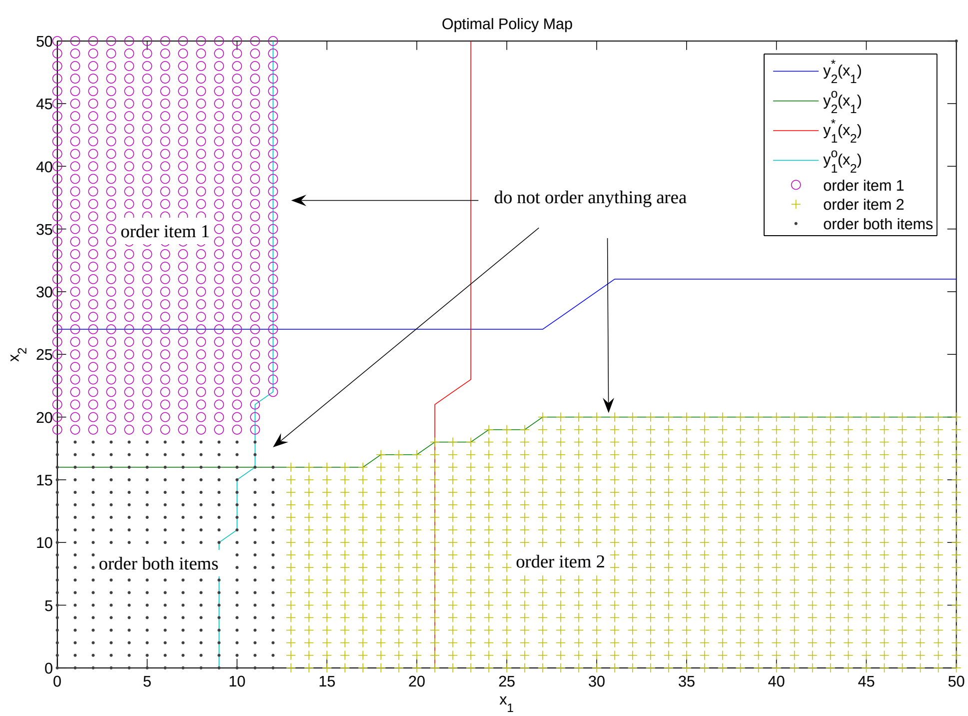


Figure 2.13. Optimal policy map

Theorem 2.2 *The optimal policy has the form $(s_1(x_2), S_1(x_2), s_2(x_1), S_2(x_1))$ which is defined as :*

For all inventory level tuples (x_1, x_2):

Order $(S_1(x_2) - x_1)$ units of component 1, if $x_1 < s_1(x_2)$.

Order $(S_2(x_1) - x_2)$ units of component 2, if $x_2 < s_2(x_1)$.

$S_i(x_j)$ is the function of the optimal target inventory level of component i, given the current inventory level x_i of the component i is less than the control function $s_i(x_j)$ for $i = 1, 2,\ j \neq i$. If current inventory level of component i is greater than or equal to $s_i(x_j)$, then the best policy is not to order component i.

The $s_1(x_2), s_2(x_1), S_1(x_2), S_2(x_1)$ functions can be defined as :

$$s_1(x_2) = \begin{cases} y_1^{o*}, & for\ x_2 \leq y_2^{o*} \\ y_1^c(x_2), & for\ y_2^{o*} \leq x_2 \leq y_2^{*o} \\ y_1^o(x_2), & for\ x_2 > y_2^{*o} \end{cases}$$

where $y_1^c(x_2)$ is a function defined as :

$y_1^c(x_2) = \{x_1 | g(x_1, x_2, x_1, x_2) + K_1 + K_2 = g(y_1^*, y_2^*, y_1^*, y_2^*, x_1 < y_1^*)\}$

for $y_2^{o*} \leq x_2 \leq y_2^{*o}$.

$$s_2(x_1) = \begin{cases} y_2^{*o}, & for\ x_1 \leq y_1^{*o} \\ y_2^c(x_1), & for\ y_1^{*o} \leq x_1 \leq y_1^{o*} \\ y_2^o(x_1), & for\ x_1 > y_1^{o*} \end{cases}$$

where $y_2^c(x_1)$ is a function defined as :

$y_2^c(x_1) = \{x_2 | g(x_1, x_2, x_1, x_2) + K_1 + K_2 = g(y_1^*, y_2^*, y_1^*, y_2^*), x_2 < y_2^*\}$

for $y_1^{*o} \leq x_1 \leq y_1^{o*}$.

$$S_1(x_2) = \begin{cases} y_1^*, & for\ x_2 \leq y_2^{*o} \\ y_1^*(x_2), & for\ x_2 > y_2^{*o} \end{cases}$$

$$S_2(x_1) = \begin{cases} y_2^*, & for\ x_1 \leq y_1^{o*} \\ y_2^*(x_1), & for\ x_1 > y_1^{o*} \end{cases}$$

Proof: For any $(x_1, x_2) < (y_1^*, y_2^*)$ the upper bound cost of the optimal policy is $g(0, 0, y_1^*, y_2^*) + K_1 + K_2$. Because for any inventory position it is always possible to order up to (y_1^*, y_2^*). If $g(x_1, x_2, x_1, x_2) \leq g(x_1, x_2, y_1^*, y_2^*) + K_1 + K_2$ then action 4 (staying) is the best alternative. By the convexity of the cost function, the inventory positions where action 4 is optimal, are in an area which is limited by the boundary functions $y_2^c(x_1), y_1^c(x_2)$, where $y_2^c(x_1)$ is the inverse function of $y_1^c(x_2)$. For any $(x_1, x_2) < (y_1^*, y_2^*)$, where $x_2 < y_2^c(x_1)$ or $x_1 < y_1^c(x_2)$, a better alternative is ordering only one component if $g(x_1, x_2, y_1^*(x_2), x_2) + K_1 < g(x_1, x_2, y_1^*, y_2^*) + K_1 + K_2$ or $g(x_1, x_2, y^*x_1, y_2^*(x_1)) + K_2 < g(x_1, x_2, y_1^*, y_2^*) + K_1 + K_2$. According to the previous proofs the optimality of actions 2 and 3 (order only one component actions) can be checked using $y_1^o(x_2)$ and $y_1^o(x_2)$ functions. For any (x_1, x_2) where $x_1 > y_1^*$ or $x_2 > y_2^*$, ordering both components is not possible anymore. The decision is made between action 4 (staying) and actions 2,3, using $y_1^o(x_2)$ and $y_1^o(x_2)$ functions.

By combining $y_1^o(x_2), y_1^o(x_2), y_2^c(x_1)$ and $y_1^c(x_2)$ functions it is possible to characterize $s_1(x_2), s_2(x_1)$ order decision functions and their respective target inventory functions $S_1(x_2), S_2(x_1)$.

Please note than in our example Figure 2.14, $S_1(x_2)$ function is not continuous, as $s_1(x_2)$,$s_2(x_1)$,$S_2(x_1)$ functions are continuous.
However this does not affect the continuity of the cost function $g(0, x_2, S_1(x_2), x_2)$. Based on the optimal policies, expected cost is characterized as :

1. Order both components area

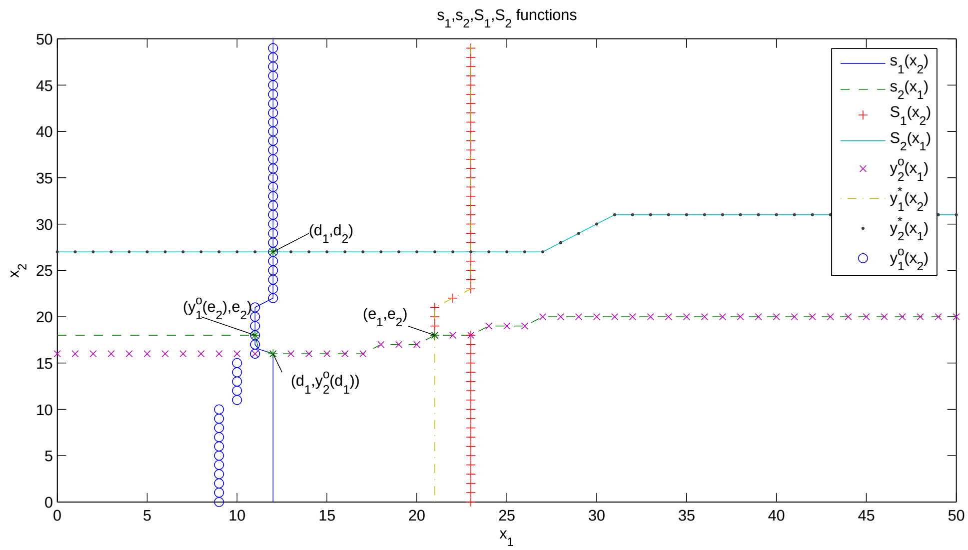


Figure 2.14. s_1, s_2, S_1, S_2 functions

Optimal policy cost is $g(0, 0, y_1^*, y_2^*) + K_1 + K_2$ for all points belonging to this area.

2. Order component 1 only area

   Optimal policy cost is $g(0, x_2, y_1^*(x_2), x_2) + K_1$ for all points (x_1, x_2) belonging to this area.

3. Order component 2 only area

   Optimal policy cost is $g(x_1, 0, x_1, y_2^*(x_1)) + K_2$ for all points (x_1, x_2) belonging to this area.

4. Do not order anything area

   Optimal policy cost is $g(x_1, x_2, x_1, x_2)$ for all points (x_1, x_2), belonging to this area.

The resulting cost function is not a convex function anymore. The differences between the optimal policy cost function and the convex cost function where the policy is not to order anything can be seen in Figures 2.15 and 2.16.

The optimal policy cost function is not a convex function. It is a K^2-convex function. The properties of K^2-convex function will be introduced in the next chapter.

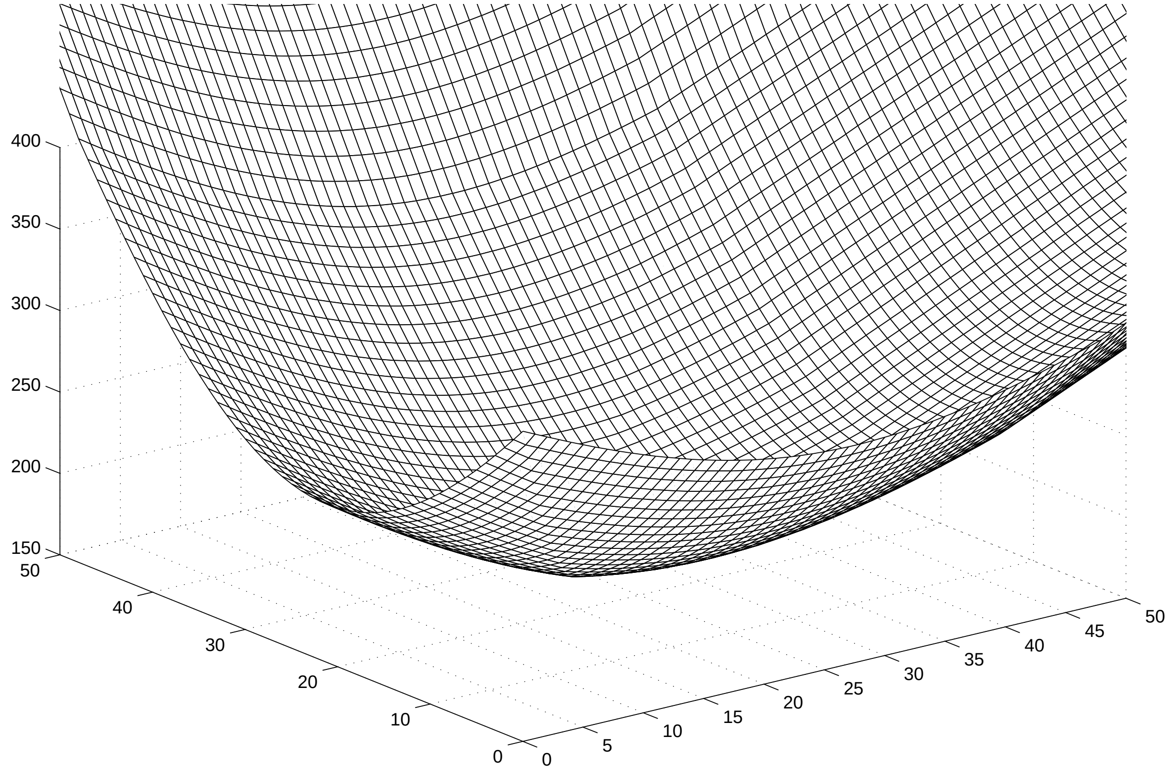


Figure 2.15. Do not order policy cost function

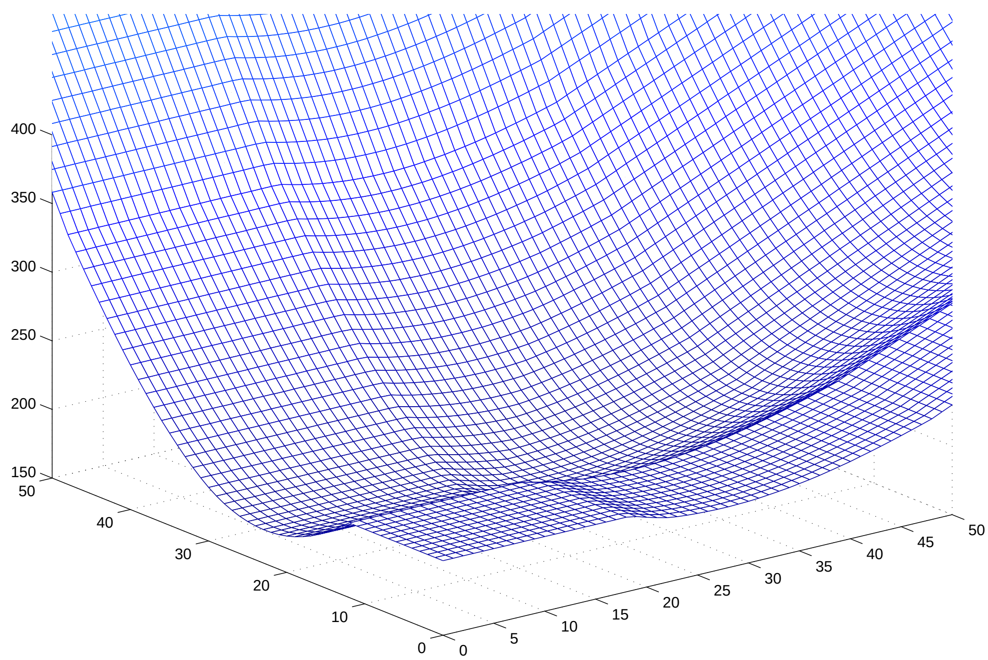


Figure 2.16. Optimal policy cost function

3. MULTIPLE PERIOD MODEL

3.1. Model Definition

In this chapter, we consider a model in which transactions continue for N periods. We introduce time period index n and time discount factor α into the notation (table 3.1).

Table 3.1. Notation in Multi Period model

y_{in}	inventory position chosen for regular production of unit i at period n
$\tilde{y}_{in}$	inventory position chosen after expediting of unit i at period n
x_{in}	inventory of unit i at the start of period n
$\tilde{x}_{in}$	inventory of unit i before expediting in period n
d_n	random demand for the end-product during period n
$h^r_{in}(x_i)$	adjusted holding cost function for excess inventory amount of x_i at the end of period n
α	time discount for costs occurring in the next period.

In the multi-period model procured components which are not used at end of the period may be used in the following periods, except last period. The definition of the adjusted holding cost changes to:

$$h^r_{in}(x_i) = h_i(x_i) + (1-\alpha)c_i x_i \qquad \text{for all } n = 1, 2, \cdots, N-1$$
$$h^r_{in}(x_i) = h_i(x_i) + c_i x_i \qquad \text{only for } n = N$$

We do not lose the excess inventory at the end of period, but we charge a extra holding cost for the excess inventory, that could be produced at the next period.

From now on h_i will be used instead of h^r_{in} except the last period. The single period solution in Chapter 2 is valid for the last period. We refer to the cost of the last period as $g_1(x_{11}, x_{21}, y_{11}, y_{21})$. We count the period from the last one to the first one. Last period has index number 1, first period has index number N.

We may replace $\tilde{x}_{in}$ and $\tilde{y}_{in}$ using inventory balancing equations :

$$\tilde{x}_{in} = y_{in} - d_n$$
$$\tilde{y}_{in} = \tilde{x}_{in}^{+}$$

These equations imply that it is not possible to expedite more than needed at the current period. This is probably optimal when there is no setup cost. With setup costs it is not necessarily optimal. For a single component model where it is allowed to order more than needed $\tilde{y}_{in} \geq \tilde{x}_{in}^{+}$, to meet also the demand for the next period $d_{(n-1)}$ by expedition for avoiding regular production setup cost in the next period, you may refer to Huggins and Olsen[2]. Inventory balancing equations for period transitions are :

$$x_{i(n-1)} = y_{in} - d_n, \qquad \text{for } n = 2 \cdots N$$

3.2. Expected Cost Function for the Multi Period Model

The cost function, $g_n(x_{1n}, x_{2n}, y_{1n}, y_{2n})$ for n periods is :

$$E_{d_n}\left[\sum_{i=1}^{2}[K_i\delta(y_{in} - x_{in}) + h_i(y_{in} - d_n)^{+}] + e((d_n - y_{1n})^{+}, (d_n - y_{2n})^{+}) + \alpha g_{n-1}((y_{1n} - d_n)^{+}, (y_{2n} - d_n)^{+}, y_{1(n-1)}, y_{2(n-1)})\right], \qquad \text{for } n \geq 2$$

Last period $n = 1$ is defined as :

$$g_1(x_{11}, x_{21}, y_{11}, y_{21}) = E_{d_1}\left[\sum_{i=1}^{2}[K_i\delta(y_{i1} - x_{i1}) + h_{i1}^{r}(y_{i1} - d_1)] + e((d_1 - y_{11})^{+}, (d_1 - y_{21})^{+})\right]$$

Similarly the optimal policy cost function is :

$$\begin{aligned} g_n^*(x_{1n}, x_{2n}) \;&= \min_{y_{1n}^* \geq x_{1n}, y_{2n}^* \geq x_{2n}} E_{d_n} \Bigg[\sum_{i=1}^{2} [K_i \delta(y_{in}^* - x_{in}) + h_i(y_{in}^* - d_n)^+] + \\ & e((d_n - y_{1n}^*)^+, (d_n - y_{2n}^*)^+) + \\ & \alpha g_{n-1}^*((y_{1n}^* - d_n)^+, (y_{2n}^* - d_n)^+) \Bigg], \qquad \text{for } n \geq 2 \end{aligned}$$

and

$$\begin{aligned} g_1^*(x_{11}, x_{21}) = \; & \min_{y_{11}^* \geq x_{11}, y_{21}^* \geq x_{21}} \; E_{d_1} \Bigg[\sum_{i=1}^{2} [K_i \delta(y_{i1}^* - x_{i1}) + h_{i1}^r (y_{i1}^* - d_1)^+] + \\ & e((d_1 - y_{11}^*)^+, (d_1 - y_{21}^*)^+) \Bigg] \end{aligned}$$

Please note if setup costs are zero, the costs related only with the n^{th} period are convex holding and expediting costs. In Section 3.3 we cover the case with zero setup costs and use the convexity of non setup costs at the n^{th} period.

3.3. Optimal Policies in Multi Period Model with Positive Initial Inventory and no Setup Costs

When setup costs are zero, the cost function, $g_n(x_{1n}, x_{2n}, y_{1n}, y_{2n})$ for n periods is :

$$\begin{aligned} & E_{d_n} \Bigg[\sum_{i=1}^{2} [h_i(y_{in} - d_n)^+] + e((d_n - y_{1n})^+, (d_n - y_{2n})^+) + \\ & \alpha g_{n-1}((y_{1n} - d_n)^+, (y_{2n} - d_n)^+, y_{1(n-1)}, y_{2(n-1)}) \Bigg], \qquad \text{for } n \geq 2 \end{aligned}$$

Last period $n = 1$ is defined as :

$$g_1(x_{11}, x_{21}, y_{11}, y_{21}) = E_{d_1} \left[\sum_{i=1}^{2} [h_{i1}^r (y_{i1} - d_1)] + e((d_1 - y_{11})^+, (d_1 - y_{21})^+) \right]$$

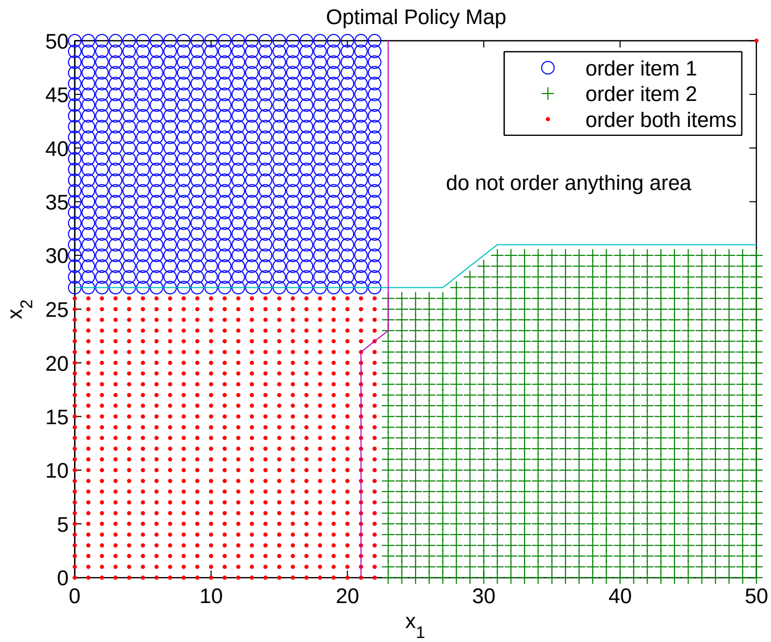


Figure 3.1. Optimal policy map for no setup model

Similarly the optimal policy cost function is :

$$\begin{aligned} g_n^*(x_{1n}, x_{2n}) \;&= \min_{y_{1n}^* \geq x_{1n}, y_{2n}^* \geq x_{2n}} E_{d_n} \left[\sum_{i=1}^{2} [h_i(y_{in}^* - d_n)^+] + \right. \\ &\quad e((d_n - y_{1n}^*)^+, (d_n - y_{2n}^*)^+) + \\ &\quad \left. \alpha g_{n-1}^*((y_{1n}^* - d_n)^+, (y_{2n}^* - d_n)^+) \right], \text{ for } n \geq 2 \end{aligned}$$

and

$$g_1^*(x_{11}, x_{21}) = \min_{y_{11}^* \geq x_{11}, y_{21}^* \geq x_{21}} E_{d_1} \left[\sum_{i=1}^{2} [h_{i1}^r (y_{i1}^* - d_1)^+] + e((d_1 - y_{11}^*)^+, (d_1 - y_{21}^*)^+) \right] \quad (3.1)$$

The optimal policy for the model with no setup costs in n periods is a modified base stock policy, given that the optimal cost function for $n-1$ periods is convex. This can be proven by an induction based on convexity. The initial step starts with the last period of the model (with index number n=1), which is a single period model analyzed in Section 2.4. The optimal policy map is shown in Figure 3.1.

Proposition 3.1 *Let $n = 1$ be the last period of the multi-period model. Let $y^*_{11}(x_{21})$ and $y^*_{21}(x_{11})$ be functions giving minimum of the cost function at given initial inventory of x_{21} and x_{11} respectively.*

$$y^*_{11}(x_{21}) = argmin_{y_{11}}\{g_1(0, 0, y_{11}, x_{21})\}$$
$$y^*_{21}(x_{11}) = argmin_{y_{21}}\{g_1(0, 0, x_{11}, y_{21})\}$$

Then $g_1(0, 0, y^*_{11}(x_{21}), x_{21})$ and $g_1(0, 0, x_{11}, y^*_{21}(x_{11}))$ are convex functions.

Proof: Assume $g_1(0, 0, y^*_{11}(x_{21}), x_{21})$ is not convex. Then there exists at least three points $(a_1, a_2), (b_1, b_2), (c_1, c_2)$, where $y^*_{11}(a_2) = a_1$, $y^*_{11}(b_2) = b_1$, $y^*_{11}(c_2) = c_1$, $b_2 = \lambda a_2 + (1 - \lambda)c_2$ for $\lambda \in (0, 1)$ and :

$$g_1(0, 0, b_1, b_2) > \lambda g_1(0, 0, a_1, a_2) + (1 - \lambda)g_1(0, 0, c_1, c_2)$$

However $\lambda g_1(0, 0, a_1, a_2) + (1 - \lambda)g_1(0, 0, c_1, c_2) \geq g_1(0, 0, \lambda a_1 + (1 - \lambda)c_1, b_2)$ by the convexity of $g_1(.)$ and $g_1(0, 0, \lambda a_1 + (1 - \lambda)c_1, b_2) \geq g_1(0, 0, b_1, b_2)$ by the definition of b_1 as $b_1 = y^*_{11}(b_2) = argmin_{y_{11}}\{g_1(0, 0, y_{11}, b_2)\}$, which disproves the existence of at least three points violating the convexity of $g_1(0, 0, y^*_{11}(x_{21}), x_{21})$ functions. Hence $g_1(0, 0, y^*_{11}(x_{21}), x_{21})$ is convex.
The convexity of $g_1(0, 0, x_{11}, y^*_{21}(x_{11}))$ is proven in a similar way by contradiction.

Proposition 3.2 *The single period optimal policy cost function $g^*_1(x_{11}, x_{21})$ is convex.*

Proof: We have to show that the single period optimal policy cost function is convex. The domain of the function consists of four areas. If they are taken as separate domains:

1. Order both components area

   The value of the optimal policy cost function in the "order both components" domain is constant. So it is convex.

2. Order component 1 only area

   In x_2 direction the value of the optimal policy cost function is constant. In x_1 direction the function takes the values of $g_1(0, x_{21}, y_{11}^*(x_{21}), x_{21})$, which is equal to $g_1(0, 0, y_{11}^*(x_{21}), x_{21})$, for all points (x_1, x_2) belonging to this area. The convexity of $g_1(0, 0, y_{11}^*(x_{21}), x_{21})$ is proven in Proposition 3.1. The optimal policy cost function is convex, if only area 2 is taken as the domain.

3. Order component 2 only area

   Similar to order component 1 area, the optimal cost function is convex.

4. Do not order anything area

   The optimal policy cost function is equal to g_1 which is convex.

The function is convex in all areas, if only one area is declared as the function domain. For all points a, b, c in a single area, where $c = \lambda a + (1 - \lambda)b$ and $a > b$ we may claim convexity. To show the convexity of the optimal policy cost function, in the union of all areas, we need to show that convexity applies for any a, b, c in different areas. We will show here only an example where a is in area 1, b is in area 4, and c is in area 2 (figure 3.2).

To prove by contradiction, we assume that there exists at least one c, for $a \leq b$, where $c = \lambda_a a + (1 - \lambda_a)b$, for $\lambda_a \in (0, 1)$ such that:

$$g_1(c_1, c_2, y_{11}^*(c_2), y_{21}^*(c_1)) > \lambda_a g_1(a_1, a_2, y_{11}^*(a_2), y_{21}^*(a_1)) + (1 - \lambda_a) g_1(b_1, b_2, y_{11}^*(b_2), y_2^*(b_1)) \quad (3.2)$$

We know the optimal policy for all points a in area 1 is (y_{11}^*, y_{21}^*), for all points c in area 2 is ordering only component 2, and for all points b in area 4 is to stay at the

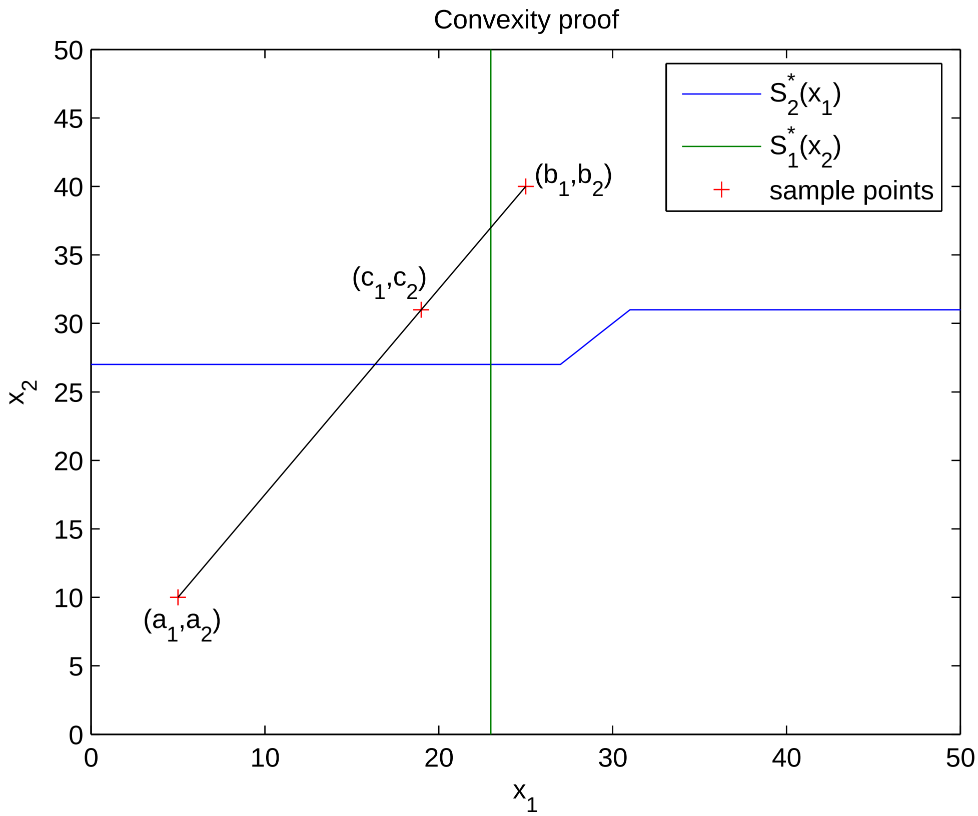


Figure 3.2. Sample proof of convexity

level.

$$
\begin{aligned}
g_1(c_1, c_2, y_{11}^*(c_2), y_{21}^*(c_1)) &= g_1(0, 0, y_{11}^*(c_2), c_2) \\
g_1(a_1, a_2, y_{11}^*(a_2), y_{21}^*(a_1)) &= g_1(0, 0, y_{11}^*, y_{21}^*) \\
g_1(b_1, b_2, y_{11}^*(b_2), y_{21}^*(b_1)) &= g_1(0, 0, b_1, b_2)
\end{aligned} \tag{3.3}
$$

Substituting equations 3.3 in 3.2:

$$
\begin{aligned}
g_1(0, 0, y_{11}^*(c_2), c_2) > \lambda_a g_1(0, 0, y_{11}^*, y_{21}^*) + \\
(1 - \lambda_a) g_1(0, 0, b_1, b_2)
\end{aligned} \tag{3.4}
$$

Since $g_1(0, 0, b_1, b_2)$ is greater than or equal to $g_1(0, 0, y_{11}^*(b_2), b_2)$, $(y_{11}^*(b_2), b_2)$, being the minimum point of the optimal cost function where $y_{21} = b_2$, then RHS of equation 3.4 is greater than:

$$\begin{aligned} &\lambda_a g_1(0,0,y_{11}^*,y_{21}^*) + (1-\lambda_a) g_1(0,0,b_1,b_2) \\ &\geq \lambda_a g_1(0,0,y_{11}^*,y_{21}^*) + (1-\lambda_a) g_1(0,0,y_{11}^*(b_2),b_2) \end{aligned} \tag{3.5}$$

Our initial assumption is that c is in area 2 and a linear combination of a in area 1 and b in area 4. c_2 can be defined as: $c_2 = \lambda_a a_2 1 + (1-\lambda_a) b_2$. Being in area 2, c_2 is also a linear combination of y_{21}^* and b_2, because (y_{11}^*, y_{21}^*) determines the boundary between area 1 and 2: $c_2 = \lambda_y y_{21}^* + (1-\lambda_y) b_2$. Furthermore $\lambda_y > \lambda_a$, because y_{21}^* is closer to c_2, than a_2 coordinate of any point a in area 1.

Please recall that $y_{11}^* = y_{11}^*(y_{21}^*)$. By the convexity of $g_1(0,0,y_{11}^*(y_{21}),y_{21})$:

$$\begin{aligned} &\lambda_y g_1(0,0,y_{11}^*,y_{21}^*) + (1-\lambda_y) g_1(0,0,y_{11}^*(b_2),b_2) \\ &\geq g_1(0,0,y_{11}^*(c_2),c_2) \end{aligned} \tag{3.6}$$

$\lambda_a < \lambda_y$ and $g_1(0,0,y_{11}^*(b_2),b_2) \geq g_1(0,0,y_{11}^*,y_{21}^*)$, leads to the following inequality :

$$\begin{aligned} &\lambda_y g_1(0,0,y_{11}^*,y_{21}^*) + (1-\lambda_y) g_1(0,0,y_{11}^*(b_2),b_2) \\ &< \lambda_a g_1(0,0,y_{11}^*,y_{21}^*) + (1-\lambda_a) g_1(0,0,y_{11}^*(b_2),b_2) \end{aligned} \tag{3.7}$$

$g_1(0,0,y_{11}^*(c_2),c_2)$ is less than or equal to LHS of equation 3.6, which is less than RHS of equation 3.7. However in equation 3.4, $g_1(0,0,y_{11}^*(c_2),c_2)$ is more than RHS of equation 3.7 which leads to a contradiction. This contradiction proves the convexity g_1^* function

Theorem 3.1 *If $g_{n-1}^*(x_{1n-1}, x_{2n-1})$ is a convex function then $g_n^*(x_{1n}, x_{2n})$ is also a convex function. Optimal policy for n periods is a generalized base stock policy, where functions $S_{1n}(x_{2n})$ and $S_{2n}(x_{1n})$ determine optimal target inventory levels $y_{1n}^*(x_{1n}, x_{2n})$*

and $y^*_{2n}(x_{1n}, x_{2n})$.

$$\begin{aligned} y^*_{1n}(x_{1n}, x_{2n}) &= \max(x_{1n}, S_{1n}(x_{2n})) \\ y^*_{2n}(x_{1n}, x_{2n}) &= \max(x_{2n}, S_{2n}(x_{1n})) \end{aligned} \tag{3.8}$$

Let (y^*_{1n}, y^*_{2n}) *be the point which minimizes the* $g_n(0, 0, y_{1n}, y_{2n})$ *function.*

$$(y^*_{1n}, y^*_{2n}) = argmin_{(y_{1n}, y_{2n})}\{g_n(0, 0, y_{1n}, y_{2n})\}$$

Let $y^*_{1n}(x_{2n}), y^*_{2n}(x_{1n})$ *be the functions minimizers at given* x_{2n} *and* x_{1n} *respectively:*

$$\begin{aligned} y^*_{1n}(x_{2n}) &= argmin_{(y_{1n})}\{g_n(0, 0, y_{1n}, x_{2n})\} \\ y^*_{2n}(x_{1n}) &= argmin_{(y_{2n})}\{g_n(0, 0, x_{1n}, y_{2n})\} \end{aligned}$$

Then $S_{1n}(x_{2n})$ *and* $S_{2n}(x_{1n})$ *are characterized as :*

$$S_{1n}(x_{2n}) = \begin{cases} y^*_{1n} & \text{for } x_{2n} \leq y^*_{2n} \\ y^*_{1n}(x_{2n}) & \text{for } x_{2n} > y^*_{2n} \end{cases}, S_{2n}(x_{1n}) = \begin{cases} = y^*_{2n} & \text{for } x_{1n} \leq y^*_{1n} \\ y^*_{2n}(x_{1n}) & \text{for } x_{1n} > y^*_{1n} \end{cases}$$

Proof: $g_n(0, 0, y_{1n}, y_{2n})$ includes expectation of holding, expediting costs at period n and $g^*_{n-1}(x_{1n-1}, x_{2n-1})$ given that the optimal policy is applied in $n-1$ periods. If $g^*_{n-1}(x_{1n-1}, x_{2n-1})$ is convex, since remaining components of nth period cost function are convex and their expectation over demand d then all cost functions are convex and their expectation over demand d is also convex. Hence $g_n(0, 0, y_{1n}, y_{2n})$ is convex. Using this convexity the results of Proposition 2.2 can be proven for (y^*_{1n}, y^*_{2n}) point and $y^*_{1n}(x_{2n}), y^*_{2n}(x_{1n})$ functions. Resulting optimal cost function $g^*_n(x_{1n}, x_{2n})$ is convex which can be proven similar to Proposition 3.2

3.4. $[K_1K_2]$-convexity

When there are positive setup costs K_1 and K_2, we claim the optimal policy is a modified (s, S) policy as in the single period model. The optimality of the modified (s, S) policy for the n^{th} period is related with the $[K_1K_2]$-convexity of the optimal cost function in the $n-1$ th period.

To show $[K_1K_2]-$convexity of the optimal policy cost function in the single period is an essential step for claiming $(s_1(x_2), S_1(x_2), s_2(x_1), S_2(x_1))$ policy is also optimal in multi periods.

A two dimensional function f(x_1,x_2) is $[K_1, K_2]$-convex, if the following rule holds:

For any $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2)$ in the domain of function f, where c equals $\lambda a + (1-\lambda)b$, for $\lambda \in [0, 1]$ and $a \leq b$:

$$f(c) \leq \lambda g(a) + (1-\lambda)[f(b) + \sum_{i=1}^{2} \delta(b_i - a_i)K_i] \tag{3.9}$$

where $\delta(a) = 1$, for $a > 0$ and $\delta(a) = 0$, for $a \leq 0$.

$[K_1, K_2]$-convex functions have following properties [7]:

1. If f is $[n_1, n_2]$-convex it is also $[m_1, m_2]$-convex, where $m_1 \geq n_1, m_2 \geq n_2$. Thus a convex function f is $[n_1, n_2]$-convex, for $n_1, n_2 \in R^+$.
2. If f is $[n_1, n_2]$-convex, g is $[m_1, m_2]$-convex, Function h which is defined as $h = af + bg$, for $a, b \in R^+$ is $[an_1 + bm_1, an_2 + bm_2]$-convex.
3. If f is $[K_1, K_2]$-convex, and R=(r_1, r_2) is random vector, such that $E[(x-r)] < \infty$ for all x then $E[(x-r)]$ is also $[K_1, K_2]$-convex.

Proposition 3.3 *The single period optimal policy cost function is $[K_1, K_2]$-convex.*

Proof: The domain of the function consists of four areas:

1. Order both components area

   The value of the optimal policy cost function in the "order both components" domain is constant. So it is convex.

2. Order component 1 only area

   In x_2 direction the value of the optimal policy cost function is constant. In x_1 direction the function takes the values of $g(0, x_2, y_1^*(x_2), x_2) + K_1$ for all points (x_1, x_2), belonging to this area. The convexity of the optimal policy cost function depends on g which is a convex function (Section 2.3.1). The optimal policy cost function is convex.

3. Order component 2 only area

   Similar to order component 1 area the optimal cost function is convex.

4. Do not order anything area

   The optimal policy cost function is equal to g which is convex.

The function is convex in all areas, if only one area is declared as the function domain. For all points a, b, c in a single area, where $c = \lambda a + (1-\lambda)b$ and $a > b$ we may claim $[K_1, K_2]$-convexity. To show the $[K_1, K_2]$-convexity of the optimal policy cost function, in the union of all areas, we need to show that 3.9 applies for any a, b, c in different areas. We will show here only an example where a is in area 1, b is in area 4, and c is in area 2 (figure 3.3).

To prove equation 3.9 by contradiction, we assume the reverse of this statement. There exist at least one c, for $a \leq b$, where $c = \lambda a + (1-\lambda)b$ such that:

$$g(c_1, c_2, y_1^*(c_2), y_2^*(c_1)) > \lambda g(a_1^*, a_2^*, y_1^*(a_2), y_2^*(a_1)) + \\ (1-\lambda)[g(b_1^*, b_2^*, y_1^*(b_2), y_2^*(b_1)) + \sum_{i=1}^{2} \delta(y_i(b_{2-i}) - y_i(a_{2-i}))K_i] \tag{3.10}$$

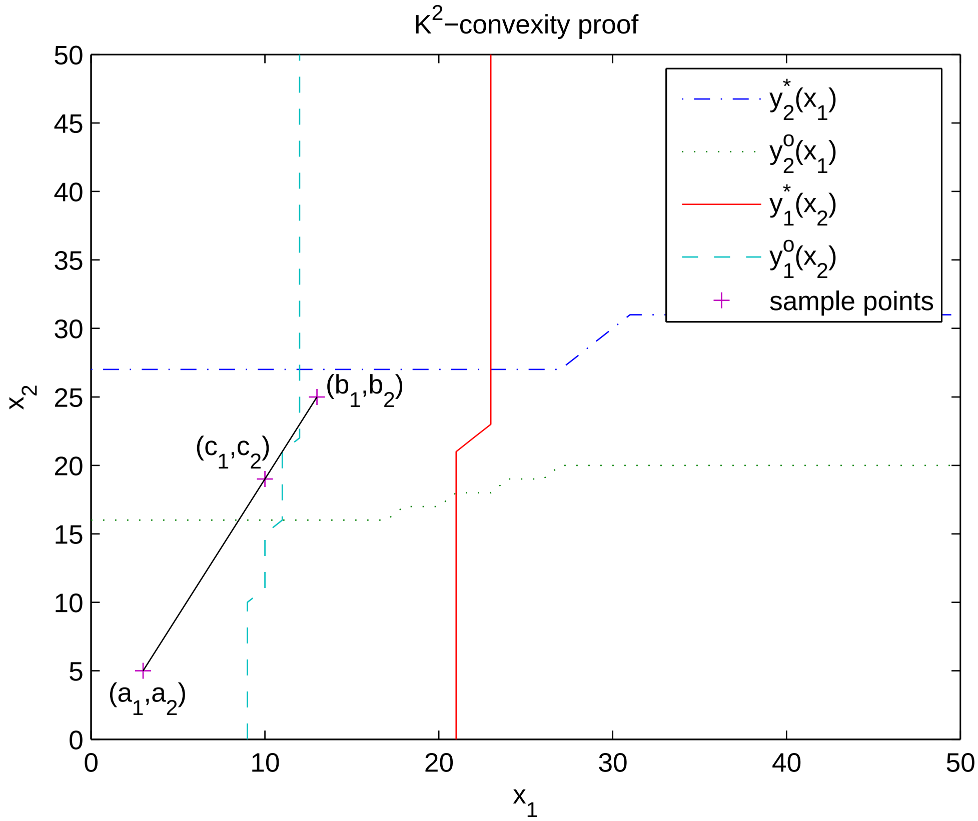


Figure 3.3. Sample proof of $[K_1K_2]$-convexity

We know the optimal policy for all points a in area 1 is (y_1^*, y_2^*), for all points c in area 2 is ordering only component 2, and for all points b in area 4 is to stay at the level.

$$\begin{aligned} g(c_1, c_2, y_1^*(c_2), y_2^*(c_1)) &= g(c_1, c_2, y_1^*(c_2), c_2) \\ g(a_1, a_2, y_1^*(a_2), y_2^*(a_1)) &= g(y_1^*, y_2^*, y_1^*, y_2^*) + K_1 + K_2 \\ g(b_1, b_2, y_1^*(b_2), y_2^*(b_1)) &= g(b_1, b_2, b_1, b_2) \end{aligned} \tag{3.11}$$

Substituting equations 3.11 in 3.10:

$$\begin{aligned} g(c_1, c_2, y_1^*(c_2), c_2) > \lambda[g(y_1^*, y_2^*, y_1^*, y_2^*) + K_1 + K_2] + \\ (1-\lambda)[g(b_1, b_2, b_1, b_2) + K_1 + K_2] \end{aligned} \tag{3.12}$$

$g(b_1, b_2, b_1, b_2)$ is more than or equal to $g(y_1^*, y_2^*, y_1^*, y_2^*)$, (y_1^*, y_2^*), being the minimum point of the optimal cost function. Taking the RHS of equation 3.12 :

$$\lambda[g(y_1^*, y_2^*, y_1^*, y_2^*) + K_1 + K_2] + (1 - \lambda)[g(b_1, b_2, b_1, b_2) + K_1 + K_2] \\ \geq g(y_1^*, y_2^*, y_1^*, y_2^*) + K_1 + K_2 \tag{3.13}$$

For all points c in area 2, $g(c_1, c_2, y_1^*(c_2), c_2)$ is strictly less than $g(y_1^*, y_2^*, y_1^*, y_2^*) + K_1 + K_2$, otherwise ordering both components would be better than ordering only component 1 and c would be in area 1. By the transitivity :

$$\lambda g(y_1^*, y_2^*, y_1^*, y_2^*) + (1 - \lambda)g(b_1, b_2, b_1, b_2) + K_1 + K_2 \\ > g(c_1, c_2, y_1^*(c_2), c_2) \tag{3.14}$$

Which contradicts with equation 3.12 and completes the proof of the $[K_1K_2]$-convexity in this case. Hence the cost function satisfies for $\lambda \in [0, 1]$:

$$g(c_1, c_2, y_1^*(c_2), y_2(c_1)^*) \leq \lambda g(a_1, a_2, y_1^*(a_2), y_2^*(a_1)) + \\ (1 - \lambda)[g(b_1, b_2, y_1^*(b_2), y_2^*(b_1)) + \sum_{i=1}^{2} \delta(y_i(b_{3-i}) - y_i(a_{3-i}))K_i] \tag{3.15}$$

3.5. Optimal Policy in the Two Period Model

In the previous section we have shown that the modified (s, S) policy is the optimal policy in single period model. We claim it is also optimal for two period model. The optimality of the modified (s, S) policy depends on the $[K_1K_2]$-convexity of the cost function. To be able to understand whether $[K_1K_2]$-convexity is preserved in two periods we have to show that if the last period cost function is $[K_1K_2]$-convex then the first period optimal cost function is also $[K_1K_2]$-convex. The optimal policy cost function in two period model is defined as :

$$g_2^*(x_{12}, x_{22}) \quad = \min_{y_{12}^* \geq x_{12}, y_{22}^* \geq x_{12}} E_{d_2} \quad \left[\sum_{i=1}^{2} [K_i \delta(y_{i2}^* - x_{i2}) + h_i(y_{i2}^* - d_2)^+] + \right.$$
$$e((d_2 - y_{12}^*)^+, (d_2 - y_{22}^*)^+) +$$
$$\left. \alpha g_1^*((y_{12}^* - d_2)^+, (y_{22}^* - d_2)^+) \right]$$

To find the optimal policy we start with the policy cost function where only action 4 (do not order policy) is used at the second period and optimal policy is used in the first period, which is defined as g_2^o cost function.

$$g_2^o(x_{12}, x_{22}, x_{12}, x_{22}) \quad = E_{d_2} \quad \left[\sum_{i=1}^{2} [h_i(x_{i2} - d_2)^+] + \right.$$
$$e((d_2 - x_{12})^+, (d_2 - x_{22})^+) +$$
$$\left. \alpha g_1^*((x_{11} - d_2)^+, (x_{21} - d_2)^+) \right]$$

The $[K_1 K_2]$-convexity of $g_1^*(x_{11}, x_{21})$ is proven in the previous section. The remaining part of the $g_2^o(x_{12}, x_{22}, x_{12}, x_{22})$ is convex. Sum of a convex and $[K_1 K_2]$-convex function is again a $[K_1 K_2]$-convex function, and this in preserved under expectation, hence $g_2^o(x_{12}, x_{22}, x_{12}, x_{22})$ is a $[K_1 K_2]$-convex function.

The $[K_1 K_2]$-convexity of two period model guarantees the optimality of the modified (s, S) policy. $s_{12}, s_{22}, S_{12}, S_{22}$ functions for the second period can be identified. Four areas of the optimal policy are identified using these functions:

1. Order both components area
   If $x_{12} < s_{12}(x_{22})$ and $x_{22} < s_{22}(x_{12})$ both components are ordered. $S_{12}(x_{22}) = y_{12}^*, S_{22}(x_{12}) = y_{22}^*$. The optimal cost in this area is constant: $g_2^*(x_{12}, x_{22}) = g_2^*(y_{12}^*, y_{22}^*) + K_1 + K_2$.
2. Order component 1 only area

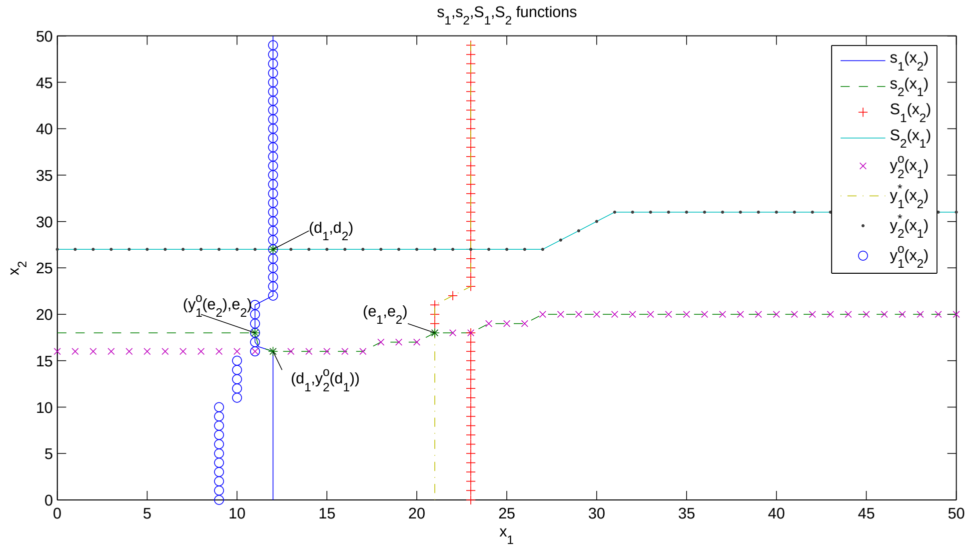


Figure 3.4. s_1, s_2, S_1, S_2 functions for $n = 2$

If $x_{12} < s_{12}(x_{22})$ and $x_{22} \geq s_{22}(x_{12})$ only component one is ordered $S_{12}(x_{22}) = y^*_{12}(x_{22})$. The optimal cost in this area is : $g^*_2(x_{12}, x_{22}) = g^*_2(y^*_{12}(x_{22}), x_{22}) + K_1$.

3. Order component 2 only area

   If $x_{12} \geq s_{12}(x_{22})$ and $x_{22} < s_{22}(x_{12})$ only component two is ordered $S_{22}(x_{12}) = y^*_{22}(x_{12})$. The optimal cost in this area is : $g^*_2(x_{12}, x_{22}) = g^*_2(x_{12}, y^*_{22}(x_{12})) + K_2$.

4. Do not order anything area

   If $x_{12} \geq s_{12}(x_{22})$ and $x_{22} \geq s_{22}(x_{12})$ no components are ordered. The optimal cost in this area is : $g^*_2(x_{12}, x_{22}) = g_2(x_{12}, x_{22}, x_{12}, x_{22})$.

For n periods ($n > 2$), it would be possible to claim modified (s, S) policy as the optimal policy if $[K_1K_2]$-convexity of the optimal two period policy g^*_2 could be proven. In this proof it is necessary to show $g^*_2(y^*_{12}(x_{22}), x_{22})$ is K_2-convex and $g^*_2(x_{12}, y^*_{22}(x_{12}))$ is K_1-convex. We fail to show any proof or disproof for this proposition. In numerical test results we are able to show some counter examples showing that modified (s, S) policy is not optimal for all infinite horizon multi-period models.

4. ALGORITHMS AND COMPUTATIONAL RESULTS

In this chapter, we report our computational experience with the most general version of the problem: two-component assemble-to-order problem with positive setup costs. We are interested in discounted infinite horizon costs and formulate the problem as a Markov Decision Process.

The computational work presented in this chapter has two objectives:

- Is the modified (s, S) policy as conjectured at the end of Chapter 3 indeed optimal? This cannot be answered in positive using a computational study but a counter-example will answer it in the negative.
- Even if the modified (s, S) policy is not optimal, it might be a reasonable heuristic for the overall problem. A computational study with a wide range of parameters can shed light into this.

4.1. Defining the Model as a Markov Decision Process

The infinite horizon model can be defined as a Markov Decision Process (MDP). Depending on the target inventory level decisions, the system moves among inventory levels randomly at each stage (period). Multi-dimensional inventory levels can be described as different states. Decision depends only on the last state visited, and each decision incurs rewards (costs). MDPs can be solved using policy iteration and value iteration methods. For more information about MDPs and related solution methods; policy iteration and value iteration, please refer to Puterman [22]. For the MDP formulation of our problem, notation in Table 4.1 is used.

The Markov state of the model is defined as the inventory levels of components before expediting $H = \{(x_1, x_2) | x_1, x_2 \in [-d_{max}, \infty)\}$, where d_{max} is the possible maximum demand at a period. Possible actions are target inventory levels before expediting: $A = \{(y_1, y_2) | y_1, y_2 \in [0, \infty)\}$. Possible actions of the state (x_1, x_2) is all inventory lev-

Table 4.1. MDP Notation

H	:	set of all states in MDP
A	:	set of all possible actions
$A(h)$	:	set of all possible actions available to state h
T	:	transition matrix
R	:	reward matrix

els equal or above $A(x_1, x_2) = \{(y_1, y_2)|y_1 \geq x_1^+, y_2 \geq x_2^+\}$. Transition matrix T has $|H| \times |A| \times |H|$ elements. The probabilities in T are defined as :

$$T((x_1, x_2), (y_1, y_2), (y_1 - d, y_2 - d)) = \begin{cases} \Pr(d) & \text{for } y_1 \geq x_1^+, y_2 \geq x_2^+ \\ 0 & \text{otherwise} \end{cases}$$

Reward matrix R has $|H| \times |A|$ elements. Setup, expediting and holding costs related with states are defined as *negative rewards:*

$$R((x_1, x_2), (y_1, y_2)) = \\ - \left(\sum_{i=1}^{2} K_i \delta(y_i - x_i) + E_d[\sum_{i=1}^{2} h_i (y_i - d)^+ + e((y_1 - d)^-, (y_2 - d)^-)] \right) \\ \text{for } y_1 \geq x_1^+, y_2 \geq x_2^+$$

For actions $(y_1, y_2) \notin A((x_1, x_2))$, $R((x_1, x_2), (y_1, y_2)) = -M$, where M is sufficiently large number prohibiting selection of impossible actions.

4.2. Policy Iteration Algorithms

4.2.1. Implementation Details

In our algorithms policy iteration method is used for finding optimal inventory levels. Kevin Murphy's MDP toolbox written for MATLAB is used [23]. Some of the functions are modified for decreasing time and space complexity of algorithms: For X

Table 4.2. Space Complexity of Tests

	Number of
Possible Demand Levels	9
Possible Inventory Levels	33
Possible Target Inv. Levels	25
States	1089
Actions	625
Reward Matrix Elements	680625
Transition Matrix Elements	741200625

possible levels of inventory levels and Y possible levels of target inventory levels for each component, X^2 states, and Y^2 actions are required. Then reward matrix R has $X^2 \times Y^2$ elements and transition matrix T has $X^4 \times Y^2$ elements. Total numbers in tests are shown in Table 4.2. In algorithms transition matrix T is not created. Required iteration values for each possible policy of each state is calculated dynamically using possible demand levels, decreasing space complexity to $O(X^2 \times Y^2)$ to the size of reward matrix R.

4.2.2. Algorithm Details

Three different algorithms are implemented for finding or approximating the optimal policy. The main difference between algorithms is in selection of possible actions for each state during the policy iteration state:

1. **Optimal Inventory Policy (OPT)**: This algorithm scans all available policies for each state, which makes $O(Y^2)$ different policies for each of X^2 states. This is the slowest algorithm but it is guaranteed to find the optimal solution.
2. **Independent Single Component Policy (IND)**: This algorithm assumes the joint expediting discount α_d is negligible and finds independent optimal policies for each single component. It solves single component model for each of two components scanning only $O(Y)$ possible policies for possible X states of each

component. It is the fastest of three algorithms but looses accuracy if the discount α_d is significant. It is used as a fast heuristic which provides an upper-bound to benchmark other algorithms.

3. **Best Modified** (s,S) **Policy (MOD)**: It assumes optimal policy for infinite horizon model is an $(s_1(x_2), s_2(x_1), S_1(x_2), S_2(x_1))$ policy, as in Theorem 2.2. If this assumption is true then instead of scanning best actions for each of all X^2 states at each iteration step, only states related with $s_1(x_2), s_2(x_1)$ functions can be checked, which are at least $4\times X$ states and have $O(\log_2(X)\times X)$ complexity in the worst case. MOD starts with initial $(s_{10}(x_2), S_{10}(x_2), s_{20}(x_1), S_{20}(x_1))$ vectors. Let s_1, S_1, s_2, S_2 be policy parameters calculated in IND. Initially vectors are set as:

$$s_{10}(x_2) = s_1, S_{10}(x_2) = S_1 \text{ for all } x_2$$
$$s_{20}(x_1) = s_2, S_{20}(x_1) = S_2 \text{ for all } x_1$$

Initial policy for each state (x_1, x_2) is to order $S_{10}(x_2) - x_1$ units of component 1, if $x_1 < s_{10}(x_2)$ and to order $S_{20}(x_1) - x_2$ units of component 2, if $x_2 < s_{20}(x_1)$. At each policy iteration step n, it is necessary to check the values of $s_{1n}(x_2), s_{2n}(x_1)$ functions. For each inventory level x_1, optimal action for states $(x_1, s_{2n-1}(x_2)-1)$ and $(x_1, s_{2n-1}(x_2))$ are checked. If optimal component 2 action for $(x_1, s_{2n-1}(x_2)-1)$ is ordering component 2; $y^*_{2n}(x_1, s_{2n-1}(x_1)-1) > s_{2n-1}(x_2)-1$ and optimal component 2 action for $(x_1, s_{2n-1}(x_2)-1)$ is not to order component 2; $y^*_{2n}(x_1, s_{2n-1}(x_2)) = s_{2n-1}(x_2))$ then, $s_{2n}(x_1) = s_{2n-1}(x_1)$. If both optimal policies are order component 2 then $s_{2n}(x_1) > s_{2n-1}(x_1)$, state $(x_1, s_{2n-1}(x_2)+1)$ is checked with state $(x_1, s_{2n-1}(x_2))$ and search for $s_{2n}(x_1)$ goes to the states with higher component 2 inventory levels. If both optimal policies are not to order then $s_{2n}(x_1) < s_{2n-1}(x_1)$, state $(x_1, s_{2n-1}(x_2) - 1)$ is checked with $(x_1, s_{2n-1}(x_2))$ and search $s_{2n}(x_1)$ goes to the states with lower component 2 inventory levels. Checks end when one of checked adjacent states has different optimal policy. Search for $s_{2n}(x_1)$ may be implemented using bisection for $O(\log_2 X)$ time complexity. In the tests we have implemented search sequentially because, number of inventory levels are small. $S_{2n}(x_1)$ is set to optimal order quantity found. $S_{2n}(x_1) = y^*_{2n}(x_1, s_{2n}(x_1))$. Having searched $s_{2n}(x_1)$ for each x_1 same process is applied for searching $s_{1n}(x_2)$ function value for each x_2. Number of states checked for determining optimal policy in an iteration has $O(\log_2(X) \times X)$ complexity.

4.3. Test Settings

We have taken 576 different test runs for identifying the behaviour of the optimal policy (OPT), the single item policy (IND) and the modified s, S heuristic (MOD).

4.3.1. Demand Distribution

The demand varies between 0 and 8 in discrete steps. In all cases average demand is taken as 4. Three different demand distributions are used:

1. Uniform demand : Each demand has equal probability $p = 0.1111$. This distribution has maximal variance.
2. Discrete Normal demand with high variance: Demand is normally distributed with mean 4 and standard deviation 2. This distribution has relatively high variance.
3. Discrete Normal demand with low variance: Demand is normally distributed with mean 4 and standard deviation 1.2. This distribution has relatively low variance.

Demand probabilities for each distribution are shown in Table 4.3

4.3.2. Test Parameters

The values taken by the parameters during the tests are shown in Table 4.4. There are a total of 192 combinations. For the sake of simplicity, expediting costs, holding costs and joint expediting discount factor is varied using adjusted costs. The full list of all test parameters including exact expediting costs, holding costs, and joint sale

Table 4.3. Probability for Demands

Demand	Uniform	Normal High	Normal Low
0	0.11111	0.02763	0.00129
1	0.11111	0.06628	0.01461
2	0.11111	0.12383	0.08291
3	0.11111	0.18017	0.23495
4	0.11111	0.20416	0.33249
5	0.11111	0.18017	0.23495
6	0.11111	0.12383	0.08291
7	0.11111	0.06628	0.01461
8	0.11111	0.02763	0.00129

discount factor values are given in Appendix B. Please note that the joint expediting discount rate α_d can have quite high values as $0.775, 0.6$. These high values have been taken for finding examples where $lim_{n\to\infty} g_n^*(x_{1n}, x_{2n})$ is not $[K_1K_2]-$convex, hence, modified (s, S) policy is not the optimal policy for $n > 2$.

4.4. Test Results

4.4.1. Policy Costs

In the results, the average optimal cost of all tests are the highest for the uniform distribution demand. Second highest is normal distribution with high variance. Normal distribution with low variance has the lowest optimal long run cost. Demand distributions with higher variances have higher uncertainty and higher expected costs. IND ignores joint expediting discounts and deviates from optimal solution about 5-8 percent on the average. The deviation is higher in demand distribution with higher variances. Average costs of MOD is very close to the optimal policy. In all tests with different demand distributions less than 0.5% average deviation is observed. The maximum deviation observed is between $97 - 120\%$ for IND. Maximum deviation also increases with the variance of demand. The maximum deviation of MOD from the

Table 4.4. Parameter Values Tested

Parameter	Values
K_1	50, 200
K_2	150
e_1	5, 15
e_2	10, 20, 30
h_1	1.5, 2.5
h_2	0.5, 1
α_d	0.775, 0.6, 0.425, 0.25

optimal solution shows no similar pattern. However maximum deviation observed for MOD is less than 19% which is quite encouraging for a heuristic (Table 4.5).

Table 4.5. Test Results: Average and Maximum Deviation from the Optimal

	Uniform Distribution			High Variance Normal			Low Variance Normal		
pol.	OPT	IND	MOD	OPT	IND	MOD	OPT	IND	MOD
cost	239.56	254.39	240.08	223.84	236.52	224.67	210.51	220.39	210.99
avg.%	0	7.55	0.25	0	6.61	0.4	0	5.41	0.25
max.%	0	119.89	18.63	0	108.23	14.06	0	97.82	15.8

In most of the tests, MOD gives the optimal policy cost. In some tests it can not converge to the optimal solution [2] . And in some cases it can not converge to any policy and stops after an iteration limit is breached. However the last policy before the stopping criteria is met can still be very close to the optimal result. (Table 4.6). In all cases where modified policy can not converge to the optimal solution the optimal cost function is not $[K_1 K_2]-$convex. The average deviation of suboptimal results can be seen in Table 4.7. One remarkable observation is about the assumption that joint expediting discount is less than any single unit expediting cost ($\alpha_d(e_1+e_2) < e_1$ and $\alpha_d(e_1+e_2) < e_2$). MOD always found optimal solutions in the tests obeying this

[2] If the policy cost is approximately 0.1% close to the optimal cost, the policy is considered optimal.

assumption.

Table 4.6. Test Results: Convergence of MOD

Uniform Distribution			High Variance Normal			Low Variance Normal		
Opt. Results	Subopt. Results	can't Conv.	Opt. Results	Subopt. Results	can't Conv.	Opt. Results	Subopt. Results	can't Conv.
179	13	2	174	18	3	183	9	12

Table 4.7. Suboptimal Results for MOD

Uniform Distribution		High Variance Normal		Low Variance Normal	
Subopt. Results	Average Difference	Subopt. Results	Average Difference	Subopt. Results	Average Difference
13	3.71%	18	4.30%	9	5.45%

4.4.2. Time Performance

Average total time performances of algorithms are given in Table 4.8. In this table the durations related with MOD test, where policy iteration technique can only be stopped with limiting number of iterations, are excluded from average times. IND has the best performance. OPT takes about $80-95\%$ longer than IND. MOD takes $20-30\%$ longer than IND. Combined with very accurate results close to OPT, MOD gives a trade-off between time and accuracy. Time performance can be analyzed in

Table 4.8. Average Total Time Performance of Algorithms in Seconds and their Deviation from Best

	Uniform Distribution		High Variance Normal		Low Variance Normal	
Policy	time	dev.	time	dev.	time	dev.
OPT	49.94	80.94%	51.71	83.89%	54.06	94.11%
MOD	35.41	28.30%	34.26	21.83%	34.76	24.81%
IND	27.6	0.00%	28.12	0.00%	27.85	0.00%

more detail. Regarding the time performance, the algorithms can be separated into three main stages:

1. Preparation of the Reward Matrix : For the policy iteration technique and for calculating long run cost of the policy it is necessary to define the reward matrix. Calculating the reward matrix and determining the initial policy is done in this stage. The time cost of determining initial policy is negligible so it is added to this stage.
2. Policy iteration : In the optimal algorithm and the modified (s, S) algorithm better policies are yielded in this stage. Independent single policy algorithm is fixed as the initial policy which is optimal policy for each component ignoring joint expediting discount. Therefore independent single policy algorithm do not consume time at this stage.
3. Calculation of total costs : Using reward matrix and final policy yielded in stage 2, long run cost of the system is calculated and the results are printed in the files.

Detailed time values are shown in Table 4.9. During reward matrix preparation and calculation of average long run costs, all algorithms have approximately same time values, because the algorithms are similar at these stages. IND algorithm has advantage over other algorithms because it omits multi-component policy iteration stage at the cost of suboptimal solutions. MOD policy, which assumes that the optimal policy has $s_1(x_1), s_2(x_2), S_1(x_1), S_2(x_2)$ form, benefits from much faster policy iteration stage. However it might yield suboptimal results if this assumption is not true.

Table 4.9. Average Time Distribution of Algorithms in Seconds

	Uniform Distribution			High Variance Normal			Low Variance Normal		
Policy	rew.	pol.	calc.	rew.	pol.	calc.	rew.	pol.	calc.
OPT	21.33	22.31	6.30	21.34	23.97	6.40	22.01	25.37	6.67
MOD	21.91	6.60	6.90	21.04	6.91	6.30	21.51	6.83	6.42
IND	21.31	0.00	6.29	21.61	0.00	6.51	21.46	0.00	6.39

5. CONCLUSION

We have investigated a two component assemble-to-order model with joint expediting discount. In the model without setup costs, we have fully characterized the optimal policy as a modified base stock policy for single and multiple periods, where target inventory levels $S_1(x_2)$ and $S_2(x_1)$ are functions of other component's inventory level. The optimality of the modified base stock policy depends on the convexity of the optimal cost function, which is preserved when new periods are added to the model.

In the model with setup costs, we proved that the optimal policy is a modified (s, S) policy for single and two periods: Policy parameters $s_1(x_2), s_2(x_1), S_1(x_2), S_2(x_1)$ are functions of other component's inventory levels. Optimality of modified policy depends on the $[K_1K_2]$-convexity of the optimal cost function. However, for more than two periods, $[K_1K_2]$-convexity is not guaranteed. But in most of our test cases in infinite horizon we have observed that the optimal policy has the modified (s, S) form. Thus we have developed an algorithm exploiting the modified (s, S) structure of the optimal policy (MOD), which has less time complexity than exact optimal policy algorithm OPT.

In the model without setup costs, it is proven that, MOD gives exact optimal solutions in less time than OPT. However this is not guaranteed in the model with setup costs. In setup cost settings to compare MOD's accuracy and time performance we have compared it with OPT and IND, which is a simple heuristic completely ignoring effects of joint expediting discount. Accuracy of MOD is very close to the optimal solutions of OPT and time performance of MOD is only $20 - 30\%$ greater than IND, where as OPT spends at least 80% more time than IND. In most of the cases MOD can find the optimal solution (536 out of 576), although there are counter-examples that MOD is not the optimal policy algorithm. However in the tests where total joint expediting discount is less than any single unit expediting cost ($e_1 > \alpha_d(e_1 + e_2)$ and $e_2 > \alpha_d(e_1 + e_2)$), no counter-examples disproving optimality of MOD can be found. MOD can be proposed as an efficient heuristic not only for our setting but also for other assemble-to-order systems with joint costs and convex single period cost functions.

As a managerial insight we have observed that independent single optimal inventory control policies of components may deviate from joint optimal inventory policy significantly. In our examples even a small incentive in joint expediting cost may totally change target optimal inventory level of a component depending on the holding and expediting cost of the other component. In a system with more than 100 components it may be impossible to find the overall optimal policy, even with the approximation algorithm we have proposed. However, for improving the performances of independent single optimal policies, state dependent (s, S) policies may be used where states may summarize the overall performance of the system (number of components in shortage, number of expediting flights planned, number of orders pending etc.). Future research may be directed in two ways : Number of components in the system may be increased. In order to decrease the dimensional complexity of more than two components, the interaction of only selected significant components may be investigated. Other research direction is to investigate different joint interaction effects on costs other than unit expediting cost; joint setup cost, joint holding and procurement incentives or disincentives. Algorithms based on modified (s, S) policy may find near optimal or optimal solutions in different joint cost structures with less time complexity, provided that single period cost of the models are convex.

APPENDIX A: CONVEXITY OF THE EXPECTED COST

In the initial assumptions, it is assumed that joint expediting discount is always less than any unit expediting cost; $e_1 > \alpha_d(e_1+e_2)$ and $e_2 > \alpha_d(e_1+e_2)$. However in the tests values violating this assumption are used. The question is whether convexity of the single period cost function can be claimed without this assumption. The convexity of single period model depends on the convexity of the holding cost and the expediting cost. Since the cost function is the sum of holding and expediting costs, if each of these cost components are convex then the sum of convex functions is convex and expectation preserves convexity.

The convexity of the holding cost can be proven relatively easily. The function $h_i(x_i)$ which is defined as: $h_i(x_i) = h_i x_i$ for $x_i > 0$, $h_i(x_i) = 0$ for $x_i \geq 0$ is convex because linear and constant functions are convex.

The convexity of expediting cost requires some work. Let $x = (x_1, x_2)$. If $x_1 \leq 0$ or $x_2 \leq 0$ then the expediting function $e(x_1, x_2)$ acts like the holding cost function. The non-trivial proof is where $x_1 > 0$ and $x_2 > 0$.

Say we have x,y,$z \in \mathbb{R}^2$ where y is the linear combination of x and y:

$$y = \lambda x + (1 - \lambda)z \text{ for } 0 < \lambda < 1$$

We need to prove : $e(y) \leq \lambda e(x) + (1 - \lambda)e(z)$.

$$e(y) = e(y_1, y_2) = e_1 y_1 + e_2 y_2 - \min(y_1, y_2)\alpha_d(e_1 + e_2)$$

$$e(x) = e(x_1, x_2) = e_1 x_1 + e_2 x_2 - \min(x_1, x_2)\alpha_d(e_1 + e_2) \tag{A.1}$$

$$e(z) = e(z_1, z_2) = e_1 z_1 + e_2 z_2 - \min(z_1, z_2)\alpha_d(e_1 + e_2) \tag{A.2}$$

Substituting we have:

$$\begin{aligned} e(y) &= e_1(\lambda x_1 + (1-\lambda)z_1) + e_2(\lambda x_2 + (1-\lambda)z_2) - \\ &\quad \min(\lambda x_1 + (1-\lambda)z_1, \lambda x_2 + (1-\lambda)z_2)\alpha_d(e_1 + e_2) \end{aligned} \tag{A.3}$$

that needs to be less than or equal to : $\lambda e(x) + (1-\lambda)e(z)$ which is expanded as:

$$\begin{aligned} &= \lambda[e_1 x_1 + e_2 x_2 - \min(x_1, x_2)\alpha_d(e_1 + e_2)] + \\ &\quad (1-\lambda)[e_1 z_1 + e_2 z_2 - \min(z_1, z_2)\alpha_d(e_1 + e_2)] \\ &= e_1(\lambda x_1 + (1-\lambda)z_1) + e_2(\lambda x_2 + (1-\lambda)z_2) - \\ &\quad [\min(\lambda x_1, \lambda x_2) + \min((1-\lambda)z_1, (1-\lambda)z_2)]\alpha_d(e_1 + e_2) \end{aligned} \tag{A.4}$$

Then the condition (A.3) $\leq$ (A.4) simplifies to:

$$\begin{aligned} &\min(\lambda x_1 + (1-\lambda)z_1, \lambda x_2 + (1-\lambda)z_2) \geq \\ &\min(\lambda x_1, \lambda x_2) + \min((1-\lambda)z_1, (1-\lambda)z_2) \end{aligned} \tag{A.5}$$

Without loss of generality if $x_1 \leq x_2$ and $z_1 \leq z_2$ the LHS and RHS are equal. The non-trivial case is , if $x_1 > x_2$ and $z_2 > z_1$. For this case define a and b as : $a = x_1 - x_2$, $b = z_2 - z_1$. Both a and b are greater than 0. Replacing x_1 and z_2 with $x_2 + a$ and $z_1 + b$:

$$\begin{aligned} &\min(\lambda(x_2 + a) + (1-\lambda)z_1, \lambda x_2 + (1-\lambda)(z_1 + b) \geq \\ &\qquad \min(\lambda(x_2 + a), \lambda x_2) + \min((1-\lambda)z_1, (1-\lambda)(z_1 + b)) \\ &\min(\lambda(x_2 + a) + (1-\lambda)z_1, \lambda x_2 + (1-\lambda)(z_1 + b) \geq \lambda x_2 + (1-\lambda)z_1 \\ &\lambda x_2 + (1-\lambda)z_1 + \min(\lambda a, (1-\lambda)b) \geq \lambda x_2 + (1-\lambda)z_1 \end{aligned}$$

Then

$$\min(\lambda a, (1-\lambda)b) \geq 0$$

which is true for $0 < \lambda < 1$.

APPENDIX B: TEST PARAMETERS

For calculating test parameters unit regular production cost $c_1, c2$ are taken as 10 and 5. These parameters are embedded in the model by adjusting $h_1, h_2, e_1, e_2, \alpha_d$ parameters. Actual values of $h_1, h_2, e_1, e_2, \alpha_d$ and their adjusted values are for each test run is listed here.

Table B.1: Test Runs

Test No	K_1	K_2	ae_1	ae_2	ah_1	ah_2	$a\alpha_d$	e_1	e_2	h_1	h_2	α_d
1	50	150	5	10	1.5	0.5	0.775	15	15	1	0.25	0.387
2	50	150	5	10	1.5	0.5	0.6	15	15	1	0.25	0.3
3	50	150	5	10	1.5	0.5	0.425	15	15	1	0.25	0.212
4	50	150	5	10	1.5	0.5	0.25	15	15	1	0.25	0.125
5	50	150	5	10	1.5	1	0.775	15	15	1	0.75	0.387
6	50	150	5	10	1.5	1	0.6	15	15	1	0.75	0.3
7	50	150	5	10	1.5	1	0.425	15	15	1	0.75	0.212
8	50	150	5	10	1.5	1	0.25	15	15	1	0.75	0.125
9	50	150	5	10	2.5	0.5	0.775	15	15	2	0.25	0.387
10	50	150	5	10	2.5	0.5	0.6	15	15	2	0.25	0.3
11	50	150	5	10	2.5	0.5	0.425	15	15	2	0.25	0.212
12	50	150	5	10	2.5	0.5	0.25	15	15	2	0.25	0.125
13	50	150	5	10	2.5	1	0.775	15	15	2	0.75	0.387
14	50	150	5	10	2.5	1	0.6	15	15	2	0.75	0.3
15	50	150	5	10	2.5	1	0.425	15	15	2	0.75	0.212
16	50	150	5	10	2.5	1	0.25	15	15	2	0.75	0.125
17	50	150	5	20	1.5	0.5	0.775	15	25	1	0.25	0.484
18	50	150	5	20	1.5	0.5	0.6	15	25	1	0.25	0.375
19	50	150	5	20	1.5	0.5	0.425	15	25	1	0.25	0.266
20	50	150	5	20	1.5	0.5	0.25	15	25	1	0.25	0.156
21	50	150	5	20	1.5	1	0.775	15	25	1	0.75	0.484
22	50	150	5	20	1.5	1	0.6	15	25	1	0.75	0.375
23	50	150	5	20	1.5	1	0.425	15	25	1	0.75	0.266
24	50	150	5	20	1.5	1	0.25	15	25	1	0.75	0.156
25	50	150	5	20	2.5	0.5	0.775	15	25	2	0.25	0.484
26	50	150	5	20	2.5	0.5	0.6	15	25	2	0.25	0.375
27	50	150	5	20	2.5	0.5	0.425	15	25	2	0.25	0.266

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Test No	K_1	K_2	ae_1	ae_2	ah_1	ah_2	$a\alpha_d$	e_1	e_2	h_1	h_2	α_d
28	50	150	5	20	2.5	0.5	0.25	15	25	2	0.25	0.156
29	50	150	5	20	2.5	1	0.775	15	25	2	0.75	0.484
30	50	150	5	20	2.5	1	0.6	15	25	2	0.75	0.375
31	50	150	5	20	2.5	1	0.425	15	25	2	0.75	0.266
32	50	150	5	20	2.5	1	0.25	15	25	2	0.75	0.156
33	50	150	5	30	1.5	0.5	0.775	15	35	1	0.25	0.542
34	50	150	5	30	1.5	0.5	0.6	15	35	1	0.25	0.42
35	50	150	5	30	1.5	0.5	0.425	15	35	1	0.25	0.297
36	50	150	5	30	1.5	0.5	0.25	15	35	1	0.25	0.175
37	50	150	5	30	1.5	1	0.775	15	35	1	0.75	0.542
38	50	150	5	30	1.5	1	0.6	15	35	1	0.75	0.42
39	50	150	5	30	1.5	1	0.425	15	35	1	0.75	0.297
40	50	150	5	30	1.5	1	0.25	15	35	1	0.75	0.175
41	50	150	5	30	2.5	0.5	0.775	15	35	2	0.25	0.542
42	50	150	5	30	2.5	0.5	0.6	15	35	2	0.25	0.42
43	50	150	5	30	2.5	0.5	0.425	15	35	2	0.25	0.297
44	50	150	5	30	2.5	0.5	0.25	15	35	2	0.25	0.175
45	50	150	5	30	2.5	1	0.775	15	35	2	0.75	0.542
46	50	150	5	30	2.5	1	0.6	15	35	2	0.75	0.42
47	50	150	5	30	2.5	1	0.425	15	35	2	0.75	0.297
48	50	150	5	30	2.5	1	0.25	15	35	2	0.75	0.175
49	50	150	15	10	1.5	0.5	0.775	25	15	1	0.25	0.484
50	50	150	15	10	1.5	0.5	0.6	25	15	1	0.25	0.375
51	50	150	15	10	1.5	0.5	0.425	25	15	1	0.25	0.266
52	50	150	15	10	1.5	0.5	0.25	25	15	1	0.25	0.156
53	50	150	15	10	1.5	1	0.775	25	15	1	0.75	0.484
54	50	150	15	10	1.5	1	0.6	25	15	1	0.75	0.375
55	50	150	15	10	1.5	1	0.425	25	15	1	0.75	0.266
56	50	150	15	10	1.5	1	0.25	25	15	1	0.75	0.156

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Test No	K_1	K_2	ae_1	ae_2	ah_1	ah_2	$a\alpha_d$	e_1	e_2	h_1	h_2	α_d
57	50	150	15	10	2.5	0.5	0.775	25	15	2	0.25	0.484
58	50	150	15	10	2.5	0.5	0.6	25	15	2	0.25	0.375
59	50	150	15	10	2.5	0.5	0.425	25	15	2	0.25	0.266
60	50	150	15	10	2.5	0.5	0.25	25	15	2	0.25	0.156
61	50	150	15	10	2.5	1	0.775	25	15	2	0.75	0.484
62	50	150	15	10	2.5	1	0.6	25	15	2	0.75	0.375
63	50	150	15	10	2.5	1	0.425	25	15	2	0.75	0.266
64	50	150	15	10	2.5	1	0.25	25	15	2	0.75	0.156
65	50	150	15	20	1.5	0.5	0.775	25	25	1	0.25	0.542
66	50	150	15	20	1.5	0.5	0.6	25	25	1	0.25	0.42
67	50	150	15	20	1.5	0.5	0.425	25	25	1	0.25	0.297
68	50	150	15	20	1.5	0.5	0.25	25	25	1	0.25	0.175
69	50	150	15	20	1.5	1	0.775	25	25	1	0.75	0.542
70	50	150	15	20	1.5	1	0.6	25	25	1	0.75	0.42
71	50	150	15	20	1.5	1	0.425	25	25	1	0.75	0.297
72	50	150	15	20	1.5	1	0.25	25	25	1	0.75	0.175
73	50	150	15	20	2.5	0.5	0.775	25	25	2	0.25	0.542
74	50	150	15	20	2.5	0.5	0.6	25	25	2	0.25	0.42
75	50	150	15	20	2.5	0.5	0.425	25	25	2	0.25	0.297
76	50	150	15	20	2.5	0.5	0.25	25	25	2	0.25	0.175
77	50	150	15	20	2.5	1	0.775	25	25	2	0.75	0.542
78	50	150	15	20	2.5	1	0.6	25	25	2	0.75	0.42
79	50	150	15	20	2.5	1	0.425	25	25	2	0.75	0.297
80	50	150	15	20	2.5	1	0.25	25	25	2	0.75	0.175
81	50	150	15	30	1.5	0.5	0.775	25	35	1	0.25	0.581
82	50	150	15	30	1.5	0.5	0.6	25	35	1	0.25	0.45
83	50	150	15	30	1.5	0.5	0.425	25	35	1	0.25	0.319
84	50	150	15	30	1.5	0.5	0.25	25	35	1	0.25	0.188
85	50	150	15	30	1.5	1	0.775	25	35	1	0.75	0.581

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Test No	K_1	K_2	ae_1	ae_2	ah_1	ah_2	$a\alpha_d$	e_1	e_2	h_1	h_2	α_d
86	50	150	15	30	1.5	1	0.6	25	35	1	0.75	0.45
87	50	150	15	30	1.5	1	0.425	25	35	1	0.75	0.319
88	50	150	15	30	1.5	1	0.25	25	35	1	0.75	0.188
89	50	150	15	30	2.5	0.5	0.775	25	35	2	0.25	0.581
90	50	150	15	30	2.5	0.5	0.6	25	35	2	0.25	0.45
91	50	150	15	30	2.5	0.5	0.425	25	35	2	0.25	0.319
92	50	150	15	30	2.5	0.5	0.25	25	35	2	0.25	0.188
93	50	150	15	30	2.5	1	0.775	25	35	2	0.75	0.581
94	50	150	15	30	2.5	1	0.6	25	35	2	0.75	0.45
95	50	150	15	30	2.5	1	0.425	25	35	2	0.75	0.319
96	50	150	15	30	2.5	1	0.25	25	35	2	0.75	0.188
97	200	150	5	10	1.5	0.5	0.775	15	15	1	0.25	0.387
98	200	150	5	10	1.5	0.5	0.6	15	15	1	0.25	0.3
99	200	150	5	10	1.5	0.5	0.425	15	15	1	0.25	0.212
100	200	150	5	10	1.5	0.5	0.25	15	15	1	0.25	0.125
101	200	150	5	10	1.5	1	0.775	15	15	1	0.75	0.387
102	200	150	5	10	1.5	1	0.6	15	15	1	0.75	0.3
103	200	150	5	10	1.5	1	0.425	15	15	1	0.75	0.212
104	200	150	5	10	1.5	1	0.25	15	15	1	0.75	0.125
105	200	150	5	10	2.5	0.5	0.775	15	15	2	0.25	0.387
106	200	150	5	10	2.5	0.5	0.6	15	15	2	0.25	0.3
107	200	150	5	10	2.5	0.5	0.425	15	15	2	0.25	0.212
108	200	150	5	10	2.5	0.5	0.25	15	15	2	0.25	0.125
109	200	150	5	10	2.5	1	0.775	15	15	2	0.75	0.387
110	200	150	5	10	2.5	1	0.6	15	15	2	0.75	0.3
111	200	150	5	10	2.5	1	0.425	15	15	2	0.75	0.212
112	200	150	5	10	2.5	1	0.25	15	15	2	0.75	0.125
113	200	150	5	20	1.5	0.5	0.775	15	25	1	0.25	0.484
114	200	150	5	20	1.5	0.5	0.6	15	25	1	0.25	0.375

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Test No	K_1	K_2	ae_1	ae_2	ah_1	ah_2	$a\alpha_d$	e_1	e_2	h_1	h_2	α_d
115	200	150	5	20	1.5	0.5	0.425	15	25	1	0.25	0.266
116	200	150	5	20	1.5	0.5	0.25	15	25	1	0.25	0.156
117	200	150	5	20	1.5	1	0.775	15	25	1	0.75	0.484
118	200	150	5	20	1.5	1	0.6	15	25	1	0.75	0.375
119	200	150	5	20	1.5	1	0.425	15	25	1	0.75	0.266
120	200	150	5	20	1.5	1	0.25	15	25	1	0.75	0.156
121	200	150	5	20	2.5	0.5	0.775	15	25	2	0.25	0.484
122	200	150	5	20	2.5	0.5	0.6	15	25	2	0.25	0.375
123	200	150	5	20	2.5	0.5	0.425	15	25	2	0.25	0.266
124	200	150	5	20	2.5	0.5	0.25	15	25	2	0.25	0.156
125	200	150	5	20	2.5	1	0.775	15	25	2	0.75	0.484
126	200	150	5	20	2.5	1	0.6	15	25	2	0.75	0.375
127	200	150	5	20	2.5	1	0.425	15	25	2	0.75	0.266
128	200	150	5	20	2.5	1	0.25	15	25	2	0.75	0.156
129	200	150	5	30	1.5	0.5	0.775	15	35	1	0.25	0.542
130	200	150	5	30	1.5	0.5	0.6	15	35	1	0.25	0.42
131	200	150	5	30	1.5	0.5	0.425	15	35	1	0.25	0.297
132	200	150	5	30	1.5	0.5	0.25	15	35	1	0.25	0.175
133	200	150	5	30	1.5	1	0.775	15	35	1	0.75	0.542
134	200	150	5	30	1.5	1	0.6	15	35	1	0.75	0.42
135	200	150	5	30	1.5	1	0.425	15	35	1	0.75	0.297
136	200	150	5	30	1.5	1	0.25	15	35	1	0.75	0.175
137	200	150	5	30	2.5	0.5	0.775	15	35	2	0.25	0.542
138	200	150	5	30	2.5	0.5	0.6	15	35	2	0.25	0.42
139	200	150	5	30	2.5	0.5	0.425	15	35	2	0.25	0.297
140	200	150	5	30	2.5	0.5	0.25	15	35	2	0.25	0.175
141	200	150	5	30	2.5	1	0.775	15	35	2	0.75	0.542
142	200	150	5	30	2.5	1	0.6	15	35	2	0.75	0.42
143	200	150	5	30	2.5	1	0.425	15	35	2	0.75	0.297

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Test No	K_1	K_2	ae_1	ae_2	ah_1	ah_2	$a\alpha_d$	e_1	e_2	h_1	h_2	α_d
144	200	150	5	30	2.5	1	0.25	15	35	2	0.75	0.175
145	200	150	15	10	1.5	0.5	0.775	25	15	1	0.25	0.484
146	200	150	15	10	1.5	0.5	0.6	25	15	1	0.25	0.375
147	200	150	15	10	1.5	0.5	0.425	25	15	1	0.25	0.266
148	200	150	15	10	1.5	0.5	0.25	25	15	1	0.25	0.156
149	200	150	15	10	1.5	1	0.775	25	15	1	0.75	0.484
150	200	150	15	10	1.5	1	0.6	25	15	1	0.75	0.375
151	200	150	15	10	1.5	1	0.425	25	15	1	0.75	0.266
152	200	150	15	10	1.5	1	0.25	25	15	1	0.75	0.156
153	200	150	15	10	2.5	0.5	0.775	25	15	2	0.25	0.484
154	200	150	15	10	2.5	0.5	0.6	25	15	2	0.25	0.375
155	200	150	15	10	2.5	0.5	0.425	25	15	2	0.25	0.266
156	200	150	15	10	2.5	0.5	0.25	25	15	2	0.25	0.156
157	200	150	15	10	2.5	1	0.775	25	15	2	0.75	0.484
158	200	150	15	10	2.5	1	0.6	25	15	2	0.75	0.375
159	200	150	15	10	2.5	1	0.425	25	15	2	0.75	0.266
160	200	150	15	10	2.5	1	0.25	25	15	2	0.75	0.156
161	200	150	15	20	1.5	0.5	0.775	25	25	1	0.25	0.542
162	200	150	15	20	1.5	0.5	0.6	25	25	1	0.25	0.42
163	200	150	15	20	1.5	0.5	0.425	25	25	1	0.25	0.297
164	200	150	15	20	1.5	0.5	0.25	25	25	1	0.25	0.175
165	200	150	15	20	1.5	1	0.775	25	25	1	0.75	0.542
166	200	150	15	20	1.5	1	0.6	25	25	1	0.75	0.42
167	200	150	15	20	1.5	1	0.425	25	25	1	0.75	0.297
168	200	150	15	20	1.5	1	0.25	25	25	1	0.75	0.175
169	200	150	15	20	2.5	0.5	0.775	25	25	2	0.25	0.542
170	200	150	15	20	2.5	0.5	0.6	25	25	2	0.25	0.42
171	200	150	15	20	2.5	0.5	0.425	25	25	2	0.25	0.297
172	200	150	15	20	2.5	0.5	0.25	25	25	2	0.25	0.175

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Test No	K_1	K_2	ae_1	ae_2	ah_1	ah_2	$a\alpha_d$	e_1	e_2	h_1	h_2	α_d
173	200	150	15	20	2.5	1	0.775	25	25	2	0.75	0.542
174	200	150	15	20	2.5	1	0.6	25	25	2	0.75	0.42
175	200	150	15	20	2.5	1	0.425	25	25	2	0.75	0.297
176	200	150	15	20	2.5	1	0.25	25	25	2	0.75	0.175
177	200	150	15	30	1.5	0.5	0.775	25	35	1	0.25	0.581
178	200	150	15	30	1.5	0.5	0.6	25	35	1	0.25	0.45
179	200	150	15	30	1.5	0.5	0.425	25	35	1	0.25	0.319
180	200	150	15	30	1.5	0.5	0.25	25	35	1	0.25	0.188
181	200	150	15	30	1.5	1	0.775	25	35	1	0.75	0.581
182	200	150	15	30	1.5	1	0.6	25	35	1	0.75	0.45
183	200	150	15	30	1.5	1	0.425	25	35	1	0.75	0.319
184	200	150	15	30	1.5	1	0.25	25	35	1	0.75	0.188
185	200	150	15	30	2.5	0.5	0.775	25	35	2	0.25	0.581
186	200	150	15	30	2.5	0.5	0.6	25	35	2	0.25	0.45
187	200	150	15	30	2.5	0.5	0.425	25	35	2	0.25	0.319
188	200	150	15	30	2.5	0.5	0.25	25	35	2	0.25	0.188
189	200	150	15	30	2.5	1	0.775	25	35	2	0.75	0.581
190	200	150	15	30	2.5	1	0.6	25	35	2	0.75	0.45
191	200	150	15	30	2.5	1	0.425	25	35	2	0.75	0.319
192	200	150	15	30	2.5	1	0.25	25	35	2	0.75	0.188

APPENDIX C: TEST RESULTS

There different algorithms OPT, IND and MOD have been tested in the tests. There there different demand distributions: uniform, normal with high variance and normal with low variance. For test specific $h_1, h_2, e_1, e_2, \alpha_d$ parameters please refer to the table in Appendix B. Find the parameter row, having the same test number.

Table C.1: Test Results

	Uniform Distribution			High Variance Normal			Low Variance Normal		
Test	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.
1	135.60	169.36	135.60	135.60	156.58	135.60	135.60	147.27	135.60
2	161.99	170.17	161.99	150.98	157.54	150.98	142.29	147.74	142.29
3	168.06	171.00	168.06	156.02	158.51	156.02	146.27	148.21	146.27
4	171.17	171.81	171.17	158.96	159.46	158.96	148.38	148.68	148.38
5	135.60	201.20	135.60	135.60	187.29	135.60	135.60	175.87	135.60
6	188.12	203.00	190.33	175.15	188.66	175.15	165.24	176.88	165.24
7	198.18	204.81	198.18	184.12	190.05	184.12	173.38	177.90	173.38
8	204.84	206.61	204.84	189.22	191.42	189.22	177.23	178.92	177.23
9	135.60	190.42	135.60	135.60	170.29	135.60	135.60	157.59	135.60
10	184.28	191.80	184.28	166.24	171.67	166.24	153.33	158.05	153.33
11	190.85	193.20	190.85	171.84	173.07	171.84	156.91	158.52	156.91
12	194.20	194.59	194.20	174.17	174.44	174.17	158.34	158.99	158.34
13	135.60	219.76	135.60	135.60	200.43	135.60	135.60	186.18	135.60
14	207.94	222.69	207.94	189.64	202.35	194.77	177.94	187.19	182.35
15	220.32	225.66	220.32	199.31	204.29	199.31	183.90	188.22	183.90
16	227.22	228.59	227.22	205.10	206.21	205.10	187.59	189.23	187.59
17	160.91	173.08	165.33	150.20	160.13	150.20	141.44	150.48	141.44
18	170.04	173.84	170.04	157.72	161.05	157.72	147.39	150.86	147.39
19	173.55	174.60	173.55	161.10	161.97	161.10	149.60	151.24	149.60
20	175.04	175.36	175.04	162.50	162.90	162.50	151.26	151.63	151.26
21	185.91	211.53	185.91	173.58	194.66	185.73	164.06	180.75	164.06
22	201.74	212.66	201.74	186.77	196.05	186.77	175.35	181.75	175.35
23	209.33	213.78	209.33	193.68	197.43	193.68	180.44	182.74	180.44
24	213.58	214.91	213.58	197.84	198.82	197.84	182.80	183.75	182.80
25	182.86	194.75	186.55	165.52	173.86	165.52	152.63	160.79	152.63
26	192.89	195.94	192.89	173.09	175.20	173.09	157.79	161.17	157.79

Continued on next page

Table C.1: Test Results

	Uniform Distribution			High Variance Normal			Low Variance Normal		
Test	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.
27	196.56	197.14	196.56	176.57	176.54	176.57	159.55	161.55	159.55
28	198.43	198.34	198.43	177.89	177.89	177.89	161.21	161.94	161.21
29	205.51	233.33	205.51	187.80	207.82	193.68	175.82	191.06	203.61
30	224.22	234.86	224.22	202.67	209.75	202.67	186.20	192.06	186.20
31	232.28	236.39	232.28	208.91	211.67	208.91	190.25	193.06	190.25
32	236.52	237.93	236.52	213.32	213.62	213.32	192.61	194.06	192.61
33	166.66	176.01	166.66	155.26	163.25	155.26	145.46	150.68	145.46
34	173.29	176.45	173.29	160.90	163.82	160.90	149.46	151.21	149.46
35	175.44	176.89	175.44	162.92	164.39	162.92	151.78	151.75	151.78
36	177.17	177.33	177.17	164.79	164.96	164.79	152.28	152.28	152.28
37	195.96	213.05	203.36	181.54	197.24	181.54	171.50	184.76	171.50
38	208.88	214.62	208.88	193.57	198.92	193.57	180.22	185.50	180.22
39	214.86	216.21	214.86	198.86	200.61	198.86	183.54	186.26	183.54
40	217.14	217.78	217.14	201.60	202.29	201.60	186.32	187.01	186.32
41	189.42	199.61	189.42	170.66	178.83	170.66	156.17	160.99	156.17
42	196.31	200.05	196.31	176.29	179.40	176.29	159.42	161.52	159.42
43	198.82	200.49	198.82	178.32	179.98	178.32	161.74	162.06	161.74
44	200.56	200.93	200.56	180.19	180.55	180.19	162.59	162.59	162.59
45	217.23	234.12	217.23	197.18	209.49	197.18	182.31	195.07	182.31
46	231.90	236.26	231.90	208.55	211.92	208.55	190.03	195.82	190.03
47	237.79	238.42	237.79	214.22	214.37	214.22	193.35	196.57	193.35
48	240.56	240.56	240.56	216.93	216.80	216.93	196.59	197.32	196.59
49	166.17	183.68	166.17	156.84	172.53	156.84	147.05	158.90	147.05
50	178.29	183.97	178.82	166.37	173.06	166.37	153.96	159.16	153.96
51	182.97	184.27	182.97	171.09	173.59	171.09	158.13	159.42	158.13
52	184.82	184.56	184.82	173.69	174.13	173.69	159.23	159.68	159.23

Continued on next page

Table C.1: Test Results

	Uniform Distribution			High Variance Normal			Low Variance Normal		
Test	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.
53	187.05	218.44	187.05	177.11	204.08	192.31	168.17	188.88	181.44
54	207.82	219.06	207.82	192.74	204.84	192.74	179.71	189.38	179.71
55	216.57	219.69	216.57	201.56	205.60	201.56	186.35	189.87	186.35
56	219.99	220.32	219.99	205.70	206.36	205.70	189.65	190.37	189.65
57	194.98	217.79	194.98	179.00	195.25	190.74	161.35	170.42	161.35
58	210.74	218.79	210.74	190.15	196.34	190.15	168.91	171.13	168.91
59	217.42	219.80	217.42	195.54	197.42	195.54	171.68	171.85	171.68
60	220.58	220.82	220.58	198.15	198.51	198.15	172.57	172.57	172.57
61	209.65	250.49	209.65	196.89	225.88	196.89	181.05	198.69	193.73
62	239.02	252.29	239.02	215.90	227.40	222.50	195.59	200.02	195.59
63	249.55	254.09	249.55	224.75	228.91	224.75	200.60	201.36	200.60
64	254.81	255.90	254.81	229.80	230.44	229.80	202.70	202.70	202.70
65	173.42	187.31	173.42	163.13	175.90	163.13	151.67	162.11	151.67
66	182.75	187.56	183.04	170.96	176.44	170.96	158.14	162.28	158.14
67	186.01	187.83	186.01	175.04	176.98	175.04	160.44	162.46	160.44
68	187.94	188.09	187.94	177.00	177.51	177.00	162.31	162.63	162.31
69	201.79	226.69	239.38	186.95	211.35	202.92	175.52	193.74	186.86
70	216.24	227.10	216.24	201.02	212.14	208.02	185.83	194.23	185.83
71	223.54	227.52	223.54	207.74	212.94	207.74	191.46	194.71	191.46
72	227.18	227.93	227.18	212.93	213.72	212.93	194.37	195.19	194.37
73	204.93	221.63	215.06	185.92	198.59	196.02	166.37	174.05	171.51
74	217.00	222.56	217.00	195.46	199.68	198.75	171.54	174.58	171.54
75	222.07	223.48	222.07	199.34	200.79	199.34	173.92	175.11	173.92
76	224.20	224.41	224.20	201.64	201.88	201.64	175.94	175.65	175.94
77	228.31	259.86	228.31	207.91	233.00	207.91	189.95	203.81	189.95
78	249.03	261.19	249.03	224.15	234.58	224.15	200.70	205.07	200.70

Continued on next page

Table C.1: Test Results

	Uniform Distribution			High Variance Normal			Low Variance Normal		
Test	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.
79	257.60	262.54	257.60	232.68	236.18	232.68	204.81	206.34	204.81
80	262.62	263.88	262.62	236.96	237.76	236.96	207.64	207.60	207.64
81	178.60	189.47	182.95	166.59	178.28	173.93	154.11	162.76	154.11
82	185.47	189.58	185.47	173.64	178.63	173.64	159.62	162.98	159.62
83	188.00	189.69	188.00	177.02	178.99	177.02	162.05	163.21	162.05
84	189.33	189.80	189.33	178.99	179.34	178.99	162.91	163.43	162.91
85	208.21	229.67	226.61	193.04	214.77	193.04	179.93	197.42	179.93
86	221.56	230.20	221.56	205.90	215.67	205.90	190.31	197.73	190.31
87	227.12	230.74	227.12	212.81	216.57	212.81	194.49	198.04	194.49
88	230.25	231.27	230.25	216.48	217.47	216.48	197.39	198.34	197.39
89	210.84	224.73	210.84	190.37	201.75	190.37	169.06	174.24	169.06
90	220.78	225.30	220.78	198.26	202.48	198.26	173.11	174.93	173.11
91	224.04	225.86	224.04	201.52	203.22	201.52	175.83	175.61	175.83
92	226.31	226.42	226.31	203.93	203.96	203.93	176.30	176.30	176.30
93	239.48	261.66	239.48	216.44	235.79	246.86	195.80	208.04	195.80
94	254.95	263.39	254.95	230.28	237.62	230.28	203.68	209.00	203.68
95	262.37	265.11	262.37	236.69	239.46	236.69	207.84	209.97	207.84
96	266.07	266.83	266.07	240.71	241.29	240.71	210.90	210.93	210.90
97	135.60	234.55	135.60	135.60	222.14	135.60	135.60	215.30	135.60
98	220.41	235.26	220.41	212.10	222.82	212.10	206.99	215.63	206.99
99	231.63	235.98	231.63	221.44	223.51	221.44	214.70	215.97	214.70
100	235.72	236.69	235.72	224.14	224.20	224.14	216.23	216.29	216.23
101	135.60	255.11	135.60	135.60	246.09	135.60	135.60	244.88	135.60
102	240.00	259.34	240.00	231.03	248.71	231.03	225.23	245.54	225.23
103	257.79	263.62	257.79	244.69	251.36	244.69	237.06	246.21	237.06
104	267.07	267.85	267.07	252.00	253.98	252.00	242.84	246.86	242.84

Continued on next page

Table C.1: Test Results

	Uniform Distribution			High Variance Normal			Low Variance Normal		
Test	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.
105	135.60	263.86	135.60	135.60	250.05	135.60	135.60	241.74	135.60
106	240.00	264.83	240.00	240.00	251.06	240.00	240.00	242.22	240.00
107	262.34	265.81	262.34	249.59	252.07	249.59	240.16	242.71	240.16
108	266.29	266.78	266.29	252.67	253.07	252.67	242.63	243.19	242.63
109	135.60	298.17	135.60	135.60	282.36	135.60	135.60	268.24	135.60
110	240.00	299.57	240.00	240.00	283.42	240.00	240.00	269.74	240.00
111	291.95	300.99	291.95	275.30	284.49	275.30	265.56	271.26	265.56
112	300.74	302.38	300.74	282.33	285.54	282.33	271.20	272.76	271.20
113	225.60	238.75	225.60	210.11	226.10	210.11	205.23	218.93	205.23
114	233.87	239.30	233.87	223.95	226.66	223.95	216.35	219.08	216.35
115	237.94	239.85	237.94	226.62	227.21	226.62	218.23	219.22	218.23
116	240.32	240.40	240.32	227.80	227.77	227.80	219.22	219.37	219.22
117	225.60	276.88	225.60	225.60	260.57	225.60	225.60	250.92	225.60
118	263.18	277.86	263.18	249.36	261.60	249.36	240.93	251.30	240.93
119	272.82	278.85	272.82	257.10	262.63	257.10	247.60	251.68	247.60
120	276.98	279.84	276.98	261.51	263.67	261.51	249.57	252.06	249.57
121	225.60	268.57	225.60	225.60	254.56	225.60	225.60	245.77	225.60
122	264.82	269.26	264.82	251.57	255.31	251.57	241.69	245.98	241.69
123	268.63	269.95	268.63	254.86	256.06	254.86	244.32	246.19	244.32
124	270.51	270.65	270.51	256.52	256.82	256.52	245.69	246.40	245.69
125	225.60	309.64	225.60	225.60	290.91	225.60	225.60	275.37	225.60
126	297.34	310.10	297.34	279.39	291.71	279.39	269.19	276.35	269.19
127	306.36	310.57	306.36	286.85	292.51	286.85	275.39	277.33	275.39
128	309.99	311.04	309.99	291.77	293.32	291.77	278.80	278.31	278.80
129	229.68	241.85	229.68	220.24	228.95	220.24	213.77	219.54	213.77
130	237.67	242.04	237.67	226.66	229.22	226.66	218.20	219.74	218.20

Continued on next page

Table C.1: Test Results

	Uniform Distribution			High Variance Normal			Low Variance Normal		
Test	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.
131	240.75	242.23	240.75	228.63	229.48	228.63	219.73	219.95	219.73
132	242.59	242.42	242.59	229.77	229.75	229.77	220.20	220.16	220.20
133	253.53	278.64	253.53	241.67	258.26	241.67	234.81	252.22	234.81
134	272.14	280.02	272.14	256.74	260.69	256.74	247.05	252.97	247.05
135	278.26	281.41	278.26	262.91	263.13	262.91	250.31	253.72	250.31
136	281.23	282.79	281.23	265.56	265.56	265.56	253.55	254.47	253.55
137	260.72	272.00	260.72	247.90	257.96	247.90	238.98	246.27	238.98
138	268.57	272.26	268.57	254.62	258.30	254.62	244.26	246.57	244.26
139	270.88	272.52	270.88	257.12	258.64	257.12	246.18	246.86	246.18
140	272.37	272.78	272.37	258.58	258.98	258.58	247.12	247.15	247.12
141	288.05	312.32	288.05	272.11	293.73	272.11	263.01	280.98	263.01
142	306.30	312.97	306.30	286.45	294.78	286.45	275.61	281.35	275.61
143	311.04	313.63	311.04	292.43	295.83	292.43	279.55	281.72	279.55
144	313.25	314.29	313.25	295.13	296.87	295.13	282.84	282.09	282.84
145	225.60	261.90	225.60	210.11	243.64	210.11	205.23	232.91	205.23
146	239.99	262.31	239.99	228.83	244.26	238.90	220.10	233.17	230.48
147	251.86	262.73	258.08	237.85	244.88	237.85	227.36	233.43	227.36
148	258.94	263.15	258.94	244.95	245.51	244.95	233.32	233.69	233.32
149	225.60	268.57	225.60	225.60	274.51	225.60	225.60	261.22	225.60
150	263.81	275.64	263.81	249.97	275.51	249.97	241.55	262.10	241.55
151	279.80	282.71	279.80	262.08	276.51	262.08	251.10	262.97	251.10
152	289.09	289.85	289.09	271.36	277.52	271.36	257.83	263.85	257.83
153	225.60	309.43	225.60	225.60	287.01	225.60	225.60	268.57	225.60
154	284.88	310.15	284.88	268.88	288.06	268.88	257.70	269.14	257.70
155	301.25	310.88	301.25	281.65	289.12	281.65	264.99	269.70	264.99
156	309.13	311.62	309.13	287.52	290.18	287.52	268.83	270.28	268.83

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Table C.1: Test Results

	Uniform Distribution			High Variance Normal			Low Variance Normal		
Test	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.
157	225.60	338.90	225.60	225.60	319.96	225.60	225.60	295.56	225.60
158	303.59	341.14	303.59	285.18	320.93	285.18	274.15	297.04	274.15
159	326.27	343.39	326.27	303.99	321.89	313.00	289.61	298.51	289.54
160	342.23	345.66	342.23	315.13	322.86	315.13	296.57	300.00	296.57
161	231.56	265.74	231.56	221.28	247.51	221.28	214.66	236.29	214.66
162	251.56	266.07	251.22	237.15	248.02	237.15	226.88	236.43	226.88
163	261.19	266.41	261.19	246.93	248.54	246.93	235.20	236.56	235.20
164	265.59	266.74	265.59	248.87	249.05	248.87	236.27	236.69	236.27
165	253.53	302.07	253.90	241.67	274.38	241.82	234.81	268.59	234.62
166	278.62	303.25	278.62	261.32	277.07	261.32	250.55	268.89	250.55
167	291.70	304.43	291.70	274.30	279.79	274.30	259.96	269.18	259.96
168	301.43	305.60	301.43	280.75	282.49	280.75	265.54	269.48	265.54
169	271.76	314.15	273.49	258.22	291.80	258.22	250.51	272.79	250.51
170	300.49	314.60	300.49	281.23	292.54	281.23	264.79	273.04	264.79
171	311.25	315.04	311.25	289.75	293.28	289.75	269.81	273.29	269.81
172	315.44	315.49	315.44	293.23	294.02	293.23	272.52	273.55	272.52
173	287.68	352.03	287.68	273.02	328.24	273.02	263.74	302.55	263.74
174	324.88	352.97	324.88	302.44	329.01	302.55	288.33	303.53	288.33
175	347.11	353.92	347.11	317.53	329.78	317.53	298.77	304.52	298.77
176	353.42	354.85	353.42	326.76	330.55	326.76	306.25	305.50	306.25
177	240.58	268.26	240.39	229.33	250.30	229.55	220.48	237.00	221.65
178	257.71	268.36	257.71	245.02	250.53	245.02	232.29	237.17	232.29
179	265.48	268.47	265.48	249.42	250.77	249.42	237.00	237.34	237.00
180	267.86	268.57	267.86	250.75	251.01	250.75	237.75	237.51	237.75
181	264.46	304.09	264.46	250.51	287.45	253.36	242.01	268.57	244.64
182	287.67	305.60	287.67	269.76	288.09	269.76	256.67	269.53	256.67

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Table C.1: Test Results

	Uniform Distribution			High Variance Normal			Low Variance Normal		
Test	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.	Opt.	Ind.	Mod.
183	301.08	307.11	301.08	280.32	288.72	280.32	265.33	270.49	265.33
184	307.04	308.63	307.04	285.80	289.35	285.80	269.50	271.46	269.50
185	285.61	316.75	285.61	269.48	295.11	269.48	258.09	273.34	258.09
186	308.56	316.95	308.56	286.90	295.45	286.90	268.26	273.66	268.26
187	315.20	317.15	315.20	293.33	295.80	293.33	272.43	273.99	272.43
188	317.32	317.35	317.32	295.33	296.14	295.33	274.38	274.31	274.38
189	304.43	354.35	304.43	285.79	331.69	285.79	274.65	307.93	274.65
190	340.37	355.56	340.37	313.12	332.56	313.12	296.23	308.35	296.23
191	353.46	356.77	353.46	326.32	333.43	326.32	306.02	308.78	306.02
192	357.82	357.97	357.82	332.38	334.30	332.38	308.54	309.20	308.54

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