FURTHER REGULARITY OF SOLUTIONS FOR ALMOST CUBIC NLS EQUATION

by

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ABSTRACT

FURTHER REGULARITY OF SOLUTIONS FOR ALMOST CUBIC NLS EQUATION

This thesis consists of two major parts. In the first one, we try to give the preliminary local well-posedness results for the ACNLS, and $L^2 - H^1$ regularity result which is an easy and straightforward consequence of the equation, since the norm of the gradientof a function can be estimated by difference quotients.

In the second part, we prove some regularity results for ACNLS. First, we prove H^s local well-posedness, where the continuous dependence is weakened; and an improvement of it by obtaining the continuous dependence with an additional condition. At the end, we prove local $X_{s,b}$ local existence result using Banach fixed point theorem, where the interval of existence is not taken to be maximal. The interval depends closely on the arguments of the high-low frequency decomposition.

ÖZET

NEREDEYSE KÜBİK DOĞRUSAL OLMAYAN SCHRÖDINGER DENKLEMİ'NİN ÇÖZÜMLERİNİN İLERİ TÜREVLENEBİLİRLİK ÖZELLİKLERİ

Bu tez iki ana kısımdan oluşmaktadır. Birinci kısımda, neredeyse kübik Schrödinger denklemi ile ilgili bazı önbilgi niteliğindeki yerel iyi-tanımlılık neticelerini, ve bir fonksiyonun türevinin normunun fark-oranlar yardımıyla yakınsanabilmesi yüzünden, denklemin kolay ve doğrudan getirisi olan $L^2 - H^1$ türevlenebilirlik sonucunu ispatlayacağız.

İkinci kısımda, ACNLS için bazı türevlenebilirlik sonuçları ispatlayacağız. İlk olarak, sürekli bağımlılığın daha zayıf bir formunun kullanıldığı H^s yerel iyi-tanımlanmışlığı ve akabinde bunun fazladan bir koşul daha eklenerek sürekli bağımlılığın elde edildiği geliştirilmiş bir halini ispatlayacağız. Son olarak, varlık aralığının maximal alınmadığı, Banach sabit nokta teoremini kullanarak $X_{s,b}$ uzaylarında yerel varlık teoremini ispatlayacağız. Bu aralık ağırlıklı olarak yüksek-alçak frekans ayrışımı argümanlarına dayanarak belirlenmektedir.

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LIST OF SYMBOLS

$\langle \cdot, \cdot \rangle_{-1,1}$	duality pairing between H^{-1} and H^1
$\langle \cdot, \cdot \rangle_{\mathcal{A},\mathcal{B}}$	duality product between $\mathcal{A}$ and $\mathcal{B}$
$\langle x \rangle$	$(1+\|x\|^2)^{1/2}$
$B^s_{a,b}$	$B^s_{a,b}(\mathbb{R}^2)$, the Besov space of functions
$C(E;F)$	space of continuous functions from the topological space E to the topological space F
$C^k(\Omega)$	space of continuous functions $u : \Omega \to \mathbb{C}$ such that $D^\alpha u \in C(\Omega)$ for all $\alpha \in \mathbb{N}^2$ with $\|\alpha\| \leq k$
$\mathcal{D}$	denotes $\mathcal{D}(\mathbb{R}^2)$, space of infinitely differentiable compactly supported functions
D^α	$\dfrac{\partial^{\|\alpha\|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and $\|\alpha\| = \alpha_1 + \alpha_2$
D^s_a	the Riesz potential with respect to the indicated variable a
$E \hookrightarrow F$	E is continuously imbedded in F
$\hat{f}$	Fourier transform of f with respect to the space variable, defined to be $\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} \mathrm{d}\boldsymbol{x}$ for $f \in L^1(\mathbb{R}^2)$
$\mathcal{F}_a$	also denotes the Fourier transform with respect to the indicated variable a.
H^s	$W^{s,2}(\mathbb{R}^2)$, Banach space of elements $u \in \mathcal{S}'(\mathbb{R}^2)$ such that $(1+\|\xi\|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^2)$ for $s \in \mathbb{R}$. $H^s(\mathbb{R}^2)$ is equipped with the norm $\\|u\\|_{H^s} = \\|(1+\|\xi\|^2)^{s/2}\hat{u}\\|_2$
$H^{0,s}$	$H^{0,s}(\mathbb{R}^2)$, the Banach space of functions $u \in L^2$ with $\|.\|u(.) \in L^2$ equipped with the norm $\\|u\\|_{H^{0,s}} = \\|(1+\|x\|^2)^{1/2} u(x)\\|_{L^2}$
$\operatorname{Im} z$	imaginary part of z
J^s_a	the Bessel potential with respect to the indicated variable a
L^p	$L^p(\mathbb{R}^2)$
$L^p(I;B)$	Banach space of classes of measurable functions $u : I \to B$ such that $\\|u(t)\\|_B \in L^p(I)$ where $I \subset \mathbb{R}$

$L^p(\Omega)$	Banach space of classes of measurable functions $u : \Omega \to \mathbb{C}$ such that $\|u\|_p < \infty$, where $$\|u\|_p = \begin{cases} \left(\int_\Omega |u(\boldsymbol{x})| \mathrm{d}\boldsymbol{x}\right)^{1/p}, & \text{if } p < \infty \\ \text{ess sup}_\Omega |u|, & \text{if } p = \infty \end{cases}$$
Mf	the maximal function of f
p'	conjugate of $p \geq 1$ given by $1/p + 1/p' = 1$
Re z	real part of z
$\mathcal{S}$	denotes $\mathcal{S}(\mathbb{R}^2)$, Schwartz space
$\mathcal{S}'$	denotes tempered distributions $\mathcal{S}'(\mathbb{R}^2)$, topological dual of $\mathcal{S}(\mathbb{R}^2)$
$\boldsymbol{u} \in L^p(I; L^q)$	$u_k \in L^p(I; L^q)$ for each component u_k of $\boldsymbol{u}$
u_r	$\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\boldsymbol{x} \cdot \nabla u$
u_t	partial derivative of $u(t, \boldsymbol{x})$ with respect to t
u_{x_k}	$\dfrac{\partial u}{\partial x_k}$
$W^{m,p}$	$W^{m,p}(\mathbb{R}^2)$, the Sobolev space of functions such that $\dfrac{d^k u}{dt^k} \in L^p(\mathbb{R}^2)$ for $k = 0, \ldots, m$
$W^{-m,p'}$	$W^{-m,p'}(\mathbb{R}^2)$, dual of $W^{m,p}(\mathbb{R}^2)$
$W^{k,p}$	$W^{k,p}(\mathbb{R}^2)$ for $k \in \mathbb{Z}$
$W^{m,p}(I; B)$	Banach space of classes of measurable functions $u : I \to B$ such that $\dfrac{d^k u}{dt^k} \in L^p(I; B)$ for $k = 0, \ldots, m$ where B is any Banach space
$X_{a,b}$	$X_{a,b}(\mathbb{R}^2)$, the Bourgain space of functions
$X^\delta_{a,b}$	$X^\delta_{a,b}(\mathbb{R}^2)$, the restriction of the Bourgain space to the time interval $[0, \delta]$
$x+$	$x + \epsilon$ for ϵ chosen small
χ_A	the characteristic function of the set A
∇u	gradient of u, (u_{x_1}, u_{x_2}) in $\mathbb{R}^2$
Δu	$\sum_{k=1}^2 u_{x_k x_k}$ in $\mathbb{R}^2$

1. INTRODUCTION

A class of equations that generalized the two dimensional cubic NLS was introduced and called almost cubic NLS in a paper of Eden and Kuz. The original intention was to carry the standard results of the cubic NLS to the case of almost cubic NLS. Since this class includes some case of the Zakharov-Schulman equations (see [1]) as well as the purely elliptic case and the hyperbolic-elliptic-elliptic cases of the generalized Davey-Stewartson equations as introduced in Babaoglu and Erbay, [2], all the results that are obtained for ACNLS would have been applicable to these equations as well. In particular, the question when the solutions of the initial value problem for the generalized Davey-Stewartson equations has global existence and when solutions blow-up was left open in Babaoglu et.al., [3]. Despite various attempts in [4], [5] a complete answer was not reached. It was only in the paper of Eden-Gurel-Kuz utilizing the abstract framework of the ACNLS that a complete answer to the question was furnished, see [6, Section 7] by analyzing the sign of the symbol that is used to define the cubic non-local nonlinearity. As is well-known, a well-developed theory exists for the cubic NLS (see e.g. [7], [8] and [9]), still leaving the question of global existence of solutions with finite initial mass open. Various attempts were made to obtain results that produce global existence of solutions for "rough data", i.e. data with infinite energy, starting with the seminal work, [10], of Bourgain which introduced the high-low frequency method. Using this method Bourgain was able to show that for the initial data in H^s for $s > 3/5$ one can still have global existence. In fact, in his book Bourgain also discusses some scattering results for a similar class of initial data, see [9, Prop. 3.53]. Later on, in order to improve these results with the hope of achieving the finite mass case, the I-method was introduced by Colliander, Keel, Staffilani, Takaoka and Tao. To the best of our knowledge in the two dimensional case the best result so far is the one by Grillakis and Fang, in [11], that shows global existence when the data is in $H^1/2$. This thesis started with the ambitious hope of adapting the high-low frequency argument of Bourgain in [10] to the case of the ACNLS. This hope is yet to be realized. In the first part of this thesis, we gave the preliminary local well-posedness results and $L^2 - H^1$ regularity result in details. The second part of the thesis contains some original contributions to the theory of ACNLS that we were fortunate to obtain as a result of our struggle with Bourgain's paper. Namely, we are able to show that there are classes of rough initial data for which the Cauchy problem for ACNLS is locally well-posed, these classes include H^s and Bourgain spaces. For the local well-posedness in H^s we will make use of the ideas of Cazenave-Weissler, [12], and will concentrate on the estimates needed for the non-local non-linear term in the appropriate Besov spaces. This is a reoccuring theme in the study of ACNLS as can be witnessed in [13, Thm 3.5, 4.4, 5.2] as well. For the latter class, we make use of a fixed point argument that is given in [14] for the case of NLS equation. (a paper that to our opinion seems to contain some inaccuracies that we also overcame in the process) This section can be seen as a prelude to our more ambitious and unrealized goal to obtain global existence results for rough data. The results in the second part of this paper is already published in [15].

2. PRELIMINARY RESULTS FOR ACNLS

Consider the following two dimensional NLS equation:

$$\begin{aligned} iu_t + \beta u_{xx} + u_{yy} - K(|u|^2)u &= 0 \\ u(0,x) &= \varphi(x)\ , \qquad \varphi \in H^s \end{aligned} \tag{2.1}$$

where $\beta = \pm 1$ and $\widehat{K(f)}(\xi) = \alpha(\xi)\hat{f}(\xi)$ for $f \in L^2$, and the symbol satisfies the following:

(H1) α is even and homogeneous of degree 0,

(H2) $\alpha \in C^\infty(\mathbb{R}^2 \setminus \{0,0\})$.

We call this equation almost cubic nonlinear Schrödinger (ACNLS) equation and classify the cases $\beta = \pm 1$ as the elliptic and the hyperbolic cases respectively. In the proceeding arguements we will establish local well posedness of the corresponding Cauchy problem in H^s. In the local and the global theory of this equation, we will be using the conservation of some quantities, called the mass and the energy defined as:

$$M(u) = \int_{\mathbb{R}^2} |u|^2 \, \mathrm{d}x \, \mathrm{d}y, \quad E(u) = \int_{\mathbb{R}^2} \left[\beta|u_x|^2 + |u_y|^2 + 1/2K(|u|^2)|u|^2\right] \, \mathrm{d}x \, \mathrm{d}y, \tag{2.2}$$

respectively and mass is naturally defined for L^2-solutions whereas it is possible to define energy (See [13, Corollary 4.5, Proposition 6.1]).

For the mass conservation, if we begin with H^1-solutions, considering $H^{-1} - H^1$ duality product of (2.1) with $2u$ gives

$$2i\langle u_t, u\rangle_{-1,1} = 2(\beta\|u_x\|_2^2 + \|u_y\|_2^2) + 2\int_{\mathbb{R}^2} K(|u|^2)|u|^2 \, \mathrm{d}\boldsymbol{x}.$$

Since the right hand side is real, we obtain the mass conservation on $[0, T_{max})$. For the conservation of energy, multiplying (2.1) by $2\bar{u}_t$ and then taking real parts give $2\mathrm{Re}\,\bar{u}_t(\beta u_{xx} + u_{yy}) = K(|u|^2)(|u|^2)_t$ from which it follows that

$$\begin{aligned} 0 &= \frac{d}{dt} \int\limits_{\mathbb{R}^2} (\beta|u_x|^2 + |u_y|^2)\,\mathrm{d}x\,\mathrm{d}y + \mathrm{Re} \int\limits_{\mathbb{R}^2} \alpha(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi})\overline{\widehat{(f_t)}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \\ &= \frac{d}{dt}\left[\int\limits_{\mathbb{R}^2} (\beta|u_x|^2 + |u_y|^2)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int\limits_{\mathbb{R}^2} K(|u|^2)|u|^2\,\mathrm{d}x\,\mathrm{d}y\right], \end{aligned}$$

by using Parseval identity and the fact that α is even. These formal computations make sense for H^2-solutions. By using continuous dependence results, one can approximate L^2 and H^1-solutions with H^1 and H^2-solutions, respectively to obtain the necessary conservation laws. The definition of K with the assumptions (H1) and (H2) allows us to deduce the facts that $\mathrm{Im}\,(\mathrm{K}(|\mathrm{u}|^2)|\mathrm{u}|^2) = 0$ and for $G(u) \equiv 1/4 \int_{\mathbb{R}^2} K(|u|^2)|u|^2\mathrm{d}\boldsymbol{x}$, $G \in C^1(L^4; \mathbb{R})$ with $G'(u)(v) = \mathrm{Re} \int_{\mathbb{R}^2} K(|u|^2)u\bar{v}\,\mathrm{d}\boldsymbol{x}$ for every $v \in L^4$.

2.1. L^2 local well-posedness

Theorem 2.1.1. *For the ACNLS, for any $\phi \in L^2$, there exist $T_{max}, T_{min} \in (0, \infty]$ and a unique maximal solution $u \in C((-T_{min}, T_{max}), L^2) \cap L^4_{loc}((-T_{min}, T_{max}), L^4)$ of the ACNLS. Moreover, the following properties hold:*

- *(P1) If $T_{max} < \infty$ (respectively $T_{min} < \infty$), then $\|u\|_{L^q((0,T_{max}),L^r)} = \infty$ (respectively $\|u\|_{L^q((-T_{min},0),L^r)} = \infty$) for every admissible pair (q, r) with $r > 4$.*
- *(P2) $u \in L^q_{loc}((-T_{min}, T_{max}), L^r)$ for very admissible pair (q, r).*
- *(P3) $\|u(t)\|_{L^2} = \|\phi\|_{L^2}$ for all t in $(-T_{min}, T_{max})$.*
- *(P4) The mappings $\phi \mapsto T_{min}, T_{max}$ are lower semi-continuous $L^2 \to (0, \infty]$. If $\phi_n \to \phi$ in L^2 and if u_n are the corresponding solutions of the ACNLS, then $u_n \to u$ in $L^q(I, L^r)$ for every interval $I \Subset (-T_{min}, T_{max})$ and for every admissible pair (q,r).*

Proof. If I is an interval $I \subset \mathbb{R}$ with $0 \in I$ and if $u, v \in L^4(I, L^4)$, we have that,

$$\|K(|u|^2)u - K(|v|^2)v\|_{L^4(I,L^4)} \leq C(\|u\|^2_{L^4(I,L^4)} + \|v\|^2_{L^4(I,L^4)})\|u - v\|_{L^4(I,L^4)}. \tag{2.3}$$

Then if we write

$$G(u)(t) = \int_0^t \mathcal{T}_\beta(t-s)K(|u(s)|^2)u(s)\mathrm{d}s,$$

from (2.3) and the Strichartz' estimates, we get,

$$G(u) \in C(I, L^2) \cap L^q(I, L^r) \ \text{ for every admissible pair}(q, r),$$

$$\|G(u)\|_{L^q(I,L^r)} \leq C\|u\|^3_{L^4(I,L^4)} \ \ \mathit{and} \tag{2.4}$$

$$\|G(u) - G(v)\|_{L^q(I,L^r)} \leq C(\|u\|^2_{L^4(I,L^4)} + \|v\|^2_{L^4(I,L^4)})\|u - v\|_{L^4(I,L^4)}, \tag{2.5}$$

for some constant C independent of I. First we will prove the existence and uniqueness of the solution.

We will prove that there exist a $\delta > 0$ such that if $\phi \in L^2$ satisfies

$$\|\mathcal{T}_\beta(.)\phi\|_{L^4(I,L^4)} < \delta, \tag{2.6}$$

for some interval $I \subset \mathbb{R}$ containing 0, then there exits a unique solution $u \in C(I, L^2) \cap L^4(I, L^4)$ of ACNLS. In addition, $u \in L^q(I, L^r)$ for every admissible pair (q, r). Moreover, if $\phi, \psi \in L^2$ both satisfy (2.6) and if u, v are the corresponding solutions of the ACNLS equation, then

$$\|u - v\|_{L^\infty(I,L^2)} + \|u - v\|_{L^4(I,L^4)} \leq C\|\phi - \psi\|_{L^2}, \tag{2.7}$$

for some constant C independent of $I, u \ \mathit{and} \ \ v$.

Now fix a $\delta > 0$, to be chosen later, and let $\phi \in L^2$ satisfy (2.6). Consider the set

$$E = \{u \in L^4(I, L^4) : \|u\|_{L^4(I,L^4)} \leq 2\delta\},$$

so that (E, d) is a complete metric space with $d(u, v) = \|u - v\|_{L^4(I,L^4)}$.

For $u \in E$, from Duhamel's principle, for $t \in I$, write,

$$H(u)(t) = \mathcal{T}_\beta(t)\phi + i \int_0^t \mathcal{T}_\beta(t - s)K(|u(s)|^2)u(s)\mathrm{d}s. \tag{2.8}$$

It follows easily from (2.6), (2.4) and (2.5) that if δ is small enough (independently of ϕ and I), then H is a strict contraction on E. Thus H has a unique fixed point u, which is a unique solution of ACNLS in E.

Using (2.4) and the Strichartz' estimates, we get that $u \in C(I, L^2) \cap L^q(I, L^r)$ for every admissible pair (q, r). The inequality (2.7) follows easily from (2.5) and the Strichartz' estimates.

Now we shall prove uniqueness (without the assumption (2.6)). Let I be an interval with $0 \in I$ and consider two solutions $u, v \in L^4(I, L^4)$ of ACNLS. We show that if $0 \in J \subset I$ with $|J|$ sufficiently small, then $u = v$ on J. Suppose that this assumption is true, then we may define $0 < \theta \leq T_2$ where $I = [T_1, T_2]$, by

$$\theta = \sup\{0 < \tau \leq T_2 : u = v \text{ on } (0, \tau)\}.$$

It follows that $u = v$ on $[0, \theta]$. If $\theta = T_2$, then the uniqueness follows for the positive times. So assume for a contradiction that $\theta < T_2$. We see that $u_1(\cdot) = u(\theta + \cdot)$ and $v_1(\cdot) = v(\theta + \cdot)$ are solutions of ACNLS with ϕ replaced by $u(\theta) = v(\theta)$ on the interval $(0, T_2 - \theta)$. By uniqueness for the small time, we can say that $u_1 = v_1$ on some small interval $[0, \epsilon]$ with $0 < \epsilon < T_2 - \theta$. But this means that $u = v$ on $[0, \theta + \epsilon]$, which contradicts with the definition of θ. Hence small time uniqueness gives uniqueness for the positive times. Similarly we can show that the same holds for the negative times.

Now we will prove our assumption on small time uniqueness, namely, if $0 \in J \subset I$ with $|J|$ sufficiently small, then $u = v$ on J. We have,

$$\|u\|^2_{L^4(J,L^4)} + \|v\|^2_{L^4(J,L^4)} \to 0 \quad as \quad |J| \downarrow 0. \tag{2.9}$$

From (2.7), we can deduce that,

$$\|u - v\|_{L^4(J,L^4)} \leq C(\|u\|^2_{L^4(J,L^4)} + \|v\|^2_{L^4(J,L^4)})\|u - v\|_{L^4(J,L^4)}. \tag{2.10}$$

Then using (2.9), for $|J|$ sufficiently small, we have

$$C(\|u\|^2_{L^4(J,L^4)} + \|v\|^2_{L^4(J,L^4)}) < 1,$$

and we can conclude that

$$\|u - v\|_{L^4(J,L^4)} \leq d\|u - v\|_{L^4(J,L^4)}, \quad with \quad some \quad 0 < d < 1.$$

Thus $\|u - v\|_{L^4(J,L^4)} = 0$, meaning, $u = v$ on J.

The property $(P3)$, the mass conservation, is proven for smooth solutions. For the general case we take a sequence $\phi_n \in H^1$ such that $\phi_n \to \phi$ in L^2, for which we will show that the ACNLS is locally well-posed in H^1. From Strichartz' estimates we get that for n sufficiently large, ϕ_n satisfies (2.6), so that by (2.7), u_n denoting the solutions associated to ϕ_n, $u_n \to u$ in $C(I, L^2)$. Then we see that u_n are H^1 solutions of the ACNLS, so that the mass conservation is obtained for u_n for n sufficiently large, namely, $\|u_n(t)\|_{L^2} = \|\phi_n\|_{L^2}$ for all $t \in I$. Then passing to limit as $n \to \infty$ we get $\|u(t)\|_{L^2} = \|\phi\|_{L^2}$ for all $t \in I$.

We are now able to construct the maximal solution (and consequently property $(P1)$, the blow-up alternative) and show property $(P4)$, the continuous dependence.

Let $\phi \in L^2$. By Strichartz' estimates we know that $\mathcal{T}_\beta(\cdot)\phi \in L^4(\mathbb{R}, L^4)$. Then we have $\|\mathcal{T}_\beta(\cdot)\phi\|_{L^4((-T,T),L^4)} \to 0$ as $T \to 0$. Hence ϕ satisfies (2.6) and hence we can construct the unique local solution u. Again first we will prove arguments for the positive times, then the proof for the negative times will be done similarly, now we can define $T_{max}(\phi) = \sup\{T > 0 :$ solution to the ACNLS exists on $[0,T]\}$. Then by uniqueness, there exists a solution $u \in C((0,T_{max}), L^2) \cap L^4((0,T_{max}), L^4)$ of ACNLS. If $T_{max} < \infty$ and $\|u\|_{L^4((0,T_{max}),L^4)} < \infty$. Let $0 \le t \le t+s < T_{max}$. Then we have, by Duhamel's principle and uniqueness of the solution, that

$$\mathcal{T}_\beta(s)u(t) = u(t+s) - i\int_0^s \mathcal{T}_\beta(s-\tau)K(|u(t+\tau)|^2)u(t+\tau)\mathrm{d}\tau. \tag{2.11}$$

Then by (2.4) we get,

$$\|\mathcal{T}_\beta(.)u(t)\|_{L^4((0,T_{max}-t),L^4)} \le \|u\|_{L^4((t,T_{max}),L^4)} + C\|u\|^3_{L^4((t,T_{max}),L^4)}. \tag{2.12}$$

Hence choosing t close enough to T_{max} gives us that

$$\|\mathcal{T}_\beta(.)u(t)\|_{L^4((0,T_{max}-t),L^4)} < \delta/2.$$

This says that u can be extended after T_{max}, but T_{max} was chosen to be maximal. Thus $\|u\|_{L^4((0,T_{max}),L^4)} = \infty$, and for any admissible pair (q,r) where $r > 4$, from Hölder's inequality, we have

$$\|u\|_{L^4((0,T),L^4)} \le \|u\|_{L^\infty((0,T),L^2)}\mu\|u\|^{1-\mu}_{L^q((0,T),L^r)} \le \|\phi\|^\mu_{L^2}\|u\|^{1-\mu}_{L^q((0,T),L^r)},$$

for any $T < T_{max}$, with $\mu = 2(r-4)/4(r-2)$. Letting $T \to T_{max}$, we get

$$\|u\|_{L^q((0,T_{max}),L^r)} = \infty.$$

Thus properties $(P1)$ and $(P2)$ are proven.

To show the continuous dependence, consider $T \in \mathbb{R}$ such that $T < T_{max}$. Since $u \in C([0,T], L^2)$, it follows from Strichartz' estimates and compactness of $[0,T]$ that there exists $\sigma > 0$ such that $\|\mathcal{T}_\beta(.)u(t)\|_{L^4((0,\sigma),L^4)} \leq \delta/2$ for all $t \in [0,T]$. Let n be an integer such that $T < n\sigma$, let $C \geq 1$ be the constant in (2.7), and M be such that $\|\mathcal{T}_\beta(.)v\|_{L^4(\mathbb{R},L^4)} \leq M\|v\|_{L^2}$. Let $\epsilon > 0$ be small enough so that $MK^{n-1}\epsilon < \delta/2$. We will show that if $\|\phi-\psi\|_{L^2} \leq \epsilon$, then $T_{max}(\psi) > T$ and $\|u-v\|_{C([0,T],L^2)} + \|u-v\|_{L^4((0,T),L^4)} \leq nK^n\|\phi-\psi\|_{L^2}$, where v is the corresponding solution to the initial datum ψ. The inequality $\|\phi-\psi\|_{L^2} \leq \epsilon$, implies that

$$\begin{aligned}
\|\mathcal{T}_\beta(.)\psi\|_{L^4((0,T/n),L^4)} &\leq \|\mathcal{T}_\beta(.)\phi\|_{L^4((0,T/n),L^4)} + \|\mathcal{T}_\beta(.)(\phi-\psi)\|_{L^4((0,T/n),L^4)} \\
&\leq \delta/2 + M\epsilon \\
&\leq \delta.
\end{aligned}$$

Thus, we get $T_{max}(\psi) > T/n$ and that

$$\|u-v\|_{C([0,T/n],L^2)} + \|u-v\|_{L^4((0,T/n),L^4)} \leq K\|\phi-\psi\|_{L^2}.$$

Hence, $\|u(T/n) - v(T/n)\|_{L^2} \leq K\epsilon$, and iterating this argument n times gives the result. □

2.2. H^1 local well-posedness and regularity

Theorem 2.2.1. *Given $\phi \in H^1$, there exists a unique maximal solution*

$$u \in C([0,T_{max}),H^1) \cap C^1([0,T_{max}),H^{-1}),$$

of the ACNLS on $[0,T_{max})$ with the following properties:

(P1) $\nabla u \in L^4([0,t],L^4)$ *for every* $t < T_{max}$.

(P2) If $T_{max} < \infty$, *then* $\|u\|_{L^\infty([0,T_{max}),H^1)} = \infty$

(P3) If $\phi_n \to \phi$ *in* H^1 *and* u_n*'s are the corresponding solutions, then for any* $I \Subset [0,T_{max})$, $u_n \to u$ *in* $C(I,H^1)$

Proof. The proof is similar to the proof of the L^2 local well-posedness theorem, after replacing the L^2-norm with H^1 norm and $L^q(I, L^r)$-norms with $L^q(I, W^{1,r})$-norms for all admissible pairs (q, r). □

One of the immediate questions to ask about the resemblance of the proof is that, since the H^1 solutions are also L^2 solutions, do the solutions in L^2 framework and in H^1 framework also resemble? The answer to that question is not immediate. We know that if the initial datum is in H^1, then both local well-posedness theorems give us a maximal solution and by uniqueness, we can conclude that the solutions coincide in the maximal intervals of existence. So we can pose the question in another form: If the initial datum ϕ is in H^1, do the L^2 maximal interval of existence and H^1 maximal interval of existence coincide?

Proposition 2.2.2 ($L^2 - H^1$ regularity)**.** *Consider the ACNLS with the initial datum $\phi \in L^2$, then by theorem* (2.1.1)*, there exist T_{max} and T_{min} in* $(0, \infty]$ *and a unique maximal solution $u \in C((-T_{min}, T_{max}), L^2)$. If ϕ is moreover in H^1, then $u \in C((-T_{min}, T_{max}), H^1)$.*

Proof. First we can easily see that the maximal interval of existence we obtain in theorem (2.2.1) is not larger than the maximal interval of existence we obtain in theorem (2.1.1) since all H^1 solutions are also L^2 solutions.

Now let u be the L^2 solution of the ACNLS, where we considered the initial datum in L^2 and used the theorem (2.1.1) to construct u. We will show that if $\phi \in H^1$, then $u \in C(I, H^1)$, where I is an interval containing 0. Considering theorem (2.2.1), we can construct a solution $v \in C((-T_*, T^*), H^1)$, where $T_*, T^* > 0$ and $(-T_*, T^*)$ is the maximal interval of existence we obtain by theorem (2.2.1). Since v is also an L^2 solution, by uniqueness we can say that $u = v$ as long as both are defined. Thus it is enough for us to show that $I \subset (-T_*, T^*)$. If we assume $I = (a, b)$, and $b > T^*$ since the ACNLS is invariant under space translations and gradient is the limit of the finite differences quotient, the inequality (2.7) yields

$$\|\nabla v\|_{L^\infty((0,T^*),L^2)} \leq C\|\nabla \phi\|_{L^2},$$

which contradicts with the blow-up alternative for the H^1 solutions, namely the property $(P2)$ of theorem (2.2.1). Hence $b < T^*$. Similarly we can show that $a > T_*$. $\square$

3. FURTHER REGULARITY RESULTS FOR ACNLS

In this chapter we will state some further regularity results for the solutions of the ACNLS equation. First we will give H^s local well-posedness result and finally we will prove $X^{\delta}_{s,b}$ local existence result.

3.1. H^s local well-posedness and regularity

This section consists of two H^s local well-posedness results. In the first one continuous dependence is weakened, and in the second one we achieve the desired continuous dependence by requiring an additional condition on the solution.

Theorem 3.1.1 (H^s local well-posedness 1)**.** *Let $s < 1$ and let (γ, ρ) be the admissible pair defined by $\rho = 4/(1+s)$, $\gamma = 4/(1-s)$. Given $\varphi \in H^s$, there exist $T_{max}, T_{min} \in (0, \infty]$ and a unique maximal solution*

$$u \in C((-T_{min}, T_{max}), H^s) \cap L^{\gamma}((-T_{min}, T_{max}), B^s_{\rho,2}) \tag{3.1}$$

of the ACNLS, and the following hold:

(P1) $u \in L^q_{loc}((-T_{min}, T_{max}), B^s_{r,2})$ *for all admissible* (q, r).

(P2) (Blow-up) If $T_{max} < \infty$ *(or* $T_{min} < \infty$*), then* $\|u(t)\|_{H^s} \to \infty$ *as* $t \to T_{max}$ *(or* $t \to T_{min}$*).*

(P3) u *depends continuously on* φ *in the following sense: There exists* $0 < T < T_{max}, T_{min}$ *such that if* $\varphi_n \to \varphi$ *in* H^s *and if* u_n *denotes the solution of ACNLS with the initial datum* φ_n, *then* $0 < T < T_{max}(\varphi_n), T_{min}(\varphi_n)$ *for all sufficiently large* n *and* u_n *is bounded in* $L^q((-T, T), B^s_{r,2})$ *for all admissible* (q, r). *Moreover,* $u_n \to u$ *as* $n \to \infty$ *in* $L^q((-T, T), L^r)$ *and in* $C((-T, T), H^{s-\epsilon})$ *for all* $\epsilon > 0$.

(P4) If, moreover, $\varphi \in H^1$ *and* $\beta = 1$ *then we have that the energy,*

$$E(u) = \int_{\mathbb{R}^2} \left[\beta |u_x|^2 + |u_y|^2 + 1/2K(|u|^2)|u|^2\right] \, \mathrm{d}x \, \mathrm{d}y,$$

is conserved in every compact subset of $(-T_{min}, T_{max})$.

After obtaining Strichartz' type estimates and some estimates for the nonlinearity, the result will follow in the spirit of the proof in [12, Theorem 1.3]. Since Strichartz' type estimates hold for the solution of the linear equation $iu_t + \beta u_{xx} + u_{yy} = 0$ (see [13] for details), what we need to show is the estimates on the nonlinearity to recall the proof of the theorem in [12].

Lemma 3.1.2. *Let* K *satisfies* $(H1)(H2)$*. Let* $0 < s < 1$*,* $1 \leq q \leq \infty$*,* $1 \leq p \leq r \leq \infty$*. If* $\sigma = 2pr/(r-p)$*, then*

$$\|K(|f|^2)f\|_{\dot{B}^s_{p,q}} \leq C\|f\|^2_{L^\sigma}\|f\|_{\dot{B}^s_{r,q}},$$

and

$$\|K(|f|^2)f)\|_{B^s_{p,q}} \leq C\|f\|^2_{L^\sigma}\|f\|_{B^s_{r,q}} \qquad \forall f \in B^s_{r,q} \cap L^\sigma.$$

Proof. From Hölder's inequality we have $\|fg\|_{L^p} \leq \|f\|_{L^{\sigma/2}}\|g\|_{L^r}$. Since $2/\sigma + 1/r = 1/p$. Then

$$\|K(|f|^2)f\|_{L^p} \leq \|K(|f|^2)\|_{L^{\sigma/2}}\|f\|_{L^r}.$$

Since K is a Calderon-Zygmund type operator (see [16],[17]) and $\sigma > 2$ then K is bounded on $L^{\sigma/2}$, which gives

$$\|K(|f|^2)f\|_{L^p} \leq C\|(|f|^2)\|_{L^{\rho/2}}\|f\|_{L^r} = C\|f\|^2_{L^\sigma}\|g\|_{L^r}.$$

n Now set $\tau_y f = f(\cdot - y)$. By the Remark 1.4.4 ((ii),(iii)) in [7], if we can show for $y \in \mathbb{R}^2$, $\|K(|f|^2)\tau_y f - K(|f|^2)f\|_{L^p} \leq C\|f\|^2_{L^\sigma}\|\tau_y f - f\|_{L^r}$, first inequality of the lemma will be obtained. So

$$\begin{aligned}\|\tau_y(K(|f|^2)f) - K(|f|^2)f\|_{L^p} &= \|\tau_y K(|f|^2)\tau_y f - K(|f|^2)f\|_{L^p} \\ &\leq \|\tau_y K(|f|^2)f - K(|f|^2)f\|_{L^p} \qquad (3.2)\\ &+\|\tau_y K(|f|^2)\tau_y f - \tau_y K(|f|^2)f\|_{L^p} \qquad (3.3)\end{aligned}$$

and

$$\begin{aligned}(3.2) \quad &\leq \|f\|_{L^\sigma}\|K(\tau_y|f|^2) - K(|f|^2)\|_{L^{2pr/(p+r)}} \\ &\leq \|f\|_{L^\sigma}\|K(\tau_y|f|^2 - |f|^2)\|_{L^{2pr/(p+r)}} \\ &\leq C\|f\|_{L^\sigma}\|\tau_y|f|^2 - |f|^2\|_{L^{2pr/(p+r)}} \\ &\leq C\|f\|_{L^\sigma}(\|\tau_y\bar{f}(\tau_y f - f)\|_{L^{2pr/(p+r)}} + \|f(\tau_y\bar{f} - \bar{f})\|_{L^{2pr/(p+r)}}) \\ &\leq C\|f\|_{L^\sigma}(\|f\|_{L^\sigma}\|\tau_y f - f\|_{L^r}) \\ &\leq C\|f\|^2_{L^\sigma}\|\tau_y f - f\|_{L^r}\end{aligned}$$

where we obtain the second line by the $L^{2pr/(p+r)} \to L^{2pr/(p+r)}$ boundedness of K. We estimate (3.3) by:

$$\begin{aligned}(3.3) \leq \|\tau_y K(|f|^2)(\tau_y f - f)\|_{L^p} &\leq \|\tau_y K(|f|^2)\|_{L^{\sigma/2}}\|\tau_y f - f\|_{L^r} \\ &\leq C\||f|^2\|_{L^{\sigma/(2)}}\|\tau_y f - f\|_{L^r} \\ &\leq C\|f\|^2_{L^\sigma}\|\tau_y f - f\|_{L^r}\end{aligned}$$

as desired. Hence the inequalities of the lemma are proved. □

Lemma 3.1.3. *: For $f, g \in C(I, H^s) \cap L^\gamma(I, B^s_{\rho,2})$ and $I = (0, T)$ for $T > 0$ to be chosen later, we have:*

$$\|K(|f|^2)f - K(|g|^2)g\|_{L^{\gamma'}(I,L^{\rho'})} \leq C(\|f\|^2_{L^\gamma(I,B^s_{\rho,2})} + \|g\|^2_{L^\gamma(I,B^s_{\rho,2})})\|f - g\|_{L^p(I,B^s_{\rho,2})},$$

where $1/p = s + 1/\gamma$.

Proof. For such f, g we have

$$\|K(|f(t)|^2)f(t) - K(|g(t)|^2)g(t)\|_{L^{\rho'}} \leq \|K(|f(t)|^2)(f(t) - g(t))\|_{L^{\rho'}} \tag{3.4}$$

$$+ \|g(t)K(|f(t)|^2 - |g(t)|^2)\|_{L^{\rho'}}. \tag{3.5}$$

We estimate (3.4) and (3.5) as follows:

$$\begin{aligned}
(3.4) &\leq \|f(t) - g(t)\|_{L^\rho}\|K(|f(t)|^2)\|_{L^{2/(1-s)}} \\
&\leq \|f(t) - g(t)\|_{L^\rho}\||f(t)|^2\|_{L^{2/(1-s)}} \\
&\leq C\|f(t) - g(t)\|_{L^\rho}\|f(t)\|^2_{L^{4/(1-s)}} \\
&\leq C\|f(t) - g(t)\|_{L^\rho}\|f(t)\|^2_{B^s_{\rho,2}}
\end{aligned}$$

by the Sobolev embedding, (see [18]), $B^s_{\rho,2} \hookrightarrow L^{4/(1-s)}$. For (3.5):

$$\begin{aligned}
(3.5) &\leq C\|g(t)\|_{L^{4/(1-s)}}\||f(t)|^2 - |g(t)|^2\|_{L^2} \\
&\leq C\|g(t)\|_{L^{4/(1-s)}}\||f(t)| + |g(t)|\|_{L^{4/(1-s)}}\|f(t) - g(t)\|_{L^\rho} \\
&\leq C\|g(t)\|_{L^{4/(1-s)}}(\|f(t)\|_{L^{4/(1-s)}} + \|g(t)\|_{L^{4/(1-s)}})\|f(t) - g(t)\|_{L^\rho}.
\end{aligned}$$

We obtain by the Cauchy inequality ($ab \leq (a^2 + b^2)/2$) that

$$\|K(|f(t)|^2)f(t) - K(|g(t)|^2)g(t)\|_{L^{\rho'}} \leq C(\|f(t)\|^2_{B^s_{\rho,2}} + \|g(t)\|^2_{B^s_{\rho,2}})\|f(t) - g(t)\|_{L^\rho},$$

then Hölder's inequality in time yields

$$\|K(|f|^2)f - K(|g|^2)g\|_{L^{\gamma'}(I,L^{\rho'})} \leq C(\|f\|^2_{L^\gamma(I,B^s_{\rho,2})} + \|g\|^2_{L^\gamma(I,B^s_{\rho,2})})\|f - g\|_{L^\rho(I,L^\rho)}$$

which is the estimate we look for.

Hence we get all the crucial estimates in the proof given in the [12], and the proof follows similarly. □

Theorem 3.1.4. *(H^s Regularity) Let (γ, ρ) be defined as in Theorem* 3.1.1.

For $\varphi \in H^s$, let $u \in C((-T_{min}, T_{max}), H^s) \cap L^\gamma((-T_{min}, T_{max}), B^s_{\rho,2})$ be the maximal H^s solution of the ACNLS. Then if, in addition, $\varphi \in H^{s'}$ for some $s < s' < 1$, for any admissible (q, r)

$$u \in C((-T_{min}, T_{max}), H^{s'}) \cap L^q((-T_{min}, T_{max}), B^{s'}_{r,2}).$$

Proof. Since there is local well posedness in $H^{s'}$ we have u as an $H^{s'}$ solution on some maximal interval $[0, T)$ where $T \leq T_{max}$ (since all $H^{s'}$ solutions are also H^s solutions). Now we want to show that $T = T_{max}$.

To obtain a contradiction, suppose $T \neq T_{max}$, then $T < T_{max}$ and $\|u(t)\|_{H^{s'}} \to \infty$ as $t \to T$. Since $T < T_{max}$, we have

$$\|u\|_{L^\gamma((-T_{min},T_{max}),B^s_{\rho,2})} + \sup_{0 \leq t \leq T} \|u\|_{H^s} < \infty. \tag{3.6}$$

Then by previous calculations, we get

$$\| \left[K(|u|^2)u\right](t)\|_{B^{s'}_{\rho',2}} \leq C\|u\|^2_{L_{4/(1-s)}} \|u\|_{B^{s'}_{\rho,2}} \leq C\|u\|^2_{B^s_{\rho,2}} \|u\|_{B^{s'}_{\rho,2}}, \tag{3.7}$$

by Sobolev embedding $W^{m,p} \hookrightarrow L^q \quad for \quad p \leq q \leq np/(n - mp)$.

Now let $I \subset (0, T)$. Then

$$\|K(|u|^2)u(t)\|_{L^{\gamma'}(I,B^{s'}_{\rho',2})} \leq C\|u\|^2_{L^\gamma(I,L^{4/(1-s)})} \|u\|_{L^p(I,B^{s'}_{\rho,2})}, \tag{3.8}$$

where $1/p = 1/\gamma + s$.

Then by Strichartz' estimates we get,

$$\|u\|_{L^\infty(I,H^{s'})} + \|u\|_{L^\gamma(I,B^{s'}_{\rho,2})} \leq C\|\varphi\|_{H^{s'}} + C\|u\|_{L^1(I,H^{s'})} + C\|u\|_{L^p(I,B^{s'}_{\rho,2})}, \tag{3.9}$$

for each I with $0 \in I \subset (0,T)$. Now let $0 < \epsilon < \tau < T$ and consider $I = (0,\tau)$. Then

$$\begin{aligned}\|u\|_{L^1(I,H^{s'})} &\leq \|u\|_{L^1((0,\tau-\epsilon),H^{s'})} + \|u\|_{L^1((\tau-\epsilon,\tau),H^{s'})}\\ &\leq \|u\|_{L^1((0,\tau-\epsilon),H^{s'})} + \epsilon\|u\|_{L^\infty((\tau-\epsilon,\tau),H^{s'})}\\ &\leq C_\epsilon + \epsilon\|u\|_{L^\infty(I,H^{s'})}\end{aligned}$$

and similarly

$$\|u\|_{L^p(I,B^{s'}_{\rho,2})} \leq C_\epsilon + \epsilon^s\|u\|_{L^\gamma(I,B^{s'}_{\rho,2})}$$

which we obtain by Hölder inequality (here, we want to get positive power of ϵ which we did by the choice of (γ,ρ)).

$$\|u\|_{L^\infty(I,H^{s'})} + \|u\|_{L^\gamma(I,B^{s'}_{\rho,2})} \leq C + C_\epsilon + \epsilon C\|u\|_{L^\infty(I,H^{s'})} + \epsilon^s C\|u\|_{L^\gamma(I,B^{s'}_{\rho,2})}$$

then choosing ϵ small enough such that, $\epsilon C < 1/2$ *and* $\epsilon^s C < 1/2$ we get

$$\|u\|_{L^\infty(I,H^{s'})} + \|u\|_{L^\gamma(I,B^{s'}_{\rho,2})} \leq C,$$

where C is independent of τ. We obtain a contradiction if we let $\tau \to T$ with T being the maximal time of existence. Hence the result follows. $\square$

As we can see, the H^s well posedness theorem does not state the continuous dependence. But we can state (as in [19, Theorem 1.4]):

Theorem 3.1.5 (H^s local well-posedness 2)**.** *For $\phi \in H^s$, the L^2 solution of the ACNLS and the L^2 local well-posedness time T_0 satisfy:*

(P1) $u \in L^\infty(I,H^s)$.

(P2) There exist δ_2 such that if $\|u\|_{L^4(I,L^4)} \leq \delta_2$ then $\|D^s u\|_{L^q(I,L^r)} \leq C\|\phi\|_{H^s}$ for all admissible pairs (q,r).

(P3) There is a $\delta_3 > 0$ such that, if $\|u\|_{L^4(I,L^4)} \leq \delta_2$, u' is a solution of the ACNLS

with the initial datum $\phi' \in H^s$, and if $\|\phi-\phi'\|_{L^2} \leq \delta_3$, then $\|D^s(u-u')\|_{L^q(I,L^r)} \leq C\|\phi-\phi'\|_{H^s}$ for all admissible pairs (q,r). Again where C is only dependent on dimension, s and the operator K.

Proof. For the proof of 1. we need to prove two claims;

Claim 1: The operator K commutes with D^s whenever $\|D^sK(u)\|_{L^2} < \infty$.

Proof of Claim 1: Since K is a singular integral operator, we have that K is bounded from L^p to L^p if $p > 1$, and for $u \in L^2$, $K(u)$ is in L^2 and $\int (1+|\xi|^2)^s|\alpha(\xi)\hat{u}(\xi)|^2 d\xi \leq M^2 \int (1+|\xi|^2)^s|\hat{u}(\xi)|^2 \mathrm{d}\xi = M^2\|u\|^2_{H^s}$ where $M = \sup|\alpha(\xi)|$, which gives that $K(u)$ is in H^s for $u \in H^s$. So $D^sK(u) \in L^2$ whence $D^sK(u) = K(D^su)$ since Fourier transforms of both sides are equal. For $|u|^2$, consider a sequence of functions (u_n) such that $u_n \in H^s$ for all n and $K(u_n) \to K(|u|^2)$ in $\dot{H}^s$ then, by Parseval identity, $\||\xi|^s\alpha(\xi)(\hat{u_n} - \hat{|u|^2})\|_{L^2} \to 0$. So $\||\xi|^s(\hat{u_n} - \hat{|u|^2})\|_{L^2} \to 0$ which gives that $u_n \to u$ in $\dot{H}^s$. Then since for such u_n's $D^sK(u_n) = K(D^su_n)$, we get $\|K(D^s|u(t)|^2)\|_{L^2} = \|D^sK(|u(t)|^2)\|_{L^2}$

Claim 2: $\|D^s(fg)\|_{L^p} \leq C[\|D^sf\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|D^sg\|_{L^{p_3}}\|f\|_{L^{p_4}}]$ where $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$

Proof of Claim 2: For the functions ψ and ϕ in L^2, if we define operators $\Delta_j f = \boldsymbol{F}^{-1}(\varphi_j\hat{f})$ *for* $f \in \boldsymbol{S}$, then by Littlewood-Paley theorem (for details, see [20],[8, Appendix A]) we know that

$$\|f\|_{\dot{W}^{s,p}} \approx \|(\sum_{-\infty}^{\infty} 2^{2js}\Delta_j f)\|_{L^p},$$

and consequently:

$$\begin{aligned}\|D^s(fg)\|_{L^p} &= \|\sum_{j,k} D^s(\Delta_j f\Delta_k g)\|_{L^p}\\ &\leq \|\sum_j\sum_{k<j} D^s(\Delta_j f\Delta_k g)\|_{L^p} + \|\sum_{j=k} D^s(\Delta_j f\Delta_k g)\|_{L^p} + \|\sum_k\sum_{j<k} D^s(\Delta_j f\Delta_k g)\|_{L^p}\\ &= S_1 + S_2 + S_3.\end{aligned}$$

Since S_1 and S_3 are almost symmetric if we can find an estimate for S_1 then interchanging the roles of f and g we will get the estimate for S_3. Now

$$\begin{aligned}
S_1 &\leq C\|(\sum_j 2^{2js}|\Delta_j f|^2|\sum_{k<j}\Delta_k g|^2)^{1/2}\|_{L^p} \\
&\leq C\|(\sum_j 2^{2js}|\Delta_j f|^2)^{1/2} Mg\|_{L^p} \\
&\leq C\|(\sum_j 2^{2js}|\Delta_j f|^2)^{1/2}\|_{L^{p_1}}\|Mg\|_{L^{p_2}} \\
&\leq C\|D^s f\|_{L^{p_1}}\|g\|_{L^{p_2}}
\end{aligned}$$

where $1/p = 1/p_1 + 1/p_2$ and M is the Hardy-Littlewood maximal operator which is bounded in L^p where $1 < p$. But to have the second line, we have to consider the basic Fourier transform property that if $g(x) = \lambda^n f(\lambda^{-1}x)$ then $\hat{g}(\xi) = \hat{f}(\lambda x)$. For f being the function ψ and so $f(\lambda x)$ being the $\varphi(x)$, since $\check{\varphi} \in L^1$, $\sum_{k<j}\Delta_k$ becomes a radially bounded approximate identity for $j > 0$ and we know that such approximate identities are bounded by M.

Obtaining an estimate for S_2 is much easier:

$$\begin{aligned}
S_2 &\leq C\|(\sum_j 2^{2js}|\Delta_j f\Delta_j g|^2)^{1/2}\|_{L^p} \\
&\leq C\|(\sum_j 2^{js}|\Delta_j f\Delta_j g|)\|_{L^p} \\
&\leq C\|(\sum_j 2^{2js}|\Delta_j f|^2)^{1/2}(\sum_j \Delta_j g|^2)^{1/2}\|_{L^p} \\
&\leq C\|(\sum_j 2^{2js}|\Delta_j f|^2)^{1/2}\|_{L^{p_1}}\|(\sum_j |\Delta_j g|^2)^{1/2}\|_{L^{p_2}} \\
&\leq C\|D^s f\|_{L^{p_1}}\|g\|_{L^{p_2}}
\end{aligned}$$

by Cauchy-Schwarz inequality. Hence we get the result since p_1, p_2, p_3, p_4 are chosen to satisfy the condition in the claim. Now the proof of property 1 will be done similarly as done in [7, Theorem 4.9.1]. For that, let (ρ, γ) be the admissible pair

$(4/(1+s), 4/(1-s))$ and we have by Hölder's inequality and Claim 2

$$\|D^s(K(|u(t)|^2)u(t))\|_{L^{\rho'}} \leq C\|D^sK(|u(t)|^2)\|_{L^2}\|u(t)\|_{L^\gamma} + C\|D^su(t)\|_{L^\rho}\|K(|u(t)|^2)\|_{L^{2/(1-s)}}.$$

Also by Claim 1:

$$\begin{aligned}\|D^s(K(|u(t)|^2)u(t))\|_{L^{\rho'}} &\leq C\|K(D^s|u(t)|^2)\|_{L^2}\|u(t)\|_{L^\gamma} + C\|D^su(t)\|_{L^\rho}\||u(t)|^2\|_{L^{2/(1-s)}} \\ &\leq C\|(D^s|u(t)|^2)\|_{L^2}\|u(t)\|_{L^\gamma} + C\|D^su(t)\|_{L^\rho}\||u(t)|^2\|_{L^{2/(1-s)}} \\ &\leq C\|(D^su(t))\|_{L^\rho}\|u(t)\|_{L^\gamma}\|u(t)\|_{L^\gamma} + C\|D^su(t)\|_{L^\rho}\||u(t)|^2\|_{L^{2/(1-s)}} \\ &\leq C\|(D^su(t))\|_{L^\rho}\|u(t)\|_{L^\gamma}^2\end{aligned}$$

since $\gamma = 4/(1-s)$
Then by Sobolev embedding we reach

$$\|D^s(K(|u(t)|^2)u(t))\|_{L^{\rho'}} \leq C\|(D^su(t))\|_{L^\rho}^3.$$

Applying Hölder's inequality in time:

$$\|D^s(K(|u|^2)u)\|_{L^{\gamma'}(I,L^{\rho'})} \leq CT_0^s\|(D^su)\|_{L^\gamma(I,L^\rho)}^3.$$

Now we apply Strichartz' estimates to $u = H(u)$ by Duhamel's principle where $H(u)$ is the integral representation of the solution of the ACNLS equation, to end up with

$$\|D^sH(u)\|_{L^q(I,L^r)} \leq C\|\phi\|_{H^s} + CT_0^s\|(D^su)\|_{L^\gamma(I,L^\rho)}^3,$$

for every admissible pair (q, r) which gives the property 1.

To prove property 2 consider the metric space

$$E = \{u \in L^q(I, W^{s,r}) : \|D^s u\|_{L^q(I,L^r)} \leq M_1, \|u\|_{L^q(I,L^r)} \leq M_2$$
$$, u(0) = \phi\}$$

defined for each admissible pair (q, r) equiped with the metric

$$d(u, v) = \sup_{(q,r) admissible} \|u - v\|_{L^q(I,L^r)}.$$

First note that completeness of the metric space with respect to the metric follows directly from Theorem 1.2.5 in [7] (where X denotes $W^{s,q}$ and Y denotes L^r).

Hence to show the existence, what we have to show is that the integral representation is a contraction mapping in the metric space E. First we show $H : E \to E$. Pick $u \in E$. Strichartz' estimates give,

$$\|H(u)\|_{L^q(I,L^r)} \leq C\|\phi\|_{L^2} + \left\| \int_0^t \mathcal{T}_\beta(t - \tau)[K(|u|^2)u](\tau)\mathrm{d}\tau \right\|_{L^q(I,L^r)}$$

and since

$$\left\| \int_0^t \mathcal{T}_\beta(t - \tau)[K(|u|^2)u](\tau)\mathrm{d}\tau \|_{L^q(I,L^r)} \leq C\|K(|u|^2)u \right\|_{L^{4/3}(I,L^{4/3})}$$

we have, by Hölder inequality and $L^p \to L^p$ boundedness of K

$$\|K(|u|^2)u\|_{L^{4/3}(I,L^{4/3})} \quad \leq \|K(|u|^2)\|_{L^2(I,L^2)}\|u\|_{L^4(I,L^4)} \leq C\|u\|^3_{L^4(I,L^4)}.$$

Hence

$$\|H(u)\|_{L^q(I,L^r)} \leq C\|\phi\|_{L^2} + CM_2^3,$$

so if M_2 is sufficiently small we have

$$\|H(u)\|_{L^q(I,L^r)} \leq M_2,$$

for (q,r) admissible. Here we also used the fact that if $\|u\|_{L^4(I,L^4)}$ is small $(<< 1)$ then $\|\phi\|_{L^2} \approx \|u\|_{L^4(I,L^4)}$.

Moreover we have

$$\|D^s H(u)\|_{L^q(I,L^r)} \leq C\|D^s\phi\|_{L^2} + \left\|\int_0^t \mathcal{T}_\beta(t-\tau)D^s[K(|u|^2)u](\tau)\mathrm{d}\tau\right\|_{L^q(I,L^r)}.$$

The second summand can be estimated as follows,

$$\begin{aligned}\left\|\int_0^t \mathcal{T}_\beta(t-\tau)D^s[K(|u|^2)u](\tau)\mathrm{d}\tau\right\|_{L^q(I,L^r)} &\leq C\|D^s[K(|u|^2)u]\|_{L^{4/3}(I,L^{4/3})} \\ &\leq C\|K(|u|^2)\|_{L^2(I,L^2)}\|D^s u\|_{L^4(I,L^4)} \\ &\qquad + C\|u\|_{L^4(I,L^4)}\|D^s K(|u|^2)\|_{L^2(I,L^2)} \\ &\leq C\||u|^2\|_{L^2(I,L^2)}\|D^s u\|_{L^4(I,L^4)} \\ &\qquad + C\|u\|_{L^4(I,L^4)}\|K(D^s|u|^2)\|_{L^2(I,L^2)} \\ &\leq C\|u\|^2_{L^4(I,L^4)}\|D^s u\|_{L^4(I,L^4)} \\ &\qquad + C\|u\|_{L^4(I,L^4)}\|(D^s|u|^2)\|_{L^2(I,L^2)} \\ &\leq C\|u\|^2_{L^4(I,L^4)}\|D^s u\|_{L^4(I,L^4)} \\ &\qquad + C\|u\|_{L^4(I,L^4)}\|u\|_{L^4(I,L^4)}\|D^s u\|_{L^4(I,L^4)} \\ &\leq C\|D^s u\|_{L^4(I,L^4)}\|u\|^2_{L^4(I,L^4)}\end{aligned}$$

where the constant C is modified at each step. Thus we have

$$\|D^s H(u)\|_{L^q(I,L^r)} \leq C\|D^s\phi\|_{L^2} + C\|D^s u\|_{L^4(I,L^4)}\|u\|^2_{L^4(I,L^4)},$$

so for $2C\|\phi\|_{H^s} = M_1$ and again for M_2 small enough, we get $H : E \to E$ is into.

Remark 3.1.6. $I = [0, T_0]$ *so that everything done here is for the* L^2 *local well-*

posedness time T_0, and the smallness condition for $\|u\|_{L^4(I,L^4)}$ is independent of T_0.

Thus what is left to show is that H is a contraction mapping. Now for $u, v \in E$ we have:

$$\begin{aligned}
\|H(u) - H(v)\|_{L^q(I,L^r)} &\le C\|K(|u|^2)u - K(|v|^2)v\|_{L^{4/3}(I,L^{4/3})} \\
&\le C\|K(|u|^2)(u-v)\|_{L^{4/3}(I,L^{4/3})} \\
&\quad + C\|v(K(|u|^2 - |v|^2))\|_{L^{4/3}(I,L^{4/3})} \\
&\le C\|K(|u|^2)\|_{L^2(I,L^2)}\|u-v\|_{L^4(I,L^4)} \\
&\quad + C\|v\|_{L^4(I,L^4)}\|K(|u|^2 - |v|^2)\|_{L^2(I,L^2)} \\
&\le \|u\|^2_{L^4(I,L^4)}\|u-v\|_{L^4(I,L^4)} + C\|v\|_{L^4(I,L^4)}\||u|^2 - |v|^2\|_{L^2(I,L^2)} \\
&\le C\|u\|^2_{L^4(I,L^4)}\|u-v\|_{L^4(I,L^4)} \\
&\quad + C\|v\|_{L^4(I,L^4)}\|u-v\|_{L^4(I,L^4)}\|u+v\|_{L^4(I,L^4)} \qquad (3.10) \\
&\le \|v\|_{L^4(I,L^4)}(C\|u\|_{L^4(I,L^4)} + C\|v\|_{L^4(I,L^4)})\|u-v\|_{L^4(I,L^4)} \\
&\quad + C\|u\|^2_{L^4(I,L^4)}\|u-v\|_{L^4(I,L^4)} \qquad (3.11) \\
&\le (C\|v\|^2_{L^4(I,L^4)} + C\|u\|_{L^4(I,L^4)}\|v\|_{L^4(I,L^4)})\|u-v\|_{L^4(I,L^4)} \\
&\quad + C\|u\|^2_{L^4(I,L^4)}\|u-v\|_{L^4(I,L^4)} \qquad (3.12) \\
&\le (C\|u\|^2_{L^4(I,L^4)} + C\|v\|^2_{L^4(I,L^4)})\|u-v\|_{L^4(I,L^4)} \qquad (3.13)
\end{aligned}$$

where the last inequality follows from the Cauchy inequality. For M_2 small and (q, r) an admissible pair, we have

$$\|H(u) - H(v)\|_{L^q(I,L^r)} \le 1/2d(u, v),$$

which implies H is a contraction mapping from E into E. Since there is a uniqueness in L^2, if $\|u\|_{L^4(I,L^4)} \le \delta_2'$ for some δ_2' which makes u satisfy the above conditions, u coincides with the L^2 solution of the equation. Moreover we have $\|u\|_{L^4(I,L^4)} \le \delta_2'$ implies $\|D^s u\|_{L^q(I,L^r)} \le C\|\phi\|_{H^s}$ for each (q, r) admissible, which gives the property 2. Call $\delta_2 = \delta_2'/2$. For the property 3 we must mention a couple of things first. We know from L^2 local well-posedness theorem that if $\|\phi - \phi'\|_{L^2} \le \delta_3$ for some δ_3 then, if u' is the solution of the ACNLS equation with the initial datum ϕ' then $\|u-u'\|_{L^4(I,L^4)} \leq \delta_2$ where δ_2 is as given above. Thus $\|u'\| \leq 2\delta_2$ so u' satisfies the assertion in property 2. For any admissible pair (q,r):

$$\begin{aligned}\|D^s(u-v)\|_{L^q(I,L^r)} &\leq C\|D^s(\phi-\phi')\|_L^2 + C\|D^s[K(|u|^2)u - K(|v|^2)v]\|_{L^{4/3}(I,L^{4/3})}\\ &\leq C\|D^s(\phi-\phi')\|_L^2 + II\end{aligned}$$

and

$$\begin{aligned}II &\leq C\|(u-v)D^s(K(|u|^2))\|_{L^{4/3}(I,L^{4/3})} + C\|vD^s(K(|u|^2-|v|^2))\|_{L^{4/3}(I,L^{4/3})}\\
&\leq C\|K(D^s|u|^2)\|_{L^2(I,L^2)}\|u-v\|_{L^4(I,L^4)} + C\|D^s(u-v)\|_{L^4(I,L^4)}\|u\|^2_{L^4(I,L^4)}\\
&\quad + C\|D^s v\|_{L^4(I,L^4)}\|K(|u|^2-|v|^2)\|_{L^2(I,L^2)}\\
&\qquad + C\|v\|_{L^4(I,L^4)}\|K(D^s(|u|^2-|v|^2))\|_{L^2(I,L^2)}\\
&\leq C\|D^s u\|_{L^4(I,L^4)}\|u\|_{L^4(I,L^4)}\|u-v\|_{L^4(I,L^4)} + C\|D^s(u-v)\|_{L^4(I,L^4)}\|u\|^2_{L^4(I,L^4)}\\
&\quad + C\|D^s v\|_{L^4(I,L^4)}\||u|^2-|v|^2\|_{L^2(I,L^2)} + C\|v\|_{L^4(I,L^4)}\|D^s(|u|^2-|v|^2)\|_{L^2(I,L^2)}\\
&\leq C\|D^s u\|_{L^4(I,L^4)}\|u\|_{L^4(I,L^4)}\|u-v\|_{L^4(I,L^4)} + C\|D^s(u-v)\|_{L^4(I,L^4)}\|u\|^2_{L^4(I,L^4)}\\
&\quad + C\|v\|_{L^4(I,L^4)}\left(\|D^s(|u|+|v|)\|_{L^4(I,L^4)}\|u-v\|_{L^4(I,L^4)}\right)\\
&\qquad + C\|v\|_{L^4(I,L^4)}\left(\|D^s(u-v)\|_{L^4(I,L^4)}\|u+v\|_{L^4(I,L^4)}\right)\\
&\qquad\quad + C\|D^s v\|_{L^4(I,L^4)}\||u|+|v|\|_{L^4(I,L^4)}\|u-v\|_{L^4(I,L^4)}.\end{aligned}$$

Upon writing $\|D^s v\|_{L^4(I,L^4)} \leq \|D^s(u-v)\|_{L^4(I,L^4)} + \|D^s u\|_{L^4(I,L^4)}$ by linearity of D^s and triangle inequality, and using the L^2 local well-posedness result above, we get

$$\begin{aligned}II \quad &\leq C\|\phi\|_{H^s}\delta_2 2\|\phi-\phi'\|_{L^2} + C\|D^s(u-v)\|_{L^4(I,L^4)}\delta_2^2 + C\|D^s(u-v)\|_{L^4(I,L^4)}2\delta_2\delta_3\\
&\quad +C\|\phi\|_{H^s}2\delta_2\|\phi-\phi'\|_{L^2} + C\delta_2(2\|\phi\|_{H^s}\|\phi-\phi'\|_{L^2}\\
&\qquad +C\|\phi\|_{H^s}\|D^s(u-v)\|_{L^4(I,L^4)}\delta_3 + C\|D^s(u-v)\|_{L^4(I,L^4)}2\delta_2).\end{aligned}$$

Therefore

$$II \leq C\|D^s(u-v)\|_{L^4(I,L^4)}(\delta_2^2 + 2\delta_2\delta_3 + \|\phi\|_{H^s}(\delta_2\delta_3) + 2\delta_2) + 6C\|\phi\|_{H^s}\delta_2\|\phi-\phi'\|_{L^2}.$$

Choosing δ_3 small and δ_2 even possibly smaller (δ_2 depending on $\|\phi\|_{H^s}$ and δ_3 depending on both $\|\phi\|_{H^s}$ and δ_2) we arrive at

$$1/2d(D^s u, D^s v) \leq \|\phi - \phi'\|_{\dot{H}^s} + 1/2\|\phi - \phi'\|_{L^2}.$$

By the previous calculations and L^2 local well-posedness, we already have

$$d(u, v) \leq 2\|\phi - \phi'\|_{L^2}.$$

We combine these results to get the property 3, which is the H^s local well-posedness result under the given assumptions. □

3.2. Bourgain spaces $X^{\delta}_{s,b}$ and Strichartz type estimates

For $s, b \in \mathbb{R}$ the Bourgain space $X_{s,b}$ is the closure of the Schwartz functions $\boldsymbol{S}_{t,x}(\mathbb{R} \times \mathbb{R}^2)$ under the norm

$$\|u\|_{X_{s,b}} = \|\langle \xi \rangle^s \langle \tau - |\xi|^2 \rangle^b \hat{u}(\tau, \xi)\|_{L^2_\tau L^2_\xi(\mathbb{R} \times \mathbb{R}^2)}, \tag{3.14}$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$ and the Fourier transform is not only taken in space but is taken in both space and time. As we can see the definition is given in $\mathbb{R} \times \mathbb{R}^2$, but we can define the restriction of the Bourgain space on $I \times \mathbb{R}^2$ for some time interval $[0, \delta]$ as

$$\|u\|_{X^{\delta}_{s,b}} = inf\{\|f\|_{X_{s,b}} : f \in X_{s,b}, \quad f(t) = u(t) \ \ \forall t \in [0, \delta]\}. \tag{3.15}$$

This norm (3.14) can be written in another form using the solution operator of the Schrödinger equation and Bessel potentials, as follows,

$$\|u\|_{X_{s,b}} = \|J^b_t J^s_x \mathcal{T}_1(-t) u(x, t)\|_{L^2(\mathbb{R}, L^2)}. \tag{3.16}$$

For these spaces, we can see that there is a trivial embedding

$$X_{s',b'} \subset X_{s,b}, \tag{3.17}$$

for $s' \leq s$ and $b' \leq b$. Also from Parseval's identity and Cauchy-Schwarz inequality we have the duality relationship

$$(X_{s,b})^* = X_{-s,-b}, \tag{3.18}$$

These spaces behave well under interpolation in both indices s and b. One of the most problematic property of these spaces is that although they are invariant under translations in space and time, they are not invariant under conjugation. This means even though a function u is in a Bourgain space $X_{s,b}$, this does not imply that its conjugate $\bar{u}$ is in that Bourgain space.

In order to use Bourgain spaces we need to give Strichartz-like estimates, namely we need to give estimates concerning the solution operator and embeddings into spaces more common and whose theory are much widely explored. Although the study of Bourgain spaces are rapidly developing we will only focus on the estimates and embeddings closely related to the Schrödinger equation.

The basic estimates of the Bourgain spaces can be asserted as follows;

Proposition 3.2.1. *For any admissible pair (q, r), any $s \in \mathbb{R}$, time interval $I = [0, \delta]$ and ϵ sufficiently small, we have;*

$$\|u\|_{L^q(I,L^r)} \leq C\|u\|_{X^{\delta}_{s,1/2+\epsilon}}, \tag{3.19}$$

and

$$\|u\|_{L^{q_\theta}(I,L^{r_\theta})} \leq C\|u\|_{X^{\delta}_{s,\theta/2+\epsilon}}, \tag{3.20}$$

where $\theta \in [0,1]$, $\quad 1/q_\theta = \theta/q + (1-\theta)/2 \quad$ *and* $\quad 1/r_\theta = \theta/r + (1-\theta)/2$, *where* C *is independent of* I

Proof. Since Bourgain norms behave well under time restrictions, we only need to show that for any $b > 1/2$

$$\|u\|_{L^q(\mathbb{R},L^r)} \leq C\|u\|_{X_{0,b}}.$$

Because once we showed that for any $u \in X^\delta_{0,b}$, we can find a sequence $u_n \in X_{0,b}$ such that $u_n\chi_{\{[0,\delta]\times\mathbb{R}^2\}} = u$ and

$$\|u_n\|_{X_{0,b}} \leq \|u\|_{X^\delta_{0,b}} + 1/n \quad for \quad n \in \mathbb{N},$$

and hence conclude that

$$\|u\|_{L^q(I,L^r)} = \|u_n\|_{L^q(I,L^r)} \leq C\|u_n\|_{X_{0,b}} = C\|u\|_{X^\delta_{0,b}} + 1/n, \quad \forall n \in \mathbb{N},$$

and as $n \to \infty$, which implies the inequality (3.19). So it is enough for us to show (3.2). Now let (q,r) be admissible and $b > 1/2$. We know that

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^2\times\mathbb{R}} \mathcal{F}_{x,t}(u)(\xi,\lambda)e^{i(\xi.x+\lambda t)}\mathrm{d}\xi\mathrm{d}\lambda \\ &= \int_{\mathbb{R}^2\times\mathbb{R}} \mathcal{F}_{x,t}(u)(\xi,\lambda-|\xi|^2)e^{i(\xi.x+(\lambda-|\xi|^2)t)}\mathrm{d}\xi\mathrm{d}\lambda. \end{aligned} \tag{3.21}$$

We define

$$\mathcal{F}_x(u_\lambda)(\xi) = \langle\lambda\rangle^b\mathcal{F}_{x,t}(u)(\xi,\lambda-|\xi|^2), \tag{3.22}$$

and

$$f(x,t,\lambda) = \int e^{i(x.\xi+t|\xi|^2)}\mathcal{F}_x(u_\lambda)(\xi)d\xi = \mathcal{T}_1(t)u_\lambda. \tag{3.23}$$

Then we get $\|u\|_{X_{0,b}} = \left(\int_{\mathbb{R}^2\times\mathbb{R}} |\mathcal{F}_x(u_\lambda)(\xi)|^2 \mathrm{d}\xi\mathrm{d}\lambda\right)^{1/2}$ and

$$u(x,t) = \int e^{it\lambda} f(x,t,\lambda)\langle\lambda\rangle^{-b} d\lambda,$$

by (3.21). So we have,

$$\|u\|_{L^q(\mathbb{R},L^r)} \le \int \|f(\cdot,\cdot,\lambda)\|_{L^q(\mathbb{R},L^r)}\langle\lambda\rangle^{-b} d\lambda, \tag{3.24}$$

and Strichartz' estimates give:

$$\|f(\cdot,\cdot,\lambda)\|_{L^q(\mathbb{R},L^r)} = \|\mathcal{T}_1(t)u_\lambda\|_{L^q(\mathbb{R},L^r)} \le C\|u_\lambda\|_{L^2} = C\|\mathcal{F}_x(u_\lambda)\|_{L^2}. \tag{3.25}$$

Thus combining (3.24) and(3.25) we get

$$\begin{aligned}\|u\|_{L^q(\mathbb{R},L^r)} &\le C\int \|\mathcal{F}_x(u_\lambda)\|_{L^2}\langle\lambda\rangle^{-b} d\lambda \le C\left(\int \|\mathcal{F}_x(u_\lambda)\|_{L^2}^2 d\lambda\right)^{1/2} \\ &\le C\|u\|_{X_{0,b}},\end{aligned}$$

where for the second inequality we use Hölder's inequality and the fact that $b > 1/2$. Thus we have showed (3.2).

For the second part of the theorem we only have to observe that

$$\|u\|_{L^2(I,L^2)} = \|u\|_{X^\delta_{0,0}},$$

and the result follows by interpolation. □

Corollary 3.2.2. *For any sufficiently small ϵ and admissible (q,r), the following in-*

equalities hold true:

$$\|u\|_{L^4(I,L^4)} \leq C\|u\|_{X^{\delta}_{0,1/2+\epsilon}}, \tag{3.26}$$

$$\|u\|_{L^q(I,W^{s,r})} \leq C\|u\|_{X^{\delta}_{s,1/2+\epsilon}}, \tag{3.27}$$

$$\|u\|_{L^{4-\epsilon}(I,L^{4-\epsilon})} \leq C\|u\|_{X^{\delta}_{0,1/2-\epsilon/8}} \tag{3.28}$$

$$\|u\|_{X_{0,-1/2+\epsilon}} \leq C\|u\|_{L^{p_\epsilon}(I,L^{p_\epsilon})}, \tag{3.29}$$

where $p_\epsilon = (4-8\epsilon)/(3-8\epsilon)$.

Proof. We easily see that (3.26) follows from the previous proposition since $(4,4)$ is an admissible pair. In the proof of the proposition, if we change u with $J^s u$, where J^s is the Bessel potential, we get (3.27). Also (3.28) is immediate by letting $(q,r) = (4,4)$ and $\theta = (4-2\epsilon)/(4-\epsilon)$ in (3.20) and observing that $\theta/2 < 1/2 - \epsilon/8$.

For the inequality (3.29), we use the duality argument and Hölder inequality:

$$\|u\|_{X^{\delta}_{-1/2+\epsilon}} \leq \sup_{\|v\|_{X^{\delta}_{1/2-\epsilon}} \leq 1} |\langle v, u\rangle| \leq \sup_{\|v\|_{X^{\delta}_{1/2-\epsilon}} \leq 1} \|v\|_{L^{4-8\epsilon}(I,L^{4-8\epsilon})}\|u\|_{L^{p_\epsilon}(I,L^{p_\epsilon})},$$

where p_ϵ is as above. Because $\|v\|_{L^{4-8\epsilon}(I,L^{4-8\epsilon})} \leq C\|v\|_{X^{\delta}_{0,1/2-\epsilon}}$, we obtain (3.29). □

These embeddings are important in the study of the Bourgain spaces as L^p spaces are much easier to work with. But to study nonlinear Schrödinger equations, one has to make use of the following Strichartz type estimates, which are proved by Gou and Cui in [14].

Proposition 3.2.3. *For $s \in \mathbb{R}$, we have*

$$\|\mathcal{T}_1(t)\phi\|_{X^\delta_{s,b}} \leq C\|\phi\|_{H^s}, \quad for \quad -\infty < b < \infty, \tag{3.30}$$

$$\|u\|_{X^\delta_{s,-b_1}} \leq C\delta^{b_1-b_2-}\|u\|_{X^\delta_{s,-b_2}}, \quad for \quad 0 \leq b_2 \leq b_1 < 1/2, \tag{3.31}$$

$$\left\|\int_0^t \mathcal{T}_1(t-\lambda)u(\lambda)d\lambda\right\|_{X^\delta_{s,b}} \leq C\delta^{1/2-b}\|u\|_{X^\delta_{s,b-1}}, \quad for \quad 1/2 < b \leq 1, \tag{3.32}$$

where C is independent of δ.

Proof. First we will prove (3.30). To this end, take a compactly supported $C^\infty(\mathbb{R})$ function ψ such that $\psi(t) = 1$ for $0 \leq t \leq 1$ and $\psi(t) = 0$ for $t \leq -1$ and $t \geq 2$. Then we have

$$\begin{aligned}\|\mathcal{T}_1(t)\phi\|_{X^\delta_{s,b}} &\leq \|\psi(t/\delta)\mathcal{T}_1(t)\phi\|_{X_{s,b}} = \|J^b_t J^s_x \mathcal{T}_1(-t)(\psi(t/\delta)\mathcal{T}_1(t)\phi)\|_{L^2(\mathbb{R},L^2)} \\ &\leq \|J^b_t\psi\|_{L^2(\mathbb{R})}\|J^s_x\phi\|_{L^2(\mathbb{R})} \leq C\|\phi\|_{H^s},\end{aligned}$$

and this is (3.30).

As the time localization arguments done in the previous proposition, (3.31) would be shown if we could show

$$\|\psi(t/\delta)u\|_{X_{s,-b_1}} \leq C\delta^{b_1-b_2-}\|u\|_{X_{s,-b_2}}. \tag{3.33}$$

By duality, it is enough to show

$$\|\psi(t/\delta)u\|_{X_{s,b_2}} \leq C\delta^{b_1-b_2-}\|u\|_{X_{s,b_1}}. \tag{3.34}$$

To prove (3.34), set $f(x,t) = J^{b_1}_t J^s_x \mathcal{T}_1(-t)u(x,t)$, so:

$$\|\psi(t/\delta)u\|_{X_{s,b_2}} = \left\|J^{b_2}_t\left(\psi(t/\delta)J^{-b_1}_t f\right)\right\|_{L^2(\mathbb{R},L^2)}. \tag{3.35}$$

Since $\|u\|_{X_{s,b_1}} = \|f\|_{L^2(\mathbb{R},L^2)}$, putting $J_t^{-b_1}\|f\|_{L^2(\mathbb{R}^2)} = g$, the inequality (3.31) will follow if we can show

$$\|\psi(t/\delta)g\|_{H^{b_2}(\mathbb{R})} \leq C\delta^{b_1-b_2-}\|g\|_{H^{b_1}(\mathbb{R})}. \tag{3.36}$$

By [21, (3,6)], we have

$$\|\psi(t/\delta)g\|_{H^a(\mathbb{R})} \leq C\delta^{1-2a}\|g\|_{H^a(\mathbb{R})}, \quad for \quad 1/2 < a \leq 1.$$

Since

$$\|\psi(t/\delta)g\|_{L^2(\mathbb{R})} \leq C\left(\int_{-\delta}^{2\delta} |g(t)|^2 \mathrm{d}t\right)^{1/2} \leq C\delta^{1/2-1/q}\|g\|_{L^q(\mathbb{R})},$$

and by Sobolev embedding theorem, $\|g\|_{L^q(\mathbb{R})} \leq C\|g\|_{H^b}$ for $2 \leq q < \infty$ and $b = 1/2 - 1/q$, then get

$$\|\psi(t/\delta)g\|_{L^2(\mathbb{R})} \leq C\delta^b\|g\|_{H^b}, \qquad 0 \leq b < 1/2. \tag{3.37}$$

For sufficiently small $\epsilon > 0$ we let $a = 1/2 + \epsilon$, $b = (b_1 - b_2)(1 + 2\epsilon)/(1 - 2b_2 + 2\epsilon)$ and $\theta = 2b_2/(1 + 2\epsilon)$ and interpolate between (3.2) and (3.37) to get,

$$\|\psi(t/\delta)g\|_{H^{b_2}} = \|\psi(t/\delta)g\|_{H^{a\theta+b(1-\theta)}(\mathbb{R})} \leq C\delta^{\theta(1-2a)+(1-\theta)b}\|g\|_{H^{a\theta+b(1-\theta)}(\mathbb{R})}.$$

Similarly, (3.32) follows from the inequality

$$\left\|\psi(t/\delta)\int_0^t \mathcal{T}_1(t-\lambda)u(\lambda)\mathrm{d}\lambda\right\|_{X_{s,b}} \leq C\delta^{1/2-b}\|u\|_{X_{s,b-1}}, \qquad 1/2 < b \leq 1,$$

which was shown in [21] for the one dimensional case, and for the n dimension case, the proof is similar. □

This proposition has an important corollary for the study of Schrödinger equations:

Corollary 3.2.4. *For $s \in \mathbb{R}$ and sufficiently small $\epsilon > 0$:*

$$\left\| \int_0^t \mathcal{T}_1(t-\lambda)u(\lambda)d\lambda \right\|_{X^\delta_{s,1/2+\epsilon}} \leq C\|u\|_{X^\delta_{s,-1/2+3\epsilon}}, \tag{3.38}$$

where C is independent of δ.

Proof. In (3.31) and (3.32). Setting $b = 1/2 + \epsilon$, $b_1 = 1/2 - \epsilon = -(b-1)$ and $b_2 = 1/2 - 3\epsilon$, we get

$$\begin{aligned} \left\| \int_0^t \mathcal{T}_1(t-\lambda)u(\lambda)d\lambda \right\|_{X^\delta_{s,1/2+\epsilon}} &\leq C\delta^{-\epsilon}\|u\|_{X^\delta_{s,-b_1}} \leq C\delta^{-\epsilon}\delta^{b_1-b_2-\epsilon}\|u\|_{X^\delta_{s,-b_2}} \\ &\leq C\|u\|_{X_{s,-1/2+3\epsilon}}, \end{aligned}$$

which is (3.38) □

3.3. Local existence in Bourgain spaces

In this section we give the proof of the local existence theorem which was stated in the first chapter for ACNLS which also includes the case of cubic NLS for $\alpha(\xi) = 1$ for $\xi \in \mathbb{R}^2$.

$$\begin{aligned} iu_t + \triangle u &= K(|u|^2)u \\ u(0,x) &= \phi(x) \in H^s \qquad\qquad 0 \leq s \leq 1, \end{aligned} \tag{3.39}$$

which is (2.1) for $\beta = 1$. As done in the previous chapter, we decompose the initial datum into its low and high frequencies ϕ_0 and ϕ_1 and write the corresponding initial value problem:

$$
\begin{aligned}
iu_{ot} + \triangle u_0 &= K(|u_0|^2)u_0 \\
u_0(0,x) &= \phi_0(x) \in H^1
\end{aligned}
\tag{3.40}
$$

By the theory of the ACNLS, we know that u_0 exists globally, see [13, Theorem 4.4]. Recalling that $\phi = \phi_0 + \phi_1$, consider the difference of the equations (3.39) and (3.40) and obtain

$$
\begin{aligned}
iv_t + \triangle v &= K(|u_0 + v|^2)(u_0 + v) - K(|u_0|^2)u_0 \\
v(0,x) &= \phi_1 \in H^s.
\end{aligned}
\tag{3.41}
$$

since global existence of the solution to (3.41) is much to ask, we can only expect that the equation (3.41) is locally well-posed. The following theorem states the local existence result which is what we require for the proof of the global existence result of Bourgain.

Theorem 3.3.1. *If there exist C_1, C_2, C_3 such that $\|\phi_0\|_{L^2} \leq C_1$, $E(\phi_0) \leq C_2 N^{2(1-s)}$, $\|\phi_1\|_{L^2} \leq C_3 N^{-s}$ and $\|\phi_1\|_{H^s} \leq C_3$, then for $\delta = cN^{-2(1-s)-}$, there exist $N_0 > 1$ and $c_0 > 0$ such that for each $N \geq N_0$ and $c \leq c_0$, the solutions to the initial value problems* (3.40) *and* (3.41) *both exist in [0,δ]*

Proof. In the proof of this theorem, we will use the Banach fixed point theorem for the integral equation that the solution of the ACNLS satisfies in the metric space in which we want the solutions to reside.

Write

$$
w = \mathcal{T}_1(t)\phi_2 - i\int_0^t \mathcal{T}_1(t-\tau)F(u_0(\tau), v(\tau))d\tau,
$$

where $F(u_0, v) = K(|u_0 + v|^2)(u_0 + v) - K(|u_0|^2)u_0$ and let M be a positive number to be specified later. We define the set

$$B_{M,N}^{\sigma} = \{v \in X_{s,1/2+\epsilon'}^{\delta} : [v] \equiv N^s \|v\|_{X_{0,1/2+\epsilon'}^{\delta}} + \|v\|_{X_{s,1/2+\epsilon'}^{\delta}} \leq M\},$$

and the map

$$H(v)(t) = \mathcal{T}_1(t)\phi_2 - i\int_0^t \mathcal{T}_1(t-\tau)F(u_0(\tau), v(\tau))\mathrm{d}\tau,$$

for $v \in X_{s+1/2+\epsilon'}^{\delta}$.

So we need to show S is well defined in the metric space and maps the space into itself.

First we have

$$\begin{aligned}\|H(v)\|_{X_{0,1/2+\epsilon'}^{\delta}} &\leq \|\mathcal{T}_1(t)\phi_2\|_{X_{0,1/2+\epsilon'}^{\delta}} + \left\|\int_0^t \mathcal{T}_1(t-\tau)F(u_0(\tau), v(\tau))\mathrm{d}\tau\right\|_{X_{0,1/2+\epsilon'}^{\delta}} \\ &\leq C\|\phi_2\|_{L^2} + C\|F(u_0, v)\|_{X_{0,-1/2+3\epsilon'}^{\delta}} \\ &\leq CN^{-s} + C\|F(u_0, v)\|_{X_{0,-1/2+3\epsilon'}^{\delta}},\end{aligned}$$

and by the linearity of the operator K estimating each term in the sum will reveal the desired result. First call $1/r_i = 1/4 + (3-i)/(4+\epsilon)$ and let $s \geq 2/3$, then:

$$\|K(u_0\bar{v})v\|_{X_{0,-1/2+3\epsilon'}^{\delta}} \leq C\|K(u_0\bar{v})v\|_{L_{t,\delta}^{r_1}L_x^{r_1}} \tag{3.42}$$

$$\leq C\|v\|_{L_{t,\delta}^4 L_x^4}\|K(u_0\bar{v})\|_{L_{t,\delta}^{(4+\epsilon)/2}L_x^{(4+\epsilon)/2}} \tag{3.43}$$

$$\leq C\|v\|_{L^4L^4}\|u_0\bar{v}\|_{L_{t,\delta}^{(4+\epsilon)/2}} \tag{3.44}$$

$$\leq C\|v\|_{X_{0,1/2+\epsilon'}^{\delta}}\|u_0\|_{L_{t,\delta}^{4+\epsilon}L_x^{4+\epsilon}}\|v\|_{L_{t,\delta}^{4+\epsilon}L_x^{4+\epsilon}},$$

where in the inequality (3.42) we used the Bourgain embeddings and to pass from (3.43) to (3.44) we used the boundedness of the operator K and the last inequality follow from the Hölder's inequality. Now having $\|u_0\|_{L^{4+\epsilon}_{t,\delta}L^{4+\epsilon}_x} \leq c^{1/(4+\epsilon)}$ and the Bourgain norm of v, we need to estimate $\|v\|_{L^{4+\epsilon}_{t,\delta}L^{4+\epsilon}_x}$ and this is achieved by Sobolev embedding theorem with $\mu_\epsilon = 2((2+\epsilon)/(4+\epsilon) - 1/(4+\epsilon)) = (2+2\epsilon)/(4+\epsilon)$ and Bourgain embedding as follows:

$$\begin{aligned}
\|v\|_{L^{4+\epsilon}_{t,\delta}L^{4+\epsilon}_x} &\leq \|J^{\mu_\epsilon}v\|_{L^{4+\epsilon}_{t,\delta}L^{(4+\epsilon)/(2+\epsilon)}_x} \\
&\leq C\|J^{\mu_\epsilon}v\|_{X^\delta_{0,1/2+\epsilon'}} \\
&\leq CN^{(2+2\epsilon)/(4+\epsilon)} \\
&\leq CN^{(1-2s)/2+O(\epsilon)}.
\end{aligned}$$

Hence, we reach the bound

$$\|K(u_0\bar{v})v\|_{X^\delta_{0,-1/2+3\epsilon'}} \leq CN^{-s}(c^{1/(4+\epsilon)}N^{(1-2s)/2+O(\epsilon)}) \leq Cc^{1/(4+\epsilon)}N^{-s}.$$

This calculation also says that $\|K(v\bar{u}_0)v\|_{X^\delta_{0,-1/2+3\epsilon'}}$ has the same bound.

Now consider $\|K(v\bar{v})u_0\|_{X^\delta_{0,-1/2+3\epsilon'}}$:

$$\begin{aligned}
\|K(v\bar{v})u_0\|_{X^\delta_{0,-1/2+3\epsilon'}} &\leq C\|K(v\bar{v}u_0)\|_{L^{r_1}_{t,\delta}L^{r_1}_x} \\
&\leq \|u_0\|_{L^{4+\epsilon}_{t,\delta}L^{4+\epsilon}_x}\|v\bar{v}\|_{L^{r_2}_{t,\delta}L^{r_2}_x} \\
&\leq \|u_0\|_{L^{4+\epsilon}_{t,\delta}L^{4+\epsilon}_x}\|v\|_{L^{4+\epsilon}_{t,\delta}L^{4+\epsilon}_x}\|v\|_{L^4_{t,\delta}L^4_x} \\
&\leq Cc^{1/(4+\epsilon)}\|v\|_{X^\delta_{0,1/2+\epsilon'}}\|v\|_{L^{4+\epsilon}_{t,\delta}L^{4+\epsilon}_x},
\end{aligned} \tag{3.45}$$

but by the preceding calculation (3.45) has the same bound.

Now for $\|K(v\bar{v})v\|_{X^\delta_{0,-1/2+3\epsilon'}}$ we have, again by the above arguments that

$$\|K(v\bar{v})v\|_{X^\delta_{0,-1/2+3\epsilon'}} \leq CN^{-s}N^{((1-2s/2+O(\epsilon))2}, \tag{3.46}$$

which implies $\|K(v\bar{v})v\|_{X^{\delta}_{0,-1/2+3\epsilon'}} \leq CN^{-s}N^{1-2s+O(\epsilon)}$, which is the desired bound.

For $\|K(u_0\bar{u_0}v)\|_{X_{0,-1/2+3\epsilon'}}$ we have:

$$\begin{aligned}
\|K(u_0\bar{u_0}v)\|_{X_{0,-1/2+3\epsilon'}} &\leq \|K(u_0\bar{u_0})v\|_{L^{r_1}_{t,\delta}L^{r_1}_x} &(3.47)\\
&\leq \|v\|_{L^4_{t,\delta}L^4_x}\|K(u_0\bar{u_0})\|_{L^{(4+\epsilon)/2}_{t,\delta}L^{(4+\epsilon)/2}_x} &(3.48)\\
&\leq \|v\|_{L^4_{t,\delta}L^4_x}\|u_0\|^2_{L^{4+\epsilon}_{t,\delta}L^{4+\epsilon}_x} &(3.49)\\
&\leq Cc^{2/(4+\epsilon)}.N^{-s} &(3.50)
\end{aligned}$$

which is what we aimed at. So what is left to bound are $K(u_0\bar{v})u_0$ and $K(v\bar{u_0})u_0$. For $\|K(u_0\bar{v})u_0\|_{X^{\delta}_{0,-1/2+3\epsilon'}}$:

$$\begin{aligned}
\|K(u_0\bar{v})u_0\|_{X^{\delta}_{0,-1/2+3\epsilon'}} &\leq \|K(u_0\bar{v})u_0\|_{L^{r_1}_{t,\delta}L^{r_1}_x}\\
&\leq \|u_0\|_{L^{4+\epsilon}_{t,\delta}L^{4+\epsilon}_x}\|K(u_0\bar{v})\|_{L^{\gamma_\epsilon}_{t,\delta}L^{\gamma_\epsilon}_x}\\
&\leq \|u_0\|_{L^{4+\epsilon}_{t,\delta}L^{4+\epsilon}_x}\|u_0\bar{v}\|_{L^{\gamma_\epsilon}_{t,\delta}L^{\gamma_\epsilon}_x}\\
&\leq Cc^{1/(4+\epsilon)}\|u_0\|_{L^{4+\epsilon}_{t,\delta}L^{4+\epsilon}_x}\|v\|_{L^4_{t,\delta}L^4_x}\\
&\leq Cc^{2/(4+\epsilon)}N^{-s}.
\end{aligned}$$

This is the desired bound and it also holds for the last term.

We need moreover to show $\|H(v)\|_{X^{\delta}_{s,1/2+\epsilon'}} \leq C$ to conclude H, maps $B^{\delta}_{M,N}$ into itself. Firstly

$$\begin{aligned}
\|H(v)\|_{X^{\delta}_{s,1/2+\epsilon'}} &\leq \|\mathcal{T}_1(t)\phi_2\|_{X^{\delta}_{s,1/2+\epsilon'}} + \|\int_0^t \mathcal{T}_1(t-\tau)(u_0(\tau),v(\tau))d\tau\|_{X^{\delta}_{s,1/2+\epsilon'}} &(3.51)\\
&\leq C\|\phi_2\|_{H^s} + C\|F(u_0,v)\|_{X^{\delta}_{s,-1/2+3\epsilon'}}, &(3.52)
\end{aligned}$$

and $\|F(u_0,v)\|_{X^{\delta}_{s,-1/2+3\epsilon'}} = \|J^s_xF(u_0,v)\|_{X^{\delta}_{0,-1/2+3\epsilon'}} \leq C\|J^s_xF(u_0,v)\|_{L^{r_\epsilon}_{t,\delta}L^{r_\epsilon}_x}$.

Hence we have to estimate each summand in the sum $F(u_0,v)$, where F is defined at the beginning of the proof.

$$\begin{aligned}\|J_x^s K(u_0\bar{v})v\|_{L^{r_1}_{t,\delta}L^{r_1}_x} &\le \|J_x^s v\|_{L^4_{t,\delta}L^4_x}\|K(u_0\bar{v})\|_{L^{(4+\epsilon')/2}_{t,\delta}L^{(4+\epsilon')/2}_x}\\ &\qquad + \|v\|_{L^{4+\epsilon'}_{t,\delta}L^{4+\epsilon'}_x}\|J_x^s K(u_0\bar{v})\|_{L^{r_2}_{t,\delta}L^{r_2}_x}\\ &= I + II. \end{aligned} \tag{3.53}$$

We consider I and II separately:

$$\begin{aligned} I &\le C\|v\|_{X^\delta_{s,1/2+\epsilon'}}\|u_0\bar{v}\|_{L^{(4+\epsilon')/2}_{t,\delta}L^{(4+\epsilon')/2}_x}\\ &\le C\|v\|_{X^\delta_{s,1/2+\epsilon'}}\|v\|_{L^{4+\epsilon'}_{t,\delta}L^{4+\epsilon'}_x}\|u_0\|_{L^{4+\epsilon'}_{t,\delta}L^{4+\epsilon'}_x}\\ &\le \|v\|_{X^\delta_{s,1/2+\epsilon'}}Cc^{1/(4+\epsilon)}N^{(1-2s)/2+O(\epsilon)}. \end{aligned}$$

Since $s \ge 1/2$ the order of N is negative. For II we write:

$$\begin{aligned} II &\le CN^{(1-2s)/2+O(\epsilon)}\|K(J_x^s u_0\bar{v})\|_{L^{r_2}_{t,\delta}L^{r_2}_x}\\ &\le CN^{(1-2s)/2+O(\epsilon)}\|J_x^s u_0\bar{v}\|_{L^{r_2}_{t,\delta}L^{r_2}_x}\\ &\le CN^{(1-2s)/2+O(\epsilon)}[\|J_x^s u_0\|_{L^4_{t,\delta}L^4_x}\|v\|_{L^{4+\epsilon'}_{t,\delta}L^{4+\epsilon'}_x} + \|u_0\|_{L^{4+\epsilon'}_{t,\delta}}\|J_x^s v\|_{L^4_{t,\delta}L^4_x}]\\ &\le CN^{(1-2s)/2+O(\epsilon)}(\|v\|_{X_{s,1/2+\epsilon'}}N^{s(1-s)} + c^{1/(4+\epsilon)}\|v\|_{X^\delta_{s,1/2+\epsilon'}}), \end{aligned}$$

by the Bourgain embeddings and interpolation in the Bourgain spaces. Hence

$$\begin{aligned} II &\le CN^{1/2+O(\epsilon)}\|v\|_{X^\delta_{s,1/2+\epsilon'}}N^{s(1-s)}\\ &\le C\|v\|_{X^\delta_{s,1/2+\epsilon'}}N^{1/2-s^2}. \end{aligned}$$

Thus $\|J_x^s K(u_0\bar{v})v\|_{L^{r_\epsilon}_{t,\delta}L^{r_\epsilon}_x} \le C\|v\|_{X^\delta_{s,1/2+\epsilon'}}$. Likely $\|J_x^s K(v\bar{u}_0)v\|_{L^{r_1}_{t,\delta}L^{r_1}_x}$ admits the same bound.

For $\|J_x^s K(v\bar{v})u_0\|_{L_{t,\delta}^{r_1}L_x^{r_1}}$ we have:

$$\begin{aligned}\|J_x^s K(v\bar{v})u_0\|_{L_{t,\delta}^{r_1}L_x^{r_1}} &\leq \|J_x^s u_0\|_{L_{t,\delta}^4 L_x^4}\|K(v\bar{v})\|_{L_{t,\delta}^{(4+\epsilon')/2}L_x^{(4+\epsilon')/2}} \\ &\quad + \|u_0\|_{L_{t,\delta}^{4+\epsilon'}L_x^{4+\epsilon'}}\|J_x^s K(v\bar{v})\|_{L_{t,\delta}^{r_2}L_x^{r_2}} \\ &= I + II\end{aligned}$$

And

$$\begin{aligned}I &\leq cN^{s(1-s)}\|v\bar{v}\|_{L_{t,\delta}^{(4+\epsilon')/2}L_x^{(4+\epsilon')/2}} \\ &\leq CN^{s(1-s)}\|v\|^2_{L_{t,\delta}^{4+\epsilon'}L_x^{4+\epsilon'}} \\ &\leq CN^{s(1-s)}N^{1/2-s+O(\epsilon)}\|v\|_{X^\delta_{s,1/2+\epsilon'}} \\ &\leq C\|v\|_{X^\delta_{s,1/2+\epsilon'}}N^{1/2-s^2+O(\epsilon)} \\ &\leq C\|v\|_{X^\delta_{s,1/2+\epsilon'}},\end{aligned}$$

and since $\|v\|_{L_{t,\delta}^{4+\epsilon'}L_x^{4+\epsilon'}} \leq \|v\|_{X^\delta_{s,1/2+\epsilon'}}$ we get:

$$\begin{aligned}II &\leq Cc^{1/(4+\epsilon)}\|K(J_x^s v\bar{v})\|_{L_{t,\delta}^{r_2}L_x^{r_2}} \\ &\leq Cc^{1/(4+\epsilon)}\|v\|_{L_{t,\delta}^{4+\epsilon'}L_x^{4+\epsilon'}}\|J_x^s v\|_{L_{t,\delta}^4 L_x^4} \\ &\leq Cc^{1/(4+\epsilon)}N^{1/2-s+O(\epsilon)}\|v\|_{X^\delta_{s,1/2+\epsilon'}} \\ &\leq Cc^{1/(4+\epsilon)}\|v\|_{X^\delta_{s,1/2+\epsilon'}}.\end{aligned}$$

Therefore $\|J_x^s K(v\bar{v})u_0\|_{L_{t,\delta}^{r_\epsilon}L_x^{r_\epsilon}}$ has the required bound and so does $\|J_x^s K(v\bar{v})v\|_{L_{t,\delta}^{r_\epsilon}L_x^{r_\epsilon}}$. For $\|J_x^s K(u_0\bar{u_0})v\|_{L_{t,\delta}^{r_\epsilon}L_x^{r_\epsilon}}$ we have:

$$\begin{aligned}\|J_x^s K(u_0\bar{u_0})v\|_{L_{t,\delta}^{r_\epsilon}L_x^{r_\epsilon}} &\leq \|J_x^s v\|_{L_{t,\delta}^4 L_x^4}\|K(u_0\bar{u_0})\|_{L_{t,\delta}^{(4+\epsilon')/2}L_x^{(4+\epsilon')/2}} \\ &\quad + \|v\|_{L_{t,\delta}^{4+\epsilon'}L_x^{4+\epsilon'}}\|K(J_x^s u_0\bar{u_0})\|_{L_{t,\delta}^{r_2}L_x^{r_2}} \\ &= I + II\end{aligned}$$

We bound I and II as follows:

$$\begin{aligned} I &\leq C\|J_x^s v\|_{X^\delta_{0,1/2+\epsilon'}}\|u_0\|^2_{L^{4+\epsilon'}_{t,\delta}L^{4+\epsilon'}_x} \\ &\leq Cc^{2/(4+\epsilon)}\|v\|_{X^\delta_{s,1/2+\epsilon'}}, \end{aligned}$$

and

$$\begin{aligned} II &\leq C\|v\|_{X^\delta_{s,1/2+\epsilon'}}[\|J_x^s u_0\|_{L^4_{t,\delta}L^4_x}\|u_0\|_{L^{4+\epsilon'}_{t,\delta}L^{4+\epsilon'}_x}] \\ &\leq Cc^{1/(4+\epsilon)}N^{s(1-s)}\|v\|_{X^\delta_{s,1/2+\epsilon'}} \\ &\leq Cc^{1/(4+\epsilon)}\|v\|_{X^\delta_{s,1/2+\epsilon'}}. \end{aligned}$$

Finally we treat the last terms:

$$\begin{aligned} \|J_x^s K(u_0\bar{v})u_0\|_{L^{r_\epsilon}_{t,\delta}L^{r_\epsilon}_x} &\leq \|J_x^s u_0\|_{L^4_{t,\delta}L^4_x}\|K(u_0\bar{v})\|_{L^{(4+\epsilon')/2}_{t,\delta}L^{(4+\epsilon')/2}_x} \\ &\quad + \|u_0\|_{L^{4+\epsilon'}_{t,\delta}L^{4+\epsilon'}_x}\|K(J_x^s u_0\bar{v})\|_{L^{r_2}_{t,\delta}L^{r_2}_x} \\ &= I + II. \end{aligned}$$

Similar to what we have been doing:

$$\begin{aligned} I &\leq CN^{s(1-s)}\|u_0\bar{v}\|_{L^{(4+\epsilon')/2}_{t,\delta}L^{(4+\epsilon')/2}_x} \\ &\leq CN^{s(1-s)}\|u_0\|_{L^{4+\epsilon'}_{t,\delta}L^{4+\epsilon'}_x}\|v\|_{L^{4+\epsilon'}_{t,\delta}L^{4+\epsilon'}_x} \\ &\leq Cc^{1/(4+\epsilon)}N^{s(1-s)}\|v\|_{X^\delta_{s,1/2+\epsilon'}} \\ &\leq Cc^{1/(4+\epsilon)}\|v\|_{X^\delta_{s,1/2+\epsilon'}}, \end{aligned}$$

and $II \leq Cc^{1/(4+\epsilon)}\|v\|_{X^\delta_{s,1/2+\epsilon'}}$ which is already done in the sequel. The last term $\|J_x^s K(v\bar{u_0})u_0\|_{L^{r_\epsilon}_{t,\delta}L^{r_\epsilon}_x}$ also has the same bound.

So we found that $\|H(v)\|_{X^\delta_{s,1/2+\epsilon'}} \leq C\|v\|_{X^\delta_{s,1/2+\epsilon'}}N^a c^b$ where $a < 0$ and $b > 0$ thus we conclude that H is well defined that H maps $B^\delta_{M,N}$ into itself. A similar and straight forward argument shows that

$$[H(u_0) - H(v)] \leq CN^{a'}c^{b'}[u_0 - v],$$

for some $a' < 0$ and $b' > 0$ and hence H is a contraction mapping from $B^{\delta}_{M,N}$ into itself for large M, large N and small c, which is the local well-posedness result. $\square$

4. CONCLUSIONS

The aim of this work was to prove global existence for ACNLS with the initial datum below energy space using high-low frequency decomposition, which we couldn't accomplish. The reason why we couldn't do it was that we couldn't find the appropriate way to estimate the H^1 norm of the nonlinear term which appears in the integral equation. The problem was that Bourgain, making good use of the Bourgain spaces and the duality product, splits the derivative in two parts so that the less regular term is exposed to "less derivative", then he uses the bilinear estimates and shows that the nonlinear term is in H^1. While having three terms to differentiate, gave Bourgain the freedom to choose the regular term in differentiating, in our case for ACNLS, we couldn't find a way to put less derivative on the less regular term since there are only two terms . The nonlocal operator K binds two terms together so that, especially for the terms containing both the regular and the less regular terms, it becomes difficult to come up with a way to put less derivative to the less regular term.

Using the same technique plus the pseudo-conformal transformation, Bourgain also shows that there is scattering for the cubic NLS in L^2 with the initial datum in $H^{0,s}$ for $s > 2/3$. For ACNLS one can show global existence in H^s for the initial datum $\phi \in H^s$ with sufficiently small norm, and we believe that as one can show scattering in L^2 for ACNLS, for the initial datum with small L^4-norm, one can also show scattering in H^s or in $H^{0,s}$, under some well-posedness and smallness conditions.

APPENDIX A: BASIC DEFINITIONS AND ESTIMATES

Definition A.0.1. *[22] Bessel potential (J^s) and Riesz potential (D^s) are the operators defined as*

$$J^s u = \mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\hat{u} \quad and \qquad D^s u = \mathcal{F}^{-1}|\xi|^s \hat{u} \tag{A.1}$$

where the $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and whenever the latter makes sense.

Since we will use and there does not exist a unique definition of it, it may be useful to define Hardy-Littlewood Maximal function

Definition A.0.2. *For $f \in L^1_{loc}$ a Hardy-Littlewood maximal function is defined as follows*

$$Mf(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |f(y)|dy$$

where I is open and $|I| = \int \chi_I(y)dy$.

Definition A.0.3. *A function, f, is called a Schwartz function if it is infinitely differentiable and $x^\mu D^\gamma f \to 0$ as $|x| \to \infty$ for all nonnegative multiindices $\mu = (\mu_1, \mu_2)$ and $\gamma = (\gamma_1, \gamma_2)$ ($\mu_i, \gamma_i \geq 0$ for $i \in \{1, 2\}$), where $x^\mu = x_1^{\mu_1} x_2^{mu_2}$ and $D^\gamma f = \frac{d^{\gamma_1}}{dx_1^{\gamma_1}} \frac{d^{\gamma_2}}{dx_2^{\gamma_2}} f$.*

Definition A.0.4. *The space $\mathcal{S}'$ is the space of tempered distributions on $\mathcal{R}^2$, which means that $\mathcal{S}'$ is the topological dual of $\mathcal{S}$.*

Definition A.0.5. *For $m \in \mathbb{N}$, the Sobolev space $W^{m,p}$ is given by*

$$W^{m,p} = \{f \in L^p : D^\alpha u \in L^p \quad \forall\alpha \quad multiindex \quad such \quad that \quad |\alpha| \leq m\}$$

with the norm

$$\|u\|_{W^{m,p}} = \sum_{\substack{|\alpha|\leq m \\ \alpha \ multiindex}} \|D^{\alpha}u\|_{L^p}. \tag{A.2}$$

For $p = 2$ we call $W^{m,2} = H^m$ and since $p = 2$ we can characterize the Sobolev space using the Fourier transform, namely; given $m \in \mathbb{N}$ we can define

$$H^s = \{u \in \boldsymbol{S'} : (1 + |\xi|^2)^{s/2}\hat{u} \in L^2\}$$

with the norm

$$\|u\|_{H^s} = \|(1 + |\xi|^2)^{s/2}\hat{u}\|_{L^2} \tag{A.3}$$

where $\boldsymbol{S'}$ is the dual of Schwartz space. We can see that the requirement $s \in \mathbb{N}$ is just to make the definition consistent with the previous one, and we can extend this definition for the noninteger real positive number as the interpolation between the integer indiced Sobolev spaces; and to negative numbers by taking the dual of the positive indiced Sobolev spaces, see for further details [22].

Now, to define Besov spaces, consider a compactly supported function $\psi \in C_c^{\infty}(\mathbb{R}^2)$ such that, $supp(\psi) \subset \mathbb{R}^2 - \{0\}$ and $\sum_{-\infty}^{\infty} \psi(2^{-j}x) = 1$ and call $\psi_j(\xi) = \psi(2^{-j}\xi)$, namely; consider a radial function $\phi \in C_c^{\infty}(\mathbb{R}^2)$ such that

$$\phi(\xi) = 1 \ for \ |\xi| \leq 1 \ and \ \phi(\xi) = 0 \ for \ |\xi| \geq 2$$

then, define $\psi(\xi) = \phi(\xi) - \phi(2\xi)$, which satisfies the above conditions.

We are now, ready to define Besov spaces.

Definition A.0.6. *[22] The Besov space $B^s_{p,q}$ for $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ is the* *closure of* $\boldsymbol{S}'$ *with respect to the norm*

$$\|u\|_{B^s_{p,q}} = \|\mathcal{F}^{-1}(\phi\hat{u})\|_{L^p} + \begin{cases} \left(\sum_{j=1}^{\infty}(2^{sj}\|\mathcal{F}^{-1}(\psi_j\hat{u})\|_{L^p})^q\right)^{1/q} & if\ q < \infty \\ \sup_{j\geq 1} 2^{js}\|\mathcal{F}^{-1}(\psi_j\hat{u})\|_{L^p} & if\ q = \infty \end{cases} \tag{A.4}$$

The homogeneous Sobolev($\dot{H}^s$) and Besov($\dot{B}^s_{p,q}$) spaces are the closure of the Schwartz space, $\mathcal{S}$, under the seminorms

$$\|u\|_{\dot{H}^s} = \|u\|_{H^s} - \|u\|_{L^2}, \qquad \|u\|_{\dot{B}^s_{p,q}} = \|u\|_{B^s_{p,q}} - \|\mathcal{F}^{-1}(\phi\hat{u})\|_{L^p} \tag{A.5}$$

respectively.

There are important embedding results concerning these Sobolev and Besov spaces. For the Sobolev and Besov spaces we will need the following embedding results.

Theorem A.0.2. [23, Theorem 2.4.5] *Let* $m \geq 1$ *be an integer and* $1 \leq p < \infty$. *Then*

(1) if $1/p - m/n > 0$, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ *with* $1/q = 1/p - m/n$,

(2) if $1/p - m/n = 0$, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, *for* $p \leq q < \infty$,

(3) if $1/p - m/n < 0$, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$.

For the proof see [18] and [22].

Theorem A.0.3. *For* $s \in \mathbb{R}$, $2 \leq p < \infty$, *we have* $B^s_{p,2} \hookrightarrow W^{s,p}$

Again see [22] for details.

The basic space-time estimates, essential for solving the ACLNS are the Strichartz estimates, and to define it we should first introduce an admissible pair.

Definition A.0.7. *A pair* (q, r) *is admissible in* $\mathbb{R}^2$ *if*

$$1/q = 1/2 - 1/r$$

and $2 \leq r < \infty$. *We see that* $(\infty, 2)$ *is admissible, and the pair will correspond to some conserved quantity, so is important.*

Now if we denote $(\mathcal{T}_\beta(t))_{t\in\mathbb{R}}$ as the solution semigroup for the linear equation

$$iu_t + \beta u_{xx} + u_{yy} = 0,$$

we have;

Theorem A.0.4 (Strichartz' Estimates). *[7] If* (q, r) *is admissible, then the following properties hold;*

(P1) For every $\varphi \in L^2$*, the function* $t \mapsto \mathcal{T}_\beta(t)\phi$ *belongs to*

$$L^q(\mathbb{R}, L^r) \cap C(\mathbb{R}, L^2).$$

Moreover, there exist a constant C *such that*

$$\|\mathcal{T}_\beta(.)\varphi\|_{L^q(\mathbb{R},L^r)} \leq C\|\varphi\|_{L^2}.$$

(P2) Let I *be an interval of* $\mathbb{R}$, $J = \bar{I}$, *and* $0 \in J$. *If* (γ, ρ) *is an admissible pair and* $f \in L^{\gamma'}(I, L^{\rho'})$*, then for every* (q, r)*, the function* $t \mapsto \int_0^t \mathcal{T}_\beta(t-s)f(s)ds$ *for* $t \in I$*, belongs to* $L^q(\mathbb{R}, L^r) \cap C(\mathbb{R}, L^2)$ *and there exists a constant* C *depending on q,r,*γ *and* ρ *and is independent of* I *such that*

$$\| \int_0^t \mathcal{T}_\beta(t-s)f(s)ds\|_{L^q(I,L^r)} \leq C\|f\|_{L^{\gamma'}(I,L^{\rho'})}$$

see [7] for details.

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