ANALYSIS OF SINGLE AND TWO-ECHELON INVENTORY SYSTEMS UNDER DISRUPTIONS IN SUPPLY

by

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ANALYSIS OF SINGLE AND TWO-ECHELON INVENTORY SYSTEMS UNDER DISRUPTIONS IN SUPPLY

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ABSTRACT

ANALYSIS OF SINGLE AND TWO-ECHELON INVENTORY SYSTEMS UNDER DISRUPTIONS IN SUPPLY

In this thesis, we analyze two different models. In the first model, we consider a two-echelon supply chain with a supplier, a manufacturer and two retailers. The manufacturer is subject to non-stationary supply disruptions. The length of a supply unavailability duration is a non-stationary geometric type random variable. In every period the manufacturer places an order with the supplier by taking into account any possible supply disruptions in the planning horizon, and subsequently makes an allocation of available stock to retailers. At the retailer level, customer demand is observed and it is assumed to be deterministic but time-dependent. The aim is to find the optimal ordering policy for the manufacturer and the optimal allocation amounts to the retailers that will minimize expected system-wide costs over a finite planning horizon. We present a dynamic programming model and structural properties of the optimal ordering policy under a simplified allocation rule. The structural results that we obtain lead to an easy computational procedure for the optimal system-wide order-up-to level. We also discuss the effectiveness of the allocation rule through a numerical study.

In the second model, the environment is very similar to the first model, except we have a single echelon system. In the second model, we have a supplier and a manufacturer. The manufacturer is subject to stochastic demand and stochastic supplier availability. The supplier's availability structure is same as the supplier availability structure in the first model. Demand uncertainty is also modeled similar to supplier availability. Demand is either a fixed amount represented by d, or zero, with respective probabilities. On the contrary to the first model, there is no retailer in this model and demand is observed at manufacturer. The objective is to minimize expected holding and backlogging costs over a finite planning horizon considering stochastic demand amounts under the supply uncertainty. We present a dynamic programming model and a formula which explicitly determines the order-up-to levels. An algorithm is developed to compute the optimal inventory levels over the planning horizon using the formula. We also present a numerical study for the model.

ÖZET

TEK VE İKİ KATMANLI ENVANTER SİSTEMİNİN TEDARİK KESİNTİSİNDE ANALİZİ

Bu tez çalışmasında, iki farklı model inceledik. İlk modelde, bir tedarikçi, bir üretici ve üreticinin iki tane perakendecisinden oluşan iki kademeli bir tedarik zincirini araştırdık. Üretici durağan olmayan tedarik kesintilerine sahiptir. Tedarik kesintisinin süresinin uzunluğu durağan olmayan geometrik türde rassal bir değişkendir. Her dönem, üretici tedarikçiye planlama dönemi içindeki olası tedarik kesintilerini dikkate alarak sipariş verir ve daha sonra elindeki mevcut stoğun perakendicilere paylaştırmasını yapar. Müşteri talebi perakendeci düzeyinde oluşmaktadır ve müşteri talebinin deterministik ama zamana bağımlılığı olduğunu varsaymaktayız. Amacımız, sonlu bir planlama dönemi boyunca üreticinin tedarikçiye vereceği siparişler için optimal bir sipariş politikası bulmak ve sistem genelinde beklenen maliyetleri en aza indiren perakendiciler için optimum stok paylaştırma miktarlarını tespit etmektir. Basitleştirilmiş bir stok paylaştırma kuralı kullanarak bu model için dinamik programlama modelini sunduk ve bu modeli kullanarak optimal sipariş miktarının karakteristik özelliklerini gösterdik. Bu bulduğumuz karakteristiklerden yola çıkarak sistem için optimal sipariş verme düzeyini bulan bir algoritma geliştirdik. Sayısal bir çalışma ile basitleştirilmiş paylaştırma kuralının geçerliliğini de tartıştık.

Tek katmanlı bir envanter sistemi olması dışında ikinci modeldeki ortam ve parametreler ilk model ile çok benzerdir. İkinci modelde bir tedarikçi ve bir üretici vardır. Üretici stokastik talep ve tedariğe sahiptir. Tedariğin yapısı ilk model ile aynıdır. Talep belirsizliği de tedarik belirsizliğine benzemektedir. Talep belli olasılıklarla ile ya sabit d değerindedir ya da sıfırdır. Sistemde herhangi bir perakendeci yoktur ve talep ile üretici seviyesinde karşılaşılmaktadır. Amaç tedarik kısıntıları altında sonlu bir planlama dönemi için yok satma ve stok tutma maliyetlerini en aza indirmektir.

Bir dinamik programlama modeli oluşturduk ve sipariş verme düzlerini açıkça veren bir formül belirledik. Bu formülden yararlanarak optimum stok seviyelerini belirleyen bir algoritma geliştirdik. Model için sayısal bir analiz de sunduk.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

$b_{i,n}$	unit backlogging cost of Retailer i for unsatisfied demand at the end of period n where $i = 1, 2$ and $n = 1, 2, \ldots, N$
b_n	unit backlogging cost of manufacturer for unsatisfied demand at the end of period n, $n = 1, 2, \ldots, N$
D_n	demand observed at manufacturer in period n, $D_n = d$ with probability α_n and $D_n = 0$ with probability $1 - \alpha_n$ where $n = 1, 2, \ldots, N$
$D_i(n, n+k)$	total demand observed at Retailer i from period n till $n + k$
$D(n, n+k)$	total demand observed in the system from period n till $n+k$, $D(n, n+k) = D_1(n, n+k) + D_2(n, n+k)$ in chapter 3 and $D(n, n+k) = \sum_{i=0}^{+k} D_{n+i}$ in chapter 4
$h_{i,n}$	unit holding cost per unit residing at location i at the end of period n, where $i = 0$ for the manufacturer, $i = 1, 2$ for the retailers and $n = 1, 2, \ldots, N$
h_n	unit holding cost residing at manufacturer at the end of period $n = 1, 2, \ldots, N$
$I_{i,n}$	inventory position of Retailer i at the beginning of period n for $i = 1, 2$ and $n = 1, 2, \ldots, N$
I_n	inventory position of manufacturer at the beginning of period n for $n = 1, 2, \ldots, N$.
$\mathbf{I_n}$	$\mathbf{I_n} = (I_{0,n}, I_{1,n}, I_{2,n})$ is a vector that shows the system-wide inventory position and Retailer's inventory positions at the beginning of period n for $n = 1, 2, \ldots, N$.
$I_{0,n}$	system-wide inventory position at the beginning of period n for $n = 1, 2, \ldots, N$
n	indicates the period, $n = 1, 2, \ldots, N$
N	length of the planning horizon
p_n	probability that supplier is available in period n where $n = 1, 2, \ldots, N$, $0 \leq p_n \leq 1$
q_n	probability that supplier is unavailable in period n where $n = 1, 2, \ldots, N$, $q_n = 1 - p_n$

$S_n(u_n)$	realized order amount from supplier when order quantity is u_n in period n where $n = 1, 2, \ldots, N$
$u_{i,n}$	stock amount that is sent from manufacturer to Retailer i in period n where $i = 0, 1, 2$ and $n = 1, 2, \ldots, N$
u_n	order amount of manufacturer in period $n = 1, 2, \ldots, N$
$\mathbf{u_n}$	$\mathbf{u_n} = (u_{1,n}, u_{2,n})$ is a vector that indicates the stock allocation amounts of manufacturer to retailers in period n where $n = 1, 2, \ldots, N$
$Y_{i,n}$	inventory position of Retailer i after the replenishment from manufacturer and before the demand realization in period n, $Y_{i,n} = I_{i,n} + u_{i,n}$ where $i = 1, 2$ and $n = 1, 2, \ldots, N$
Y_n	inventory position at manufacturer after the replenishment from supplier and before the demand realization in period $n = 1, 2, \ldots, N$, $Y_n = I_n + S_n(u_n)$
$Y_{0,n}$	system-wide inventory position after the replenishment from supplier and before the demand realization in period $n = 1, 2, \ldots, N$, $Y_{0,n} = I_{0,n} + S_n(u_n)$
α_n	probability that demand value at the manufacturer is d in period n where $n = 1, 2, \ldots, N$
τ_n	random variable indicating that after period n at the first time inventory position is increased to its optimal value

1. INTRODUCTION

In typical studies that analyze optimal inventory policies, usually the customer demand, rather than the supply is the main focus of the study. In recent years, due to a more global and integrated manufacturing environment, analysis of supply uncertainty became important. Moreover, the availability of supply and the customer demand can be correlated. That is, if the customer demand is high in a season, due to raw material scarcity or capacity restrictions of the supplier, a disruption in supply may become more likely. Our interest in this thesis is to investigate inventory models under supply disruptions.

Global companies, in order to improve customer service levels and decrease lead time, operate with several distribution centers which are supplied by a central warehouse or a manufacturing facility. This type of distribution systems can be controlled centrally, or information about distribution centers can be monitored centrally. Although, these multi-echelon supply chain problems have been investigated for many years, the effect of supply uncertainty is not considered extensively. One of our aims in this thesis is to analyze a two-echelon supply chain which is supplied by an unreliable supplier. We consider a two-echelon supply chain with a supplier, a manufacturer and two retailers. Retailers are assumed to serve different customer segments, and therefore they are differentiated mainly by their unit backlogging costs. At the beginning of each period, the manufacturer places an order with the outside supplier. The main feature of our research is that, the supplier is unreliable and there is a possibility that the manufacturer may not be able to receive the quantity that it orders. Subsequent to manufacturer's order from the supplier, and after the realization of its receipt (if any), the manufacturer sends goods (after possible manufacturing and/or assembly activities) to the retailers. The manufacturer as well as the retailers are allowed to carry inventories. After receipt of the shipment by the retailers, customer demand occurs at retailer level and holding (at each echelon) and backlogging (at retailer level) costs are charged. The uncertainty structure of the supplier is simple, yet allows interesting modeling possibilities. Supply is either fully available or unavailable with certain probabilities. The supply availability probabilities are time-dependent, reflecting non-stationary supply disruptions or capacity shortages. Our supply process also resembles a supply system where the inter-delivery times for the quantities ordered are non-stationary random variables, and the supplier keeps track of the system-wide inventory position for delivering the total outstanding orders when supply becomes available. Demand values for the retailers over the planning horizon are assumed to be deterministic, but non-stationary. This assumption is reasonable for a number of applications, as retailers project their anticipated demands over a planning horizon, and then the manufacturer makes a plan for the ordering and distribution of goods based on these projected demand values. The system operates under a centralized decision making, and accordingly the manufacturer has full information on stock levels, cost and demand parameters of the retailers. In any period, when making an ordering and allocation decision, the manufacturer takes into account the supply uncertainty structure, and the manufacturer's aim is to minimize the total expected ordering and inventory related costs over a finite planning horizon.

We also consider a periodic review inventory problem for a manufacturer who has both stochastic demand and supply. As in the previous model the supplier is unreliable, and the supply disruption structure is same. However, unlike the previous model the manufacturer faces stochastic demand. This problem is a single stage inventory problem with stochastic supply and demand. Demand has a binomial type distribution. In other words, in each period with a certain probability demand is a fixed value d, or zero. At the beginning of each period manufacturer places an order with supplier not only considering the supply disruptions but also considering the demand possibilities. During the ordering decision manufacturer tries to minimize stock holding and backlogging costs. We assume that there is not any fixed ordering cost and unit purchasing cost. After the receipt of the order (if any) from the supplier manufacturer makes the production. Then, the demand is observed at the manufacturer (if any) and inventory related costs are incurred at the end of the period at the manufacturer. We assume that the lead time from supplier to manufacturer is zero and also production lead time is zero.

To the best of our knowledge, the literature on the inventory models treat supply uncertainty in a multi-echelon environment is very scarce. In the first model that we consider, we make three major contributions: i. We model a two-echelon supply chain under randomly distributed periods of supply disruptions. ii. We provide characterization and an easy-to-implement computational procedure for the optimal system-wide order-up-to level when a simple rule is used for allocating the stock to retailers. iii. We present a numerical study to test the effectiveness of the proposed allocation rule.

In the second model that we consider, the major contributions are as follows: i. We model an inventory system under random supply disruptions and demand. ii. We provide a new reformulation of the expected cost function. iii. We develop an algorithm to compute the optimal order up-to-levels for the manufacturer. iv. We present a numerical study.

In Chapter 2, we give a literature survey about inventory system with supply uncertainty and multi-echelon inventory systems. The details of the two-echelon inventory system with supply disruptions is given in Chapter 3. In Chapter 4, we analyze the single stage inventory problem with stochastic supply and demand parameters. In Chapter 5, we summarize our work and state some extensions for the problems.

2. LITERATURE SURVEY

In this thesis, we treat a two-echelon centralized supply chain with supply uncertainty and an inventory system with both supply and demand uncertainty in a single stage. Therefore, our research is related to two main tracks in the literature: models with supply uncertainty, and multi-echelon inventory systems. In what follows we review the most relevant papers from these two tracks.

The literature on supply or capacity uncertainty can roughly be categorized into three groups. One of these groups handles supply uncertainty as randomness in yield. Yano and Lee, 1995 provide a comprehensive review of literature on determining lot sizes when production or procurement yield is random in both continuous and periodic review systems. Gerchak et al., 1988 study a finite horizon problem with stationary, stochastic demand and stationary, stochastically-proportional random yield. They prove that for a single period problem, order point is independent of yield uncertainty and it is same as in constant yield model. They also show that the finite horizon problem converges the infinite problem as number of period increases and the order point in infinite horizon problem is not smaller than the order point of the certain yield problem. Henig and Gerchak, 1990 extend Gerchak et al., 1988 and prove the existence of critical order levels for both finite and infinite horizon problems and their independence from random yield. Erdem and Ozekici, 2002 model a system where yield is uncertain due to randomness in capacity of supplier. The major finding of Erdem and Ozekici is that a base stock policy is optimal for single, multiple and infinite periods problems and the order-up-to level depends on the state of the environment. In a recent paper Arifoglu and Ozekici, 2010 consider a model with fixed capacity and Markov modulated random yield.

The second group of papers for supply uncertainty treats supply uncertainty as capacity uncertainty. Gullu, 1998 considers the capacity uncertainty under stochastic demand and utilizes the analogy between the class of base-stock production/inventory policies that operate under demand/capacity uncertainty, and the $G/G/1$ queues. Ciarallo et al., 1994 show the optimality of order-up-to type policies in the presence of random capacity in single, multiple and infinite periods. Extending the model of Ciarallo et al., 1994, Wang and Gerchak, 1996 explore the implications of random yields and variable capacity jointly for a finite-horizon periodic review inventory system and show that the optimal policy is re-order type. Iida, 2002 develops upper and lower bounds on the optimal policies for the infinite horizon non-stationary production-inventory problem with an uncertain capacity constraint.

The third group of papers consider random disruptions in supply, i.e. models the supply uncertainty as two different availability state which are randomly realized. Parlar and Berkin, 1991 analyze the continuous time problem which is very hard to structure. Assuming that the supply and no supply periods are exponentially distributed or supply horizon is exponentially distributed and no supply period is deterministic, they find optimal order quantities by using the renewal theory. Parlar and Perry, 1995 add cost for learning the state of the supplier to Parlar and Berkin, 1991 problem environment. The problem in this case is not only when and how much stock to order but also how much to wait to order again. Parlar and Perry, 1996 extend Parlar and Berkin, 1991 to multiple suppliers and observe that as the number of suppliers gets larger, the problem converges to classical EOQ model. Parlar, 1997, Mohebbi, 2004 and Mohebbi and Hao, 2006 extend Parlar and Berkin's, 1991 paper by adding non-zero random lead time under random demand. Parlar et al., 1995 investigate the stochastic demand problem and impose dependency between available and unavailable periods and show that optimal policy is (s,S) type. Ozekici and Parlar, 1999 show the optimality of base stock policy when fixed ordering cost is zero, and (s,S) policy when fixed ordering cost is non-zero. Song and Zipkin, 1996 analyze the Markovian model of supply and random lead time and find out that classical policies are still optimal with dynamically changing parameters. Gullu et al., 1997 analyze Bernoulli distributed supplier availability. They show that the order up to level policy is optimal and find newsboy problem similar formula to find optimal inventory level. In 1999, they extend the study to partial availability. The first model in this thesis can be considered as an extension of Gullu et al., 1997, and Gullu et al., 1999 where the general set-up of the problem is similar but a multi-echelon system is considered. The second model in this thesis is another extension of Gullu et al., 1997, and Gullu et al., 1999 where the customer demand is stochastic.

The details of multi-echelon inventory systems with deterministic demand and no supply uncertainty can be found in Muckstadt et al., 1993. In their pioneering work, Clark and Scarf, 1960 study a periodic review serial installation system. They define "echelon stock" as sum of all stock in the given node and all stock in lower nodes with including transit stock. As in our study of Chapter 3, demand occurs at the lowest echelon and excess demand is backordered. They prove the optimality of an order-up-to type policy in each echelon. Federgruen and Zipkin, 1984a approximate a central depot and several retailers with stochastic demand by a single location inventory system problem. In their setting, central depot does not hold any inventory and just distributes the stock to retailers. Similar to our work, they use a myopic stock allocation rule. Federgruen, 1993 gives a comprehensive review for multi-echelon inventory systems. Diks and Kok, 1998 analyze a continuous review divergent multi-echelon inventory system. Their system is similar to Clark and Scharf's, 1960 and they use the equal probability of stock-out allocation approach similar to Eppen and Schrage, 1981. Using a decomposition approach they find the optimal replenishment policy for each echelon which is again order-up-level type. Muckstadt and Roundy, 1993 analyze serial, assembly and distribution systems with constant demand rate and focus on reorder intervals rather than lot sizes. A review on multi-echelon assembly and distribution systems is Houtum et al., 1996.

Distribution systems can be categorized according to the stage where the excess stock is kept and also according to transshipment types between downstream nodes. Our model can be considered as an extension of Eppen and Schrage, 1981 to a setting with supplier uncertainty and deterministic demand. In their paper depot does not hold any stock where in our case depot holds excess stock. According to the decision structure distribution systems can be categorized as centralized systems and decentralized systems. Eppen, 1979 compares a decentral system with two retailers normally distributed random demand, that they give their orders independently from each other considering minimization of their own backorder and holding costs, with a centralized system with two retailer and a central warehouse. In the centralized setting, ordering decision is given together to minimize the system backorder and holding cost to satisfy the total system demand (cumulation of two normally distributed demand) from the central warehouse. Then, he concludes that centralized problem outperforms the decentral system. Chen and Lin, 1989 extend Eppen, 1979 with concave holding and penalty cost. Cherikh, 2000 shows that the central system in classical newsboy problem environment for single period is more profitable rather than less costly. Gross, 1963 analyzes ordering and transshipment rules in a two location distribution system. In this paper transshipment is possible before demand realization. The rationale for making this transshipment is to have a balanced stock among the locations. So, backorder cost is also minimized. Das, 1975 modifies Gross's work to include transshipment possibility in the middle of a period. Another paper, in which transshipment is allowed, is Tagaras', 1989 paper. In this problem setting, the source of two lower locations has infinite capacity and meet the order of both locations. However, if the realized stochastic demand of a location is higher than its order up-to level but the realized demand is smaller than the order up-to level of other location, then the excess stock in the second location can be transferred to first location which not only decreases the average expected cost but it also increases the service levels in locations. However, the marginal effect of pooling decreases as the number of locations increases, by Cai and Du, 2009. Gerchak and He, 2003 and Berman et al., 2010 analyze the effect of demand variability on the benefit of pooling. As a result of these papers on risk/stock pooling, it can be stated that pooling stocks into a central warehouse decreases the expected cost as well as increases the profitability since it decreases the risk of uncertain demand. Having a higher demand in a location can be counterbalanced by having a lower demand in another location. Additionally, Axsäter, 2003 gives a recent review of multi-echelon distribution systems. Since in the literature centralized systems is referred as efficient systems, we also consider a centralized distribution system in this thesis.

In multi-echelon problems, two or more different retailers are supplied from a common manufacturer who has limited stock. Thus we have stock allocation problem, i.e, we should make an allocation of the limited stocks to retailers. In literature, stock allocation problem is investigated in several articles in different contexts. Frank et al., 1999 find optimal ordering policy for two demand classes where first one has deterministic demand structure whose demand should be satisfied fully but the second demand class has uncertainty whose demand can be satisfied partially resulting in lost sales. This approach is similar to our allocation policy in the sense that first demand class has priority over the second but we have back-orders instead of lost sales. Nahmias and Demmy, 1981 study high and low demand classes stock rationing considering a support level K for high priority demand class. When stock on hand is equal to or smaller than the support level K, then the low priority demand class starts to be backordered. Fill rates of systems with and without support level is compared in the paper and expected backorder rate expressions are derived. Similarly, Ha, 1997 has also two demand classes and first one has priority over the second one. Again, Ha, 1997 applies stock rationing rule and even there is stock or production the low priority demand class may not be satisfied in order to reserve some stock to high demand class customers. He applies the queueing theory and using the monotone switching curve solves the problem. De Véricourt, 2002 extends Ha, 1997 to N demand classes and gives the characterization of stock allocation policy in a make-to-stock environment with several customers and also provide an algorithm to find optimal parameters to be used in stock allocation. Arslan et al., 2007 analyze a similar problem to Nahmias and Demmy, 1981. They change allocation rule in such a way that when a replenishment comes even the stock on-hand inventory does not reach the critical ratio, stock is not reserved to higher demand classes but stock is firstly allocated to satisfy the backorders. Backorders are satisfied in the order of occurrence, i.e, first backorder is satisfied firstly. They show the equivalence of this system to serial inventory system and develop an heuristic. Cachon and Lariviere, 1999 treat stock allocation from a different perspective than other papers, in the sense that in their setting supplier does not know the demand figures of the retailers. Supplier only makes the allocation to retailers according to the received order quantities from the retailers. They find out that in this setting retailers can distort their orders to get a higher portion from available stock. Additionally, not only allocated quantity is manipulated but also supplier's capacity estimation is misled. They propose to use pricing mechanism to have a undistorted order quantities from retailers.

Based on our review of the literature above, we combine ideas from supply uncertainty, multi-echelon systems and stock rationing literature in a novel manner.

3. ANALYSIS OF A TWO-ECHELON SUPPLY CHAIN WITH DISRUPTIONS IN SUPPLY AND DETERMINISTIC DEMAND

In the first part of this Chapter 3, we develop the dynamic model of the problem. In the second part we propose a simplified stock allocation policy for the problem. Then we modify the dynamic model that we develop in the first part according to the simplified allocation policy. In the fourth part, we construct an algorithm to calculate the order-up-to level for each period. Finally, in the fifth part we make the numerical analysis on the problem.

3.1. Development of the Dynamic Model

At the beginning of any period n, the manufacturer observes the inventory positions of Retailer 1, $I_{1,n}$ and Retailer 2, $I_{2,n}$, and also the system-wide inventory level $I_{0,n}$. Then, the manufacturer places an order of size u_n with the supplier. Let $S_n(u_n)$ be the random variable denoting the amount received from the supplier. In this thesis, the supply uncertainty structure assumes that $S_n(u_n) = u_n$ with probability p_n and $S_n(u_n) = 0$ with probability $q_n = 1 - p_n$. The manufacturer pays c per each received unit. After the realization of the supply, the system-wide inventory raises to $Y_{0,n} := I_{0,n} + S_n(u_n)$. Then, the manufacturer makes a stock allocation to the retailers by amounts $u_{1,n}$ and $u_{2,n}$, which increases the stock level of retailer i to a level $Y_{i,n} := I_{i,n} + u_{i,n}$, $i = 1, 2$. Let $D_i(n, n+k)$ be the total demand observed at retailer i over the periods $n, n+1, \ldots, n+k$. In particular, $D_i(n, n)$ is the demand of retailer i in period n. We also let $D(n, n+k) = D_1(n, n+k) + D_2(n, n+k)$ be the total system-wide demand over the periods $n, n+1, \ldots, n+k$. After the receipt of the allocation quantities by the retailers, demands are observed, and relevant end-of-period inventory holding and backlogging costs are charged at each location. Let $h_{i,n}$ be the holding cost per unit residing at location i at the end of period n, where $i = 0$ for the manufacturer and $i = 1, 2$ for the retailers. Also let $b_{i,n}$ be the unit backlogging cost for Retailer i,

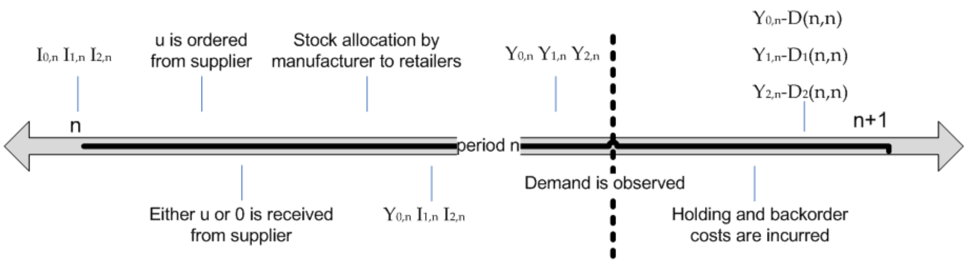

Figure 3.1. Time and Order of Events

charged for the units which are short at the end of period n. Backlogging cost is only charged at the retailer level. We assume that the lead-time from the supplier to the manufacturer and the lead-times from the manufacturer to the retailers are negligible. Accordingly, let $L_n(Y_{0,n}, \mathbf{Y_n})$ be the single period cost function associated with the decision $\mathbf{Y_n} = (Y_{1,n}, Y_{2,n})$:

$$
\begin{aligned}
L_n(Y_{0,n}, \mathbf{Y_n}) &= h_{0,n}(Y_{0,n} - \sum_{i=1}^{2} Y_{i,n})^+ + \sum_{i=1}^{2} h_{i,n}(Y_{i,n} - D_i(n,n))^+ \\
&+ \sum_{i=1}^{2} b_{i,n}(D_i(n,n) - Y_{i,n})^+. \qquad (3.1)
\end{aligned}
$$

Figure 3.1 illustrates the timing and order of various events and decisions within a period.

For notational convenience, set $\mathbf{I_n} = (I_{0,n}, I_{1,n}, I_{2,n})$, and let $C_n(\mathbf{I_n})$ be the minimum expected cost of operating the system in periods $n, n+1, \ldots, N+1$ with a starting inventory vector of $\mathbf{I_n}$, where N is the length of the planning horizon with $C_{N+1}(\mathbf{I_{N+1}}) \equiv 0$. Furthermore, let $\mathbf{u_n} = (u_{1,n}, u_{2,n})$ and define the set $A(\mathbf{I_n}, u_n)$ as:

$$
A(\mathbf{I_n}, u_n) = \{\mathbf{u_n} : \mathbf{u_n} \geq 0 \ \ , \sum_{i=1}^{2}(I_{i,n} + u_{i,n}) \leq I_{0,n} + S_n(u_n)\}.
$$

Essentially, given the inventory levels, $\mathbf{I_n}$, and the order amount, u_n, $A(\mathbf{I_n}, u_n)$ defines the set of feasible allocation amounts to the retailers. Then, $C_n(\mathbf{I_n})$ can be written as:

$$\begin{aligned} C_n(\mathbf{I_n}) &= \min_{u_n \geq 0} E[cS_n(u_n) + \min_{\mathbf{u_n} \in A(\mathbf{I_n}, u_n)} \{L_n(I_{0,n} + S_n(u_n), I_{1,n} + u_{1,n}, I_{2,n} + u_{2,n}) \\ &+ C_{n+1}(I_{0,n} + S_n(u_n) - D(n,n), I_{1,n} + u_{1,n} - D_1(n,n), I_{2,n} + u_{2,n} - D_2(n,n))\}]. \end{aligned} \tag{3.2}$$

At the beginning of the period n, the manufacturer places an order of size u_n (the outer minimization), receives $S_n(u_n)$, and pays the purchasing cost $cS_n(u_n)$. Then, the manufacturer decides on the allocation amounts (the inner minimization) and the single period cost L_n is incurred. The new state $\mathbf{I_{n+1}} = (I_{0,n} + S_n(u_n) - D(n,n), I_{1,n} + u_{1,n} - D_1(n,n), I_{2,n} + u_{2,n} - D_2(n,n))$ is observed and C_{n+1} is incurred. The expectation is taken over $S_n(u_n)$ since it is a random variable. Letting $Y_{0,n} = I_{0,n} + u_n$ and $Y_{i,n} = I_{i,n} + u_{i,n}$, and taking the expectation in the above expression yields

$$\begin{aligned} C_n(\mathbf{I_n}) &= \min_{Y_{0,n} \geq I_{0,n}} \{cY_{0,n}p_n - cI_{0,n}p_n \\ &+ p_n \min_{\mathbf{Y_n} \in B(Y_{0,n}, I_{1,n}, I_{2,n})} \{L_n(Y_{0,n}, \mathbf{Y_n}) \\ &+ C_{n+1}(Y_{0,n} - D(n,n), Y_{1,n} - D_1(n,n), Y_{2,n} - D_2(n,n))\} \\ &+ q_n \min_{\mathbf{Y_n} \in B(I_{0,n}, I_{1,n}, I_{2,n})} \{L_n(I_{0,n}, \mathbf{Y_n}) \\ &+ C_{n+1}(I_{0,n} - D(n,n), Y_{1,n} - D_1(n,n), Y_{2,n} - D_2(n,n))\}\}, \end{aligned} \tag{3.3}$$

where

$$B(I, I_{1,n}, I_{2,n}) = \{\mathbf{Y_n} : Y_{i,n} \geq I_{i,n} \; i = 1, 2 \;, \sum_{i=1}^{2} Y_{i,n} \leq I\}.$$

We define an auxiliary function $G_n(Y_{0,n}, I_{1,n}, I_{2,n})$ for $n = 1, 2, \ldots, N$ as:

$$\begin{aligned} G_n(Y_{0,n}, I_{1,n}, I_{2,n}) &= \min_{\mathbf{Y_n} \in B(Y_{0,n}, I_{1,n}, I_{2,n})} \{L_n(Y_{0,n}, \mathbf{Y_n}) + cY_{0,n} + C_{n+1}(Y_{0,n} - D(n,n), \\ & Y_{1,n} - D_1(n,n), Y_{2,n} - D_2(n,n))\}. \end{aligned} \tag{3.4}$$

By using the auxiliary function (3.4), and the fact that $cp_nY_{0,n} + cq_nI_{0,n} - cI_{0,n} =$

$cp_nY_{0,n} - cp_nI_{0,n}$, (3.3) can be rewritten as

$$C_n(\mathbf{I_n}) = p_n \min_{Y_{0,n} \geq I_{0,n}} \{G_n(Y_{0,n}, I_{1,n}, I_{2,n})\} + q_nG_n(\mathbf{I_n}) - cI_{0,n}. \tag{3.5}$$

It is certainly possible to evaluate the optimal policy of the problem numerically, by using a backward dynamic programming algorithm which utilizes (3.5), (3.4), and the starting solution $C_{N+1}(\mathbf{I_{N+1}}) = 0$. However, it is very difficult to obtain analytical insight and derive structural properties of the optimal solution due to complex nature of the problem. The main difficulty is the existence of the inner minimization solving the allocation problem. In the next section we propose a plausible simplified model that resolves the allocation problem in a particular way.

3.2. A Simplified Allocation Policy

In this section we present a simple allocation policy for the problem in Section 1. Specifically this policy assumes that

i. The manufacturer tries to satisfy the current period's demand as much as possible using its inventory on-hand. That is, there is no intentionally reserving stock for the future periods at the expense of a retailer.

ii. The holding cost of the manufacturer is smaller than the holding costs of retailers, $h_{0,n} \leq h_{i,n}$ for $i = 1, 2$. Therefore, there is no economies of scale in transferring the excess amount to the retailers as it is more costly to keep stock at the retailer level. Therefore, the manufacturer increases stock levels of the retailers at most up to their current demand values, $Y_{i,n} \leq D_i(n, n)$ for $i = 1, 2$. The implication of this assumption is that the single period cost function changes as

$$L_n(Y_{0,n}, \mathbf{Y_n}) = h_{0,n}(Y_{0,n} - D(n, n))^+ + \sum_{i=1}^{2} b_{i,n}(D_i(n, n) - Y_{i,n})^+. \tag{3.6}$$

Since $Y_{i,n} \leq D_i(n, n)$ for $i = 1, 2$ and provided that the retailers have zero stock at the beginning of the planning horizon, inventory levels of the retailers at the beginning of any period can be at most zero. That is, $I_{i,n+1} = Y_{i,n} - D_i(n,n) \leq 0$.

iii. The manufacturer makes a prioritization between retailers based on their unit backlogging costs. Without loss of generality assume that $b_{1,n} \geq b_{2,n}$. The manufacturer, accordingly first tries to meet the demand of Retailer 1. After satisfying the demand of Retailer 1, if manufacturer has still on-hand stock then the demand of Retailer 2 is tried to be met as much as possible.

Although the allocation policy is quite simple and myopically decides on the quantities to be sent to retailers, in many industries it is very common to define customer or retailer priorities in case of shortages in capacity or available inventory. Most of the time, because a customer is more profitable or has more trade volume, or simply has a strict contractual agreement, he can have a higher priority than others. When allocating available stock to orders, customer priority is essential, and we adopt the same customer priority logic into our allocation rule. We assume that the first retailer is more important than the second one, and this importance is quantified with a higher backlogging cost. The optimal policy may be such that, after satisfying the demand of Retailer 1, if there is still stock, the manufacturer reserves some or all of this stock to satisfy the possible demands of Retailer 1 in subsequent periods, by deliberately forcing a backlog at Retailer 2. Even though such a policy may be favorable from a cost point of view, it would be quite difficult to implement it in an actual distribution setting, as this would cause a considerable loss of customer goodwill at Retailer 2. Therefore, the simple allocation rule that we impose is reasonable from the implementation point of view.

We should note that under our allocation rule, the "inner optimization" problem in $C_n(\mathbf{I_n})$ is eliminated and the manufacturer only decides on how many units to order, and then the inventory on hand is allocated to retailers by the policy explained above. Our aim is to provide a restructuring of the problem under this allocation policy, and then to present a characterization of the optimal system-wide stock replenishment quantity.

3.3. A Restructuring of the Optimization Model

We first note that, by using the proposed allocation policy a state reduction is possible. Mainly, $I_{1,n}$ can be eliminated from the state description of the dynamic programming model (3.5). That is, by only knowing $Y_{0,n}$ and $I_{2,n}$, we can make the allocation and determine stock levels of retailers before demand realization. Suppose that after replenishment from the supplier, the system-wide inventory level is $Y_{0,n}$. Then, after the allocation the stock level of Retailer 1 will raise to $Y_{1,n} = min(D_1(n,n), Y_{0,n} - I_{2,n})$. Similarly, the post-allocation stock level of Retailer 2 will be $Y_{2,n} = min(D_2(n,n), Y_{0,n} - Y_{1,n}) = min(D_2(n,n), Y_{0,n} - min(D_1(n,n), Y_{0,n} - I_{2,n})) = max(I_{2,n}, min(Y_{0,n} - D_1(n,n), D_2(n,n)))$. Therefore, it is enough to know the total system-wide stock after replenishment from the supplier, $Y_{0,n}$, and inventory level of Retailer 2 at the beginning of the period, $I_{2,n}$, in order to find the post-allocation inventory levels, $Y_{1,n}$ and $Y_{2,n}$, of retailers. With that in mind, the auxiliary function (3.4) can be redefined as $G_n(Y_{0,n}, I_{2,n})$ for $n = 1, 2, \ldots, N$, having state variables as the system-wide inventory level after replenishment from the supplier, $Y_{0,n}$ and inventory level at Retailer 2 at the beginning of period n, $I_{2,n}$:

$$G_n(Y_{0,n}, I_{2,n}) = \begin{cases} L_n(Y_{0,n}, Y_{0,n} - I_{2,n}, I_{2,n}) + C_{n+1}(Y_{0,n} - D(n,n), I_{2,n} - D_2(n,n)) \\ \quad +cY_{0,n}\ , \qquad \text{if } Y_{0,n} \le D_1(n,n) + I_{2,n} \\ L_n(Y_{0,n}, D_1(n,n), Y_{0,n} - D_1(n,n)) + C_{n+1}(Y_{0,n} - D(n,n), Y_{0,n} - \\ \quad D(n,n)) + cY_{0,n}\ , \qquad \text{if } D_1(n,n) + I_{2,n} \le Y_{0,n} \le D_1(n,n) \\ L_n(Y_{0,n}, D_1(n,n), Y_{0,n} - D_1(n,n)) + C_{n+1}(Y_{0,n} - D(n,n), Y_{0,n} \\ \quad -D(n,n)) + cY_{0,n}\ , \qquad \text{if } D(n,n) \ge Y_{0,n} \ge D_1(n,n) \\ L_n(Y_{0,n}, D_1(n,n), D_2(n,n)) + C_{n+1}(Y_{0,n} - D(n,n), 0) + cY_{0,n}\ , \\ \qquad \text{if } Y_{0,n} \ge D(n,n) \end{cases} \tag{3.7}$$

where analogous to (3.5)

$$C_n(I_{0,n}, I_{2,n}) = p_n \min_{Y_{0,n} \ge I_{0,n}} \{G_n(Y_{0,n}, I_{2,n})\} + q_n G_n(I_{0,n}, I_{2,n}) - cI_{0,n}, \tag{3.8}$$

and $C_{N+1}(I_{0,N+1}, I_{2,N+1}) = 0$. The four regions over which $G_n(Y_{0,n}, I_{2,n})$ is defined can be explained as follows:

- Case 1: If $Y_{0,n} \leq D_1(n,n) + I_{2,n}$, then the manufacturer cannot satisfy the demand of Retailer 1 since $Y_{0,n} - I_{2,n} \leq D_1(n,n)$. So, the whole stock at the manufacturer is sent to Retailer 1, increasing its inventory level to $Y_{0,n} - I_{2,n}$.
- Case 2: If $Y_{0,n} \leq D_1(n,n)$ and $Y_{0,n} - I_{2,n} \geq D_1(n,n)$, then the manufacturer can satisfy the demand of Retailer 1, and increases its inventory level to $D_1(n,n)$. The remaining stock is sent to Retailer 2, increasing its inventory level to $Y_{0,n} - D_1(n,n)$.
- Case 3: If $D(n,n) \geq Y_{0,n} \geq D_1(n,n)$, then the on-hand stock at the manufacturer is not enough to satisfy the total demand of both retailers. According to our allocation rule, the manufacturer fully satisfies the demand of Retailer 1, that is increasing its stock level to its demand value, $D_1(n,n)$, then sends the remaining stock on-hand to Retailer 2, increasing its stock level to $Y_{0,n} - D_1(n,n)$.
- Case 4: If $Y_{0,n} \geq D(n,n)$, then the total system-wide stock is enough to satisfy the total demand in period n, and the manufacturer increases the stock levels of retailers to their respective demand values.

We should note that the post-allocation stock levels of the retailers are independent of $I_{2,n}$, as long as $Y_{0,n} \geq D_1(n,n)$.

3.4. Characterization of the Optimal Ordering Policy

Our first observation is that $G_n(Y_{0,n}, I_{2,n})$ is not convex in $Y_{0,n}$ for a given $I_{2,n}$. Considering a problem with $N = 5$, $c = 1$, and $h_{0,n} = 3$, $b_{1,n} = 20$, $b_{2,n} = 2$, $D_1(n,n) = 5$, $D_2(n,n) = 3$, $p_n = 0,40$ for all $n = 1, 2, \ldots, 5$, Figure 3.2 presents $G_1(Y_{0,1}, I_{2,1})$ for different values of $Y_{0,1}$ when $I_{2,n} = 0$, and we can clearly observe the kinks in the function.

A closer inspection of Figure 3.2 also reveals that the kinks in G_1 occur at points $5, 8, 13, 16, 21, \ldots$, and these points correspond to cumulative retailer demands D_1, D_1+

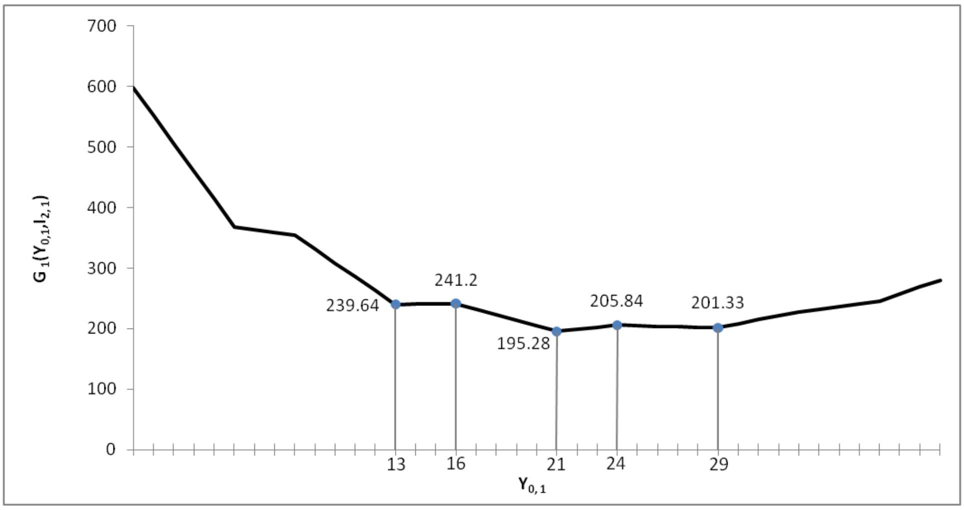

Figure 3.2. Non-convexity of Function G_1

$D_2, 2D_1 + D_2, 2D_1 + 2D_2, 3D_1 + 2D_2, \ldots$. In fact, for the last period (period N) in the planning horizon, G_N can be written as:

$$G_N(Y_{0,N}, I_{2,N}) = \begin{cases} L_N(Y_{0,N}, Y_{0,N} - I_{2,N}, I_{2,N}) + cY_{0,N} \\ \quad \text{if } Y_{0,N} \leq D_1(N,N) + I_{2,N} \\ L_N(Y_{0,N}, D_1(N,N), Y_{0,N} - D_1(N,N)) + cY_{0,N} \\ \quad \text{if } D_1(N,N) + I_{2,N} \leq Y_{0,N} \leq D_1(N,N) \\ L_N(Y_{0,N}, D_1(N,N), Y_{0,N} - D_1(N,N)) + cY_{0,N} \\ \quad \text{if } D(N,N) \geq Y_{0,N} \geq D_1(N,N) \\ L_N(Y_{0,N}, D_1(N,N), D_2(N,N)) + cY_{0,N} \\ \quad \text{if } Y_{0,N} \geq D(N,N) \end{cases} . \quad (3.9)$$

and it can easily be seen that $G_N(Y_{0,N}, I_{2,N})$ is minimized at $Y^*_{0,N} = D(N,N)$ *for any* given $I_{2,N} \leq 0$ since $c \leq b_{i,N}$ for $i = 1, 2$. Note that, the minimizer $Y^*_{0,N} = D(N,N)$ is independent of $I_{2,N}$, and it corresponds to a cumulative retailer demand. Also, from $C_N(I_{0,N}, I_{2,N}) = p_N \min_{Y_{0,N} \geq I_{0,N}}\{G_N(Y_{0,N}, I_{2,N})\} + q_N G_N(I_{0,N}, I_{2,N}) - cI_{0,N}$, it can be concluded that $C_N(I_{0,N}, I_{2,N})$ is minimized at $I^*_{0,N} = Y^*_{0,N} = D(N,N)$. Our next aim is to utilize these observations for an easily computable characterization of the optimal system-wide order-up-to level. Our exposition consists of the following steps:

i. We inductively show that the optimal system-wide order-up-to level of period n is greater than or equal to the total demand of period n. Therefore, this optimal level, $Y^*_{0,n}$ is independent of the inventory level of the retailers.

ii. We also show that $Y^*_{0,n} \leq Y^*_{0,n+1} + D(n,n)$. That is, there is a bound on the system-wide order-up-to level for period n, and this bound depends on the order-up-to level of the succeeding period.

iii. We obtain slopes of G_n in between the kink points (see Figure 3.2) and utilize these slopes for finding the optimal solution. For instance, in Figure 3.2, the slope of G_1 for $13 \leq Y_{0,1} \leq 16$ is $0.52 = (241.2 - 239.6)/3$. Let δ be the slope of G_n for a given interval. Note that these slopes very much depend on the regions of $Y_{0,n}$ in equation (3.7). In order to be able to obtain δ values for different values of $Y_{0,n}$, we need to be able to rewrite $G_n(Y_{0,n}, I_{2,n})$ in a different form. In the following theorem, we provide these alternative expressions for $G_n(Y_{0,n}, I_{2,n})$, for $Y_{0,n} \leq Y^*_{0,n+1} + D(n,n)$ (equations (3.10)-(3.14)).

Proposition 3.1. *Define τ_n as the random variable that indicates the first period after n that the supplier becomes fully available. Therefore, $\{\tau_n = i\}$ means that the supplier becomes available the first time in period $n+i$. Note that, $P(\tau_n = l) = p_{n+l}\prod_{j=1}^{l-1} q_{n+j}$ for $l = 1, 2, \ldots$ with $\prod_{j=1}^{0} = 1$. Suppose that $Y_{0,n} \leq Y^*_{0,n+1} + D(n,n)$. Then, $G_n(Y_{0,n}, I_{2,n})$ can be reexpressed by using the random variable τ_n as in equations (3.10)-(3.14) for different intervals of $Y_{0,n}$ for period n. Note that $\sum_{i=1}^{0} = 0$.*

- *Case 1: If $Y_{0,n} \leq D_1(n,n) + I_{2,n}$:*

$$
\begin{aligned}
G_n(Y_{0,n}, I_{2,n}) &= L_n(Y_{0,n}, Y_{0,n} - I_{2,n}, I_{2,n}) \\
&+ \sum_{i=1}^{N-n} L_{n+i}(Y_{0,n} - D(n,n+i-1), Y_{0,n} - I_{2,n} - D_1(n,n+i-1), \\
&I_{2,n} - D_2(n,n+i-1))P(\tau_n > i) \\
&+ \sum_{i=1}^{N-n} G_{n+i}(Y^*_{0,n+i}, I_{2,n} - D_2(n,n+i-1))P(\tau_n = i) \\
&+ \sum_{i=1}^{N-n} cD(n,n+i-1)P(\tau_n = i) \\
&+ cY_{0,n}P(\tau_n > N-n).
\end{aligned} \quad (3.10)
$$

- *Case 2: If $D_1(n,n) + I_{2,n} \leq Y_{0,n} \leq D_1(n,n)$:*

$$\begin{aligned}
G_n(Y_{0,n}, I_{2,n}) = {} & L_n(Y_{0,n}, D_1(n,n), Y_{0,n} - D_1(n,n)) \\
& + \sum_{i=1}^{N-n} L_{n+i}(Y_{0,n} - D(n,n+i-1), D_1(n,n) - D_1(n,n+i-1), \\
& Y_{0,n} - D_2(n,n+i-1) - D_1(n,n))P(\tau_n > i) \\
& + \sum_{i=1}^{N-n} G_{n+i}(Y^*_{0,n+i}, Y_{0,n} - D_2(n,n+i-1) - D_1(n,n))P(\tau_n = i) \\
& + \sum_{i=1}^{N-n} cD(n,n+i-1)P(\tau_n = i) \\
& + cY_{0,n}P(\tau_n > N-n).
\end{aligned} \tag{3.11}$$

- *Case 3: If $D_1(n,n) \leq Y_{0,n} \leq D(n,n)$:*

$$\begin{aligned}
G_n(Y_{0,n}, I_{2,n}) = {} & L_n(Y_{0,n}, D_1(n,n), Y_{0,n} - D_1(n,n)) \\
& + \sum_{i=1}^{N-n} L_{n+i}(Y_{0,n} - D(n,n+i-1), D_1(n,n) - D_1(n,n+i-1), \\
& Y_{0,n} - D_2(n,n+i-1) - D_1(n,n))P(\tau_n > i) \\
& + \sum_{i=1}^{N-n} G_{n+i}(Y^*_{0,n+i}, Y_{0,n} - D_2(n,n+i-1) - D_1(n,n))P(\tau_n = i) \\
& + \sum_{i=1}^{N-n} cD(n,n+i-1)P(\tau_n = i) \\
& + cY_{0,n}P(\tau_n > N-n).
\end{aligned} \tag{3.12}$$

- *Case 4: If $Y_{0,n} = D(n,n+k) + x \quad k = 0,1,2,\ldots,N-n$ and $0 \leq x \leq D_1(n+k+1,n+k+1)$:*

$$\begin{aligned}
G_n(Y_{0,n}, I_{2,n}) = {} & L_n(Y_{0,n}, D_1(n,n), D_2(n,n)) \\
& + \sum_{i=1}^{k} L_{n+i}(Y_{0,n} - D(n,n+i-1), D_1(n+i,n+i), D_2(n+i,n+i))P(\tau_n > i) \\
& + L_{n+k+1}(Y_{0,n} - D(n,n+k), Y_{0,n} - D(n,n+k), 0)P(\tau_n > k+1) \\
& + \sum_{i=k+2}^{N-n} L_{n+i}(Y_{0,n} - D(n,n+i-1), x - D_1(n+k+1,n+i-1), \\
& - D_2(n+k+1,n+i-1))P(\tau_n > i).
\end{aligned}$$

$$+\sum_{i=1}^{k+1} G_{n+i}(Y^*_{0,n+i},0)P(\tau_n = i)$$
$$+\sum_{i=k+2}^{N-n} G_{n+i}(Y^*_{0,n+i},-D_2(n+k+1,n+i-1))P(\tau_n = i)$$
$$+\sum_{i=1}^{N-n} cD(n,n+i-1)P(\tau_n = i)$$
$$+cY_{0,n}P(\tau_n > N-n). \tag{3.13}$$

- *Case 5: If* $Y_{0,n} = D(n,n+k)+x \quad k = 0,1,2,\ldots,N-n$ *and* $D_1(n+k+1,n+k+1) < x < D(n+k+1,n+k+1)$*:*

$$G_n(Y_{0,n},I_{2,n}) = L_n(Y_{0,n},D_1(n,n),D_2(n,n))$$
$$+\sum_{i=1}^{k} L_{n+i}(Y_{0,n}-D(n,n+i-1),D_1(n+i,n+i),D_2(n+i,n+i))P(\tau_n > i)$$
$$+L_{n+k+1}(Y_{0,n}-D(n,n+k),D_1(n+k+1,n+k+1),$$
$$x-D_1(n+k+1,n+k+1))P(\tau_n > k+1)$$
$$+\sum_{i=k+2}^{N-n} L_{n+i}(Y_{0,n}-D(n,n+i-1),-D_1(n+k+2,n+i-1),$$
$$x-D_1(n+k+1,n+k+1)-D_2(n+k+1,n+i-1))P(\tau_n > i)$$
$$+\sum_{i=1}^{k+1} G_{n+i}(Y^*_{0,n+i},0)P(\tau_n = i)$$
$$+\sum_{i=k+2}^{N-n} G_{n+i}(Y^*_{0,n+i},$$
$$x-D_1(n+k+1,n+k+1)-D_2(n+k+1,n+i-1))P(\tau_n = i)$$
$$+\sum_{i=1}^{N-n} cD(n,n+i-1)P(\tau_n = i)$$
$$+cY_{0,n}P(\tau_n > N-n). \tag{3.14}$$

Proof. Proof of the proposition is given in Appendix A. □

$\delta_{i,n,k},\ i = 1,2,2',\ n = 1,2,\ldots,N,\ k = -1,0,1,\ldots,N-n$ gives the slope of $G_n(Y_{0,n},I_{2,n})$ for different intervals of $Y_{0,n}$ in period n. When $k=-1$, $Y_{0,n} \le D(n,n)$. For $k \ge 0$, $D(n,(k+2)n) \ge Y_{0,n} \ge D(n,(k+1)n)$. For a given interval for $Y_{0,n}$, let

$\beta > 0$ be such that $Y_{0,n} - \beta$ also belongs to the same interval. The slope of a particular interval indexed by (i, n, k) is given as:

$$\delta_{i,n,k} = (G_n(Y_{0,n}, I_{2,n}) - G_n(Y_{0,n} - \beta, I_{2,n}))/\beta,$$

where $G_n(Y_{0,n}, I_{2,n})$ and $G_n(Y_{0,n} - \beta, I_{2,n})$ terms are given as in equations (3.10)-(3.14). Slopes between the kink points is found by applying the definition of $\delta_{i,n,k}$. Basically it is just simple difference and division. For $Y_{0,n} \leq D_1(n,n) + I_{2,n}$:

$$\begin{aligned}
\delta_{1,n,0} &= (G_n(Y_{0,n}, I_{2,n}) - G_n(Y_{0,n} - \beta, I_{2,n}))/\beta, \\
&= \{\{L_n(Y_{0,n}, Y_{0,n} - I_{2,n}, I_{2,n}) + \sum_{i=1}^{N-n} G_{n+i}(Y^*_{0,n+i}, I_{2,n} - D_2(n, n+i-1))P(\tau_n = i) \\
&\quad + \sum_{i=1}^{N-n} L_{n+i}(Y_{0,n} - D(n, n+i-1), Y_{0,n} - I_{2,n} - D_1(n, n+i-1), \\
&\quad I_{2,n} - D_2(n, n+i-1))P(\tau_n > i) \\
&\quad + \sum_{i=1}^{N-n} cD(n, n+i-1)P(\tau_n = i) + cY_{0,n}P(\tau_n > N-n)\} \\
&\quad -\{L_n(Y_{0,n} - \beta, Y_{0,n} - \beta - I_{2,n}, I_{2,n}) \\
&\quad + \sum_{i=1}^{N-n} G_{n+i}(Y^*_{0,n+i}, I_{2,n} - D_2(n, n+i-1))P(\tau_n = i) \\
&\quad + \sum_{i=1}^{N-n} L_{n+i}(Y_{0,n} - \beta - D(n, n+i-1), Y_{0,n} - \beta - I_{2,n} - D_1(n, n+i-1), \\
&\quad I_{2,n} - D_2(n, n+i-1))P(\tau_n > i) \\
&\quad + \sum_{i=1}^{N-n} cD(n, n+i-1)P(\tau_n = i) + cY_{0,n}P(\tau_n > N-n)\}\}/\beta, \\
&= \{\beta b_{1,n} + \beta \sum_{i=1}^{N-n} b_{1,n+i}P(\tau_n > i)\}/\beta, \\
&= \sum_{i=0}^{N-n} b_{1,n+i}P(\tau_n > i). \qquad (3.15)
\end{aligned}$$

Other slope equations (3.17) - (3.20) can be derived similarly. The slope expressions for different intervals are as follows:

- Case 1: If $Y_{0,n} \leq D_1(n,n) + I_{2,n}$:

$$\delta_{1,n,-1} = -\sum_{i=0}^{N-n} b_{1,n+i} P(\tau_n > i) + cP(\tau_n > N-n). \quad (3.16)$$

- Case 2: If $D_1(n,n) + I_{2,n} \leq Y_{0,n} \leq D_1(n,n)$:

$$\delta_{2,n,-1} = -\sum_{i=0}^{N-n} b_{2,n+i} P(\tau_n > i) + cP(\tau_n > N-n). \quad (3.17)$$

- Case 3: If $D_1(n,n) \leq Y_{0,n} \leq D(n,n)$:

$$\delta_{2',n,-1} = -\sum_{i=0}^{N-n} b_{2,n+i} P(\tau_n > i) + cP(\tau_n > N-n). \quad (3.18)$$

- Case 4: If $Y_{0,n} = D(n, n+k) + x \quad k = 0, 1, 2, \ldots, N-n$ and $0 < x \leq D_1(n+k+1, n+k+1)$:

$$\delta_{1,n,k} = \sum_{i=0}^{k} h_{0,n+i} P(\tau_n > i) - \sum_{i=k+1}^{N-n} b_{1,n+i} P(\tau_n > i) + cP(\tau_n > N-n). \quad (3.19)$$

- Case 5: If $Y_{0,n} = D(n, n+k) + x \quad k = 0, 1, 2, \ldots, N-n$ and $D_1(n+k+1, n+k+1) < x \leq D(n+k+1, n+k+1)$:

$$\delta_{2,n,k} = \sum_{i=0}^{k} h_{0,n+i} P(\tau_n > i) - \sum_{i=k+1}^{N-n} b_{2,n+i} P(\tau_n > i) + cP(\tau_n > N-n). \quad (3.20)$$

In the following theorem we will present a key result which will characterize an interval for optimal system wide order up to level of a period. This result shows that our assumption in proposition 3.1., $Y^*_{0,n} \geq D(n,n)$ for $n = 1, 2, \ldots, N$, is correct which means that indeed proposition 3.1. always holds.

Theorem 3.1. *i.* $Y^*_{0,n} \geq D(n,n)$ *for* $n = 1, 2, \ldots, N$.

ii. $C_n(I_{0,n}, I_{2,n})$ *is minimized at* $I_{0,n} = Y^*_{0,n}$ *where* $Y^*_{0,n}$ *minimizes* $G_n(Y_{0,n}, I_{2,n})$ *for* $n = 1, 2, \ldots, N$.

iii. $Y^*_{0,n} \leq Y^*_{0,n+1} + D(n,n)$ *for* $n = 1, 2, \ldots, N$.

Proof. We prove the theorem by induction. For period N, $Y^*_{0,N} = D(N,N)$ as shown previously, and therefore part i holds. We have shown that part ii also holds for period N. Part iii follows trivially with the convention that $Y^*_{0,N+1} = 0$ (recall that $C_{N+1} = 0$). Suppose that these assertions hold for $n+1$. In particular, if part ii holds for period $n+1$, then for $Y_{0,n} \geq D(n,n) + Y^*_{0,n+1}$, we have:

$$\begin{aligned} G_n(Y_{0,n}, I_{2,n}) &= L_n(Y_{0,n}, D_1(n,n), D_2(n,n)) + C_{n+1}(Y_{0,n} - D(n,n), 0), \\ &\geq L_n(Y^*_{0,n+1} + D(n,n), D_1(n,n), D_2(n,n)) + C_{n+1}(Y_{0,n} - D(n,n), 0), \\ &\geq L_n(Y^*_{0,n+1} + D(n,n), D_1(n,n), D_2(n,n)) + C_{n+1}(Y^*_{0,n+1}, 0), \\ &= G_n(Y^*_{0,n+1} + D(n,n), I_{2,n}). \end{aligned} \tag{3.21}$$

The first inequality follows from the fact that $L_n(Y_{0,n}, D_1(n,n), D_2(n,n))$ increases for $Y_{0,n} \geq D(n,n)$. The second inequality is due to the induction hypothesis for part ii. Therefore, part iii holds for period n. Then, it follows that we can obtain the slopes of $G_n(Y_{0,n}, I_{2,n})$ as in equations (3.16)-(3.20). Note from these equations that, as $b_{i,n} > c$ the slope is negative for any $Y_{0,n} \leq D(n,n)$ (also see Figure 3.2 for $Y_{0,1} \leq 8$). Hence, $G_n(Y_{0,n}, I_{2,n})$ is decreasing over this region, and therefore $Y^*_{0,n} \geq D(n,n)$, which is part i, and part i also follows. This completes the induction. □

In what follows we present observations on the slopes of the function $G_n(Y_{0,n}, I_{2,n})$ that will lead to an efficient computational procedure for the optimal system-wide order-up-to levels:

- Consider the region $Y_{0,n} \geq D(n,n)$ and equations (3.19) and (3.20) for a given k, and note that:

$$\begin{aligned} \delta_{1,n,k} - \delta_{2,n,k} &= -\sum_{i=k+1}^{N-n} b_{1,n+i} P(\tau_n > i) + \sum_{i=k+1}^{N-n} b_{2,n+i} P(\tau_n > i), \\ &\leq 0 \text{ since } b_{1,n+i} \geq b_{2,n+i}. \end{aligned} \tag{3.22}$$

So, if it is not beneficial to order to Retailer 1, $\delta_{1,n,k} > 0$, then it is also not beneficial to Retailer 2, $\delta_{2,n,k} > 0$, for the same subsequent period $n+k+1$. Conversely, if it is not beneficial to keep stock to Retailer 2 for the future period $n+k+1$, $\delta_{2,n,k} > 0$, it can still be beneficial to keep stock for the future period $n+k+1$ for Retailer 1. This is quite intuitive since the backlogging cost of Retailer 1 is at least equal to the backlogging cost of Retailer 2.

- As seen from equations (3.16)-(3.20), slope expressions are independent from the particular demand values. Therefore, when it is beneficial to increase the system-wide inventory level for a small amount, that is, when the slope is negative, it is beneficial to increase it up to the next cumulative demand point (up to the point where the slope expression changes). So, the optimal inventory level is one of the cumulative retailer demand points given by: $\{D(n,n), D(n,n)+D_1(n+1,n+1), D(n,n+1), D(n,n+1)+D_1(n+2,n+2), D(n,n+3), \ldots, D(n,N-1)+D_1(N,N), D(n,N)\}$, and it can be characterized as: $Y_{0,n} = D_1(n,n+K_1)+D_2(n,n+K_2)$ where $K_2 \in \{K_1, K_1-1\}$, and $K_1 \in \{1,2,\ldots,N-n\}$.
- Fix a retailer i and observe that the marginal benefit of keeping stock for further periods decreases for that retailer:

$$\begin{aligned} \delta_{i,n,k} - \delta_{i,n,k+1} &= -h_{0,n+k+1}P(\tau_n > k+1) - b_{i,n+k+1}P(\tau_n > k+1), \\ &\leq 0. \end{aligned} \tag{3.23}$$

  So, if it is not beneficial to keep stock for a future period $n+k+1$ for a retailer, $\delta_{i,n,k} > 0$, then it is also not beneficial for further periods $k'+1 \geq k+1$, since $\delta_{i,n,k'} \geq \delta_{i,n,k} > 0$
- If $\delta_{1,n,k} \geq 0$ for some k then $Y^*_{0,n} \leq D(n,n+k)$. This follows from the fact that for $k' \geq k$ $\delta_{1,n,k'} \geq 0$. Additionally, since $\delta_{1,n,k} \geq \delta_{2,n,k}$, $\delta_{2,n,k'} \geq 0$ for $k' \geq k$.
- We already know that $Y^*_{0,n} \geq D(n,n)$. In order to understand if one needs to hold more stock than $D(n,n)$, it is sufficient to check $\delta_{1,n,0}$, note that if $\delta_{1,n,0} > 0$, then $\delta_{2,n,0} > 0$ too. Then if $\delta_{1,n,0} > 0$ $Y^*_{0,n} = D(n,n)$. But if $\delta_{1,n,0} \leq 0$, then by at least keeping stock for the demand of Retailer 1 in period $n+1$ one can decrease the cost. So, in this case $Y^*_{0,n} \geq D(n,n) + D_1(n+1,n+1)$.

Suppose that $Y^*_{0,n+1} = D_1(n+1, n+K_1) + D_2(n+1, n+K_2)$ for some $1 \leq K_1 \leq N-n$, $K_2 \in \{K_1, K_1-1\}$ and $n \leq N-1$. Trivially, for $n = N-1$, $Y^*_{0,N} = D(N,N)$, so $K_1 = K_2 = 1$. First, we should check if $\delta_{1,n,0} > 0$. If this is the case, then $Y^*_{0,n} = D(n,n)$, as there is no benefit of keeping more stock. Otherwise, consider raising the inventory level to $Y'_{0,n} = D(n,n) + D_1(n+1, n+1)$. The marginal change in the cost function due to keeping an extra stock of $D_1(n+1, n+1)$ on top of $D(n,n)$ is given by $\delta_{1,n,0}D_1(n+1, n+1)$. Similarly, consider raising the inventory level to $Y''_{0,n} = D(n, n+1)$. The marginal change in the cost function in this case would be $\delta_{1,n,0}D_1(n+1, n+1) + \delta_{2,n,0}D_2(n+1, n+1)$. Obviously, $Y''_{0,n}$ is a better decision than $Y'_{0,n}$ if $\delta_{2,n,0}D_2(n+1, n+1) < 0$. We can extend this idea and define for $k_2 \in \{k_1-1, k_1\}$:

$$R(k_1, k_2) = \sum_{i=0}^{k_1} \delta_{1,n,i} D_1(n+i+1, n+i+1) + \sum_{i=0}^{k_2} \delta_{2,n,i} D_2(n+i+1, n+i+1), \quad (3.24)$$

where $\sum_{i=0}^{-1} = 0$. $R(k_1, k_2)$ is the marginal change in the cost function for period n when the system-wide order-up-to level is increased from $D(n,n)$ (the minimum needs to be hold) to $D_1(n, n+k_1+1) + D_2(n, n+k_2+1)$. In order this move to be viable, we should have $R(k_1, k_2) \leq 0$. One can simply search for all k_1 and $k_2 \in \{k_1-1, k_1\}$ values that minimize $R(k_1, k_2)$. Let

$$(K^*_1, K^*_2) = \underset{\substack{k_1 \in \{0,1,\ldots,K_1-1\} \\ k_2 \in \{k_1-1, k_1\}}}{\arg\min} \{R(k_1, k_2)\}. \quad (3.25)$$

Note that, k_1 in (3.25) is restricted to be less than or equal to $K_1 - 1$ as we know that $Y^*_{0,n} \leq D(n,n) + Y^*_{0,n+1}$. Suppose that $R(K^*_1, K^*_2) > 0$. Then, we set $Y^*_{0,n} = D(n,n)$, as there is no marginal benefit of increasing the stock level beyond its lower bound $D(n,n)$. Otherwise, we let $Y^*_{0,n} = D_1(n, n+K^*_1+1) + D_2(n, n+K^*_2+1)$.

3.4.1. Algorithm

The following algorithm calculates the optimal system-wide order-up-to levels for all periods $n = 1, 2, \ldots, N$.

Step 0. $K_1 = K_2 = 1$ ($Y^*_{0,N} = D(N, N)$).

Step 1. For $n = N - 1$ to 1:

Check $\delta_{1,n,0} \leq 0$. If this is true, then continue with step 2, otherwise set $K^*_1 = K^*_2 = -1$ and go to step 3.

Step 2. For $n = N - 1$ to 1 and $\delta_{1,n,1} \leq 0$:

Solve (3.25) to find K^*_1 and K^*_2.

Step 3. $Y^*_{0,n} = D_1(n, n + K^*_1 + 1) + D_2(n, n + K^*_2 + 1)$

Set $K_1 = K^*_1 + 2$ and $K_2 = K^*_2 + 2$.

3.5. Numerical Analysis

In this section we conduct a numerical analysis on the behavior of the order-up-to level and compare the problem result under the simplified allocation rule and without allocation rule.

We analyze the optimal order up to levels in different parameter combinations where $N = 8$. The different patterns for demand values, backorder costs, availability probabilities of supplier and unit purchasing cost is summarized in Table 3.1. The average demand per period is 9 in demand patterns dp_1, dp_2 and dp_3 and it is 12 in dp'_1, dp'_2 and dp'_3. The demand patterns are same except values between dp_1 and dp'_1, dp_2 and dp'_2, dp_3 and dp'_3. We have fixed the demand pattern of Retailer 2 to dp_3 and the backorder cost pattern to bp_1 in all scenarios. The holding cost of manufacturer is fixed to $h_{0,n} = 1$ for all $n = 1, 2 \ldots, 8$.

The scenarios which are generated by the combinations of above demand, supply probability, backorder cost and unit cost patterns are listed in Table B.1 in Appendix B. We have examined 144 many different scenarios. Optimal order-up-to levels for each

Table 3.1. Parameter Patterns

	Demand						Availability prob.				Backorder cost			Unit cost	
n	dp_1	dp_2	dp_3	dp'_1	dp'_2	dp'_3	pp_1	pp_2	pp_3	pp_4	bp_1	bp_2	bp_3	cp_1	cp_2
1	6	6	9	9	9	12	0.1	0.9	0.9	0.5	5	10	20	0	2
2	6	12	9	9	15	12	0.1	0.9	0.1	0.5	5	10	20	0	2
3	6	6	9	9	9	12	0.1	0.9	0.9	0.5	5	10	20	0	2
4	6	12	9	9	15	12	0.1	0.9	0.1	0.5	5	10	20	0	2
5	12	6	9	15	9	12	0.1	0.9	0.9	0.5	5	10	20	0	2
6	12	12	9	15	15	12	0.1	0.9	0.1	0.5	5	10	20	0	2
7	12	6	9	15	9	12	0.1	0.9	0.9	0.5	5	10	20	0	2
8	12	12	9	15	15	12	0.1	0.9	0.1	0.5	5	10	20	0	2

period for the general problem, i.e. when there is not any simplified allocation rule, and for the problem with simplified allocation rule is shown in Table C.1 in Appendix C. In the total cost column of the table, optimal cost values of the scenarios are listed which are the expected total cost of operating the system from the beginning of first period to end of planning horizon with initial inventory level zero i.e., $C_1(\mathbf{0})$ where $\mathbf{0}$ is a zero vector, $(0, 0, 0)$ for the general problem and $C_1(0, 0)$ for the problem with simplified allocation rule.

The results of two sample problem with allocation rule is shown in Table 3.2. As seen from the table, in the scenario 25 the optimal system-wide inventory level in terms of number of period kept gets larger as goes to the beginning of the planning horizon. However, in the scenario 43 it fluctuates because in this scenario the backlogging cost of Retailer 1 is not large enough to counterbalance the cost increase of keeping stock for Retailer 2. We observe that when supplier availability probability is constant over the periods, i.e, when it is all 0.1, pp_1, or 0.9, pp_2, or 0.5, pp_4, number of period kept, k, monotonically increases as goes to the beginning of the planning horizon. But when the supplier availability probability is cyclic between 0.1 and 0.9 it can fluctuate as described above. Another observation is that when the backlogging cost is 5 for both retailers, since the retailers become identical in terms of their costs even their demand values are different manufacturer does not hold any stock for Retailer 1, i.e, kept stock is multiples of period demands exactly as expected. Problem resembles the one retailer case.

Table 3.2. Resulsts of Sample Problems

			With allocation rule
Scenario	n	$Y^*_{0,n}$	in terms of next period num.
43	8	21	0
	7	21	0
	6	42	1
	5	33	1 for Retailer 1
	4	36	1
	3	30	1
	2	30	1
	1	30	1
25	8	21	0
	7	33	1 for Retailer 1
	6	42	1
	5	54	1 for Retailer 1
	4	48	1 for Retailer 1
	3	42	1 for Retailer 1
	2	36	1 for Retailer 1
	1	36	1 for Retailer 1

In numerical analysis, it is seen that only in 29 scenarios the total cost starting from period $n = 1$ to end of $n = N = 8$ with initial zero inventory level differs between general problem and the problem with simplified allocation rule out of 144 scenarios. Additionally, the biggest deviation is smaller than 0.7 percent. The biggest differences occur in pp_4 parameter. Out of these 29 scenarios 12 of them has pp_4 and these are the top 12 biggest differences. If we ignore these scenarios with pp_4, then only in 17 scenarios the total cost is different and the maximum deviation is less than 0.015 percent. So, we can say that the allocation rule performance is worst when supply probability is 0.5 percent. It performs better when supplier probability is higher or smaller or even it fluctuates between high and small values since in extreme cases manufacturer keeps or does not keep stock for both retailers. However, when supply probability is in the middle, manufacturer may want to keep stock for first retailer and not for the second which contradicts each other and decreases the performance of the algorithm.

4. ANALYSIS OF AN INVENTORY SYSTEM WITH DISRUPTIONS IN SUPPLY WITH STOCHASTIC DEMAND

In the first part of Chapter 4, we develop the dynamic programming model of the problem and in the second part, we structure the order-up-to levels of the problem and develop an algorithm to calculate the optimal inventory levels of the periods. In the last part, we give numerical examples for the problem.

4.1. Development of the Dynamic Model and Structural Results

At the beginning of any period n, the manufacturer observes the inventory position, I_n, and places an order of size u_n with the supplier. Let $S_n(u_n)$ be the random variable denoting the amount received from the supplier in period n. The supply uncertainty structure that we consider assumes that $S_n(u_n) = u_n$ with probability p_n and, $S_n(u_n) = 0$ with probability $q_n = 1 - p_n$. There is not any fix order cost or unit purchasing cost. After the realization of supply, the inventory level at manufacturer becomes Y_n, $Y_n = I_n + S_n(u_n)$, and then the demand in period n is observed as $D_n = d$ with probability α_n and $D_n = 0$ with probability $1 - \alpha_n$. Afterwards, relevant end-of-period inventory holding and backlogging costs are charged. Let h_n be the holding cost per unit residing at the end of period n and b_n be the unit backlogging cost that is charged for the units short at the end of period n. We assume that the lead-time from the supplier to the manufacturer and production time at manufacturer are negligible. As a summary in a period the following events are realized in the following order:

i. Initial inventory position I_n is observed.
ii. Order u_n is given to supplier.
iii. Order is realized as $S_n(u_n)$, $S_n(u_n) = 0$ with probability $1 - p_n$ and $S_n(u_n) = u_n$ with probability p_n. Lead time is assumed as zero. Inventory position becomes $Y_n = I_n + S_n(u_n)$.

vi. Demand of the period, D_n, is realized. It is either 0 or d. Inventory position becomes $Y_n - D_n \in \{Y_n, Y_n - d\}$.

v. Holding or backlogging cost is charged.

Let $\mathcal{L}_n(Y_n)$ be the single period cost and $L_n(Y_n)$ be the expected single period cost function associated with the inventory position Y_n after the supply realization but before the demand realization in period n:

$$\mathcal{L}_n(Y_n) = h_n(Y_n - D_n)^+ + b_n(D_n - Y_n)^+. \quad (4.1)$$

$$\begin{aligned} L_n(Y_n) &= E_{D_n}[\mathcal{L}_n(Y_n)], \\ &= (1-\alpha_n)(h_n(Y_n)^+ + b_n(-Y_n)^+) + \alpha_n(h_n(Y_n - d)^+ + b_n(d - Y_n)^+). \end{aligned} \quad (4.2)$$

As seen from the above equation $L_n(Y_n)$ is a convex function since it is combination of convex functions.

We define $C_n(I_n)$ as the minimum expected cost of operating the system in periods $n, n+1, \ldots, N+1$ with a starting inventory I_n, where N is the length of the planning horizon with $C_{N+1}(I_{N+1}) \equiv 0$. $C_n(I_n)$ can be written as:

$$C_n(I_n) = \min_{u_n \geq 0}\{E_{S_n(u_n)}[E_{D_n}[\mathcal{L}_n(I_n + S_n(u_n)) + C_{n+1}(I_n + S_n(u_n) - D_n)]]\}. \quad (4.3)$$

The outer expectation in equation (4.3) is for the supplier availability and the inner one is for demand uncertainty. If we interchange the order of the expectations and take the expectation for supplier availability uncertainty, $S_n(u_n)$, we get equation (4.4).

$$C_n(I_n) = E_{D_n}[p_n \min_{u_n \geq 0}\{\mathcal{L}_n(I_n+u_n)+C_{n+1}(I_n+u_n-D_n)\}+q_n\{\mathcal{L}_n(I_n)+C_{n+1}(I_n-D_n)\}]. \quad (4.4)$$

When we move the demand expectation into the minimization term, we obtain equation (4.5).

$$C_n(I_n) = p_n \min_{u_n \geq 0}\{E_{D_n}[\mathcal{L}_n(I_n+u_n)+C_{n+1}(I_n+u_n-D_n)]\}+q_n E_{D_n}[\mathcal{L}_n(I_n)+C_{n+1}(I_n-D_n)]. \quad (4.5)$$

To simplify the notation we define an auxiliary function $G_n(Y_n)$ which corresponds to expected cost of operating the system after the supply realization but before demand realization in period n to the end of planning horizon where the inventory level after the supply realization is Y_n in period n.

$$G_n(Y_n) = E_{D_n}[\mathcal{L}_n(Y_n) + C_{n+1}(Y_n - D_n)], \quad (4.6)$$

$$= L_n(Y_n) + (1-\alpha_n)C_{n+1}(Y_n) + \alpha_n C_{n+1}(Y_n - d). \quad (4.7)$$

If we plug equation (4.6) into equation (4.5) and use definiton $Y_n = I_n + u_n$, we get equation (4.8).

$$\begin{aligned} C_n(I_n) &= p_n \min_{u_n \geq 0} G_n(I_n + u_n) + q_n G_n(I_n), \\ &= p_n \min_{Y_n \geq I_n} G_n(Y_n) + q_n G_n(I_n). \end{aligned} \quad (4.8)$$

Theorem 4.1. *i. $G_n(Y_n)$ is convex in Y_n and let Y_n^* be its smallest minimizer in period n.*

ii. $C_n(I_n)$ is convex in I_n and it is minimized at $I_n = I_n^* = Y_n^*$ in period n where Y_n^* minimizes $G_n(Y_n)$.

iii. Order-up-to policy is optimal. An amount of $u_n = Y_n^* - I_n$ is ordered whenever the inventory level at the beginning of period n, I_n, is less than or equal to Y_n^*. Otherwise no order is given, $u_n = 0$. In other words, $u_n = max(Y_n^* - I_n, 0)$.

Proof. For last period N, we have:

$$G_N(Y_N) = L_N(Y_N) + (1-\alpha_N)C_{N+1}(Y_N) + \alpha_N C_{N+1}(Y_N - d) = L_N(Y_N), \quad (4.9)$$

since $C_{N+1}(I_N) = 0$ by assumption and $L_N(Y_N)$ is convex, $G_N(Y_N)$ is convex. Since it is convex, there is at least a value which minimizes $G_N(Y_N)$. Let Y_N^* be its smallest minimizer.

$$\begin{aligned} C_N(I_N) &= p_N \min_{Y_N \geq I_N} G_N(Y_N) + q_N G_N(I_N), \\ &= \begin{cases} p_N G_N(Y_N^*) + q_N G_N(I_N) & \text{if } I_N \leq Y_N^* \\ p_N G_N(I_N) + q_N G_N(I_N) = G_N(I_N) & \text{if } I_N > Y_N^* \end{cases}. \end{aligned} \tag{4.10}$$

As seen from above, the order-up-to level form of ordering policy is optimum in period N. The convexity of $C_N(I_N)$ follows from the convexity of $G_N(Y_N)$. So, propositions hold in period N. Assume they also hold in period $n+1$.Then, $G_n(Y_n) = L_n(Y_n) + (1-\alpha_n)C_{n+1}(Y_n) + \alpha_n C_{n+1}(Y_n - d)$ is convex from summation of convex functions. Since $G_n(Y_n)$ is convex, it has at least a minimizer. Let us assume that the smallest one is Y_n^*. Now, we check $C_n(I_n)$:

$$\begin{aligned} C_n(I_n) &= p_n \min_{Y_n \geq I_n} G_n(Y_n) + q_n G_n(I_n), \\ &= \begin{cases} p_n G_n(Y_n^*) + q_n G_n(I_n) & \text{if } I_n \leq Y_n^* \\ p_n G_n(I_n) + q_n G_n(I_n) = G_n(I_n) & \text{if } I_n > Y_n^* \end{cases}. \end{aligned} \tag{4.11}$$

So, optimal ordering policy is seen as order-up-to type in period n. Convexity of $C_n(I_n)$ follows from the convexity of $G_n(Y_n)$. Then, propositions hold in period n.

As a result, by induction they hold for all periods $n = 1, 2, \ldots, N$ which completes the proof. □

For last period N, the optimal order-up-to level is the point where $L_N(Y_N)$ is minimized as stated before. Let us find the optimal order-up-to level for period N:

$$
\begin{aligned}
G_N(Y_N) &= L_N(Y_N), \\
&= (1-\alpha_N)(h_N(Y_N)^+ + b_N(-Y_N)^+) + \alpha_N(h_N(Y_N-d)^+ + b_N(d-Y_N)^+), \\
&= (1-\alpha_N)h_N Y_N + \alpha_N b_N(d-Y_N) \text{ for } 0 \le Y_N \le d, \\
&= ((1-\alpha_N)h_N - \alpha_N b_N)Y_N. \qquad (4.12)
\end{aligned}
$$

Therefore,

$$
\min_{0\le Y_N \le d} G_N(Y) = \min_{0\le Y_N\le d}\{((1-\alpha_N)h_N - \alpha_N b_N)Y_N\}. \qquad (4.13)
$$

As a result, if $(1-\alpha_N)h_N - \alpha_N b_N > 0$, $G_N(Y_N)$ is minimized at $Y_N = Y_N^* = 0$. If $(1-\alpha_N)h_N - \alpha_N b_N < 0$, $G_N(Y_N)$ is minimized at $Y_N = Y_N^* = d$. When $(1-\alpha_N)h_N - \alpha_N b_N = 0$, $G_N(Y_N)$ is minimized at $Y_N = 0$ or $Y_N = d$, then the optimal is $Y_N^* = min(0,d) = 0$ by the definition.

Theorem 4.2. *i.* $Y_n^* \ge 0$ *for* $n = 1, 2, \ldots, N$.

ii. $Y_n^* \le d + Y_{n+1}^*$ *for* $n = 1, 2, \ldots, N$.

iii. If $(1-\alpha_n)h_n \le \alpha_n b_n$ *for all* $n = 1, 2, \ldots, N$, *then* $Y_n^* \ge d$ *for all* $n = 1, 2, \ldots, N$.

Proof. As explained above, since $Y_N^* = 0$ or $Y_N^* = d$, propositions hold in period N. Suppose that they also hold in period $n+1$.

i. We will check if $G_n(0) \le G_n(Y_n)$ for $Y_n \le 0$:

$$
\begin{aligned}
G_n(0) &= L_n(0) + (1-\alpha_n)C_{n+1}(0) + \alpha_n C_{n+1}(-d), \\
&\le L_n(Y_n) + (1-\alpha_n)C_{n+1}(0) + \alpha_n C_{n+1}(-d), \qquad (4.14)
\end{aligned}
$$

since $L_n(0) \le L_n(Y_n)$. Because $C_{n+1}(I_{n+1})$ is minimized at Y_{n+1}^* and $Y_{n+1}^* \ge 0 \ge Y_n$ and $Y_{n+1}^* \ge 0 \ge Y_n - d$,

$$G_n(0) \leq L_n(Y_n) + (1-\alpha_n)C_{n+1}(Y_n) + \alpha_n C_{n+1}(Y_n - d), \\ = G_n(Y_n). \tag{4.15}$$

So, proposition holds in period n. By induction, $Y_n^* \geq 0$ for $n = 1, 2, \ldots, N$.

ii. We will check if $G_n(Y_{n+1}^* + d) \leq G_n(Y_n)$ for $Y \geq d + Y_{n+1}^*$:

$$G_n(Y_{n+1}^* + d) = L_n(Y_{n+1}^* + d) + (1-\alpha_n)C_{n+1}(Y_{n+1}^* + d) + \alpha_n C_{n+1}(Y_{n+1}^*), \\ \leq L_n(Y_n) + (1-\alpha_n)C_{n+1}(Y_{n+1}^* + d) + \alpha_n C_{n+1}(Y_{n+1}^*), \tag{4.16}$$

since $L_n(0) \leq L_n(Y_n)$. Then,

$$G_n(Y_{n+1}^* + d) \leq L_n(Y_n) + (1-\alpha_n)C_{n+1}(Y_n) + \alpha_n C_{n+1}(Y_n - d), \\ = G_n(Y_n). \tag{4.17}$$

So, item ii holds for period n and by induction $Y_n \geq d + Y_{n+1}^*$ holds for $n = 1, 2, \ldots, N$.

iii. We will check the validity of $G_n(d) \leq G_n(Y_n)$ for $0 \leq Y_n \leq d$ when $(1-\alpha_n)h_n + c \leq \alpha_n b_n$ holds in period n. Note that since $0 \leq Y_n \leq d$ when $(1-\alpha_n)h_n + c \leq \alpha_n b_n$, $L_n(d) \leq L_n(Y_n)$ for $Y_n \leq d$:

$$G_n(d) = L_n(d) + (1-\alpha_n)C_{n+1}(d) + \alpha_n C_{n+1}(0), \\ \leq L_n(Y_n) + (1-\alpha_n)C_{n+1}(d) + \alpha_n C_{n+1}(0), \\ \leq L_n(Y_n) + (1-\alpha_n)C_{n+1}(Y_n) + \alpha_n C_{n+1}(Y_n - d), \\ = G_n(Y_n). \tag{4.18}$$

So, item iii holds. By induction, $Y_n^* \geq d$ for $n = 1, 2, \ldots, N$ if $(1-\alpha_n)h_n + c \leq \alpha_n b_n$ for $n = 1, 2, \ldots, N$.

□

4.2. Reformulation of the Objective Function Based on Hitting Time Analysis

We define τ_n such a period that for the first time after period n both the supplier is available, $Q_{n+\tau_n} = \infty$, and the ordering quantity is non-negative, i.e, the beginning inventory level is at most the optimal order-up-to level. If the inventory level is Y_n in period n after replenishment, $Y_n - D(n, n+\tau_n-1) \leq Y^*_{n+\tau_n}$. We call τ_n as the hitting time.

We rewrite function $G_n(Y_n)$ using the definition of τ_n. From period n to $n+\tau_n$, the replenishment from supplier is zero since either supplier is unavailable or order quantity is zero by the definition of τ_n. So, during this time period inventory level decreases stochastically by D_{n+i} in each period and the single period cost is incurred accordingly. In other words, in period $n+i$, for $0 \leq i < \tau_n$, $L_{n+i}(Y_n - D(n, n+i-1))$ is incurred with probability $P(\tau_n > i)$. However, in period $\tau_n + n$ manufacturer has optimal order-up-to inventory position after replenishment by the definition of τ_n thus $G_{n+i}(Y^*_{n+i})$ is incurred with probability $P(\tau_n = i)$ from period $\tau_n + n$ to end of planning horizon N. Thus, we can reexpress $G_n(Y_n)$ as in equation (4.19) using τ_n:

$$G_n(Y_n) = L_n(Y_n) + \sum_{i=1}^{N-n} E[\mathcal{L}_{n+i}(Y_n - D(n, n+i-1))1_{\{\tau_n > i\}}] + \sum_{i=1}^{N-n} E[G_{n+i}(Y^*_{n+i})1_{\{\tau_n = i\}}]. \tag{4.19}$$

If it is beneficial to increase inventory position from a demand multiple value, manufacturer continues increase inventory position to next demand multiple, since slope value does not change between two demand multiple points. So, order-up-to levels are one of $\{0, d, 2d, \ldots, (N-n+1)d\}$. Let for some $K \in \{0, 1, \ldots, N-n\}$, $Y^*_{n+1} = Kd$. Since we know that $Y^*_n \leq Y^*_{n+1} + d$, we define $y(\eta) = jd - \eta$ for $0 \leq \eta \leq d$ and $j \in \{0, 1, \ldots, K+1\}$ by guaranteeing non-negative $y(\eta)$, $y(\eta) \geq 0$, and $y = y(0) = jd$. We define a new function, $\mathcal{G}_n(y, \eta)$ which represents the slopes between demand multiples, $\mathcal{G}_n(y, \eta) = G_n(y) - G_n(y(\eta))$. By using (4.19) we explore $\mathcal{G}_n(y, \eta)$ and then we use this to check the benefit of increasing or decreasing the inventory level from a demand multiple point, y. Note that, τ_n also depends on η through $y(\eta)$, but since $d \geq \eta \geq 0$ and $Y^*_{n+k} \in \{0, d, 2d, \ldots, (N-n-k+1)d\}$, if $y - D(n, n+k-1) \geq Y^*_{n+k}$ then $y(\eta) - D(n, n+k-1) \geq Y^*_{n+k}$. So, $\{\tau_n = i\}$ has the same probability for $y(\eta)$ for $d \geq \eta \geq 0$. Additionally, to rearrange and combine the similar terms and to have a simpler form, we assume stationary and constant holding and backlogging costs in this reformulation. In other words, $h_n = h$ and $b_n = b$ for all $n = 1, 2, 3, \ldots, N$.

$$\begin{aligned} \mathcal{G}_n(y,\eta) &= G_n(y) - G_n(y(\eta)), \\ &= L_n(y) - L_n(y(\eta)) + \sum_{i=1}^{N-n} E[1_{\{\tau_n > i\}}(\mathcal{L}_{n+i}(y - D(n, n+i-1)), \\ &\quad -\mathcal{L}_{n+i}(y(\eta) - D(n, n+i-1)))]. \end{aligned} \tag{4.20}$$

For any i in (4.20), by fixing $D_n, D_{n+1}, \ldots, D_{n+i-1}$ we fix $D(n, n+i-1)$ and we fix $Q_{n+1}, Q_{n+2}, \ldots, Q_{n+i-1}$. We define $V_n(y,\eta) = \mathcal{L}_{n+i}(y - D(n, n+i-1)) - \mathcal{L}_{n+i}(y(\eta) - D(n, n+i-1))$. We have three partition for $V_n(y,\eta)$ according to value $D(n, n+i-1)$:

- Case 1: $y - D(n, n+i-1) < d$, i.e, $y - D(n, n+i-1) \in \{0, -d, -2d, \ldots\}$ and $y(\eta) - D(n, n+i-1) < d$, i.e, $y - \eta - D(n, n+i-1) \in \{-\eta, -d-\eta, -2d-\eta, \ldots\}$: Since inventory level is zero or there is backorder, either manufacturer increase its backorder level by d with probability α_{n+i} or remain as it is with probability $1 - \alpha_{n+i}$. In other words, with inventory level $y(\eta)$, we have more backorder cost:

$$\begin{aligned} V_{n+i}(y,\eta) &= \mathcal{L}_{n+i}(y - D(n, n+i-1)) - \mathcal{L}_{n+i}(y(\eta) - D(n, n+i-1)), \\ &= \alpha_{n+i}(d - y + D(n, n+i-1))b + (1-\alpha_{n+i})b(D(n, n+i-1) - y) \\ &\quad -\alpha_{n+i}(d - y + D(n, n+i-1) + \eta)b \\ &\quad -(1-\alpha_{n+i})b(D(n, n+i-1) - y + \eta), \\ &= -b\eta. \end{aligned} \tag{4.21}$$

- Case 2: $y - D(n, n+i-1) = d$ and $y(\eta) - D(n, n+i-1) = d - \eta$: If demand is d with probability α_{n+i}, manufacturer with inventory level y has $b\eta$ much less backorder cost but if demand is zero with probability $1 - \alpha_{n+i}$, it has $h\eta$ more holding cost:

$$\begin{aligned} V_{n+i}(y,\eta) &= \mathcal{L}_{n+i}(y - D(n, n+i-1)) - \mathcal{L}_{n+i}(y(\eta) - D(n, n+i-1)), \\ &= (1-\alpha_{n+i})hd - (1-\alpha_{n+i})h(d-\eta) - \alpha_{n+i}b\eta, \\ &= -\alpha_{n+i}b\eta + (1-\alpha_{n+i})h\eta. \end{aligned} \tag{4.22}$$

- Case 3: $y - D(n, n+i-1) > d$ namely $y - D(n, n+i-1) \geq 2d$ and i.e, $y - D(n, n+i-1) \in \{2d, 3d, \ldots\}$ and $y(\eta) - D(n, n+i-1) \geq 2d$, i.e, $y - \eta - D(n, n+i-1) \in \{2d - \eta, -3d - \eta, \ldots\}$: For both manufacturers with inventory level y and $y(\eta)$ since there are enough inventory for demand, they only incur holding cost. Since inventory level for manufacturer with inventory level y is more than $y(\eta)$ by η much, he has more backorder cost by $h\eta$:

$$\begin{aligned} V_{n+i}(y,\eta) &= \mathcal{L}_{n+i}(y - D(n, n+i-1)) - \mathcal{L}_{n+i}(y(\eta) - D(n, n+i-1)), \\ &= \alpha_{n+i}(y-d)h + (1-\alpha_{n+i})yh \\ &\quad -\alpha_{n+i}(y-\eta-d)h - (1-\alpha_{n+i})(y-\eta)h, \\ &= h\eta. \end{aligned} \tag{4.23}$$

As a result,

$$\mathcal{G}_n(y,\eta) = L_n(y) - L_n(y(\eta)) + \sum_{i=1}^{N-n} E[V_{n+i}(y,\eta)1_{\{\tau_n > i\}}], \tag{4.24}$$

where

$$\begin{aligned} V_{n+i}(y,\eta) &= E_{D_{n+i}}[\mathcal{L}_{n+i}(y - D(n, n+i-1)) - \mathcal{L}_{n+i}(y(\eta) - D(n, n+i-1))], \\ &= \begin{cases} -b\eta & \text{if } y - D(n, n+i-1) < d \\ -\alpha_{n+i}b\eta + (1-\alpha_{n+i})h\eta & \text{if } y - D(n, n+i-1) = d \\ h\eta & \text{if } y - D(n, n+i-1) > d \end{cases}. \end{aligned} \tag{4.25}$$

As seen from above equation, the individual values of demand realizations are not important but the total is important. Now we will try to calculate the probability of $\tau_n > i$. As explained before, τ_n not only depends on supplier availability but also the inventory level at the beginning of the period. For a fixed i, all possible combinations of supplier availability is given by Ω_i which is 2^i subsets of $\{1, 2, \dots, i\}$. Elements of Ω_i is represented by A_j for $j = 1, 2, \dots, 2^i$, $\Omega_i = \cup_{j=1}^{2^i} A_j$. Elements of a subset A_j indicates the infinite supply periods, i.e the periods when the supplier is available. If $k \in A_j$ then $Q_{n+k} = \infty$. Note that, A_1, the first subset of Ω_i is empty set, $A_1 = \emptyset$, which means that there is not any period in which supplier is available. For $A_i \in \Omega_i$, $P_Q(A_i) = \prod_{k \in A_i} p_{n+k} \prod_{k \notin A_i} q_{n+k}$ is the probability of occurrence of this set. Up to now, we are interested in supplier availability probability. However, even supplier is available if the inventory level is higher than the optimal order-up-to level, manufacturer cannot reach the optimal. In other words, if $Y^*_{n+k} < y - D(n, n+k-1)$, if the total demand from period n to $n+k$, excluding $n+k$, is less than the difference between the initial inventory level and the optimal order-up-to level, manufacturer cannot reach Y^*_{n+k}, he has much inventory than Y^*_{n+k}. Then the condition for possible values of cumulative demand, and the beginning inventory level, together with supply availability can be expressed in the following sets:

$$
\begin{aligned}
U_i^1(A_j) &= \{k \in A_j : D(n, n+k-1) < y - Y^*_{n+k}, y - D(n, n+i-1) < d\} \\
U_i^2(A_j) &= \{k \in A_j : D(n, n+k-1) < y - Y^*_{n+k}, y - D(n, n+i-1) = d\} \\
U_i^3(A_j) &= \{k \in A_j : D(n, n+k-1) < y - Y^*_{n+k}, y - D(n, n+i-1) > d\}
\end{aligned}
$$

Since we know the subsets related with supplier availability and probability of demand, we can calculate the probabilities of these sets, $P(U_i^1(A_j))$, $P(U_i^2(A_j))$ and $P(U_i^3(A_j))$. As a result we can calculate 4.24 as:

$$
\begin{aligned}
\mathcal{G}_n(y, \eta) &= L_n(y) - L_n(y(\eta)) + \sum_{i=1}^{N-n} \sum_{A_j \in \Omega_i} P_Q(A_j) \sum_{k=1}^{3} (P(U_i^k(A_j)) V_{n+i}(y, \eta)), \\
&= L_n(y) - L_n(y(\eta)) + \sum_{i=1}^{N-n} \sum_{A_j \in \Omega_i} P_Q(A_j)[P(U_i^1(A_j))(-b\eta) \\
&\quad + P(U_i^2(A_j))(-\alpha b + (1-\alpha)h)\eta + P(U_i^3(A_j))h\eta],
\end{aligned}
$$

$$= \quad L_n(y) - L_n(y(\eta)) - b\sum_{i=1}^{N-n}\sum_{A_j\in\Omega_i} P_Q(A_j)[P(U_i^1(A_j)) + \alpha P(U_i^2(A_j))]$$
$$+h\sum_{i=1}^{N-n}\sum_{A_j\in\Omega_i} P_Q(A_j)[P(U_i^2(A_j))(1-\alpha) + P(U_i^3(A_j))], \tag{4.26}$$

where,

$$L_n(y) - L_n(y(\eta)) \quad = \quad \begin{cases} -b\eta & \text{if } y < d \\ -\alpha b\eta + (1-\alpha)h\eta & \text{if } y = d \\ h\eta & \text{if } y > d \end{cases}. \tag{4.27}$$

As a result, if this difference, (4.26), is negative, then the optimal order-up-to level is y otherwise manufacturer decrease the inventory level by d and check the difference again. He will increase the inventory level up to point where this difference is positive.

We will state all these finding in the following theorem. No additional proof is given for the theorem since it is derived from the results of above discussions.

Theorem 4.3. *The optimal order-up-to level for period* $n \in \{1, 2, \ldots, N\}$, Y_n^* *is equal to* $K_n d$ *for* $0 \le K_n \le N - n + 1$ *with* $K_n = 1$. *If* $(1-\alpha)h - \alpha b \le 0$ *then* $Y_N^* = d$, *otherwise* $Y_N^* = 0$. *Given that* $Y_{n+1}^* = K_{n+1}d$ *for* $0 \le N - n$ *and* $n < N$, *let*

$$K' = max\{i = 1, 2, \ldots, K_{n+1} + 1 : \mathcal{G}_n(y, \eta) < 0\}. \tag{4.28}$$

Then, $Y_n^* = K'd$. *If no such* K' *exists, then* $Y_n^* = 0$.

4.2.1. Algorithm

The following algorithm determines the optimal order-up-to levels.

Step 0. Check $(1-\alpha)h - \alpha b \le 0$. If this is true $Y_N* = d$ and $K_N = 1$, otherwise set $Y_N^* = 0$ and $K_N = 0$

Step 1. For $n = N - 1$ to 1:

Find K' which satisfy 4.28. If no such K' exists, set $K' = 0$

Step 2. Set $Y_n^* = K'd$ and $K = K'$.

4.3. Numerical Analysis

In this section we make numerical analysis on the behavior of the order-up-to level. We check the optimal order-up-to levels in different parameter combinations where $N = 10$. The different patterns for demand and supplier availability probabilities, backorder cost and holding cost are shown in Table 4.1. We take fix demand value as $d = 10$ and we fix the backorder cost to $b_n = 20$ for all $n = 1, 2, \ldots, N$ in all scenarios .

Table 4.1. Parameter Patterns for Single Stage Inventory Problem

	Holding cost		Demand/Supply probability				
n	hp_1	hp_2	pp_1	pp_2	pp_3	pp_4	pp_5
1	1	5	0.1	0.5	0.9	0.1	0.9
2	1	5	0.1	0.5	0.9	0.9	0.1
3	1	5	0.1	0.5	0.9	0.1	0.9
4	1	5	0.1	0.5	0.9	0.9	0.1
5	1	5	0.1	0.5	0.9	0.1	0.9
6	1	5	0.1	0.5	0.9	0.9	0.1
7	1	5	0.1	0.5	0.9	0.1	0.9
8	1	5	0.1	0.5	0.9	0.9	0.1
9	1	5	0.1	0.5	0.9	0.1	0.9
10	1	5	0.1	0.5	0.9	0.9	0.1

We generate 26 different scenarios using these patterns. Details of the scenarios are listed in Table D.1 in Appendix D. Optimal order-up-to levels for each period and total cost of operating the inventory system from period $n = 1$ to $n = 10$ with initial inventory level zero are given in Table E.1 in Appendix E.

As seen from the results, when the probability patterns are constant over the planning horizon for demand and supply, the stock kept on-hand is monotonically increases as going from the end of planning horizon to beginning. However, when the probabilities are not constant the optimal order up-to level fluctuates as in scenario 19.

In scenarios 13 and 16, since the holding cost is relatively high and the probability of non-zero demand is very low, the manufacturer does not hold any inventory in any period. However, when the supply probability is also very low as in scenario 10, manufacturer holds inventory in periods $n = 1, 2, \ldots, 7$. When we decrease the holding cost from 5 to 1, then we end up with scenario 4 (compared to 13) and scenario 7 (compared to 16). In scenarios 4 and 7, manufacturer holds stock in all periods $n = 1, 2, \ldots, 10$ since holding cost gets very low compared to backlogging cost.

When supply probability is pp_3 and holding cost is hp_2, optimal stock level for periods does not change and it stays as constant over the planning horizon. Since the supply probability and the holding cost are high, the risk does not change from the beginning of planning horizon to the end of it, thus optimal inventory level does not change over the horizon regardless of demand possibility. However, as in scenario 9, if the holding cost is very low, the manufacturer plans to hold more inventory at the beginning of the planning horizon.

Highest total cost occurs in scenarios 3 and 12, when the demand probability is high and the supply probability is low as expected. Also, in these scenarios the optimal inventory levels are higher. The lowest cost occurs in scenario 7, when the supply probability is high and the demand probability is low.

5. CONCLUSIONS

In this thesis, firstly we consider a two-echelon, single item, periodic review, deterministic demand inventory system under non-stationary supplier availability in finite planning horizon where we have a supplier, manufacturer and two retailers. Retailers differentiated from each other by their backlogging costs. Supplier availability has a binomial structure namely supplier is either fully available or unavailable in each period. After the delivery from supplier to manufacturer, manufacturer makes the stock transfer to retailers. The problem is to find the optimal system-wide inventory level considering the supplier unavailability.

Firstly we developed the dynamic programming formulation of the problem without restricting the stock allocation rule. However, stock allocation decision in determining the system-wide inventory level complicates the problem very much. Then, we described a simplified stock allocation rule and modified the dynamic programming accordingly. In the simplified allocation rule, we assumed that manufacturer firstly tries to satisfy the demand of the retailer with higher backlogging cost as much as possible. After satisfying the demand of this retailer, the demand of the other retailer is tried to be satisfied. Although, the dynamic programming recursive function is not convex by using the special characteristics of its slopes, we developed an approach to determine the optimal system-wide inventory level of the system. Finally, we gave computational results for this policy. As the supplier unavailability decreases and the backlogging costs of retailers approach to each other, the optimal system-wide inventory levels in the model with simplified allocation rule approaches to the optimal system-wide inventory levels in the model without allocation rule.

Secondly in this thesis, we consider a manufacturer who has an unreliable supplier and stochastic demand. Both demand and supply have binomial structure. Manufacturer give order with supplier and after the receipt of the order demand is realized at the manufacturer. The aim of the manufacturer is to minimize total holding and backlogging costs over the finite planning horizon.

To solve this inventory problem, we developed the dynamic programming model. Then using hitting time, we reformulated the problem and found the slope expression of total cost function. By the help of the slope expression, we developed an algorithm to calculate the optimal order-up-to levels.

This thesis can be further extended to advance supplier information area. Additionally, different supplier availability or demand patterns can be considered. We can add fixed ordering cost and non-zero lead times, lead time from supplier to manufacturer, from manufacturer to retailers or manufacturing lead time. More complicated supply chain structures can be modeled like stock keeping at retailers, demand realization at manufacturer or lower holding cost at retailers. Contracting with supplier to overcome the uncertainty can be analyzed.

APPENDIX A: Proof of Proposition 3.1.

Alternative expressions are obtained by simple conditioning, substitution and induction. For Case 1 $Y_{0,n} \leq D_1(n,n) + I_{2,n}$ at period $n = N$ equation (3.10) gives:

$$
\begin{aligned}
G_N(Y_{0,N}, I_{2,N}) &= L_N(Y_{0,N}, Y_{0,N} - I_{2,N}, I_{2,N}) \\
&+ \sum_{i=1}^{N-N} L_{N+i}(Y_{0,N} - D(N, n+i-1), Y_{0,N} - I_{2,N} - D_1(N, N+i-1), \\
&I_{2,N} - D_2(N, N+i-1))P(\tau_N > i) \\
&+ \sum_{i=1}^{N-N} G_{N+i}(Y^*_{0,N+i}, I_{2,n} - D_2(N, N+i-1))P(\tau_N = i) \\
&+ \sum_{i=1}^{N-N} cD(N, N+i-1)P(\tau_N = i) \\
&+ cY_{0,N}P(\tau_N > N-N), \\
&= L_N(Y_{0,N}, Y_{0,N} - I_{2,N}, I_{2,N}) + cY_{0,N}. \qquad \text{(A.1)}
\end{aligned}
$$

The equation (A.1) shows that the equation (3.10) holds in period N. Assume that it also holds in period $n+1$. From the equations (3.7) and (3.8) for $Y_{0,n} \leq D_1(n,n) + I_{2,n}$ we have:

$$
\begin{aligned}
G_n(Y_{0,n}, I_{2,n}) &= L_n(Y_{0,n}, Y_{0,n} - I_{2,n}, I_{2,n}) \\
&\quad + C_{n+1}(Y_{0,n} - D(n,n), I_{2,n} - D_2(n,n)) + cY_{0,n}, \\
&= L_n(Y_{0,n}, Y_{0,n} - I_{2,n}, I_{2,n}) \\
&\quad + p_{n+1} \min_{Y_{0,n+1} \geq I_{0,n+1}} \{G_{n+1}(Y_{0,n+1}, I_{2,n+1})\} + q_{n+1} G_{n+1}(I_{0,n+1}, I_{2,n+1}) \\
&\quad - cI_{0,n+1} + cY_{0,n}, \qquad \text{(A.2)}
\end{aligned}
$$

where $I_{0,n+1} = Y_{0,n} - D(n,n)$. By assumption $Y_{0,n} \leq Y^*_{0,n+1} + D(n,n)$ in period $n+1$, so $I_{0,n+1} \leq Y^*_{0,n+1}$. This means that we can increase the net inventory level to $Y^*_{0,n+1}$

in period $n+1$ with the probability p_{n+1}.

$$\begin{aligned} G_n(Y_{0,n}, I_{2,n}) &= L_n(Y_{0,n}, Y_{0,n} - I_{2,n}, I_{2,n}) \\ &+ p_{n+1} G_{n+1}(Y^*_{0,n+1}, I_{2,n+1}) + q_{n+1} G_{n+1}(I_{0,n+1}, I_{2,n+1}) \\ &- cI_{0,n+1} + cY_{0,n}. \end{aligned} \tag{A.3}$$

Replacing $I_{0,n+1}$ and $I_{2,n+1}$ with $Y_{0,n} - D(n,n)$ and $I_{2,n} - D_2(n,n)$ respectively, we get:

$$\begin{aligned} G_n(Y_{0,n}, I_{2,n}) &= L_n(Y_{0,n}, Y_{0,n} - I_{2,n}, I_{2,n}) \\ &+ p_{n+1} G_{n+1}(Y^*_{0,n+1}, I_{2,n} - D_2(n,n)) \\ &+ q_{n+1} G_{n+1}(Y_{0,n} - D(n,n), I_{2,n} - D_2(n,n)) + cD(n,n). \end{aligned} \tag{A.4}$$

Using (3.10) for period $n+1$ to $G_{n+1}(Y_{0,n} - D(n,n), I_{2,n} - D_2(n,n))$, we get:

$$\begin{aligned} G_n(Y_{0,n}, I_{2,n}) &= L_n(Y_{0,n}, Y_{0,n} - I_{2,n}, I_{2,n}) \\ &+ p_{n+1} G_{n+1}(Y^*_{0,n+1}, I_{2,n} - D_2(n,n)) \\ &+ q_{n+1}\{L_{n+1}(Y_{0,n} - D(n,n), Y_{0,n} - I_{2,n} - D_1(n,n), I_{2,n} - D_2(n,n)) \\ &+ \sum_{i=1}^{N-n-1} L_{n+1+i}(Y_{0,n} - D(n,n) - D(n+1, n+i), \\ &Y_{0,n+1} - I_{2,n} - D_1(n,n) - D_1(n+1, n+i), \\ &I_{2,n} - D_2(n,n) - D_2(n+1, n+i)) P(\tau_{n+1} > i) \\ &+ \sum_{i=1}^{N-n-1} G_{n+1+i}(Y^*_{0,n+1+i}, I_{2,n} - D_2(n,n) - D_2(n, n+i)) P(\tau_{n+1} = i) \\ &+ \sum_{i=1}^{N-n-1} cD(n+1, n+i) P(\tau_{n+1} = i) \\ &+ c(Y_{0,n} - D(n,n)) P(\tau_{n+1} > N-n-1)\} + cD(n,n). \end{aligned} \tag{A.5}$$

We replace p_{n+1} with $P(\tau_n = 1)$ and q_{n+1} with $P(\tau_n > 1)$ and then rearrange the terms to get:

$$\begin{aligned} G_n(Y_{0,n}, I_{2,n}) &= L_n(Y_{0,n}, Y_{0,n} - I_{2,n}, I_{2,n}) \\ &+P(\tau_n = 1)G_{n+1}(Y^*_{0,n+1}, I_{2,n} - D_2(n,n)) + P(\tau_n > 1) \\ &\{L_{n+1}(Y_{0,n} - D(n,n), Y_{0,n} - I_{2,n} - D_1(n,n), I_{2,n} - D_2(n,n)) \\ &+ \sum_{i=1}^{N-n-1} L_{n+1+i}(Y_{0,n} - D(n, n+i), \\ &Y_{0,n+1} - I_{2,n} - D_1(n, n+i), I_{2,n} - D_2(n, n+i))P(\tau_{n+1} > i) \\ &+ \sum_{i=1}^{N-n-1} G_{n+1+i}(Y^*_{0,n+1+i}, I_{2,n} - D_2(n, n+i))P(\tau_{n+1} = i) \\ &+ \sum_{i=1}^{N-n-1} cD(n+1, n+i)P(\tau_{n+1} = i) + cY_{0,n}P(\tau_{n+1} > N-n-1) \\ &-cD(n,n)P(\tau_{n+1} > N-n-1))\} + cD(n,n). \end{aligned} \tag{A.6}$$

Let us combine the similar terms noting that $P(\tau_n > 1)P(\tau_{n+1} > i) = P(\tau_n > i+1)$, $P(\tau_n > 1)P(\tau_{n+1} = i) = P(\tau_n = i+1)$ and $P(\tau_n > 1)P(\tau_{n+1} > N-n-1) = P(\tau_n > N-n)$.

$$\begin{aligned} G_n(Y_{0,n}, I_{2,n}) &= L_n(Y_{0,n}, Y_{0,n} - I_{2,n}, I_{2,n}) \\ &+P(\tau_n = 1)G_{n+1}(Y^*_{0,n+1}, I_{2,n} - D_2(n,n)) + P(\tau_n > 1) \\ &L_{n+1}(Y_{0,n} - D(n,n), Y_{0,n} - I_{2,n} - D_1(n,n), I_{2,n} - D_2(n,n)) \\ &+ \sum_{i=1}^{N-n-1} L_{n+1+i}(Y_{0,n} - D(n, n+i), \\ &Y_{0,n+1} - I_{2,n} - D_1(n, n+i), I_{2,n} - D_2(n, n+i))P(\tau_n > i+1) \\ &+ \sum_{i=1}^{N-n-1} G_{n+1+i}(Y^*_{0,n+1+i}, I_{2,n} - D_2(n, n+i))P(\tau_n = i+1) \\ &+ \sum_{i=1}^{N-n-1} cD(n+1, n+i)P(\tau_n = i+1) \\ &+cY_{0,n}P(\tau_n > N-n) - cD(n,n)P(\tau_n > N-n) + cD(n,n). \end{aligned} \tag{A.7}$$

Rearrange the terms and $-cD(n,n)P(\tau_n > N-n) + cD(n,n) = cD(n,n)\{1 - P(\tau_n > N-n) = \sum_{i=1}^{N-n} cD(n,n)P(\tau_n = i)$:

$$\begin{aligned} G_n(Y_{0,n}, I_{2,n}) \;=\; & L_n(Y_{0,n}, Y_{0,n} - I_{2,n}, I_{2,n}) \\ & + \sum_{i=1}^{N-n} G_{n+i}(Y^*_{0,n+i}, I_{2,n} - D_2(n, n+i-1))P(\tau_n = i) \\ & + \sum_{i=1}^{N-n} L_{n+i}(Y_{0,n} - D(n, n+i-1), \\ & Y_{0,n} - I_{2,n} - D_1(n, n+i-1), \\ & I_{2,n} - D_2(n, n+i-1))P(\tau_n > i) \\ & + \sum_{i=1}^{N-n} cD(n, n+i-1)P(\tau_n = i) \\ & + cY_{0,n}P(\tau_n > N-n). \end{aligned} \tag{A.8}$$

Which is the same equation with (3.10) for period n. So, by induction it holds for all period $n = 1, 2, \ldots, N$.

The alternative expressions for other intervals can be proved similarly.

APPENDIX B: Parameters of Scenarios for Two-Echelon Problem

Table B.1. Parameter Sets of Scenarios for Two-Echelon Problem

	Parameter Set					Parameter Set			
Scenarios	cp	bp	pp	dp	Scenarios	cp	bp	pp	dp
1	cp_1	bp_1	pp_1	dp_1	21	cp_1	bp_3	pp_1	dp_3
2	cp_1	bp_1	pp_1	dp_2	22	cp_1	bp_3	pp_2	dp_1
3	cp_1	bp_1	pp_1	dp_3	23	cp_1	bp_3	pp_2	dp_2
4	cp_1	bp_1	pp_2	dp_1	24	cp_1	bp_3	pp_2	dp_3
5	cp_1	bp_1	pp_2	dp_2	25	cp_1	bp_3	pp_3	dp_1
6	cp_1	bp_1	pp_2	dp_3	26	cp_1	bp_3	pp_3	dp_2
7	cp_1	bp_1	pp_3	dp_1	27	cp_1	bp_3	pp_3	dp_3
8	cp_1	bp_1	pp_3	dp_2	28	cp_2	bp_1	pp_1	dp_1
9	cp_1	bp_1	pp_3	dp_3	29	cp_2	bp_1	pp_1	dp_2
10	cp_1	bp_2	pp_1	dp_1	30	cp_2	bp_1	pp_1	dp_3
11	cp_1	bp_2	pp_1	dp_2	31	cp_2	bp_1	pp_2	dp_1
12	cp_1	bp_2	pp_1	dp_3	32	cp_2	bp_1	pp_2	dp_2
13	cp_1	bp_2	pp_2	dp_1	33	cp_2	bp_1	pp_2	dp_3
14	cp_1	bp_2	pp_2	dp_2	34	cp_2	bp_1	pp_3	dp_1
15	cp_1	bp_2	pp_2	dp_3	35	cp_2	bp_1	pp_3	dp_2
16	cp_1	bp_2	pp_3	dp_1	36	cp_2	bp_1	pp_3	dp_3
17	cp_1	bp_2	pp_3	dp_2	37	cp_2	bp_2	pp_1	dp_1
18	cp_1	bp_2	pp_3	dp_3	38	cp_2	bp_2	pp_1	dp_2
19	cp_1	bp_3	pp_1	dp_1	39	cp_2	bp_2	pp_1	dp_3
20	cp_1	bp_3	pp_1	dp_2	40	cp_2	bp_2	pp_2	dp_1

Table B.1. Parameter Sets of Scenarios for Two-Echelon Problem continue

	Parameter Set					Parameter Set			
Scenarios	cp	bp	pp	dp	Scenarios	cp	bp	pp	dp
41	cp_2	bp_2	pp_2	dp_2	69	cp_2	bp_2	pp_4	dp_3
42	cp_2	bp_2	pp_2	dp_3	70	cp_2	bp_3	pp_4	dp_1
43	cp_2	bp_2	pp_3	dp_1	71	cp_2	bp_3	pp_4	dp_2
44	cp_2	bp_2	pp_3	dp_2	72	cp_2	bp_3	pp_4	dp_3
45	cp_2	bp_2	pp_3	dp_3	73	cp_1	bp_1	pp_1	dp'_1
46	cp_2	bp_3	pp_1	dp_1	74	cp_1	bp_1	pp_1	dp'_2
47	cp_2	bp_3	pp_1	dp_2	75	cp_1	bp_1	pp_1	dp'_3
48	cp_2	bp_3	pp_1	dp_3	76	cp_1	bp_1	pp_2	dp'_1
49	cp_2	bp_3	pp_2	dp_1	77	cp_1	bp_1	pp_2	dp'_2
50	cp_2	bp_3	pp_2	dp_2	78	cp_1	bp_1	pp_2	dp'_3
51	cp_2	bp_3	pp_2	dp_3	79	cp_1	bp_1	pp_3	dp'_1
52	cp_2	bp_3	pp_3	dp_1	80	cp_1	bp_1	pp_3	dp'_2
53	cp_2	bp_3	pp_3	dp_2	81	cp_1	bp_1	pp_3	dp'_3
54	cp_2	bp_3	pp_3	dp_3	82	cp_1	bp_2	pp_1	dp'_1
55	cp_1	bp_1	pp_4	dp_1	83	cp_1	bp_2	pp_1	dp'_2
56	cp_1	bp_1	pp_4	dp_2	84	cp_1	bp_2	pp_1	dp'_3
57	cp_1	bp_1	pp_4	dp_3	85	cp_1	bp_2	pp_2	dp'_1
58	cp_1	bp_2	pp_4	dp_1	86	cp_1	bp_2	pp_2	dp'_2
59	cp_1	bp_2	pp_4	dp_2	87	cp_1	bp_2	pp_2	dp'_3
60	cp_1	bp_2	pp_4	dp_3	88	cp_1	bp_2	pp_3	dp'_1
61	cp_1	bp_3	pp_4	dp_1	89	cp_1	bp_2	pp_3	dp'_2
62	cp_1	bp_3	pp_4	dp_2	90	cp_1	bp_2	pp_3	dp'_3
63	cp_1	bp_3	pp_4	dp_3	91	cp_1	bp_3	pp_1	dp'_1
64	cp_2	bp_1	pp_4	dp_1	92	cp_1	bp_3	pp_1	dp'_2
65	cp_2	bp_1	pp_4	dp_2	93	cp_1	bp_3	pp_1	dp'_3
66	cp_2	bp_1	pp_4	dp_3	94	cp_1	bp_3	pp_2	dp'_1
67	cp_2	bp_2	pp_4	dp_1	95	cp_1	bp_3	pp_2	dp'_2
68	cp_2	bp_2	pp_4	dp_2	96	cp_1	bp_3	pp_2	dp'_3

Table B.1. Parameter Sets of Scenarios for Two-Echelon Problem continue

	Parameter Set					Parameter Set			
Scenarios	cp	bp	pp	dp	Scenarios	cp	bp	pp	dp
97	cp_1	bp_3	pp_3	dp'_1	121	cp_2	bp_3	pp_2	dp'_1
98	cp_1	bp_3	pp_3	dp'_2	122	cp_2	bp_3	pp_2	dp'_2
99	cp_1	bp_3	pp_3	dp'_3	123	cp_2	bp_3	pp_2	dp'_3
100	cp_2	bp_1	pp_1	dp'_1	124	cp_2	bp_3	pp_3	dp'_1
101	cp_2	bp_1	pp_1	dp'_2	125	cp_2	bp_3	pp_3	dp'_2
102	cp_2	bp_1	pp_1	dp'_3	126	cp_2	bp_3	pp_3	dp'_3
103	cp_2	bp_1	pp_2	dp'_1	127	cp_1	bp_1	pp_4	dp'_1
104	cp_2	bp_1	pp_2	dp'_2	128	cp_1	bp_1	pp_4	dp'_2
105	cp_2	bp_1	pp_2	dp'_3	129	cp_1	bp_1	pp_4	dp'_3
106	cp_2	bp_1	pp_3	dp'_1	130	cp_1	bp_2	pp_4	dp'_1
107	cp_2	bp_1	pp_3	dp'_2	131	cp_1	bp_2	pp_4	dp'_2
108	cp_2	bp_1	pp_3	dp'_3	132	cp_1	bp_2	pp_4	dp'_3
109	cp_2	bp_2	pp_1	dp'_1	133	cp_1	bp_3	pp_4	dp'_1
110	cp_2	bp_2	pp_1	dp'_2	134	cp_1	bp_3	pp_4	dp'_2
111	cp_2	bp_2	pp_1	dp'_3	135	cp_1	bp_3	pp_4	dp'_3
112	cp_2	bp_2	pp_2	dp'_1	136	cp_2	bp_1	pp_4	dp'_1
113	cp_2	bp_2	pp_2	dp'_2	137	cp_2	bp_1	pp_4	dp'_2
114	cp_2	bp_2	pp_2	dp'_3	138	cp_2	bp_1	pp_4	dp'_3
115	cp_2	bp_2	pp_3	dp'_1	139	cp_2	bp_2	pp_4	dp'_1
116	cp_2	bp_2	pp_3	dp'_2	140	cp_2	bp_2	pp_4	dp'_2
117	cp_2	bp_2	pp_3	dp'_3	141	cp_2	bp_2	pp_4	dp'_3
118	cp_2	bp_3	pp_1	dp'_1	142	cp_2	bp_3	pp_4	dp'_1
119	cp_2	bp_3	pp_1	dp'_2	143	cp_2	bp_3	pp_4	dp'_2
120	cp_2	bp_3	pp_1	dp'_3	144	cp_2	bp_3	pp_4	dp'_3

APPENDIX C: Results of Scenarios for Two-Echelon Problem

Table C.1. Results of Scenarios for Two-Echelon Problem

		Period								
Sc. Num.	Problem	1	2	3	4	5	6	7	8	Cost
1	with allocation rule	123	108	93	78	84	63	42	21	1797.91
	without allocation rule	123	108	93	78	84	63	42	21	1797.91
2	with allocation rule	123	108	87	72	72	57	36	21	1904.95
	without allocation rule	123	108	87	72	72	57	36	21	1904.95
3	with allocation rule	126	108	90	72	72	54	36	18	1943.35
	without allocation rule	126	108	90	72	72	54	36	18	1943.35
4	with allocation rule	15	15	15	15	21	21	21	21	78.7
	without allocation rule	15	15	15	15	21	21	21	21	78.7
5	with allocation rule	15	21	15	21	15	21	15	21	78.74
	without allocation rule	15	21	15	21	15	21	15	21	78.74
6	with allocation rule	18	18	18	18	18	18	18	18	78.89
	without allocation rule	18	18	18	18	18	18	18	18	78.89
7	with allocation rule	30	30	30	36	21	42	21	21	225.28
	without allocation rule	30	30	30	36	21	42	21	21	225.28
8	with allocation rule	36	36	36	36	15	36	15	21	220.73
	without allocation rule	36	36	36	36	15	36	15	21	220.73
9	with allocation rule	36	36	36	36	18	36	18	18	239.17
	without allocation rule	36	36	36	36	18	36	18	18	239.17

Table C.1. Results of Scenarios for Two-Echelon Problem continue

		Period								
Sc. Num.	Problem	1	2	3	4	5	6	7	8	Cost
10	with allocation rule	123	120	105	90	84	63	42	21	2557.01
	without allocation rule	123	120	105	90	84	63	42	21	2557.01
11	with allocation rule	123	120	99	84	72	57	36	21	2783.24
	without allocation rule	123	120	99	84	72	57	36	21	2783.24
12	with allocation rule	126	117	99	81	72	54	36	18	2860.4
	without allocation rule	126	117	99	81	72	54	36	18	2860.4
13	with allocation rule	21	21	21	27	33	33	33	21	112.7
	without allocation rule	21	21	21	27	33	33	33	21	112.7
14	with allocation rule	27	27	27	27	27	27	27	21	112.7
	without allocation rule	27	27	27	27	27	27	27	21	112.7
15	with allocation rule	27	27	27	27	27	27	27	18	113.03
	without allocation rule	27	27	27	27	27	27	27	18	113.03
16	with allocation rule	30	30	30	36	33	42	33	21	301.6
	without allocation rule	30	30	30	36	33	42	33	21	301.6
17	with allocation rule	36	36	36	36	27	36	27	21	302.08
	without allocation rule	36	36	36	36	27	36	27	21	302.08
18	with allocation rule	36	36	36	36	27	36	27	18	331.76
	without allocation rule	36	36	36	36	27	36	27	18	331.76
19	with allocation rule	135	120	105	90	84	63	42	21	4062
	without allocation rule	135	120	105	90	84	63	42	21	4062
20	with allocation rule	135	120	99	84	72	57	36	21	4526.61
	without allocation rule	135	120	99	84	72	57	36	21	4526.61
21	with allocation rule	135	117	99	81	72	54	36	18	4684.57
	without allocation rule	135	117	99	81	72	54	36	18	4684.57
22	with allocation rule	21	21	21	27	33	33	33	21	126.55
	without allocation rule	21	21	21	27	33	33	33	21	126.55

Table C.1. Results of Scenarios for Two-Echelon Problem continue

		Period								
Sc. Num.	Problem	1	2	3	4	5	6	7	8	Cost
23	with allocation rule	27	27	27	27	27	27	27	21	126.56
	without allocation rule	27	27	27	27	27	27	27	21	126.56
24	with allocation rule	27	27	27	27	27	27	27	18	129.92
	without allocation rule	27	27	27	27	27	27	27	18	129.92
25	with allocation rule	36	36	42	48	45	42	33	21	421.92
	without allocation rule	36	36	42	48	54	42	33	21	421.93
26	with allocation rule	42	48	42	48	33	36	27	21	423.7
	without allocation rule	42	48	42	48	42	36	27	21	423.71
27	with allocation rule	45	45	45	45	36	36	27	18	484.78
	without allocation rule	45	45	45	45	45	36	27	18	484.78
28	with allocation rule	102	108	93	78	63	63	42	21	1950.12
	without allocation rule	102	108	93	78	63	63	42	21	1950.12
29	with allocation rule	108	108	87	72	51	57	36	21	2057.51
	without allocation rule	108	108	87	72	51	57	36	21	2057.51
30	with allocation rule	108	108	90	72	54	54	36	18	2097.24
	without allocation rule	108	108	90	72	54	54	36	18	2097.24
31	with allocation rule	15	15	15	15	21	21	21	21	362.04
	without allocation rule	15	15	15	15	21	21	21	21	362.04
32	with allocation rule	15	21	15	21	15	21	15	21	362.19
	without allocation rule	15	21	15	21	15	21	15	21	362.19
33	with allocation rule	18	18	18	18	18	18	18	18	362.89
	without allocation rule	18	18	18	18	18	18	18	18	362.89
34	with allocation rule	30	30	30	36	21	42	21	21	508.27
	without allocation rule	30	30	30	36	21	42	21	21	508.27
35	with allocation rule	36	36	36	36	15	36	15	21	503.82
	without allocation rule	36	36	36	36	15	36	15	21	503.82

Table C.1. Results of Scenarios for Two-Echelon Problem continue

		Period								
Sc. Num.	Problem	1	2	3	4	5	6	7	8	Cost
36	with allocation rule	36	36	36	36	18	36	18	18	522.86
	without allocation rule	36	36	36	36	18	36	18	18	522.86
37	with allocation rule	114	108	105	90	75	63	42	21	2714.3
	without allocation rule	114	108	105	90	75	63	42	21	2714.3
38	with allocation rule	114	108	99	84	63	57	36	21	2940.53
	without allocation rule	114	108	99	84	63	57	36	21	2940.53
39	with allocation rule	117	108	99	81	63	54	36	18	3018.1
	without allocation rule	117	108	99	81	63	54	36	18	3018.1
40	with allocation rule	21	21	21	27	33	33	21	21	396.27
	without allocation rule	21	21	21	27	33	33	21	21	396.27
41	with allocation rule	27	27	27	27	27	27	15	21	396.29
	without allocation rule	27	27	27	27	27	27	15	21	396.29
42	with allocation rule	27	27	27	27	27	27	18	18	397.21
	without allocation rule	27	27	27	27	27	27	18	18	397.21
43	with allocation rule	30	30	30	36	33	42	21	21	584.6
	without allocation rule	30	30	30	36	33	42	21	21	584.6
44	with allocation rule	36	36	36	36	27	36	15	21	585.19
	without allocation rule	36	36	36	36	27	36	15	21	585.19
45	with allocation rule	36	36	36	36	27	36	18	18	615.47
	without allocation rule	36	36	36	36	27	36	18	18	615.47
46	with allocation rule	126	120	105	90	75	63	42	21	4220.96
	without allocation rule	135	120	105	90	75	63	42	21	4221.49
47	with allocation rule	126	120	99	84	63	57	36	21	4685.57
	without allocation rule	135	120	99	84	63	57	36	21	4686.11
48	with allocation rule	126	117	99	81	63	54	36	18	4843.53
	without allocation rule	135	117	99	81	63	54	36	18	4844.07

Table C.1. Results of Scenarios for Two-Echelon Problem continue

		Period								
Sc. Num.	Problem	1	2	3	4	5	6	7	8	Cost
49	with allocation rule	21	21	21	27	33	33	33	21	412.28
	without allocation rule	21	21	21	27	33	33	33	21	412.28
50	with allocation rule	27	27	27	27	27	27	27	21	412.31
	without allocation rule	27	27	27	27	27	27	27	21	412.31
51	with allocation rule	27	27	27	27	27	27	27	18	415.72
	without allocation rule	27	27	27	27	27	27	27	18	415.72
52	with allocation rule	36	36	42	48	45	42	33	21	705.37
	without allocation rule	36	36	42	48	54	42	33	21	705.39
53	with allocation rule	42	48	42	48	33	36	27	21	707.26
	without allocation rule	42	48	42	48	42	36	27	21	707.28
54	with allocation rule	45	45	45	45	36	36	27	18	768.83
	without allocation rule	45	45	45	45	45	36	27	18	768.85
55	with allocation rule	45	45	51	57	63	42	42	21	376.3
	without allocation rule	45	45	51	57	63	42	42	21	376.3
56	with allocation rule	51	57	51	57	51	36	36	21	381.29
	without allocation rule	51	57	51	57	51	36	36	21	381.29
57	with allocation rule	54	54	54	54	54	36	36	18	388.9
	without allocation rule	54	54	54	54	54	36	36	18	388.9
58	with allocation rule	51	57	63	69	63	54	42	21	474.2
	without allocation rule	51	57	63	69	63	54	42	21	474.2
59	with allocation rule	63	63	63	63	51	48	36	21	488.82
	without allocation rule	63	63	63	63	51	48	36	21	488.82
60	with allocation rule	63	63	63	63	54	45	36	18	505.37
	without allocation rule	63	63	63	63	54	45	36	18	505.37
61	with allocation rule	63	69	75	69	75	54	42	21	630.01
	without allocation rule	72	57	63	69	75	54	42	21	634.31

Table C.1. Results of Scenarios for Two-Echelon Problem continue

		Period								
Sc. Num.	Problem	1	2	3	4	5	6	7	8	Cost
62	with allocation rule	69	75	69	63	63	48	36	21	670.42
	without allocation rule	63	63	63	63	63	48	36	21	673.7
63	with allocation rule	72	72	72	63	63	45	36	18	705.9
	without allocation rule	63	63	63	63	63	45	36	18	709.42
64	with allocation rule	45	45	51	57	63	42	42	21	648.62
	without allocation rule	45	45	51	57	63	42	42	21	648.62
65	with allocation rule	51	57	51	57	51	36	36	21	654.41
	without allocation rule	51	57	51	57	51	36	36	21	654.41
66	with allocation rule	54	54	54	54	54	36	36	18	663.26
	without allocation rule	54	54	54	54	54	36	36	18	663.26
67	with allocation rule	51	57	63	69	63	54	42	21	750.88
	without allocation rule	51	57	63	69	63	54	42	21	750.88
68	with allocation rule	63	63	63	63	51	48	36	21	765.88
	without allocation rule	63	63	63	63	51	48	36	21	765.88
69	with allocation rule	63	63	63	63	54	45	36	18	783.04
	without allocation rule	63	63	63	63	54	45	36	18	783.04
70	with allocation rule	63	69	63	69	75	54	42	21	908.47
	without allocation rule	72	57	63	69	75	54	42	21	912.66
71	with allocation rule	69	75	63	63	63	48	36	21	949.21
	without allocation rule	63	63	63	63	63	48	36	21	952.25
72	with allocation rule	72	72	63	63	63	45	36	18	984.9
	without allocation rule	63	63	63	63	63	45	36	18	988.21
73	with allocation rule	144	126	108	90	96	72	48	24	2121.8
	without allocation rule	144	126	108	90	96	72	48	24	2121.8
74	with allocation rule	144	126	102	84	84	66	42	24	2228.84
	without allocation rule	144	126	102	84	84	66	42	24	2228.84

Table C.1. Results of Scenarios for Two-Echelon Problem continue

		Period								
Sc. Num.	Problem	1	2	3	4	5	6	7	8	Cost
75	with allocation rule	147	126	105	84	84	63	42	21	2267.24
	without allocation rule	147	126	105	84	84	63	42	21	2267.24
76	with allocation rule	18	18	18	18	24	24	24	24	91.85
	without allocation rule	18	18	18	18	24	24	24	24	91.85
77	with allocation rule	18	24	18	24	18	24	18	24	91.89
	without allocation rule	18	24	18	24	18	24	18	24	91.89
78	with allocation rule	21	21	21	21	21	21	21	21	92.04
	without allocation rule	21	21	21	21	21	21	21	21	92.04
79	with allocation rule	36	36	36	42	24	48	24	24	265.15
	without allocation rule	36	36	36	42	24	48	24	24	265.15
80	with allocation rule	42	42	42	42	18	42	18	24	260.59
	without allocation rule	42	42	42	42	18	42	18	24	260.59
81	with allocation rule	42	42	42	42	21	42	21	21	279.03
	without allocation rule	42	42	42	42	21	42	21	21	279.03
82	with allocation rule	144	141	123	105	96	72	48	24	3186.58
	without allocation rule	144	141	123	105	96	72	48	24	3186.58
83	with allocation rule	144	141	117	99	84	66	42	24	3412.81
	without allocation rule	144	141	117	99	84	66	42	24	3412.81
84	with allocation rule	147	138	117	96	84	63	42	21	3489.97
	without allocation rule	147	138	117	96	84	63	42	21	3489.97
85	with allocation rule	27	27	27	33	39	39	39	24	137.23
	without allocation rule	27	27	27	33	39	39	39	24	137.23
86	with allocation rule	33	33	33	33	33	33	33	24	137.23
	without allocation rule	33	33	33	33	33	33	33	24	137.23
87	with allocation rule	33	33	33	33	33	33	33	21	137.56
	without allocation rule	33	33	33	33	33	33	33	21	137.56

Table C.1. Results of Scenarios for Two-Echelon Problem continue

		Period								
Sc. Num.	Problem	1	2	3	4	5	6	7	8	Cost
88	with allocation rule	36	36	36	42	39	48	39	24	372.32
	without allocation rule	36	36	36	42	39	48	39	24	372.32
89	with allocation rule	42	42	42	42	33	42	33	24	372.8
	without allocation rule	42	42	42	42	33	42	33	24	372.8
90	with allocation rule	42	42	42	42	33	42	33	21	402.49
	without allocation rule	42	42	42	42	33	42	33	21	402.49
91	with allocation rule	159	141	123	105	96	72	48	24	5299.63
	without allocation rule	159	141	123	105	96	72	48	24	5299.63
92	with allocation rule	159	141	117	99	84	66	42	24	5764.25
	without allocation rule	159	141	117	99	84	66	42	24	5764.25
93	with allocation rule	159	138	117	96	84	63	42	21	5922.21
	without allocation rule	159	138	117	96	84	63	42	21	5922.21
94	with allocation rule	27	27	27	33	39	39	39	24	156.71
	without allocation rule	27	27	27	33	39	39	39	24	156.71
95	with allocation rule	33	33	33	33	33	33	33	24	156.72
	without allocation rule	33	33	33	33	33	33	33	24	156.72
96	with allocation rule	33	33	33	33	33	33	33	21	160.08
	without allocation rule	33	33	33	33	33	33	33	21	160.08
97	with allocation rule	45	45	51	57	54	48	39	24	543.65
	without allocation rule	45	45	51	57	63	48	39	24	543.66
98	with allocation rule	51	57	51	57	42	42	33	24	545.43
	without allocation rule	51	57	51	57	51	42	33	24	545.44
99	with allocation rule	54	54	54	54	45	42	33	21	606.51
	without allocation rule	54	54	54	54	54	42	33	21	606.52
100	with allocation rule	120	126	108	90	72	72	48	24	2299.66
	without allocation rule	120	126	108	90	72	72	48	24	2299.66

Table C.1. Results of Scenarios for Two-Echelon Problem continue

		Period								
Sc. Num.	Problem	1	2	3	4	5	6	7	8	Cost
101	with allocation rule	126	126	102	84	60	66	42	24	2407.05
	without allocation rule	126	126	102	84	60	66	42	24	2407.05
102	with allocation rule	126	126	105	84	63	63	42	21	2446.79
	without allocation rule	126	126	105	84	63	63	42	21	2446.79
103	with allocation rule	18	18	18	18	24	24	24	24	422.52
	without allocation rule	18	18	18	18	24	24	24	24	422.52
104	with allocation rule	18	24	18	24	18	24	18	24	422.67
	without allocation rule	18	24	18	24	18	24	18	24	422.67
105	with allocation rule	21	21	21	21	21	21	21	21	423.37
	without allocation rule	21	21	21	21	21	21	21	21	423.37
106	with allocation rule	36	36	36	42	24	48	24	24	595.41
	without allocation rule	36	36	36	42	24	48	24	24	595.41
107	with allocation rule	42	42	42	42	18	42	18	24	590.96
	without allocation rule	42	42	42	42	18	42	18	24	590.96
108	with allocation rule	42	42	42	42	21	42	21	21	610
	without allocation rule	42	42	42	42	21	42	21	21	610
109	with allocation rule	135	126	123	105	87	72	48	24	3370.8
	without allocation rule	135	126	123	105	87	72	48	24	3370.8
110	with allocation rule	135	126	117	99	75	66	42	24	3597.02
	without allocation rule	135	126	117	99	75	66	42	24	3597.02
111	with allocation rule	138	126	117	96	75	63	42	21	3674.6
	without allocation rule	138	126	117	96	75	63	42	21	3674.6
112	with allocation rule	27	27	27	33	39	39	24	24	468.19
	without allocation rule	27	27	27	33	39	39	24	24	468.19
113	with allocation rule	33	33	33	33	33	33	18	24	468.21
	without allocation rule	33	33	33	33	33	33	18	24	468.21

Table C.1. Results of Scenarios for Two-Echelon Problem continue

		Period								
Sc. Num.	Problem	1	2	3	4	5	6	7	8	Cost
114	with allocation rule	33	33	33	33	33	33	21	21	469.14
	without allocation rule	33	33	33	33	33	33	21	21	469.14
115	with allocation rule	36	36	36	42	39	48	24	24	702.61
	without allocation rule	36	36	36	42	39	48	24	24	702.61
116	with allocation rule	42	42	42	42	33	42	18	24	703.2
	without allocation rule	42	42	42	42	33	42	18	24	703.2
117	with allocation rule	42	42	42	42	33	42	21	21	733.48
	without allocation rule	42	42	42	42	33	42	21	21	733.48
118	with allocation rule	150	141	123	105	87	72	48	24	5485.93
	without allocation rule	159	141	123	105	87	72	48	24	5486.46
119	with allocation rule	150	141	117	99	75	66	42	24	5950.54
	without allocation rule	159	141	117	99	75	66	42	24	5951.08
120	with allocation rule	150	138	117	96	75	63	42	21	6108.5
	without allocation rule	159	138	117	96	75	63	42	21	6109.04
121	with allocation rule	27	27	27	33	39	39	39	24	490.37
	without allocation rule	27	27	27	33	39	39	39	24	490.37
122	with allocation rule	33	33	33	33	33	33	33	24	490.4
	without allocation rule	33	33	33	33	33	33	33	24	490.4
123	with allocation rule	33	33	33	33	33	33	33	21	493.81
	without allocation rule	33	33	33	33	33	33	33	21	493.81
124	with allocation rule	45	45	51	57	54	48	39	24	874.51
	without allocation rule	45	45	51	57	63	48	39	24	874.53
125	with allocation rule	51	57	51	57	42	42	33	24	876.39
	without allocation rule	51	57	51	57	51	42	33	24	876.41
126	with allocation rule	54	54	54	54	45	42	33	21	937.96
	without allocation rule	54	54	54	54	54	42	33	21	937.98

Table C.1. Results of Scenarios for Two-Echelon Problem continue

		Period								
Sc. Num.	Problem	1	2	3	4	5	6	7	8	Cost
127	with allocation rule	54	54	60	66	72	48	48	24	441.12
	without allocation rule	54	54	60	66	72	48	48	24	441.12
128	with allocation rule	60	66	60	66	60	42	42	24	446.11
	without allocation rule	60	66	60	66	60	42	42	24	446.11
129	with allocation rule	63	63	63	63	63	42	42	21	453.71
	without allocation rule	63	63	63	63	63	42	42	21	453.71
130	with allocation rule	63	69	75	81	72	63	48	24	577.84
	without allocation rule	63	69	75	81	72	63	48	24	577.84
131	with allocation rule	75	75	75	75	60	57	42	24	592.46
	without allocation rule	75	75	75	75	60	57	42	24	592.46
132	with allocation rule	75	75	75	75	63	54	42	21	609.01
	without allocation rule	75	75	75	75	63	54	42	21	609.01
133	with allocation rule	78	84	90	81	87	63	48	24	800.5
	without allocation rule	87	69	75	81	87	63	48	24	805.27
134	with allocation rule	84	90	84	75	75	57	42	24	840.9
	without allocation rule	75	75	75	75	75	57	42	24	845.36
135	with allocation rule	87	87	87	75	75	54	42	21	876.39
	without allocation rule	96	75	75	75	75	54	42	21	880.69
136	with allocation rule	54	54	60	66	72	48	48	24	759.16
	without allocation rule	54	54	60	66	72	48	48	24	759.16
137	with allocation rule	60	66	60	66	60	42	42	24	764.95
	without allocation rule	60	66	60	66	60	42	42	24	764.95
138	with allocation rule	63	63	63	63	63	42	42	21	773.8
	without allocation rule	63	63	63	63	63	42	42	21	773.8
139	with allocation rule	63	69	75	81	72	63	48	24	901.35
	without allocation rule	63	69	75	81	72	63	48	24	901.35

Table C.1. Results of Scenarios for Two-Echelon Problem continue

		Period								
Sc. Num.	Problem	1	2	3	4	5	6	7	8	Cost
140	with allocation rule	75	75	75	75	60	57	42	24	916.35
	without allocation rule	75	75	75	75	60	57	42	24	916.35
141	with allocation rule	75	75	75	75	63	54	42	21	933.5
	without allocation rule	75	75	75	75	63	54	42	21	933.5
142	with allocation rule	78	84	75	81	87	63	48	24	1126.23
	without allocation rule	87	69	75	81	87	63	48	24	1130.84
143	with allocation rule	84	90	75	75	75	57	42	24	1166.96
	without allocation rule	75	75	75	75	75	57	42	24	1171.11
144	with allocation rule	87	87	75	75	75	54	42	21	1202.66
	without allocation rule	96	75	75	75	75	54	42	21	1206.84

APPENDIX D: Parameters of Scenarios for Single Stage Inventory Problem

Table D.1. Parameter Set of Scenarios for Inventory Problem

n	Holding cost	Demand probability	Supply probability
1	hp_1	pp_1	pp_1
2	hp_1	pp_2	pp_1
3	hp_1	pp_3	pp_1
4	hp_1	pp_1	pp_2
5	hp_1	pp_2	pp_2
6	hp_1	pp_3	pp_2
7	hp_1	pp_1	pp_3
8	hp_1	pp_2	pp_3
9	hp_1	pp_3	pp_3
10	hp_2	pp_1	pp_1
11	hp_2	pp_2	pp_1
12	hp_2	pp_3	pp_1
13	hp_2	pp_1	pp_2
14	hp_2	pp_2	pp_2
15	hp_2	pp_3	pp_2
16	hp_2	pp_1	pp_3
17	hp_2	pp_2	pp_3
18	hp_2	pp_3	pp_3
19	hp_1	pp_4	pp_4
20	hp_1	pp_4	pp_5

Table D.1. Parameter Set of Scenarios for Inventory Problem continue

n	Holding cost	Demand probability	Supply probability
21	hp_1	pp_5	pp_4
22	hp_1	pp_5	pp_5
23	hp_2	pp_4	pp_4
24	hp_2	pp_4	pp_5
25	hp_2	pp_5	pp_4
26	hp_2	pp_5	pp_5

APPENDIX E: Results of Scenarios for Single Stage Inventory Problem

Table E.1. Results of Scenarios for Inventory Problem

	Period										
Sc. Num.	1	2	3	4	5	6	7	8	9	10	Cost
1	20	20	20	10	10	10	10	10	10	10	612.94
2	60	50	50	40	40	30	30	20	20	10	2852.61
3	90	80	70	60	60	50	40	30	20	10	5046.71
4	10	10	10	10	10	10	10	10	10	10	137.49
5	30	30	30	30	30	20	20	20	20	10	399.49
6	40	40	40	40	40	40	30	30	20	10	609.43
7	10	10	10	10	10	10	10	10	10	10	92.72
8	10	10	10	10	10	10	10	10	10	10	110.49
9	20	20	20	20	20	20	20	20	20	10	127.5
10	10	10	10	10	10	10	10	0	0	0	736.33
11	40	30	30	30	30	20	20	20	10	10	3142.77
12	70	60	60	50	40	40	30	30	20	10	5452.07
13	0	0	0	0	0	0	0	0	0	0	360.04
14	20	20	20	20	20	10	10	10	10	10	786.91
15	30	30	30	30	20	20	20	20	20	10	1101.85
16	0	0	0	0	0	0	0	0	0	0	219.75
17	10	10	10	10	10	10	10	10	10	10	298.97
18	10	10	10	10	10	10	10	10	10	10	227.04
19	10	20	10	20	10	20	10	20	10	10	163.98
20	20	20	20	20	20	20	20	20	10	10	176.25

Table E.1. Results of Scenarios for Inventory Problem continue

	Period										
Sc. Num.	1	2	3	4	5	6	7	8	9	10	Cost
21	20	20	20	20	20	20	20	10	10	10	350.48
22	20	10	20	10	20	10	20	10	10	10	164.99
23	10	10	10	10	10	10	10	10	0	10	300.29
24	10	10	10	10	10	10	10	10	10	10	417.34
25	10	10	10	10	10	10	10	10	10	0	554.4
26	10	10	10	10	10	10	10	10	10	0	316.82

REFERENCES

Arifoglu, K. and S. Ozekici, 2010, *Optimal policies for inventory systems with finite capacity and partially observed Markov-modulated demand and supply processes, European Journal of Operational Research,* Vol. 204, pp. 421-438.

Arslan H., C. S. Graves and T. Roemer, 2007, *A single-product inventory model for multiple demand classes, Management Science,* Vol.00, pp. 1-15.

Axs äter S., 2003, *A new decision rule for lateral transshipments in inventory systems, Management Science,* Vol. 49, pp. 1168-1179.

Berman, Krass and Tajbakhsh, 2010, *On the benefits of risk pooling in inventory management, Production and Operations Management,* pp. 1-15.

Cachon, G. and A. M. Lariviere, 1999, *Capacity choice and allocation: strategic behavior and supply chain performance, Management Science,* Vol. 45, pp. 1091-1108.

Cai X. and D. Du, 2009, *On the effects of risk pooling in supply chain management: Review and extensions, Acta Mathematica Applicatae Sinica,* Vol. 25, pp. 709-722.

Chen, M. and C. Lin, 1989, *Effects of centralisation on expected costs in a multi-location newsboy problem, Journal of the Operational Research Society,* Vol. 40, pp. 597-602.

Cherikh, M., 2000, *On the effect of centralisation on expected profits in a multi-location newsboy problem, Journal of the Operational Research Society,* Vol. 51, pp. 755-761.

Ciarallo, F.W., R. Akella and T.E. Morton, 1994, *A periodic review production planning model with uncertain capacity and uncertain demand-optimality of extended myopic policies, Management Science,* Vol. 40, No. 3, pp. 320-332.

Clark, A.J. and H. Scarf, 1960, *Optimal policies for a multi-echelon inventory problem, Management Science, Vol. 50, pp. 475-490.*

Das, C., 1975, Supply and redistribution rules for two-location inventory systems: One-period analysis, Management Science, Vol. 21, pp. 765-776.

de Véricourt, F., F. Karaesmen and Y. Dallery, 2002, *Optimal stock allocation for a capacitated supply system, Management Science,* Vol. 48, pp. 1486-1501.

Diks, E.B. and Kok de A.G., 1998, *Optimal control of a divergent multi-echelon inventory system, EJOR,* Vol. 111, pp. 75-97.

Eppen, G. and L. Schrage, 1981, *Centralized ordering policies in a multi-warehouse system with lead times and random demand, TIMS Studies in the Management Sciences,* Vol. 16, pp. 51-67.

Eppen, G., 1979, *Effects of centralisation on expected costs in a multi-location newsboy problem, Management Science,* Vol. 25, pp. 498-501.

Erdem, A.S. and S. Ozekici, 2002, *Inventory models with random yield in a random environment, International Journal of Production Economics,* Vol. 78, pp. 239-253.

Federgruen, A. and P. H. Zipkin, 1984a, *Approximations of dynamic multilocation production and inventory problems, Management Science,* Vol. 30, pp. 69-84.

Federgruen, A., 1993, *Centralized planning models for multi-echelon inventory systems under uncertainty, in: S. C. Graves et al. (eds.), Handbooks in OR & MS,* Vol. 4, North Amsterdam, pp. 133-173.

Frank, K.C., R.Q. Zhang, I. Duenyas, 1999, *Optimal policies for inventory systems with priority demand classes, Operations Research,* Vol. 51, pp. 993-1002.

Gullu, R., 1998, *Base stock policies for production/inventory problems with uncertain* *capacity levels, European Journal of Operations Research,* Vol. 105, pp. 43-51.

Gullu, R., E. Onal, and N. Erkip, 1997, *Analysis of a deterministic demand production/inventory system under non-stationary supply uncertainty, IIE Transactions,* Vol. 29, No. 8, pp. 703-709.

Gullu, R., E. Onal, and N. Erkip, 1999, *Analysis of an inventory system under supply uncertainty, International Journal of Production Economics,* Vol. 59, pp. 377-385.

Gerchak Y. and Q. He, 2003, *On the relation between the benefits of risk pooling and the variability of demand, IIE Transactions,* Vol. 35, pp. 1027-1031.

Gerchak, Y., R.G. Vickson and M. Parlar, 1988, *Periodic review production models with variable yield and uncertain demand, IIE Transactions,* Vol. 20, No. 2, pp.144-150.

Gross, D., 1963, *Centralized inventory control in multi-location supply systems, H. Scarf, D. Gilford, M. Shelly, eds, Multi-stage Inventory Models and Techniques,* pp. 47-84, Stanford University Press, Stanford, CA.

Ha A., 1997, *Stock-rationing policy for a make-to-stock production system with two priority classes and backordering, Naval Research Logistics,* Vol. 44, pp. 457-472.

Henig, M. and Y. Gerchak, 1990, *The structure of periodic review policies in the presence of random yield, Operations Research,* Vol.38, No.4, pp. 634-643.

Houtum, Van G.J. et al., 1996, *Materials coordination in stochastic multi-echelon systems, EJOR,* Vol. 95, pp. 1-23.

Iiada, T., 2002, *A non-stationary periodic review production-inventory model with uncertain production capacity and uncertain demand, European Journal of Operational Research,* Vol. 140, pp. 670-683.

Mohebbi, E., 2004, *A replenishment model for the supply-uncertainty problem, Inter-*

national Journal of Production Economics, Vol. 87, pp. 25-37.

Mohebbi, E., and D. Hao, 2006, *When supplier's availability affects the replenishment lead time - An extension of the supply-interruption problem, European Journal of Operational Research,* Vol. 175, pp. 992-1008.

Muckstadt, J. A. and R. Roundy, 1993, *Analysis of multistage production systems, in: S. C. Graves et al. (eds.), Handbooks in OR & MS,* Vol. 4, pp. 59-131, North Amsterdam.

Muckstadt, J. A. and R. Roundy, 1993, *Analysis of multistage production systems, in: S. C. Graves et al. (eds.), Handbooks in OR & MS,* Vol. 4, pp. 59-131, North Amsterdam.

Nahmias, S. and W.S. Demmy, 1981, *Operating characteristics of an inventory system with rationing, Management Science,* Vol. 51, pp. 1236-1245.

Ozekici, S. and M. Parlar, 1999, *Inventory models with unreliable suppliers in a random environment, Analysis of Operations Research,* Vol. 91, pp. 123-136.

Parlar, M. and D. Berkin, 1991, *Future supply uncertainty in EOQ models, Naval Research Logistics,* Vol. 38, pp. 107-121.

Parlar, M. and D. Perry, 1995, *Analysis of a (Q,r,T) inventory policy with deterministic and random yields when future supply is uncertain, European Journal of Operational Research,* vol. 84, pp. 431-443.

Parlar, M. and D. Perry, 1996, *Inventory models of future supply uncertainty with single and multiple suppliers, Naval Research Logistics,* Vol. 43, pp. 191-210.

Parlar, M., 1997, *Continuous-review inventory problem with random supply interruptions, European Journal of Operational Research,* Vol. 99, pp. 366-385.

Parlar, M., Y. Wang and Y. Gerchak, 1995, *A periodic review inventory model with*

Markovian supply availability, International Journal of Production Economics, Vol. 42, pp. 131-136.

Song, J.S. and P.H. Zipkin, 1996, *Inventory control with information about supply conditions, Management Science,* Vol. 42, No. 10, pp. 1409-1419.

Tagaras, G., 1989, *Effects of pooling on the optimization and service levels of two-location inventory systems, IIE Transactions,* Vol. 21, pp. 250-257.

Wang, Y. and Y. Gerchak, 1996, *Periodic review production models with variable capacity, random yield, and uncertain demand, Management Science,* Vol. 42, No. 1, pp. 130-137.

Yano, C.A. and H.L. Lee, 1995, *Lot sizing with random yields: A review, Operations Research,* Vol. 43, No. 2, pp.311-334.