UNCERTAINTY AND CERTAINTY RELATIONS FOR THE Q-OSCILLATOR

by

Fatma Nur Türkmen

B.S., in Physics, Boğaziçi University, 2004

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Physics
Boğaziçi University
2008

UNCERTAINTY AND CERTAINTY RELATIONS FOR THE Q-OSCILLATOR

APPROVED BY:

Prof. Metin Arık (Thesis Supervisor)	....
Assist. Prof. Tonguç Rador	....
Assoc. Prof. Ali Yıldız	....

DATE OF APPROVAL: 13.11.2008

ACKNOWLEDGEMENTS

I would like to thank my advisor Prof. Dr. Metin Arık for his guidance, encouragement, patience, support and helpful comments and suggestions for the improvement of this thesis. I thank TÜBİTAK (Scientific and Technical Research Council of Turkey) for support through the Basic Sciences Unit Research Project 106T593.

ABSTRACT

UNCERTAINTY AND CERTAINTY RELATIONS FOR THE Q-OSCILLATOR

The uncertainty and certainty relations for the momentum and position operators for the q-oscillator and Fibonacci oscillator are investigated in this thesis. The one-dimensional q-oscillator, the two-dimensional q-oscillator which is invariant under the action of the unitary q-deformed quantum group and Fibonacci oscillator are studied.

We study the one-dimensional q-oscillator. Firstly, by using the commutation relation for the momentum and position operators, the uncertainty relations for the energy eigenstates and any state which is a superposition of energy eigenstates are calculated. By calculating the upper limit of the expectation value of the hamiltonian, the upper limits of ΔP and ΔX and the certainty relations are obtained in the case in which $0 < q < 1$. Then further uncertainty relations for the momentum and position are obtained. Secondly, by calculating ΔP and ΔX directly and by finding their lower and upper limits, the uncertainty and certainty relations for the energy eigenstates are again obtained. As a result, the two ways of finding the uncertainty and certainty relations for the energy eigenstates are true but the most informative results are selected from the two different sets of results obtained by these two methods. Thus the further uncertainty relations and the certainty relations are obtained for the energy eigenstates and an arbitrary state. The classical limits of $(\varepsilon_{n+1} - \varepsilon_n)/\varepsilon_n$ where ε_n are the energy eigenvalues are calculated for the different intervals of q. It is shown that the classical limit of this quantity in the case in which $q \geq \sqrt{2}$ is unreasonable.

A similar procedure is repeated for the two-dimensional q-oscillator and Fibonacci oscillator.

ÖZET

Q-SALINIMCISINA AİT BELİRSİZLİK VE BELİRLİLİK BAĞINTILARI

Bu tezde, q-salınımcısına ait momentum ve konum işlemcilerinin sağladıkları belirsizlik ve belirlilik bağıntıları incelenmiştir. Bir boyutlu q-salınımcısı, üniter q-deforme kuantum grubu altında değişmez olan iki boyutlu q-salınımcısı ve Fibonacci salınımcısı çalışılmıştır.

Bir boyutlu q-salınımcısını inceledik. İlk olarak, momentum işlemcisi ve konum işlemcisine ait komütasyon bağıntıları kullanılarak, enerji özvektörleri için ve enerji özvektörlerinin bir kombinasyonu olan her hangi bir durum için belirsizlik bağıntıları hesaplanmıştır. Hamiltonyenin beklenen değerinin üst limiti hesaplanarak, $0 < q < 1$ koşulunda ΔP'nin ve ΔX'in üst limitleri ve belirsizlik bağıntıları elde edilmiştir. Sonra ileri belirsizlik bağıntıları elde edilmiştir. İkinci olarak, ΔP ve ΔX doğrudan hesaplanarak ve bunların alt ve üst limitleri bulunarak, enerji özvektörleri için belirsizlik bağıntıları ve belirlilik bağıntıları tekrar elde edilmiştir. Son olarak, enerji özvektörleri için belirsizlik bağıntıları ve belirlilik bağıntıları bulmanın iki yolu da doğrudur. Bu iki metodla elde edilen iki farklı çözüm setinden en çok bilgi veren sonuçlar seçilmiştir. Böylece, enerji özvektörleri için ve herhangi bir durum için ileri belirsizlik bağıntıları ve belirlilik bağıntıları elde edilmiştir. q'nun farklı değer aralıkları için $(\varepsilon_{n+1} - \varepsilon_n)/\varepsilon_n$'in klasik limitleri hesaplanmıştır. ε_n'in enerji özdeğerleri olduğu bu ifadenin klasik limitinin $q \geq \sqrt{2}$ koşulunda makul olmadığı gösterilmiştir.

Benzer bir yöntem iki boyutlu q-salınımcısı ve Fibonacci salınımcısı için tekrarlanmıştır.

TABLE OF CONTENTS

1. INTRODUCTION

The importance of the uncertainty relations and the certainty relations for the q-oscillator is related to the importance of the topics which are the Heisenberg uncertainty relation[1] and the quantum harmonic oscillator.

The Heisenberg uncertainty principle is significant because it is the basis of quantum mechanics. This principle says that it is not possible to measure the momentum and the position of a particle simultaneously with a greater accuracy than the maximum accuracy determined by the uncertainty relation which is $\Delta P \Delta X \geq \frac{\hbar}{2}$. If it were possible, then quantum mechanics would collapse.

The quantum harmonic oscillator is a significant subject because it is exactly solvable and is related to all branches of physics. Most of the physical systems excited near their ground states behave like harmonic oscillators. Some of the examples are molecular bound states and quantized radiation. The harmonic oscillator is applied to nuclear and hadronic bound states. When the electromagnetic field is quantized one again basically obtains the harmonic oscillator: The electrical energy behaves as the kinetic energy of the oscillator and the magnetic energy behaves as the potential energy of the oscillator. Quantum field theory generalizes this phenomenon to any particle or field which is quantized. With the discovery of quantum groups[2-5] (q-groups) which are one-parameter generalizations (deformations) of the familiar Lie groups, similar generalizations of the quantum harmonic oscillator have attracted a lot of scientists' attention[6-15]. The first appearance of these generalizations, now called q-oscillators, predated the discovery of quantum groups by a decade[16-20]. The simplest one-dimensional version of the q-oscillator is defined by the commutation relation[20]

$$aa^\dagger - q^2 a^\dagger a = 1 \tag{1.1}$$

where $a^\dagger$ and a are the creation operator and the annihilation operator respectively and q is the real deformation parameter. There are different kinds of multi-dimensional extensions of the q-oscillator.[16-24] The most interesting one which is invariant under the action of the unitary quantum group has been constructed by Pusz and Woronowicz[22]. The physical significance of this invariance is that there is a one to one correspondence[25] between the excited states of the q-oscillator and the excited states of the ordinary oscillator.

In section (1.1), we study the uncertainty relations. The derivation of the uncertainty relations will occur. This method will be used through chapter (2).

In section (1.2), we study the physical importance of the harmonic oscillator. Our purpose here is to realize that the applications of the harmonic oscillator thus the q-oscillator or the Fibonacci oscillator are possible in most areas of physics.

In section (2.1), we study the one-dimensional q-oscillator. If we define the position operator and the momentum operator linearly in terms of the creation operator and the annihilation operator, then the commutation relation satisfied is different than the canonical commutation relation which we calculate. The commutation relations for the hamiltonian, the creation operator and the annihilation operator are then considered to obtain the energy eigenvalues. It is examined under which conditions there must be a ground state. Under the condition that there is a ground state, the uncertainty relation for the energy eigenstates can be calculated. After that, for the case in which $0 < q < 1$ and the energy eigenvalue ε is less than $\frac{1}{1-q^2}$, the upper limit of the energy eigenvalues is found. By using this limit, the upper limits of ΔP, ΔX and $\Delta P \Delta X$ can be obtained. Then one can find the lower limits of ΔP and ΔX. On the other hand, the lower limits and the upper limit of the expectation value of the hamiltonian are studied. Therefore we have the uncertainty relation and the certainty relation for any state. To get these relations in a different way, we calculate ΔP and ΔX for the energy eigenstates and we examine their lower limits and the upper limits. After comparing these two sets of results obtained by the two methods, the most informative ones are selected. In addition, $(\varepsilon_{n+1} - \varepsilon_n)/\varepsilon_n$ and its classical limits are calculated for the different intervals of q.

In section (2.2), we study the two-dimensional q-oscillator. There are various kinds of multi-dimensional extensions of the q-oscillator. Here we will study the one which has been constructed by Pusz and Woronowicz. This construction is invariant under the quantum group $U_q(2)$. The degeneracy of the bound state energies of the ordinary oscillator is conserved. The hamiltonian for the two-dimensional q-oscillator is defined so that it has two properties which will be explained in this section. In addition to the energy eigenvalues, the spectra of $a_1^\dagger a_1$ and $a_2^\dagger a_2$ are calculated. However, in general, the same procedure is followed here as section (2.1).

In section (2.3), we study Fibonacci oscillators. A similar procedure to the one in section (2.1) is followed here. Moreover, the spectrum of the deformed number operator $a^\dagger a$ is examined by two methods. So it has two different forms.

In summary, through chapter (1), we will be interested in the importance of the subject of this thesis. Through chapter (2), we will focus on the uncertainty and certainty relations. Besides, some other important aspects of the q-oscillator or Fibonacci oscillator will be mentioned. In the end, the conclusion part will appear.

1.1. THE UNCERTAINTY RELATIONS

Here we will recall the well-known method of finding the uncertainty relations. The Heisenberg uncertainty principle is obtained by this way.

Now, let A and B be two Hermitian operators satisfying

$$[A, B] = iC. \tag{1.2}$$

To compute the uncertainty relation for A and B, we write

$$(\Delta A)^2(\Delta B)^2 = \langle\Psi|(A - \langle A\rangle)^2|\Psi\rangle\langle\Psi|(B - \langle B\rangle)^2|\Psi\rangle \tag{1.3}$$

where

$$\langle A\rangle = \langle \Psi|A|\Psi\rangle \tag{1.4}$$

and

$$\langle B\rangle = \langle \Psi|B|\Psi\rangle. \tag{1.5}$$

To simplify the notation, we define the pair

$$\hat{A} = A - \langle A\rangle \tag{1.6}$$

$$\hat{B} = B - \langle B\rangle. \tag{1.7}$$

We next use the equalities

$$\hat{A}^\dagger = \hat{A} \tag{1.8}$$

and

$$\hat{B}^\dagger = \hat{B} \tag{1.9}$$

to rewrite the equation (1.3) in terms of $\hat{A}$ and $\hat{B}$. Therefore we get

$$(\Delta A)^2(\Delta B)^2 = \langle \Psi|\hat{A}^\dagger\hat{A}|\Psi\rangle\langle \Psi|\hat{B}^\dagger\hat{B}|\Psi\rangle. \tag{1.10}$$

Evidently,

$$(\Delta A)^2(\Delta B)^2 = \langle \hat{A}\Psi|\hat{A}\Psi\rangle\langle \hat{B}\Psi|\hat{B}\Psi\rangle. \tag{1.11}$$

If we apply the Schwartz inequality which is

$$|V_1|^2|V_2|^2 \geq |\langle V_1|V_2\rangle|^2, \tag{1.12}$$

we get

$$\langle\hat{A}\Psi|\hat{A}\Psi\rangle\langle\hat{B}\Psi|\hat{B}\Psi\rangle \geq |\langle\hat{A}\Psi|\hat{B}\Psi\rangle|^2. \tag{1.13}$$

It is obvious that

$$(\Delta A)^2(\Delta B)^2 \geq |\langle\hat{A}\Psi|\hat{B}\Psi\rangle|^2. \tag{1.14}$$

substituting Eq. (1.11) in it. By using the fact that

$$\langle\hat{A}\Psi|\hat{B}\Psi\rangle = \langle\Psi|\hat{A}\hat{B}|\Psi\rangle, \tag{1.15}$$

we rewrite the above inequality as

$$(\Delta A)^2(\Delta B)^2 \geq |\langle\Psi|\hat{A}\hat{B}|\Psi\rangle|^2. \tag{1.16}$$

To be able to use the equation (1.2), let us write

$$\hat{A}\hat{B} = \frac{1}{2}[\hat{A},\hat{B}]_+ + \frac{1}{2}[\hat{A},\hat{B}]. \tag{1.17}$$

where

$$[\hat{A},\hat{B}]_+ = \hat{A}\hat{B} + \hat{B}\hat{A}. \tag{1.18}$$

Then the inequality (1.16) becomes

$$(\Delta A)^2(\Delta B)^2 \geq |\langle\Psi|\frac{1}{2}[\hat{A},\hat{B}]_+ + \frac{1}{2}[\hat{A},\hat{B}]|\Psi\rangle|^2. \tag{1.19}$$

Clearly,

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle\Psi|[\hat{A},\hat{B}]_+|\Psi\rangle + \boldsymbol{i}\langle\Psi|C|\Psi\rangle|^2. \tag{1.20}$$

Since $[\hat{A},\hat{B}]_+$ and C are hermitian operators, their eigenvalues are real. Using the fact that

$$|x+iy|^2 = x^2+y^2, \tag{1.21}$$

we get

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\langle\Psi|[\hat{A},\hat{B}]_+|\Psi\rangle^2 + \frac{1}{4}\langle\Psi|C|\Psi\rangle^2. \tag{1.22}$$

We know that the first term is certainly positive, i.e.

$$\langle\Psi|[\hat{A},\hat{B}]_+|\Psi\rangle \geq 0. \tag{1.23}$$

So we have

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\langle\Psi|C|\Psi\rangle^2. \tag{1.24}$$

Taking the square root of this inequality, we obtain

$$\Delta A\Delta B \geq \frac{1}{2}|\langle\Psi|C|\Psi\rangle| \tag{1.25}$$

for any state $|\Psi\rangle$. This is the famous uncertainty relation.

1.2. THE PHYSICAL IMPORTANCE OF THE HARMONIC OSCILLATOR

The linear harmonic oscillator is important in both classical and quantum physics. Its importance stems from the property of its potential energy and its applications to most continuous physical systems.

Firstly, let us examine the special property of the potential energy given by

$$V(x) = \frac{1}{2}mw^2x^2 \tag{1.26}$$

where m, w and x denote the mass, angular frequency and position of the harmonic oscillator respectively. This potential is very significant since any arbitrary smooth potential near a stable equilibrium position is approximately equal to it. To prove this fact, let us express $V(x)$ near x_0 as

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + V''(x_0)(x - x_0)^2 + ... \tag{1.27}$$

using the Taylor expansion. Next, let us consider x_0 as a stable equilibrium point. Thus $V(x)$ has a minimum at this point. Then we can obviously say

$$V'(x_0) = 0 \tag{1.28}$$

and

$$V''(x_0) > 0. \tag{1.29}$$

For simplicity, let us choose

$$x_0 = 0 \tag{1.30}$$

and

$$V(x_0) = 0 \tag{1.31}$$

without loss of generality. We also neglect the third and higher order terms since they approximate to zero. Finally, we have

$$V(x) = \frac{1}{2}V''(0)x^2 \tag{1.32}$$

which has the same form as Eq. (1.26). As an example, we can mention the oscillations of the atoms in a diatomic molecule here.

On the other hand, the behavior of most continuous physical systems, such as the vibrations of an elastic medium or the electromagnetic field in a cavity can be explained by the linear combination of many linear harmonic oscillators. The quantization of these physical systems are also described by the quantum mechanics of many harmonic oscillators. So that is why all modern field theories use the results of the study of the harmonic oscillation.

2. THE UNCERTAINTY RELATIONS AND THE CERTAINTY RELATIONS

2.1. THE ONE-DIMENSIONAL Q-OSCILLATOR

To begin with, let us choose the units so that

$$\hbar = 1, \tag{2.1}$$

$$m = 1 \tag{2.2}$$

and

$$w = 1. \tag{2.3}$$

In terms of the momentum and position operators, the annihilation and creation operators are defined as

$$a \equiv \frac{1}{\sqrt{2}}(X + iP) \tag{2.4}$$

and

$$a^\dagger \equiv \frac{1}{\sqrt{2}}(X - iP) \tag{2.5}$$

respectively. These are so called because they allow us to decrease and increase the energy.

The hamiltonian is defined as

$$H \equiv \frac{1}{2}P^2 + \frac{1}{2}X^2. \tag{2.6}$$

As one can easily see the hamiltonian of the q-oscillator has the same structure as the hamiltonian of the ordinary oscillator.

If we calculate the momentum and position operators in terms of the annihilation and creation operators, we have

$$P = \frac{i}{\sqrt{2}}(a^\dagger - a) \tag{2.7}$$

and

$$X = \frac{1}{\sqrt{2}}(a^\dagger + a), \tag{2.8}$$

using definitions (2.4) and (2.5). Evidently, substituting these two equations into Eq. (2.6), we get

$$H = \frac{1}{2}(aa^\dagger + a^\dagger a) \tag{2.9}$$

This is a new form of the hamiltonian in terms of the annihilation and creation operators. Now, let us use the Eq. (1.1) to rewrite the above equation as

$$H = \frac{1}{2}(1 + q^2 a^\dagger a + a^\dagger a). \tag{2.10}$$

Clearly,

$$H = (\frac{1 + q^2}{2})a^\dagger a + \frac{1}{2}. \tag{2.11}$$

This equation involves the deformed number operator alone which is defined as

$$N = a^\dagger a. \tag{2.12}$$

For $q = 1$, the hamiltonian becomes

$$H = a^\dagger a + \frac{1}{2}. \tag{2.13}$$

In terms of $aa^\dagger$, the hamiltonian changes form as

$$H = q^{-2}\{(\frac{1+q^2}{2})aa^\dagger - \frac{1}{2}\}. \tag{2.14}$$

using Eqs. (2.9), (1.1) and eliminating the term $a^\dagger a$.

We will now search for the commutation relations for the hamiltonian, the annihilation and creation operators. Our purpose here is to find the eigenvalues of the hamiltonian, i.e. the energy eigenvalues. We start by multiplying the both sides by a from the right:

$$Ha = q^{-2}\{(\frac{1+q^2}{2})aa^\dagger - \frac{1}{2}\}a. \tag{2.15}$$

From the associativity property of the matrices, it is clear that

$$Ha = q^{-2}a\{(\frac{1+q^2}{2})a^\dagger a - \frac{1}{2}\} \tag{2.16}$$

So the commutation relation for the hamiltonian and the annihilation operator is

$$Ha = q^{-2}a(H-1) \tag{2.17}$$

according to Eq. (2.11). It can be similarly shown that

$$Ha^\dagger = \{(\frac{1+q^2}{2})a^\dagger a + \frac{1}{2}\}a^\dagger \tag{2.18}$$

and

$$Ha^\dagger = a^\dagger\{(\frac{1+q^2}{2})aa^\dagger + \frac{1}{2}\}, \tag{2.19}$$

taking into account Eqs. (2.11) and (2.18) respectively. The commutation relation for the hamiltonian and the creation operator is

$$Ha^\dagger = a^\dagger(q^2 H + 1) \tag{2.20}$$

using Eq. (2.14).

We are now ready to obtain the energy eigenvalues. The eigenvalue problem is

$$H|n\rangle = \varepsilon_n|n\rangle. \tag{2.21}$$

By multiplication of Eq. (2.17) on the right with the energy eigenstate $|n\rangle$, we find

$$Ha|n\rangle = q^{-2}a(H-1)|n\rangle. \tag{2.22}$$

Evidently, we get

$$Ha|n\rangle = q^{-2}a(\varepsilon_n - 1)|n\rangle \tag{2.23}$$

and

$$H(a|n\rangle) = q^{-2}(\varepsilon_n - 1)(a|n\rangle). \tag{2.24}$$

On the other hand, we write

$$Ha^{\dagger}|n\rangle = a^{\dagger}(q^2 H + 1)|n\rangle \tag{2.25}$$

multiplying Eq. (2.20) by $|n\rangle$. Clearly, we have

$$Ha^{\dagger}|n\rangle = a^{\dagger}(q^2 \varepsilon_n + 1)|n\rangle \tag{2.26}$$

and

$$H(a^{\dagger}|n\rangle) = (q^2 \varepsilon_n + 1)(a^{\dagger}|n\rangle). \tag{2.27}$$

ε_n is an eigenvalue of H then $q^{-2}(\varepsilon_n - 1)$ and $(q^2\varepsilon_n + 1)$ are also the eigenvalues of H because they satisfy the Eq. (2.21) and correspond to different eigenstates. Let us consider

$$a|n\rangle = C_n|n-1\rangle \tag{2.28}$$

where C_n are n dependent coefficients. The recursion formula is

$$\varepsilon_{n-1} = q^{-2}\varepsilon_n - q^{-2} \tag{2.29}$$

according to Eqs. (2.24), (2.28) and (2.21). By using it, we get

$$\varepsilon_{n-2} = q^{-4}\varepsilon_n - q^{-4} - q^{-2}, \tag{2.30}$$

$$\varepsilon_{n-3} = q^{-6}\varepsilon_n - q^{-6} - q^{-4} - q^{-2} \tag{2.31}$$

and so on. Expressing these in a more contact form, we get

$$\varepsilon_{n-m} = q^{-2m}\varepsilon_n - q^{-2m}(\frac{1-q^{2m}}{1-q^2}) \tag{2.32}$$

or

$$\varepsilon_{n-m} = q^{-2m}\{\varepsilon_n - (\frac{1-q^{2m}}{1-q^2})\} \tag{2.33}$$

where $m = 0, 1, 2, 3, ...$ Now, from Eq. (2.29) we get

$$\varepsilon_{n+1} = q^2\varepsilon_n + 1. \tag{2.34}$$

So Eq. (2.27) reads

$$H(a^\dagger|n\rangle) = \varepsilon_{n+1}(a^\dagger|n\rangle). \tag{2.35}$$

One can easily show that

$$a^\dagger|n\rangle = D_n|n+1\rangle. \tag{2.36}$$

If we climb up in energy using Eq. (2.34), we have

$$\varepsilon_{n+2} = q^4\varepsilon_n + q^2 + 1, \tag{2.37}$$

$$\varepsilon_{n+3} = q^6\varepsilon_n + q^4 + q^2 + 1 \tag{2.38}$$

and so on. Again,

$$\varepsilon_{n+m} = q^{2m}\varepsilon_n + (\frac{1-q^{2m}}{1-q^2}) \tag{2.39}$$

is the generalized form of the energy eigenvalues.

In summary, we have

$$\varepsilon_{n-m} = q^{-2m}\{\varepsilon_n - (\frac{1-q^{2m}}{1-q^2})\}, \tag{2.40}$$

$$\varepsilon_{n+m} = q^{2m}\varepsilon_n + (\frac{1-q^{2m}}{1-q^2}) \tag{2.41}$$

where $m = 0, 1, 2, ...$

Let us evaluate these eigenvalues in the limit $q \to 1$. Then we get

$$\varepsilon_{n-m} = \varepsilon_n - m \tag{2.42}$$

and

$$\varepsilon_{n+m} = \varepsilon_n + m. \tag{2.43}$$

To evaluate D_n, we can write

$$a|n+1\rangle = C_{n+1}|n\rangle \tag{2.44}$$

evidently. It follows that

$$\langle n+1|a^\dagger = C^*_{n+1}\langle n|, \tag{2.45}$$

$$a^\dagger a|n+1\rangle = C_{n+1}a^\dagger|n\rangle \tag{2.46}$$

applying the creation operator to Eq. (2.44) and

$$a^\dagger a|n+1\rangle = C_{n+1}D_n|n+1\rangle \tag{2.47}$$

from Eq. (2.36). Taking the inner product with $\langle n+1|$ and exploiting the orthonormality of the basis, i.e.

$$\langle n|m\rangle = \delta_{nm} \tag{2.48}$$

and Eqs. (2.44), (2.45), we get

$$(C^*_{n+1}\langle n|)(C_{n+1}|n\rangle) = C_{n+1}D_n \tag{2.49}$$

and

$$C^*_{n+1}C_{n+1} = C_{n+1}D_n. \tag{2.50}$$

Hence we have

$$D_n = C^*_{n+1} \tag{2.51}$$

and Eq. (2.36) reads

$$a^\dagger|n\rangle = C^*_{n+1}|n+1\rangle. \tag{2.52}$$

The first recursion formula was Eq. (2.29). We will now search for its validity for the cases. We begin by recalling our fist assumption. It says that a and $a^\dagger$ are the annihilation and creation operators respectively. To testify this assumption, we must examine whether ε_n is an increasing function of n or not. If ε_n is an increasing function of n, then our assumption is true. If we find out that ε_n is a decreasing function of n, then there is a contradiction. To get rid of this contradiction, we redefine the annihilation and creation operators so that ε_n is again an increasing function of n. We start by considering

$$\varepsilon_n - \varepsilon_{n-1} = q^{-2}\{(q^2-1)\varepsilon_n + 1\} \tag{2.53}$$

from Eq. (2.29). Hence, to decide whether $\varepsilon_n - \varepsilon_{n-1}$ is positive or negative, it is sufficient to look at the term $(q^2-1)\varepsilon_n + 1$.

For $q > 1$, we have

$$q^2 - 1 > 0, \tag{2.54}$$

and then

$$(q^2-1)\varepsilon_n + 1 > 0. \tag{2.55}$$

Since

$$\varepsilon_n - \varepsilon_{n-1} > 0, \tag{2.56}$$

we can safely conclude that ε_n is an increasing function of n.

For $q = 1$, it is obvious that

$$\varepsilon_n - \varepsilon_{n-1} > 0. \tag{2.57}$$

Here again, ε_n is an increasing function of n.

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$, we get

$$(1-q^2)\varepsilon_n > 1 \tag{2.58}$$

and thus

$$(q^2 - 1)\varepsilon_n + 1 < 0. \tag{2.59}$$

Therefore we find

$$\varepsilon_n - \varepsilon_{n-1} < 0 \tag{2.60}$$

and ε_n is a decreasing function of n. It means that the role of "a" changes. It behaves as a creation operator. This situation requires to make some modifications. We will rearrange the notations and we will not accept the validity of the first assumption for this case. Let us consider b and $b^\dagger$ as our new annihilation and creation operators. They satisfy

$$b = a^\dagger \tag{2.61}$$

and

$$b^\dagger = a. \tag{2.62}$$

Let us consider

$$b|n\rangle = F_n|n-1\rangle. \tag{2.63}$$

Then we obtain

$$\varepsilon_{n-1} = q^2\varepsilon_n + 1 \tag{2.64}$$

from Eq. (2.27) and it follows that

$$\varepsilon_{n+1} = q^{-2}\varepsilon_n - q^{-2}. \tag{2.65}$$

This and Eq. (2.24) require

$$b^{\dagger}|n\rangle = G_n|n+1\rangle. \tag{2.66}$$

If we generalize the recursion formula in Eq. (2.64), we find

$$\varepsilon_{n-m} = q^{2m}\varepsilon_n + (\frac{1-q^{2m}}{1-q^2}) \tag{2.67}$$

and then we get

$$\varepsilon_{n+m} = q^{-2m}\{\varepsilon_n - (\frac{1-q^{2m}}{1-q^2})\} \tag{2.68}$$

from (2.65) where $m = 0, 1, 2, ...$

Now, we will search for the behavior of ε_n for the remaining cases.

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$, it is clear that

$$\varepsilon_n - \varepsilon_{n-1} = 0. \tag{2.69}$$

In fact, ε_n neither increases nor decreases.

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$, one can show that

$$(1-q^2)\varepsilon_n < 1 \tag{2.70}$$

and then

$$(q^2-1)\varepsilon_n + 1 > 0. \tag{2.71}$$

Hence

$$\varepsilon_n - \varepsilon_{n-1} > 0 \tag{2.72}$$

and ε_n is an increasing function of n.

We have found the eigenvalues but we must check whether they are positive or negative to be able to conclude that they are definitely the energy eigenvalues. As we know, the negative energy is not allowed in quantum physics. If we find out that the eigenvalues are negative for some value of m, then we will say that there must be a ground state such that $a|0\rangle = 0$. In this way, we will get rid of the negative eigenvalues. We begin by using the fact that ε_n is the energy measured in an experiment. So it is positive.

For the real parameter q, we have

$$q^{2m}\varepsilon_n > 0 \tag{2.73}$$

and

$$\frac{1-q^{2m}}{1-q^2} \geq 0. \tag{2.74}$$

Therefore we find

$$\varepsilon_{n+m} > 0 \tag{2.75}$$

for $q > 0$, where $m = 0, 1, 2, ...$

For $q > 1$, we obviously have

$$1 - (1-q^2)\varepsilon_n > 0. \tag{2.76}$$

So $\ln\{1-(1-q^2)\varepsilon_n\}$ is well-defined. If

$$m > \frac{\ln\{1-(1-q^2)\varepsilon_n\}}{\ln(q^2)}, \tag{2.77}$$

then

$$m\ln(q^2) > \ln\{1-(1-q^2)\varepsilon_n\} \tag{2.78}$$

and thus

$$q^{2m} > 1-(1-q^2)\varepsilon_n. \tag{2.79}$$

Clearly, we get

$$(1-q^2)\varepsilon_n - (1-q^{2m}) > 0 \tag{2.80}$$

and

$$\varepsilon_n - (\frac{1-q^{2m}}{1-q^2}) < 0. \tag{2.81}$$

Therefore we find

$$\varepsilon_{n-m} < 0. \tag{2.82}$$

So there must be a ground state.

For $q=1$, if $m > \varepsilon_n$, then

$$\varepsilon_{n-m} < 0. \tag{2.83}$$

So there must be a ground state.

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$, it is obvious that

$$\varepsilon_n - (\frac{1}{1-q^2}) > 0. \tag{2.84}$$

Thus we can write

$$\varepsilon_n - (\frac{1}{1-q^2}) + (\frac{q^{2m}}{1-q^2}) > 0 \tag{2.85}$$

and

$$\varepsilon_n - (\frac{1-q^{2m}}{1-q^2}) > 0. \tag{2.86}$$

Finally, we have

$$\varepsilon_{n-m} > 0 \tag{2.87}$$

for every m where $m = 0, 1, 2, ...$ So there is no ground state.

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$, we have

$$\varepsilon_{n+m} = \varepsilon_n \tag{2.88}$$

and

$$\varepsilon_{n-m} = \varepsilon_n \tag{2.89}$$

where $m = 0, 1, 2, ...$ It means that we have only one energy eigenvalue in this case.

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$, it is clear that

$$(1 - q^2)\varepsilon_n < 1 \tag{2.90}$$

and

$$1 - (1 - q^2)\varepsilon_n > 0. \tag{2.91}$$

Hence $\ln\{1 - (1 - q^2)\varepsilon_n\}$ is well-defined. If

$$m > \frac{\ln\{1 - (1 - q^2)\varepsilon_n\}}{\ln(q^2)}, \tag{2.92}$$

then

$$m \ln(q^2) < \ln\{1 - (1 - q^2)\varepsilon_n\}. \tag{2.93}$$

Obviously, we find

$$q^{2m} < 1 - (1 - q^2)\varepsilon_n \tag{2.94}$$

and thus

$$(1 - q^2)\varepsilon_n - (1 - q^{2m}) < 0. \tag{2.95}$$

The next step is to write

$$\varepsilon_n - (\frac{1 - q^{2m}}{1 - q^2}) < 0. \tag{2.96}$$

Evidently, we can conclude that

$$\varepsilon_{n-m} < 0 \tag{2.97}$$

for some m. So there must be a ground state.

If there is a ground state, then we consider

$$a|0\rangle = 0. \tag{2.98}$$

Here $|0\rangle$ is called the ground state of the system. To determine the energy of this state, we write

$$a^{\dagger}a|0\rangle = 0 \tag{2.99}$$

by applying the creation operator to it. Evidently, we get

$$H|0\rangle = \varepsilon_0|0\rangle. \tag{2.100}$$

We will call ε_0 the ground state energy. Inserting Eq. (2.11) into the above equation, we get

$$\{(\frac{1+q^2}{2})a^{\dagger}a + \frac{1}{2}\}|0\rangle = \varepsilon_0|0\rangle \tag{2.101}$$

and therefore

$$\frac{1}{2}|0\rangle = \varepsilon_0|0\rangle. \tag{2.102}$$

So the ground state energy is found as

$$\varepsilon_0 = \frac{1}{2}. \tag{2.103}$$

For the cases in which there must occur a ground state, the eigenvalues are computed again. For this aim, we evaluate Eqs. (2.41) and (2.43) for $n = 0$ in the following three cases. We use the above equation. Then we change the variable m to n. Here n are nonnegative integers.

For $q > 1$,

$$\varepsilon_n = \frac{1}{2}q^{2n} + (\frac{1 - q^{2n}}{1 - q^2}). \tag{2.104}$$

For $q = 1$,

$$\varepsilon_n = \frac{1}{2} + n. \tag{2.105}$$

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$\varepsilon_n = \frac{1}{2}q^{2n} + (\frac{1 - q^{2n}}{1 - q^2}). \tag{2.106}$$

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$, without loss of generality we can consider

$$\varepsilon_n = \varepsilon_0 \tag{2.107}$$

and then write n instead of m. So we have

$$\varepsilon_n = q^{-2n}\{\varepsilon_0 - (\frac{1 - q^{2n}}{1 - q^2})\} \tag{2.108}$$

where $n = 0, \pm 1, \pm 2, ...$

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$, we get

$$\varepsilon_n = \frac{1}{1 - q^2} \tag{2.109}$$

similarly where $n = 0, \pm 1, \pm 2, ...$

In summary, we have the following energy eigenvalues.

For $q > 1$,

$$\varepsilon_n = \frac{1}{2}q^{2n} + (\frac{1 - q^{2n}}{1 - q^2}) \tag{2.110}$$

where $n = 0, 1, 2, ...$

For $q = 1$,

$$\varepsilon_n = \frac{1}{2} + n \tag{2.111}$$

where $n = 0, 1, 2, ...$

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$,

$$\varepsilon_n = q^{-2n}\{\varepsilon_0 - (\frac{1 - q^{2n}}{1 - q^2})\} \tag{2.112}$$

where $n = 0, \pm 1, \pm 2, ...$

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$,

$$\varepsilon_n = \frac{1}{1 - q^2} \tag{2.113}$$

where $n = 0, \pm 1, \pm 2, ...$

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$\varepsilon_n = \frac{1}{2}q^{2n} + (\frac{1 - q^{2n}}{1 - q^2}) \tag{2.114}$$

where $n = 0, 1, 2, ...$

So far, we have obtained the energy eigenvalues. We are ready now to study the exact relation between the states.

For $q > 1$, we get

$$\{(\frac{1+q^2}{2})a^\dagger a + \frac{1}{2}\}|n\rangle = \{\frac{1}{2}q^{2n} + (\frac{1-q^{2n}}{1-q^2})\}|n\rangle \tag{2.115}$$

substituting Eqs. (2.11) and (2.110) into Eq. (2.21). If we solve this, we can easily see that

$$(\frac{1+q^2}{2})|C_n|^2 + \frac{1}{2} = \frac{1}{2}q^{2n} + (\frac{1-q^{2n}}{1-q^2}) \tag{2.116}$$

from Eq. (2.28). It follows that

$$|C_n|^2 = \frac{1-q^{2n}}{1-q^2}. \tag{2.117}$$

Conventionally, we choose C_n as real. So

$$C_n = \sqrt{\frac{1-q^{2n}}{1-q^2}}. \tag{2.118}$$

Therefore Eq. (2.28) reads

$$a|n\rangle = \sqrt{\frac{1-q^{2n}}{1-q^2}}|n-1\rangle \tag{2.119}$$

and Eq. (2.52) reads

$$a^\dagger|n\rangle = \sqrt{\frac{1-q^{2n+2}}{1-q^2}}|n+1\rangle. \tag{2.120}$$

The deformed number operator satisfies

$$a^\dagger a|n\rangle = (\frac{1-q^{2n}}{1-q^2})|n\rangle. \tag{2.121}$$

For $q = 1$, we similarly write

$$(a^\dagger a + \frac{1}{2})|n\rangle = (n + \frac{1}{2})|n\rangle \tag{2.122}$$

substituting Eqs. (2.13) and (2.111) into Eq. (2.21). Then we get

$$|C_n|^2 = n \tag{2.123}$$

and

$$C_n = \sqrt{n}. \tag{2.124}$$

Hence Eq. (2.28) reads

$$a|n\rangle = \sqrt{n}|n-1\rangle \tag{2.125}$$

and Eq. (2.52) reads

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{2.126}$$

for this case. The number operator satisfies

$$a^\dagger a|n\rangle = n|n\rangle. \tag{2.127}$$

Now, to study the uncertainty relations, let us first obtain a commutation relation for the momentum and position operators. Therefore that is the beginning of finding the uncertainty relations. For this purpose, we rewrite Eq. (1.1) by using definitions (2.4) and (2.5). In this case,

$$\frac{1}{2}(X + iP)(X - iP) - \frac{q^2}{2}(X - iP)(X + iP) = 1. \tag{2.128}$$

If we solve this equation step by step, we get

$$\frac{1}{2}(X^2 + P^2 + i[P, X]) - \frac{q^2}{2}(X^2 + P^2 - i[P, X]) = 1 \tag{2.129}$$

and then

$$(1 - q^2)(X^2 + P^2) + i(1 + q^2)[P, X] = 2. \tag{2.130}$$

In the end, we obtain

$$[P, X] = i\frac{(1 - q^2)(X^2 + P^2) - 2}{1 + q^2}. \tag{2.131}$$

Now, to compute the uncertainty relation, we write

$$\Delta P \Delta X \geq \frac{1}{2}|\langle\Psi|\frac{(1 - q^2)(X^2 + P^2) - 2}{1 + q^2}|\Psi\rangle|, \tag{2.132}$$

according to Eqs. (1.25) and (2.131). Obviously, we have

$$\Delta P \Delta X \geq |(\frac{1 - q^2}{1 + q^2})\langle\Psi|H|\Psi\rangle - (\frac{1}{1 + q^2})|. \tag{2.133}$$

First of all, let us evaluate it for the energy eigenvalues. In this case,

$$|\Psi\rangle = |n\rangle. \tag{2.134}$$

Obviously, the inequality becomes

$$\Delta P \Delta X \geq |(\frac{1 - q^2}{1 + q^2})\langle n|H|n\rangle - (\frac{1}{1 + q^2})|. \tag{2.135}$$

One can easily see that

$$\langle n|H|n\rangle = \varepsilon_n. \tag{2.136}$$

Substituting Eq. (2.110) into Eq. (2.135), we get

$$\Delta P\Delta X \geq (\frac{1}{1+q^2})|(1-q^2)\{\frac{1}{2}q^{2n} + (\frac{1-q^{2n}}{1-q^2})\} - 1|. \tag{2.137}$$

It follows

$$\Delta P\Delta X \geq (\frac{q^{2n}}{1+q^2})|(\frac{1-q^2}{2}) - 1| \tag{2.138}$$

and thus

$$\Delta P\Delta X \geq \frac{1}{2}q^{2n}. \tag{2.139}$$

Therefore we have obtained the uncertainty relation for the energy eigenvalues and for the case in which there must be a ground state. It is seen that this relation is q and state dependent. Let us evaluate this inequality for $q = 1$. Then we have

$$\Delta P\Delta X \geq \frac{1}{2}. \tag{2.140}$$

We will now search for the certainty relation. This is a new concept in quantum physics actually. This concept tells us that there must be an upper limit for the uncertainties in the momentum and position operators. We begin by examining the relation between these uncertainties and the expectation value of the hamiltonian. Meanwhile, the limit for the energy eigenvalues will be necessary.

To derive the certainty relations, let us start by analyzing uncertainties. For any observable A, the uncertainty for any state, $|\Psi\rangle$ is given by

$$\Delta A \equiv \{\langle\Psi|A^2|\Psi\rangle - (\langle\Psi|A|\Psi\rangle)^2\}^{1/2}. \tag{2.141}$$

We write the form

$$(\Delta A)^2 \equiv \langle\Psi|A^2|\Psi\rangle - (\langle\Psi|A|\Psi\rangle)^2 \tag{2.142}$$

to study it simply. As we know,

$$(\langle\Psi|A|\Psi\rangle)^2 \geq 0. \tag{2.143}$$

It follows

$$(\Delta A)^2 \leq \langle\Psi|A^2|\Psi\rangle. \tag{2.144}$$

So, using this reality, we get

$$(\Delta P)^2 \leq \langle\Psi|P^2|\Psi\rangle \tag{2.145}$$

and

$$(\Delta X)^2 \leq \langle\Psi|X^2|\Psi\rangle. \tag{2.146}$$

Next, let us write

$$\langle\Psi|P^2|\Psi\rangle + \langle\Psi|X^2|\Psi\rangle = 2\langle\Psi|H|\Psi\rangle \tag{2.147}$$

sandwiching Eq. (2.6) between $\langle\Psi|$ and $|\Psi\rangle$. Since we can write

$$\langle\Psi|X^2|\Psi\rangle = \langle\Psi|X^\dagger X|\Psi\rangle, \tag{2.148}$$

we find

$$\langle\Psi|X^2|\Psi\rangle = (X|\Psi\rangle)^\dagger (X|\Psi\rangle). \tag{2.149}$$

The right of the equation gives the square of the norm of the vector, $X|\Psi\rangle$ exactly. Hence we have

$$\langle\Psi|X^2|\Psi\rangle \geq 0. \tag{2.150}$$

Using this fact, we can obviously say

$$\langle\Psi|P^2|\Psi\rangle \leq 2\langle\Psi|H|\Psi\rangle. \tag{2.151}$$

In a similar way, one can easily show that

$$\langle\Psi|X^2|\Psi\rangle \leq 2\langle\Psi|H|\Psi\rangle \tag{2.152}$$

since

$$\langle\Psi|P^2|\Psi\rangle \geq 0. \tag{2.153}$$

Now, combining Eqs. (2.145) and (2.151), we obtain

$$(\Delta P)^2 \leq 2\langle\Psi|H|\Psi\rangle. \tag{2.154}$$

On the other hand, we find

$$(\Delta X)^2 \leq 2\langle\Psi|H|\Psi\rangle \tag{2.155}$$

combining Eqs. (2.146) and (2.152).

Here, let us first evaluate the certainty relations for the energy eigenstates. In this case, Eqs. (2.154) and (2.155) become

$$\Delta P \leq \sqrt{2\varepsilon_n} \tag{2.156}$$

and

$$\Delta X \leq \sqrt{2\varepsilon_n} \tag{2.157}$$

respectively.

In the present case, finding the limits of the energy eigenvalues ε_n is an essential task. Therefore let us study this subject. To obtain the certainty relations, we must calculate the upper limits of ε_n but for later use we also calculate the lower limits of ε_n. We will see that there is an upper limit for ε_n in the case in which $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$. Otherwise there is no upper limit for ε_n. ε_n increases as n increases and ε_n decreases as n decreases. In addition, ε_n approximates to its maximum value in the limit $n \to \infty$. We will use these facts in the following cases to determine the limits of ε_n.

For $q > 1$, ε_n takes the minimum value at $n = 0$. Since

$$\varepsilon_0 = \frac{1}{2} \tag{2.158}$$

and

$$\lim_{n\to\infty} \varepsilon_n = \infty, \tag{2.159}$$

we can obviously say

$$\varepsilon_n \geq \frac{1}{2}. \tag{2.160}$$

For $q = 1$, here again ε_n takes the minimum value at $n = 0$. Since

$$\varepsilon_0 = \frac{1}{2} \tag{2.161}$$

and

$$\lim_{n\to\infty} \varepsilon_n = \infty, \tag{2.162}$$

we can clearly see that

$$\varepsilon_n \geq \frac{1}{2}. \tag{2.163}$$

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$, ε_n approximates to its minimum value in the limit $n \to -\infty$. Since

$$\lim_{n\to-\infty} \varepsilon_n = \frac{1}{1-q^2} \tag{2.164}$$

and

$$\lim_{n\to\infty} \varepsilon_n = \infty, \tag{2.165}$$

we obtain

$$\varepsilon_n > \frac{1}{1-q^2} \tag{2.166}$$

evidently.

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$, we have only one energy eigenvalue which is

$$\varepsilon_n = \frac{1}{1-q^2}. \tag{2.167}$$

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$, ε_n takes the minimum value at $n = 0$. Using the fact

$$\varepsilon_0 = \frac{1}{2} \tag{2.168}$$

and

$$\lim_{n\to\infty} \varepsilon_n = \frac{1}{1-q^2}, \tag{2.169}$$

we conclude

$$\frac{1}{2} \leq \varepsilon_n < \frac{1}{1-q^2}. \tag{2.170}$$

In summary, we have the following relations.

For $q > 1$,

$$\varepsilon_n \geq \frac{1}{2}. \tag{2.171}$$

For q=1,

$$\varepsilon_n \geq \frac{1}{2}. \tag{2.172}$$

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$,

$$\varepsilon_n > \frac{1}{1-q^2}. \tag{2.173}$$

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$,

$$\varepsilon_n = \frac{1}{1-q^2}. \tag{2.174}$$

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$\frac{1}{2} \leq \varepsilon_n < \frac{1}{1-q^2}. \tag{2.175}$$

What we have just done is finding the limits of ε_n. As we mentioned before, we will use the results of this study to obtain the certainty relations. We will examine only the case in which $0 < q < 1$ and $\varepsilon < \frac{1}{1-q^2}$ because it is the unique case in which ε_n has a finite upper limit. We will combine this upper limit and Eqs. (2.156) and (2.157). So we get

$$\Delta P < \sqrt{\frac{2}{1-q^2}}, \tag{2.176}$$

$$\Delta X < \sqrt{\frac{2}{1-q^2}} \tag{2.177}$$

and thus

$$\Delta P \Delta X < \frac{2}{1-q^2}. \tag{2.178}$$

It follows that

$$\frac{1}{2} q^{2n} \frac{1}{\Delta P} \leq \Delta X. \tag{2.179}$$

Combining this and Eq. (2.177), we have

$$\frac{1}{2}q^{2n}\frac{1}{\Delta P} < \sqrt{\frac{2}{1-q^2}} \tag{2.180}$$

and

$$\Delta P > q^{2n}\sqrt{\frac{1-q^2}{8}}. \tag{2.181}$$

Similarly,

$$\Delta X > q^{2n}\sqrt{\frac{1-q^2}{8}}. \tag{2.182}$$

So we have

$$q^{2n}\sqrt{\frac{1-q^2}{8}} < \Delta P < \sqrt{\frac{2}{1-q^2}} \tag{2.183}$$

and

$$q^{2n}\sqrt{\frac{1-q^2}{8}} < \Delta X < \sqrt{\frac{2}{1-q^2}} \tag{2.184}$$

in summary.

We will now summarize the uncertainty and certainty relations for the energy eigenstates that we have obtained by the first method.

The uncertainty and certainty relations for the momentum and position:

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$q^{2n}\sqrt{\frac{1-q^2}{8}} < \Delta P, \Delta X < \sqrt{\frac{2}{1-q^2}}. \tag{2.185}$$

The uncertainty and certainty relations for $\Delta P \Delta X$:

For $q > 1$,

$$\Delta P \Delta X \geq \frac{1}{2} q^{2n}. \tag{2.186}$$

For $q = 1$,

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.187}$$

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$\frac{1}{2} q^{2n} \leq \Delta P \Delta X < \frac{2}{1-q^2}. \tag{2.188}$$

Up to this point, we have studied the uncertainty and certainty relations for the energy eigenstates. From now on, we will discuss the uncertainty and certainty relations for any state $|\Psi\rangle$.

As we know, any state can be expressed as

$$|\Psi\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \tag{2.189}$$

where C_n satisfies

$$\sum_{n=0}^{\infty} |C_n|^2 = 1. \tag{2.190}$$

Let us write

$$\langle\Psi|H|\Psi\rangle = (\sum_{m=0}^{\infty} C_m^* \langle m|)H(\sum_{n=0}^{\infty} C_n|n\rangle) \tag{2.191}$$

for later use. Exploiting the orthonormality of the basis, we obtain

$$\langle\Psi|H|\Psi\rangle = \sum_{n=0}^{\infty} |C_n|^2 \varepsilon_n. \tag{2.192}$$

In our calculations, we will use the limits of $\langle\Psi|H|\Psi\rangle$ actually. So our first task is to evaluate these limits. Let us treat separately the five cases.

For $q > 1$, we get

$$\sum_{n=0}^{\infty} |C_n|^2 \varepsilon_n \geq \sum_{n=0}^{\infty} |C_n|^2 \frac{1}{2}, \tag{2.193}$$

if we multiply Eq. (2.171) by $|C_n|^2$ and then sum over all terms. It follows that

$$\langle\Psi|H|\Psi\rangle \geq \frac{1}{2} \tag{2.194}$$

from the last equation and Eqs. (2.190), (2.192).

For $q = 1$, similarly one can find

$$\langle\Psi|H|\Psi\rangle \geq \frac{1}{2} \tag{2.195}$$

from Eq. (2.172).

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$, we have

$$\sum_{n=0}^{\infty} |C_n|^2 \varepsilon_n > \sum_{n=0}^{\infty} |C_n|^2 (\frac{1}{1-q^2}) \tag{2.196}$$

and then

$$\langle\Psi|H|\Psi\rangle > \frac{1}{1-q^2} \tag{2.197}$$

using Eq. (2.173).

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$, we have

$$\langle\Psi|H|\Psi\rangle = \frac{1}{1-q^2} \tag{2.198}$$

expressly.

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$, we obtain

$$\sum_{n=0}^{\infty}|C_n|^2\frac{1}{2} \leq \sum_{n=0}^{\infty}|C_n|^2\varepsilon_n < \sum_{n=0}^{\infty}|C_n|^2(\frac{1}{1-q^2}) \tag{2.199}$$

and thus

$$\frac{1}{2} \leq \langle\Psi|H|\Psi\rangle < \frac{1}{1-q^2} \tag{2.200}$$

using Eq. (2.175).

In summary, we have the following relations.

For $q > 1$,

$$\langle\Psi|H|\Psi\rangle \geq \frac{1}{2}. \tag{2.201}$$

For $q = 1$,

$$\langle\Psi|H|\Psi\rangle \geq \frac{1}{2}. \tag{2.202}$$

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$,

$$\langle\Psi|H|\Psi\rangle > \frac{1}{1-q^2}. \tag{2.203}$$

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$,

$$\langle\Psi|H|\Psi\rangle = \frac{1}{1-q^2}. \tag{2.204}$$

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$\frac{1}{2} \leq \langle\Psi|H|\Psi\rangle < \frac{1}{1-q^2}. \tag{2.205}$$

We have obtained the limits of $\langle\Psi|H|\Psi\rangle$. Hereafter, we will evaluate Eqs. (2.133), (2.154) and (2.155) for any state.

For $q > 1$, we write

$$(\frac{1-q^2}{1+q^2})\langle\Psi|H|\Psi\rangle - (\frac{1}{1+q^2}) \leq -\frac{1}{2} \tag{2.206}$$

multiplying Eq. (2.201) by $\frac{1-q^2}{1+q^2}$ and subtracting $\frac{1}{1+q^2}$ from it. Then we get

$$|(\frac{1-q^2}{1+q^2})\langle\Psi|H|\Psi\rangle - (\frac{1}{1+q^2})| \geq \frac{1}{2} \tag{2.207}$$

expressly. So, combining this equation and Eq. (2.133), we can conclude that

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.208}$$

For $q = 1$, we obtain

$$\Delta P \Delta X \geq \frac{1}{2} \tag{2.209}$$

in the same way.

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$, we write

$$|(\frac{1-q^2}{1+q^2})\langle\Psi|H|\Psi\rangle - (\frac{1}{1+q^2})| > 0 \tag{2.210}$$

taking into account Eq. (2.203). So it is obvious that

$$\Delta P \Delta X > 0 \tag{2.211}$$

from Eqs. (2.210) and (2.133).

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$, it is clear that

$$|(\frac{1-q^2}{1+q^2})\langle\Psi|H|\Psi\rangle - (\frac{1}{1+q^2})| = 0. \tag{2.212}$$

From it and Eq. (2.133), we have

$$\Delta P \Delta X \geq 0. \tag{2.213}$$

Finally, for $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$, one can easily show that

$$-\frac{1}{2} \leq (\frac{1-q^2}{1+q^2})\langle\Psi|H|\Psi\rangle - (\frac{1}{1+q^2}) < 0 \tag{2.214}$$

by using Eq. (2.205). It follows that

$$|(\frac{1-q^2}{1+q^2})\langle\Psi|H|\Psi\rangle - (\frac{1}{1+q^2})| > 0. \tag{2.215}$$

As a result, we find

$$\Delta P \Delta X > 0 \tag{2.216}$$

by combining Eqs. (2.215) and (2.133).

Our next task is to find the certainty relations for any state $|\Psi\rangle$. To get the certainty relations, let us proceed as follows.

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$, we can see that

$$\Delta P < \sqrt{\frac{2}{1-q^2}} \tag{2.217}$$

according to Eqs. (2.154) and (2.205). Similarly, we get

$$\Delta X < \sqrt{\frac{2}{1-q^2}} \tag{2.218}$$

from Eqs. (2.155) and (2.205). Furthermore, we obtain

$$\Delta P \Delta X < \frac{2}{1-q^2}. \tag{2.219}$$

We will now summarize the uncertainty and certainty relations for any state Ψ.

The certainty relations for the momentum and position:

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$\Delta P, \Delta X < \sqrt{\frac{2}{1-q^2}}. \tag{2.220}$$

The uncertainty and certainty relations for $\Delta P \Delta X$:

For $q > 1$,

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.221}$$

For $q = 1$,

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.222}$$

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$,

$$\Delta P \Delta X > 0. \tag{2.223}$$

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$,

$$\Delta P \Delta X \geq 0. \tag{2.224}$$

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$0 < \Delta P \Delta X < \frac{2}{1-q^2}. \tag{2.225}$$

We know that the Heisenberg uncertainty principle provides a lower bound on the product of the uncertainties in the momentum and position. In other words, it requires

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.226}$$

Let us assume that it is also valid for this case. Then we get

$$\Delta X \geq \frac{1}{2(\Delta P)} \tag{2.227}$$

from the above inequality. Therefore this equation and Eq. (2.220) imply

$$\frac{1}{2(\Delta P)} < \sqrt{\frac{2}{1-q^2}} \tag{2.228}$$

and

$$\Delta P > \sqrt{\frac{1-q^2}{8}}. \tag{2.229}$$

In a similar way, one can prove that

$$\Delta X > \sqrt{\frac{1-q^2}{8}} \tag{2.230}$$

using Eqs. (2.226) and (2.220). As a consequence, we call these two equations the further uncertainty relations.

Now, we have come to the other part of this section. This time, we will work for only the energy eigenstates. In order to get the uncertainty and certainty relations, we will apply another method. We will first evaluate ΔP and ΔX for the energy eigenstates exactly. Then we will find their limits. In this way, we will reach the uncertainty and certainty relations.

We begin by writing

$$(\Delta P)^2 = \langle n|P^2|n\rangle - (\langle n|P|n\rangle)^2. \tag{2.231}$$

By using the expression of the momentum in Eq. (2.7), we rewrite Eq. (2.231) as

$$(\Delta P)^2 = -\frac{1}{2}\langle n|(a^\dagger)^2 + a^2 - a^\dagger a - aa^\dagger|n\rangle + \frac{1}{2}(\langle n|a^\dagger - a|n\rangle)^2. \tag{2.232}$$

Since

$$\langle n|(a^\dagger)^2|n\rangle = 0, \tag{2.233}$$

$$\langle n|a^2|n\rangle = 0, \tag{2.234}$$

$$\langle n|(a^\dagger)|n\rangle = 0 \tag{2.235}$$

and

$$\langle n|a|n\rangle = 0 \tag{2.236}$$

according to the orthonormality of the basis, Eq. (2.232) reads

$$(\Delta P)^2 = \langle n|\frac{1}{2}(aa^\dagger + a^\dagger a)|n\rangle. \tag{2.237}$$

From Eq. (2.9), one can show that

$$(\Delta P)^2 = \langle n|H|n\rangle. \tag{2.238}$$

It follows that

$$\Delta P = \sqrt{\varepsilon_n}. \tag{2.239}$$

Following a similar way, we write

$$(\Delta X)^2 = \langle n|X^2|n\rangle - (\langle n|X|n\rangle)^2. \tag{2.240}$$

Inserting Eq. (2.8) into it, we get

$$(\Delta X)^2 = \frac{1}{2}\langle n|(a^\dagger)^2 + a^2 + a^\dagger a + aa^\dagger|n\rangle - \frac{1}{2}(\langle n|a^\dagger + a|n\rangle)^2. \tag{2.241}$$

Again, using Eqs. (2.233)-(2.236), we obtain

$$(\Delta X)^2 = \langle n|\frac{1}{2}(aa^\dagger + a^\dagger a)|n\rangle. \tag{2.242}$$

It can be easily seen that this expression is the same as Eq. (2.237). Hence we can conclude that

$$\Delta X = \sqrt{\varepsilon_n}. \tag{2.243}$$

In summary, we have

$$\Delta P = \sqrt{\varepsilon_n}, \tag{2.244}$$

$$\Delta X = \sqrt{\varepsilon_n}. \tag{2.245}$$

At this stage, the limits of the energy eigenvalues ε_n are necessary to complete our study. So Eqs. (2.171)-(2.175) play a crucial role here. Taking into account these equations, we have the following relations that are for only the energy eigenstates and obtained by the second method.

The uncertainty and certainty relations for the momentum and position:

For $q > 1$,

$$\Delta P, \Delta X \geq \frac{1}{\sqrt{2}}. \tag{2.246}$$

For $q = 1$,

$$\Delta P, \Delta X \geq \frac{1}{\sqrt{2}}. \tag{2.247}$$

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$,

$$\Delta P, \Delta X > \frac{1}{\sqrt{1-q^2}}. \tag{2.248}$$

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$,

$$\Delta P, \Delta X = \frac{1}{\sqrt{1-q^2}}. \tag{2.249}$$

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$\frac{1}{\sqrt{2}} \leq \Delta P, \Delta X < \frac{1}{\sqrt{1-q^2}}. \tag{2.250}$$

The uncertainty and certainty relations for $\Delta P \Delta X$:

For $q > 1$,

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.251}$$

For $q = 1$,

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.252}$$

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$,

$$\Delta P \Delta X > \frac{1}{1-q^2}. \tag{2.253}$$

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$,

$$\Delta P \Delta X = \frac{1}{1-q^2}. \tag{2.254}$$

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$\frac{1}{2} \leq \Delta P \Delta X < \frac{1}{1-q^2}. \tag{2.255}$$

In the present case, we will make a comparison between Eqs. (2.185)-(2.188) and Eqs. (2.246)-(2.255).

Taking the following relations into consideration will help us make the comparison. While

$$\frac{1}{2}q^n \geq \frac{1}{2} \tag{2.256}$$

for $q > 1$,

$$\frac{1}{2}q^n \leq \frac{1}{2} \tag{2.257}$$

for $0 < q < 1$.

As a consequence, we select the ones which give the most information. However, we must keep in mind that these are the relations for only the energy eigenstates. Here comes the final results.

The uncertainty and certainty relations for the momentum and position:

For $q > 1$,

$$\Delta P, \Delta X \geq \frac{1}{\sqrt{2}}. \tag{2.258}$$

For $q = 1$,

$$\Delta P, \Delta X \geq \frac{1}{\sqrt{2}}. \tag{2.259}$$

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$,

$$\Delta P, \Delta X > \frac{1}{\sqrt{1-q^2}}. \tag{2.260}$$

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$,

$$\Delta P, \Delta X = \frac{1}{\sqrt{1-q^2}}. \tag{2.261}$$

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$\frac{1}{\sqrt{2}} \leq \Delta P, \Delta X < \frac{1}{\sqrt{1-q^2}}. \tag{2.262}$$

The uncertainty and certainty relations for $\Delta P \Delta X$:

For $q > 1$,

$$\Delta P \Delta X \geq \frac{1}{2} q^n. \tag{2.263}$$

For $q = 1$,

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.264}$$

For $0 < q < 1$ and $\varepsilon_n > \frac{1}{1-q^2}$,

$$\Delta P \Delta X > \frac{1}{1-q^2}. \tag{2.265}$$

For $0 < q < 1$ and $\varepsilon_n = \frac{1}{1-q^2}$,

$$\Delta P \Delta X = \frac{1}{1-q^2}. \tag{2.266}$$

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$\frac{1}{2} \leq \Delta P \Delta X < \frac{1}{1-q^2}. \tag{2.267}$$

Now, we wander the behavior of $(\varepsilon_{n+1} - \varepsilon_n)/\varepsilon_n$ in the classical limit $n \to \infty$ for the cases in which there is a ground state. For this aim, we write

$$\frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_n} = \frac{\frac{1}{2}q^{2n+2} + (\frac{1-q^{2n+2}}{1-q^2}) - \frac{1}{2}q^{2n} - (\frac{1-q^{2n}}{1-q^2})}{\frac{1}{2}q^{2n} + (\frac{1-q^{2n}}{1-q^2})} \tag{2.268}$$

using Eq. (2.110) or Eq. (2.114) which are the same equations. Cancelling some terms, we get

$$\frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_n} = \frac{(\frac{1+q^2}{2})q^{2n}}{\frac{1}{2}q^{2n} + (\frac{1-q^{2n}}{1-q^2})} \tag{2.269}$$

and

$$\frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_n} = \frac{(\frac{1+q^2}{2})}{\frac{1}{2} + (\frac{q^{-2n}-1}{1-q^2})} \tag{2.270}$$

but now in another form. Let us study for the cases separately:

For $q > 1$,

$$\lim_{n\to\infty} \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_n} = q^2 - 1 \tag{2.271}$$

by using the form in Eq. (2.270). If we evaluate this limit for $q \geq \sqrt{2}$, then we have

$$\lim_{n\to\infty} \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_n} > 1. \tag{2.272}$$

This tells us that

$$\varepsilon_{n+1} > 2\varepsilon_n \tag{2.273}$$

for $q \geq \sqrt{2}$ in the limit $n \to \infty$. Hence we can easily comment that the classical limit of this quantity is unreasonable when q is around $\sqrt{2}$ or greater than it.

For $q = 1$,

$$\lim_{n\to\infty} \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_n} = 0. \tag{2.274}$$

It is seen that in the classical limit, the continuity of energy is satisfied.

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$,

$$\lim_{n\to\infty} \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_n} = 0 \tag{2.275}$$

by using the form in Eq. (2.269). Thus the continuity of energy is also satisfied in the classical limit for this case.

Hence this is the end of this section. In the next section, we will work in two dimension.

2.2. THE TWO-DIMENSIONAL Q-OSCILLATOR

The easy method to construct a two-dimensional q-oscillator is to take two commuting copies of the q-oscillator[26]. Then we can write

$$a_1 = a \otimes I \tag{2.276}$$

$$a_2 = I \otimes a \tag{2.277}$$

so that one has the commutation relations

$$a_i a_i^\dagger - q^2 a_i^\dagger a_i = 1 \tag{2.278}$$

where $i = 1, 2$ and

$$a_i a_j = a_j a_i, \tag{2.279}$$

$$a_i a_j^\dagger = a_j^\dagger a_i \tag{2.280}$$

where $i \neq j$.

However a more important construction has been discovered by Pusz and Woronowicz and has the property of being invariant under the quantum group $U_q(2)$[22] and having a degenerate spectrum for the total q-deformed number operator by

$$N_q = a_1^\dagger a_1 + a_2^\dagger a_2. \tag{2.281}$$

These commutation relations in this case are given by

$$a_2 a_1 - q^{-1} a_1 a_2 = 0, \tag{2.282}$$

$$a_1 a_2^\dagger - q a_2^\dagger a_1 = 0, \tag{2.283}$$

$$a_1 a_1^\dagger - q^2 a_1^\dagger a_1 = 1 \tag{2.284}$$

and

$$a_2 a_2^\dagger - q^2 a_2^\dagger a_2 = a_1 a_1^\dagger - a_1^\dagger a_1. \tag{2.285}$$

We can define the annihilation and creation operators in terms of the hermitean momentum operators P_k and position operators X_k by

$$a_k \equiv \frac{1}{\sqrt{2}}(X_k + iP_k) \tag{2.286}$$

and

$$a_k^\dagger \equiv \frac{1}{\sqrt{2}}(X_k - iP_k). \tag{2.287}$$

The hamiltonian is defined as

$$H = \frac{1}{2}(P_1^2 + X_1^2) + \frac{1}{2}(\frac{1+q^2}{2})(P_2^2 + X_2^2). \tag{2.288}$$

This definition satisfies two conditions. The first condition is that when the hamiltonian is reduced to one-dimension it should be the familiar non-deformed harmonic oscillator hamiltonian which is $\frac{1}{2}(P_2^2 + X_2^2)$. The second condition is that when expressed in terms of the creation and annihilation operators the hamiltonian should be a linear function of the deformed number operator (2.281). This ensures that the spectrum of the hamiltonian is also degenerate. As we will show, the factor $\frac{1+q^2}{2}$ in this equation ensures that this condition is satisfied.

Now, let us calculate the momentum and position operators in terms of the annihilation and creation operators. For this aim, let us use the definitions in Eqs.

(2.286) and (2.287) to write

$$P_k = \frac{i}{\sqrt{2}}(a_k^\dagger - a_k) \tag{2.289}$$

and

$$X_k = \frac{1}{\sqrt{2}}(a_k^\dagger + a_k) \tag{2.290}$$

where $k = 1, 2$. If we write these equations more explicitly, we get

$$P_1 = \frac{i}{\sqrt{2}}(a_1^\dagger - a_1), \tag{2.291}$$

$$X_1 = \frac{1}{\sqrt{2}}(a_1^\dagger + a_1), \tag{2.292}$$

$$P_2 = \frac{i}{\sqrt{2}}(a_2^\dagger - a_2) \tag{2.293}$$

and

$$X_2 = \frac{1}{\sqrt{2}}(a_2^\dagger + a_2). \tag{2.294}$$

Reexpressing the hamiltonian in terms of the creation and annihilation operators, we have

$$H = \frac{1}{2}(a_1 a_1^\dagger + a_1^\dagger a_1) + \frac{1}{2}(\frac{1+q^2}{2})(a_2 a_2^\dagger + a_2^\dagger a_2) \tag{2.295}$$

using Eqs. (2.291)-(2.294). If we use the commutation relations in Eqs. (2.284) and (2.285), we get another form of the hamiltonian. Thus we write

$$H = \frac{1}{2}(1 + q^2 a_1^\dagger a_1 + a_1^\dagger a_1) + \frac{1}{2}(\frac{1+q^2}{2})(1 + q^2 a_1^\dagger a_1 - a_1^\dagger a_1 + q^2 a_2^\dagger a_2 + a_2^\dagger a_2). \tag{2.296}$$

If we rearrange it, we get

$$H = (\frac{1+q^2}{2})a_1^\dagger a_1 + \frac{1}{2}(\frac{1+q^2}{2})(q^2-1)a_1^\dagger a_1 + (\frac{1+q^2}{2})^2 a_2^\dagger a_2 + \frac{1}{2} + \frac{1}{2}(\frac{1+q^2}{2}), \quad (2.297)$$

$$H = (\frac{1+q^2}{2})^2(a_1^\dagger a_1 + a_2^\dagger a_2) + \frac{1}{2}\{1 + (\frac{1+q^2}{2})\} \quad (2.298)$$

and therefore

$$H = (\frac{1+q^2}{2})^2(a_1^\dagger a_1 + a_2^\dagger a_2) + (\frac{3+q^2}{4}). \quad (2.299)$$

This form of the hamiltonian is more compact and involves only the total q-deformed number operator in Eq. (2.281). Here we can also define the q-deformed number operators as

$$N_1 = a_1^\dagger a_1 \quad (2.300)$$

and

$$N_2 = a_2^\dagger a_2. \quad (2.301)$$

For $q = 1$, the hamiltonian becomes

$$H = a_1^\dagger a_1 + a_2^\dagger a_2 + 1, \quad (2.302)$$

as we expect.

Hereafter we will find the commutation relations for the hamiltonian, the annihilation and creation operators because we will use them to obtain the eigenvalues of the hamiltonian. The eigenvalue problem that we want to solve is

$$H|n_1n_2\rangle = \varepsilon_{n_1,n_2}|n_1n_2\rangle. \tag{2.303}$$

Let us begin by multiplying Eq. (2.299) on the right by a_1. So we write

$$Ha_1 = (\frac{1+q^2}{2})^2(a_1^\dagger a_1 + a_2^\dagger a_2)a_1 + (\frac{3+q^2}{4})a_1. \tag{2.304}$$

Using Eqs. (2.282)-(2.284), it becomes

$$Ha_1 = (\frac{1+q^2}{2})^2(\frac{a_1a_1^\dagger a_1 - a_1}{q^2} + q^{-2}a_1a_2^\dagger a_2) + (\frac{3+q^2}{4})a_1. \tag{2.305}$$

We next obtain

$$Ha_1 = (\frac{1+q^2}{2})^2 q^{-2}a_1(a_1^\dagger a_1 + a_2^\dagger a_2) - (\frac{1+q^2}{2})^2 q^{-2}a_1 + (\frac{3+q^2}{4})a_1 \tag{2.306}$$

from the associativity property of the matrices. At this point, we must use the definition of the hamiltonian in Eq. (2.299) because we want to find the relation between Ha_1 and a_1H. Hence we get

$$Ha_1 = q^{-2}a_1\{H - (\frac{3+q^2}{4})\} - (\frac{1+q^2}{2})^2 q^{-2}a_1 + (\frac{3+q^2}{4})a_1. \tag{2.307}$$

If we rearrange it, then we have

$$Ha_1 = q^{-2}a_1H + \{(\frac{3+q^2}{4})(q^2-1)q^{-2} - (\frac{1+q^2}{2})^2 q^{-2}\}a_1 \tag{2.308}$$

and

$$Ha_1 = q^{-2}a_1H + \frac{1}{4}q^{-2}(q^4 + 2q^2 - 3 - q^4 - 2q^2 - 1)a_1. \tag{2.309}$$

Therefore the commutation relation for the hamiltonian and the annihilation operator a_1 is

$$Ha_1 = q^{-2}a_1(H-1). \tag{2.310}$$

As we mentioned before, we will use it to obtain the energy eigenvalues. By multiplication of the above equation on the right with $|n_1n_2\rangle$, one can easily find

$$Ha_1|n_1n_2\rangle = q^{-2}a_1(H-1)|n_1n_2\rangle. \tag{2.311}$$

From Eq. (2.303), it is obvious that

$$Ha_1|n_1n_2\rangle = q^{-2}a_1(\varepsilon_{n_1,n_2}-1)|n_1n_2\rangle. \tag{2.312}$$

Then we can rearrange it as

$$H(a_1|n_1n_2\rangle) = q^{-2}(\varepsilon_{n_1,n_2}-1)(a_1|n_1n_2\rangle). \tag{2.313}$$

As one can easily see, the hamiltonian has another eigenstate with another eigenvalue. Let us define this state as

$$a_1|n_1n_2\rangle = C_{n_1,n_2}|n_1-1,n_2\rangle \tag{2.314}$$

where C_{n_1,n_2} are n_1 and n_2 dependent coefficients. Inserting it into Eq. (2.313), we have

$$\varepsilon_{n_1-1,n_2} = q^{-2}(\varepsilon_{n_1,n_2}-1). \tag{2.315}$$

This is a recursion formula. To deduce its general form from it, we write

$$\varepsilon_{n_1-2,n_2} = q^{-2}(\varepsilon_{n_1-1,n_2}-1). \tag{2.316}$$

It is clear that

$$\varepsilon_{n_1-2,n_2} = q^{-4}\{\varepsilon_{n_1,n_2} - (1+q^2)\}. \tag{2.317}$$

We continue by writing

$$\varepsilon_{n_1-3,n_2} = q^{-4}\{\varepsilon_{n_1-1,n_2} - (1+q^2)\} \tag{2.318}$$

and then

$$\varepsilon_{n_1-3,n_2} = q^{-6}\{\varepsilon_{n_1,n_2} - \{1+q^2+(q^2)^2\}\}. \tag{2.319}$$

From Eqs. (2.315), (2.317) and (2.319), we can conclude that the general form is that

$$\varepsilon_{n_1-m_1,n_2} = (q^{-2})^{m_1}\{\varepsilon_{n_1,n_2} - \{1+q^2+(q^2)^2+...+(q^2)^{m_1-1}\}\}. \tag{2.320}$$

To express it more compactly, we write

$$\varepsilon_{n_1-m_1,n_2} = q^{-2m_1}\{\varepsilon_{n_1,n_2} - (\frac{1-q^{2m_1}}{1-q^2})\}. \tag{2.321}$$

This time, let us multiply Eq. (2.299) on the right by $a_1^\dagger$. Then we get

$$Ha_1^\dagger = (\frac{1+q^2}{2})^2(a_1^\dagger a_1 + a_2^\dagger a_2)a_1^\dagger + (\frac{3+q^2}{4})a_1^\dagger \tag{2.322}$$

obviously. By using Eqs. (2.282) and (2.283), one can easily show that

$$Ha_1^\dagger = (\frac{1+q^2}{2})^2(a_1^\dagger a_1 a_1^\dagger + q^2 a_1^\dagger a_2^\dagger a_2) + (\frac{3+q^2}{4})a_1^\dagger. \tag{2.323}$$

Let us rewrite it as

$$Ha_1^\dagger = (\frac{1+q^2}{2})^2 a_1^\dagger(1 + q^2 a_1^\dagger a_1 + q^2 a_2^\dagger a_2) + (\frac{3+q^2}{4})a_1^\dagger \tag{2.324}$$

using Eq. (2.284). Then it becomes

$$Ha_1^\dagger = (\frac{1+q^2}{2})^2 q^2 a_1^\dagger(a_1^\dagger a_1 + a_2^\dagger a_2) + (\frac{1+q^2}{2})^2 a_1^\dagger + (\frac{3+q^2}{4})a_1^\dagger. \tag{2.325}$$

We can now use Eq. (2.299) to obtain the relation between $Ha_1^\dagger$ and $a_1^\dagger H$. So we have

$$Ha_1^\dagger = q^2 a_1^\dagger\{H - (\frac{3+q^2}{4})\} + \frac{1}{4}\{(1+q^2)^2 + q^2 + 3\}a_1^\dagger \tag{2.326}$$

and

$$Ha_1^\dagger = q^2 a_1^\dagger H + a_1^\dagger. \tag{2.327}$$

As a result, the second commutation relation is

$$Ha_1^\dagger = a_1^\dagger(q^2 H + 1). \tag{2.328}$$

Again we will use it to find the rest of the energy eigenvalues. If we multiply it on the right with $|n_1 n_2\rangle$, we write

$$Ha_1^\dagger|n_1 n_2\rangle = a_1^\dagger(q^2 H + 1)|n_1 n_2\rangle. \tag{2.329}$$

Then it becomes

$$H(a_1^\dagger|n_1 n_2\rangle) = (q^2\varepsilon_{n_1,n_2} + 1)(a_1^\dagger|n_1 n_2\rangle) \tag{2.330}$$

from Eq. (2.303). The next step is to change the form of the recursion formula in Eq.

(2.315) as

$$\varepsilon_{n_1+1,n_2} = q^2\varepsilon_{n_1,n_2} + 1. \tag{2.331}$$

If we use it, Eq. (2.330) reads

$$H(a_1^\dagger|n_1n_2\rangle) = \varepsilon_{n_1+1,n_2}(a_1^\dagger|n_1n_2\rangle). \tag{2.332}$$

Hence it is evident that

$$a_1^\dagger|n_1n_2\rangle = D_{n_1,n_2}|n_1+1,n_2\rangle. \tag{2.333}$$

If we climb up in energy using it, we obtain

$$\varepsilon_{n_1+2,n_2} = q^2\varepsilon_{n_1+1,n_2} + 1. \tag{2.334}$$

Substituting Eq. (2.331) into it, we find

$$\varepsilon_{n_1+2,n_2} = q^4\varepsilon_{n_1,n_2} + (1+q^2). \tag{2.335}$$

Similarly, we get

$$\varepsilon_{n_1+3,n_2} = q^4\varepsilon_{n_1+1,n_2} + (1+q^2) \tag{2.336}$$

and then

$$\varepsilon_{n_1+3,n_2} = q^6\varepsilon_{n_1,n_2} + \{1+q^2+(q^2)^2\}. \tag{2.337}$$

As a consequence, we get the generalized form as

$$\varepsilon_{n_1+m_1,n_2} = (q^2)^{m_1}\varepsilon_{n_1,n_2} + \{1+q^2+(q^2)^2+...+(q^2)^{m_1-1}\}. \tag{2.338}$$

If we rewrite it in a more compact way, we have

$$\varepsilon_{n_1+m_1,n_2} = q^{2m_1}\varepsilon_{n_1,n_2} + (\frac{1-q^{2m_1}}{1-q^2}). \tag{2.339}$$

To continue our study of the energy eigenvalues, let us now multiply Eq. (2.299) on the right by a_2. Then we write

$$Ha_2 = (\frac{1+q^2}{2})^2(a_1^\dagger a_1 + a_2^\dagger a_2)a_2 + (\frac{3+q^2}{4})a_2. \tag{2.340}$$

After that we use Eqs. (2.282)-(2.285) to write

$$Ha_2 = (\frac{1+q^2}{2})^2\{a_2a_1^\dagger a_1 + q^{-2}\{(1-q^2)a_1^\dagger a_1 + a_2a_2^\dagger - 1\}a_2\} + (\frac{3+q^2}{4})a_2 \tag{2.341}$$

and then

$$Ha_2 = (\frac{1+q^2}{2})^2\{a_2a_1^\dagger a_1 + q^{-2}a_2\{(1-q^2)a_1^\dagger a_1 + a_2^\dagger a_2 - 1\}\} + (\frac{3+q^2}{4})a_2. \tag{2.342}$$

If we rearrange it so that it involves the total q-deformed number operator, we have

$$Ha_2 = (\frac{1+q^2}{2})^2q^{-2}a_2(a_1^\dagger a_1 + a_2^\dagger a_2) - (\frac{1+q^2}{2})^2q^{-2}a_2 + (\frac{3+q^2}{4})a_2. \tag{2.343}$$

We again use Eq. (2.299) to rewrite it as

$$Ha_2 = q^{-2}a_2\{H - (\frac{3+q^2}{4})\} + \frac{1}{4}\{-q^{-2}(1+q^2)^2 + 3 + q^2\}a_2. \tag{2.344}$$

If we calculate this equation, we obtain

$$Ha_2 = q^{-2}a_2H + \frac{1}{4}\{-q^{-2}(1+q^2)^2 + 3 + q^2 - q^{-2}(3+q^2)\}a_2. \tag{2.345}$$

In the end, we find the third commutation relation as

$$Ha_2 = q^{-2} a_2 (H - 1). \tag{2.346}$$

To continue, we multiply it on the right with $|n_1 n_2\rangle$. So we have

$$Ha_2|n_1 n_2\rangle = q^{-2} a_2 (H - 1)|n_1 n_2\rangle. \tag{2.347}$$

It becomes

$$H(a_2|n_1 n_2\rangle) = q^{-2}(\varepsilon_{n_1,n_2} - 1)(a_2|n_1 n_2\rangle) \tag{2.348}$$

if we use Eq. (2.303). Next we consider

$$a_2|n_1 n_2\rangle = F_{n_1,n_2}|n_1, n_2 - 1\rangle. \tag{2.349}$$

So we find the following recursion formula. It is

$$\varepsilon_{n_1,n_2-1} = q^{-2}(\varepsilon_{n_1,n_2} - 1). \tag{2.350}$$

Since this formula is very similar to Eq. (2.315) mathematically, then we can immediately say that

$$\varepsilon_{n_1,n_2-m_2} = q^{-2m_2}\{\varepsilon_{n_1,n_2} - (\frac{1-q^{2m_2}}{1-q^2})\}. \tag{2.351}$$

Now, it remains to determine the last commutation relation. For this purpose, we will follow a similar procedure as before. Let us multiply Eq. (2.299) on the right by $a_2^\dagger$. Then we get

$$Ha_2^\dagger = (\frac{1+q^2}{2})^2(a_1^\dagger a_1 + a_2^\dagger a_2)a_2^\dagger + (\frac{3+q^2}{4})a_2^\dagger. \tag{2.352}$$

This equation becomes

$$Ha_2^\dagger = (\frac{1+q^2}{2})^2(a_2^\dagger a_1^\dagger a_1 + a_2^\dagger a_2 a_2^\dagger) + (\frac{3+q^2}{4})a_2^\dagger \tag{2.353}$$

if we use Eqs. (2.282) and (2.283). From Eqs. (2.284) and (2.285), we get

$$Ha_2^\dagger = (\frac{1+q^2}{2})^2 a_2^\dagger\{q^2(a_1^\dagger a_1 + a_2^\dagger a_2) + 1\} + (\frac{3+q^2}{4})a_2^\dagger. \tag{2.354}$$

By using Eq. (2.299), we have

$$Ha_2^\dagger = q^2 a_2^\dagger\{H - (\frac{3+q^2}{4})\} + (\frac{1+q^2}{2})^2 a_2^\dagger + (\frac{3+q^2}{4})a_2^\dagger \tag{2.355}$$

and then

$$Ha_2^\dagger = a_2^\dagger\{q^2 H + \frac{1}{4}(-q^4 - 3q^2 + q^4 + 2q^2 + 1 + 3 + q^2)\}. \tag{2.356}$$

So the fourth commutation relation is

$$Ha_2^\dagger = a_2^\dagger(q^2 H + 1). \tag{2.357}$$

Next if we multiply it on the right with $|n_1 n_2\rangle$, we find

$$Ha_2^\dagger|n_1 n_2\rangle = a_2^\dagger(q^2 H + 1)|n_1 n_2.\rangle \tag{2.358}$$

Let us use Eq. (2.303) to write

$$H(a_2^\dagger|n_1 n_2\rangle) = (q^2\varepsilon_{n_1,n_2} + 1)(a_2^\dagger|n_1 n_2\rangle). \tag{2.359}$$

Since another form of Eq. (2.350) is

$$\varepsilon_{n_1,n_2+1} = q^2\varepsilon_{n_1,n_2} + 1, \tag{2.360}$$

we can immediately conclude that

$$a_2^\dagger|n_1 n_2\rangle = G_{n_1,n_2}|n_1, n_2+1\rangle. \tag{2.361}$$

Since Eq. (2.360) is very similar to Eq. (2.331) mathematically, we can safely say that

$$\varepsilon_{n_1,n_2+m_2} = q^{2m_2}\varepsilon_{n_1,n_2} + (\frac{1-q^{2m_2}}{1-q^2}). \tag{2.362}$$

In summary, we have

$$\varepsilon_{n_1-m_1,n_2} = q^{-2m_1}\{\varepsilon_{n_1,n_2} - (\frac{1-q^{2m_1}}{1-q^2})\}, \tag{2.363}$$

$$\varepsilon_{n_1+m_1,n_2} = q^{2m_1}\varepsilon_{n_1,n_2} + (\frac{1-q^{2m_1}}{1-q^2}), \tag{2.364}$$

$$\varepsilon_{n_1,n_2-m_2} = q^{-2m_2}\{\varepsilon_{n_1,n_2} - (\frac{1-q^{2m_2}}{1-q^2})\}, \tag{2.365}$$

$$\varepsilon_{n_1,n_2+m_2} = q^{2m_2}\varepsilon_{n_1,n_2} + (\frac{1-q^{2m_2}}{1-q^2}). \tag{2.366}$$

Now, let us evaluate these recursion formulas in the limit $q = 1$. Then Eqs. (2.363), (2.364), (2.365), (2.366) read

$$\varepsilon_{n_1-m_1,n_2} = \varepsilon_{n_1,n_2} - m_1, \tag{2.367}$$

$$\varepsilon_{n_1+m_1,n_2} = \varepsilon_{n_1,n_2} + m_1, \tag{2.368}$$

$$\varepsilon_{n_1,n_2-m_2} = \varepsilon_{n_1,n_2} - m_2, \tag{2.369}$$

$$\varepsilon_{n_1,n_2+m_2} = \varepsilon_{n_1,n_2} + m_2 \tag{2.370}$$

respectively.

Up to this point, we have obtained the generalized forms of the recursion formulas for the energy eigenvalues. The similarity between the energy eigenvalues in Eqs. (2.40), (2.41) and these energy eigenvalues attracts our attention at this stage. So it means that they also share some important properties. We will mention these properties one by one.

Firstly, from the study in section (2.1), we can surely conclude that

$$D_{n_1,n_2} = C^*_{n_1+1,n_2} \tag{2.371}$$

and

$$G_{n_1,n_2} = F^*_{n_1,n_2+1}. \tag{2.372}$$

So Eqs. (2.333) and (2.361) read

$$a_1^\dagger|n_1 n_2\rangle = C^*_{n_1+1,n_2}|n_1+1, n_2\rangle \tag{2.373}$$

and

$$a_2^\dagger|n_1 n_2\rangle = F^*_{n_1,n_2+1}|n_1, n_2+1\rangle \tag{2.374}$$

respectively.

Secondly,

$$\varepsilon_{n_1,n_2} - \varepsilon_{n_1-1,n_2} \geq 0 \tag{2.375}$$

and

$$\varepsilon_{n_1,n_2} - \varepsilon_{n_1,n_2-1} \geq 0 \tag{2.376}$$

for all cases of q and ε_{n_1,n_2} except the case in which $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$. So ε_{n_1,n_2} decreases as n_1 or n_2 increases in this case. It requires to redefine the annihilation and creation operators as

$$b_1 = a_1^\dagger, \tag{2.377}$$

$$b_1^\dagger = a_1, \tag{2.378}$$

$$b_2 = a_2^\dagger, \tag{2.379}$$

and

$$b_2^\dagger = a_2. \tag{2.380}$$

Let us next consider

$$b_1|n_1 n_2\rangle = J_{n_1,n_2}|n_1 - 1, n_2\rangle \tag{2.381}$$

and

$$b_2|n_1 n_2\rangle = K_{n_1,n_2}|n_1, n_2 - 1\rangle. \tag{2.382}$$

Therefore we have

$$\varepsilon_{n_1-m_1,n_2} = q^{2m_1}\varepsilon_{n_1,n_2} + (\frac{1-q^{2m_1}}{1-q^2}), \tag{2.383}$$

$$\varepsilon_{n_1+m_1,n_2} = q^{-2m_1}\{\varepsilon_{n_1,n_2} - (\frac{1-q^{2m_1}}{1-q^2})\}, \tag{2.384}$$

$$\varepsilon_{n_1,n_2-m_2} = q^{2m_2}\varepsilon_{n_1,n_2} + (\frac{1-q^{2m_2}}{1-q^2}) \tag{2.385}$$

and

$$\varepsilon_{n_1,n_2+m_2} = q^{-2m_2}\{\varepsilon_{n_1,n_2} - (\frac{1-q^{2m_2}}{1-q^2})\}. \tag{2.386}$$

Thirdly, it is necessary to know whether these all recursion formulas are negative or positive to decide that they are exactly energy eigenvalues. In other words, for which cases there must occur a ground state? So again, we can confidently say that there must be a ground state for the following three cases. They are $(q > 1)$, $(q = 1)$ and $(0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2})$. In addition, in the case in which $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$ there is only one energy eigenvalue which is $\frac{1}{1-q^2}$.

If there is a ground state, then we can say

$$a_1|00\rangle = 0 \tag{2.387}$$

and

$$a_2|00\rangle = 0. \tag{2.388}$$

We consider $|00\rangle$ as the ground state of the system. To calculate the ground state energy, we write

$$(\frac{1+q^2}{2})^2(a_1^\dagger a_1 + a_2^\dagger a_2)|00\rangle = 0 \tag{2.389}$$

from Eqs. (2.387) and (2.388). The next step is to write

$$\{H - (\frac{3+q^2}{4})\}|00\rangle = 0 \tag{2.390}$$

using Eq. (2.299). So it is obvious that

$$H|00\rangle = (\frac{3+q^2}{4})|00\rangle. \tag{2.391}$$

This and Eq. (2.303) imply that

$$\varepsilon_{0,0} = \frac{3+q^2}{4}. \tag{2.392}$$

This is the ground state energy.

At this point, we must evaluate the eigenvalues again according to the final facts. In other words, for the cases in which there must be a ground state, we will compute the energy eigenvalues. Let us first write

$$\varepsilon_{n_1+m_1,n_2+m_2} = q^{2m_1}\varepsilon_{n_1,n_2+m_2} + (\frac{1-q^{2m_1}}{1-q^2}) \tag{2.393}$$

using Eq. (2.364). Then we can write

$$\varepsilon_{n_1+m_1,n_2+m_2} = q^{2(m_1+m_2)}\varepsilon_{n_1,n_2} + (\frac{1-q^{2(m_1+m_2)}}{1-q^2}) \tag{2.394}$$

substituting Eq. (2.366) into it. We rearrange it as

$$\varepsilon_{n_1,n_2} = q^{2(n_1+n_2)}\varepsilon_{0,0} + (\frac{1-q^{2(n_1+n_2)}}{1-q^2}) \tag{2.395}$$

by letting $n_1 = 0$, $n_2 = 0$ and inverting m_1, m_2 to n_1, n_2 respectively. As a result, substituting Eq. (2.392) into it, we get

$$\varepsilon_{n_1,n_2} = (\frac{3+q^2}{4})q^{2(n_1+n_2)} + (\frac{1-q^{2(n_1+n_2)}}{1-q^2}). \tag{2.396}$$

Accordingly, for the following three cases we rewrite the energy eigenvalues.

For $q > 1$,

$$\varepsilon_{n_1,n_2} = (\frac{3+q^2}{4})q^{2(n_1+n_2)} + (\frac{1-q^{2(n_1+n_2)}}{1-q^2}). \tag{2.397}$$

For $q = 1$,

$$\varepsilon_{n_1,n_2} = 1 + n_1 + n_2. \tag{2.398}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\varepsilon_{n_1,n_2} = (\frac{3+q^2}{4})q^{2(n_1+n_2)} + (\frac{1-q^{2(n_1+n_2)}}{1-q^2}). \tag{2.399}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$, we can consider

$$\varepsilon_{n_1,n_2} = \varepsilon_{0,0} \tag{2.400}$$

without loss of generality. Then we invert m_1, m_2 to n_1, n_2 respectively. Let us first write

$$\varepsilon_{n_1+m_1,n_2+m_2} = q^{-2m_1}\{\varepsilon_{n_1,n_2+m_2} - (\frac{1-q^{2m_1}}{1-q^2})\} \tag{2.401}$$

from Eq. (2.384). We next obtain

$$\varepsilon_{n_1+m_1,n_2+m_2} = q^{-2(m_1+m_2)}\{\varepsilon_{n_1,n_2} - (\frac{1-q^{2(m_1+m_2)}}{1-q^2})\} \tag{2.402}$$

using Eq. (2.386). Then we get

$$\varepsilon_{n_1,n_2} = q^{-2(n_1+n_2)}\{\varepsilon_{0,0} - (\frac{1-q^{2(n_1+n_2)}}{1-q^2})\} \tag{2.403}$$

using it where $n_1, n_2 = 0, \pm1, \pm2, ...$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$, we can similarly get

$$\varepsilon_{n_1,n_2} = \frac{1}{1-q^2} \tag{2.404}$$

where $n_1, n_2 = 0, \pm1, \pm2, ...$

In summary, we have the following energy eigenvalues.

For $q > 1$,

$$\varepsilon_{n_1,n_2} = (\frac{3+q^2}{4})q^{2(n_1+n_2)} + (\frac{1-q^{2(n_1+n_2)}}{1-q^2}) \tag{2.405}$$

where $n_1, n_2 = 0, 1, 2, ...$

For $q = 1$,

$$\varepsilon_{n_1,n_2} = 1 + n_1 + n_2 \tag{2.406}$$

where $n_1, n_2 = 0, 1, 2, ...$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\varepsilon_{n_1,n_2} = q^{-2(n_1+n_2)}\{\varepsilon_{0,0} - (\frac{1-q^{2(n_1+n_2)}}{1-q^2})\} \tag{2.407}$$

where $n_1, n_2 = 0, \pm 1, \pm 2, ...$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\varepsilon_{n_1,n_2} = \frac{1}{1-q^2} \tag{2.408}$$

where $n_1, n_2 = 0, \pm 1, \pm 2, ...$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\varepsilon_{n_1,n_2} = (\frac{3+q^2}{4})q^{2(n_1+n_2)} + (\frac{1-q^{2(n_1+n_2)}}{1-q^2}) \tag{2.409}$$

where $n_1, n_2 = 0, 1, 2, ...$

Thus we have completed our study of finding the energy eigenvalues partly. Hence this study is not complete exactly. We will see that the study of the eigenvalues of the q-deformed number operators N_1, N_2 will bring us some extra cases in which there must be a ground state. Here, let us recall that the definitions of N_1 and N_2 were given in Eqs. (2.300) and (2.301) respectively. We will call these extra cases the anomalous cases because they are unexpected and extraordinary facts.

To begin with, we want to get the commutation relations between N_1, N_2 and a_1, $a_1^\dagger$, a_2, $a_2^\dagger$.

Let us write

$$N_1 a_1 = a_1^\dagger a_1 a_1 \tag{2.410}$$

using the definition in Eq. (2.300). From Eq. (2.284), we get

$$N_1 a_1 = q^{-2}(a_1 a_1^\dagger - 1)a_1. \tag{2.411}$$

This implies that

$$N_1 a_1 = q^{-2} a_1 (a_1^\dagger a_1 - 1). \tag{2.412}$$

Therefore the commutation relation for N_1 and a_1 is

$$N_1 a_1 = q^{-2} a_1 (N_1 - 1). \tag{2.413}$$

We next introduce the eigenvalue problem. It is

$$N_1 |n_1 n_2\rangle = N^{(1)}_{n_1,n_2} |n_1 n_2\rangle. \tag{2.414}$$

$N^{(1)}_{n_1,n_2}$ denotes the eigenvalues of N_1 here. We note that Eq. (2.413) is very similar to Eq. (2.17) mathematically. In addition, Eq. (2.414) is similar to Eq. (2.21). Hence we can reach some results directly. We have

$$N^{(1)}_{n_1-m_1,n_2} = q^{-2m_1}\{N^{(1)}_{n_1,n_2} - (\frac{1-q^{2m_1}}{1-q^2})\} \tag{2.415}$$

and

$$N^{(1)}_{n_1+m_1,n_2} = q^{2m_1} N^{(1)}_{n_1,n_2} + (\frac{1-q^{2m_1}}{1-q^2}). \tag{2.416}$$

Let us continue finding the eigenvalues of N_1 by writing

$$N_1 a_2 = a_1^\dagger a_1 a_2 \tag{2.417}$$

from Eq. (2.300). Then it is obvious that

$$N_1 a_2 = a_2 a_1^\dagger a_1 \tag{2.418}$$

from Eqs. (2.282) and (2.283). So we can see that

$$N_1 a_2 = a_2 N_1. \tag{2.419}$$

It tells us that N_1 and a_2 commute. To find the eigenvalues, let us multiply it on the right with the energy eigenstate $|n_1 n_2\rangle$. Hence we get

$$N_1 a_2 |n_1 n_2\rangle = a_2 N_1 |n_1 n_2\rangle. \tag{2.420}$$

It follows that

$$N_1 (a_2 |n_1 n_2\rangle) = N^{(1)}_{n_1,n_2} (a_2 |n_1 n_2\rangle)\rangle. \tag{2.421}$$

Using Eqs. (2.349) and (2.414), we have

$$N^{(1)}_{n_1,n_2-1} = N^{(1)}_{n_1,n_2} \tag{2.422}$$

and thus

$$N^{(1)}_{n_1,n_2-m_2} = N^{(1)}_{n_1,n_2}. \tag{2.423}$$

By using Eq. (2.422), we find

$$N^{(1)}_{n_1,n_2+1} = N^{(1)}_{n_1,n_2}. \tag{2.424}$$

We generalize it as

$$N^{(1)}_{n_1,n_2+m_2} = N^{(1)}_{n_1,n_2}. \tag{2.425}$$

In this context, the second eigenvalue problem is

$$N_2|n_1n_2\rangle = N^{(2)}_{n_1,n_2}|n_1n_2\rangle. \tag{2.426}$$

To analyze the relation between N_2a_2 and a_2N_2, we proceed as follows. We get

$$N_2a_2 = a_2^\dagger a_2a_2 \tag{2.427}$$

from Eq. (2.301). We rearrange it as

$$N_2a_2 = q^{-2}\{(1-q^2)a_1^\dagger a_1 + a_2a_2^\dagger - 1\}a_2 \tag{2.428}$$

by using Eqs. (2.285) and (2.284). One can easily show that

$$N_2a_2 = q^{-2}a_2\{(1-q^2)N_1 + N_2 - 1\} \tag{2.429}$$

from the definitions of N_1 and N_2. By multiplication of it on the right with the energy eigenstate $|n_1n_2\rangle$ we have

$$N_2a_2|n_1n_2\rangle = q^{-2}a_2\{(1-q^2)N_1 + N_2 - 1\}|n_1n_2\rangle. \tag{2.430}$$

Then we get

$$N_2(a_2|n_1n_2\rangle) = q^{-2}\{(1-q^2)N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - 1\}(a_2|n_1n_2\rangle) \tag{2.431}$$

taking account of Eqs. (2.414) and (2.426). Using Eqs. (2.349) and (2.426), we obtain the last recursion formula. This recursion formula is

$$N^{(2)}_{n_1,n_2-1} = q^{-2}\{(1-q^2)N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - 1\}. \tag{2.432}$$

However, we want to reach its generalized form. For this reason, we write

$$N^{(2)}_{n_1,n_2-2} = q^{-2}\{(1-q^2)N^{(1)}_{n_1,n_2-1} + N^{(2)}_{n_1,n_2-1} - 1\}. \tag{2.433}$$

We substitute Eqs. (2.422) and (2.432) into it to get

$$N^{(2)}_{n_1,n_2-2} = q^{-2}\{(1+q^{-2})(1-q^2)N^{(1)}_{n_1,n_2} + q^{-2}N^{(2)}_{n_1,n_2} - (1+q^{-2})\}. \tag{2.434}$$

In a similar way, we proceed by writing

$$N^{(2)}_{n_1,n_2-3} = q^{-2}\{(1+q^{-2}+q^{-4})(1-q^2)N^{(1)}_{n_1,n_2} + q^{-4}N^{(2)}_{n_1,n_2} - (1+q^{-2}+q^{-4})\}. \tag{2.435}$$

At this point, the above equations give an idea to generalize the recursion formula in Eq. (2.432). As a result, we obtain

$$N^{(2)}_{n_1,n_2-m_2} = q^{-2}\{\{1+q^{-2}+(q^{-2})^2+...+(q^{-2})^{m_2-1}\}(1-q^2)N^{(1)}_{n_1,n_2}$$

$$+(q^{-2})^{m_2-1}N^{(2)}_{n_1,n_2} - \{1+q^{-2}+(q^{-2})^2+...+(q^{-2})^{m_2-1}\}\}. \tag{2.436}$$

If we tidy up this equation, we reach

$$N^{(2)}_{n_1,n_2-m_2} = q^{-2m_2}\{(1-q^{2m_2})N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1-q^{2m_2}}{1-q^2})\}. \tag{2.437}$$

By writing n_2+1 instead of n_2, Eq. (2.432) becomes

$$N^{(2)}_{n_1,n_2+1} = (q^2-1)N^{(1)}_{n_1,n_2+1} + q^2N^{(2)}_{n_1,n_2} + 1. \tag{2.438}$$

Substituting Eq. (2.424) into it, we have

$$N^{(2)}_{n_1,n_2+1} = (q^2-1)N^{(1)}_{n_1,n_2} + q^2 N^{(2)}_{n_1,n_2} + 1. \qquad (2.439)$$

Evidently, we get

$$N^{(2)}_{n_1,n_2+2} = (1+q^2)(q^2-1)N^{(1)}_{n_1,n_2} + q^4 N^{(2)}_{n_1,n_2} + (1+q^2) \qquad (2.440)$$

from it and Eq. (2.424). Let us continue by writing

$$N^{(2)}_{n_1,n_2+3} = (1+q^2+q^4)(q^2-1)N^{(1)}_{n_1,n_2} + q^6 N^{(2)}_{n_1,n_2} + (1+q^2+q^4). \qquad (2.441)$$

The last three equations give us an idea to obtain the generalized recursion formula. Accordingly, one can show that

$$N^{(2)}_{n_1,n_2+m_2} = \{1+q^2+(q^2)^2+...+(q^2)^{m_2-1}\}(q^2-1)N^{(1)}_{n_1,n_2}$$

$$+(q^2)^{m_2} N^{(2)}_{n_1,n_2} + \{1+q^2+(q^2)^2+...+(q^2)^{m_2-1}\}. \qquad (2.442)$$

Finally, if we write it more compactly, we have

$$N^{(2)}_{n_1,n_2+m_2} = (q^{2m_2}-1)N^{(1)}_{n_1,n_2} + q^{2m_2} N^{(2)}_{n_1,n_2} + (\frac{1-q^{2m_2}}{1-q^2}). \qquad (2.443)$$

Now, we will discuss the commutation relation for N_2 and a_1. We start by writing

$$N_2 a_1 = a_2^\dagger a_2 a_1 \qquad (2.444)$$

from Eq. (2.301). Then we use Eqs. (2.282) and (2.283) to have

$$N_2 a_1 = q^{-2} a_1 a_2^\dagger a_2. \tag{2.445}$$

This implies that

$$N_2 a_1 = q^{-2} a_1 N_2. \tag{2.446}$$

The next step is to write

$$N_2 a_1 |n_1 n_2\rangle = q^{-2} a_1 N_2 |n_1 n_2\rangle \tag{2.447}$$

by multiplying Eq. (2.446) on the right with $|n_1 n_2\rangle$. It follows that

$$N_2(a_1|n_1 n_2\rangle) = q^{-2} N^{(2)}_{n_1,n_2}(a_1|n_1 n_2\rangle) \tag{2.448}$$

according to Eq. (2.426). We get the recursion formula

$$N^{(2)}_{n_1-1,n_2} = q^{-2} N^{(2)}_{n_1,n_2} \tag{2.449}$$

by using Eqs. (2.314) and (2.426). If we generalize it, we have

$$N^{(2)}_{n_1-m_1,n_2} = q^{-2m_1} N^{(2)}_{n_1,n_2}. \tag{2.450}$$

Eq. (2.449) implies that

$$N^{(2)}_{n_1+1,n_2} = q^{2} N^{(2)}_{n_1,n_2}. \tag{2.451}$$

From here, we obtain

$$N^{(2)}_{n_1+m_1,n_2} = q^{2m_1} N^{(2)}_{n_1,n_2}. \tag{2.452}$$

In the present case, let us combine some of our results to obtain more compact expressions. In other words, let us first write $n_2 + m_2$ instead of n_2 in Eq. (2.416). Then we get

$$N^{(1)}_{n_1+m_1,n_2+m_2} = q^{2m_1} N^{(1)}_{n_1,n_2+m_2} + (\frac{1-q^{2m_1}}{1-q^2}). \tag{2.453}$$

After that we write

$$N^{(1)}_{n_1+m_1,n_2+m_2} = q^{2m_1} N^{(1)}_{n_1,n_2} + (\frac{1-q^{2m_1}}{1-q^2}) \tag{2.454}$$

inserting Eq. (2.425) into it. Similarly, we get

$$N^{(1)}_{n_1-m_1,n_2-m_2} = q^{-2m_1}\{N^{(1)}_{n_1,n_2-m_2} + (\frac{1-q^{2m_1}}{1-q^2})\} \tag{2.455}$$

by putting $n_2 - m_2$ instead of n_2 in Eq. (2.415). Next, we substitute Eq. (2.423) into it to have

$$N^{(1)}_{n_1-m_1,n_2-m_2} = q^{-2m_1}\{N^{(1)}_{n_1,n_2} - (\frac{1-q^{2m_1}}{1-q^2})\}. \tag{2.456}$$

Since we also want to find the combined forms of the eigenvalues of $N^{(2)}_{n_1,n_2}$, we proceed writing $n_2 + m_2$ instead of n_2 in Eq. (2.452). Then we get

$$N^{(2)}_{n_1+m_1,n_2+m_2} = q^{2m_1} N^{(2)}_{n_1,n_2+m_2}. \tag{2.457}$$

The next step is to substitute Eq. (2.443) into it. So we obtain

$$N^{(2)}_{n_1+m_1,n_2+m_2} = q^{2m_1}\{(q^{2m_2}-1)N^{(1)}_{n_1,n_2} + q^{2m_2} N^{(2)}_{n_1,n_2} + (\frac{1-q^{2m_2}}{1-q^2})\}. \tag{2.458}$$

Similarly, we have

$$N^{(2)}_{n_1-m_1,n_2-m_2} = q^{-2m_1} N^{(2)}_{n_1,n_2-m_2} \tag{2.459}$$

if we write $n_2 - m_2$ instead of n_2 in Eq. (2.450). Then let us substitute Eq. (2.437) into it to obtain

$$N^{(2)}_{n_1-m_1,n_2-m_2} = q^{-2(m_1+m_2)}\{(1-q^{2m_2})N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1-q^{2m_2}}{1-q^2})\}. \tag{2.460}$$

In summary, we have

$$N^{(1)}_{n_1-m_1,n_2} = q^{-2m_1}\{N^{(1)}_{n_1,n_2} - (\frac{1-q^{2m_1}}{1-q^2})\}, \tag{2.461}$$

$$N^{(1)}_{n_1+m_1,n_2} = q^{2m_1}N^{(1)}_{n_1,n_2} + (\frac{1-q^{2m_1}}{1-q^2}), \tag{2.462}$$

$$N^{(1)}_{n_1,n_2-m_2} = N^{(1)}_{n_1,n_2}, \tag{2.463}$$

$$N^{(1)}_{n_1,n_2+m_2} = N^{(1)}_{n_1,n_2}, \tag{2.464}$$

$$N^{(2)}_{n_1-m_1,n_2} = q^{-2m_1}N^{(2)}_{n_1,n_2}, \tag{2.465}$$

$$N^{(2)}_{n_1+m_1,n_2} = q^{2m_1}N^{(2)}_{n_1,n_2}, \tag{2.466}$$

$$N^{(2)}_{n_1,n_2-m_2} = q^{-2m_2}\{(1-q^{2m_2})N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1-q^{2m_2}}{1-q^2})\}, \tag{2.467}$$

$$N^{(2)}_{n_1,n_2+m_2} = (q^{2m_2}-1)N^{(1)}_{n_1,n_2} + q^{2m_2}N^{(2)}_{n_1,n_2} + (\frac{1-q^{2m_2}}{1-q^2}) \tag{2.468}$$

and

$$N^{(1)}_{n_1-m_1,n_2-m_2} = q^{-2m_1}\{N^{(1)}_{n_1,n_2} - (\frac{1-q^{2m_1}}{1-q^2})\}, \tag{2.469}$$

$$N^{(1)}_{n_1+m_1,n_2+m_2} = q^{2m_1}N^{(1)}_{n_1,n_2} + (\frac{1-q^{2m_1}}{1-q^2}), \tag{2.470}$$

$$N^{(2)}_{n_1-m_1,n_2-m_2} = q^{-2(m_1+m_2)}\{(1-q^{2m_2})N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1-q^{2m_2}}{1-q^2})\}, \tag{2.471}$$

$$N^{(2)}_{n_1+m_1,n_2+m_2} = q^{2m_1}\{(q^{2m_2}-1)N^{(1)}_{n_1,n_2} + q^{2m_2}N^{(2)}_{n_1,n_2} + (\frac{1-q^{2m_2}}{1-q^2})\}. \tag{2.472}$$

If there is a ground state such that $a_1|00\rangle = 0$ and $a_2|00\rangle = 0$, we can use Eqs. (2.387) and (2.414) to say

$$N^{(1)}_{0,0}|00\rangle = 0. \tag{2.473}$$

Therefore it is obvious that

$$N^{(1)}_{0,0} = 0. \tag{2.474}$$

In this case, Eq. (2.470) becomes

$$N^{(1)}_{n_1,n_2} = \frac{1-q^{2n_1}}{1-q^2} \tag{2.475}$$

when we write $n_1 = 0$, $n_2 = 0$ and then change m_1, m_2 into n_1, n_2 respectively. In addition, Eq. (2.414) reads

$$a_1^\dagger a_1|n_1 n_2\rangle = (\frac{1-q^{2n_1}}{1-q^2})|n_1 n_2\rangle. \tag{2.476}$$

If we follow a similar way, we write

$$N_{0,0}^{(2)}|00\rangle = 0 \tag{2.477}$$

according to Eqs. (2.388) and (2.426). Clearly, we have

$$N_{0,0}^{(2)} = 0. \tag{2.478}$$

We recalculate Eq. (2.472) by writing $n_1 = 0$, $n_2 = 0$ and then changing m_1, m_2 into n_1, n_2 respectively. Accordingly, we conclude

$$N_{n_1,n_2}^{(2)} = q^{2n_1}(\frac{1-q^{2n_2}}{1-q^2}). \tag{2.479}$$

In addition, Eq. (2.426) reads

$$a_2^\dagger a_2|n_1 n_2\rangle = q^{2n_1}(\frac{1-q^{2n_2}}{1-q^2})|n_1 n_2\rangle. \tag{2.480}$$

In summary, we have

$$N_{n_1,n_2}^{(1)} = \frac{1-q^{2n_1}}{1-q^2}, \tag{2.481}$$

$$N_{n_1,n_2}^{(2)} = q^{2n_1}(\frac{1-q^{2n_2}}{1-q^2}). \tag{2.482}$$

What we have just done is to find the eigenvalues of $a_1^\dagger a_1$ and $a_2^\dagger a_2$ for the cases in which there must be a ground state such that $a_1|00\rangle = 0$ and $a_2|00\rangle = 0$. Now, to find the eigenvalues of $a_1^\dagger a_1$ and $a_2^\dagger a_2$ for the case in which $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$, we follow a different way. Let us recall that new annihilation and creation operators are defined for this case. Accordingly, let us write these eigenvalues again. Eqs. (2.462), (2.461), (2.464), (2.463), (2.466), (2.465), (2.468) and (2.467) change as

$$N^{(1)}_{n_1-m_1,n_2} = q^{2m_1} N^{(1)}_{n_1,n_2} + (\frac{1-q^{2m_1}}{1-q^2}), \tag{2.483}$$

$$N^{(1)}_{n_1+m_1,n_2} = q^{-2m_1}\{N^{(1)}_{n_1,n_2} - (\frac{1-q^{2m_1}}{1-q^2})\}, \tag{2.484}$$

$$N^{(1)}_{n_1,n_2-m_2} = N^{(1)}_{n_1,n_2}, \tag{2.485}$$

$$N^{(1)}_{n_1,n_2+m_2} = N^{(1)}_{n_1,n_2}, \tag{2.486}$$

$$N^{(2)}_{n_1-m_1,n_2} = q^{2m_1} N^{(2)}_{n_1,n_2}, \tag{2.487}$$

$$N^{(2)}_{n_1+m_1,n_2} = q^{-2m_1} N^{(2)}_{n_1,n_2}, \tag{2.488}$$

$$N^{(2)}_{n_1,n_2-m_2} = (q^{2m_2}-1) N^{(1)}_{n_1,n_2} + q^{2m_2} N^{(2)}_{n_1,n_2} + (\frac{1-q^{2m_2}}{1-q^2}) \tag{2.489}$$

and

$$N^{(2)}_{n_1,n_2+m_2} = q^{-2m_2}\{(1-q^{2m_2}) N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1-q^{2m_2}}{1-q^2})\} \tag{2.490}$$

respectively.

At this stage, we focus on the fact that $N^{(1)}_{n_1,n_2}$ and $N^{(2)}_{n_1,n_2}$ can not take the negative values. To confirm this, let us write

$$N^{(1)}_{n_1,n_2} = \langle n_1 n_2 | a_1^\dagger a_1 | n_1 n_2 \rangle \tag{2.491}$$

using Eq. (2.414). We can rewrite it as

$$N^{(1)}_{n_1,n_2} = (a_1 | n_1 n_2 \rangle)^\dagger (a_1 | n_1 n_2 \rangle). \tag{2.492}$$

We can easily see that this is exactly the square of the norm of a vector. Therefore it can not be negative definitely. Eventually, $N^{(1)}_{n_1,n_2}$ must be nonnegative. The same is true for $N^{(2)}_{n_1,n_2}$. This can be also proved in a similar way. For that reason, we wander whether there are some extra cases in which there must be a ground state.

Before starting, we notice that Eqs. (2.461) and (2.462) are mathematically similar to Eqs. (2.40) and (2.41) respectively. Hence we reach the following result directly. For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$, there must be a state such that

$$a_1 | 0 n_2 \rangle = 0. \tag{2.493}$$

We will use this fact later.

Now, we will analyze the following five cases. Some of these cases have some subcases.

Firstly, for $q > 1$, it is obvious that

$$N^{(2)}_{n_1,n_2+m_2} > 0. \tag{2.494}$$

Secondly, for $q = 1$, since

$$N^{(2)}_{n_1,n_2} + m_2 > 0, \qquad (2.495)$$

we get

$$N^{(2)}_{n_1,n_2+m_2} > 0. \qquad (2.496)$$

Thirdly, for $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$, we have

$$(\frac{1+q^2}{2})(N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2}) + (\frac{3+q^2}{4}) > \frac{1}{1-q^2} \qquad (2.497)$$

if we sandwich Eq. (2.299) between $\langle n_1 n_2|$ and $|n_1 n_2\rangle$ and then use Eqs. (2.303), (2.414) and (2.426). It follows that

$$(\frac{1+q^2}{2})(N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2}) > (\frac{1+q^2}{2})^2(\frac{1}{1-q^2}) \qquad (2.498)$$

and then

$$N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} > \frac{1}{1-q^2}. \qquad (2.499)$$

Now, the following three subcases of this case will be examined. So we will use this inequality.

For $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$, it is obvious that

$$\frac{N^{(1)}_{n_1,n_2} - (\frac{1}{1-q^2})}{N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1}{1-q^2})} > 0. \qquad (2.500)$$

This implies that $\ln\{\frac{N^{(1)}_{n_1,n_2}-(\frac{1}{1-q^2})}{N^{(1)}_{n_1,n_2}+N^{(2)}_{n_1,n_2}-(\frac{1}{1-q^2})}\}$ is well-defined. If

$$m_2 > \frac{\ln\{\frac{N^{(1)}_{n_1,n_2}-(\frac{1}{1-q^2})}{N^{(1)}_{n_1,n_2}+N^{(2)}_{n_1,n_2}-(\frac{1}{1-q^2})}\}}{\ln q^2} \tag{2.501}$$

then

$$m_2 \ln q^2 < \ln\{\frac{N^{(1)}_{n_1,n_2} - (\frac{1}{1-q^2})}{N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1}{1-q^2})}\} \tag{2.502}$$

and thus

$$q^{2m_2} < \frac{N^{(1)}_{n_1,n_2} - (\frac{1}{1-q^2})}{N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1}{1-q^2})}. \tag{2.503}$$

Evidently, we have

$$q^{2m_2}\{N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1}{1-q^2})\} < N^{(1)}_{n_1,n_2} - (\frac{1}{1-q^2}) \tag{2.504}$$

and then

$$(q^{2m_2}-1)N^{(1)}_{n_1,n_2} + q^{2m_2}N^{(2)}_{n_1,n_2} + (\frac{1-q^{2m_2}}{1-q^2}) < 0. \tag{2.505}$$

Therefore we conclude that

$$N^{(2)}_{n_1,n_2-m_2} < 0. \tag{2.506}$$

This means that there must be a state such that

$$b_2|n_1 0\rangle = 0. \tag{2.507}$$

If we look at this case from the angle of the energy, there is no ground state because n_1 does not have a lower bound. However, it is obvious that there is a greatest lower bound for the energy. Furthermore, we can say that there is a ground state that is only related with $N^{(2)}_{n_1,n_2}$. Accordingly, we call this state an anomalous ground state.

Now, it is clear that

$$N^{(1)}_{n_1,n_2} - (\frac{1}{1-q^2}) > 0 \tag{2.508}$$

and thus

$$(1-q^{2m_2})\{N^{(1)}_{n_1,n_2} - (\frac{1}{1-q^2})\} > 0. \tag{2.509}$$

Evidently, we get

$$(1-q^{2m_2})N^{(1)}_{n_1,n_2} - (\frac{1-q^{2m_2}}{1-q^2}) > 0. \tag{2.510}$$

We can surely write

$$N^{(2)}_{n_1,n_2+m_2} > 0. \tag{2.511}$$

For $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$, we obtain

$$N^{(2)}_{n_1,n_2-m_2} = q^{2m_2} N^{(2)}_{n_1,n_2}. \tag{2.512}$$

One can easily see that

$$N^{(2)}_{n_1,n_2-m_2} > 0. \tag{2.513}$$

Similarly, we have

$$N^{(2)}_{n_1,n_2+m_2} = q^{-2m_2} N^{(2)}_{n_1,n_2}. \tag{2.514}$$

So we conclude that

$$N^{(2)}_{n_1,n_2+m_2} > 0. \tag{2.515}$$

For $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$, we have

$$q^{2m_2}\{N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1}{1-q^2})\} > 0 \tag{2.516}$$

from Eq. (2.499) and we obviously have

$$(\frac{1}{1-q^2}) - N^{(1)}_{n_1,n_2} > 0. \tag{2.517}$$

Adding these two inequalities, we obtain

$$q^{2m_2}\{N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1}{1-q^2})\} + (\frac{1}{1-q^2}) - N^{(1)}_{n_1,n_2} > 0. \tag{2.518}$$

It follows that

$$(q^{2m_2} - 1)N^{(1)}_{n_1,n_2} + q^{2m_2}N^{(2)}_{n_1,n_2} + (\frac{1-q^{2m_2}}{1-q^2}) > 0. \tag{2.519}$$

Therefore we get

$$N^{(2)}_{n_1,n_2-m_2} > 0. \tag{2.520}$$

Now, we have

$$\{(\frac{1}{1-q^2}) - N^{(1)}_{n_1,n_2}\}q^{2m_2} > 0. \tag{2.521}$$

Then we get

$$-(\frac{1-q^{2m_2}}{1-q^2})+(\frac{1}{1-q^2})-N^{(1)}_{n_1,n_2}q^{2m_2}>0, \tag{2.522}$$

if we add $\frac{1}{1-q^2}$ to it and subtract $\frac{1}{1-q^2}$ from it. If we add

$$N^{(1)}_{n_1,n_2}+N^{(2)}_{n_1,n_2}>\frac{1}{1-q^2} \tag{2.523}$$

to the above inequality, we find

$$-(\frac{1-q^{2m_2}}{1-q^2})-N^{(1)}_{n_1,n_2}q^{2m_2}+N^{(1)}_{n_1,n_2}+N^{(2)}_{n_1,n_2}>0. \tag{2.524}$$

If we tidy up it, we get

$$(1-q^{2m_2})N^{(1)}_{n_1,n_2}+N^{(2)}_{n_1,n_2}-(\frac{1-q^{2m_2}}{1-q^2})>0. \tag{2.525}$$

So it follows that

$$N^{(2)}_{n_1,n_2+m_2}>0. \tag{2.526}$$

In the beginning of the study of finding extra cases, we implicitly mentioned that there must be a state such that

$$b_1^\dagger|0n_2\rangle=0. \tag{2.527}$$

If we say it more explicitly, Eq. (2.493) necessitates this. Here again, there is a ground state that is related with only $N^{(1)}_{n_1,n_2}$. So this state is again an anomalous ground state.

Fourthly, for $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$, we find

$$N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} = \frac{1}{1-q^2} \tag{2.528}$$

if we sandwich Eq. (2.299) between $\langle n_1 n_2|$ and $|n_1 n_2\rangle$ and then use Eqs. (2.303), (2.414) and (2.426).

Now, let us study the following two subcases of this case.

For $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$, we have

$$N^{(2)}_{n_1,n_2} = 0 \tag{2.529}$$

from Eq. (2.528). So Eq. (2.467) reads

$$N^{(2)}_{n_1,n_2-m_2} = q^{-2m_2} N^{(2)}_{n_1,n_2}. \tag{2.530}$$

So we have

$$N^{(2)}_{n_1,n_2-m_2} = 0. \tag{2.531}$$

In addition, Eq. (2.468) reads

$$N^{(2)}_{n_1,n_2+m_2} = q^{2m_2} N^{(2)}_{n_1,n_2}. \tag{2.532}$$

Hence

$$N^{(2)}_{n_1,n_2+m_2} = 0. \tag{2.533}$$

For $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$, we certainly have

$$q^{2m_2}\{N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1}{1-q^2})\} = 0 \tag{2.534}$$

from Eq. (2.528). Then it is obvious that

$$q^{2m_2}\{N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1}{1-q^2})\} - N^{(1)}_{n_1,n_2} + (\frac{1}{1-q^2}) = -N^{(1)}_{n_1,n_2} + (\frac{1}{1-q^2}). \tag{2.535}$$

The next step is to look at Eq. (2.489) to decide

$$N^{(2)}_{n_1,n_2-m_2} = (\frac{1}{1-q^2}) - N^{(1)}_{n_1,n_2}. \tag{2.536}$$

We know that

$$N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1}{1-q^2}) = 0 \tag{2.537}$$

from Eq. (2.528). It follows that

$$q^{-2m_2}\{N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1}{1-q^2}) + q^{2m_2}\{(\frac{1}{1-q^2}) - N^{(1)}_{n_1,n_2}\}\} = q^{-2m_2}q^{2m_2}\{(\frac{1}{1-q^2}) - N^{(1)}_{n_1,n_2}\}. \tag{2.538}$$

Using Eq. (2.490), we conclude that

$$N^{(2)}_{n_1,n_2+m_2} = (\frac{1}{1-q^2}) - N^{(1)}_{n_1,n_2}. \tag{2.539}$$

Again here we can see that there must be a state such that

$$a_1|0n_2\rangle = 0 \tag{2.540}$$

for this case if we recall Eq. (2.493). This is an anomalous ground state that is related with only $N^{(1)}_{n_1,n_2}$.

Fifthly and finally, for $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$, we have

$$N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} < \frac{1}{1-q^2}. \tag{2.541}$$

For this case, we may have only the subcase in which $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$. Since

$$q^{2m_2} - 1 \leq 0 \tag{2.542}$$

and

$$N^{(1)}_{n_1,n_2} - (\frac{1}{1-q^2}) < 0, \tag{2.543}$$

we conclude that

$$(q^{2m_2} - 1)\{N^{(1)}_{n_1,n_2} - (\frac{1}{1-q^2})\} + q^{2m_2} N^{(2)}_{n_1,n_2} > 0. \tag{2.544}$$

Therefore we have

$$N^{(2)}_{n_1,n_2+m_2} > 0. \tag{2.545}$$

In summary, we have the following anomalous cases.

For $0 < q < 1$, $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$, there must be an anomalous ground state such that

$$b_2|n_1 0\rangle = 0. \tag{2.546}$$

For $0 < q < 1$, $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$, there must be an anomalous ground state such that

$$b_1^\dagger|0n_2\rangle = 0. \tag{2.547}$$

For $0 < q < 1$, $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$, there must be an anomalous ground state such that

$$a_1|0n_2\rangle = 0. \tag{2.548}$$

We have finished the study of finding anomalous cases. Now, we want to recalculate $N^{(1)}_{n_1,n_2}$ and $N^{(2)}_{n_1,n_2}$ for these cases.

The first case is the one in which $0 < q < 1$, $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$. According to Eq. (2.546) we can write

$$b_2^\dagger b_2|n_1 0\rangle = 0. \tag{2.549}$$

This gives us that

$$a_2 a_2^\dagger|n_1 0\rangle = 0 \tag{2.550}$$

if we use Eqs. (2.379) and (2.380). We will write this equation in terms of only $N^{(1)}_{n_1,n_2}$ and $N^{(2)}_{n_1,n_2}$ because we want to find the eigenvalues of $N^{(1)}_{n_1,n_2}$ and $N^{(2)}_{n_1,n_2}$ corresponding to the anomalous ground state. Therefore we use Eqs. (2.285) and (2.284) to write

$$\{(q^2-1)a_1^\dagger a_1 + q^2 a_2^\dagger a_2 + 1\}|n_1 0\rangle = 0. \tag{2.551}$$

From Eqs. (2.300), (2.301), (2.414) and (2.426) we can write

$$\{(q^2-1)N^{(1)}_{n_1,0} + q^2 N^{(2)}_{n_1,0} + 1\}|n_1 0\rangle = 0. \tag{2.552}$$

It follows that

$$(q^2-1)N^{(1)}_{n_1,0}+q^2N^{(2)}_{n_1,0}+1=0. \tag{2.553}$$

For $n_1=0$, this equation reads

$$(q^2-1)N^{(1)}_{0,0}+q^2N^{(2)}_{0,0}+1=0. \tag{2.554}$$

Let us solve it for $N^{(2)}_{0,0}$. Then we get

$$N^{(2)}_{0,0}=(q^{-2}-1)N^{(1)}_{0,0}-q^{-2}. \tag{2.555}$$

Now, let us combine Eqs. (2.487) and (2.488) to get

$$N^{(2)}_{n_1+m_1,n_2}=q^{-2m_1}N^{(2)}_{n_1,n_2} \tag{2.556}$$

where $m_1=0,\pm1,\pm2,....$ If we write n_2+m_2 instead of n_2, we obtain

$$N^{(2)}_{n_1+m_1,n_2+m_2}=q^{-2m_1}N^{(2)}_{n_1,n_2+m_2}. \tag{2.557}$$

where $m_2=0,1,2,....$ For $n_1=0$ and $n_2=0$ it becomes

$$N^{(2)}_{m_1,m_2}=q^{-2m_1}N^{(2)}_{0,m_2}. \tag{2.558}$$

For $n_1=0$ and $n_2=0$ Eq. (2.490) reads

$$N^{(2)}_{0,m_2}=q^{-2m_2}\{(1-q^{2m_2})N^{(1)}_{0,0}+N^{(2)}_{0,0}-(\frac{1-q^{2m_2}}{1-q^2})\}. \tag{2.559}$$

Feeding Eq. (2.555) into this equation, we find

$$N^{(2)}_{0,m_2}=q^{-2m_2}\{(q^{-2}-q^{2m_2})N^{(1)}_{0,0}-(\frac{q^{-2}-q^{2m_2}}{1-q^2})\}. \tag{2.560}$$

If we tidy up it, we get

$$N^{(2)}_{0,m_2} = (q^{-2(m_2+1)} - 1)\{N^{(1)}_{0,0} - (\frac{1}{1-q^2})\}. \tag{2.561}$$

Let us feed this into Eq. (2.558) and change m_1, m_2 into n_1, n_2 respectively. Then we get

$$N^{(2)}_{n_1,n_2} = q^{-2n_1}(q^{-2(n_2+1)} - 1)\{N^{(1)}_{0,0} - (\frac{1}{1-q^2})\} \tag{2.562}$$

where $n_1 = 0, \pm1, \pm2, ...$ and $n_2 = 0, 1, 2, ...$ In addition, let us combine Eqs. (2.483) and (2.484) to write

$$N^{(1)}_{n_1+m_1,n_2} = q^{-2m_1}\{N^{(1)}_{n_1,n_2} - (\frac{1-q^{2m_1}}{1-q^2})\}. \tag{2.563}$$

where $m_1 = 0, \pm1, \pm2, ...$ If we write $n_2 + m_2$ instead of n_2, we obtain

$$N^{(1)}_{n_1+m_1,n_2+m_2} = q^{-2m_1}\{N^{(1)}_{n_1,n_2+m_2} - (\frac{1-q^{2m_1}}{1-q^2})\} \tag{2.564}$$

where $m_2 = 0, 1, 2, ...$ Let us substitute Eq. (2.486) into it and then evaluate it for $n_1 = 0$ and $n_2 = 0$ and finally change m_1, m_2 into n_1, n_2 respectively. Therefore it is clearly seen that

$$N^{(1)}_{n_1,n_2} = q^{-2n_1}\{N^{(1)}_{0,0} - (\frac{1-q^{2n_1}}{1-q^2})\} \tag{2.565}$$

where $n_1 = 0, \pm1, \pm2, ...$ and $n_2 = 0, 1, 2, ...$

The second anomalous case is the one in which $0 < q < 1$, $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$. If we take Eq. (2.547) into account, we get

$$b_1 b_1^\dagger |0 n_2\rangle = 0. \tag{2.566}$$

When we use Eqs. (2.377) and (2.378), we have

$$a_1^\dagger a_1|0n_2\rangle = 0. \tag{2.567}$$

Then we get

$$N^{(1)}_{0,n_2}|0n_2\rangle = 0 \tag{2.568}$$

from Eqs. (2.300) and (2.414). So it is obvious that

$$N^{(1)}_{0,n_2} = 0. \tag{2.569}$$

For $n_2 = 0$, it becomes

$$N^{(1)}_{0,0} = 0. \tag{2.570}$$

The next step is to combine Eqs. (2.485) and (2.486), to have

$$N^{(1)}_{n_1,n_2+m_2} = N^{(1)}_{n_1,n_2} \tag{2.571}$$

where $m_2 = 0, \pm 1, \pm 2, ...$ If we write $n_1 + m_1$ instead of n_1, we obtain

$$N^{(1)}_{n_1+m_1,n_2+m_2} = N^{(1)}_{n_1+m_1,n_2} \tag{2.572}$$

where $m_1 = 0, -1, -2, ...$ For $n_1 = 0$ and $n_2 = 0$, this equation reads

$$N^{(1)}_{m_1,m_2} = N^{(1)}_{m_1,0}. \tag{2.573}$$

Now, let us find $N^{(1)}_{m_1,0}$. For $n_1 = 0$ and $n_2 = 0$, Eq. (2.483) reads

$$N^{(1)}_{m_1,0} = q^{-2m_1} N^{(1)}_{0,0} + (\frac{1 - q^{-2m_1}}{1 - q^2}) \tag{2.574}$$

where $m_1 = 0, -1, -2, ...$ Let us substitute Eq. (2.570) into it to get

$$N^{(1)}_{m_1,0} = \frac{1-q^{-2m_1}}{1-q^2}. \tag{2.575}$$

We can now insert this into Eq. (2.573) and change m_1, m_2 into n_1, n_2 respectively, to have

$$N^{(1)}_{n_1,n_2} = \frac{1-q^{-2n_1}}{1-q^2} \tag{2.576}$$

where $n_1 = 0, -1, -2, ...$ and $n_2 = 0, \pm1, \pm2, ...$ To get $N^{(2)}_{n_1,n_2}$ for this case, let us write Eq. (2.487) as

$$N^{(2)}_{n_1+m_1,n_2+m_2} = q^{-2m_1} N^{(2)}_{n_1,n_2+m_2} \tag{2.577}$$

where $m_1 = 0, -1, -2, ...$ and $m_2 = 0, \pm1, \pm2, ...$ Now, we can combine Eqs. (2.489) and (2.490) as

$$N^{(2)}_{n_1,n_2+m_2} = q^{-2m_2}\{(1-q^{2m_2})N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1-q^{2m_2}}{1-q^2})\} \tag{2.578}$$

where $m_2 = 0, \pm1, \pm2, ...$ Let us substitute it into Eq. (2.577) to have

$$N^{(2)}_{n_1+m_1,n_2+m_2} = q^{-2(m_1+m_2)}\{(1-q^{2m_2})N^{(1)}_{n_1,n_2} + N^{(2)}_{n_1,n_2} - (\frac{1-q^{2m_2}}{1-q^2})\}. \tag{2.579}$$

If we evaluate it for $n_1 = 0$ and $n_2 = 0$ and then change m_1, m_2 into n_1, n_2 respectively, we find

$$N^{(2)}_{n_1,n_2} = q^{-2(n_1+n_2)}\{(1-q^{2n_2})N^{(1)}_{0,0} + N^{(2)}_{0,0} - (\frac{1-q^{2n_2}}{1-q^2})\}. \tag{2.580}$$

According to Eq. (2.570), it becomes

$$N^{(2)}_{n_1,n_2} = q^{-2(n_1+n_2)}\{N^{(2)}_{0,0} - (\frac{1-q^{2n_2}}{1-q^2})\} \tag{2.581}$$

where $n_1 = 0, -1, -2, ...$ and $n_2 = 0, \pm 1, \pm 2, ...$

Finally, the third anomalous case is the one in which $0 < q < 1$, $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$. We can write

$$a_1^{\dagger} a_1 |0 n_2\rangle = 0 \tag{2.582}$$

using Eq. (2.548). Then we have

$$N^{(1)}_{0,n_2} |0 n_2\rangle = 0 \tag{2.583}$$

from Eqs. (2.300) and (2.414). It is clearly seen that

$$N^{(1)}_{0,n_2} = 0. \tag{2.584}$$

For $n_2 = 0$, it becomes

$$N^{(1)}_{0,0} = 0. \tag{2.585}$$

Now, let us combine Eqs. (2.463) and (2.464) to write

$$N^{(1)}_{n_1,n_2+m_2} = N^{(1)}_{n_1,n_2} \tag{2.586}$$

where $m_2 = 0, \pm 1, \pm 2, ...$ Next, we write $n_1 + m_1$ instead of n_1 to obtain

$$N^{(1)}_{n_1+m_1,n_2+m_2} = N^{(1)}_{n_1+m_1,n_2} \tag{2.587}$$

where $m_1 = 0, 1, 2, ...$ For $n_1 = 0$ and $n_2 = 0$ this equation becomes

$$N^{(1)}_{m_1,m_2} = N^{(1)}_{m_1,0}. \tag{2.588}$$

To find $N^{(1)}_{m_1,0}$, we rewrite Eq. (2.462) for $n_1 = 0$ and $n_2 = 0$. Hence we have

$$N^{(1)}_{m_1,0} = q^{2m_1} N^{(1)}_{0,0} + (\frac{1 - q^{2m_1}}{1 - q^2}) \tag{2.589}$$

where $m_1 = 0, 1, 2, ...$ If we use Eq. (2.585), we find

$$N^{(1)}_{m_1,0} = \frac{1 - q^{2m_1}}{1 - q^2}. \tag{2.590}$$

Inserting this into Eq. (2.588) and changing m_1, m_2 into n_1, n_2 respectively, we obtain

$$N^{(1)}_{n_1,n_2} = \frac{1 - q^{2n_1}}{1 - q^2} \tag{2.591}$$

where $n_1 = 0, 1, 2, ...$ and $n_2 = 0, \pm 1, \pm 2, ...$ To find $N^{(2)}_{n_1,n_2}$, we use Eq. (2.466). Then we get

$$N^{(2)}_{n_1+m_1,n_2} = q^{2m_1} N^{(2)}_{n_1,n_2}. \tag{2.592}$$

where $m_1 = 0, 1, 2, ...$ From it, we can write

$$N^{(2)}_{n_1+m_1,n_2+m_2} = q^{2m_1} N^{(2)}_{n_1,n_2+m_2}. \tag{2.593}$$

Now, let us combine Eqs. (2.467) and (2.468) to get

$$N^{(2)}_{n_1,n_2+m_2} = (q^{2m_2} - 1) N^{(1)}_{n_1,n_2} + q^{2m_2} N^{(2)}_{n_1,n_2} + (\frac{1 - q^{2m_2}}{1 - q^2}) \tag{2.594}$$

where $m_2 = 0, \pm 1, \pm 2, ...$ Let us feed this into Eq. (2.593) to have

$$N^{(2)}_{n_1+m_1,n_2+m_2} = q^{2m_1} \{(q^{2m_2} - 1) N^{(1)}_{n_1,n_2} + q^{2m_2} N^{(2)}_{n_1,n_2} + (\frac{1 - q^{2m_2}}{1 - q^2})\}. \tag{2.595}$$

For $n_1 = 0$ and $n_2 = 0$, it becomes

$$N^{(2)}_{m_1,m_2} = q^{2m_1}\{(q^{2m_2} - 1)N^{(1)}_{0,0} + q^{2m_2}N^{(2)}_{0,0} + (\frac{1 - q^{2m_2}}{1 - q^2})\}. \tag{2.596}$$

If we use Eq. (2.585) and change m_1, m_2 into n_1, n_2 respectively, we get

$$N^{(2)}_{n_1,n_2} = q^{2n_1}\{q^{2n_2}N^{(2)}_{0,0} + (\frac{1 - q^{2n_2}}{1 - q^2})\} \tag{2.597}$$

where $n_1 = 0, 1, 2, ...$ and $n_2 = 0, \pm1, \pm2, ...$

Let us summarize $N^{(1)}_{n_1,n_2}$ and $N^{(2)}_{n_1,n_2}$ for the three anomalous cases.

For $0 < q < 1$, $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$N^{(1)}_{n_1,n_2} = q^{-2n_1}\{N^{(1)}_{0,0} - (\frac{1 - q^{2n_1}}{1 - q^2})\} \tag{2.598}$$

where $n_1 = 0, \pm1, \pm2, ...$ and $n_2 = 0, 1, 2, ...,$

$$N^{(2)}_{n_1,n_2} = q^{-2n_1}(q^{-2(n_2+1)} - 1)\{N^{(1)}_{0,0} - (\frac{1}{1 - q^2})\} \tag{2.599}$$

where $n_1 = 0, \pm1, \pm2, ...$ and $n_2 = 0, 1, 2, ...$

For $0 < q < 1$, $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$N^{(1)}_{n_1,n_2} = \frac{1 - q^{-2n_1}}{1 - q^2} \tag{2.600}$$

where $n_1 = 0, -1, -2, ...$ and $n_2 = 0, \pm1, \pm2, ...,$

$$N^{(2)}_{n_1,n_2} = q^{-2(n_1+n_2)}\{N^{(2)}_{0,0} - (\frac{1 - q^{2n_2}}{1 - q^2})\} \tag{2.601}$$

where $n_1 = 0, -1, -2, ...$ and $n_2 = 0, \pm1, \pm2, ...$

For $0 < q < 1$, $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$N^{(1)}_{n_1,n_2} = \frac{1-q^{2n_1}}{1-q^2} \tag{2.602}$$

where $n_1 = 0, 1, 2, ...$ and $n_2 = 0, \pm1, \pm2, ...$,

$$N^{(2)}_{n_1,n_2} = q^{n_1}\{q^{2n_2}N^{(2)}_{0,0} + (\frac{1-q^{2n_2}}{1-q^2})\} \tag{2.603}$$

where $n_1 = 0, 1, 2, ...$ and $n_2 = 0, \pm1, \pm2, ...$

Hereafter, we will be interested in the uncertainty relations for the momentum and position operators. Here we will examine the uncertainty relations for the energy eigenstates. We will use Eqs. (2.291)-(2.294) to evaluate the following commutation relations. Firstly, we write

$$[P_1, P_2] = [\frac{i}{\sqrt{2}}(a_1^\dagger - a_1), \frac{i}{\sqrt{2}}(a_2^\dagger - a_2)]. \tag{2.604}$$

Evidently, we get

$$[P_1, P_2] = -\frac{1}{2}([a_1^\dagger, a_2^\dagger] - [a_1^\dagger, a_2] - [a_1, a_2^\dagger] + [a_1, a_2]). \tag{2.605}$$

Since we have

$$\langle n_1 n_2 | a_1 a_2 | n_1 n_2 \rangle = 0, \tag{2.606}$$

$$\langle n_1 n_2 | a_1^\dagger a_2 | n_1 n_2 \rangle = 0, \tag{2.607}$$

$$\langle n_1 n_2 | a_1 a_2^\dagger | n_1 n_2 \rangle = 0 \tag{2.608}$$

and

$$\langle n_1 n_2 | a_1^\dagger a_2^\dagger | n_1 n_2 \rangle = 0, \tag{2.609}$$

we reach

$$\langle n_1 n_2 | [P_1, P_2] | n_1 n_2 \rangle = 0. \tag{2.610}$$

Similarly, we write

$$[P_1, X_2] = [\frac{i}{\sqrt{2}}(a_1^\dagger - a_1), \frac{1}{\sqrt{2}}(a_2^\dagger + a_2)]. \tag{2.611}$$

Obviously, we have

$$[P_1, X_2] = \frac{i}{2}([a_1^\dagger, a_2^\dagger] + [a_1^\dagger, a_2] - [a_1, a_2^\dagger] - [a_1, a_2]) \tag{2.612}$$

and then

$$\langle n_1 n_2 | [P_1, X_2] | n_1 n_2 \rangle = 0 \tag{2.613}$$

from Eqs. (2.606)-(2.609). We continue writing

$$[P_2, X_1] = [\frac{i}{\sqrt{2}}(a_2^\dagger - a_2), \frac{1}{\sqrt{2}}(a_1^\dagger + a_1)]. \tag{2.614}$$

It follows that

$$[P_2, X_1] = \frac{i}{2}([a_2^\dagger, a_1^\dagger] + [a_2^\dagger, a_1] - [a_2, a_1^\dagger] - [a_2, a_1]) \tag{2.615}$$

and then

$$\langle n_1 n_2 | [P_2, X_1] | n_1 n_2 \rangle = 0 \tag{2.616}$$

from Eqs. (2.606)-(2.609). Finally, we have

$$[X_1, X_2] = [\frac{1}{\sqrt{2}}(a_1^\dagger + a_1), \frac{1}{\sqrt{2}}(a_2^\dagger + a_2)] \tag{2.617}$$

and

$$[X_1, X_2] = \frac{1}{2}([a_1^\dagger, a_2^\dagger] + [a_1^\dagger, a_2] + [a_1, a_2^\dagger] + [a_1, a_2]). \tag{2.618}$$

Therefore one can show that

$$\langle n_1 n_2 | [X_1, X_2] | n_1 n_2 \rangle = 0 \tag{2.619}$$

from Eqs. (2.606)-(2.609). If we use Eq. (1.25), we obtain

$$\Delta P_1 \Delta P_2 \geq 0, \tag{2.620}$$

$$\Delta P_1 \Delta X_2 \geq 0, \tag{2.621}$$

$$\Delta P_2 \Delta X_1 \geq 0 \tag{2.622}$$

and

$$\Delta X_1 \Delta X_2 \geq 0 \tag{2.623}$$

according to Eqs. (2.610), (2.613), (2.616) and (2.619) respectively. Let us calculate the remaining commutation relations to obtain the corresponding uncertainty relations. It is evident that

$$[P_1, X_1] = \frac{i}{2}[a_1^\dagger - a_1, a_1^\dagger + a_1]. \tag{2.624}$$

It follows that

$$[P_1, X_1] = i[a_1^\dagger, a_1]. \tag{2.625}$$

We use Eq. (2.284) to write

$$[P_1, X_1] = i(a_1^\dagger a_1 - q^2 a_1^\dagger a_1 - 1). \tag{2.626}$$

If we tidy up it, we get

$$[P_1, X_1] = i\{(1 - q^2)a_1^\dagger a_1 - 1\}. \tag{2.627}$$

From Eq. (1.25), one can immediately get

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}|(1 - q^2)\langle\Psi|a_1^\dagger a_1|\Psi\rangle - 1|. \tag{2.628}$$

Similarly, we have

$$[P_2, X_2] = \frac{i}{2}[a_2^\dagger - a_2, a_2^\dagger + a_2]. \tag{2.629}$$

Then it is obvious that

$$[P_2, X_2] = i[a_2^\dagger, a_2]. \tag{2.630}$$

We use Eqs. (2.285) and (2.284) to obtain

$$[P_2, X_2] = i(a_2^\dagger a_2 - 1 - q^2 a_1^\dagger a_1 + a_1^\dagger a_1 - q^2 a_2^\dagger a_2). \tag{2.631}$$

If we tidy up it, we find

$$[P_2, X_2] = i\{(1 - q^2)(a_1^\dagger a_1 + a_2^\dagger a_2) - 1\}. \tag{2.632}$$

In terms of the hamiltonian, it becomes

$$[P_2, X_2] = i\{(\frac{2}{1+q^2})^2(1-q^2)\{H-(\frac{3+q^2}{4})\}-1\}. \tag{2.633}$$

from Eq. (2.299). More compactly, we have

$$[P_2, X_2] = i(\frac{2}{1+q^2})^2\{(1-q^2)H-1\}. \tag{2.634}$$

Therefore we can write

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}(\frac{2}{1+q^2})^2|(1-q^2)\langle\Psi|H|\Psi\rangle - 1| \tag{2.635}$$

from Eq. (1.25).

At this stage, we will recalculate Eqs. (2.628) and (2.635) for the energy eigenstates $|n_1 n_2\rangle$. In this case, we consider as

$$|\Psi\rangle = |n_1 n_2\rangle. \tag{2.636}$$

Firstly, Eq. (2.628) reads

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}|(1-q^2)N^{(1)}_{n_1,n_2} - 1|. \tag{2.637}$$

Let us evaluate this inequality for the following five cases. For this aim, we will substitute Eq. (2.481) into the above inequality for the following three cases. We will follow a different way for the remaining two cases. Thus we will have found the uncertainty relations for the energy eigenstates.

For $q > 1$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2} q^{2n_1}. \tag{2.638}$$

For $q = 1$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}. \tag{2.639}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2} q^{2n_1}. \tag{2.640}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\langle n_1 n_2 | a_1^\dagger a_1 | n_1 n_2 \rangle > \frac{1}{1 - q^2} \tag{2.641}$$

if we take Eqs. (2.483)-(2.486) into consideration. Since

$$\frac{1}{2} |(1 - q^2) \langle n_1 n_2 | a_1^\dagger a_1 | n_1 n_2 \rangle - 1| > 0, \tag{2.642}$$

we can confidently say that

$$\Delta P_1 \Delta X_1 > 0 \tag{2.643}$$

looking at Eq. (2.637).

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$, we have

$$\langle n_1 n_2 | a_1^\dagger a_1 | n_1 n_2 \rangle = \frac{1}{1-q^2} \tag{2.644}$$

from Eqs. (2.469) and (2.470). Finally, we substitute it into Eq. (2.637) to have

$$\Delta P_1 \Delta X_1 \geq 0. \tag{2.645}$$

Secondly, Eq. (2.635) reads

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}(\frac{2}{1+q^2})^2 |(1-q^2)\varepsilon_{n_1,n_2} - 1| \tag{2.646}$$

from Eq. (2.303). In the present case, we substitute Eq. (2.396) into the above inequality for the following three cases to have the uncertainty relations for the energy eigenstates. For the remaining two cases we will follow a different way.

For $q > 1$,

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}(\frac{2}{1+q^2})^2 q^{2(n_1+n_2)} |(1-q^2)(\frac{3+q^2}{4}) - 1|. \tag{2.647}$$

If we tidy up it, we get

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2} q^{2(n_1+n_2)}. \tag{2.648}$$

For $q = 1$, the above inequality reads

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}. \tag{2.649}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$, we again have

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2} q^{2(n_1+n_2)}. \tag{2.650}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$, we obviously have

$$|(1-q^2)\varepsilon_{n_1,n_2} - 1| > 0. \tag{2.651}$$

$$\Delta P_2 \Delta X_2 > 0. \tag{2.652}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$, we obtain

$$\Delta P_2 \Delta X_2 \geq 0 \tag{2.653}$$

if we use the value of ε_{n_1,n_2} in Eq. (2.646).

Now, we want to find the certainty relations. Let us first recall Eq. (2.144). According to it, we can write the following relations. It is clear that

$$(\Delta P_1)^2 \leq \langle \Psi | P_1^2 | \Psi \rangle, \tag{2.654}$$

$$(\Delta X_1)^2 \leq \langle \Psi | X_1^2 | \Psi \rangle, \tag{2.655}$$

$$(\Delta P_2)^2 \leq \langle \Psi | P_2^2 | \Psi \rangle \tag{2.656}$$

and

$$(\Delta X_2)^2 \leq \langle\Psi|X_2^2|\Psi\rangle. \tag{2.657}$$

Then we sandwich Eq. (2.288) between $\langle\Psi|$ and $|\Psi\rangle$ to write

$$\langle\Psi|P_1^2|\Psi\rangle + \langle\Psi|X_1^2|\Psi\rangle$$

$$+(\frac{1+q^2}{2})(\langle\Psi|P_2^2|\Psi\rangle + \langle\Psi|X_2^2|\Psi\rangle) = 2\langle\Psi|H|\Psi\rangle. \tag{2.658}$$

Since

$$\langle\Psi|P_1^2|\Psi\rangle = \langle\Psi|P_1^\dagger P_1|\Psi\rangle, \tag{2.659}$$

we find

$$\langle\Psi|P_1^2|\Psi\rangle = (P_1|\Psi\rangle)^\dagger(P_1|\Psi\rangle). \tag{2.660}$$

The right of the equation is the square of the length of the vector, $P_1|\Psi\rangle$. So it must be nonnegative. We express it as

$$\langle\Psi|P_1^2|\Psi\rangle \geq 0 \tag{2.661}$$

mathematically. Similarly, one can easily show that

$$\langle\Psi|X_1^2|\Psi\rangle \geq 0, \tag{2.662}$$

$$\langle\Psi|P_2^2|\Psi\rangle \geq 0 \tag{2.663}$$

and

$$\langle\Psi|X_2^2|\Psi\rangle \geq 0. \tag{2.664}$$

If we take these four equations and Eq. (2.658) into consideration, we obtain

$$\langle\Psi|P_1^2|\Psi\rangle \leq 2\langle\Psi|H|\Psi\rangle, \tag{2.665}$$

$$\langle\Psi|X_1^2|\Psi\rangle \leq 2\langle\Psi|H|\Psi\rangle, \tag{2.666}$$

$$\langle\Psi|P_2^2|\Psi\rangle \leq (\frac{4}{1+q^2})\langle\Psi|H|\Psi\rangle \tag{2.667}$$

and

$$\langle\Psi|X_2^2|\Psi\rangle \leq (\frac{4}{1+q^2})\langle\Psi|H|\Psi\rangle. \tag{2.668}$$

Let us combine Eqs. (2.654) and (2.665) to get

$$(\Delta P_1)^2 \leq 2\langle\Psi|H|\Psi\rangle. \tag{2.669}$$

Similarly, we get

$$(\Delta X_1)^2 \leq 2\langle\Psi|H|\Psi\rangle \tag{2.670}$$

according to Eqs. (2.655) and (2.666). Next, we have

$$(\Delta P_2)^2 \leq (\frac{4}{1+q^2})\langle\Psi|H|\Psi\rangle, \tag{2.671}$$

if we combine Eqs. (2.656) and (2.667). Finally, we obtain

$$(\Delta X_2)^2 \leq (\frac{4}{1+q^2})\langle\Psi|H|\Psi\rangle \tag{2.672}$$

from Eqs. (2.657) and (2.668).

First of all, we will evaluate the certainty relations for only the energy eigenstates. So we have

$$\langle n_1 n_2|H|n_1 n_2\rangle = \varepsilon_{n_1,n_2} \tag{2.673}$$

by multiplying Eq. (2.303) on the left by $\langle n_1 n_2|$ and exploiting the orthonormality of the basis, i.e.

$$\langle n_1 n_2|m_1 m_2\rangle = \delta_{n_1 m_1}\delta_{n_2 m_2}. \tag{2.674}$$

If we use Eq. (2.673), then Eqs. (2.669), (2.670), (2.671) and (2.672) read

$$(\Delta P_1)^2 \leq 2\varepsilon_{n_1,n_2}, \tag{2.675}$$

$$(\Delta X_1)^2 \leq 2\varepsilon_{n_1,n_2}, \tag{2.676}$$

$$(\Delta P_2)^2 \leq (\frac{4}{1+q^2})\varepsilon_{n_1,n_2} \tag{2.677}$$

and

$$(\Delta X_2)^2 \leq (\frac{4}{1+q^2})\varepsilon_{n_1,n_2} \tag{2.678}$$

respectively.

At this point, it is clear that we need the limits of ε_{n_1,n_2}. Here it is enough to know the upper limits of ε_{n_1,n_2} but for later use we will also compute the lower limits of ε_{n_1,n_2}. In the following calculations, we will use the fact that ε_{n_1,n_2} is an increasing function of n_1 and n_2. For all cases, ε_{n_1,n_2} approximate to its maximum value in the limit $n_1, n_2 \to \infty$.

For $q > 1$, ε_{n_1,n_2} takes the minimum value at $n_1 = 0$ and $n_2 = 0$. Since

$$\varepsilon_{0,0} = \frac{3+q^2}{4} \tag{2.679}$$

and

$$\lim_{n_1,n_2\to\infty} \varepsilon_{n_1,n_2} = \infty, \tag{2.680}$$

we have

$$\varepsilon_{n_1,n_2} \geq \frac{3+q^2}{4}. \tag{2.681}$$

For $q = 1$, ε_{n_1,n_2} takes the minimum value at $n_1 = 0$ and $n_2 = 0$. In this case, we obtain

$$\varepsilon_{0,0} = 1 \tag{2.682}$$

and

$$\lim_{n_1,n_2\to\infty} \varepsilon_{n_1,n_2} = \infty. \tag{2.683}$$

Therefore it is obvious that

$$\varepsilon_{n_1,n_2} \geq 1. \tag{2.684}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$, ε_{n_1,n_2} approximates to its minimum value in the limit $n_1, n_2 \to -\infty$. So we get

$$\lim_{n_1,n_2 \to -\infty} \varepsilon_{n_1,n_2} = \frac{1}{1-q^2} \tag{2.685}$$

and

$$\lim_{n_1,n_2 \to \infty} \varepsilon_{n_1,n_2} = \infty. \tag{2.686}$$

Accordingly, we have

$$\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}. \tag{2.687}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$, we have only one energy eigenvalue. Hence we get

$$\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}. \tag{2.688}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$, ε_{n_1,n_2} takes the minimum value at $n_1 = 0$ and $n_2 = 0$. Since

$$\varepsilon_{0,0} = \frac{3+q^2}{4} \tag{2.689}$$

and

$$\lim_{n_1,n_2 \to \infty} \varepsilon_{n_1,n_2} = \frac{1}{1-q^2}, \tag{2.690}$$

we can obviously write

$$\frac{3+q^2}{4} \leq \varepsilon_{n_1,n_2} < \frac{1}{1-q^2}. \tag{2.691}$$

In summary, we have the following limits of the energy eigenvalues.

For $q > 1$,

$$\varepsilon_{n_1,n_2} \geq \frac{3+q^2}{4}. \tag{2.692}$$

For q=1,

$$\varepsilon_{n_1,n_2} \geq 1. \tag{2.693}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}. \tag{2.694}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}. \tag{2.695}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{3+q^2}{4} \leq \varepsilon_{n_1,n_2} < \frac{1}{1-q^2}. \tag{2.696}$$

Now, let us continue to evaluate the certainty relations. As we can see, there is an upper limit for ε_{n_1,n_2} for only one case in which $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$. So we will find the certainty relations for only this case. Hence Eqs. (2.675), (2.676), (2.677) and (2.678) read

$$\Delta P_1 < \sqrt{\frac{2}{1-q^2}}, \tag{2.697}$$

$$\Delta X_1 < \sqrt{\frac{2}{1-q^2}}, \tag{2.698}$$

$$\Delta P_2 < \sqrt{\frac{4}{1-q^4}} \tag{2.699}$$

and

$$\Delta X_2 < \sqrt{\frac{4}{1-q^4}} \tag{2.700}$$

respectively. Then we use these four equations to obtain

$$\Delta P_1 \Delta P_2 < (\frac{2}{1-q^2})\sqrt{\frac{2}{1+q^2}}, \tag{2.701}$$

$$\Delta P_1 \Delta X_2 < (\frac{2}{1-q^2})\sqrt{\frac{2}{1+q^2}}, \tag{2.702}$$

$$\Delta P_2 \Delta X_1 < (\frac{2}{1-q^2})\sqrt{\frac{2}{1+q^2}}, \tag{2.703}$$

$$\Delta X_1 \Delta X_2 < (\frac{2}{1-q^2})\sqrt{\frac{2}{1+q^2}}, \tag{2.704}$$

$$\Delta P_1 \Delta X_1 < \frac{2}{1-q^2} \tag{2.705}$$

and

$$\Delta P_2 \Delta X_2 < \frac{4}{1-q^4}. \tag{2.706}$$

We will now summarize the uncertainty and certainty relations for the energy eigenstates that we have obtained by the first method.

The uncertainty and certainty relations for the momentum and position:

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\Delta P_1 < \sqrt{\frac{2}{1-q^2}}, \tag{2.707}$$

$$\Delta X_1 < \sqrt{\frac{2}{1-q^2}}, \tag{2.708}$$

$$\Delta P_2 < \sqrt{\frac{4}{1-q^4}}, \tag{2.709}$$

$$\Delta X_2 < \sqrt{\frac{4}{1-q^4}}. \tag{2.710}$$

The uncertainty and certainty relations for $\Delta P_1 \Delta P_2$, $\Delta P_1 \Delta X_2$, $\Delta P_2 \Delta X_1$ and $\Delta X_1 \Delta X_2$:

$$\Delta P_1 \Delta P_2 \geq 0, \tag{2.711}$$

$$\Delta P_1 \Delta X_2 \geq 0, \tag{2.712}$$

$$\Delta P_2 \Delta X_1 \geq 0, \tag{2.713}$$

$$\Delta X_1 \Delta X_2 \geq 0. \tag{2.714}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta P_2 < (\frac{2}{1-q^2})\sqrt{\frac{2}{1+q^2}}, \tag{2.715}$$

$$\Delta P_1 \Delta X_2 < (\frac{2}{1-q^2})\sqrt{\frac{2}{1+q^2}}, \tag{2.716}$$

$$\Delta P_2 \Delta X_1 < (\frac{2}{1-q^2})\sqrt{\frac{2}{1+q^2}}, \tag{2.717}$$

$$\Delta X_1 \Delta X_2 < (\frac{2}{1-q^2})\sqrt{\frac{2}{1+q^2}}. \tag{2.718}$$

The uncertainty and certainty relations for $\Delta P_1 \Delta X_1$:

For $q > 1$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2} q^{2n_1}. \tag{2.719}$$

For $q = 1$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}. \tag{2.720}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 > 0. \tag{2.721}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 \geq 0. \tag{2.722}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2} q^{2n_1}. \tag{2.723}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 < \frac{2}{1-q^2}. \tag{2.724}$$

The uncertainty and certainty relations for $\Delta P_2 \Delta X_2$:

For $q > 1$,

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2} q^{2(n_1+n_2)}. \tag{2.725}$$

For $q = 1$,

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}. \tag{2.726}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_2 \Delta X_2 > 0. \tag{2.727}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_2 \Delta X_2 \geq 0. \tag{2.728}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{2} q^{2(n_1+n_2)} \leq \Delta P_2 \Delta X_2 < \frac{4}{1-q^4}. \tag{2.729}$$

Up to now, we have studied the uncertainty and certainty relations for the energy eigenstates. However, from now on, we will generalize them to any state $|\Psi\rangle$. An arbitrary state can be expressed as

$$|\Psi\rangle = \sum_{n_1,n_2=0}^{\infty} C_{n_1,n_2} |n_1 n_2\rangle \tag{2.730}$$

where C_{n_1,n_2} satisfies

$$\sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 = 1. \tag{2.731}$$

For later use, we will calculate some expressions. Firstly, we write

$$\langle\Psi|a_1^\dagger a_1|\Psi\rangle = (\sum_{m_1,m_2=0}^{\infty} C^*_{m_1,m_2}\langle m_1m_2|)a_1^\dagger a_1(\sum_{n_1,n_2=0}^{\infty} C_{n_1,n_2}|n_1n_2\rangle). \tag{2.732}$$

After that, we exploit the orthonormality of the basis to have

$$\langle\Psi|a_1^\dagger a_1|\Psi\rangle = \sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 N^{(1)}_{n_1,n_2}. \tag{2.733}$$

Secondly, we write

$$\langle\Psi|H|\Psi\rangle = (\sum_{m_1,m_2=0}^{\infty} C^*_{m_1,m_2}\langle m_1m_2|)H(\sum_{n_1,n_2=0}^{\infty} C_{n_1,n_2}|n_1n_2\rangle). \tag{2.734}$$

Then we exploit the orthonormality of the basis to get

$$\langle\Psi|H|\Psi\rangle = \sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 \varepsilon_{n_1,n_2}. \tag{2.735}$$

To evaluate Eq. (2.637), finding the limits of $N^{(1)}_{n_1,n_2}$ is an essential task.

For $q > 1$, we look at Eq. (2.481) to have

$$N^{(1)}_{n_1,n_2} \geq 0. \tag{2.736}$$

For $q = 1$, it is clear that

$$N^{(1)}_{n_1,n_2} \geq 0 \tag{2.737}$$

according to Eq. (2.481).

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$, we take Eqs. (2.483)-(2.486) into consideration to decide

$$N^{(1)}_{n_1,n_2} = q^{-2n_1}\{N^{(1)}_{0,0} - (\frac{1-q^{2n_1}}{1-q^2})\} \tag{2.738}$$

where $n_1 = 0, \pm 1, \pm 2, ...$ This equation tells us that

$$N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}. \tag{2.739}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$, we have

$$N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2} \tag{2.740}$$

from Eqs. (2.461) - (2.464).

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$, we use Eq. (2.481) to get

$$0 \leq N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}. \tag{2.741}$$

In summary, we have the following limits of $N^{(1)}_{n_1,n_2}$:

For $q > 1$,

$$N^{(1)}_{n_1,n_2} \geq 0. \tag{2.742}$$

For $q = 1$,

$$N^{(1)}_{n_1,n_2} \geq 0. \tag{2.743}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}. \tag{2.744}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}. \tag{2.745}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$0 \leq N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}. \tag{2.746}$$

We are ready now to find the limits of $\langle\Psi|a_1^\dagger a_1|\Psi\rangle$. We use Eq. (2.733) to evaluate these limits for the following five cases.

For $q > 1$, since we have

$$\sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 N^{(1)}_{n_1,n_2} \geq 0 \tag{2.747}$$

from Eq. (2.742), we can conclude that

$$\langle\Psi|a_1^\dagger a_1|\Psi\rangle \geq 0. \tag{2.748}$$

For $q = 1$, similarly, we get

$$\sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 N^{(1)}_{n_1,n_2} \geq 0 \tag{2.749}$$

from Eq. (2.743). Then it follows that

$$\langle \Psi | a_1^\dagger a_1 | \Psi \rangle \geq 0. \tag{2.750}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$, we obtain

$$\sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 N^{(1)}_{n_1,n_2} > \sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 (\frac{1}{1-q^2}) \tag{2.751}$$

from Eq. (2.744). Then we use Eq. (2.731) to write

$$\langle \Psi | a_1^\dagger a_1 | \Psi \rangle > \frac{1}{1-q^2}. \tag{2.752}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$, we can obviously write

$$\sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 N^{(1)}_{n_1,n_2} = \sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 (\frac{1}{1-q^2}) \tag{2.753}$$

if we use Eq. (2.745). If we take Eq. (2.731) into account, we get

$$\langle \Psi | a_1^\dagger a_1 | \Psi \rangle = \frac{1}{1-q^2}. \tag{2.754}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$, we use Eq. (2.746) to get

$$0 \leq \sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 N^{(1)}_{n_1,n_2} < \sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 (\frac{1}{1-q^2}). \tag{2.755}$$

Then we obtain

$$0 \leq \langle\Psi|a_1^\dagger a_1|\Psi\rangle < \frac{1}{1-q^2} \tag{2.756}$$

using Eq. (2.731).

In summary, we have the following limits of $\langle\Psi|a_1^\dagger a_1|\Psi\rangle$:

For $q > 1$,

$$\langle\Psi|a_1^\dagger a_1|\Psi\rangle \geq 0. \tag{2.757}$$

For $q = 1$,

$$\langle\Psi|a_1^\dagger a_1|\Psi\rangle \geq 0. \tag{2.758}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\langle\Psi|a_1^\dagger a_1|\Psi\rangle > \frac{1}{1-q^2}. \tag{2.759}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\langle\Psi|a_1^\dagger a_1|\Psi\rangle = \frac{1}{1-q^2}. \tag{2.760}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$0 \leq \langle\Psi|a_1^\dagger a_1|\Psi\rangle < \frac{1}{1-q^2}. \tag{2.761}$$

Now, we will evaluate Eq. (2.628) for the following five cases.

For $q > 1$, we can write

$$(1-q^2)\langle\Psi|a_1^\dagger a_1|\Psi\rangle - 1 \leq -1 \tag{2.762}$$

using Eq. (2.757). It follows that

$$\frac{1}{2}|(1-q^2)\langle\Psi|a_1^\dagger a_1|\Psi\rangle - 1| \geq \frac{1}{2}. \tag{2.763}$$

Then we obviously have

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}. \tag{2.764}$$

For $q = 1$, Eq. (2.628) reads

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}. \tag{2.765}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$, using Eq. (2.759), we have

$$\frac{1}{2}|(1-q^2)\langle\Psi|a_1^\dagger a_1|\Psi\rangle - 1| > 0. \tag{2.766}$$

This gives us that

$$\Delta P_1 \Delta X_1 > 0. \tag{2.767}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$, we evidently get

$$\Delta P_1 \Delta X_1 \geq 0 \tag{2.768}$$

by using Eq. (2.760).

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$, since we have

$$-1 \leq (1-q^2)\langle\Psi|a_1^\dagger a_1|\Psi\rangle - 1 < 0 \tag{2.769}$$

from Eq. (2.761) and then

$$\frac{1}{2}|(1-q^2)\langle\Psi|a_1^\dagger a_1|\Psi\rangle - 1| > 0, \tag{2.770}$$

we can surely say that

$$\Delta P_1 \Delta X_1 > 0. \tag{2.771}$$

Now, we will compute Eq. (2.635). For this purpose, we will use the limits of ε_{n_1,n_2} to calculate the limits of $\langle\Psi|H|\Psi\rangle$. To calculate them, we need the expression in Eq. (2.735). So let us study for the following cases.

For $q > 1$, we obtain

$$\sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 \varepsilon_{n_1,n_2} \geq \sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 (\frac{3+q^2}{4}) \tag{2.772}$$

from Eq. (2.692). Let us use Eq. (2.731) to write

$$\langle\Psi|H|\Psi\rangle \geq \frac{3+q^2}{4}. \tag{2.773}$$

For $q = 1$, using Eq. (2.693) we write

$$\sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 \varepsilon_{n_1,n_2} \geq \sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2. \tag{2.774}$$

Then let us use Eq. (2.731) to obtain

$$\langle\Psi|H|\Psi\rangle \geq 1. \tag{2.775}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$, we have

$$\sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 \varepsilon_{n_1,n_2} > \sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 (\frac{1}{1-q^2}) \tag{2.776}$$

from Eq. (2.694). This gives us that

$$\langle\Psi|H|\Psi\rangle > \frac{1}{1-q^2} \tag{2.777}$$

if we use Eq. (2.731).

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$, we have

$$\sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 \varepsilon_{n_1,n_2} = \sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 (\frac{1}{1-q^2}) \tag{2.778}$$

from Eq. (2.695). Then we have

$$\langle\Psi|H|\Psi\rangle = \frac{1}{1-q^2} \tag{2.779}$$

from Eq. (2.731).

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$, we can write

$$\sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 (\frac{3+q^2}{4}) \leq \sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 \varepsilon_{n_1,n_2} < \sum_{n_1,n_2=0}^{\infty} |C_{n_1,n_2}|^2 (\frac{1}{1-q^2}) \tag{2.780}$$

from Eq. (2.696). If we use Eq. (2.731), we evidently have

$$\frac{3+q^2}{4} \leq \langle\Psi|H|\Psi\rangle < \frac{1}{1-q^2}. \tag{2.781}$$

In summary, we have the following limits of $\langle\Psi|H|\Psi\rangle$:

For $q > 1$,

$$\langle\Psi|H|\Psi\rangle \geq \frac{3+q^2}{4}. \tag{2.782}$$

For $q = 1$,

$$\langle\Psi|H|\Psi\rangle \geq 1. \tag{2.783}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\langle\Psi|H|\Psi\rangle > \frac{1}{1-q^2}. \tag{2.784}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\langle\Psi|H|\Psi\rangle = \frac{1}{1-q^2}. \tag{2.785}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{3+q^2}{4} \leq \langle\Psi|H|\Psi\rangle < \frac{1}{1-q^2}. \tag{2.786}$$

This is the end of studying the limits of $\langle\Psi|H|\Psi\rangle$. So we are ready now to evaluate Eq. (2.635).

For $q > 1$, we get

$$(1-q^2)\langle\Psi|H|\Psi\rangle - 1 \leq -(\frac{1+q^2}{2})^2 \tag{2.787}$$

if we use Eq. (2.782). It follows that

$$\frac{1}{2}(\frac{2}{1+q^2})^2|(1-q^2)\langle\Psi|H|\Psi\rangle - 1| \geq \frac{1}{2}. \tag{2.788}$$

Obviously, we obtain

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}. \tag{2.789}$$

For $q = 1$, Eq. (2.635) reads

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}. \tag{2.790}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$, we find

$$\frac{1}{2}(\frac{2}{1+q^2})^2|(1-q^2)\langle\Psi|H|\Psi\rangle - 1| > 0 \tag{2.791}$$

if we take Eq. (2.784) into consideration. Hence we have

$$\Delta P_2 \Delta X_2 > 0. \tag{2.792}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$, Eq. (2.635) reads

$$\Delta P_2 \Delta X_2 \geq 0 \tag{2.793}$$

if we use Eq. (2.785).

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$, we can write

$$-(\frac{1+q^2}{2})^2 \leq (1-q^2)\langle\Psi|H|\Psi\rangle - 1 < 0 \tag{2.794}$$

from Eq. (2.786). It follows that

$$0 < \frac{1}{2}(\frac{2}{1+q^2})^2|(1-q^2)\langle\Psi|H|\Psi\rangle - 1| < 6(\frac{1}{1+q^2})^2. \tag{2.795}$$

Therefore we get

$$\Delta P_2 \Delta X_2 > 0. \tag{2.796}$$

We have obtained the uncertainty relations for any state $|\Psi\rangle$. It remains to compute the certainty relations. The case in which $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$ is the unique case that we can study the certainty relations.

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$, let us recall Eq. (2.786). So using Eqs. (2.669)-(2.672) we obtain

$$\Delta P_1 < \sqrt{\frac{2}{1-q^2}}, \tag{2.797}$$

$$\Delta X_1 < \sqrt{\frac{2}{1-q^2}}, \tag{2.798}$$

$$\Delta P_2 < \sqrt{\frac{4}{1-q^4}} \tag{2.799}$$

and

$$\Delta X_2 < \sqrt{\frac{4}{1-q^4}} \tag{2.800}$$

respectively.

We will now summarize the uncertainty and certainty relations for any state $|\Psi\rangle$.

The certainty relations for the momentum and position:

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\Delta P_1, \Delta X_1 < \sqrt{\frac{2}{1-q^2}}, \tag{2.801}$$

$$\Delta P_2, \Delta X_2 < \sqrt{\frac{4}{1-q^4}}. \tag{2.802}$$

The uncertainty and certainty relations for $\Delta P_1 \Delta X_1$:

For $q > 1$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}. \tag{2.803}$$

For $q = 1$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}. \tag{2.804}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 > 0. \tag{2.805}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 \geq 0. \tag{2.806}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 > 0. \tag{2.807}$$

The uncertainty and certainty relations for $\Delta P_2 \Delta X_2$:

For $q > 1$,

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}. \tag{2.808}$$

For $q = 1$,

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}. \tag{2.809}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_2 \Delta X_2 > 0. \tag{2.810}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_2 \Delta X_2 \geq 0. \tag{2.811}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\Delta P_2 \Delta X_2 > 0. \tag{2.812}$$

We have finished the first part of this section. The second part will contain the calculations of ΔP_1, ΔX_1, ΔP_2, ΔX_2 and finding their limits. This is the second method to find the uncertainty and certainty relations. However, we will work for only the energy eigenstates here.

Let us start by recalling the implicit expression for an uncertainty. For an operator A, we have

$$(\Delta A)^2 = \langle n_1 n_2 | A^2 | n_1 n_2 \rangle - (\langle n_1 n_2 | A | n_1 n_2 \rangle)^2. \tag{2.813}$$

Using this expression, we will calculate the following uncertainties.

Now, we use Eq. (2.291) to write

$$(\Delta P_1)^2 = \langle n_1 n_2 | \{ -\frac{1}{2}(a_1^\dagger - a_1)^2 \} | n_1 n_2 \rangle - \{ \langle n_1 n_2 | \{ \frac{i}{\sqrt{2}}(a_1^\dagger - a_1) \} | n_1 n_2 \rangle \}^2. \tag{2.814}$$

We know that

$$\langle n_1 n_2 | a_1 | n_1 n_2 \rangle = 0, \tag{2.815}$$

$$\langle n_1 n_2 | a_1^\dagger | n_1 n_2 \rangle = 0, \tag{2.816}$$

$$\langle n_1 n_2 | a_1^2 | n_1 n_2 \rangle = 0 \tag{2.817}$$

and

$$\langle n_1 n_2 | (a_1^\dagger)^2 | n_1 n_2 \rangle = 0. \tag{2.818}$$

Accordingly, Eq. (2.814) reads

$$(\Delta P_1)^2 = \frac{1}{2} \langle n_1 n_2 | (a_1 a_1^\dagger + a_1^\dagger a_1) | n_1 n_2 \rangle. \tag{2.819}$$

To obtain it in terms of the q-deformed number operator N_1, we write

$$\Delta P_1 = \{ \frac{1}{2} + (\frac{1+q^2}{2}) \langle n_1 n_2 | a_1^\dagger a_1 | n_1 n_2 \rangle \}^{1/2} \tag{2.820}$$

using Eq. (2.284). This obviously gives us that

$$\Delta P_1 = \{ \frac{1}{2} + (\frac{1+q^2}{2}) N_{n_1,n_2}^{(1)} \}^{1/2}. \tag{2.821}$$

Let us now calculate the uncertainty for X_1. We can write

$$(\Delta X_1)^2 = \langle n_1 n_2 | \{ \frac{1}{2} (a_1^\dagger + a_1)^2 \} | n_1 n_2 \rangle - \{ \langle n_1 n_2 | \{ \frac{1}{\sqrt{2}} (a_1^\dagger + a_1) \} | n_1 n_2 \rangle \}^2 \tag{2.822}$$

using Eq. (2.292). Next using Eqs. (2.815)-(2.818), we obtain

$$(\Delta X_1)^2 = \frac{1}{2} \langle n_1 n_2 | (a_1 a_1^\dagger + a_1^\dagger a_1) | n_1 n_2 \rangle. \tag{2.823}$$

We note that

$$\Delta P_1 = \Delta X_1 \tag{2.824}$$

if we look at Eqs. (2.819) and (2.823).

To continue, let us look at Eq. (2.293). Then we write

$$(\Delta P_2)^2 = \langle n_1n_2|\{-\frac{1}{2}(a_2^\dagger - a_2)^2\}|n_1n_2\rangle - \{\langle n_1n_2|\{\frac{i}{\sqrt{2}}(a_2^\dagger - a_2)\}|n_1n_2\rangle\}^2. \quad (2.825)$$

Since we know that

$$\langle n_1n_2|a_2|n_1n_2\rangle = 0, \quad (2.826)$$

$$\langle n_1n_2|a_2^\dagger|n_1n_2\rangle = 0, \quad (2.827)$$

$$\langle n_1n_2|a_2^2|n_1n_2\rangle = 0 \quad (2.828)$$

and

$$\langle n_1n_2|(a_2^\dagger)^2|n_1n_2\rangle = 0, \quad (2.829)$$

we can clearly see that

$$(\Delta P_2)^2 = \frac{1}{2}\langle n_1n_2|(a_2a_2^\dagger + a_2^\dagger a_2)|n_1n_2\rangle. \quad (2.830)$$

Let us use Eqs. (2.285) and (2.284) to write

$$(\Delta P_2)^2 = \frac{1}{2}\langle n_1n_2|(1 + q^2a_1^\dagger a_1 - a_1^\dagger a_1 + q^2a_2^\dagger a_2 + a_2^\dagger a_2)|n_1n_2\rangle. \quad (2.831)$$

If we tidy up it, we get

$$\Delta P_2 = \{(\frac{q^2-1}{2})\langle n_1n_2|a_1^\dagger a_1|n_1n_2\rangle + (\frac{1+q^2}{2})\langle n_1n_2|a_2^\dagger a_2|n_1n_2\rangle + \frac{1}{2}\}^{1/2}. \quad (2.832)$$

Finally, we have

$$\Delta P_2 = \{(\frac{q^2-1}{2})N^{(1)}_{n_1,n_2} + (\frac{1+q^2}{2})N^{(2)}_{n_1,n_2} + \frac{1}{2}\}^{1/2}. \tag{2.833}$$

Now, let us write

$$(\Delta X_2)^2 = \langle n_1 n_2|\{\frac{1}{2}(a_2^\dagger + a_2)^2\}|n_1 n_2\rangle - \{\langle n_1 n_2|\{\frac{1}{\sqrt{2}}(a_2^\dagger + a_2)\}|n_1 n_2\rangle\}^2 \tag{2.834}$$

using Eq. (2.294). If we use Eqs. (2.826)-(2.829), we get

$$(\Delta X_2)^2 = \frac{1}{2}\langle n_1 n_2|(a_2 a_2^\dagger + a_2^\dagger a_2)|n_1 n_2\rangle. \tag{2.835}$$

We can easily see that

$$\Delta P_2 = \Delta X_2 \tag{2.836}$$

if we look at Eqs. (2.830) and (2.835).

In summary, we have

$$\Delta P_1, \Delta X_1 = \{\frac{1}{2} + (\frac{1+q^2}{2})N^{(1)}_{n_1,n_2}\}^{1/2}, \tag{2.837}$$

$$\Delta P_2, \Delta X_2 = \{(\frac{q^2-1}{2})N^{(1)}_{n_1,n_2} + (\frac{1+q^2}{2})N^{(2)}_{n_1,n_2} + \frac{1}{2}\}^{1/2}. \tag{2.838}$$

We are ready now to calculate the uncertainty and certainty relations for the energy eigenstates $|n_1 n_2\rangle$.

Firstly, we will keep Eq. (2.837) in mind while we are studying for the following five cases.

For $q > 1$, we have

$$\Delta P_1, \Delta X_1 \geq \frac{1}{\sqrt{2}} \tag{2.839}$$

from Eq. (2.742).

For $q = 1$, we get

$$\Delta P_1, \Delta X_1 \geq \frac{1}{\sqrt{2}} \tag{2.840}$$

using Eq. (2.743).

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$, we have

$$\{\frac{1}{2} + (\frac{1+q^2}{2})N^{(1)}_{n_1,n_2}\}^{1/2} > \sqrt{\frac{1}{1-q^2}} \tag{2.841}$$

from Eq. (2.744). This gives us that

$$\Delta P_1, \Delta X_1 > \sqrt{\frac{1}{1-q^2}}. \tag{2.842}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$, we get

$$\Delta P_1 = \Delta X_1 = \sqrt{\frac{1}{1-q^2}} \tag{2.843}$$

from Eq. (2.745).

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$, we obtain

$$\frac{1}{2} \leq \frac{1}{2} + (\frac{1+q^2}{2})N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2} \tag{2.844}$$

using Eq. (2.746). Therefore we have

$$\frac{1}{\sqrt{2}} \leq \Delta P_1, \Delta X_1 < \sqrt{\frac{1}{1-q^2}}. \tag{2.845}$$

Secondly, we will use Eq. (2.838) to study for P_2 and X_2. Here it is necessary to find the limits of $N^{(2)}_{n_1,n_2}$. We look at Eqs. (2.465) and (2.466) to evaluate the lower limit. So we conclude that

$$N^{(2)}_{n_1,n_2} \geq 0 \tag{2.846}$$

for all cases. In addition, we want to find the upper limits.

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$, we get

$$N^{(2)}_{n_1,n_2} \leq \frac{1}{1-q^2} \tag{2.847}$$

from Eq. (2.528).

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$, $N^{(2)}_{n_1,n_2}$ approximates to its maximum value at $n_1 = 0$ and in the limit $n_2 \to \infty$. Accordingly, we find

$$N^{(2)}_{n_1,n_2} < \frac{1}{1-q^2} \tag{2.848}$$

from Eq. (2.482).

In summary, we have the following limits of $N^{(2)}_{n_1,n_2}$.

For all of the cases,

$$N^{(2)}_{n_1,n_2} \geq 0. \tag{2.849}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$,

$$N^{(2)}_{n_1,n_2} \leq \frac{1}{1-q^2}. \tag{2.850}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$N^{(2)}_{n_1,n_2} < \frac{1}{1-q^2}. \tag{2.851}$$

Using Eq. (2.849), we can safely say that

$$(\Delta P_2)^2, (\Delta X_2)^2 \geq (\frac{q^2-1}{2})N^{(1)}_{n_1,n_2} + \frac{1}{2}. \tag{2.852}$$

Let us evaluate it for the following five cases.

For $q > 1$, we have

$$\Delta P_2, \Delta X_2 \geq \frac{1}{\sqrt{2}} \tag{2.853}$$

using Eq. (2.742).

For $q = 1$, Eq. (2.852) reads

$$\Delta P_2, \Delta X_2 \geq \frac{1}{\sqrt{2}}. \tag{2.854}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$, we have

$$(\frac{q^2-1}{2})N^{(1)}_{n_1,n_2} + \frac{1}{2} < 0 \tag{2.855}$$

from Eq. (2.744). So we conclude that

$$\Delta P_2, \Delta X_2 \geq 0. \tag{2.856}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$, Eq. (2.852) reads

$$\Delta P_2, \Delta X_2 \geq 0 \tag{2.857}$$

if we use Eq. (2.745).

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$, Eq. (2.852) reads

$$(\Delta P_2)^2, (\Delta X_2)^2 \geq \frac{1}{2}q^{2n_1} \tag{2.858}$$

if we use Eq. (2.481). Then we have

$$\Delta P_2, \Delta X_2 \geq \frac{1}{\sqrt{2}}q^{n_1} \tag{2.859}$$

In addition to these uncertainty relations we want to find the certainty relations. For this purpose, we will search for an upper limit for ΔP_1, ΔX_1, ΔP_2 and ΔX_2.

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$, we obtain

$$\frac{1}{2} + (\frac{1+q^2}{2})N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2} \tag{2.860}$$

if we use Eq. (2.746). Evidently, we find

$$\Delta P_1, \Delta X_1 < \sqrt{\frac{1}{1-q^2}} \tag{2.861}$$

if we take Eq. (2.837) into account.

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$, since $N^{(1)}_{n_1,n_2}$ is nonnegative, we can conclude that

$$(\frac{q^2-1}{2})N^{(1)}_{n_1,n_2} \leq 0. \tag{2.862}$$

We also have

$$(\frac{1+q^2}{2})N^{(2)}_{n_1,n_2} + \frac{1}{2} \leq \frac{1}{1-q^2} \tag{2.863}$$

from Eq. (2.850). Adding these two inequalities, we find

$$(\frac{q^2-1}{2})N^{(1)}_{n_1,n_2} + (\frac{1+q^2}{2})N^{(2)}_{n_1,n_2} + \frac{1}{2} \leq \frac{1}{1-q^2}. \tag{2.864}$$

It follows that

$$\Delta P_2, \Delta X_2 \leq \sqrt{\frac{1}{1-q^2}} \tag{2.865}$$

from Eq. (2.838).

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$, we can similarly conclude that

$$\Delta P_2, \Delta X_2 < \sqrt{\frac{1}{1-q^2}} \tag{2.866}$$

from Eq. (2.851).

We will now summarize the uncertainty and certainty relations for the energy eigenstates that we have obtained by the second method.

The uncertainty and certainty relations for ΔP_1, ΔX_1:

For $q > 1$,

$$\Delta P_1, \Delta X_1 \geq \frac{1}{\sqrt{2}}. \tag{2.867}$$

For $q = 1$,

$$\Delta P_1, \Delta X_1 \geq \frac{1}{\sqrt{2}}. \tag{2.868}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_1, \Delta X_1 > \sqrt{\frac{1}{1-q^2}}. \tag{2.869}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_1, \Delta X_1 = \sqrt{\frac{1}{1-q^2}}. \tag{2.870}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{\sqrt{2}} \leq \Delta P_1, \Delta X_1 < \sqrt{\frac{1}{1-q^2}}. \tag{2.871}$$

The uncertainty and certainty relations for ΔP_2, ΔX_2:

For $q > 1$,

$$\Delta P_2, \Delta X_2 \geq \frac{1}{\sqrt{2}}. \tag{2.872}$$

For $q = 1$,

$$\Delta P_2, \Delta X_2 \geq \frac{1}{\sqrt{2}}. \tag{2.873}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_2, \Delta X_2 \geq 0. \tag{2.874}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_2, \Delta X_2 \geq 0. \tag{2.875}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\Delta P_2, \Delta X_2 \geq \frac{1}{\sqrt{2}} q^{n_1}. \tag{2.876}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_2, \Delta X_2 \leq \sqrt{\frac{1}{1-q^2}}. \tag{2.877}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\Delta P_2, \Delta X_2 < \frac{1}{1-q^2}. \tag{2.878}$$

For simplicity, we consider that F denotes any quantity from $\Delta P_1 \Delta P_2$, $\Delta P_1 \Delta X_2$, $\Delta P_2 \Delta X_1$ and $\Delta X_1 \Delta X_2$. The uncertainty and certainty relations for F:

For $q > 1$,

$$F \geq \frac{1}{2}. \tag{2.879}$$

For $q = 1$,

$$F \geq \frac{1}{2}. \tag{2.880}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$F \geq 0. \tag{2.881}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$F \geq 0. \tag{2.882}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$F \geq \frac{1}{2}q^{n_1}. \tag{2.883}$$

The uncertainty and certainty relations for $\Delta P_1 \Delta X_1$:

For $q > 1$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}. \tag{2.884}$$

For $q = 1$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}. \tag{2.885}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 > \frac{1}{1-q^2}. \tag{2.886}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 = \frac{1}{1-q^2}. \tag{2.887}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{2} \leq \Delta P_1 \Delta X_1 < \frac{1}{1-q^2}. \tag{2.888}$$

The uncertainty and certainty relations for $\Delta P_2 \Delta X_2$:

For $q > 1$,

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}. \tag{2.889}$$

For $q = 1$,

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}. \tag{2.890}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_2 \Delta X_2 \geq 0. \tag{2.891}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_2 \Delta X_2 \geq 0. \tag{2.892}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2} q^{2n_1}. \tag{2.893}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_2 \Delta X_2 \leq \frac{1}{1-q^2}. \tag{2.894}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\Delta P_2 \Delta X_2 < \frac{1}{1-q^2}. \quad (2.895)$$

Now, we will make a comparison between the uncertainty and certainty relations obtained by the first method and the ones obtained by the second method. As we know, to be able to compare them, they must be in the same category. If we say it more explicitly, we must rearrange some of the results obtained by the second method so that they are categorized according to the energy eigenvalues, not $N^{(1)}_{n_1,n_2}$. However, we will lose some information due to this rearrangement. After the categorization, we will be able to select the most informative ones. In this way, we will have the best conclusions for the energy eigenstates. As an example, let us categorize the relations for $\Delta P_2 \Delta X_2$. It is clear that Eqs. (2.889) and (2.890) remain unchanged. We know that for the case in which $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$, there are three cases. They are ($N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$), ($N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$) and ($N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$). So we look at Eqs. (2.891), (2.892) and (2.893) to conclude that these three cases depending on $N^{(1)}_{n_1,n_2}$ share that

$$\Delta P_2 \Delta X_2 \geq 0. \quad (2.896)$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$, there are two cases which are ($N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$) and ($N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$). Hence we use Eqs. (2.892) and (2.893) to decide that the two cases depending on $N^{(1)}_{n_1,n_2}$ share that

$$\Delta P_2 \Delta X_2 \geq 0. \quad (2.897)$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$, we have only one case which is ($N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$). Therefore Eq. (2.893) is also valid.

Let us now summarize the uncertainty and certainty relations for the energy eigenstates after the categorization.

The uncertainty and certainty relations for ΔP_1, ΔX_1:

For $q > 1$,

$$\Delta P_1, \Delta X_1 \geq \frac{1}{\sqrt{2}}. \tag{2.898}$$

For $q = 1$,

$$\Delta P_1, \Delta X_1 \geq \frac{1}{\sqrt{2}}. \tag{2.899}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_1, \Delta X_1 > \sqrt{\frac{1}{1-q^2}}. \tag{2.900}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_1, \Delta X_1 = \sqrt{\frac{1}{1-q^2}}. \tag{2.901}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{\sqrt{2}} \leq \Delta P_1, \Delta X_1 < \sqrt{\frac{1}{1-q^2}}. \tag{2.902}$$

The uncertainty and certainty relations for ΔP_2, ΔX_2:

For $q > 1$,

$$\Delta P_2, \Delta X_2 \geq \frac{1}{\sqrt{2}}. \tag{2.903}$$

For $q = 1$,

$$\Delta P_2, \Delta X_2 \geq \frac{1}{\sqrt{2}}. \tag{2.904}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_2, \Delta X_2 \geq 0. \tag{2.905}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$,

$$0 \leq \Delta P_2, \Delta X_2 \leq \sqrt{\frac{1}{1-q^2}}. \tag{2.906}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{\sqrt{2}} q^{n_1} \leq \Delta P_2, \Delta X_2 < \sqrt{\frac{1}{1-q^2}}. \tag{2.907}$$

The uncertainty and certainty relations for F:

For $q > 1$,

$$F \geq \frac{1}{2}. \tag{2.908}$$

For $q = 1$,

$$F \geq \frac{1}{2}. \tag{2.909}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$F \geq 0. \tag{2.910}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$F \geq 0. \tag{2.911}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{2}q^{n_1} \leq F < \frac{1}{1-q^2}. \tag{2.912}$$

The uncertainty and certainty relations for $\Delta P_1 \Delta X_1$:

For $q > 1$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}. \tag{2.913}$$

For $q = 1$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}. \tag{2.914}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 > \frac{1}{1-q^2}. \tag{2.915}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 = \frac{1}{1-q^2}. \tag{2.916}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{2} \leq \Delta P_1 \Delta X_1 < \frac{1}{1-q^2}. \tag{2.917}$$

The uncertainty and certainty relations for $\Delta P_2 \Delta X_2$:

For $q > 1$,

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}. \tag{2.918}$$

For $q = 1$,

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}. \tag{2.919}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_2 \Delta X_2 \geq 0. \tag{2.920}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$,

$$0 \leq \Delta P_2 \Delta X_2 \leq \frac{1}{1-q^2}. \tag{2.921}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{2}q^{2n_1} \leq \Delta P_2 \Delta X_2 < \frac{1}{1-q^2}. \tag{2.922}$$

At this point, we are ready to make a comparison between these results and the ones obtained by the first method. For this purpose, we will need the following relations.

For $q > 1$,

$$\frac{1}{2} \leq \frac{1}{2}q^{2n_1}, \tag{2.923}$$

$$\frac{1}{2} \leq \frac{1}{2}q^{2(n_1+n_2)}. \tag{2.924}$$

For $q < 1$,

$$\frac{1}{2} \geq \frac{1}{2}q^{2n_1}, \tag{2.925}$$

$$\frac{1}{2}q^{2n_1} \geq \frac{1}{2}q^{2(n_1+n_2)}, \tag{2.926}$$

$$\frac{1}{1-q^2} < \frac{4}{1-q^4}. \tag{2.927}$$

Here comes the most informative results.

The uncertainty and certainty relations for ΔP_1, ΔX_1:

For $q > 1$,

$$\Delta P_1, \Delta X_1 \geq \frac{1}{\sqrt{2}}. \tag{2.928}$$

For $q = 1$,

$$\Delta P_1, \Delta X_1 \geq \frac{1}{\sqrt{2}}. \tag{2.929}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_1, \Delta X_1 > \sqrt{\frac{1}{1-q^2}}. \tag{2.930}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_1, \Delta X_1 = \sqrt{\frac{1}{1-q^2}}. \tag{2.931}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{\sqrt{2}} \leq \Delta P_1, \Delta X_1 < \sqrt{\frac{1}{1-q^2}}. \tag{2.932}$$

The uncertainty and certainty relations for ΔP_2, ΔX_2:

For $q > 1$,

$$\Delta P_2, \Delta X_2 \geq \frac{1}{\sqrt{2}}. \tag{2.933}$$

For $q = 1$,

$$\Delta P_2, \Delta X_2 \geq \frac{1}{\sqrt{2}}. \tag{2.934}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_2, \Delta X_2 \geq 0. \tag{2.935}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$,

$$0 \leq \Delta P_2, \Delta X_2 \leq \sqrt{\frac{1}{1-q^2}}. \tag{2.936}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{\sqrt{2}} q^{n_1} \leq \Delta P_2, \Delta X_2 < \sqrt{\frac{1}{1-q^2}}. \tag{2.937}$$

The uncertainty and certainty relations for F:

For $q > 1$,

$$F \geq \frac{1}{2}. \tag{2.938}$$

For $q = 1$,

$$F \geq \frac{1}{2}. \tag{2.939}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$F \geq 0. \tag{2.940}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$F \geq 0. \tag{2.941}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{2}q^{n_1} \leq F < \frac{1}{1-q^2}. \tag{2.942}$$

The uncertainty and certainty relations for $\Delta P_1 \Delta X_1$:

For $q > 1$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}q^{2n_1}. \tag{2.943}$$

For $q = 1$,

$$\Delta P_1 \Delta X_1 \geq \frac{1}{2}. \tag{2.944}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 > \frac{1}{1-q^2}. \tag{2.945}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} = \frac{1}{1-q^2}$,

$$\Delta P_1 \Delta X_1 = \frac{1}{1-q^2}. \tag{2.946}$$

For $0 < q < 1$ and $N^{(1)}_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{2} \leq \Delta P_1 \Delta X_1 < \frac{1}{1-q^2}. \tag{2.947}$$

The uncertainty and certainty relations for $\Delta P_2 \Delta X_2$:

For $q > 1$,

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2} q^{2(n_1+n_2)}. \tag{2.948}$$

For $q = 1$,

$$\Delta P_2 \Delta X_2 \geq \frac{1}{2}. \tag{2.949}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} > \frac{1}{1-q^2}$,

$$\Delta P_2 \Delta X_2 > 0. \tag{2.950}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} = \frac{1}{1-q^2}$,

$$0 \leq \Delta P_2 \Delta X_2 \leq \frac{1}{1-q^2}. \tag{2.951}$$

For $0 < q < 1$ and $\varepsilon_{n_1,n_2} < \frac{1}{1-q^2}$,

$$\frac{1}{2} q^{2n_1} \leq \Delta P_2 \Delta X_2 < \frac{1}{1-q^2}. \tag{2.952}$$

Finally, we will calculate the classical limits of $(\varepsilon_{n_1+1,n_2} - \varepsilon_{n_1,n_2})/\varepsilon_{n_1,n_2}$ for the cases in which there must occur a ground state. In other words, the behavior of this quantity in the limit n_1 and $n_2 \to \infty$ will be examined now. So let us write

$$\frac{\varepsilon_{n_1+1,n_2} - \varepsilon_{n_1,n_2}}{\varepsilon_{n_1,n_2}} = \frac{(\frac{3+q^2}{4})q^{2(n_1+n_2)+2} + (\frac{1-q^{2(n_1+n_2)+2}}{1-q^2}) - (\frac{3+q^2}{4})q^{2(n_1+n_2)} - (\frac{1-q^{2(n_1+n_2)}}{1-q^2})}{(\frac{3+q^2}{4})q^{2(n_1+n_2)} + (\frac{1-q^{2(n_1+n_2)}}{1-q^2})} \tag{2.953}$$

by using Eq. (2.405). If we tidy up it, we get

$$\frac{\varepsilon_{n_1+1,n_2} - \varepsilon_{n_1,n_2}}{\varepsilon_{n_1,n_2}} = \frac{(\frac{1+q^2}{2})^2 q^{2(n_1+n_2)}}{(\frac{3+q^2}{4})q^{2(n_1+n_2)} + (\frac{1-q^{2(n_1+n_2)}}{1-q^2})}. \tag{2.954}$$

Its another form is

$$\frac{\varepsilon_{n_1+1,n_2} - \varepsilon_{n_1,n_2}}{\varepsilon_{n_1,n_2}} = \frac{(\frac{1+q^2}{2})^2}{(\frac{3+q^2}{4}) + (\frac{q^{-2(n_1+n_2)}-1}{1-q^2})}. \tag{2.955}$$

For $q > 1$, we obtain

$$\lim_{n_1,n_2 \to \infty} \frac{\varepsilon_{n_1+1,n_2} - \varepsilon_{n_1,n_2}}{\varepsilon_{n_1,n_2}} = q^2 - 1 \tag{2.956}$$

if we take Eq. (2.955) into consideration.

For $q \geq \sqrt{2}$, we conclude that

$$\lim_{n_1,n_2\to\infty} \frac{\varepsilon_{n_1+1,n_2} - \varepsilon_{n_1,n_2}}{\varepsilon_{n_1,n_2}} \geq 1 \tag{2.957}$$

using the above equation. It follows that

$$\varepsilon_{n_1+1,n_2} \geq 2\varepsilon_{n_1,n_2}. \tag{2.958}$$

This tells us that the energy behaves unreasonably in the classical limit because the continuity of the energy is not seen here.

For $q = 1$, we have

$$\frac{\varepsilon_{n_1+1,n_2} - \varepsilon_{n_1,n_2}}{\varepsilon_{n_1,n_2}} = \frac{1}{1 + n_1 + n_2} \tag{2.959}$$

from Eq. (2.406). So one can easily see that

$$\lim_{n_1,n_2\to\infty} \frac{\varepsilon_{n_1+1,n_2} - \varepsilon_{n_1,n_2}}{\varepsilon_{n_1,n_2}} = 0. \tag{2.960}$$

Therefore, in the classical limit, the continuity condition is satisfied for this case.

For $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$, we use Eq. (2.954) to write

$$\lim_{n_1,n_2\to\infty} \frac{\varepsilon_{n_1+1,n_2} - \varepsilon_{n_1,n_2}}{\varepsilon_{n_1,n_2}} = 0. \tag{2.961}$$

It means that the energy is continuous in the classical limit as it must be.

We have come to the end of this section. In the next section, we will study the Fibonacci oscillators.

2.3. FIBONACCI OSCILLATORS

We will investigate Fibonacci oscillator with the two parameters in this section.

To begin with, we will define the annihilation and creation operators and the hamiltonian in terms of the momentum and position operators. For this purpose, let us look at the beginning of section (2.1). We will consider that Eqs. (2.4)-(2.9) are also valid here.

Let us now introduce some new concepts. The most general form of generalized integers is a sequence where an integer is generalized to the corresponding term in the sequence. Here we can mention a generalized Fibonacci sequence as an example. Each term of this sequence is a linear combination of the two previous terms with fixed weights.

Now, we are ready to describe the Fibonacci oscillators. Fibonacci oscillator[13, 27, 28] is the oscillator whose spectrum is given by a generalized Fibonacci sequence. This deformation of the quantum harmonic oscillator algebra is similar to the q-deformation of Lie groups and Lie algebras. This deformation is also the most general deformation of the quantum harmonic oscillator algebra whose spectrum is given by the natural numbers n. Fibonacci basic integers are defined as

$$[n] = \frac{q_1^n - q_2^n}{q_1 - q_2} \tag{2.962}$$

with the choice of initial conditions

$$[0] = 0, \tag{2.963}$$

$$[1] = 1 \tag{2.964}$$

and the condition

$$a|0\rangle = 0 \tag{2.965}$$

where $[n]$ also satisfies

$$a^{\dagger}a|n\rangle = [n]|n\rangle. \tag{2.966}$$

Here, the constants q_1 and q_2 are called the real parameters of the Fibonacci basic integers. Now, we want to introduce Fibonacci oscillators in a different way. So let us write

$$[N] = a^{\dagger}a = \frac{q_1^N - q_2^N}{q_1 - q_2} \tag{2.967}$$

and

$$[N+1] = aa^{\dagger} = \frac{q_1^{N+1} - q_2^{N+1}}{q_1 - q_2}. \tag{2.968}$$

Here, N satisfies that

$$N|n\rangle = n|n\rangle \tag{2.969}$$

where $n = 0, 1, 2, ...$ Then we can write

$$a^{\dagger}a|n\rangle = (\frac{q_1^n - q_2^n}{q_1 - q_2})|n\rangle \tag{2.970}$$

and

$$aa^{\dagger}|n\rangle = (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})|n\rangle. \tag{2.971}$$

One can easily show that Eq. (2.967) satisfies the generalized Fibonacci sequence which is

$$[N+2] = \alpha[N+1] + \beta[N] \tag{2.972}$$

where

$$\alpha = q_1 + q_2 \tag{2.973}$$

and

$$\beta = -q_1 q_2. \tag{2.974}$$

So that is why this oscillator is called the Fibonacci oscillator. For $\alpha = 1$ and $\beta = 1$, the sequence $[n]$ yields the well-known Fibonacci numbers which are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, ... \tag{2.975}$$

Now, let us study the algebra of Fibonacci oscillator. Using Eqs. (2.967), (2.968) and the definition of the Fibonacci basic integer, we obtain

$$aa^\dagger - q_1 a^\dagger a = q_2^N \tag{2.976}$$

and

$$aa^\dagger - q_2 a^\dagger a = q_1^N. \tag{2.977}$$

Then let us write

$$aN|n\rangle = na|n\rangle \tag{2.978}$$

using Eq. (2.969). If we consider that

$$a|n\rangle = F_n|n-1\rangle \tag{2.979}$$

where F_n are n dependent constants, then Eq. (2.978) becomes

$$aN|n\rangle = F_n n|n-1\rangle. \tag{2.980}$$

Since we have

$$(N+1)|n-1\rangle = n|n-1\rangle \tag{2.981}$$

from Eq. (2.969), Eq. (2.980) reads

$$aN|n\rangle = F_n(N+1)|n-1\rangle. \tag{2.982}$$

We again use Eq. (2.979) to write

$$\{aN\}|n\rangle = \{(N+1)a\}|n\rangle. \tag{2.983}$$

So we get

$$aN = (N+1)a \tag{2.984}$$

from it. Similarly, we have

$$af(N)|n\rangle = F_n f(n)|n-1\rangle \tag{2.985}$$

from Eqs. (2.969) and (2.979). Then we get

$$f(N+1)|n-1\rangle = f(n)|n-1\rangle \tag{2.986}$$

from Eq. (2.969). If we substitute it into Eq. (2.985) and use Eq. (2.979), we find

$$\{af(N)\}|n\rangle = \{f(N+1)a\}|n\rangle. \tag{2.987}$$

It follows that

$$af(N) = f(N+1)a. \tag{2.988}$$

At this stage, we will combine Eqs. (2.976) and (2.977) to see another aspect of the algebra of Fibonacci oscillator. Firstly, we multiply Eqs. (2.976) and (2.977) on the left by a. So we have

$$aaa^\dagger - q_1 aa^\dagger a = aq_2^N \tag{2.989}$$

and

$$aaa^\dagger - q_2 aa^\dagger a = aq_1^N \tag{2.990}$$

respectively. Let us now add these two equations to get

$$2aaa^\dagger - (q_1 + q_2)aa^\dagger a = a(q_1^N + q_2^N). \tag{2.991}$$

Next, we multiply Eq. (2.976) on the right by $q_2 a$ to have

$$q_2 aa^\dagger a - q_1 q_2 a^\dagger aa = q_2^{N+1} a \tag{2.992}$$

and we multiply Eq. (2.977) on the right by $q_1 a$ to have

$$q_1 aa^\dagger a - q_1 q_2 a^\dagger aa = q_1^{N+1} a. \tag{2.993}$$

Now, let us add these two equations to write

$$(q_1 + q_2)aa^\dagger a - 2q_1q_2a^\dagger aa = (q_1^{N+1} + q_2^{N+1})a. \tag{2.994}$$

We can easily see that

$$aaa^\dagger - (q_1 + q_2)aa^\dagger a + q_1q_2a^\dagger aa = \frac{1}{2}\{a(q_1^N + q_2^N) - (q_1^{N+1} + q_2^{N+1})a\} \tag{2.995}$$

if we subtract Eq. (2.994) from Eq. (2.991). Evidently, we get

$$aq_1^N = q_1^{N+1}a \tag{2.996}$$

from Eq. (2.988). Inserting this into Eq. (2.995), we obtain

$$aaa^\dagger - (q_1 + q_2)aa^\dagger a + q_1q_2a^\dagger aa = 0. \tag{2.997}$$

This is the most compact form that can be obtained from the combination of Eqs. (2.976) and (2.977).

At this point, we want to find the representations from Eq. (2.997). For this aim, we consider that

$$a^\dagger a|n\rangle = B_n|n\rangle \tag{2.998}$$

and

$$aa^\dagger|n\rangle = C_n|n\rangle. \tag{2.999}$$

If we multiply it on the left by $a^\dagger$, we find

$$a^\dagger(aa^\dagger|n\rangle) = a^\dagger(C_n|n\rangle). \tag{2.1000}$$

Then it is obvious that

$$a^\dagger a(a^\dagger|n\rangle) = C_n(a^\dagger|n\rangle). \tag{2.1001}$$

If we consider that

$$a^\dagger|n\rangle = G_n|n+1\rangle, \tag{2.1002}$$

then it is evident that

$$C_n = B_{n+1} \tag{2.1003}$$

from Eqs. (2.998) and (2.1001). So Eq. (2.999) reads

$$aa^\dagger|n\rangle = B_{n+1}|n\rangle. \tag{2.1004}$$

From Eq. (2.997), we get

$$a^\dagger aa = (\frac{1}{q_1q_2})\{(q_1+q_2)aa^\dagger a - aaa^\dagger\}. \tag{2.1005}$$

Then by multiplication of this equation on the right with $|n\rangle$, we obtain

$$a^\dagger a(a|n\rangle) = (\frac{1}{q_1q_2})\{(q_1+q_2)a(a^\dagger a|n\rangle) - a(aa^\dagger|n\rangle)\}. \tag{2.1006}$$

So it is clear that

$$a^\dagger a(a|n\rangle) = (\frac{1}{q_1q_2})\{(q_1+q_2)B_n - B_{n+1}\}(a|n\rangle) \tag{2.1007}$$

if we use Eqs. (2.998) and (2.1004). This immediately gives us that

$$B_{n-1} = B_n(\frac{q_1+q_2}{q_1q_2}) - B_{n+1}(\frac{1}{q_1q_2}) \tag{2.1008}$$

if we use Eqs. (2.979) and (2.998).

At first glance, to change the variables seems to simplify the calculations. On the contrary, we will see that it will make difficult the calculations. However, let us first change the variables as follows to experience it. We consider

$$b = q_1 + q_2 \tag{2.1009}$$

$$c = q_1 q_2. \tag{2.1010}$$

If we insert these two equations into Eq. (2.1008), we find

$$B_{n-1} = B_n(\frac{b}{c}) - B_{n+1}(\frac{1}{c}). \tag{2.1011}$$

This is the recursion formula for Fibonacci oscillator. Then we write $n-1$ instead of n in it to get

$$B_{n-2} = B_{n-1}(\frac{b}{c}) - B_n(\frac{1}{c}). \tag{2.1012}$$

This gives us that

$$B_{n-2} = B_n(\frac{b^2}{c^2} - \frac{1}{c}) - B_{n+1}(\frac{b}{c^2}) \tag{2.1013}$$

if we use Eq. (2.1011). Continuing in this way, we obtain

$$B_{n-3} = B_n(\frac{b^3}{c^3} - 2\frac{b}{c^2}) - B_{n+1}(\frac{b^2}{c^3} - \frac{1}{c^2}), \tag{2.1014}$$

$$B_{n-4} = B_n(\frac{b^4}{c^4} - 3\frac{b^2}{c^3} + \frac{1}{c^2}) - B_{n+1}(\frac{b^3}{c^4} - 2\frac{b}{c^3}), \tag{2.1015}$$

$$B_{n-5} = B_n(\frac{b^5}{c^5} - 4\frac{b^3}{c^4} + 3\frac{b}{c^3}) - B_{n+1}(\frac{b^4}{c^5} - 3\frac{b^2}{c^4} + \frac{1}{c^3}), \qquad (2.1016)$$

$$B_{n-6} = B_n(\frac{b^6}{c^6} - 5\frac{b^4}{c^5} + 6\frac{b^2}{c^4} - \frac{1}{c^3}) - B_{n+1}(\frac{b^5}{c^6} - 4\frac{b^3}{c^5} + 3\frac{b}{c^4}), \qquad (2.1017)$$

$$B_{n-7} = B_n(\frac{b^7}{c^7} - 6\frac{b^5}{c^6} + 10\frac{b^3}{c^5} - 4\frac{b}{c^4}) - B_{n+1}(\frac{b^6}{c^7} - 5\frac{b^4}{c^6} + 6\frac{b^2}{c^5} - \frac{1}{c^4}), \qquad (2.1018)$$

$$B_{n-8} = B_n(\frac{b^8}{c^8} - 7\frac{b^6}{c^7} + 15\frac{b^4}{c^6} - 10\frac{b^2}{c^5} + \frac{1}{c^4}) - B_{n+1}(\frac{b^7}{c^8} - 6\frac{b^5}{c^7} + 10\frac{b^3}{c^6} - 4\frac{b}{c^5}) \qquad (2.1019)$$

and so on. Now, it seems that to generalize it is very difficult because of the coefficients. However, to write the coefficients in a different form, more explicitly, to write them as combinations will help us to see the general form of the recursion formula easily. So these equations become

$$B_{n-1} = B_n(\binom{1}{1}\frac{b}{c}) - B_{n+1}(\binom{0}{0}\frac{1}{c}), \qquad (2.1020)$$

$$B_{n-2} = B_n(\binom{2}{2}\frac{b^2}{c^2} - \binom{1}{0}\frac{1}{c}) - B_{n+1}(\binom{1}{1}\frac{b}{c^2}), \qquad (2.1021)$$

$$B_{n-3} = B_n(\binom{3}{3}\frac{b^3}{c^3} - \binom{2}{1}\frac{b}{c^2}) - B_{n+1}(\binom{2}{2}\frac{b^2}{c^3} - \binom{1}{0}\frac{1}{c^2}), \qquad (2.1022)$$

$$B_{n-4} = B_n(\binom{4}{4}\frac{b^4}{c^4} - \binom{3}{2}\frac{b^2}{c^3} + \binom{2}{0}\frac{1}{c^2}) - B_{n+1}(\binom{3}{3}\frac{b^3}{c^4} - \binom{2}{1}\frac{b}{c^3}), \qquad (2.1023)$$

$$B_{n-5} = B_n(\binom{5}{5}\frac{b^5}{c^5} - \binom{4}{3}\frac{b^3}{c^4} + \binom{3}{1}\frac{b}{c^3}) - B_{n+1}(\binom{4}{4}\frac{b^4}{c^5} - \binom{3}{2}\frac{b^2}{c^4} + \binom{2}{0}\frac{1}{c^3}), \quad (2.1024)$$

$$B_{n-6} = B_n(\binom{6}{6}\frac{b^6}{c^6} - \binom{5}{4}\frac{b^4}{c^5} + \binom{4}{2}\frac{b^2}{c^4} - \binom{3}{0}\frac{1}{c^3})$$

$$-B_{n+1}(\binom{5}{5}\frac{b^5}{c^6} - \binom{4}{3}\frac{b^3}{c^5} + \binom{3}{1}\frac{b}{c^4}), \quad (2.1025)$$

$$B_{n-7} = B_n(\binom{7}{7}\frac{b^7}{c^7} - \binom{6}{5}\frac{b^5}{c^6} + \binom{5}{3}\frac{b^3}{c^5} - \binom{4}{1}\frac{b}{c^4})$$

$$-B_{n+1}(\binom{6}{6}\frac{b^6}{c^7} - \binom{5}{4}\frac{b^4}{c^6} + \binom{4}{2}\frac{b^2}{c^5} - \binom{3}{0}\frac{1}{c^4}), \quad (2.1026)$$

$$B_{n-8} = B_n(\binom{8}{8}\frac{b^8}{c^8} - \binom{7}{6}\frac{b^6}{c^7} + \binom{6}{4}\frac{b^4}{c^6} - \binom{5}{2}\frac{b^2}{c^5} + \binom{4}{0}\frac{1}{c^4})$$

$$-B_{n+1}(\binom{7}{7}\frac{b^7}{c^8} - \binom{6}{5}\frac{b^5}{c^7} + \binom{5}{3}\frac{b^3}{c^6} - \binom{4}{1}\frac{b}{c^5}) \quad (2.1027)$$

and so on. Now, we want to write these in a general form. Hence we find

$$B_{n-m} = B_n \sum_{k=0}^{l} (-1)^k \binom{m-k}{m-2k} \frac{b^{m-2k}}{c^{m-k}}$$

$$-B_{n+1}\sum_{k=0}^{p}(-1)^k\begin{pmatrix}m-k-1\\m-2k-1\end{pmatrix}\frac{b^{m-2k-1}}{c^{m-k}}. \tag{2.1028}$$

If we consider

$$\alpha = q_1 + q_2 \tag{2.1029}$$

and

$$\beta = -q_1 q_2, \tag{2.1030}$$

Eq. (2.1028) becomes

$$B_{n-m} = (-1)^m\{B_n\sum_{k=0}^{l}\begin{pmatrix}m-k\\m-2k\end{pmatrix}\frac{\alpha^{m-2k}}{\beta^{m-k}}$$

$$-B_{n+1}\sum_{k=0}^{p}\begin{pmatrix}m-k-1\\m-2k-1\end{pmatrix}\frac{\alpha^{m-2k-1}}{\beta^{m-k}}\}. \tag{2.1031}$$

We have

$$l = \begin{cases} m/2 & \text{if } m \text{ is even} \\ (m-1)/2 & \text{if } m \text{ is odd,} \end{cases}$$

and

$$p = \begin{cases} m/2-1 & \text{if } m \text{ is even} \\ (m-1)/2 & \text{if } m \text{ is odd} \end{cases}$$

for Eqs. (2.1028) and (2.1031). Let us now climb up in the basic integers. We can write

$$B_{n+1} = B_n(b) - B_{n-1}(c) \tag{2.1032}$$

from Eq. (2.1008). Then one can easily see that

$$B_{n+2} = B_n(b^2 - c) - B_{n-1}(bc), \tag{2.1033}$$

$$B_{n+3} = B_n(b^3 - 2bc) - B_{n-1}(b^2c - c^2), \tag{2.1034}$$

$$B_{n+4} = B_n(b^4 - 3b^2c + c^2) - B_{n-1}(b^3c - 2bc^2), \tag{2.1035}$$

$$B_{n+5} = B_n(b^5 - 4b^3c + 3bc^2) - B_{n-1}(b^4c - 3b^2c^2 + c^3) \tag{2.1036}$$

and so on. We can also write them as

$$B_{n+1} = B_n(\begin{pmatrix} 1 \\ 1 \end{pmatrix} b) - B_{n-1}(\begin{pmatrix} 0 \\ 0 \end{pmatrix} c), \tag{2.1037}$$

$$B_{n+2} = B_n(\begin{pmatrix} 2 \\ 2 \end{pmatrix} b^2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix} c) - B_{n-1}(\begin{pmatrix} 1 \\ 1 \end{pmatrix} bc), \tag{2.1038}$$

$$B_{n+3} = B_n(\begin{pmatrix} 3 \\ 3 \end{pmatrix} b^3 - \begin{pmatrix} 2 \\ 1 \end{pmatrix} bc) - B_{n-1}(\begin{pmatrix} 2 \\ 2 \end{pmatrix} b^2c - \begin{pmatrix} 1 \\ 0 \end{pmatrix} c^2), \tag{2.1039}$$

$$B_{n+4} = B_n(\begin{pmatrix}4\\4\end{pmatrix} b^4 - \begin{pmatrix}3\\2\end{pmatrix} b^2c + \begin{pmatrix}2\\0\end{pmatrix} c^2) - B_{n-1}(\begin{pmatrix}3\\3\end{pmatrix} b^3c - \begin{pmatrix}2\\1\end{pmatrix} bc^2), \tag{2.1040}$$

$$B_{n+5} = B_n(\begin{pmatrix}5\\5\end{pmatrix} b^5 - \begin{pmatrix}4\\3\end{pmatrix} b^3c + \begin{pmatrix}3\\1\end{pmatrix} bc^2) - B_{n-1}(\begin{pmatrix}4\\4\end{pmatrix} b^4c - \begin{pmatrix}3\\2\end{pmatrix} b^2c^2 + \begin{pmatrix}2\\0\end{pmatrix} c^3) \tag{2.1041}$$

and so on. Therefore we can generalize them as

$$B_{n+m} = B_n \sum_{k=0}^{l} (-1)^k \begin{pmatrix}m-k\\m-2k\end{pmatrix} b^{m-2k}c^k$$

$$-B_{n-1} \sum_{k=0}^{p} (-1)^k \begin{pmatrix}m-k-1\\m-2k-1\end{pmatrix} b^{m-2k-1}c^k \tag{2.1042}$$

and

$$B_{n+m} = (-1)^m\{B_n \sum_{k=0}^{l} \begin{pmatrix}m-k\\m-2k\end{pmatrix} \alpha^{m-2k}\beta^k$$

$$-B_{n-1} \sum_{k=0}^{p} \begin{pmatrix}m-k-1\\m-2k-1\end{pmatrix} \alpha^{m-2k-1}\beta^k\} \tag{2.1043}$$

where l and p are the same as before.

If there is a ground state, we recalculate Eqs. (2.1042) and (2.1043). For this aim, we first evaluate these equations for $n = 1$. Then we use the initial conditions in Eqs. (2.963) and (2.964). After that, we change $m+1$ into n. Hence we get

$$B_n = \sum_{k=0}^{l} (-1)^k \begin{pmatrix}n-k-1\\n-2k-1\end{pmatrix} b^{n-2k-1}c^k \tag{2.1044}$$

where $n = 2, 3, 4, ...$ and

$$B_n = -(-1)^n \sum_{k=0}^{l} \binom{n-k-1}{n-2k-1} \alpha^{n-2k-1}\beta^k \tag{2.1045}$$

where $n = 2, 3, 4, ...$ from Eqs. (2.1042) and (2.1043) respectively. Here again l and p remain unchanged.

At this point, let us return to our first variables which are q_1 and q_2. Then we write Eq. (2.1008) as

$$B_{n-1} = (q_1 q_2)^{-1}\{B_n(q_1 + q_2) - B_{n+1}\}. \tag{2.1046}$$

To generalize it, let us write

$$B_{n-2} = (q_1 q_2)^{-2}\{B_n(q_1^2 + q_1 q_2 + q_2^2) - B_{n+1}(q_1 + q_2)\} \tag{2.1047}$$

from Eq. (2.1046). Similarly, we find

$$B_{n-3} = (q_1 q_2)^{-3}\{B_n(q_1^3 + q_1^2 q_2 + q_1 q_2^2 + q_2^3) - B_{n+1}(q_1^2 + q_1 q_2 + q_2^2)\} \tag{2.1048}$$

and so on. These three equations give us an idea to find the general form of the recursion formula. So it is clear that

$$B_{n-m} = (q_1 q_2)^{-m}\{B_n(q_1^m + q_1^{m-1} q_2 + q_1^{m-2} q_2^2 + ... + q_2^m)$$

$$-B_{n+1}(q_1^{m-1} + q_1^{m-2} q_2 + q_1^{m-3} q_2^2 + ... + q_2^{m-1})\}. \tag{2.1049}$$

More compactly, we get

$$B_{n-m} = (q_1 q_2)^{-m}\{B_n(\frac{q_1^{m+1} - q_2^{m+1}}{q_1 - q_2}) - B_{n+1}(\frac{q_1^m - q_2^m}{q_1 - q_2})\}. \tag{2.1050}$$

To obtain the other general form which climbs up in the basic integers, we write

$$B_{n+1} = B_n(q_1 + q_2) - B_{n-1}(q_1q_2) \tag{2.1051}$$

from Eq. (2.1046). Next, we find

$$B_{n+2} = B_{n+1}(q_1 + q_2) - B_n(q_1q_2) \tag{2.1052}$$

from it. In a similar way, we get

$$B_{n+3} = B_{n+1}(q_1^2 + q_1q_2 + q_2^2) - B_n(q_1q_2)(q_1 + q_2) \tag{2.1053}$$

and so on. To have the other general form, we write

$$B_{n+m} = B_{n+1}(\frac{q_1^m - q_2^m}{q_1 - q_2}) - B_n(q_1q_2)(\frac{q_1^{m-1} - q_2^{m-1}}{q_1 - q_2}) \tag{2.1054}$$

looking at Eqs. (2.1051)-(2.1053).

If there is a ground state, we recalculate this equation. Firstly, let us calculate this equation for $n = 0$. Then we use Eqs. (2.963), (2.964) and change m into n to get

$$B_n = \frac{q_1^n - q_2^n}{q_1 - q_2} \tag{2.1055}$$

where $n = 0, 1, 2, ...$

As we mentioned before, we have seen that the last way of finding the representations is the easiest one. So this is the end of finding the representations of the algebra of Fibonacci oscillator.

From now on, we will be interested in the uncertainty and certainty relations for Fibonacci oscillator generally. Now, let us investigate the commutation relation for the momentum and position operators. We can write

$$[P, X] = \frac{i}{2}[a^\dagger - a, a^\dagger + a] \tag{2.1056}$$

using Eqs. (2.7) and (2.8). Then it becomes

$$[P, X] = \frac{i}{2}([a^\dagger, a] - [a, a^\dagger]). \tag{2.1057}$$

If we tidy it up, we get

$$[P, X] = i[a^\dagger, a]. \tag{2.1058}$$

Then the formula in Eq. (1.25) necessitates

$$\Delta P \Delta X \geq \frac{1}{2}|\langle n|[a^\dagger, a]|n\rangle|. \tag{2.1059}$$

Next, we obviously get

$$\Delta P \Delta X \geq \frac{1}{2}|\langle n|a^\dagger a|n\rangle - \langle n|aa^\dagger|n\rangle|. \tag{2.1060}$$

Then we can immediately write

$$\Delta P \Delta X \geq \frac{1}{2}|(\frac{q_1^n - q_2^n}{q_1 - q_2}) - (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})| \tag{2.1061}$$

from Eqs. (2.967) and (2.968). So this inequality is the uncertainty relation in the cases in which there must be a ground state such that $a|0\rangle = 0$.

Let us now examine the energy eigenvalues of the hamiltonian for this system. Then let us calculate the limits of $\varepsilon(n)$. If we substitute Eqs. (2.967) and (2.968) into

Eq. (2.9), we get

$$H = \frac{1}{2}\{(\frac{q_1^N - q_2^N}{q_1 - q_2}) + (\frac{q_1^{N+1} - q_2^{N+1}}{q_1 - q_2})\}. \tag{2.1062}$$

Then let us sandwich this equation between $\langle n|$ and $|n\rangle$ to have

$$\varepsilon_n = \frac{1}{2}\{(\frac{q_1^n - q_2^n}{q_1 - q_2}) + (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})\} \tag{2.1063}$$

where $n = 0, 1, 2, ...$ At this point, we wander the behavior of these energy eigenvalues. For this reason, we will modify the equation as

$$\varepsilon(n) = \frac{1}{2}\{(\frac{q_1^n - q_2^n}{q_1 - q_2}) + (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})\} \tag{2.1064}$$

where n is a nonnegative real number. Next, we take the derivative of this continuous energy function. So we have

$$\frac{d\varepsilon}{dn} = \frac{1}{2}(\frac{1}{q_1 - q_2})\{q_1^n \ln q_1(1 + q_1) - q_2^n \ln q_2(1 + q_2)\} \tag{2.1065}$$

where $q_1 \neq q_2$. Let us now analyze this derivative for the three cases which are

$(q_1, q_2 > 1)$, $\{(q_1 > 1$ and $q_2 < 1)$ or $(q_1 < 1$ and $q_2 > 1)\}$ and $(q_1, q_2 < 1)$.

For $q_1, q_2 > 1$, we clearly see that

$$\frac{d\varepsilon}{dn} > 0. \tag{2.1066}$$

This means that ε_n is an increasing function. ε_n takes the minimum value at $n = 0$ and approximates to its maximum value in the limit $n \to \infty$. Since we have

$$\varepsilon_0 = \frac{1}{2} \tag{2.1067}$$

and

$$\lim_{n\to\infty} \varepsilon_n = \infty, \tag{2.1068}$$

we obtain

$$\varepsilon_n \geq \frac{1}{2}. \tag{2.1069}$$

For ($q_1 > 1$ and $q_2 < 1$) or ($q_1 < 1$ and $q_2 > 1$), we again find that

$$\frac{d\varepsilon}{dn} > 0. \tag{2.1070}$$

So this is a similar situation to the one in the previous case. Therefore we conclude that

$$\varepsilon_n \geq \frac{1}{2}. \tag{2.1071}$$

For $q_1, q_2 < 1$, we will be able to find the maximum point. To show this, we write

$$\frac{d\varepsilon}{dn} = 0 \tag{2.1072}$$

and then write

$$(\frac{q_1}{q_2})^n = \frac{(1+q_2)\ln q_2}{(1+q_1)\ln q_1} \tag{2.1073}$$

from Eq. (2.1065). It follows that

$$n\ln(\frac{q_1}{q_2}) = \ln\{\frac{(1+q_2)\ln q_2}{(1+q_1)\ln q_1}\}. \tag{2.1074}$$

If we solve it for n, we get

$$n_m = \ln\{\frac{(1+q_2)\ln q_2}{(1+q_1)\ln q_1}\}/\ln(\frac{q_1}{q_2}) \tag{2.1075}$$

where n_m is the point when $\varepsilon(n)$ takes the maximum value. In addition, one can show that

$$\frac{d^2\varepsilon}{dn^2} < 0 \tag{2.1076}$$

at this point. So it is another task to be sure that $\varepsilon(n)$ takes the maximum value at this point. Here $\varepsilon(n)$ takes the minimum value at $n = 0$ and takes the maximum value $n = n_m$. However, ε_n is a bit different from $\varepsilon(n)$ because ε_n is a discrete function and $\varepsilon(n)$ is a continuous function. So the exact solution for ε_n^{max} is

$$\varepsilon_n^{max} = \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}. \tag{2.1077}$$

In addition, we have

$$\varepsilon_0 = \frac{1}{2} \tag{2.1078}$$

and

$$\lim_{n\to\infty} \varepsilon_n = 0. \tag{2.1079}$$

Hence we conclude that

$$\frac{1}{2} \le \varepsilon_n \le \varepsilon_n^{max}. \tag{2.1080}$$

In summary, we have the following limits of the energy eigenvalues.

For $q_1, q_2 > 1$,

$$\varepsilon_n \geq \frac{1}{2}. \tag{2.1081}$$

For ($q_1 > 1$ and $q_2 < 1$) or ($q_1 < 1$ and $q_2 > 1$),

$$\varepsilon_n \geq \frac{1}{2}. \tag{2.1082}$$

For $q_1, q_2 < 1$,

$$\frac{1}{2} \leq \varepsilon_n \leq \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}. \tag{2.1083}$$

As it can be seen easily, we can find the certainty relation for only the case in which $q_1, q_2 < 1$ because there is an upper limit of ε_n in only this case. In this section, Eqs. (2.156) and (2.157) are valid. Hence we find

$$\Delta P \leq \sqrt{2 \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}} \tag{2.1084}$$

and

$$\Delta X \leq \sqrt{2 \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}} \tag{2.1085}$$

from Eqs. (2.156), (2.157) and (2.1083). Now, to find the lower limits of ΔP and ΔX we follow a way similar to the one in section (2.1) while we are obtaining Eq. (2.185). Therefore we conclude that

$$\Delta P \geq \frac{|(\frac{q_1^n - q_2^n}{q_1 - q_2}) - (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})|}{\sqrt{8 \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}}} \tag{2.1086}$$

and

$$\Delta X \geq \frac{|(\frac{q_1^n - q_2^n}{q_1 - q_2}) - (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})|}{\sqrt{8 \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}}}. \tag{2.1087}$$

We will now summarize the uncertainty and certainty relations for the energy eigenstates that we have obtained by the first method.

The uncertainty and certainty relations for the momentum and position:

For $q_1, q_2 < 1$,

$$\frac{|(\frac{q_1^n - q_2^n}{q_1 - q_2}) - (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})|}{\sqrt{8 \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}}} \leq \Delta P, \Delta X \leq \sqrt{2 \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}}. \tag{2.1088}$$

The uncertainty and certainty relations for $\Delta P \Delta X$:

For $q_1, q_2 > 1$,

$$\Delta P \Delta X \geq \frac{1}{2}|(\frac{q_1^n - q_2^n}{q_1 - q_2}) - (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})|. \tag{2.1089}$$

For ($q_1 > 1$ and $q_2 < 1$) or ($q_1 < 1$ and $q_2 > 1$),

$$\Delta P \Delta X \geq \frac{1}{2}|(\frac{q_1^n - q_2^n}{q_1 - q_2}) - (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})|. \tag{2.1090}$$

For $q_1, q_2 < 1$,

$$\frac{1}{2}|(\frac{q_1^n - q_2^n}{q_1 - q_2}) - (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})| \leq \Delta P \Delta X \leq 2 \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}. \tag{2.1091}$$

Hereafter, we will study the uncertainty and certainty relations for the energy eigenstates by the second method. Firstly, let us calculate ΔP and ΔX by using the formula in Eq. (2.142). So we write

$$(\Delta P)^2 = \langle n|P^2|n\rangle - (\langle n|P|n\rangle)^2. \tag{2.1092}$$

If we look at Eqs. (2.232)-(2.238), then we can easily see that

$$\Delta P = \sqrt{\varepsilon_n}. \tag{2.1093}$$

Similarly, one can show that

$$\Delta X = \sqrt{\varepsilon_n}. \tag{2.1094}$$

Using these two equations and the limits of ε_n in Eqs. (2.1081)-(2.1083), we can immediately find the uncertainty and certainty relations for the energy eigenstates that we have obtained by the second method.

The uncertainty and certainty relations for the momentum and position:

For $q_1, q_2 > 1$,

$$\Delta P, \Delta X \geq \frac{1}{\sqrt{2}}. \tag{2.1095}$$

For ($q_1 > 1$ and $q_2 < 1$) or ($q_1 < 1$ and $q_2 > 1$),

$$\Delta P, \Delta X \geq \frac{1}{\sqrt{2}}. \tag{2.1096}$$

For $q_1, q_2 < 1$,

$$\frac{1}{\sqrt{2}} \leq \Delta P, \Delta X \leq \sqrt{\max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}}. \tag{2.1097}$$

The uncertainty and certainty relations for $\Delta P \Delta X$:

For $q_1, q_2 > 1$,

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.1098}$$

For ($q_1 > 1$ and $q_2 < 1$) or ($q_1 < 1$ and $q_2 > 1$),

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.1099}$$

For $q_1, q_2 < 1$,

$$\frac{1}{2} \leq \Delta P \Delta X \leq \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}. \tag{2.1100}$$

After the comparison, we will summarize the most informative results.

The uncertainty and certainty relations for the momentum and position:

For $q_1, q_2 > 1$,

$$\Delta P, \Delta X \geq \frac{1}{\sqrt{2}}. \tag{2.1101}$$

For $(q_1 > 1$ and $q_2 < 1)$ or $(q_1 < 1$ and $q_2 > 1)$,

$$\Delta P, \Delta X \geq \frac{1}{\sqrt{2}}. \tag{2.1102}$$

For $q_1, q_2 < 1$,

$$\frac{1}{\sqrt{2}} \leq \Delta P, \Delta X \leq \sqrt{\max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}}. \tag{2.1103}$$

The uncertainty and certainty relations for $\Delta P \Delta X$:

For $q_1, q_2 > 1$,

$$\Delta P \Delta X \geq \frac{1}{2}|(\frac{q_1^n - q_2^n}{q_1 - q_2}) - (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})|. \tag{2.1104}$$

For $(q_1 > 1$ and $q_2 < 1)$ or $(q_1 < 1$ and $q_2 > 1)$,

$$\Delta P \Delta X \geq \frac{1}{2}|(\frac{q_1^n - q_2^n}{q_1 - q_2}) - (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})|. \tag{2.1105}$$

For $q_1, q_2 < 1$,

$$\frac{1}{2}|(\frac{q_1^n - q_2^n}{q_1 - q_2}) - (\frac{q_1^{n+1} - q_2^{n+1}}{q_1 - q_2})| \leq \Delta P \Delta X \leq \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}. \tag{2.1106}$$

Therefore this is the end of the study of Fibonacci oscillator with two parameters. From now on, we will examine a special case for Fibonacci oscillator. This special case is obtained in the limit $q_1 \to q_2$. For simplicity, let us consider

$$q_2 = q. \tag{2.1107}$$

This special case of Fibonacci oscillator is very significant because although, the oscillator is deformed, its invariance group is undeformed, more explicitly, the algebra of the oscillator is $U(d)$ invariant. As it can be guessed, we will take the limits of some important expressions.

Firstly, Eq. (2.962) becomes

$$[n] = nq^{n-1} \tag{2.1108}$$

in this limit with the same initial conditions in Eqs. (2.963) and (2.964). Then Eqs. (2.967) and (2.968) become

$$[N] = a^{\dagger}a = Nq^{N-1} \tag{2.1109}$$

and

$$[N+1] = aa^{\dagger} = (N+1)q^{N} \tag{2.1110}$$

respectively. Eq. (2.1109) satisfies the generalized Fibonacci sequence in Eq. (2.972) when

$$\alpha = 2q \tag{2.1111}$$

and

$$\beta = -q^2. \tag{2.1112}$$

The algebra in Eqs. (2.976) and (2.977) changes as

$$aa^{\dagger} - qa^{\dagger}a = q^{N}. \tag{2.1113}$$

Accordingly, we write Eq. (2.997) as

$$aaa^\dagger - 2qaa^\dagger a + q^2 a^\dagger aa = 0 \tag{2.1114}$$

in this special limit. This algebra gives us the following recursion formula

$$B_{n-1} = B_n \frac{2}{q} - B_{n+1} \frac{1}{q^2}. \tag{2.1115}$$

Then we can immediately write

$$B_{n-m} = q^{-(m+1)} \{B_n (m+1) q - B_{n+1} m\} \tag{2.1116}$$

from Eq. (2.1050). We also have

$$B_{n+m} = q^{m-1} \{B_{n+1} m - B_n (m-1) q\} \tag{2.1117}$$

from Eq. (2.1054). The uncertainty relation for this special case is obtained from Eq. (2.1061). When we take the limit of this equation, we find

$$\Delta P \Delta X \geq \frac{1}{2} q^{n-1} |n - (n+1) q|. \tag{2.1118}$$

The hamiltonian for this system is expressed as

$$H = \frac{1}{2} \{N q^{N-1} + (N+1) q^N\} \tag{2.1119}$$

when we look at Eq. (2.1062). So it is obvious that

$$\varepsilon_n = \frac{1}{2} q^{n-1} \{n + (n+1) q\}. \tag{2.1120}$$

To see the behavior of these energy eigenvalues, let us change the above equation as

$$\varepsilon(n) = \frac{1}{2}q^{n-1}(n + nq + q) \tag{2.1121}$$

where n is a nonnegative real number. If we take the derivative of this function, we get

$$\frac{d\varepsilon}{dn} = \frac{1}{2}q^{n-1}\{(n + nq + q)\ln q + (1 + q)\}. \tag{2.1122}$$

So let us examine this derivative for the following three cases.

For $q > 1$, we can see that

$$\frac{d\varepsilon}{dn} > 0. \tag{2.1123}$$

Hence ε_n is an increasing function and we find

$$\varepsilon_n \geq \frac{1}{2}. \tag{2.1124}$$

For $q = 1$, we again find that

$$\frac{d\varepsilon}{dn} > 0. \tag{2.1125}$$

Then ε_n is an increasing function and we have

$$\varepsilon_n \geq \frac{1}{2}. \tag{2.1126}$$

For $0 < q < 1$, we have a maximum point. To confirm this, we set this derivative equal to zero and solve for n. So we begin by writing

$$\frac{d\varepsilon}{dn} = 0. \tag{2.1127}$$

Next, we write

$$n + nq + q = -(\frac{1+q}{\ln q}) \tag{2.1128}$$

from Eq. (2.1122). It gives us that

$$n_m = -(\frac{1}{\ln q}) - (\frac{q}{1+q}). \tag{2.1129}$$

In addition, one can show that

$$\frac{d^2\varepsilon}{dn^2} < 0 \tag{2.1130}$$

at $n = n_m$. Therefore we conclude that $\varepsilon(n)$ takes its maximum value at this point exactly. If we return to our original problem, we obtain

$$\varepsilon_n^{max} = \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}. \tag{2.1131}$$

Since we have

$$\varepsilon_0 = \frac{1}{2} \tag{2.1132}$$

and

$$\lim_{n\to\infty} \varepsilon_n = 0, \tag{2.1133}$$

we conclude that

$$\frac{1}{2} \leq \varepsilon_n \leq \varepsilon_n^{max}. \tag{2.1134}$$

In summary we have the following limits of the energy eigenvalues.

For $q > 1$,

$$\varepsilon_n \geq \frac{1}{2}. \tag{2.1135}$$

For $q = 1$,

$$\varepsilon_n \geq \frac{1}{2}. \tag{2.1136}$$

For $0 < q < 1$,

$$\frac{1}{2} \leq \varepsilon_n \leq \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}. \tag{2.1137}$$

To find the certainty relation for the case in which $0 < q < 1$, we look at Eqs. (2.1084) and (2.1085). Actually, these equations are also valid here. However, the value of n_m here is different from the one in these equations. More explicitly, n_m is given in Eq. (2.1129). Then from these equations and Eq. (2.1118) we find

$$\Delta P \geq \frac{q^{n-1}|n - (n+1)q|}{\sqrt{8 \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}}} \tag{2.1138}$$

and

$$\Delta X \geq \frac{q^{n-1}|n-(n+1)q|}{\sqrt{8\max\{\varepsilon([|n_m|]),\varepsilon([|n_m+1|])\}}}. \tag{2.1139}$$

We will now summarize the uncertainty and certainty relations for the energy eigenstates that we have obtained by the first method.

The uncertainty and certainty relations for the momentum and position:

For $0 < q < 1$,

$$\frac{q^{n-1}|n-(n+1)q|}{\sqrt{8\max\{\varepsilon([|n_m|]),\varepsilon([|n_m+1|])\}}} \leq \Delta P, \Delta X \leq \sqrt{2\max\{\varepsilon([|n_m|]),\varepsilon([|n_m+1|])\}}. \tag{2.1140}$$

The uncertainty and certainty relations for $\Delta P\Delta X$:

For $q > 1$,

$$\Delta P\Delta X \geq \frac{1}{2}q^{n-1}|n-(n+1)q|. \tag{2.1141}$$

For $q = 1$,

$$\Delta P\Delta X \geq \frac{1}{2}. \tag{2.1142}$$

For $0 < q < 1$,

$$\frac{1}{2}q^{n-1}|n-(n+1)q| \leq \Delta P\Delta X \leq 2\max\{\varepsilon([|n_m|]),\varepsilon([|n_m+1|])\}. \tag{2.1143}$$

We will now study the uncertainty and certainty relations for the energy eigenstates by the second method. Eqs. (2.1093) and (2.1094) are also valid here. So the limits of ε_n in Eqs. (2.1135)-(2.1137) are essential to find the uncertainty and certainty relations. Using these limits, we will summarize the uncertainty and certainty relations for the energy eigenstates.

The uncertainty and certainty relations for the momentum and position:

For $q > 1$,

$$\Delta P, \Delta X \geq \frac{1}{\sqrt{2}}. \tag{2.1144}$$

For $q = 1$,

$$\Delta P, \Delta X \geq \frac{1}{\sqrt{2}}. \tag{2.1145}$$

For $0 < q < 1$,

$$\frac{1}{\sqrt{2}} \leq \Delta P, \Delta X \leq \sqrt{\max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}}. \tag{2.1146}$$

The uncertainty and certainty relations for $\Delta P \Delta X$:

For $q > 1$,

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.1147}$$

For $q = 1$,

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.1148}$$

For $0 < q < 1$,

$$\frac{1}{2} \leq \Delta P \Delta X \leq \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}. \tag{2.1149}$$

After the comparison, we will summarize the most informative results.

The uncertainty and certainty relations for the momentum and position:

For $q > 1$,

$$\Delta P, \Delta X \geq \frac{1}{\sqrt{2}}. \tag{2.1150}$$

For $q = 1$,

$$\Delta P, \Delta X \geq \frac{1}{\sqrt{2}}. \tag{2.1151}$$

For $0 < q < 1$,

$$\frac{1}{\sqrt{2}} \leq \Delta P, \Delta X \leq \sqrt{\max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}}. \tag{2.1152}$$

The uncertainty and certainty relations for $\Delta P \Delta X$:

For $q > 1$,

$$\Delta P \Delta X \geq \frac{1}{2} q^{n-1} |n - (n+1)q|. \tag{2.1153}$$

For $q = 1$,

$$\Delta P \Delta X \geq \frac{1}{2}. \tag{2.1154}$$

For $0 < q < 1$,

$$\frac{1}{2} q^{n-1} |n - (n+1)q| \leq \Delta P \Delta X \leq \max\{\varepsilon([|n_m|]), \varepsilon([|n_m + 1|])\}. \tag{2.1155}$$

We have come to the end of this section. Meanwhile, we have come to the end of this chapter. So the only remaining part is the conclusion part.

3. CONCLUSION

We have studied the uncertainty and certainty relations for the q-oscillator and the Fibonacci oscillator in this thesis. We have noticed some remarkable features of these oscillators. We will mention some of them.

Firstly, the q-oscillator and the Fibonacci oscillator have some extra relations that the ordinary quantum oscillator does not have. Since the hamiltonian for the q-oscillator is bounded for the case in which $0 < q < 1$ and $\varepsilon_n < \frac{1}{1-q^2}$, the momentum and position are also bounded. This fact causes extra relations that only contain ΔP or ΔX.

Secondly, after the studies in this thesis, we have got an idea about the value of the real parameter q. The value of q may be found from the greatest value of ΔX for the universe since its position is bounded or any system whose position is bounded.

Finally, in addition to the one-dimensional q-oscillator, we studied the two-dimensional q-oscillator. Actually, the study of the two-dimensional q-oscillator is the first step for the study of the multi-dimensional q-oscillator and Fibonacci oscillator.

As a result, we talked about a few of the many features of the q-oscillator and Fibonacci oscillator here. Of course, for more information, one should go through the thesis.

REFERENCES

1. W. Heisenberg, Zeitshrift für Physik, **43**, 172 (1927).

2. L.D. Fadeev, N.Y. Reshetikhin and L.A. Takhtajan. *Quantization of Lie groups and Lie Algebras*, preprint LOMI (1987).

3. V.G. Drinfeld, *Quantum groups*, Proc. Intern. Congress of Mathematicians (Berkeley, CA), **Vol. 1** (1986) pp. 798-820.

4. M. Jimbo, *Lett. Math. Phys.* **11** (1986) 247.

5. S.L. Woronowicz, *Commun. Math. Phys.* **111** (1987) 613.

6. A.J. Macfarlane, *Journal of Physics* **A 22** (1989) 4581.

7. L.C. Biedenharn, *Journal of Physics* **A 22** (1989) L873.

8. T. Hayashi, *Commun. Math. Phys.* **127** (1990) 129.

9. M. Chaichian, P. Kulish and J. Lukierski, *Phys. Lett.* **B 237** (1990) 401.

10. M. Chaichian, P. Kulish and J. Lukierski, *Phys. Lett.* **B 262** (1991) 43.

11. D.B. Fairlie and C.K. Zachos, *Phys. Lett.* **B 256** (1991) 43.

12. A. Jannussis, G. Brodimas and R. Mignani, *Journal of Physics* **A 24** (1991) L775.

13. R. Chakrabarti and R. Jagannathan, *Journal of Physics* **A 24** (1991) L711.

14. C. Daskaloyannis, *Journal of Physics* **A 24** (1991) 789.

15. D. Bonatsos, C. Daskaloyannis and K. Kokkotas, *Journal of Physics* **A 24** (1991) L795.

16. M. Baker, D.D. Coon and S. Yu, *Phys. Rev.* **D 5** (1972) 1429.

17. S. Yu, *Phys. Rev.* **D 7** (1973) 1871.

18. Nuovo Cimento **28A** (1975) 203.

19. M. Arik, D.D. Coon and Y.M. Lam, "*Operator algebra of dual resonance models*", Journal of Mathematical Physics **16** (1975) 1776.

20. Arik, M. and D. D. Coon, "*Hilbert spaces of analytic functions and generalized coherent states* ", Journal of Mathematical Physics **Vol. 17**, pp. 524-527, 1976.

21. M. Arik and M. Mungan, "*q-oscillators and relativistic position operators*", Phys. Lett. **B 282**, 101 (1992).

22. W. Pusz and S.L. Woronowicz, *Rep. Math. Phys.* **27** (1989) 231.

23. M. Chaichian and P. Kulish, *Phys. Lett.* **B 234** (1990) 72.

24. P. Kulish and E. Damaskinsky, *Journal of Physics* **A 23** (1990) L415.

25. M. Arik, "*The q-difference operator, the quantum hyperplane, Hilbert spaces of analytic functions and q-oscillators*", Zeitshrift für Physik **C 51** (1991) 627.

26. M. Arik, "*From q-oscillators to quantum groups*" in "Symmetries in Science VI : From the rotation group to quantum algebras", Ed. B. Gruber, Plenum Press N.Y., **47** (1993).

27. M. Arik, E. Demircan, T. Turgut, L. Ekinci, M. Mungan, "*Fibonacci oscillators*", Zeitshrift für Physik **C 55**, 89 (1992).

28. G. Brodimas, A. Jannussis, R. Mignani, *Journal of Physics* A:Math. Gen. **24** (1991) L775.