ON SPECIAL SOLUTIONS OF ZAKHAROV–SCHULMAN EQUATIONS

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ABSTRACT

ON SPECIAL SOLUTIONS OF ZAKHAROV–SCHULMAN EQUATIONS

In this work, two types of special solutions for Zakharov–Schulman equations are studied. Existence of standing wave solutions are established by utilizing variational methods. First set conditions on the operators for the existence of Arkadiev–Pogrebkov–Polivanov type travelling wave solutions are derived. It is observed that there exist blow-up profiles whenever either of these special solutions exist.

ÖZET

ZAKHAROV–SCHULMAN DENKLEMLERİNİN ÖZEL ÇÖZÜMLERİ ÜZERİNE

Bu çalışmada Zakharov–Schulman denklemleri için iki tip özel çözüm incelenmiştir. Duran dalga çözümlerinin varlığı varyasyonel yöntemler kullanılarak kanıtlanmıştır. Arkadiev–Pogrebkov–Polivanov tipi yürüyen dalga çözümlerinin varlığı için denklemde yer alan diferansiyel operatörler üzerinde koşullar bulunmuştur. Her iki tip özel çözümün de varlığında patlama profillerinin var olduğu gözlemlenmiştir.

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LIST OF SYMBOLS/ABBREVIATIONS

$\square$	End of proof
$A \Subset B \subset \mathbb{R}^N$	The closure of A is a compact subset of B
$B_\rho(x)$	The closed disk in $\mathbb{R}^N$ with center x and radius ρ
$C(X,Y)$	The space of continuous functions from the topological space X to the topological space Y
$C(\Omega)$	The space of continuous functions from $\Omega \subseteq \mathbb{R}^N$ to $\mathbb{R}$ (or $\mathbb{C}$)
$C_0(\Omega)$	The space of continuous functions from $\Omega \subseteq \mathbb{R}^N$ to $\mathbb{R}$ (or $\mathbb{C}$) compactly supported in Ω
$C^k(\Omega)$	The space of continuous functions u such that $D^\alpha u \in C(\Omega)$ for all $\alpha \in \mathbb{N}^n$ with $\lvert\alpha\rvert \leqslant k$, $\Omega \subseteq \mathbb{R}^N$
$C^\infty(\Omega)$	The space of infinitely differentiable functions on $\Omega \subseteq \mathbb{R}^N$
$C_c^\infty(\Omega)$	The space of infinitely differentiable functions compactly supported in $\Omega \subseteq \mathbb{R}^N$
$\mathbb{C}$	The field of complex numbers
D^α	$\dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}}$, where $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ and $\lvert\alpha\rvert = \alpha_1 + \cdots + \alpha_n$
$\operatorname{ess\,sup}_\Omega f$	The essential supremum of f over $\Omega \subseteq \mathbb{R}^N$, namely, $\inf\{a \geqslant 0 \colon \operatorname{meas}\{x \colon f(x) > a\} = 0\}$
$\hat{f}$	Fourier transform of f
$H^m(\Omega)$	$W^{m,2}(\Omega)$, see below
$H^{-m}(\Omega)$	$W^{-m,2}(\Omega)$, see below
$H^s(\mathbb{R}^N)$	The Banach space of classes of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ (or $\mathbb{C}$) such that $(1+\lvert y\rvert^s)\widehat{u} \in L^2(\mathbb{R}^2)$ for $0 < s < \infty$, with norm $$\lVert u\rVert_{H^s(\mathbb{R}^2)} = \lVert(1+\lvert y\rvert^s)\widehat{u}\rVert_2$$
$\mathfrak{Im}(z)$	Imaginary part of z

$L^p(\Omega)$ — The Banach space of classes of measurable functions $u : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$ (or $\mathbb{C}$) such that $\int_\Omega |u(x)|^p < \infty$ if $1 \leqslant p < \infty$, or $\operatorname*{ess\,sup}_\Omega |u| < \infty$ if $p = \infty$. $L^p(\Omega)$ is equipped with the norm

$$\|u\|_p = \begin{cases} \left(\int\limits_\Omega |u(x)|^p dx \right)^{1/p}, & \text{if } p < \infty, \\ \operatorname*{ess\,sup}_\Omega |u|, & \text{if } p = \infty. \end{cases}$$

$\operatorname{meas}(\Omega)$ — Lebesgue measure of the set $\Omega \subseteq \mathbb{R}^N$

p' — The conjugate of p given by $\dfrac{1}{p} + \dfrac{1}{p'} = 1$

$\mathbb{R}$ — The field of real numbers

$\mathfrak{Re}(z)$ — Real part of z

$\mathcal{S}$ — Denotes the Schwartz class $\mathcal{S}(\mathbb{R}^N)$ of rapidly decreasing functions, i.e., the functions $u\colon \mathbb{R}^N \to \mathbb{C}$ such that $\sup\limits_{\boldsymbol{x} \in \mathbb{R}^N} |\boldsymbol{x}^\beta D^\alpha u(\boldsymbol{x})| < \infty$, for all $\alpha, \beta \in \mathbb{N}^N$

$\mathcal{S}'$ — Denotes tempered distributions $\mathcal{S}'(\mathbb{R}^N)$, namely the topological dual of $\mathcal{S}$

$\mathrm{SL}_2(\mathbb{R})$ — The special linear group of 2 by 2 matrices over the field $\mathbb{R}$ with determinant 1, with the group operations of ordinary matrix multiplication and matrix inversion

u_t — Partial derivative of $u(t, x)$ with respect to t

u_{x_i} — Partial derivative of $u(t, x)$ or $u(x)$ with respect to the ith space variable x_i

$W^{m,p}(\Omega)$ — The Banach space of classes of measurable functions $u : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$ (or $\Omega \subseteq \mathbb{R}^N \to \mathbb{C}$) such that $D^\alpha u \in L^p(\Omega)$ in the sense of distributions, for every multi-index α with $|\alpha| \leqslant m$. $W^{m,p}$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \left(\sum_{|\alpha| \leqslant m} \|D^\alpha u\|_{L^p}^p \right)^{1/p}$$

$W_0^{m,p}(\Omega)$	The closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$
$W^{-m,p'}(\Omega)$	The dual of $W_0^{m,p}(\Omega)$
$\boldsymbol{x}$	Denotes $(x,y) \in \mathbb{R}^2$ or $(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$
$\boldsymbol{x}^\beta$	$\prod\limits_{1 \leqslant j \leqslant N} x_j^{\beta_j}$, where $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{N}^N$
$\lvert x\rvert$	Used interchangeably to denote the absolute value if $x \in \mathbb{R}$, the modulus of a complex number if $x \in \mathbb{C}$ and the Euclidean norm if $x \in \mathbb{R}^N$
$\lfloor x \rfloor$	Integer part of x
$x_n \rightharpoonup x$	Denotes that x_n converges to x weakly
$X \hookrightarrow Y$	Denotes that $X \subset Y$ with continuous injection
$X \subset\subset Y$	Denotes that $\overline{X} \subset Y$ and $\overline{X}$ is compact
$\mathbf{1}_\Omega$	Characteristic function of the set Ω, i.e.,

$$\mathbf{1}_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega. \end{cases}$$

Δu	Laplacian of u, i.e., $\sum\limits_{i=1}^{N} u_{x_i x_i}$ in $\mathbb{R}^N$
ACNLS	Almost-cubic nonlinear Schrödinger
APP	Arkadiev–Pogrebkov–Polivanov
DS	Davey–Stewartson
GDS	Generalized Davey–Stewartson

1. INTRODUCTION

In [1], Schulman considered the following system of equations.

$$\begin{aligned} iu_t + L_1 u + \psi u &= 0, \\ L_2 \psi &= L_3 |u|^2, \end{aligned} \tag{1.0.1}$$

where u is a complex valued function and ψ is a real valued function, both depending on $t \in (0, \infty)$ and $\boldsymbol{x} \in \mathbb{R}^N$, for $N \in \{1, 2, 3\}$; with

$$L_n = \sum_{j,k=1}^{N} C_{jk}^n \frac{\partial^2}{\partial x_j \partial x_k}, \quad n \in \{1, 2, 3\},$$

being second order linear differential operators with constant coefficients where the matrices C^n are real and symmetric. Known as Zakharov–Schulman system, the equations (1.0.1) represent a universal model for the description of interactions of small-amplitude, high frequency waves with acoustic-type water waves. As it is observed in [2], in one spatial dimension one recovers

$$iu_t + u_{xx} + \chi |u|^2 u = 0, \quad \chi \in \{0, -1, 1\},$$

which is the one dimensional Schrödinger equation - linear, repulsive, attractive depending on the value of χ. In two spatial dimensions, upon setting

$$u = A, \qquad \psi = -\chi_0 |A|^2 - \chi_1 \phi_x, \tag{1.0.2}$$

$$L_1 = \sigma \partial_x^2 + \partial_y^2, \qquad L_2 = m_1 \partial_x^2 + m_2 \partial_y^2, \qquad L_3 = -\beta \chi_1 \partial_x^2 - \chi_0 L_2, \tag{1.0.3}$$

(1.0.1) can be reduced to Davey–Stewartson (DS) system, which is introduced in [3] (see also [4]), given in suitably rescaled coordinates by

$$
\begin{aligned}
iA_t + \sigma A_{xx} + A_{yy} &= \chi_0 |A|^2 A + \chi_1 A \phi_x, \\
m_1 \phi_{xx} + m_2 \phi_{yy} &= \beta \left(|A|^2 \right)_x ,
\end{aligned}
\tag{1.0.4}
$$

with the real parameters $\sigma, \chi_0, \chi_1, m_1, m_2, \beta$, such that $|\sigma| = 1$. As Schulman states in [1] (see also [5]), DS system is known to be a reduced form of the Zakharov–Schulman system such that it is integrable for some certain parameter regime in two dimensions and that it is not integrable in three dimensions.

In this work, we study the Zakharov–Schulman system in two spatial dimensions, that is, we hereafter assume $N = 2$. Since the only cases we consider are the ones with L_1 being hyperbolic or elliptic, without loss of generality, we rewrite (1.0.1) as

$$
\begin{aligned}
iu_t + \delta u_{xx} + u_{yy} + \psi u &= 0, \\
L_2 \psi &= L_3 |u|^2, \quad \delta \in \{-1, 1\},
\end{aligned}
\tag{1.0.5}
$$

upon a suitable coordinate transformation. We assume that the solutions suitably decay at infinity. This assumption will be made more precise later on when we introduce the related Cauchy problem with the initial data u_0. In the case where L_2 is elliptic, i.e. C^2 is sign definite, the system (1.0.5) can be reduced to a single equation in u. To do so, we express ψ in terms of u by solving the Poisson equation $(1.0.5)_2$. Indeed, taking Fourier transforms of both sides of $(1.0.5)_2$ in space, we evidently have

$$
(C^2_{11}\xi_1^2 + 2C^2_{12}\xi_1\xi_2 + C^2_{22}\xi_2^2)\hat{\psi}(\boldsymbol{\xi}) = (C^3_{11}\xi_1^2 + 2C^3_{12}\xi_1\xi_2 + C^3_{22}\xi_2^2)\widehat{(|u|^2)}(\boldsymbol{\xi}),
$$

with $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$ being the Fourier variables. Then, introducing the nonlocal linear operator K defined by $\widehat{K(f)}(\boldsymbol{\xi}) = \alpha(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi})$, where

$$
\alpha(\boldsymbol{\xi}) = \frac{C^3_{11}\xi_1^2 + 2C^3_{12}\xi_1\xi_2 + C^3_{22}\xi_2^2}{C^2_{11}\xi_1^2 + 2C^2_{12}\xi_1\xi_2 + C^2_{22}\xi_2^2}, \tag{1.0.6}
$$

the system (1.0.5) reduces to the so-called *almost cubic* nonlinear Schrödinger equation (ACNLS) and so we consider the related Cauchy problem

$$\begin{aligned} iu_t + \delta u_{xx} + u_{yy} + K(|u|^2)u = 0, \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \end{aligned} \tag{1.0.7}$$

which is extensively studied in [6], [7], [8] and [9] for the cases where the initial data u_0 lie in H^1, L^2, $\Sigma = L^2(|\boldsymbol{x}|^2\, d\boldsymbol{x}) \cap H^1$. We call $\delta = 1$ the elliptic and $\delta = -1$ the hyperbolic case. At this stage, let us recall that the symbol α satisfies the following obvious yet important properties:

- α is even, real and homogeneous of degree zero,
- $\alpha \in L^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$, and in particular $\alpha(\boldsymbol{\xi}) \leqslant M_\alpha$ for all $\boldsymbol{\xi} \in \mathbb{R}^2 \setminus \{(0,0)\}$,
- $\alpha \in C^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$.

We shall as well provide the reader with an explicit expression for $M_\alpha = \max\limits_{\boldsymbol{\xi} \in \mathbb{R}^2 \setminus \{0\}} \alpha(\boldsymbol{\xi})$ in the second chapter. Upon stating such a reduction, as underlined in [2], it is worthwile to note now that in general the matrices C^n are not necessarily sign definite; in particular, the operator L_2 can be nonelliptic. In case L_2 is hyperbolic, as it is discussed for DS system in [10], it is still possible to reduce the system (1.0.5) to a single equation $iu_t + \delta u_{xx} + u_{yy} + \tilde{K}(|u|^2)u = 0$. However, since the operator $\tilde{K}$ emerges through solving a wave equation, it enjoys no regularizing effects. Therefore the usual techniques involving Sobolev space theory for semilinear Schrödinger equations do not apply to this reduced form. We do not consider such a case in this work.

Throughout the second chapter, assuming L_1 and L_2 are elliptic, we treat the system (1.0.1) in the framework of ACNLS equation, and adapt the results obtained in [7] and [6] for almost cubic nonlinear Schrödinger equation and elliptic generalized Davey–Stewartson (GDS) system which is derived by Babaoğlu and Erbay [11] to model the propagation of waves in a bulk medium composed of an elastic medium with couple stresses. We introduce the focusing and defocusing cases of the solutions of the Cauchy problem related to the system (1.0.7) with $\delta = 1$; and following [7], we discuss that in the focusing case, any given initial datum can be scaled to one with negative energy so that the corresponding solution blows up in finite time. Existence of such initial data is one of the main ideas present in [2]. On the other hand, following [7], we conclude that the focusing case is also characterized by the existence of standing wave solutions which are introduced below. Apart from this analysis, we prove the conservation laws for the quantities mass, energy and momenta as stated in [2]; and derive the conserved quantities corresponding to invariance of the solutions of the system (1.0.1) under scaling and pseudo-conformal transformation given again in [2] and [9]. We also establish virial identity which plays a crucial role in the conservation law corresponding to the scaling invariance and the sufficient conditions given in [2] and [7, Theorem 2.4] for a finite time blow-up.

In the third chapter we study the existence and regularity of the standing wave solutions, i.e. the periodic solutions of the form

$$u(\boldsymbol{x}, t) = \varphi(\boldsymbol{x}) e^{i\omega t},$$

where ω is a positive constant, φ is nonzero and lies in the energy class $H^1(\mathbb{R}^2)$. Heuristically speaking, such solutions appear due to the counterbalance between the dispersive effect of the linear part of the equation and the focusing effect of the nonlinearity. It is evident that φ, which is called the *standing wave profile*, should be a solution of

$$\Delta\varphi - \omega\varphi + K(|\varphi|^2)\varphi = 0. \tag{1.0.8}$$

By its very nature, in the Sobolev space $H^1(\mathbb{R}^2)$ we only require (1.0.8) to hold weakly. Though we later show in the regularity theorem that φ is in fact smooth and also enjoys an exponential decay rate. To prove the existence of such solutions we employ variational methods by setting up an appropriate functional J over $H^1(\mathbb{R}^2)$ so that the critical points of this functional are the solutions of (1.0.8). One such approach is to introduce the kinetic and the potential energies, then to set up and solve a constrained minimization problem via seeking minimizers of the the energy functional over a level set where the potential energy is zero (see [12]). So, it turns out that this process picks the solutions that are of minimal mass and in this regard, such solutions are called *ground states*. Through this route, the existence of standing waves is established for DS system in [13] and for semilinear Schrödinger equations in [14].

In this work, we adopt an alternative approach devised by Weinstein in [15] where an unconstrained minimization problem is set to construct the ground states for nonlinear Schrödinger equation. In particular, as Tao vividly elaborates in [16], Weinstein's approach to solving $\Delta\psi - b\psi + a|\psi|^{p-1}\psi = 0$ is based upon understanding the best constant in Gagliardo–Nirenberg–Sobolev inequality (A.4). With the nonlocal operator K in (1.0.8), we are also able to establish such a sharp estimate for the constant in a Gagliardo–Nirenberg–Sobolev type inequality by following the argument in [8].

In general, the process of minimizing a functional J over a function space involves taking a minimizing sequence $\{f_n\}$ so that $J(f_n) \to j_0 = \inf J$ and then showing that some subsequence of $\{f_n\}$ converges to an actual minimizer. At such a stage, the major obstacle arising seems to be the lack of compactness; indeed, it only follows that the minimizing sequence $\{f_n\}$ lies in a bounded set. In bounded domains, one way to eliminate this deficiency is to employ techniques concerning weak topologies. For the spaces H^1 and L^p, $1 < p < \infty$, are reflexive, we can extract a weakly convergent subsequence (see A.1) and then invoke Rellich-Kondrachov compactness theorem (A.2) to obtain strong convergence. However, in unbounded domains the imbedding of Sobolev spaces into the appropriate L^p spaces are not compact, that is to say, Rellich-Kondrachov compactness theorem does not work anymore. In $\mathbb{R}^N$, such a loss of compactness can be compensated by using the translation and rotation invariance of $\mathbb{R}^N$ and consequently obtaining some sort of "local compactness" in order to conclude that the weak convergence is also valid in the strong topology. These ideas were introduced and elaborated in Strauss' Compactness Lemma [18] for radial functions and in Lions' Concentration Compactness Principle [17]. Those compactness results are utilized in the above mentioned works [12], [13] and [14]. In [18] and in [15] the arguments go through considering radial functions lying in $H^1(\mathbb{R}^2)$, namely $H^1_r(\mathbb{R}^2)$, and utilizing the fact that $H_r^1(\mathbb{R}^2)$ is compactly imbedded in $L^p(\mathbb{R}^2)$ for all $2 < p < \infty$. Due to the nature of the nonlocal term, we cannot restrict our function space to radial functions, so we follow the arguments given in [19] and [16]. Weinstein's method is utilized by Papanicolau *et al.* in [19] to construct the ground states for DS system, by Eden and Erbay in [20] for GDS system and by Eden, Gürel and Kuz in [7] for ACNLS equation. A treatment on variational methods for nonlinear elliptic partial differential equations with nonlocal terms is present in [21].

The fourth chapter is devoted to the existence of Arkadiev–Pogrebkov–Polivanov (APP) type travelling wave solutions. Inspired by the work of Ozawa in [22], we follow [23] and obtain first set conditions on the operators so that these solutions introduced by Arkadiev *et al.* exist for the Zakharov–Schulman system. In [22], in order to to construct an explicit blow-up profile in $L^2(\mathbb{R}^2)$ for the hyperbolic-elliptic case of the Davey–Stewartson system, Ozawa used solutions of the form

$$u(x,y,t) = \frac{1}{f(x,y)}, \quad \phi(x,y,t) = \gamma \partial_x \log f(x,y), \tag{1.0.9}$$

where $f(x,y) = \frac{1}{1+\alpha x^2 + \beta y^2}$, $\gamma \in \mathbb{R}$. In a similar manner, the analogous results were obtained in [24] for the generalized Davey–Stewartson (GDS) system. As mentioned in [23], Ozawa's solution turns out to be a special case of the 1-soliton solution appears in [25] which is given by

$$u(x,y,t) = 2\bar{\nu}\frac{\exp\{2i\mathfrak{Im}(\lambda z) + 4i\mathfrak{Re}(\lambda^2)t\}}{|z+4i\lambda t+\mu|^2 + |\nu|^2}, \tag{1.0.10}$$

where $z = x + iy$ and λ, μ, ν are complex constants. Indeed, setting $\lambda = \mu = 0$ and $\nu = 1$, it is seen that (1.0.10) recovers Ozawa's solution (1.0.9). In [23], Eden and Gürel obtained the conditions on the parameters under which the solutions of the form (1.0.10) exist for the hyperbolic-elliptic GDS system and it turned out that these conditions coincide with the conditions given in [24]. Within this perspective, assuming L_1 to be hyperbolic, we derive the first set conditions on the operators L_2 and L_3 so that the solutions of the form (1.0.10) exist for the system (1.0.1). As an integrable reduced form of the Zakharov–Schulman system, we observe that upon transforming them back via (1.0.2), the conditions derived on the parameters of the DS system (1.0.4) for the existence of such solutions agree with the ones we derive for the operators in Zakharov–Schulman system. We also establish that APP type solutions exist for a different DS system

$$\begin{aligned} iA_t + \lambda A_{xx} + \mu A_{yy} &= \chi_0 |A|^2 A + \chi_1 A\phi, \\ m_1 \phi_{xx} + m_2 \phi_{yy} &= \beta(|A|^2)_{yy}, \end{aligned} \tag{1.0.11}$$

described by the equations (2.15) and (2.16) in [4]. This system is also a reduced form of the Zakharov–Schulman system such that it is integrable under some certain parameter regime. We as well observe that upon transforming them back, the conditions derived on the parameters also agree with the ones we derive on the operators in (1.0.1). So the question we address is whether the set of conditions we obtain on the operators pick only the existence results for the two DS systems, or not. Furthermore, following [22] and [24] we do obtain an explicit blow-up profile using the invariance of solutions of (1.0.1) under the pseudo-conformal transformation.

2. ZAKHAROV–SCHULMAN EQUATIONS AS AN ACNLS EQUATION

Throughout this chapter we assume L_2 to be elliptic and following [2], [23] and [7], we give a treatment of the Zakharov–Schulman system (1.0.1) in the framework of ACNLS equation. In that case, as mentioned earlier, we rewrite (1.0.5) as (1.0.7). So we find it convenient to start with carrying out our promise on the maximum value of α and so state the following assertion.

Proposition 2.0.1. [26] *The maximum value of the symbol α, as defined in (1.0.6), is equal to the greater of the roots of the equation*

$$\det C^3 - \lambda \left(C^3_{11} C^2_{22} - 2C^3_{12} C^2_{12} + C^2_{11} C^3_{22} \right) + \lambda^2 det C^2 = 0.$$

Proof. Let $Q_j(\boldsymbol{\xi})$ denote $\langle C^j \boldsymbol{\xi}, \boldsymbol{\xi} \rangle = C^j_{11}\xi_1^2 + 2C^j_{12}\xi_1\xi_2 + C^j_{22}\xi_2^2$, $j = 2, 3$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$, so that $\alpha(\boldsymbol{\xi}) = \frac{Q_3(\boldsymbol{\xi})}{Q_2(\boldsymbol{\xi})}$. As a linear algebra fact, we know that the quadratic form of a symmetric matrix attains its maximum on the unit ball at an eigenvector corresponding to the largest eigenvalue and hence this maximum is equal to that largest eigenvalue. Noting that we study the case where C^2 is sign definite, without loss of generality we may assume that C^2 is positive definite - otherwise, we multiply the numerator and the denominator of α by -1. By Spectral Theorem for symmetric operators, it is guaranteed that there exists a basis consisting of eigenvectors of the symmetric positive definite matrix C^2. Since the corresponding eigenvalues of C^2 are all positive, taking the square roots of these eigenvalues we see that there exists a symmetric positive definite matrix B such that

$$\langle C^2 \boldsymbol{\xi}, \boldsymbol{\xi} \rangle = \langle B\boldsymbol{\xi}, B\boldsymbol{\xi} \rangle = |B\boldsymbol{\xi}|^2, \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^2.$$

Changing variables $\boldsymbol{\eta} = B\boldsymbol{\xi}$, and using the fact that B is also a symmetric, positive definite matrix, it follows that

$$\alpha(\boldsymbol{\xi}) = \frac{Q_3(\boldsymbol{\xi})}{Q_2(\boldsymbol{\xi})} = \frac{\langle C^3 B^{-1}\boldsymbol{\eta}, B^{-1}\boldsymbol{\eta}\rangle}{\langle C^2 B^{-1}\boldsymbol{\eta}, B^{-1}\boldsymbol{\eta}\rangle} = \frac{\langle B^{-1}C^3 B^{-1}\boldsymbol{\eta}, \boldsymbol{\eta}\rangle}{|\boldsymbol{\eta}|^2}$$
$$= \langle B^{-1}C^3B^{-1}\frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}, \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}\rangle.$$

Therefore, we see that maximizing α over $\mathbb{R}^2$ is equivalent to maximizing the form $Q_{B^{-1}C^3B^{-1}}$ on the unit ball in $\mathbb{R}^2$. So, it is enough to look for the greater of the roots of the equation

$$\det(B^{-1}C^3B^{-1} - \lambda I) = 0.$$

Since

$$\begin{aligned}\det(B^{-1}C^3B^{-1} - \lambda I) &= \det(B^{-1}C^3B^{-1} - \lambda B^{-1}C^2B^{-1}) \\ &= \det(B^{-1}(C^3 - \lambda C^2)B^{-1}) \\ &= \frac{1}{(\det B)^2}\det(C^3 - \lambda C^2),\end{aligned}$$

it suffices to find the greater of the roots of the equation $\det(C^3 - \lambda C^2) = 0$, i.e.,

$$\left(C^3_{11}C^3_{22} - (C^3_{12})^2\right) - \lambda\left(C^3_{11}C^2_{22} - 2C^3_{12}C^2_{12} + C^2_{11}C^3_{22}\right) + \lambda^2\left(C^2_{11}C^2_{22} - (C^2_{12})^2\right) = 0,$$

and hence follows the claim. $\square$

We now discuss the evolution of some global quantities in a formal way.

2.1. Conservation Laws and Other Invariants

In [2], the real valued auxiliary functions ϕ_1, ϕ_2 satisfying $L_2\phi_j = \frac{\partial}{\partial x_j}|u|^2$, for $j = 1, 2$ are introduced to rewrite (1.0.1) as

$$iu_t + L_1 u + (\mathcal{L}_3\boldsymbol{\phi}) = 0,$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2)$, and the operator $\mathcal{L}_3$ is defined by

$$\mathcal{L}_3\boldsymbol{\phi} = \sum_{j,k=1}^{2} C^3_{jk} \frac{\partial \phi_j}{\partial x_k}.$$

Through this approach, assuming that the solutions to the Cauchy problem related to (1.0.1) decay suitably at infinity, the quantity describing the energy of the solutions u to $(1.0.1)_1$ is introduced to be

$$E(u) = \int\limits_{\mathbb{R}^2} \sum_{j,k=1}^{2} C^1_{jk} \frac{\partial u}{\partial x_j}\frac{\partial \bar{u}}{\partial x_k}\, d\boldsymbol{x} - \frac{1}{2}\int\limits_{\mathbb{R}^2} \sum_{p,q=1}^{2}\sum_{r,s=1}^{2} C^3_{pq}C^2_{rs} \frac{\partial \phi_p}{\partial x_r}\frac{\partial \phi_q}{\partial x_s}\, d\boldsymbol{x}. \tag{2.1.1}$$

However, since $\hat{\phi}_j(\boldsymbol{\xi}) = \dfrac{-i\xi_j}{(C^2\boldsymbol{\xi}, \boldsymbol{\xi})}\widehat{|u|^2}$, setting $f = |u|^2$, Plancharel's theorem yields

$$\begin{aligned}\int\limits_{\mathbb{R}^2} \sum_{p,q=1}^{2}\sum_{r,s=1}^{2} C^3_{pq}C^2_{rs} \frac{\partial \phi_p}{\partial x_r}\frac{\partial \phi_q}{\partial x_s}\, d\boldsymbol{x} &= \int\limits_{\mathbb{R}^2} \sum_{p,q=1}^{2}\sum_{r,s=1}^{2} C^3_{pq}C^2_{rs} \widehat{\left(\frac{\partial \phi_p}{\partial x_r}\right)}\widehat{\left(\frac{\partial \phi_q}{\partial x_s}\right)}\, d\boldsymbol{\xi} \\ &= \int\limits_{\mathbb{R}^2} \alpha(\boldsymbol{\xi})|\hat{f}|^2(\boldsymbol{\xi})\, d\boldsymbol{\xi}.\end{aligned}$$

Consequently, the assumption that L_2 is elliptic enables us to rewrite the energy in terms of the nonlocal operator K. So, as in [2], the quantities mass, energy and momenta for (1.0.7) are given by

$$m(u) = \int\limits_{\mathbb{R}^2} |u|^2\, dxdy \tag{2.1.2}$$

$$E(u) = \int\limits_{\mathbb{R}^2} \left(\delta|u_x|^2 + |u_y|^2\right)\, dxdy - \frac{1}{2}\int\limits_{\mathbb{R}^2} K(|u|^2)|u|^2\, dxdy \tag{2.1.3}$$

$$P_x(u) = i\int\limits_{\mathbb{R}^2} (u\bar{u}_x - \bar{u}u_x)\, dxdy, \quad P_y(u) = i\int\limits_{\mathbb{R}^2} (u\bar{u}_y - \bar{u}u_y)\, dxdy. \tag{2.1.4}$$

The above quantities all depend on t but this dependence is suppressed for the ease of notation. We now show that these quantities are conserved for sufficiently smooth solutions which suitably vanish at infinity. Multiplying (1.0.7) by $\bar{u}$ and integrate over

$\mathbb{R}^2$ we obtain

$$\int_{\mathbb{R}^2} iu_t\bar{u} + \delta u_{xx}\bar{u} + u_{yy}\bar{u} + K(|u|^2)|u|^2\, dxdy = 0,$$

and upon an integration by parts it follows that

$$i\int_{\mathbb{R}^2} (u_t\bar{u})\, dxdy + \delta\,\|u_x\|_2^2 + \|u_y\|_2^2 + \int_{\mathbb{R}^2} K(|u|^2)|u|^2\, dxdy = 0.$$

We take imaginary parts and get

$$\mathfrak{Re}\int_{\mathbb{R}^2} u_t\bar{u}\, dxdy = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |u|^2\, dxdy = 0,$$

which implies the conservation of mass (2.1.2).

Next, multiplying (1.0.7) by $2\bar{u}_t$ and taking real parts we obtain

$$2\mathfrak{Re}[\bar{u}_t(\delta u_{xx} + u_{yy})] = -K(|u|^2)(|u|^2)_t. \tag{2.1.5}$$

For the left hand side of (2.1.5), subsequent to integration by parts we have

$$2\mathfrak{Re}\int_{\mathbb{R}^2} \bar{u}_t(\delta u_{xx} + u_{yy})\, dxdy = -\frac{d}{dt}\int_{\mathbb{R}^2} (\delta|u_x|^2 + |u_y|^2)\, dxdy, \tag{2.1.6}$$

which in turn gives us

$$\frac{d}{dt}\int_{\mathbb{R}^2} \left(\delta|u_x|^2 + |u_y|^2\right)\, dxdy - \int_{\mathbb{R}^2} K(|u|^2)(|u|^2)_t\, dxdy = 0.$$

Now we set $f = |u|^2$, employ Plancharel's theorem and take real parts to get

$$\frac{d}{dt}\int_{\mathbb{R}^2} \left(\delta|u_x|^2 + |u_y|^2\right)\, dxdy - \mathfrak{Re}\int_{\mathbb{R}^2} \alpha(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi})\overline{\widehat{(f_t)}}(\boldsymbol{\xi})\, d\boldsymbol{\xi} = 0,$$

which implies

$$\frac{d}{dt}\left\{\int\limits_{\mathbb{R}^2}\left(\delta|u_x|^2+|u_y|^2\right)\,dxdy-\frac{1}{2}\int\limits_{\mathbb{R}^2}\alpha(\boldsymbol{\xi})|\hat{f}|^2(\boldsymbol{\xi})\,d\boldsymbol{\xi}\right\}=0,$$

hence follows the conservation of energy (2.1.3). We note that this quantity makes sense as long as the solutions remain in $H^1(\mathbb{R}^2)$.

Finally, we multiply (1.0.7) by $\bar{u}_x$ and obtain

$$iu_t\bar{u}_x+\bar{u}_x(\delta u_{xx}+u_{yy})+K(|u|^2)u\bar{u}_x=0, \tag{2.1.7}$$

and next, add (2.1.7) its complex conjugate to get

$$i(u_t\bar{u}_x-\bar{u}_tu_x)+2\mathfrak{Re}[\bar{u}_x(\delta u_{xx}+u_{yy})]+K(|u|^2)\left(|u|^2\right)_x=0. \tag{2.1.8}$$

Recalling that $f=|u|^2$, by we observe

$$\int\limits_{\mathbb{R}^2}K(f)f_x\,dxdy=\mathfrak{Re}\int\limits_{\mathbb{R}^2}\alpha(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi})\overline{(-i\xi_1)}\bar{\hat{f}}(\boldsymbol{\xi})\,d\boldsymbol{\xi}=\mathfrak{Re}\int\limits_{\mathbb{R}^2}i\xi_1\alpha(\boldsymbol{\xi})|\hat{f}|^2(\boldsymbol{\xi})\,d\boldsymbol{\xi}=0. \tag{2.1.9}$$

Besides, for the second term in (2.1.8) integration by parts yields

$$\int\limits_{\mathbb{R}^2}2\mathfrak{Re}[\bar{u}_x(\delta u_{xx}+u_{yy})]\,dxdy=0. \tag{2.1.10}$$

Next, we integrate by $i(u_t\bar{u}_x-\bar{u}_tu_x)$ by parts, and by (2.1.9) and (2.1.10), it turns out that

$$\int\limits_{\mathbb{R}^2}i(u_t\bar{u}_x-\bar{u}_tu_x)\,dxdy=\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^2}i\left(u\bar{u}_x-\bar{u}u_x\right)\,dxdy=0,$$

so we have the conservation of momentum P_x (2.1.4). The same result for P_y is established through exactly the same steps.

Let us now introduce the function

$$I = \int_{\mathbb{R}^2} (\delta x^2 + y^2)|u|^2 \, dxdy, \tag{2.1.11}$$

which is the quantity describes the second moment of inertia. Known as the virial identity, the below result, which is established in a formal way, plays a key role in a later blow-up argument (2.2.5).

Proposition 2.1.1. [2, Proposition 2.2] *For I as in (2.1.11), the following hold.*

$$\frac{dI}{dt} = 4\mathfrak{Im}\int_{\mathbb{R}^2} \bar{u}(xu_x + yu_y)\, dxdy, \tag{2.1.12}$$

$$\frac{d^2 I}{dt^2} = 8E(u). \tag{2.1.13}$$

Proof. To prove (2.1.12), we multiply (1.0.7) by $2\bar{u}$ and take imaginary parts to get

$$(|u|^2)_t + 2\mathfrak{Im}[\delta(u_x\bar{u})_x + (u_y\bar{u})_y] = 0,$$

and by an elementary calculation we rewrite the above line as

$$(|u|^2)_t + i[\delta(u\bar{u}_x + \bar{u}u_x)_x + (u\bar{u}_y + \bar{u}u_y)_y] = 0.$$

So, upon integration by parts we have

$$\begin{aligned}
\frac{dI}{dt} &= \int_{\mathbb{R}^2} (\delta x^2 + y^2)(|u|^2)_t \, dxdy \\
&= 2i\int_{\mathbb{R}^2} \delta^2 x(u\bar{u}_x + \bar{u}u_x) + y(u\bar{u}_y + \bar{u}u_y)\, dxdy \\
&= 4\mathfrak{Im}\int_{\mathbb{R}^2} (x\bar{u}u_x + y\bar{u}u_y)\, dxdy,
\end{aligned}$$

and hence follows (2.1.12). Then

$$\frac{d^2I}{dt^2} = 4\mathfrak{Im}\int\limits_{\mathbb{R}^2} x(\bar{u}_t u_x + \bar{u}u_{xt}) + y(\bar{u}_t u_y + \bar{u}u_{yt})\, dxdy,$$

next, integrating by parts and utilizing (1.0.7) we obtain

$$\frac{d^2I}{dt^2} = 8\mathfrak{Re}\int\limits_{\mathbb{R}^2} (-K(|u|^2) - \delta u_{xx} - u_{yy})(x\bar{u}_x + y\bar{u}_y + \bar{u})\, dxdy,$$

$$= 8\left\{\int\limits_{\mathbb{R}^2} (-K(|u|^2)|u|^2 - K(|u|^2)u(x\bar{u}_x + y\bar{u}_y))\, dxdy \right. \tag{2.1.14}$$

$$- \int\limits_{\mathbb{R}^2} \delta u_{xx}\bar{u} + u_{yy}\bar{u}\, dxdy \tag{2.1.15}$$

$$\left. -\mathfrak{Re}\int\limits_{\mathbb{R}^2} (x\bar{u}_x + y\bar{u}_y)(\delta u_{xx} + u_{yy})\, dxdy \right\}. \tag{2.1.16}$$

After several integration by parts, we observe that the last integral above vanishes and hence

$$\frac{d^2I}{dt^2} = 8\int\limits_{\mathbb{R}^2} \delta|u_x|^2 + |u_y|^2\, dxdy$$
$$- 4\int\limits_{\mathbb{R}^2} K(|u|^2)(x(|u|^2)_x + y(|u|^2)_y) + 2K(|u|^2)|u|^2\, dxdy.$$

Now we show $\int\limits_{\mathbb{R}^2} K(|u|^2)(x(|u|^2)_x + y(|u|^2)_y) + K(|u|^2)|u|^2\, dxdy = 0$. Let $f = |u|^2$, $g = |\hat{f}|^2$ and $\mathcal{J} = \int\limits_{\mathbb{R}^2} K(|u|^2)(x(|u|^2)_x + y(|u|^2)_y)\, dxdy$. Then utilizing Plancharel's theorem and integration by parts we have

$$\begin{aligned}
\mathcal{J} &= \int\limits_{\mathbb{R}^2} xK(f)f_x + yK(f)f_y)\,dxdy = \int\limits_{\mathbb{R}^2} \bar{\hat{f}}(\xi_1\widehat{K(f)}_{\xi_1} + \xi_2\widehat{K(f)}_{\xi_2})\,d\xi_1 d\xi_2 \\
&= \int\limits_{\mathbb{R}^2} \bar{\hat{f}}(\xi_1\partial_{\xi_1} + \xi_2\partial_{\xi_2})(\alpha\hat{f})\,d\xi_1 d\xi_2 \\
&= \int\limits_{\mathbb{R}^2} \alpha\bar{\hat{f}}(\xi_1\hat{f}_{\xi_1} + \xi_2\hat{f}_{\xi_2})\,d\xi_1 d\xi_2 + \int\limits_{\mathbb{R}^2} \bar{\hat{f}}(\xi_1\alpha_{\xi_1} + \xi_2\alpha_{\xi_2})\hat{f}\,d\xi_1 d\xi_2.
\end{aligned}$$

The last integral vanishes for α is homogeneous of order zero and since $\mathcal{J}$ is real, we deduce that

$$\begin{aligned}
\mathcal{J} &= \frac{1}{2}\int\limits_{\mathbb{R}^2} \alpha(\xi_1 g_{\xi_1} + \xi_2 g_{\xi_2})\,d\xi_1 d\xi_2 \\
&= \frac{1}{2}\int\limits_{\mathbb{R}^2} \alpha((\xi_1 g)_{\xi_1} + (\xi_2 g)_{\xi_2})\,d\xi_1 d\xi_2 - \int\limits_{\mathbb{R}^2} \alpha g\,d\xi_1 d\xi_2 \\
&= -\frac{1}{2}\int\limits_{\mathbb{R}^2} (\xi_1\alpha_{\xi_1} + \xi_2\alpha_{\xi_2})g\,d\xi_1 d\xi_2 - \int\limits_{\mathbb{R}^2} \alpha g\,d\xi_1 d\xi_2 \\
&= -\int\limits_{\mathbb{R}^2} \alpha g\,d\xi_1 d\xi_2 = -\int\limits_{\mathbb{R}^2} K((|u|^2)|u|^2\,d\boldsymbol{x},
\end{aligned}$$

again by utilizing Plancharel's theorem, whence follows the claim on a formal level. □

At this stage, we note that (2.1.13) and the conservation of energy yields

$$\frac{dI}{dt}(t) = 8E(u(0))t + \frac{dI}{dt}(0),$$

which in turn gives

$$I(t) = 4E(u(0))t^2 + \frac{dI}{dt}(0)t + I(0). \tag{2.1.17}$$

Following [6], we now discuss the further invariants of the Zakharov–Schulman system. Apparently, the solutions of (1.0.7) are invariant under the transformation $(\boldsymbol{x}, t, u) \mapsto$

$(\tilde{\boldsymbol{x}}, \tilde{t}, \tilde{u})$, where

$$\tilde{\boldsymbol{x}} = \frac{1}{\gamma}\boldsymbol{x}, \quad \tilde{t} = \frac{1}{\gamma^2}t, \quad \tilde{u} = \gamma u,$$

for any real parameter γ. By Noether's theorem, the conserved quantity corresponding to the above scaling symmetry is given by

$$E_{\mathrm{sc}}(u(t)) = \frac{1}{2}\frac{dI}{dt}(t) - 4tE(u(t)),$$

whose conservation is immediate by the virial identity (2.1.13). We also consider the invariance of solutions of (1.0.7) under the pseudo-conformal transformation $(\boldsymbol{x}, t, u) \mapsto (\boldsymbol{X}, T, U)$ defined in [6] by

$$\boldsymbol{X} = \frac{\boldsymbol{x}}{a+bt}, \quad ,T = \frac{c+dt}{a+bt}, \quad for \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}),$$
$$U(t, \boldsymbol{x}) = \frac{1}{a+bt}\exp\left\{ib\frac{\delta x^2 + y^2}{a+bt}\right\}u(T, \boldsymbol{X}). \tag{2.1.18}$$

where the corresponding conserved quantity is given by

$$E_{pc}(u) = \int\limits_{\mathbb{R}^2} \{\delta|xu + 2i\delta u_x|^2 + |yu + 2itu_y|^2 + 2t^2K(|u|^2)|u|^2\}\, dxdy,$$

which also reads as

$$E_{pc}(u) = I - 4t\mathfrak{Im}\int\limits_{\mathbb{R}^2} \bar{u}(xu_x + yu_y)\, dxdy + 4t^2E(u_0). \tag{2.1.19}$$

We note that this quantity stands for the energy of the solution in the transformed coordinates. It follows by (2.1.12) and (2.1.13) that

$$\frac{dE_{pc}(u)}{dt}(t) = \frac{dI}{dt}(t) - 4t\mathfrak{Im}\int_{\mathbb{R}^2} \bar{u}(xu_x + yu_y)\,dxdy$$
$$- t\frac{d}{dt}4\mathfrak{Im}\int_{\mathbb{R}^2} \bar{u}(xu_x + yu_y)\,dxdy + 8tE(u_0) = 0,$$

whence we obtain the conservation of (2.1.19). As I, this quantity makes sense as long as the solutions remain in the Hilbert space $\Sigma = H^1 \cap L^2(|\boldsymbol{x}|^2 d\boldsymbol{x})$ equipped with the norm $\|\cdot\|_\Sigma^2 = \|\cdot\|_{H^1}^2 + \||\boldsymbol{x}|\cdot\|_2^2$.

2.2. Focusing and Defocusing Cases of Elliptic-Elliptic Zakharov–Schulman System

We consider the Cauchy problem

$$\begin{aligned} iu_t + \delta u_{xx} + u_{yy} + K(|u|^2)u = 0, \quad \delta = \pm 1, \\ u(0) = u_0, \end{aligned} \tag{2.2.1}$$

which is extensively studied in [6] in the spaces $L^2(\mathbb{R}^2)$, $H^1(\mathbb{R}^2)$ and Σ. Before we introduce the focusing and defocusing cases for solutions of (2.2.1), and adapt the global existence and blow-up results in [7] depending on the assumptions on α or the initial data u_0; we state the following local existence results achieved in [6] but do not include their proofs here.

Theorem 2.2.1. [6, Theorem 4.4] *Given $u_0 \in H^1(\mathbb{R}^2)$, there exists a unique maximal solution u solving (2.2.1) on $[0, T^*)$ in $C([0,T^*); H^1(\mathbb{R}^2)) \cap C^1([0,T^*); H^{-1}(\mathbb{R}^2))$ with the following properties:*

(i) $\nabla u \in L^4([0,t]; L^4(\mathbb{R}^2))$ *for every* $t < T^*$,

(ii) $T^* < \infty$ *implies that* $\|u\|_{L^\infty([0,T^*);H^1(\mathbb{R}^2))} = \infty$,

(iii) If $\psi_n \to u_0$ *in* $H^1(\mathbb{R}^2)$ *and* u_n*'s are the corresponding solutions, then for any*

$I \Subset [0, T^*)$ *and for any* n *sufficiently large,* u_n*'s are defined on* I *and* $u_n \to u$ *in* $C(I; H^1(\mathbb{R}^2))$,

(iv) *Mass (2.1.2) and energy (2.1.3) are conserved in* $[0, T^*)$.

Theorem 2.2.2. [6, Theorem 5.2] *Given* $u_0 \in \Sigma$, *there exists a unique maximal solution* u *solving (2.2.1) on* $[0, T^*)$ *in* $C([0, T^*); \Sigma) \cap C^1([0, T^*); H^{-1}(\mathbb{R}^2))$ *with the following properties:*

(i) $|\boldsymbol{x}|u, \nabla u \in L^4([0, t]; L^4(\mathbb{R}^2))$ *for every* $t < T^*$,

(ii) $T^* < \infty$ *implies that* $\|u\|_{L^\infty([0,T^*);\Sigma)} = \infty$,

(iii) $[0, T^*)$ *coincides with the maximal interval of existence for the* H^1*-solution in Theorem (2.2.1) with initial data* u_0,

(iv) *For* $\delta = 1$, *the mapping* $t \mapsto I(t) = \int\limits_{\mathbb{R}^2} |\boldsymbol{x}|^2 |u(t, \boldsymbol{x})|^2 \, d\boldsymbol{x}$ *lies in* $C^2([0, T^*))$ *and for every* $t \in [0, T^*)$ *the identities (2.1.12) and (2.1.13) hold,*

(v) *If* $\psi_n \to u_0$ *in* Σ *and* $u_n s$ *are the corresponding solutions, then for any* $I \Subset [0, T^*)$ *and for any* n *sufficiently large,* u_n*'s are defined on* I *and* $u_n \to u$ *in* $C(I; \Sigma)$.

Leaning against the above two theorems, we proceed with a global existence result for the case where L_1 is also elliptic.

Theorem 2.2.3. [7, Theorem 2.3] *Suppose that* $\alpha(\boldsymbol{\xi}) \leqslant 0$ *for all* $\boldsymbol{\xi} \in \mathbb{R}^2 \setminus \{(0, 0)\}$. *Then* H^1*-solutions of (2.2.1), with* $\delta = 1$, *are global in time.*

Proof. We set $f = |u|^2$. The assumption on α and energy conservation yields

$$\|\nabla u(t)\|_2^2 = E(u(t)) + \frac{1}{2} \int\limits_{\mathbb{R}^2} \alpha(\boldsymbol{\xi}) |\hat{f}|^2(\boldsymbol{\xi}) \, d\boldsymbol{\xi} \leqslant E(u(t)) = E(u_0),$$

for all $t \in [0, T^*)$. Utilizing mass conservation, we obtain

$$\|u(t)\|_{H^1}^2 = \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \leqslant m(u_0) + E(u_0) < \infty$$

and hence $T^* = \infty$ by the assertion *(ii)* in Theorem (2.2.1). $\square$

We note that this result extends to Σ-solutions as well by the virtue of the assertion *(iii)* in Theorem (2.2.2). Before we state the theorem regarding sufficient conditions for a finite time blow-up, we make the following observation.

Lemma 2.2.4. *Let u be a Σ-solution for (2.2.1), with $\delta = 1$. If $I(t) = 0$ for some t, then the solution blows up in finite time.*

Proof. The mass conservation, a simple integration by parts and Cauchy-Schwarz inequality enable us to write

$$\begin{aligned} \|u_0\|_2^2 = \|u\|_2^2 &= -\frac{1}{2}\int\limits_{\mathbb{R}^2} x(u\bar{u})_x\,dxdy - \frac{1}{2}\int\limits_{\mathbb{R}^2} y(u\bar{u})_y\,dxdy \\ &\leqslant -\mathfrak{Re}\int\limits_{\mathbb{R}^2} x\bar{u}u_x\,dxdy - \mathfrak{Re}\int\limits_{\mathbb{R}^2} x\bar{u}u_x\,dxdy \\ &\leqslant \|x\bar{u}\|_2\,\|u_x\|_2 + \|y\bar{u}\|_2\,\|u_y\|_2\,. \end{aligned}$$

So, since $\|x\bar{u}\|_2^2$, $\|y\bar{u}\|_2^2 \leqslant I(t)$, we have

$$\|u_0\|_2^2 \leqslant \sqrt{I(t)}\,\left[\|u_x\|_2 + \|u_y\|_2\right],$$

Thus, if $I(t) = 0$ for some t, the H^1-norm of the solution u becomes unbounded, i.e. the solution blows up in finite time by Theorem (2.2.2) *(i)*. $\square$

We are now ready to state and prove the following theorem.

Theorem 2.2.5. [7, Theorem 2.4] *Let u be the solution of the Cauchy problem (2.2.1) with $\delta = 1$ and initial value $u_0 \in \Sigma$. If one of the conditions*

(i) $E(u_0) < 0$,

(ii) $E(u_0) = 0$ *and* $\mathfrak{Im}\int_{\mathbb{R}^2} \bar{u}_0(\boldsymbol{x}\,\boldsymbol{.}\,\nabla)u_0\,d\boldsymbol{x} < 0$,

(iii) $E(u_0) > 0$ *and* $-\mathfrak{Im}\int_{\mathbb{R}^2} \bar{u}_0(\boldsymbol{x}\,\boldsymbol{.}\,\nabla)u_0\,d\boldsymbol{x} > \sqrt{2E(u_0)I(0)}$,

holds, then $T^* < \infty$ *and so, as a result of (ii) in Theorem (2.2.1),* u *blows up in finite time.*

Proof. Suppose $E(u_0) < 0$. Then it immediately follows from (2.1.17) that for some T large enough we have $I(T) = 0$ and hence the corresponding solution blows up in finite time by Lemma 2.2.4. On the other hand, if $E(u_0) = 0$ and $\mathfrak{Im}\int_{\mathbb{R}^2} \bar{u}_0(\boldsymbol{x}\,\boldsymbol{.}\,\nabla)u_0\,d\boldsymbol{x} < 0$, then by (2.1.12) we have $I'(0) < 0$. Since $E(u(t)) = E(u_0) = 0$, for all t, (2.1.12) implies that I' is constant in time. So, since $I(0) > 0$, we see that $I(T) = I'(0)T + I(0) = 0$ for some T large enough and similarly conclude that the solution blows up in finite time. Finally, suppose that $E(u_0) > 0$ and $-\mathfrak{Im}\int_{\mathbb{R}^2} \bar{u}_0(\boldsymbol{x}\,\boldsymbol{.}\,\nabla)u_0\,d\boldsymbol{x} > \sqrt{2E(u_0)I(0)}$. Then again by (2.1.12) we have $-I'(0) > 4\sqrt{2E(u_0)I(0)}$ implying $I'(0)^2 > 32E(u_0)I(0) > 16E(u_0)I(0)$. So, since $I'(0) < 0$, $4E(u)(0)t^2 + I'(0)t + I(0) = 0$ has a positive root T. Thus $I(T) = 0$ and consequently the solution blows up in finite. $\square$

Regarding the focusing and defocusing cases for the solutions of the problem (2.2.1) with $\delta = 1$, as it is elaborated in [6], we have the following dichotomy. Either there exists some $u \in \Sigma$ such that $\langle K(|u|^2), |u|^2\rangle > 0$ whence follows the existence of initial data with negative energy and by Theorem 2.2.4 this in turn implies that the corresponding solutions blow up in finite time; or $\langle K(|u|^2), |u|^2\rangle \leqslant 0$ for every $u \in \Sigma$ so that H^1-solutions are global and so are the Σ-solutions by Theorem 2.2.2. The first situation is called the focusing case and the latter is the defocusing case. In [7], such a sharp demarcation is achieved in terms of the assumptions on the symbol α instead of the L^2 inner product $\langle K(|u|^2), |u|^2\rangle$. In the sequel we adapt these results to the problem (2.2.1).

For the case where $\alpha(\boldsymbol{\xi}) \leqslant 0$ for all $\boldsymbol{\xi} \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have already shown in Theorem 2.2.3 that H^1-solutions are global in time. Now we state two direct consequences of Theorem 2.2.3 and Theorem 2.2.5.

Proposition 2.2.6. [7, Proposition 2.5] *If $\alpha(\boldsymbol{\xi}) \leqslant 0$, for all $\boldsymbol{\xi} \in \mathbb{R}^2 \setminus \{(0,0)\}$, then the zero solution of (2.2.1) with $\delta = 1$ is stable.*

Proof. We let $\varepsilon > 0$ and consider an initial datum u_0 satisfying $\|u_0\|_{H^1} \leqslant \tilde{\delta}$ for some $\tilde{\delta} > 0$. Setting $f = |u|^2$ and $f_0 = |u_0|^2$, we utilize mass and energy conservations and obtain

$$\begin{aligned}
\|u(t)\|_{H^1}^2 = \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 &\leqslant \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 - \frac{1}{2}\int\limits_{\mathbb{R}^2} \alpha(\boldsymbol{\xi})|\hat{f}|^2(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\
&= m(u_0) + E(u_0) \\
&= \|u_0\|_2^2 + \|\nabla u_0\|_2^2 - \frac{1}{2}\int\limits_{\mathbb{R}^2} \alpha(\boldsymbol{\xi})|\hat{f}_0|^2(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\
&\leqslant \|u_0\|_2^2 + \|\nabla u_0\|_2^2 + \frac{1}{2}\|\alpha\|_\infty \int\limits_{\mathbb{R}^2} |\hat{f}_0|^2(\boldsymbol{\xi})\, d\boldsymbol{\xi} \\
&= \|u_0\|_{H^1}^2 + \frac{1}{2}\|\alpha\|_\infty \|u_0\|_4^4 .
\end{aligned}$$

Employing the Sobolev imbedding $H^1(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$, we see that

$$\|u(t)\|_{H^1}^2 \leqslant C_1\tilde{\delta}^2 + C_2\tilde{\delta}^4,$$

for some positive constants C_1, C_2, and so $\|u(t)\|_{H^1}^2 \leqslant \varepsilon$ for some suitable choice of $\tilde{\delta}$, whence follows the claim. ☐

Proposition 2.2.7. [7, Proposition 2.6] *The nontrivial standing wave solutions of (2.2.1)$_1$ with $\delta = 1$ are unstable.*

Proof. Let $u(\boldsymbol{x}, t) = \varphi(\boldsymbol{x})e^{i\omega t}$, $\omega > 0$, $\varphi \in H^1(\mathbb{R}^2)$ be a nontrivial standing wave solution for the problem (2.2.1), with $\delta = 1$. By Theorem 3.2.1 regarding the regularity of standing waves, we see that in fact $\varphi \in \Sigma$. So the virial identity (2.1.13) implies that $E(\varphi) = 0$. Then for the corresponding standing wave u we have

$$\|\nabla u\|_2^2 = \|\nabla \varphi\|_2^2 = \frac{1}{2} \int_{\mathbb{R}^2} \alpha(\boldsymbol{\xi}) |\hat{f}_0|^2(\boldsymbol{\xi}) \, d\boldsymbol{\xi}, \tag{2.2.2}$$

where $f_0 = |u|^2$. So, if we consider the initial data $(1+\varepsilon)\varphi$, for the corresponding solution it turns out that

$$\begin{aligned} E((1+\varepsilon)\varphi) &= (1+\varepsilon)^2 \|\nabla \varphi\|_2^2 - (1+\varepsilon)^4 \frac{1}{2} \int_{\mathbb{R}^2} \alpha(\boldsymbol{\xi}) |\hat{f}_0|^2(\boldsymbol{\xi}) \, d\boldsymbol{\xi} \\ &= \|\nabla \varphi\|_2^2 \left((1+\varepsilon)^2 - (1+\varepsilon)^4\right), \end{aligned}$$

by (2.2.2). Thus, $E((1+\varepsilon)\varphi) < 0$, whenever $\varepsilon > 0$ and the corresponding solution $(1+\varepsilon)u$ blows up in finite time by Theorem 2.2.5. □

As stated in [7, Remark 2.1], this proof points out the fact that standing waves exist only if there exists some $\boldsymbol{\xi} \in \mathbb{R}^2$ such that $\alpha(\boldsymbol{\xi}) > 0$, for $E(\varphi)$ cannot vanish otherwise. In other words, a standing wave solution to (2.2.1) may exist only in the focusing case.

We now recall that for a standing wave solution $u(\boldsymbol{x}, t) = \varphi(\boldsymbol{x}) e^{i\omega t}$, $\omega > 0$, $\varphi \in H^1(\mathbb{R}^2)$, φ must be a solution of (1.0.8). So for (1.0.8), setting $B(\psi) = \int_{\mathbb{R}^2} K(\psi) \bar{\psi} \, d\boldsymbol{x}$ we define the Lagrangian

$$L_\omega(\varphi) = \frac{1}{2} \|\nabla \varphi\|_2^2 - \frac{1}{4} B(|\varphi|^2) + \frac{\omega}{2} \|\varphi\|_2^2,$$

and in a standard way, we separate the Lagrangian L_ω as the difference between kinetic and the potential energies

$$T(\varphi) = \|\nabla \varphi\|_2^2, \quad V(\varphi) = \frac{1}{4} B(|\varphi|^2) - \frac{\omega}{2} \|\varphi\|_2^2, \tag{2.2.3}$$

In [2], it is set forth regardless of the spatial dimension that, in case L_2 is elliptic, i.e. C^2 sign definite, if there exists some $\boldsymbol{\xi} \in \mathbb{R}^2$ such that $(C^2\boldsymbol{\xi}, \boldsymbol{\xi})$ and $(C^3\boldsymbol{\xi}, \boldsymbol{\xi})$ are of the same sign; then there exist initial data u_0 lying in the Schwartz class such that $E(u_0) \leqslant 0$ and $\frac{dI}{dt}(0) < 0$. Moreover, they conclude that in this case the set $\Sigma_- = \{v \in \Sigma \,|\, E(v) < 0\}$ is nonempty and solutions starting in Σ_- blow up in finite time. Obviously, the above assumptions on the matrices and their quadratic forms are in agreement with the assumption made in [7] on the symbol α in order to obtain such a result. In what follows, we introduce the scaling argument utilized both for GDS system and ACNLS equation in [7] in order to obtain initial data with negative energy and hence the blow up result.

We transform $\boldsymbol{x}$ via a matrix $A(s, c)$ depending on the real parameters s, c and define

$$u^{s,c}(\boldsymbol{x}) = |\det A(s, c)|^{1/4} u(A(s, c)\boldsymbol{x}).$$

For $f = |u^2|$ as before, we see that $f^{s,c}(\boldsymbol{x}) = |\det A(s, c)|^{1/2} f(A(s, c)\boldsymbol{x})$ and directly compute $\hat{f}^{s,c}$ to be

$$\hat{f}^{s,c}(\boldsymbol{\xi}) = \frac{1}{|\det A(s, c)|^{1/2}} \hat{f}((A(s, c)^T)^{-1}\boldsymbol{\xi}).$$

Now we investigate how this transformation maps the potential energy. Using Plancharel's theorem we obtain

$$B(f^{s,c}) = \int\limits_{\mathbb{R}^2} \alpha((A(s, t))^T\boldsymbol{\xi}) |\hat{f}|^2(\boldsymbol{\xi})\, d\xi.$$

As done in [7], we now choose $A(s, c)$ in such a way that the s-limit behaviour of $\alpha((A(s, t))^T\boldsymbol{\xi})$ reveals the close kinship between $B(|u|^2)$ and $\|u\|_4^4$. As appears in [7] we let

$$A(s,c) = \begin{pmatrix} c(s+1) & s \\ cs & s+1 \end{pmatrix}$$

and immediately observe that $\det A(s,c) = c(2s+1) \neq 0$, provided that $c \neq 0$ and $s > 0$. Besides, we compute that

$$\lim_{s\to\infty} \alpha(A(s,c)^T \boldsymbol{\xi}) = \alpha(c,1),$$

and so it turns out that this transformation concentrates the Fourier transforms of the solutions on the line $\xi_1 = c\xi_2$ as s tends to infinity. Consequently, we obtain

$$\lim_{s\to\infty} B(|u^{s,c}|^2) = \alpha(c,1)\, \|u\|_4^4 \tag{2.2.4}$$

by the Lebesgue dominated convergence theorem. The below results are established for the elliptic GDS system and elliptic ACNLS equation in [7].

Lemma 2.2.8. [7, Lemma 4.1] *Let* $\omega > 0$. *If* $\alpha(c,1) > 0$ *for some* c, *then the set* $\Sigma_0 = \{v \in \Sigma \,|\, E(v) = 0\}$ *is nonempty.*

Proof. Let c_0 be the parameter such that $\alpha(c_0,1) > 0$. Then $\alpha(c_0,1)\, \|v\|_4^4 > 0$ implies

$$\lim_{s\to\infty} B(|v^{s,c_0}|^2) > 0.$$

Thus there exists some s_0 such that $B(|v^{s_0,c_0}|^2) > 0$ and then

$$V(sv^{s_0,c_0}) = \frac{1}{4} B(|v^{s_0,c_0}|^2)s^4 - \frac{\omega}{2}\, \|v^{s_0,c_0}\|_2^2\, s^2 = 0$$

has a nonzero real root, say s_1, so that we have $s_1 v^{s_0,c_0} \in \Sigma_0$.

□

Theorem 2.2.9. [7, Theorem 4.2] *Let $\omega > 0$. Then $\alpha(c,1) > 0$ for some c if and only if a standing wave solution of the form $u(\boldsymbol{x},t) = \varphi(\boldsymbol{x})e^{i\omega t}$, where $\varphi \in H^1(\mathbb{R}^2)$ solves (1.0.8) exists.*

Proof. Suppose that $\alpha(c,1) > 0$. Then Lemma 2.2.8 guarantees that Σ_0 is nonempty and the existence of standing waves follows from the constrained minimization argument in [14, Theorem 8.1.6]. On the other hand, if such a standing wave solution exists, then $V(\varphi) = 0$ by the Pohozaev identites (3.1.1) and (3.1.2), so we conclude that $\varphi \in \Sigma_0$ Moreover, $\omega > 0$ implies that $B(|\varphi|^2) > 0$ and hence $\alpha(\boldsymbol{\xi}) > 0$ for some $\boldsymbol{\xi} \in \mathbb{R}^2$, for $B(|\varphi|^2) \leqslant 0$ otherwise. If $\xi_2 \neq 0$, then $\alpha(c,1) > 0$ for $c = \xi_1/\xi_2$. In case $\xi_2 = 0$, $\dfrac{C_{11}^3}{C_{11}^2} > 0$ so letting c tend to infinity we obtain $\lim\limits_{c\to\pm\infty} \alpha(c,1) > 0$ which implies $\alpha(c_0,1) > 0$, for some $c_0 \in \mathbb{R}$. □

Theorem 2.2.10. [7, Theorem 4.3] *If $\alpha(c,1) > 0$ for some c, then for any initial datum $u_0 \in \Sigma$, there exists a suitably scaled initial datum $\tilde{u}_0$ such that local in time solutions of*

$$\begin{aligned} iu_t + \Delta u + K(|u|^2)u = 0, \\ u(\boldsymbol{x},0) = \tilde{u}_0, \end{aligned} \tag{2.2.5}$$

blow up in finite time.

Proof. We utilize scaling with the matrix $A(s,c)$ and write the energy for the scaled version $u_0^{s,c} = |\det A(s,c)|^{1/4} u_0(A(s,c)\boldsymbol{x})$. By hypothesis, there exists some c_0 such that $\alpha(c_0,1) > 0$ and by (2.2.4), we have $B(|u_0^{s_0,c_0}|^2) > 0$ for sufficiently large s_0. We set $\tilde{u}_0 = \mu u_0^{s_0,c_0}$ and observe that

$$E(\tilde{u}_0) = \mu^2 \|u_0^{s_0,c_0}\|_2^2 - \mu^4 B(|u_0^{s_0,c_0}|^2).$$

Therefore, $E(\tilde{u}_0) < 0$ for sufficiently large μ so that the solution of (2.2.5) corresponding to the initial datum $\tilde{u}_0$ blows up in finite time by Theorem 2.2.5. □

3. STANDING WAVE SOLUTIONS OF ZAKHAROV–SCHULMAN EQUATIONS

In this part of our work, we consider the case where L_1 and L_2 are both elliptic operators. So we reduce the system (1.0.1) into the single equation (1.0.7) with $\delta = 1$, and then examine the existence and regularity of the standing waves, i.e., periodic solutions of the form

$$u(\boldsymbol{x}, t) = \varphi(\boldsymbol{x})e^{i\omega t}, \tag{3.0.1}$$

where $\omega > 0$, $\varphi \in H^1(\mathbb{R}^2)$, $\varphi \neq 0$. Evidently, u is such a solution if and only if φ solves

$$\Delta\varphi - \omega\varphi + K(|\varphi|^2)\varphi = 0. \tag{3.0.2}$$

Before we proceed further, let us mention some properties that the singular integral operator K enjoys.

Lemma 3.0.11. [20, Lemma 2.1] *For $1 < p < \infty$ we have:*

(i) *K is a bounded linear operator from L^p into L^p,*
(ii) *K is self-adjoint,*
(iii) *If $f \in H^s$ then $K(f) \in H^s$, for all $s \in (0, \infty)$,*
(iv) *If $f \in W^{m,p}$ then $K(f) \in W^{m,p}$ and $\partial_j K(f) = K(\partial_j f)$, where $j = 1, 2$,*
(v) *K preserves the following operations:*
  - (translation) $K(f(\cdot + \tau))(\boldsymbol{x}) = K(f)(\boldsymbol{x} + \tau)$, *for all* $\tau \in \mathbb{R}^2$,
  - (dilatation) $K(f(\lambda\cdot))(\boldsymbol{x}) = K(f)(\lambda\boldsymbol{x})$, *for all* $\lambda > 0$,
  - (conjugation) $\overline{K(f)} = K(\overline{f})$.

Proof. Since α is homogeneous of order zero and bounded, the assertion *(i)* follows from the Calderon-Zygmund theorem [27]. The assertion *(ii)* is immediate by the definition of K. To prove *(iii)*, we invoke the characterization of H^s by Fourier transform, that is, we recall that

$$f \in H^s(\mathbb{R}^2) \text{ if and only if } \left(1+|\boldsymbol{\xi}|^2\right)^{s/2} \hat{f} \in L^2(\mathbb{R}^2).$$

So, for any $f \in H^s(\mathbb{R}^2)$ we have $(1+|\boldsymbol{\xi}|^2)^{s/2}\,\hat{f} \in L^2(\mathbb{R}^2)$. In order to conclude that $K(f) \in H^s(\mathbb{R}^2)$, it is sufficient to show that $(1+|\boldsymbol{\xi}|^2)^{s/2}\,\widehat{K(f)} \in L^2(\mathbb{R}^2)$. We easily see that

$$\left\|\left(1+|\boldsymbol{\xi}|^2\right)^{s/2}\widehat{K(f)}\right\|_2 = \left\|\left(1+|\boldsymbol{\xi}|^2\right)^{s/2}\alpha(\boldsymbol{\xi})\hat{f}\right\|_2 \leqslant \|\alpha\|_\infty \left\|\left(1+|\boldsymbol{\xi}|^2\right)^{s/2}\hat{f}\right\|_2 < +\infty,$$

and hence *(iii)* follows. We note that we do not have such a characterization using the Fourier transform for the general Sobolev spaces $W^{m,p}$. However, since the singular integral operator K is defined by the convolution $K(\,\cdot\,) = \check{\alpha} * \cdot$ on C_c^∞, we observe that $\partial_j K(f) = \partial_j(\check{\alpha} * f) = \check{\alpha} * (\partial_j f) = K(\partial_j f)$ and upon a denseness argument the assertion *(iv)* follows by *(i)*. The claim *(v)* is established by straightforward computation and using again a denseness argument. $\square$

3.1. Pohozaev Type Identites

The following identites provide us with necessary conditions for existence of standing wave solutions. Before we state the theorem, let us set $B(f) = \langle K(f), f\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product.

Theorem 3.1.1. *Suppose that φ satisfies*

$$\Delta\varphi - \omega\varphi + K(|\varphi|^2)\varphi = 0,$$

where φ is a nonzero function lying in $H^1(\mathbb{R}^2)$. Then φ satisfies the following Pohozaev *type identities:*

$$\int_{\mathbb{R}^2} (2\omega - K(|\varphi|^2))|\varphi|^2 \, dxdy = 0, \tag{3.1.1}$$

$$\int_{\mathbb{R}^2} (|\nabla\varphi|^2 - \omega|\varphi|^2) \, dxdy = 0. \tag{3.1.2}$$

Proof. We mimic the proof given in [20, Theorem 2.1]. Multiplying (3.0.2) by $x\bar{\varphi}_x$ and integrating over $\mathbb{R}^2$ we have

$$\int_{\mathbb{R}^2} x\bar{\varphi}_x\varphi_{xx} \, dxdy + \int_{\mathbb{R}^2} x\bar{\varphi}_x\varphi_{yy} \, dxdy - \omega \int_{\mathbb{R}^2} x\bar{\varphi}_x\varphi \, dxdy + \int_{\mathbb{R}^2} x\bar{\varphi}_x K(|\varphi|^2)\varphi \, dxdy = 0.$$

By the virtue of Lemma 3.0.11 and the fact that $\varphi \in H^1(\mathbb{R}^2)$, no trouble concerning the boundary conditions of the integrands arises when we employ integration by parts. We recall that $\varphi \in H^1(\mathbb{R}^2)$ in particular implies $\varphi \in L^4(\mathbb{R}^2)$ and this is essential for integrating the fourth integrand by parts. Doing so and taking real parts yields

$$\begin{aligned} -\frac{1}{2}\int_{\mathbb{R}^2} |\varphi_x|^2 \, dxdy + \frac{1}{2}\int_{\mathbb{R}^2} |\varphi_y|^2 \, dxdy + \frac{1}{2}\omega \int_{\mathbb{R}^2} |\varphi|^2 \, dxdy& \\ -\frac{1}{2}\left\{ B(|\varphi|^2) + \int_{\mathbb{R}^2} K(|\varphi|^2)_x x|\varphi|^2 \, dxdy \right\} &= 0, \end{aligned}$$

and multiplying by -2 we get

$$\int_{\mathbb{R}^2} \left(|\varphi_x|^2 - |\varphi_y|^2 - \omega|\varphi|^2\right) \, dxdy + B(|\varphi|^2) + \int_{\mathbb{R}^2} K(|\varphi|^2)_x x|\varphi|^2 \, dxdy = 0. \tag{3.1.3}$$

To ease the notation, let f stand for $|\varphi|^2$ hereafter. By Plancharel's theorem, we see that

$$\int_{\mathbb{R}^2} K(f)_x x f \, dxdy = \frac{1}{2}\int_{\mathbb{R}^2} \xi_1 \widehat{K(f)}(\boldsymbol{\xi}) \bar{\hat{f}}_{\xi_1}(\boldsymbol{\xi}) \, d\xi_1 d\xi_2 + \frac{1}{2}\int_{\mathbb{R}^2} \hat{f}_{\xi_1}(\boldsymbol{\xi}) \xi_1 \overline{\widehat{K(f)}}(\boldsymbol{\xi}) \, d\xi_1 d\xi_2$$
$$= \int_{\mathbb{R}^2} \xi_1 \alpha(\boldsymbol{\xi})(\hat{f}(\boldsymbol{\xi})\bar{\hat{f}}_{\xi_1}(\boldsymbol{\xi}) + \bar{\hat{f}}(\boldsymbol{\xi})\hat{f}_{\xi_1}(\boldsymbol{\xi})) \, d\xi_1 d\xi_2$$
$$= \frac{1}{2}\int_{\mathbb{R}^2} \xi_1 \alpha(\boldsymbol{\xi})(|\hat{f}|^2)_{\xi_1}(\boldsymbol{\xi}) \, d\xi_1 d\xi_2,$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2)$. Integrating by parts in the variable ξ_1 yields

$$\int_{\mathbb{R}^2} K(f)_x x f \, dxdy = -\frac{1}{2}\int_{\mathbb{R}^2} (\xi_1 \alpha(\boldsymbol{\xi}))_{\xi_1} |\hat{f}|^2(\boldsymbol{\xi}) \, d\boldsymbol{\xi},$$

and so we can rewrite (3.1.3) as

$$\int_{\mathbb{R}^2} \left(|\varphi_x|^2 - |\varphi_y|^2 - \omega|\varphi|^2\right) \, dxdy + B(f) - \frac{1}{2}\int_{\mathbb{R}^2} (\xi_1 \alpha(\boldsymbol{\xi}))_{\xi_1} |\hat{f}|^2(\boldsymbol{\xi}) \, d\boldsymbol{\xi} = 0,$$

which gives us

$$\int_{\mathbb{R}^2} \left(|\varphi_x|^2 - |\varphi_y|^2 - \omega|\varphi|^2\right) \, dxdy + \int_{\mathbb{R}^2} \left\{\alpha(\boldsymbol{\xi}) - \frac{1}{2}(\xi_1 \alpha(\boldsymbol{\xi}))_{\xi_1}\right\} |\hat{f}|^2(\boldsymbol{\xi}) \, d\boldsymbol{\xi} = 0. \quad (3.1.4)$$

Similarly, multiplying (3.0.2) by $y\bar{\varphi}_y$, integrating over $\mathbb{R}^2$ and following exactly the same steps above, we obtain

$$\int_{\mathbb{R}^2} \left(|\varphi_x|^2 - |\varphi_y|^2 + \omega|\varphi|^2\right) \, dxdy - \int_{\mathbb{R}^2} \left\{\alpha(\boldsymbol{\xi}) - \frac{1}{2}(\xi_2 \alpha(\boldsymbol{\xi}))_{\xi_2}\right\} |\hat{f}|^2(\boldsymbol{\xi}) \, d\boldsymbol{\xi} = 0. \quad (3.1.5)$$

Finally, we multiply (3.0.2) by $\bar{\varphi}$ and integrate over $\mathbb{R}^2$ to get

$$\int_{\mathbb{R}^2} \left(\bar{\varphi}\varphi_{xx} + \bar{\varphi}\varphi_{yy} - \omega|\varphi|^2 + K(|\varphi|^2)|\varphi|^2\right) \, dxdy,$$

and, applying integration by parts to the first and the second terms in the integrand we end up with

$$\int\limits_{\mathbb{R}^2} \left(|\varphi_x|^2 + |\varphi_y|^2 + \omega|\varphi|^2\right)\, dxdy - B(|\varphi|^2) = 0. \tag{3.1.6}$$

Subtracting (3.1.5) from (3.1.4) yields

$$\int\limits_{\mathbb{R}^2} \left(2\omega - K(|\varphi|^2)\right) |\varphi|^2\, dxdy + \frac{1}{2}\int\limits_{\mathbb{R}^2} \left(\xi_1\alpha_{\xi_1}(\boldsymbol{\xi}) + \xi_2\alpha_{\xi_2}(\boldsymbol{\xi})\right)\, d\boldsymbol{\xi} = 0.$$

Writing $\alpha(\boldsymbol{\xi}) = \dfrac{Q_3(\boldsymbol{\xi})}{Q_2(\boldsymbol{\xi})}$, where Q_j, $j = 2, 3$ is defined as in Proposition 2.0.1, we directly compute that

$$\xi_1\alpha_{\xi_1}(\boldsymbol{\xi}) + \xi_2\alpha_{\xi_2}(\boldsymbol{\xi}) = \frac{2Q_2(\boldsymbol{\xi})Q_3(\boldsymbol{\xi}) - 2Q_2(\boldsymbol{\xi})Q_3(\boldsymbol{\xi})}{Q_2^2(\boldsymbol{\xi})} = 0,$$

and hence establish

$$\int\limits_{\mathbb{R}^2} \left(2\omega - K(|\varphi|^2)\right) |\varphi|^2\, dxdy = 0.$$

Combining this with (3.1.6), we obtain

$$\int\limits_{\mathbb{R}^2} \left(|\nabla\varphi|^2 - \omega|\varphi|^2\right)\, dxdy = 0.$$

$\square$

It follows from the identites (3.1.1) and (3.1.2) that the equation (3.0.2) has a nontrivial solution only if $\omega > 0$. In this regard, we restrict our attention to the case where $\omega > 0$ throughout this chapter.

3.2. Regularity of Standing Wave Solutions

Since we later show that the equation (3.0.2) has a nonnegative solution, throughout this section we assume that φ is real valued. We now state the regularity result.

Theorem 3.2.1 (Regularity). [13, Theorem 2.4] *If* $\varphi \in H^1(\mathbb{R}^2)$ *is a weak solution of* (3.0.2)*, then the following hold.*

(i) $\varphi \in W^{2,p}(\mathbb{R}^2)$, *for all* $2 \leqslant p < \infty$,

(ii) $\lim\limits_{|\boldsymbol{x}|\to\infty} \left\{|\nabla\varphi(\boldsymbol{x})| + |\varphi(\boldsymbol{x})| + |K(\varphi^2)(\boldsymbol{x})|\right\} = 0$,

(iii) $u \in C^2$,

(iv) There exist positive constants C *and* ν *such that*

$$e^{\nu|\boldsymbol{x}|}\left\{|\varphi(\boldsymbol{x})| + |\nabla\varphi(\boldsymbol{x})|\right\} \leqslant C, \quad \textit{for all}\, \boldsymbol{x} \in \mathbb{R}^2.$$

Proof. We proceed in several steps.

Step 1: Our aim is to show that $\varphi \in L^2 \cap L^\infty$. Since $\varphi \in H^1$, we immediately have $\varphi \in L^2$. By the Sobolev imbedding theorem, $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ for all $2 \leqslant p < \infty$, so we see that $\varphi \in L^p$ for all $2 \leqslant p < \infty$ and hence $\varphi^2 \in L^{p/2}$ for all $2 \leqslant p < \infty$. Since $K \in \mathcal{L}(L^p, L^p)$ for any $1 < p < \infty$, there exists some $r > 2$ such that $\varphi \in L^r$ and so, $K(\varphi^2) \in L^{r/2}$ implying that $K(\varphi^2)\varphi \in L^r$ by Hölder inequality. Hence $\Delta\varphi \in L^r$ and by elliptic regularity $\varphi \in W^{2,r}$. Then by the Sobolev imbedding $W^{2,r}(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2)$ we have $\varphi \in W^{1,\infty}$, i.e. φ is globally Lipschitz continuous, and thus $\varphi \in L^\infty$.

Step 2: We prove *(i)*. Since $\Delta\varphi = \omega\varphi - K(|\varphi|^2)\varphi$, it is sufficient to show that the right hand side of this equation is in L^p for all $2 \leqslant p < \infty$. Now we know $\varphi \in L^p$ for all $2 \leqslant p \leqslant \infty$. Then $\varphi^2 \in L^{p/2}$ for all $2 \leqslant p \leqslant \infty$ and so $K(\varphi^2) \in L^p$ for all $1 < p < \infty$. Thereupon we have

$$\|K(\varphi^2)\varphi\|_q \leqslant \|\varphi\|_\infty \|K(\varphi^2)\|_q \leqslant \infty, \quad \text{for any}\, q \in (1, \infty),$$

so that $K(\varphi^2)\varphi \in L^q$ for all $1 < q < \infty$. Thus $\Delta\varphi \in L^p$ for all $2 \leqslant p < \infty$ and hence $\varphi \in W^{2,p}$ for all $2 \leqslant p < \infty$ by regularity of the elliptic equations.

Step 3: Since $\varphi \in L^2$, we have

$$\lim_{|\boldsymbol{x}|\to\infty} |\varphi(\boldsymbol{x})| = 0 \tag{3.2.1}$$

Step 4: As we have shown in Step 1, $\varphi \in W^{1,\infty}$. Yet $\varphi \in W^{2,p}$ for all $2 \leqslant p < \infty$ has also been established. Thus it follows that $\varphi \in W^{1,p}$ for all $2 \leqslant p \leqslant \infty$. So, for any fixed p such that $2 \leqslant p \leqslant \infty$, we also have

$$\|\varphi^2\|_p^p = \int_{\mathbb{R}^2} (|\varphi|^2)^p \, d\boldsymbol{x} = \int_{\mathbb{R}^2} |\varphi|^{2p} \, d\boldsymbol{x} = \|\varphi\|_{2p}^{2p} < \infty,$$

which implies $\varphi^2 \in L^p$ for any $2 \leqslant p \leqslant \infty$. Moreover, since $\nabla(\varphi^2) = 2\varphi\nabla\varphi$, it follows that

$$\left\|\nabla(\varphi^2)\right\|_p = 2\|\varphi\nabla\varphi\|_p \leqslant \|\varphi\|_\infty\|\nabla\varphi\|_p < \infty.$$

Therefore, $\varphi^2 \in W^{1,p}$ for all $2 \leqslant p < \infty$ and so we obtain $K(\varphi^2) \in W^{1,p}$ for all $2 \leqslant p < \infty$. Now let $q > 2$ be fixed so that $W^{1,q}$ is a Banach algebra [28, Theorem 5.23]. Then we have $K(\varphi^2)\varphi \in W^{1,q}$, whence $\Delta\varphi \in W^{1,q}$ and by elliptic regularity $\varphi \in W^{3,q}$ for any $q > 2$. The assertion *(iii)* now follows from the Sobolev imbedding $W^{3,q}(\mathbb{R}^2) \hookrightarrow C^2(\mathbb{R}^2)$. Moreover, since $\varphi \in W^{3,p}$ for all $2 < p < \infty$, we see that $\nabla\varphi \in W^{2,p}$ for all $2 < p < \infty$. It follows as a consequence of the Sobolev imbedding theorem that $\nabla\varphi \in W^{1,\infty}$ and just like in Step 3 we get

$$\lim_{|\boldsymbol{x}|\to\infty} |\nabla\varphi(\boldsymbol{x})| = 0. \tag{3.2.2}$$

Step5: We have previously shown that $\varphi \in W^{3,p}$ for all $2 < p < \infty$ and so, in particular $\varphi^2 \in W^{2,q}$ for some $q \in [2, \infty)$. Then from Lemma 3.0.11 it follows that $K(\varphi^2) \in W^{2,q}$, whence $K(\varphi^2) \in W^{1,\infty}$ by Sobolev imbedding theorem. Thereupon, we similarly conclude that $\lim\limits_{|\boldsymbol{x}|\to\infty} |K(\varphi^2)(\boldsymbol{x})| = 0$. Combining this with (3.2.1) and (3.2.2) the assertion *(ii)* follows.

Step 6: Now the only task remains is to prove *(iv)*. We first note that it is sufficient to consider the case $\omega = 1$ since ψ defined by $\varphi(\boldsymbol{x}) = \sqrt{\omega}\psi(\sqrt{\omega}\boldsymbol{x})$ satisfies $\Delta\psi - \psi + K(|\psi|^2)\psi = 0$, whenever φ is a solution of (3.0.2). Now, for any $\varepsilon > 0$, we define $\theta_\varepsilon(\boldsymbol{x}) = \exp\left(\dfrac{|\boldsymbol{x}|}{1+\varepsilon|\boldsymbol{x}|}\right)$. Then θ_ε is bounded since $\theta_\varepsilon(\boldsymbol{x}) \leqslant \exp\left(\dfrac{1}{\varepsilon}\right) = M$, for all $\boldsymbol{x} \in \mathbb{R}^2$. Furthermore, with $\boldsymbol{x} = (x_1, x_2)$,

$$|\nabla\theta_\varepsilon(\boldsymbol{x})|^2 = \theta_\varepsilon^2(\boldsymbol{x})\frac{x_1^2}{|\boldsymbol{x}|^2(1+\varepsilon|\boldsymbol{x}|)^4} + \theta_\varepsilon^2(\boldsymbol{x})\frac{x_2^2}{|\boldsymbol{x}|^2(1+\varepsilon|\boldsymbol{x}|)^4} = \theta_\varepsilon^2(\boldsymbol{x})\frac{1}{(1+\varepsilon|\boldsymbol{x}|)^4} \leqslant \theta_\varepsilon^2(\boldsymbol{x})$$

clearly states that $|\nabla\theta_\varepsilon| \leqslant \theta_\varepsilon$ almost everywhere in $\mathbb{R}^2$, and since $\theta_\varepsilon \leqslant M$ we deduce that θ_ε is globally Lipschitz continuous.

We now multiply the equation (3.0.2) by $\theta_\varepsilon\varphi \in H^1$, integrate over $\mathbb{R}^2$ and obtain

$$\int\limits_{\mathbb{R}^2} (\nabla\varphi \cdot \nabla(\theta_\varepsilon\varphi))\, d\boldsymbol{x} + \int\limits_{\mathbb{R}^2} \theta_\varepsilon\varphi^2\, d\boldsymbol{x} = \int\limits_{\mathbb{R}^2} \theta_\varepsilon\varphi^2 K(\varphi^2)\, d\boldsymbol{x},$$

subsequent to an integration by parts. Since $\nabla(\theta_\varepsilon\varphi) = \varphi\nabla\theta_\varepsilon + \theta_\varepsilon\nabla\varphi$, we see that $\nabla\varphi \cdot \nabla(\theta_\varepsilon\varphi) = \theta_\varepsilon|\nabla\varphi|^2 + \varphi(\nabla\varphi \cdot \nabla\theta_\varepsilon)$. Yet, by Cauchy–Schwarz inequality it follows that $(\nabla\theta_\varepsilon \cdot \nabla\varphi)\varphi \geqslant -|\varphi||\nabla\varphi|\theta_\varepsilon$, and thus

$$\int\limits_{\mathbb{R}^2} \theta_\varepsilon|\nabla\varphi|^2\, d\boldsymbol{x} - \int\limits_{\mathbb{R}^2} \theta_\varepsilon|\varphi||\nabla\varphi|\, d\boldsymbol{x} + \int\limits_{\mathbb{R}^2} \theta_\varepsilon\varphi^2\, d\boldsymbol{x} \leqslant \int\limits_{\mathbb{R}^2} \theta_\varepsilon K(\varphi^2)\varphi^2\, d\boldsymbol{x}.$$

Now let $\delta < \dfrac{1}{4}$. By *(ii)*, there exists some $r_1 > 0$ such that $|K(\varphi^2)(\boldsymbol{x})| < \delta$, whenever $|\boldsymbol{x}| \geqslant r_1$. On the other hand, by Cauchy inequality we have

$$\int\limits_{\mathbb{R}^2} \theta_\varepsilon|\varphi||\nabla\varphi|\, d\boldsymbol{x} \leqslant \frac{1}{2}\int\limits_{\mathbb{R}^2} \theta_\varepsilon\varphi^2\, d\boldsymbol{x} + \frac{1}{2}\int\limits_{\mathbb{R}^2} \theta_\varepsilon|\nabla\varphi|^2\, d\boldsymbol{x},$$

which in turn gives us

$$\frac{1}{2}\int_{\mathbb{R}^2} \theta_\varepsilon |\nabla\varphi|^2\, d\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^2} \theta_\varepsilon \varphi^2\, d\boldsymbol{x} \leqslant \int_{|\boldsymbol{x}|<r_1} \theta_\varepsilon K(\varphi^2)\varphi^2\, d\boldsymbol{x} + \int_{|\boldsymbol{x}|\geqslant r_1} \theta_\varepsilon K(\varphi^2)\varphi^2\, d\boldsymbol{x}.$$

Moreover, since $\theta_\varepsilon \leqslant e^{|\boldsymbol{x}|}$ it follows that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^2} \theta_\varepsilon |\nabla\varphi|^2\, d\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^2} \theta_\varepsilon \varphi^2\, d\boldsymbol{x} &\leqslant \int_{|\boldsymbol{x}|<r_1} \theta_\varepsilon K(\varphi^2)\varphi^2\, d\boldsymbol{x} + \delta \int_{|\boldsymbol{x}|\geqslant r_1} \theta_\varepsilon \varphi^2\, d\boldsymbol{x} \\ &\leqslant C_1 + \frac{1}{4}\int_{|\boldsymbol{x}|\geqslant r_1} \theta_\varepsilon \varphi^2\, d\boldsymbol{x} \leqslant C_1 + \frac{1}{4}\int_{\mathbb{R}^2} \theta_\varepsilon \varphi^2\, d\boldsymbol{x},\end{aligned}$$

and hence

$$\frac{1}{2}\int_{\mathbb{R}^2} \theta_\varepsilon |\nabla\varphi|^2\, d\boldsymbol{x} + \frac{1}{4}\int_{\mathbb{R}^2} \theta_\varepsilon |\varphi|^2\, d\boldsymbol{x} \leqslant C_1,$$

where C_1 is a constant not depending on ε. Letting $\varepsilon \to 0$ yields

$$\frac{1}{4}\int_{\mathbb{R}^2} e^{|\boldsymbol{x}|}(|\varphi|^2 + |\nabla\varphi|^2)\, d\boldsymbol{x} \leqslant C_1, \tag{3.2.3}$$

by the Monotone Convergence Theorem.

At this stage, we once again recall what we have proven in assertion *(ii)* and see that for some $r_2 > 0$, we have $|\varphi(\boldsymbol{x})| + |\nabla\varphi(\boldsymbol{x})| < 1$, whenever $|\boldsymbol{x}| \geqslant r_2$. On the other hand, for $|\boldsymbol{x}| \leqslant r_2$, it follows that

$$e^{\frac{|\boldsymbol{x}|}{2}}(|\varphi(\boldsymbol{x})| + |\nabla\varphi(\boldsymbol{x})|) \leqslant e^{\frac{r_2}{2}}\|\varphi\|_{W^{1,\infty}}. \tag{3.2.4}$$

Now we let $\boldsymbol{x} \in \mathbb{R}^2$ be fixed such that $|\boldsymbol{x}| \geqslant r_2$. Since φ and $\nabla\varphi$ are both globally Lipschitz continuous, there exists $L > 0$ such that for all $\boldsymbol{y} \in \mathbb{R}^2$ we have,

$$|\nabla\varphi(\boldsymbol{x})| - |\nabla\varphi(\boldsymbol{y})| \leqslant ||\nabla\varphi(\boldsymbol{x})| - |\nabla\varphi(\boldsymbol{y})|| \leqslant \frac{L}{\sqrt{2}}|\boldsymbol{x} - \boldsymbol{y}|,$$

and,

$$|\varphi(\boldsymbol{x})| - |\varphi(\boldsymbol{y})| \leqslant ||\varphi(\boldsymbol{x})| - |\varphi(\boldsymbol{y})|| \leqslant \frac{L}{\sqrt{2}}|\boldsymbol{x} - \boldsymbol{y}|,$$

which simply imply

$$\begin{aligned} |\nabla\varphi(\boldsymbol{y})| &\geqslant |\nabla\varphi(\boldsymbol{x})| - \frac{L}{\sqrt{2}}|\boldsymbol{x} - \boldsymbol{y}|, \\ |\varphi(\boldsymbol{y})| &\geqslant |\varphi(\boldsymbol{x})| - \frac{L}{\sqrt{2}}|\boldsymbol{x} - \boldsymbol{y}|. \end{aligned} \tag{3.2.5}$$

Thereupon we get $|\varphi(\boldsymbol{x})|^2 + |\nabla\varphi(\boldsymbol{x})|^2 \leqslant 2(|\varphi(\boldsymbol{y})|^2 + |\nabla\varphi(\boldsymbol{y})|^2 + L^2|\boldsymbol{x} - \boldsymbol{y}|^2)$. Let us now take $\rho := \dfrac{1}{2L}(|\varphi(\boldsymbol{x})|^2 + |\nabla\varphi(\boldsymbol{x})|^2)^{1/2}$. Then for any $\boldsymbol{y} \in B_\rho(\boldsymbol{x})$ it turns out that

$$|\varphi(\boldsymbol{x})|^2 + |\nabla\varphi(\boldsymbol{x})|^2 \leqslant 4(|\varphi(\boldsymbol{y})|^2 + |\nabla\varphi(\boldsymbol{y})|^2).$$

Integrating both sides of this inequality over $B_\rho(\boldsymbol{x})$ we obtain

$$\begin{aligned} (|\varphi(\boldsymbol{x})|^2 + |\nabla\varphi(\boldsymbol{x})|^2) \int\limits_{B_\rho(\boldsymbol{x})} d\boldsymbol{y} &= C_2\rho^2(|\varphi(\boldsymbol{x})|^2 + |\nabla\varphi(\boldsymbol{x})|^2) \\ &\leqslant 4 \int\limits_{B_\rho(\boldsymbol{x})} (|\varphi(\boldsymbol{y})|^2 + |\nabla\varphi(\boldsymbol{y})|^2)\, d\boldsymbol{y}, \end{aligned}$$

and plugging ρ yields

$$C_3(|\varphi(\boldsymbol{x})|^2 + |\nabla\varphi(\boldsymbol{x})|^2)^2 \leqslant 4 \int\limits_{B_\rho(\boldsymbol{x})} (|\varphi(\boldsymbol{y})|^2 + |\nabla\varphi(\boldsymbol{y})|^2)\, d\boldsymbol{y}, \tag{3.2.6}$$

where $C_3 = \dfrac{C_2}{4L^2}$. We also note that for $|\boldsymbol{x}| \geqslant r_2$ we have

$$|\varphi(\boldsymbol{x})|^2 + |\nabla\varphi(\boldsymbol{x})|^2 \leqslant (|\varphi(\boldsymbol{x})| + |\nabla\varphi(\boldsymbol{x})|)^2 < 1,$$

which implies $\rho \leqslant \dfrac{1}{2L}$, and thus

$$|\boldsymbol{y}| - |\boldsymbol{x}| + \frac{1}{2L} \geqslant 0 \quad \text{for all}\, \boldsymbol{y} \in B_\rho(x).$$

Multiplying (3.2.6) by $e^{|\boldsymbol{x}|}$ it follows from (3.2.3) that

$$\begin{aligned} C_3 e^{|\boldsymbol{x}|}(|\varphi(\boldsymbol{x})|^2 + |\nabla\varphi(\boldsymbol{x})|^2)^2 &\leqslant 4 \int\limits_{B_\rho(\boldsymbol{x})} e^{|\boldsymbol{x}|}(|\varphi(\boldsymbol{y})|^2 + |\nabla\varphi(\boldsymbol{y})|^2)\, d\boldsymbol{y} \\ &\leqslant 4 \int\limits_{B_\rho(\boldsymbol{x})} e^{\frac{1}{2L}} e^{|\boldsymbol{y}|}(|\varphi(\boldsymbol{y})|^2 + |\nabla\varphi(\boldsymbol{y})|^2)\, d\boldsymbol{y} \leqslant C_4, \end{aligned}$$

for some constant $C_4 > 0$. Thus for $|\boldsymbol{x}| \geqslant r_2$, we have

$$e^{|\boldsymbol{x}|}\left(|\varphi(\boldsymbol{x})|^2 + |\nabla\varphi(\boldsymbol{x})|^2\right)^2 \leqslant C_5, \tag{3.2.7}$$

and the assertion *(iv)* is now an immediate consequence of (3.2.4) and (3.2.7). $\square$

3.3. Existence of Standing Wave Solutions

In order to establish the existence of standing wave solutions, we adopt Weinstein's approach [15] and set up an equivalent variational problem. We briefly recall that a standing wave profile $\varphi \in H^1(\mathbb{R}^2)$ satisfies (3.0.2). As we have also noted in the previous section, it is sufficient to consider the case where $\omega = 1$, for φ is a solution of (3.0.2) if and only if ψ defined by $\varphi(\boldsymbol{x}) = \sqrt{\omega}\psi(\sqrt{\omega}\boldsymbol{x})$ is a solution of

$$\Delta\psi - \psi + K(|\psi|^2)\psi = 0. \tag{3.3.1}$$

So we hereafter assume that $\omega = 1$. Employing Pohozaev identities, we deduce that for such a solution u we have

$$E(u(t)) = E(\varphi) = \|\nabla\varphi\|_2^2 - \frac{1}{2}\int\limits_{\mathbb{R}^2} K(|\varphi|^2)|\varphi|^2\, d\boldsymbol{x} = 0, \quad \text{for all}\, t \geqslant 0.$$

Multiplying this Hamiltonian by $\|\varphi\|_2^2$ yields

$$\|\varphi\|_2^2 = \frac{2\|\varphi\|_2^2\|\nabla\varphi\|_2^2}{\langle K(|\varphi|^2), |\varphi|^2\rangle},$$

where $\langle\cdot,\cdot\rangle$ stands for the usual L^2 inner product. This allows us to define the associated Weinstein functional by

$$W(f) = \frac{2\,\|f\|_2^2\,\|\nabla f\|_2^2}{\langle K(|f|^2), |f|^2\rangle}. \tag{3.3.2}$$

Apparently, the nonlinear functional W returns the squared L^2-norm of the argument if the argument is a solution of (3.3.1). So, if there exists a minimizer for W and that minimizer is also a solution of (3.3.1), then it turns out to be a solution with minimal mass for the equation (3.3.1). At this stage, we note that it is not trivial to have control over the denominator of W; indeed, it may attain the value zero making W even undefined for some nonzero f. Inspired by [16] we consider maximizing its reciprocal $J = \frac{1}{W}$ instead of minimizing W itself and fortunately the mentioned obstacle disappears. In this regard we are to show the existence of maximizers for J and that these maximizers solve (3.3.1) in the weak sense. We first present a result that enables us to restrict our attention on the existence of nonnegative and real valued solutions for (3.3.1).

Lemma 3.3.1. [16, Lemma B.2] *Let $\psi \in H^1(\mathbb{R}^N)$. Then we have*

$$-|\nabla\psi| \leqslant \nabla|\psi| \leqslant |\nabla\psi|$$

in the weak sense. In particular, $|\psi| \in H^1(\mathbb{R}^N)$.

As a direct consequence of this lemma, it turns out that

$$J(f) = \frac{\langle K(|f|^2), |f|^2\rangle}{2\|f\|^2\|\nabla f\|^2} \leqslant \frac{\langle K(|f|^2), |f|^2\rangle}{2\|f\|^2\|\nabla |f|\|^2} = J(|f|),$$

for any $f \in H^1(\mathbb{R}^2)$, whence J has a nonnegative, real valued maximizer in case a maximizer exists. Before we advance, utilizing Plancharel's theorem and Gagliardo–Nirenberg–Sobolev inequality we note that

$$\left|\langle K(|f|^2), f^2\rangle\right| = \left|\int_{\mathbb{R}^2} \alpha(\boldsymbol{\xi})\widehat{|f|^2}(\boldsymbol{\xi})\overline{\widehat{|f|^2}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}\right| \leqslant \|\alpha\|_\infty \|f\|_4^4 \leqslant C\,\|f\|_2^2\,\|\nabla f\|_2^2\,,$$

which implies $M = \sup\limits_{f\in H^1(\mathbb{R}^2)} J(f) < \infty$. Moreover, if $J(f)$ is positive for some f, then evidently M is positive; that is, $M > 0$ only if there exists some $f \in H^1(\mathbb{R}^2)$ such that $\langle K(|f|^2), |f^2|)\rangle > 0$. The existence of such an f is equivalent to the existence of some $\boldsymbol{\xi} \in \mathbb{R}^2$ such that $\alpha(\boldsymbol{\xi}) > 0$, and this corresponds to the focusing case previously discussed in Chapter 2. Leaning against Theorem 2.2.9, we assume the existence of some f such that $\langle K(|f|^2), |f^2|\rangle > 0$ and so concentrate on the case where M is positive. We are now ready to establish that the nonnegative, real valued maximizers, upon their existence, are weak solutions of (3.3.1).

Lemma 3.3.2. [15] *Let $\varphi \in H^1(\mathbb{R}^2)$ be such that φ is nonnegative, not identically zero, and that $J(\varphi) = \max\limits_{f\in H^1(\mathbb{R}^2)} J(f)$. Then φ is a weak solution of (3.3.1).*

Proof. We show that any real valued, nonzero critical point of the nonlinear functional J solves (3.3.1) in the weak sense. This is established by directly computing the Gateaux derivative and observing that the solutions of the Euler–Lagrange equation

$$dJ(\varphi, h) = \lim_{\varepsilon\to 0} \frac{J(\varphi+\varepsilon h) - J(\varphi)}{\varepsilon} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} J(\varphi+\varepsilon h) = 0, \quad \text{for all}\, h \in C_c^\infty(\mathbb{R}^2),$$

satisfies (3.3.1). Indeed, utilizing the fact that K is a self-adjoint operator, we compute that

$$\begin{aligned} dJ(\varphi, h) = \frac{1}{4\,\|\varphi\|_2^4\,\|\nabla\varphi\|_2^4}\Bigg\{ & 8\langle K(\varphi^2), \varphi h\rangle\,\|\varphi\|_2^2\,\|\nabla\varphi\|_2^2 \\ & - 2\langle K(\varphi^2), \varphi^2\rangle \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left(\|\varphi+\varepsilon h\|_2^2\,\|\nabla\varphi+\varepsilon\nabla h\|_2^2\right) \Bigg\} = 0. \end{aligned}$$

Integration by parts yields

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left(\|\varphi+\varepsilon h\|_2^2\,\|\nabla\varphi+\varepsilon\nabla h\|_2^2\right)=2\left(\|\nabla\varphi\|_2^2\int\limits_{\mathbb{R}^2}\varphi h\,d\boldsymbol{x}+\|\varphi\|_2^2\int\limits_{\mathbb{R}^2}(-\Delta\varphi)h\,d\boldsymbol{x}\right),$$

and thus it follows that a critical point φ of J is necessarily a solution of

$$\begin{aligned}8\langle K(\varphi^2)\varphi,h\rangle\,\|\varphi\|_2^2\,\|\nabla\varphi\|_2^2+4\langle K(\varphi^2),\varphi^2\rangle\,\|\nabla\varphi\|_2^2\int\limits_{\mathbb{R}^2}(-\varphi)h\,d\boldsymbol{x}&\\ +4\langle K(\varphi^2),\varphi^2\rangle\,\|\varphi\|_2^2\int\limits_{\mathbb{R}^2}(\Delta\varphi)h\,d\boldsymbol{x}=0.&\quad(3.3.3)\end{aligned}$$

Thanks to Pohozaev identites obtained in Theorem 3.1.1, the solutions of (3.3.1) satisfy (3.3.3). On the other hand, an elementary calculation shows that J is invariant under the transformation $f\mapsto f^{a,b}=af(b\,\cdot)$, with a pair of parameters $(a,b)\in\mathbb{R}^2$, and so φ is a maximizer if and only if $\varphi^{a,b}$ is. In particular, noting that $\langle K(\varphi^2),\varphi^2\rangle$ is positive, we set

$$a=\frac{\sqrt{2}\,\|\varphi\|_2}{\sqrt{\langle K(\varphi^2),\varphi^2\rangle}}\quad\text{and}\quad b=\frac{\|\varphi\|_2}{\|\nabla\varphi\|_2}$$

and obtain

$$\int\limits_{\mathbb{R}^2}\left(\Delta\varphi^{a,b}-\varphi^{a,b}+K\left((\varphi^{a,b})^2\right)\varphi^{a,b}\right)h\,d\boldsymbol{x}=0.$$

Therefore it follows that any nonnegative and real valued maximizer φ of J, up to a scaling, is a weak solution of (3.3.1). □

Before we state the theorem on existence of maximizers we present a compactness result due to Lieb [29] which is useful to conclude that, under some condition, a bounded sequence of functions in $W^{1,p}(\mathbb{R}^N)$ can, after suitable translations, be assumed to have a weak limit that is not zero. The following result is valid for $\mathbb{R}^N$ as well, but for the sake of coherence within our work we state it in $\mathbb{R}^2$.

Lemma 3.3.3 (Compactness lemma)**.** [29, Lemma 6] *Let* $1 < p < \infty$ *and let* $\{f_k\}$ *be a uniformly bounded sequence of real valued functions in* $W^{1,p}(\mathbb{R}^2)$ *with the property that* $E_k = \{\boldsymbol{x} | f_k(\boldsymbol{x}) > \varepsilon\}$ *satisfies meas*$(E_k) \geqslant C$ *for some fixed* $\varepsilon, C > 0$*. Then there exists a sequence of translations* $\{\tau_k\}$ *of* $(\mathbb{R}^N$, $\tau_k \colon \boldsymbol{y} \mapsto \boldsymbol{y} + \boldsymbol{x}_k$, $F_k(\boldsymbol{y}) = f_k(\tau_k \boldsymbol{y}) = f_k(\boldsymbol{y} + \boldsymbol{x}_k)$*, such that* $F_{k_j} \rightharpoonup F$ *weakly in* $W^{1,p}$ *and* $F \neq 0$*, for some subsequence* k_j*.*

What remains to show is the existence of maximizers of the functional J.

Theorem 3.3.4 (Existence of maximizers)**.** [15] *There exists a nonnegative real valued function* $\varphi \in H^1(\mathbb{R}^2)$*, not identically zero, such that* $J(\varphi) = M$*, where* $M = \sup\limits_{f \in H^1(\mathbb{R}^2)} J(f)$*.*

Proof. By definition of M we may pick a sequence $\{\psi_n\} \subset H^1(\mathbb{R}^2)$ of nonzero functions such that $\{J(\psi_n)\}$ is a nondecreasing sequence with

$$\lim_{n \to \infty} J(\psi_n) = M,$$

namely, we may pick a maximizing sequence $\{\psi_n\}$. Since $M > 0$, without loss of generality, we assume $J(\psi_n)$ to be positive, and by Lemma 3.3.1, we may also assume ψ_n to be nonnegative, for each $n \in \mathbb{N}$. Similar to the preceding proof, the invariance of J under the transformation $f \mapsto f^{a,b} = af(b\,\cdot)$ enables us to define another maximizing sequence $\{\varphi_n\}$ by $\varphi_n(x) = a\psi_n(bx)$, for each n, with $a = \frac{1}{\|\nabla\psi_n\|_2}$ and $b = \frac{\|\psi_n\|_2}{\|\nabla\psi_n\|_2}$. We quickly observe that

$$\|\varphi_n\|_2 = 1 \quad \text{and} \quad \|\nabla\varphi_n\|_2 = 1, \tag{3.3.4}$$

for all n, and hence

$$\lim_{n \to \infty} J(\varphi_n) = \lim_{n \to \infty} \frac{\langle K({\varphi_n}^2), {\varphi_n}^2 \rangle}{2} = M,$$

also in a nondecreasing fashion. By (3.3.4), $\{\varphi_n\}$ is clearly a bounded sequence in $H^1(\mathbb{R}^2)$ which is reflexive. Thus there exists a subsequence which we still denote by $\{\varphi_n\}$, and some φ in $H^1(\mathbb{R}^2)$ such that $\{\varphi_n\}$ converges to φ in the weak topology (see A.1). Provided by Sobolev imbedding theorem, $\varphi_n, \varphi \in L^p(\mathbb{R}^2)$, for all $p \geqslant 2$; and we further note that $\{\varphi_n\}$ is bounded also in $L^p(\mathbb{R}^2)$, for all $p \geqslant 2$, all of which are reflexive spaces as well. We proceed by setting $\omega_n = \varphi_n - \varphi$, and clearly $\{\omega_n\}$ converges to zero weakly in $H^1(\mathbb{R}^2)$. We observe in particular that $\{\omega_n\}$ is bounded in $L^4(\mathbb{R}^2) \cap L^8(\mathbb{R}^2)$ so that the sequence $\{{\omega_n}^2\}$ remains bounded in $L^2(\mathbb{R}^2) \cap L^4(\mathbb{R}^2)$. Moreover, it follows from Lemma 3.0.11 that $\{K^2(\omega_n^2)\}$ is also bounded in $L^2(\mathbb{R}^2)$. Therefore, we can extract subsequences in an iterative manner and finally end up with a sequence which we still denote by $\{\varphi_n\}$ such that $\{\omega_n^2\}$, $\{K^2(\omega_n^2)\}$, $\{\omega_n\}$ all converge to zero weakly in $L^2(\mathbb{R}^2)$, and that $\{\omega_n\}$ converges to zero weakly in $H^1(\mathbb{R}^2)$. Our task now is nothing but to show that this convergence is also valid in the strong topology in $H^1(\mathbb{R}^2)$. Before we advance, we note that upon a Cantor diagonalization argument, we may also assume $\{\varphi_n\}$ and $\{\nabla\varphi_n\}$ to converge almost everywhere to $\{\varphi\}$ and $\{\nabla\varphi\}$, respectively. By Rellich–Kondrachov compactness theorem (A.2), for any open rectangle $R_N = (N, N)^2 \subset \mathbb{R}^2$, $\{\varphi_n\}$ has a subsequence strongly converging to φ in $L^2(R_N)$, and hence a subsequence converging to φ almost everywhere in R_N. For $N = 1$ we name this subsequence $\{\varphi_n^{(1)}\}$ and for $N = 2$, we use the same argument to extract a subsequence $\{\varphi_n^{(2)}\}$ of $\{\varphi_n^{(1)}\}$ which converges to φ almost everywhere in $L^2(R_2)$. Iterating this process, we obtain a subsequence $\{\varphi_n^{(N+1)}\}$ of $\{\varphi_n^{(N)}\}$ converging to φ almost everywhere in $L^2(R_{N+1})$. By construction, the diagonal sequence $\{\varphi_n^{(n)}\}$ converges to φ almost everywhere in $L^2(\mathbb{R}^2)$ and is clearly a subsequence of $\{\varphi_n\}$. So, up to a relabeling we assume that $\{\varphi_n\}$ also converges to φ almost everywhere in $L^2(\mathbb{R}^2)$. We now observe that the weak convergence $\varphi_n \rightharpoonup \varphi$ yields

$$\begin{aligned}\lim_{n\to\infty} \|\varphi_n - \varphi\|^2_{H^1} &= \lim_{n\to\infty} \langle \varphi_n - \varphi, \varphi_n - \varphi\rangle_{H^1} \\ &= \lim_{n\to\infty} \{\langle \varphi_n, \varphi_n\rangle_{H^1} + \langle \varphi, \varphi\rangle_{H^1} - \langle \varphi_n, \varphi\rangle_{H^1} - \langle \varphi, \varphi_n\rangle_{H^1}\} \\ &= 2 - \|\varphi\|^2_{H^1(\mathbb{R}^2)},\end{aligned}$$

where $\langle \cdot, \cdot\rangle_{H^1}$ denotes the inner product on $H^1(\mathbb{R}^2)$. This indicates that for strong convergence in $H^1(\mathbb{R}^2)$ to be established, it is sufficient to show that $\|\varphi\|_2 = 1$ and $\|\nabla\varphi\|_2 = 1$. By Lemma 3.3.3, the weak limit φ can be assumed to be nonzero. Immediate by Fatou's lemma is that we have $\|\varphi\|_2 \leqslant 1$ and $\|\nabla\varphi\|_2 \leqslant 1$ and we are to show that strict inequality cannot occur. It is guaranteed by [30, Theorem 1] that

$$\lim_{n\to\infty} \left\{\|\varphi_n\|_2^2 - \|\varphi_n - \varphi\|_2^2\right\} = \|\varphi\|_2^2 \, ,$$

which implies

$$\lim_{n\to\infty} \|\omega_n\|_2^2 = 1 - \|\varphi\|_2^2 \, , \tag{3.3.5}$$

and similarly

$$\lim_{n\to\infty} \|\nabla\omega_n\|_2^2 = 1 - \|\nabla\varphi\|_2^2 \, . \tag{3.3.6}$$

We are now ready to investigate the limit behaviour of $\langle K(\omega_n^2), \omega_n^2\rangle$. First of all, utilizing the fact that K is self-adjoint, we have

$$\begin{aligned}
\langle K(\omega_n^2), \omega_n^2\rangle &= \int\limits_{\mathbb{R}^2} K(\omega_n^2)\omega_n^2 \, d\boldsymbol{x} = \int\limits_{\mathbb{R}^2} K(\varphi_n^2 - 2\varphi_n\varphi + \varphi^2)\omega_n^2 \, d\boldsymbol{x} \\
&= \int\limits_{\mathbb{R}^2} K(\varphi_n^2)\omega_n^2 \, d\boldsymbol{x} + \int\limits_{\mathbb{R}^2} K(\varphi^2)\omega_n^2 \, d\boldsymbol{x} - 2\int\limits_{\mathbb{R}^2} K(\omega_n^2)\varphi_n\varphi \, d\boldsymbol{x} \\
&= \int\limits_{\mathbb{R}^2} K(\varphi_n^2)(\varphi_n^2 - 2\varphi_n\varphi + \varphi^2) \, d\boldsymbol{x} + \int\limits_{\mathbb{R}^2} K(\varphi^2)\omega_n^2 \, d\boldsymbol{x} - 2\int\limits_{\mathbb{R}^2} K(\omega_n^2)\varphi_n\varphi \, d\boldsymbol{x} \\
&= \int\limits_{\mathbb{R}^2} K(\varphi_n^2)\varphi_n^2 \, d\boldsymbol{x} + \int\limits_{\mathbb{R}^2} K(\varphi^2)\omega_n^2 \, d\boldsymbol{x} + \int\limits_{\mathbb{R}^2} (\varphi^2 - 2\varphi_n\varphi)K(\varphi_n^2) \, d\boldsymbol{x} \\
&\quad - 2\int\limits_{\mathbb{R}^2} \varphi_n\varphi K(\omega_n^2) \, d\boldsymbol{x}.
\end{aligned}$$

Since $\int\limits_{\mathbb{R}^2} (K(\varphi^2)\,\cdot) \, d\boldsymbol{x}$ defines a bounded linear functional on $L^2(\mathbb{R}^2)$, the weak convergence $\omega_n^2 \rightharpoonup 0$ in $L^2(\mathbb{R}^2)$ implies that $\lim\limits_{n\to\infty} \int\limits_{\mathbb{R}^2} K(\varphi^2)\omega_n^2 \, d\boldsymbol{x} = 0$, so that we have

$$\langle K(\omega_n^2), \omega_n^2 \rangle = \int\limits_{\mathbb{R}^2} K(\varphi_n^2)\varphi_n^2 \, d\boldsymbol{x} + \int\limits_{\mathbb{R}^2} (\varphi^2 - 2\varphi_n\varphi)K(\varphi_n^2) \, d\boldsymbol{x} - 2\int\limits_{\mathbb{R}^2} \varphi_n\varphi K(\omega_n^2) \, d\boldsymbol{x}.$$

We now show that

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^2} (\varphi^2 - 2\varphi_n\varphi)K(\varphi_n^2) \, d\boldsymbol{x} = -\int\limits_{\mathbb{R}^2} \varphi^2 K(\varphi^2) \, d\boldsymbol{x}, \tag{3.3.7}$$

and

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^2} \varphi_n\varphi K(\omega_n^2) \, d\boldsymbol{x} = 0, \tag{3.3.8}$$

from which it follows that $\lim\limits_{n\to\infty} \left\{\langle K(\omega_n^2), \omega_n^2\rangle - \langle K(\varphi_n^2), \varphi_n^2\rangle\right\} = -\langle K(\varphi^2), \varphi^2\rangle$. To prove (3.3.7), adding and subtracting $\langle K(\varphi_n^2), \varphi^2\rangle$ and recalling that K is self-adjoint, we have

$$\begin{aligned}
\mathcal{J} &= \left| \int\limits_{\mathbb{R}^2} (\varphi^2 - 2\varphi_n\varphi)K(\varphi_n^2) \, d\boldsymbol{x} + \int\limits_{\mathbb{R}^2} \varphi^2 K(\varphi^2) \, d\boldsymbol{x} \right| \\
&= \left| \int\limits_{\mathbb{R}^2} \varphi^2 K(\varphi_n^2) \, d\boldsymbol{x} - 2\int\limits_{\mathbb{R}^2} \varphi_n\varphi K(\varphi_n^2) \, d\boldsymbol{x} + \int\limits_{\mathbb{R}^2} \varphi^2 K(\varphi^2) \, d\boldsymbol{x} \right| \\
&= \left| \int\limits_{\mathbb{R}^2} (-\varphi_n^2 + \varphi^2)K(\varphi^2) \, d\boldsymbol{x} + \int\limits_{\mathbb{R}^2} 2(\varphi^2 - \varphi_n\varphi)K(\varphi_n^2) \, d\boldsymbol{x} \right| \\
&= \left| \int\limits_{\mathbb{R}^2} (-\omega_n)(\varphi_n + \varphi)K(\varphi^2) \, d\boldsymbol{x} + 2\int\limits_{\mathbb{R}^2} (-\omega_n)\varphi K(\varphi^2) \, d\boldsymbol{x} \right| \\
&\leqslant \left| \int\limits_{\mathbb{R}^2} (-\omega_n)(\varphi_n + \varphi)K(\varphi^2) \, d\boldsymbol{x} \right| + 2\left| \int\limits_{\mathbb{R}^2} (-\omega_n)\varphi K(\varphi^2) \, d\boldsymbol{x} \right|
\end{aligned}$$

Using Cauchy–Schwarz inequality, we get

$$\lim_{n\to\infty}\mathcal{J}\leqslant\lim_{n\to\infty}\left\{\left(\int\limits_{\mathbb{R}^2}\omega_n^2K^2(\varphi^2)\,d\boldsymbol{x}\right)^{1/2}\left(\int\limits_{\mathbb{R}^2}(\varphi_n^2+\varphi^2)^2\,d\boldsymbol{x}\right)^{1/2}\right.$$
$$\left.+2\left(\int\limits_{\mathbb{R}^2}K^2(\varphi_n^2)\,d\boldsymbol{x}\right)^{1/2}\left(\int\limits_{\mathbb{R}^2}\omega_n^2\varphi^2\,d\boldsymbol{x}\right)^{1/2}\right\}.$$

As before, we observe that $\int\limits_{\mathbb{R}^2}(K(\varphi^2)\,\cdot)\,d\boldsymbol{x}$ and $\int\limits_{\mathbb{R}^2}(\varphi^2\,\cdot)\,d\boldsymbol{x}$ define bounded linear functionals on $L^2(\mathbb{R}^2)$. So, the weak convergence $\omega_n^2\rightharpoonup 0$ in $L^2(\mathbb{R}^2)$ yields

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^2}\omega_n^2K(\omega_n^2)\,d\boldsymbol{x}=0\quad\text{and}\quad\lim_{n\to\infty}\int\limits_{\mathbb{R}^2}\omega_n^2\varphi^2\,d\boldsymbol{x}=0.$$

Thus,

$$\lim_{n\to\infty}\left|\int\limits_{\mathbb{R}^2}(\varphi^2-2\varphi_n\varphi)K(\varphi_n^2)\,d\boldsymbol{x}+\int\limits_{\mathbb{R}^2}\varphi^2K(\varphi^2)\,d\boldsymbol{x}\right|=0,$$

which proves (3.3.7). In the same spirit, we employ Cauchy–Schwarz inequality and obtain

$$\lim_{n\to\infty}\left|\int\limits_{\mathbb{R}^2}\varphi_n\varphi K(\omega_n^2)\,d\boldsymbol{x}\right|\leqslant\lim_{n\to\infty}\left\{\left(\int\limits_{\mathbb{R}^2}\varphi^2K^2(\omega_n^2)\,d\boldsymbol{x}\right)^{1/2}\left(\int\limits_{\mathbb{R}^2}\varphi_n^2\,d\boldsymbol{x}\right)^{1/2}\right\}.$$

So, (3.3.8) is now established by the weak convergence $K^2(\omega_n^2)\rightharpoonup 0$ in $L^2(\mathbb{R}^2)$, and hence

$$\lim_{n\to\infty}\left\{\langle K(\omega_n^2),\omega_n^2\rangle-\langle K(\varphi_n^2),\varphi_n^2\rangle\right\}=-\langle K(\varphi^2),\varphi^2\rangle.\tag{3.3.9}$$

Now we note that $J(\omega_n)\leqslant M$ and $J(\varphi)\leqslant M$, by definition of M. Using (3.3.5) and (3.3.6) we have

$$\lim_{n\to\infty}\langle K(\omega_n^2),\omega_n^2\rangle \leqslant 2M(1-\|\varphi\|_2^2)(1-\|\nabla\varphi\|_2^2), \tag{3.3.10}$$

and also

$$\lim_{n\to\infty}\langle K(\varphi^2),\varphi^2\rangle \leqslant 2M\,\|\varphi\|_2^2\,\|\nabla\varphi\|_2^2 \leqslant 2M. \tag{3.3.11}$$

Moreover, recalling that $\lim\limits_{n\to\infty} J(\varphi_n) = M$, clearly

$$\lim_{n\to\infty}\langle K(\varphi_n^2),\varphi_n^2\rangle = 2M. \tag{3.3.12}$$

Combining (3.3.9), (3.3.10), (3.3.11) and (3.3.12) yields

$$\begin{aligned}-2M\,\|\varphi\|_2^2\,\|\nabla\varphi\|_2^2 \leqslant -\langle K(\varphi^2),\varphi^2\rangle &= \lim_{n\to\infty}\left\{\langle K(\omega_n^2),\omega_n^2\rangle - \langle K(\varphi_n^2),\varphi_n^2\rangle\right\}\\ &\leqslant 2M(1-\|\varphi\|_2^2)(1-\|\nabla\varphi\|_2^2) - 2M,\end{aligned}$$

which in turn gives us

$$\|\varphi\|_2^2\,(1-\|\nabla\varphi\|_2^2) + \|\nabla\varphi\|_2^2\left(1-\|\varphi\|_2^2\right) \leqslant 0. \tag{3.3.13}$$

Since φ is nonzero, (3.3.13) holds if and only if $\|\varphi\|_2 = \|\nabla\varphi\|_2 = 1$ and hence the claim follows. $\square$

In the first chapter we have discussed that an alternative approach to establish existence of standing wave solutions is to minimize the Lagrangian

$$L_\omega(\psi) = \frac{1}{2}T(\psi) - V(\psi)$$

over the set $\Sigma_0 = \{\psi \in H^1(\mathbb{R}^2) \colon V(\psi) = 0\}$, where

$$T(\psi) = \|\nabla\psi\|_2^2, \quad V(\psi) = \frac{1}{4}\langle K(|\psi|^2)|\psi|^2\rangle - \frac{\omega}{2}\|\psi\|_2^2$$

are the kinetic and the potential energies, respectively. Through this constrained minimization problem we obtain the existence of a solution φ of (3.0.2) such that $L_\omega(\varphi) \leqslant L_\omega(\psi)$, for any H^1-solution ψ of (3.0.2), namely a solution belonging to the set of *ground states* defined by

$$\mathcal{G} = \{\psi \in H^1(\mathbb{R}^2) \colon \psi \,\text{solves (3.0.2) and}\, L_\omega(\psi) = j_0\},$$

where $j_0 = \inf_{\psi \in H^1(\mathbb{R}^2)} \{L_\omega(\psi) \colon \psi \,\text{solves (3.0.2)}\}$. As underlined in [8], the minimizers obtained from this constrained minimization problem and the minimizers of the Weinstein functional (3.3.2) coincide. Therefore the minimization argument in our work also yields ground states.

The main result of this chapter is now a corollary that follows from Lemma 3.3.2 and Theorem 3.3.4.

Corollary 3.3.5. *There exists a nonnegative weak solution $\varphi \in H^1(\mathbb{R}^2)$ to the equation (3.0.2), such that φ is a ground state for (3.0.2).*

Following [8] we observe that if $\alpha(\boldsymbol{\xi}) > 0$ for some $\boldsymbol{\xi} \in \mathbb{R}^2$, then we have the following Gagliardo–Nirenberg–Sobolev type inequality

$$\langle K(|f|^2), |f|^2\rangle \leqslant C_{opt}\,\|f\|_2^2\,\|\nabla f\|_2^2,$$

for any $f \in H^1(\mathbb{R}^2)$, where the best constant C_{opt} is given by $C_{opt} = \dfrac{2}{\|\varphi\|_2^2}$, with φ being a nontrivial solution of (3.3.1). As observed in [20], this estimate enables us to obtain an upper bound on the initial mass so that H^1-solutions are global in time in the focusing case as well.

Corollary 3.3.6. *For initial data* $u_0 \in H^1(\mathbb{R}^2)$ *with* $\|u_0\|_2 < \|\varphi\|_2$*, the corresponding solutions for the Cauchy problem (2.2.1) are global in time.*

4. ARKADIEV–POGREBKOV–POLIVANOV TYPE SOLUTIONS OF ZAKHAROV–SCHULMAN EQUATIONS

4.1. APP Type Travelling Wave Solutions

In this part of our work, we seek the necessary and sufficient conditions for the existence of solutions of the form

$$u(x,y,t) = 2\bar{\nu}\frac{\exp\{i\theta(x,y,t)\}}{(x-4\lambda_2 t+\mu_1)^2+(y+4\lambda_1 t+\mu_2)^2+|\nu|^2}, \qquad (4.1.1)$$

for (1.0.5), with $\delta = -1$, where $\lambda_1, \lambda_2, \mu_1, \mu_2$ are real constants, ν is a complex constant, and θ is a real valued polynomial. Before we proceed, we note that choosing $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$ and θ to be identically zero, we recover the solutions Ozawa considers in [22] for the DS system. In [22] an explicit blow-up profile is also obtained by using the invariance of the solutions for the related Cauchy problem under the pseudo-conformal invariance. In [24], the conditions on the parameters for the existence of the time-independent and radial form of the travelling wave solutions (4.1.1) for the hyperbolic-elliptic-elliptic (HEE) GDS system are derived. Moreover, following Ozawa, these solutions are utilized in [24] to develop an explicit blow up profile for the HEE GDS system. Eden and Gürel show in [23] that there actually exist time-dependent travelling wave solutions (4.1.1) for the HEE GDS system and it turns out that the conditions derived on the parameters coincide with the ones given in [24]. In the sequel, we not only obtain the first set conditions on the operators for the existence of the travelling wave solutions of the form (4.1.1) for the system (1.0.1), but also follow [22] and [24] and develop an explicit blow-up profile as well in the next section.

Before we proceed, we set $X = x + \mu_1 - 4\lambda_2 t$, $Y = y + \mu_2 + 4\lambda_1 t$, $T = t$ and $R = X^2 + Y^2 + |\nu|^2$. Doing so, u becomes

$$u(x,y,t) = 2\bar{\nu}\frac{\exp\{i\theta(x,y,t)\}}{R^2}.$$

We immediately observe that

$$\partial_x = \partial_X, \quad \partial_y = \partial_Y, \quad \partial_t = \frac{\partial X}{\partial t}\partial_X + \frac{\partial Y}{\partial t}\partial_Y + \frac{\partial T}{\partial t}\partial_T = -4\lambda_2\partial_x + 4\lambda_1\partial_Y + \partial_T,$$

and then compute

$$\begin{aligned}
u_x &= 2\bar{\nu}\exp(i\theta)\frac{i\theta_X R - 2X}{R^2}, \\
u_y &= 2\bar{\nu}\exp(i\theta)\frac{i\theta_Y R - 2Y}{R^2}, \\
u_{xx} &= \frac{2\bar{\nu}\exp(i\theta)}{R^3}\{-\theta_X^2 R^2 - 2R + 8X^2 + iR(\theta_{XX}R - 4\theta_X X)\}, \\
u_{yy} &= \frac{2\bar{\nu}\exp(i\theta)}{R^3}\{-\theta_Y^2 R^2 - 2R + 8Y^2 + iR(\theta_{YY}R - 4\theta_Y Y)\}, \\
u_t &= \frac{2\bar{\nu}\exp(i\theta)}{R^2}\{i\theta_t R - R_t\},
\end{aligned}$$

Moreover, we have $\theta_t = \theta_X X_t + \theta_Y Y_t + \theta_T = -4\lambda_2\theta_X + 4\lambda_1\theta_Y + \theta_T$ and $R_t = R_X X_t + R_Y Y_t + R_T = -8\lambda_2 X + 8\lambda_1 Y$ so that

$$u_t = \frac{2\bar{\nu}\exp(i\theta)}{R^2}\{i(-4\lambda_2\theta_X + 4\lambda_1\theta_Y + \theta_T)R + 8\lambda_2 X - 8\lambda_1 Y\}.$$

We next impose the ansatz $\psi = \dfrac{h(X,Y,T)}{R^2}$ with h being a real-valued function, and then $(1.0.5)_1$ with $\delta = -1$ yields

$$\begin{aligned}
&\frac{2\bar{\nu}\exp(i\theta)}{R^3}\big\{8(Y^2 - X^2) + R^2(4\lambda_2\theta_X - 4\lambda_1\theta_Y - \theta_T + \theta_X^2 - \theta_Y^2) \\
&\quad + iR((\theta_{YY} - \theta_{XX})R + 4\theta_X X - 4\theta_Y Y + 8\lambda_2 X - 8\lambda_1 Y) + h(X,Y,T)\big\} = 0. \quad (4.1.2)
\end{aligned}$$

The vanishing of the imaginary part entails

$$(\theta_{YY} - \theta_{XX})(X^2 + Y^2 + |\nu|^2) + 4\theta_X X - 4\theta_Y Y + 8\lambda_2 X - 8\lambda_1 Y = 0,$$

whence

$$\theta_{YY} - \theta_{XX} = 0, \tag{4.1.3}$$

$$\theta_X = -2\lambda_2 \quad \text{and} \quad \theta_Y = -2\lambda_1, \tag{4.1.4}$$

and in fact $\theta_{YY} = \theta_{XX} = 0$ by (4.1.4). Plugging these into the real part, we obtain

$$(-8\lambda_2^2 + 8\lambda_1^2 - \theta_T + 4\lambda_2^2 - 4\lambda_1^2)(X^2 + Y^2 + |\nu|^2)^2 + 8(Y^2 - X^2) + h(X, Y, T) = 0, \tag{4.1.5}$$

and it follows that $\theta_T = 4(\lambda_1^2 - \lambda_2^2)$ and hence

$$\theta(X, Y, T) = -2\lambda_2 X - 2\lambda_1 Y + 4(\lambda_1^2 - \lambda_2^2)T + C,$$

where C is a real constant. Consequently, $h(X, Y, T) = 8(X^2 - Y^2)$ and so

$$\psi(X, Y, T) = \frac{8(X^2 - Y^2)}{R^2}.$$

We now substitute x, y and t back in θ, arrange the arbitrary constant C so that we recover θ in the original variables as

$$\theta(x, y, t) = -2\lambda_2 x - 2\lambda_1 y - 4(\lambda_1^2 - \lambda_2^2)t.$$

We proceed by computing the derivatives involved in $(1.0.5)_2$.

$$
\begin{aligned}
(|u|^2)_x &= -16|\nu|^2\frac{X}{R^3},\\
(|u|^2)_y &= -16|\nu|^2\frac{Y}{R^3},\\
(|u|^2)_{xx} &= -16|\nu|^2\frac{R-6X^2}{R^4} = \frac{16|\nu|^2}{R^4}(5X^2-Y^2-|\nu|^2),\\
(|u|^2)_{yy} &= -16|\nu|^2\frac{R-6Y^2}{R^4} = \frac{16|\nu|^2}{R^4}(-X^2+5Y^2-|\nu|^2),\\
(|u|^2)_{xy} = (|u|^2)_{yx} &= 16|\nu|^2\frac{6XY}{R^3},\\
\psi_X &= \frac{8}{R^3}(-2X^3+6XY^2+2|\nu|^2X),\\
\psi_Y &= \frac{8}{R^3}(2Y^3-6X^2Y-2|\nu|^2Y),\\
\psi_{XX} &= \frac{8}{R^4}\big\{(-6X^2+6Y^2+2|\nu|^2)(X^2+Y^2+|\nu|^2)\\
&\quad -(-2X^3+6XY^2+2|\nu|^2X)6X\big\}\\
&= \frac{16}{R^4}(3X^4+3Y^4-18X^2Y^2-8|\nu|^2X^2+4|\nu|^2Y^2+|\nu|^4),\\
\psi_{YY} &= \frac{8}{R^4}\big\{(-6X^2+6Y^2-2|\nu|^2)(X^2+Y^2+|\nu|^2)\\
&\quad -(6Y^3-6X^2Y-2|\nu|^2Y)6Y\big\}\\
&= \frac{16}{R^4}(-3X^4-3Y^4+18X^2Y^2-4|\nu|^2X^2+8|\nu|^2Y^2-|\nu|^4),\\
\psi_{XY} = \psi_{YX} &= \frac{16}{R^4}(12X^3Y-12XY^3).
\end{aligned}
$$

So $(1.0.5)_2$ reads as

$$
\begin{aligned}
&C^2_{11}(3X^4+3Y^4-18X^2Y^2-8|\nu|^2X^2+4|\nu|^2Y^2+|\nu|^4)+2C^2_{12}(12X^3Y-12XY^3)\\
&+C^2_{22}(-3X^4-3Y^4+18X^2Y^2-4|\nu|^2X^2+8|\nu|^2Y^2-|\nu|^4)\\
=&C^3_{11}(5X^2-Y^2-|\nu|^2)+2C^3_{12}(6XY)+C^3_{22}(-X^2+5Y^2-|\nu|^2),
\end{aligned}
$$

and we immediately observe that X^4 and Y^4 terms yield

$$
C^2_{11}-C^2_{22}=0, \tag{4.1.6}
$$

also by X^3Y and XY^3 terms we have

$$C^2_{12} = 0. \tag{4.1.7}$$

On the other hand, collecting X^2 and Y^2 terms and utilizing (4.1.6) we have

$$-12C^2_{11} = 5C^3_{11} - C^3_{22}, \tag{4.1.8}$$

$$12C^2_{11} = -C^3_{11} + 5C^3_{22}. \tag{4.1.9}$$

Next, we add (4.1.8) and (4.1.9), and get

$$C^3_{11} + C^3_{22} = 0, \tag{4.1.10}$$

and so, by (4.1.8) and (4.1.10) it follows that

$$C^2_{11} = -\frac{1}{2}C^3_{11} \tag{4.1.11}$$

Finally, XY terms yield

$$C^3_{12} = 0. \tag{4.1.12}$$

Therefore, (4.1.6), (4.1.7), (4.1.10), (4.1.11), and (4.1.12) entail

$$C^2 = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} -2r & 0 \\ 0 & 2r \end{pmatrix}, \tag{4.1.13}$$

for some $r \in \mathbb{R}$. We conclude that for the existence of the solutions of the form (4.1.1), the coefficient matrices C^2 and C^3 of the operators L_2 and L_3 should satisfy (4.1.13).

We now proceed by investigating what is imposed on the parameters if we carry out the same analysis for the two different DS systems (1.0.4) and (1.0.11), which are known to be integrable under certain parameter regimes (see [1]). We first consider the hyperbolic DS system given by

$$
\begin{aligned}
iA_t - A_{xx} + A_{yy} &= \chi_0|A|^2A + \chi_1 A\phi_x, \\
m_1\phi_{xx} + m_2\phi_{yy} &= \beta\left(|A|^2\right)_x,
\end{aligned}
\tag{4.1.14}
$$

where χ_0, χ_1, m_1, m_2, and β are all real constants. Seeking travelling wave solutions of the form (4.1.1) with the above introduced notation, for γ to be later determined, we impose the ansatz $\phi = \gamma\partial_x \log R$. Through similar computations it turns out that $(4.1.14)_1$ yields

$$
8(Y^2 - X^2) = \chi_0 4|\nu|^2 - 2\gamma\chi_1(-X^2 + Y^2 + |\nu|^2),
$$

from which we deduce

$$
\gamma\chi_1 = 4, \tag{4.1.15}
$$

and so,

$$
\chi_0 = -2. \tag{4.1.16}
$$

On the other hand, $(4.1.14)_2$ implies

$$
\partial_x\left(\gamma m_1\frac{-2X^2 + 2Y^2 + 2|\nu|^2}{R^2} + \gamma m_2\frac{2X^2 - 2Y^2 + 2|\nu|^2}{R^2}\right) = \partial_x\beta\left(\frac{4|\nu^2|}{R^2}\right),
$$

and as a matter of fact

$$
2m_1\gamma(-X^2 + Y^2 + |\nu|^2) + 2m_2\gamma(X^2 - Y^2 + |\nu|^2) = 4\beta|\nu|^2.
$$

Therefore, we have

$$2(-m_1\gamma + m_2\gamma) = 0, \tag{4.1.17}$$

$$2(m_1\gamma + m_2\gamma) = 4\beta, \tag{4.1.18}$$

and so,

$$m_1 = m_2 = m \quad \text{and} \quad \gamma m = \beta. \tag{4.1.19}$$

Plugging (4.1.15), (4.1.16), (4.1.17), (4.1.18) and (4.1.19) into (1.0.2), it turns out that

$$C^2 = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} -2m & 0 \\ 0 & 2m \end{pmatrix},$$

and thus we conclude that the conditions imposed on the parameters involved in DS system obey (4.1.13).

We now consider a different DS system (1.0.11) which reads as

$$\begin{aligned} iA_t + \sigma A_{xx} + A_{yy} &= \chi_0 |A|^2 A + \chi_1 A\phi, \\ m_1\phi_{xx} + m_2\phi_{yy} &= \beta \left(|A|^2\right)_{yy}, \end{aligned} \tag{4.1.20}$$

in suitably scaled coordinates. We immediately observe that setting

$$u = A, \quad \psi = -\chi_0|A|^2 - \chi_1\phi, \tag{4.1.21}$$

$$L_1 = \sigma\partial_x^2 + \partial_y^2, \quad L_2 = m_1\partial_x^2 + m_2\partial_y^2, \quad L_3 = -\chi_0 L_2 - \chi_1\beta\partial_y^2, \tag{4.1.22}$$

we recover the Zakharov–Schulman system. With X, Y and R defined as above, we now impose

$$\phi(X, Y) = \frac{aX^2 + bY^2 + cXY + d}{R^2}.$$

Considering $(4.1.20)_1$ with $\sigma = -1$, through similar computations we end up with

$$8(Y^2 - X^2) = 4\chi_0|\nu|^2 + \chi_1(aX^2 + bY^2 + cXY + d),$$

whence $c = 0$ and $a = \frac{-8}{\chi_1} = -b$ so that we have

$$\phi(X, Y) = \frac{a(X^2 - Y^2) + d}{R^2}.$$

We also note that the constant terms yield

$$4\chi_0|\nu|^2 + d\chi_1 = 0. \tag{4.1.23}$$

Now, we compute the derivatives involved in $(4.1.20)_2$.

$$\begin{aligned}
\phi_x &= \frac{1}{R^3}(-2aX^3 + 6aXY^2 + (2a|\nu|^2 - 4d)X), \\
\phi_y &= \frac{1}{R^3}(2aY^3 - 6aX^2Y - (2a|\nu|^2 + 4d)Y), \\
\phi_{xx} &= \frac{1}{R^4}\left\{(-6aX^2 + 6aY^2 + (2a|\nu|^2 - 4d))R\right. \\
&\quad \left. -(-12aX^4 + 36aX^2Y^2 + 6(2a|\nu|^2 - 4d)X^2)\right\}, \\
\phi_{yy} &= \frac{1}{R^4}\left\{(6aY^2 - 6aX^2 - (2a|\nu|^2 + 4d))R\right. \\
&\quad \left. -(12aY^4 - 36aX^2Y^2 - 6(2a|\nu|^2 - 4d)Y^2)\right\},
\end{aligned}$$

Subsequently it follows that

$$\begin{aligned}
m_1 &\left[(-6aX^2 + 6aY^2 + (2a|\nu|^2 - 4d))(X^2 + Y^2 + |\nu|^2)\right. \\
&\qquad \left. -(-12aX^4 + 36aX^2Y^2 + 6(2a|\nu|^2 - 4d)X^2)\right] \\
&+ m_2\left[(6aY^2 - 6aX^2 - (2a|\nu|^2 + 4d))(X^2 + Y^2 + |\nu|^2)\right. \\
&\left. -(12aY^4 - 36aX^2Y^2 - 6(2a|\nu|^2 - 4d)Y^2)\right] = 16\beta|\nu|^2(-X^2 + 5Y^2 - |\nu|^2).
\end{aligned}$$

Collecting X^4 and Y^4 terms we obtain

$$6a(m_1 - m_2) = 0,$$

and since $a = -\dfrac{8}{\chi_1} \neq 0$, we have $m = m_1 = m_2$. On the other hand, constants, X^2 and Y^2 terms yield

$$dm = 2\beta|\nu|^2, \tag{4.1.24}$$

$$-3ma|\nu|^2 + 2dm = -2\beta|\nu|^2, \tag{4.1.25}$$

$$3ma|\nu|^2 + 2dm = 10\beta|\nu|^2. \tag{4.1.26}$$

Subtracting (4.1.25) from (4.1.26), we have $ma = 2\beta$ and since $a = -\dfrac{8}{\chi_1}$ it turns out that

$$\chi_1\beta = -4m. \tag{4.1.27}$$

On the other hand,

$$0 = 4\chi_0|\nu|^2 + d\chi_1 = 4\chi_0|\nu|^2 - d\frac{8}{a} = 4\chi_0|\nu|^2 - dm\frac{8}{2\beta} = 4\chi_0|\nu|^2 - 8|\nu|^2, \tag{4.1.28}$$

by (4.1.24), and hence $\chi_0 = 2$. Thus we have

$$L_2 = m\Delta \quad \text{and} \quad L_3 = -2m\partial_x^2 + (-2m\partial_y^2 + 4m\partial_y^2) = -2m\partial_x^2 + 2m\partial_y^2,$$

in other words,

$$C^2 = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} -2m & 0 \\ 0 & 2m \end{pmatrix},$$

and it is therefore seen that the conditions derived on the parameters for existence of APP type travelling soultions for this Davey–Stewartson system as well coincides with the conditions given in (4.1.13).

4.2. An Explicit Blow Up Profile *a la* Ozawa

As mentioned earlier, if L_2 and L_3 satisfy (4.1.13), taking $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$ and $\nu = 1$ we recover the radial solution

$$u(x, y, t) = \frac{1}{1 + \beta(x^2 + y^2)},$$

for (1.0.1) with ψ chosen as in the previous section. Since the solutions of the Zakharov–Schulman system are invariant under the pseudo-conformal transformation given in (2.1.18),

$$U(x, y, t) = \frac{1}{a + bt} \exp\left(ib\frac{y^2 - x^2}{a + bt}\right) u(T, \boldsymbol{X})$$

is also a solution of (1.0.1). A straightforward computation yields

$$\|U(t)\|_2^2 = \frac{\pi}{\beta}.$$

Following [24], we assume $ab < 0$ and let $T^* = -\frac{a}{b}$. Setting $\varepsilon = a + bt = b(t - T^*)$, we see that $|U(x, y, t)|^2 = \frac{1}{\varepsilon^2}\left|u\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\right|^2$. We note that u is a radial solution whereas the solution U obtained via utilizing the above mentioned symmetry of the Zakharov–Schulman is no longer radial. However, $|U|^2$ is radial since the exponential term simplifies and $|U|^2$ is clearly a decreasing function of $|\boldsymbol{x}|$. Indeed, the maximum value of $|U(\boldsymbol{x}, t)|^2$ is attained at the origin and

$$\max_{\boldsymbol{x} \in \mathbb{R}^2} |U(\boldsymbol{x}, t)|^2 = \frac{1}{b^2(t - T^*)^2}.$$

At this stage we observe that as t tends to T^*, the mass density $|U(\boldsymbol{x}, t)|^2$ of U converges to the Dirac distribution δ_0 at the origin in the sense of tempered distributions, namely,

$$\lim_{t \to T^{*-}} |U(\boldsymbol{x}, t)|^2 = \frac{\pi}{\beta}\delta_0, \quad \text{in } \mathcal{S}',$$

and hence occurs the mass concentration phenomenon. Consequently, the solution U blows up in finite time.

5. CONCLUSION

Utilizing the results in [2] and [7], in case L_2 is an elliptic operator we have introduced the focusing and defocusing cases of solutions for the Cauchy problem related to the Zakharov–Schulman system (2.2.1). In the defocusing case, that is when the symbol $\alpha(\boldsymbol{\xi}) \leqslant 0$ for all $\boldsymbol{\xi} \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have established the global existence of H^1-solutions for (2.2.1). On the other hand, characterized by the existence of some $\boldsymbol{\xi}^* \in \mathbb{R}^2$ such that $\alpha(\boldsymbol{\xi}^*) > 0$, the defocusing case gives rise to initial data with negative energy and hence a finite time blow-up in the corresponding solutions. These ideas were present in [2] and the existence of initial data with negative energy involved a tedious computation. The scaling argument used in [6], [7] and in our work establishes in a more simple form that any given initial datum can be scaled to one with negative energy. We have also concluded that the standing wave solutions among the special solutions we discussed in this work exist in and only in the focusing case, and that these solutions are unstable. Upon establishing the existence of such solutions in $H^1(\mathbb{R}^2)$, we have seen that the standing wave profile φ lies in $C^2(\mathbb{R}^2)$ and it enjoys an exponential decay rate. Moreover, the existence of such a ground state φ enables us to obtain the best constant for a Gagliardo–Nirenberg–Sobolev type inequality which in turn yields an upper bound for the initial mass so that the corresponding H^1-solutions are global in time as well in the focusing case.

Aside from the standing wave solutions, we have shown that the Zakharov–Schulman system admits Arkadiev–Pogrebkov–Polivanov type solutions (4.1.1) and we have derived the first set conditions on the operators so that such solutions exist. It turned out that these conditions coincide with the parameter regime necessary for the existence of such solutions for the DS systems (1.0.4) and (1.0.11) which are known to be two reduced forms of the Zakharov–Schulman system such that they are integrable under certain parameter regimes. Therefore we have concluded that when L_2 is elliptic, among its integrable reduced forms, the APP type solutions admitted by the Zakharov–Schulman system consists of the same type of solutions that exist for the DS systems. Moreover, following [22] and [24] we have obtained a finite time-blow up profile upon the existence of these time dependent travelling wave solutions. It is observed that there exist blow-up profiles whenever either of the special solutions studied in this work exist.

APPENDIX A: SOME BACKGROUND IN ANALYSIS

A.1. A Weak Convergence Characterization

Theorem A.1.1. [31, 28 Theorem] *A set in a reflexive space is weakly sequentially compact if and only if it is bounded.*

A.2. Rellich–Kondrachov Compactness Theorem

Theorem A.2.1. [32, Theorem 5.7.1] *Assume U is a bounded open subset of $\mathbb{R}^n$, and ∂U is C^1. Suppose $1 \leqslant p < N$. Then the Sobolev imbedding $W^{1,q}(U) \hookrightarrow L^q(U)$ is compact for each $1 \leqslant q < p^*$, where p^* denotes the Sobolev conjugate defined by $p^* = \frac{np}{n-p}$.*

A.3. Sobolev Imbedding Theorem

Theorem A.3.1. [33, Theorem 2.4.5] *Let $m \geqslant 1$ be an integer and $1 \leqslant p < \infty$. Then*

(i) if $\dfrac{1}{p} - \dfrac{m}{n} > 0$, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, with $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$,
(ii) if $\dfrac{1}{p} - \dfrac{m}{n} = 0$, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, for $p \leqslant q < \infty$,
(iii) if $\dfrac{1}{p} - \dfrac{m}{n} < 0$, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$.

In particular $W^{m,p}(\mathbb{R}^n) \hookrightarrow C^k(\mathbb{R}^n)$ for $m > \dfrac{n}{p}$, where $k = \left\lfloor m - \dfrac{n}{p} \right\rfloor$.

In particular we have, for $n = 2$, $m = 1$ and $p = 2$, $H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for all $2 \leqslant q < \infty$.

A.4. Gagliardo–Nirenberg–Sobolev Inequality

Theorem A.4.1. [14, Theorem 2.3.7] *Let* $1 \leqslant p, q, r \leqslant \infty$ *and let* j, m *be two integers,* $0 \leqslant j < m$. *If*

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{(1-a)}{q},$$

for some $a \in [j/m, 1]$ *(*$a < 1$ *if* $r > 1$ *and* $m - j - \frac{n}{r} = 0$*), then there exists* $C = C(n, m, j, a, q, r)$ *such that*

$$\sum_{|\alpha|=j} \|D^\alpha u\|_p \leqslant C \left(\sum_{|\alpha|=m} \|D^\alpha u\|_r\right)^a \|u\|_q^{1-a},$$

for every $u \in C_c^\infty(\mathbb{R}^n)$.

Taking $n = 2$, $p = 4$, $q = 2$, $r = 2$, $j = 0$, $m = 1$ and $a = \frac{1}{2}$ yieldst $\|u\|_4^4 \leqslant C_1\|\nabla u\|_2^2\|u\|_2^2$ for some C_1. Similarly for $n = 2$, $p = \sigma + 2$, $q = 2$, $r = 2$, $j = 0$, $m = 1$ and $a = \dfrac{\sigma}{\sigma + 2}$ there exists a constant C_2 such that $\|u\|_{\sigma+2}^{\sigma+2} \leqslant C_2\|\nabla u\|_2^\sigma\|u\|_2^2$.

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