RANDOM NUMBER GENERATION USING CHAOTIC DYNAMICAL MAPS

by

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B.S., Mathematics, Boğaziçi University, 2007

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Industrial Engineering
Boğaziçi University
2011

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DATE OF APPROVAL: 19.09.2011

To my family Yaşar, İnci, Murat Uğur and Özlem Özören

ACKNOWLEDGEMENTS

I would like to thank to Yaman Barlas for taking care of almost everything that I had to overcome during my studies. Besides his guidance and wisdom, his perspective on Dynamical Systems has changed my point of view on my life. His speed on analyzing a situation, his genius on detecting the problem on a situation, and his clever suggestions to solve the problem are really exciting and inspiring for me. I am so grateful and lucky to have a chance for studying with him.

I want to thank to Wolfgang Hörmann and Yağmur Denizhan for participating in my thesis committee. Their specializations and guidance improved the thesis.

I am very thankful to Serhat Keçici for his true friendship and helps. He, from my graduate year and on, helped me to adapt the department of industrial engineering and engineering education after my mathematics education. His point of view as an engineer is reference for me.

I also thank to İlker Evrim Çolak, who is from my undergraduate year and on. He has supported me to adapt myself to engineering education since I started the engineering education. At the earliest times of the thesis, we shared so much time to study, to discuss, and to laugh together.

And my family: my parents Yaşar Özören, İnci Özören, my brother Murat Uğur Özören and his wife Özlem Özören. I would not be me, or accomplish what I accomplished so far, if I was not so lucky to have such a family. I am very thankful to them for raising me, supporting me, and pushing me when I needed a push. They are an endless support and motivation for me.

I finally thank to Finansbank and my manager Burcu Kula for hiring me without any difficulties about my graduate education and supporting to complete my graduate education. I also tahnk to my colleagues in Retail Banking Analytics for their support.

ABSTRACT

RANDOM NUMBER GENERATION USING CHAOTIC DYNAMICAL MAPS

Random numbers are necessary basic ingredients for simulation and modeling. Currently, linear congruential generators (LCGs) are typically used as random number generators (RNGs), which generate pseudorandom numbers (PRNs) by using linear functions and modulus. In this study, we propose some chaotic functions to generate PRNs, using the unpredictability property of dynamical chaotic maps. We suggest five different RNGs that are derived from three different chaotic maps: tent map, logistic map, and family of connecting maps. The uniformity and independence of the numbers generated through the five suggested RNGs are checked in three steps. Firstly, the histograms and serial plots are visually checked. Secondly, chi-square and Kolmogorov-Smirnov tests are applied to statistically test the uniformity of the generated numbers. Finally, runs tests and autocorrelation test are applied in order to check the independence of the numbers. The same tests are applied to compare the five suggested chaotic generators with some well-known conventionally used LCGs. It is concluded that the suggested generators perform nearly as well as LCGs and can lead to an alternative way of generating random numbers. More detailed mathematical, statistical, and numerical properties of the suggested generators constitute useful further research topics.

ÖZET

KAOS ÖZELLİĞİ OLAN FONKSİYONLARLA RASTGELE SAYI ÜRETMEK

Rastgele sayılar, simülasyon ve modellemede kullanılan en gerekli unsurlardandır. Rastgele sayıları üretmek için en güncel ve yaygın yöntem doğrusal eşleşiksel üreteçlerdir. Bu üreteçler, doğrusal bir fonksiyon ve mod alma işlemini kullanarak sayıları üretir. Bu çalışmada ise, kaotik özelliği olan fonksiyonları kullanarak rastgele sayılar üretilebilmesi araştırılmaktadır. Kaotik özellikteki fonksiyonların tahmin edilemezlik özelliğine dayandırılarak; çadır fonksiyonu, lojistik fonksiyonu ve birleştirme fonksiyonundan üretilmiş beş farklı rastgele sayı üreteci önerilmektedir. Bu üreteçlerin ürettiği sayıların, $[0, 1]$ aralığında düzgün dağıldığı ve bağımsız sayılar ürettiği üç aşamada test edilmiştir. Öncelikle, üretilen sayıların histogramları ve dizisel grafikleri görsel olarak incelenmiştir. İkinci olarak, üretilen sayıların istatistiksel olarak düzgün dağıldıkları Ki-kare testi ve Kolmogorov-Smirnov testi kullanılarak test edilmiştir. Son olarak, "Run" testi ve otokorelasyon testi ile üretilen sayıların bağımsızlığı test edilmiştir. Ayrıca bu testler; önerilen beş üretecin, çok iyi bilinen ve sıklıkla kullanılan doğrusal eşleşiksel üreteçlerle kıyaslanırken de kullanılmıştır. Sonuç olarak, önerilen rastgele sayı üreteçlerinin görsel ve istatistiksel testlerde doğrusal eşleşiksel üreteçler kadar başarılı oldukları ve bu üreteçlerin rastgele sayı üretmek için kullanılabilecekleri ortaya çıkmıştır. Önerilen bu üreteçlerin; daha detaylı matematiksel, istatistiksel ve işlemsel özelliklerinin araştırılması ve incelenmesi gelecekte yapılacak yararlı araştırma konuları oluşturmaktadır.

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1. INTRODUCTION

A sequence of numbers that are chosen at random is a necessary basic ingredient in simulation, modeling and analysis in general. There are two types of random numbers: The true random numbers are completely unpredictable, and irreproducible. This kind of sequence can only be generated by a physical process, for instance, by electrical circuits or physical experiments like dice rolling. The pseudorandom numbers (PRNs) are on the other hand computed by numerical algorithms. They are supposed to appear random to someone who does not know the algorithm, and also pass the relevant statistical tests. Therefore these numbers are reproducible by the numerical algorithm, and they are most commonly used in simulations. In this work, the generation of the PRNs by constructing algorithms based on chaotic dynamical maps will be studied.

There is an enormous amount of scientific literature on random number generation. A short historical overview is presented by Law and Kelton [1]. The introduction to the properties of random numbers and the generation methods of random number are covered by Banks [2], and they are particularized by Knuth [3]. The random number generators (RNGs) have to generate nicely distributed random numbers. Nicely distributed means that the sequence of the generated random numbers is uniformly distributed between zero and one, and dense in this interval. Secondly, random numbers must be serially independent of each other. Good RNGs have also the following properties: long periodicity, repeatability, long disjoint sequences, portability, and efficiency [4].

Over the years, various linear PRN generators have been developed. They are classified into three categories. Linear congruential generator (LCG) and multiplicative linear congruential generator (MLCG) are famous and the most widely used techniques for generating random numbers. MLCG, first used by D.H. Lehmer in 1948, has a period of about approximately 10^9 on a 32-bit machine. Fibonacci random number generator and Tausworthe generators are the other two well-known generators that are based on computation of each number by performing some usual operations being addition, subtraction, and the 'exclusive-or' operation on the two preceding numbers. These generators have a period of about approximately 10^{170} on a 32-bit machine [4].

Besides such well-known linear PRN generators, three relatively new RNGs exist: Generalized feedback shift register (GFSR), twisted generalized feedback shift register (TGFSR), and Mersenne Twister (MT). GFSR is suggested in 1973 by Lewis and Payne [13]; and statistically analyzed in 1987 by H. Niederreiter [6]. TGFSR, a modified version of GFSR, is suggested in 1992 by Matsumoto and Kurita [7] and improved in 1994 by them [8]. These two generators have a period of about 10^{200} on a 32-bit machine. MT is developed in 1998 by Matsumoto and Nishimura [9]. It has a very long period of about 10^{6000} on a 32-bit machine.

The main goal of RNG research is to design robust generators without having to pay too much in terms of portability, flexibility, and efficiency. A good RNG should possess long period, high speed, and reliability. Moreover, generated random numbers have to be independent and uniformly distributed between 0 and 1. In order to check independency and uniformity of them, serial (dependence) tests and uniformity tests can be performed [2]. If the sequence of generated numbers passes these tests satisfactorily, it is considered to be random.

In this thesis we will construct PRN generators based on chaotic dynamical maps and analyze their statistical properties. We have considered tent map, logistic map and family of connecting maps as potential RNGs. These maps are very simple and they may generate random numbers faster than well-known linear RNGs. Also they provide entirely different methods of producing PRNs. There have been earlier attempts to use the logistic map as a random number generator [5]. But the chaotic maps that we consider and complete statistical analysis of results have not been previously published in the literature.

2. CHAOTIC DYNAMICAL MAPS

The motion of anything or the change in a situation implies dynamics in general. The swinging of a clock pendulum, the insulin level of blood, the flow of water in a pipe, the number of fish each spring time in a lake, and the human population of earth are basic examples of systems containing dynamics. A dynamical map encodes the law of change. It is simply defined as a map $f : V \to V$ where V represents the set of all possible states of the map. The sequence $(x, f(x), f(f(x)), f(f(f(x))), ...)$ is the orbit of dynamical map for the seed $x \in V$. The orbit is called a periodic orbit if there exists $n \in N$ such that $x = f^n(x)$, an eventually periodic orbit if there exist $n, m \in N$ such that $f^m(x) = f^n(x)$, and a dense orbit in V if there exist a point $x \in V$ such that the orbit is dense in V.

Dynamical maps can exhibit chaotic behavior. Sometimes the orbits starting at two initial states x_0 and x_1 eventually get far apart no matter how close two states x_0 and x_1 are. This makes predicting the long term future of the map very difficult if the initial state x_0 is not known exactly. Such maps are called chaotic, (see Figure 2.1). A chaotic dynamical map is generally characterized by three properties: Being sensitive to the initial states (butterfly effect), being topologically transitive, and having a dense collection of points with periodic orbits. These properties can be translated to the mathematical language. Devaney defines them as follows [14, 15]:

Definition 2.1 *Let V be a set. The dynamical map defined by $f : V \to V$ is* chaotic *on V if*

(i) f has sensitive dependence on initial conditions:
There exists $\delta > 0$ such that, for any $x \in V$ and any neighborhood $X \subset V$ around x, there exists $y \in X$ and $n \in N$ such that $|f^n(x) - f^n(y)| > \delta$. The number δ is called a sensitivity constant at x for the map f.

(ii) f is topologically transitive:
For any pair of open sets $X, Y \subset V$, there exists $k \in N$ such that $f^k(X) \cap Y \neq \emptyset$.

(iii) Periodic points of f is dense in V:

For any open subset X of V $X \cap P \neq \emptyset$, where P is the set of periodic points of f in V.

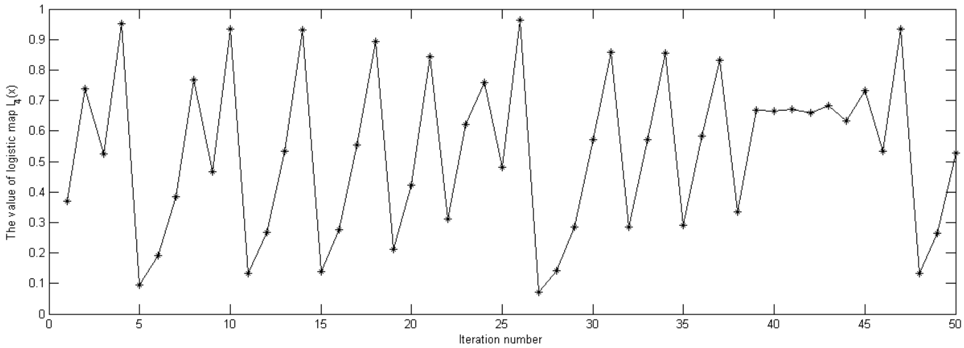


Figure 2.1. The orbit of the seed $x = 0.3$ under logistic map ($\nu = 4$).

The first condition of Devaney's definition is related with the future of the states. According to the first condition, It is not predictable how much the distance of two different close states will be after some iterations. For instance, logistic map with parameter $\nu = 4$ generates unpredictable numbers starting from the closest seeds 0.3, and 0.31. As seen in Figure 2.2, the numbers generated starting from seeds 0.3 and 0.31 get apart from each other after the 5^{th} iteration. In other words, small change in the initial states causes huge change in the following states.

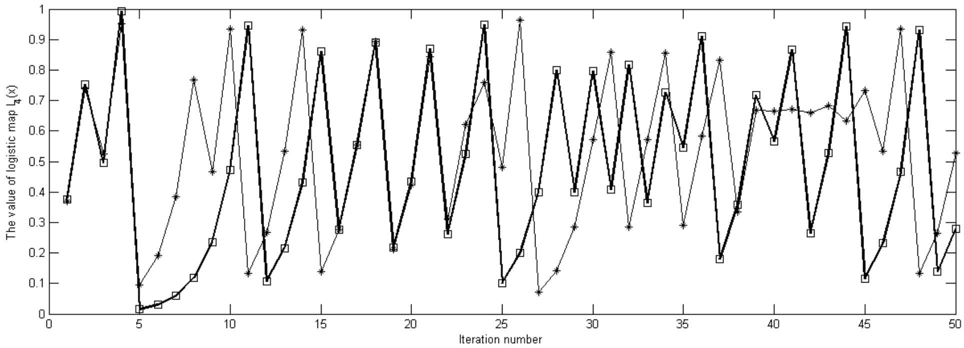


Figure 2.2. The orbit of the seeds $x = 0.3$ and $x = 0.31$ under logistic map ($\nu = 4$) are represented by asterix and squares respectively.

The second and third conditions of Devaney's definition are topological conditions. Transitivity controls whether any sequence of generated numbers contains an element, which is the element of the arbitrarily small set determined before, or not. Namely, an arbitrarily small set is first determined in the range of chaotic map, and then it is controlled that the map generates a number that is the element of the set determined. The last ingredient related with the denseness of the periodic orbits of map. It guarantees that a periodic orbit is always in any subset of the range of map. These subsets can be arbitrarily small sets.

In general, it is hard to check the chaotic behavior of dynamical maps by showing three properties directly. There are some methods to prove the chaotic behaviour of dynamical maps. One of the methods is to observe the natural behavior of map. For instance, having the wiggly iterates of a map is a way to prove the chaotic behavior of it. Before defining the wiggly iterates, we consider the definition of a one-hump map (see Figure 2.3), the definition of a m-hump map (see Figure 2.4), and a lemma related with a one-hump map which are defined by Banks [16].

Definition 2.2 *A continuous map* $f : [a, c] \to [0, 1]$ *is a* one-hump map *if strictly increases from* $f(a) = 0$ *to* $f(b) = 1$*, and then strictly decreases to* $f(c) = 0$.

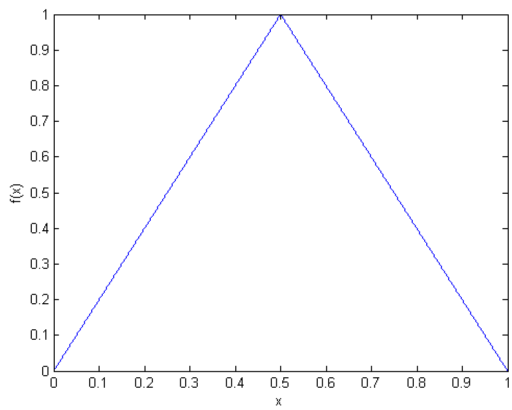


Figure 2.3. An example for a one-hump map.

Definition 2.3 f^n *is an* n-hump map *on* $[0, 1]$ *if there are* $n + 1$ *points,*

$$0 = x_0 < x_1 < ... < x_n = 1,$$

such that f^n is a one-hump map on each interval $[x_{i-1}, x_i]$. This interval is called the base of the ith hump of f^n.

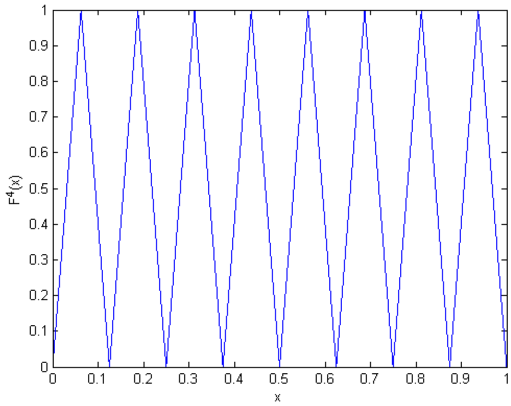


Figure 2.4. An example for an 8-hump map.

Lemma 2.1 *If f is a one-hump map on $[0, 1]$ then f^n is a 2^{n-1}- hump map on $[0, 1]$.*

Definition 2.3 and Lemma 2.1 are two ingredients of the definition of wiggly iterates [16]. A map f has wiggly iterates if f^n is a 2^{n-1}- hump map on for each positive integer n, and the length of the largest base of the humps of f^n tends to zero as n tends to infinity, (see Figure 2.5).

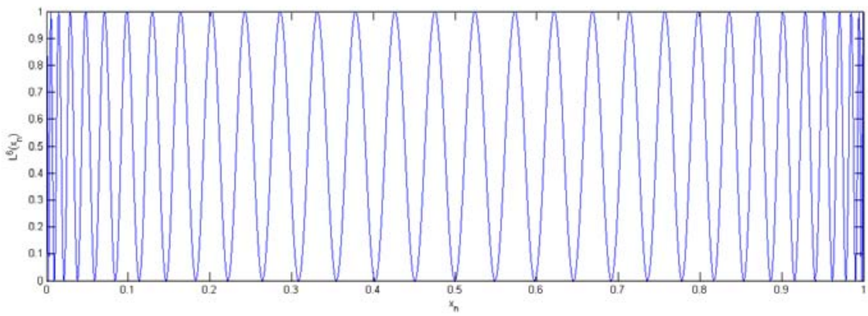


Figure 2.5. An example for wiggly iterates.

If a map $f : [0, 1] \to [0, 1]$ has wiggly iterates, then f is chaotic on the interval $[0, 1]$ [16]. The proof of the statement is easy when the following lemmas and Devaney's definition (Definition 2.1) are taken into account.

Lemma 2.2 *If $f : [0, 1] \to [0, 1]$ has wiggly iterates, then*

- *f has sensitive dependence everywhere with sensitivity constant* $\frac{1}{2}$*,*
- *f is transitive,*
- *the set of periodic points of f is dense in* $[0, 1]$*.*

The other method is to find a topologically conjugate between a chaotic dynamical map and an unknown dynamical map. In this method, two dynamical maps are connected to each other via a homeomorphism. Homeomorphism carries all properties of one map to other one. By this way, the chaotic behavior of an unknown dynamical map is ensured with homeomorphism. The definition of topologically conjugate, and the theorem related with topologically conjugate and chaos are included in Banks [16].

Definition 2.4 *A map $f : V \to V$ is* topologically conjugate *to a map $g : W \to W$ if there is homeomorphism $h : V \to W$ such that $h \circ f = g \circ h$.*

Theorem 2.1 *Let f and g be topologically conjugate via a homeomorphism h. If f is chaotic, then so is g.*

There is also a useful theorem to detect the chaotic behavior of a map. In fact, the theorem consists of a kind of simple computation [16].

Theorem 2.2 *Let $f : [0, 1] \to [0, 1]$ be a three times differentiable and symmetric one-hump map. If $f'(0) > 1$ and if f has a negative Schwarzian derivative except at $x = \frac{1}{2}$ ($Schwarzianderivative : S(f)(x) = 2f'(x)f'''(x) - 3(f''(x))^2$ for all $x \in [0, 1]$), then the restriction of f to its Cantor set* [1] *has chaotic behavior.*

Lastly, we accompany a useful theorem to establish relations with topologically transitivity of a dynamical map and a dense orbit. This theorem is important since it

[1] Cantor set is constructed by the iterations of the map f, see [16]

connects two ingredients of the definition of chaos: topologically transitivity, and denseness of periodic points. The proof of lemmas and theorems can be found in Banks [16].

Theorem 2.3 *If V is a compact metric space, then f is topologically transitive if and only if there exists a point $x \in V$ such that $\{f^n(x) : n \in N\}$ is dense in V.*

The chaotic features of maps: tent map, logistic map, and the family of connecting maps can be analyzed with the theorems and the lemmas. We will follow the specified order to show the chaotic behaviour of the maps. It is hard to show the chaotic behaviour of tent map directly. We will try to find topologically conjugacy between a chaotic dynamical map, which is shift map, and tent map. Firstly, we will consider shift map and investigate its chaotic behavior. Then, we will define tent map and determine topologically conjugate between tent map and itself. Secondly, we will examine logistic map. The chaotic behaviour of it will be shown by two ways: determining topologically conjugacy between tent map and logistic map, and computing Schwarzian derivative to apply Theorem 2.2. Lastly, we will examine the family of two connecting maps, and investigate their chaotic behaviour by using Lemma 2.2.

2.1. Shift Map

Let Σ_2 be the space of the infinite sequences on an alphabet of two symbols, denoted by 0 and 1 (i.e. $\Sigma_2 = \{(s_0 s_1 s_2 ...) : s_j \in \{0, 1\}\}$). Define the metric between two elements $s = (s_0 s_1 s_2 ...)$ and $t = (t_0 t_1 t_2 ...)$ of Σ_2 by:

$$d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

Now, we can determine the most useful property of the metric d:

Proposition 2.1 *Let $s, t \in \Sigma_2$ and suppose $s_i = t_i$ for $i = 1, 2, ..., n$. Then $d(s,t) \leq 1/2^n$. Conversely, if $d(s,t) < 1/2^n$, then $s_i = t_i$ for $i \leq n$.*

Let $s, t \in \Sigma_2$ and suppose $s_i = t_i$ for $i = 1, 2, ..., n$. Consider $d(s,t)$:

$$d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}.$$

Conversely; assume $s_k \neq t_k$ for some $k \leq n$. Then,

$$d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = \frac{1}{2^k} + \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} = \frac{1}{2^k} + \frac{1}{2^n} \geq \frac{1}{2^n}.$$

Definition 2.5 Shift Map $\sigma : \Sigma_2 \to \Sigma_2$ *is defined by*

$$\sigma(s_0 s_1 s_2 s_3 ...) = (s_1 s_2 s_3 s_4 ...)$$

where $(s_0 s_1 s_2 ...) \in \Sigma_2$.

In other words, when the shift map is applied to a sequence $s \in \Sigma_2$, the first entry s_0 of s is deleted to produce $\sigma(s)$. The fact that the shift map σ is continuous and chaotic:

Proposition 2.2 *The shift map σ is continuous on Σ_2 with respect to metric d.*

Let $\epsilon > 0$ be given and $s, t \in \Sigma_2$. We can find $n \in N$ such that $\frac{1}{2^n} < \epsilon$. Take $\delta = \frac{1}{2^{n+1}}$ such that $d(s,t) < \delta$. By Proposition 2.1, $s_i = t_i$ for $i \leq n+1$, then $d(\sigma(s), \sigma(t)) \leq \frac{1}{2^{n+1}} < \frac{1}{2^n} < \epsilon$.

Proposition 2.3 *The shift map σ is chaotic on Σ_2.*

(i) The sensitive dependence property of the shift map σ:

Let $s = (s_0 s_1 s_2 \dots)$ be in Σ_2 and define $\epsilon - ball$ around s such that $B_\epsilon(s) = \{t \in \Sigma_2 : d(s,t) < \epsilon\}$. Let $\epsilon > 0$ be given and there exists $k \in N$ such that $\frac{1}{2^k} < \epsilon < \frac{1}{2^{k-1}}$. Consider $t = (t_0 t_1 t_2 \dots)$ where $t_i \neq s_i$ if $i = k+1, k+2$ and $t_i = s_i$ otherwise. $t \in B_\epsilon(s)$, since

$$d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} < 2\frac{1}{2^{k+1}} = \frac{1}{2^k} < \epsilon.$$

Take $\delta = \frac{\epsilon}{2^3}$ and $n = k+2$, then

$$d(\sigma^n(s), \sigma^n(t)) = \sum_{i=k+2}^{\infty} \frac{|s_i - t_i|}{2^i} = \frac{1}{2^{k+2}} > \frac{\epsilon}{2^3} = \delta.$$

(ii) The transitivity of the shift map σ:

Consider the sequence s^*:

$$s^* = (\underbrace{0\,1}_{1^{st}\,block} | \underbrace{00\,01\,10\,11}_{2^{nd}\,block} | \underbrace{000\,001\,010\,011\,100\,101\,110\,111}_{3^{rd}\,block} | \underbrace{\dots}_{4^{th}\,block})$$

n^{th} block of s^* is constructed by successive listing all possible combinations of 0's and 1's of length n, and n^{th} block consists of $n2^n$ elements. Some iterations of σ applied to s^* yields a sequence which agrees with any given sequence in an arbitrarily large number of places. So, s^* is a dense orbit in Σ_2. Let $\epsilon > 0$ be given and define $S = \{s : s = (s_0 s_1 s_2 \dots s_n 00000 \dots)\}$ where $n \in N$. The cardinality of S is 2^{n+1}, and finite. For any sequence $t \in \Sigma_2$, there exists $s \in S$ such that $s_i = t_i$ for all $i = 0, 1, \dots, n$. Take $n \in N$ such that $\frac{1}{2^n} < \epsilon$, then, by Proposition 2.1, $d(s,t) \leq \frac{1}{2^n} < \epsilon$. This implies that Σ_2 is totally bounded, thus Σ_2 is compact metric space. By Theorem 2.3, the shift map is topologically transitive.

(iii) C^{per} is dense in Σ_2:

Let $s = (s_0 s_1 s_2 \dots) \in \Sigma_2$. Take $s^n \in C^{per}$ such that $s^n = (\overline{s_0 s_1 \dots s_n})$ is an

n-periodic sequence in Σ_2. By Proposition 2.1, $d\left(s^n, s\right) \leq \frac{1}{2^n}$. Consider the sequence $S = (s^n s^n s^n \ldots)$. As n tends to infinity, the sequence S converges to s.

Therefore, the shift map σ is a chaotic dynamical map.

2.2. Tent Map

Definition 2.6 *Let denote* $I := [0,1]$ *and* Tent Map *is defined by* $T_\mu : I \to R$ *where* $0 < \mu < \infty$

$$T_\mu(x) = \mu\left(\frac{1}{2} - \left|\frac{1}{2} - x\right|\right).$$

Tent map is a piecewise linear, one-dimensional map on the interval $[0,1]$. The natural invariant density of tent map is 1 [0]. The tent map is a one hump map on the interval $[0,1]$, (see Figure 2.6).

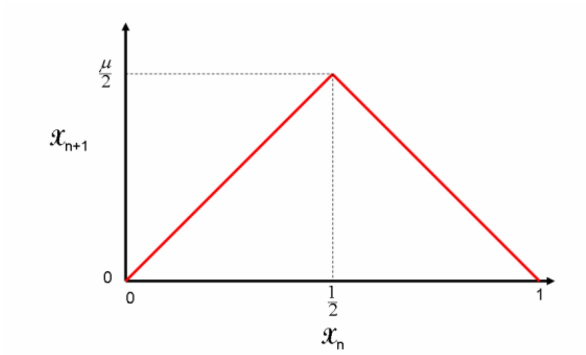


Figure 2.6. The graph of tent map.

First of all, we are interested in the tent map with parameter $\mu = 2$ on the interval $[0,1]$. The range of this tent map is $[0,1]$. It is a one hump map, and the number of humps doubles at each iteration of the tent map. As seen in Figure 2.7, there are 32 humps in the 6^{th} iteration of the tent map, and the largest base length of the humps gets smaller from 1 to $\frac{1}{32}$. Hence the nth iterate of the tent map consists of 2^{n-1} humps. The largest base length of the humps gets smaller and tends to zero as n tends to infinity. Therefore, the tent map has wiggly iterates, and it has chaotic behavior by Lemma 2.2.

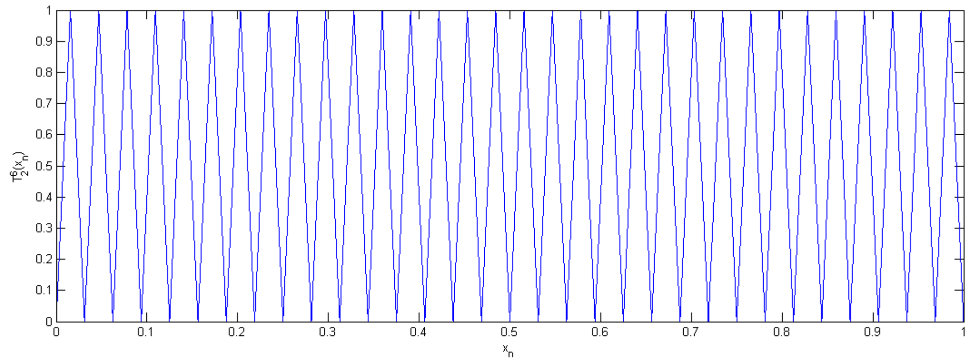


Figure 2.7. 6^{th} iteration of tent map ($\mu = 2$).

Secondly, we consider the tent map with the parameter μ is larger than 2 on the interval $[0, 1]$. The range of the tent map becomes R since the parameter μ is larger than 2. In order to show the chaotic behaviour of the tent map we will find topologically conjugate between the shift map, which is a chaotic dynamical map, and the tent map.

Proposition 2.4 *The Tent Map T_μ is chaotic when $\mu > 2$.*

To prove proposition 2.4, we will use theorem 2.1. We need to define a homeomorphism H between the shift map σ and the tent map T_μ when $\mu > 2$.

If $\mu > 2$, $T_\mu(x) \notin I$ whenever x belongs to the open interval $A_0 = (a_0, b_0)$ of I where $T_\mu(a_0) = T_\mu(b_0) = 1$. The set $I \setminus A_0$ is the union of two closed intervals $I_{1,1}$ and $I_{1,2}$. Each $I_{1,k}$ contains an open interval $A_{1,k}$ with the following property: if x belongs to $A_1 = A_{1,1} \cup A_{1,2}$, then $T_\mu(x)$ is in A_0, whence $T_\mu^2(x) \notin I$. The set $I \setminus \left(\bigcup_{k=0}^{1} A_k\right)$ is the union of four closed intervals $I_{2,i}$ for $i = 1, 2, 3, 4$ by means of which we construct a set A_2 whose points are mapped out of I by T_μ^2. Now each $I_{2,k}$ contains an open interval $A_{2,k}$ with the following property: if $x \in A_2 = A_{2,1} \cup A_{2,2} \cup A_{2,3} \cup A_{2,4}$, then $T_\mu^2(x) \in A_1$ whence $T_\mu^3(x) \notin I$. The set $I \setminus \left(\bigcup_{k=0}^{2} A_k\right)$ is the union of eight closed intervals $I_{3,j}$ for $j = 1, 2, \ldots, 8$ by means of which we construct a set A_3 whose points are mapped out of I by T_μ^3.

In the limit, by discarding the open sets A_k for which $T_\mu^n(x) \notin I$ for some n, we arrive at the uncountable set Λ,

$$\Lambda = I \setminus \left(\bigcup_{k=0}^{\infty} A_k \right)$$

which is invariant under T_μ and a Cantor Set [14].

Consider the map $H : \Lambda \to \Sigma_2$ such that $H(x) = (s_0 s_1 s_2 \dots)$ where $s_j = 0$ if $T_\mu^j(x) \in I_{1,1}$ and $s_j = 1$ if $T_\mu^j(x) \in I_{1,2}$. The fact that the map H is a homeomorphism:

(i) H is injective:

Suppose $x, y \in \Lambda$, $x \neq y$ and $H(x) = H(y)$. Then for each n, $T_\mu^n(x)$ and $T_\mu^n(y)$ are always located in the same intervals $I_{1,1}$ and $I_{1,2}$. Without loss of generality, $T_\mu^n(x)$ and $T_\mu^n(y)$ are in the intervals $I_{1,1}$ and $T_\mu^n(x) \leq T_\mu^n(y)$. Denote $x_n := T_\mu^n(x)$ and $y_n := T_\mu^n(y)$. Consider $T_\mu(x_n)$ and $T_\mu(y_n)$:

$$T_\mu(x_n) \leq \mu x_n \leq \mu y_n \leq T_\mu(y_n).$$

This implies that the tent map T_μ is monotonic in the interval between $T_\mu^n(x)$ and $T_\mu^n(y)$. By definition of H and assumption, for any $n \in N$, $T_\mu^n(x)$ and $T_\mu^n(y)$ lie on the same side of $\frac{1}{2}$; $I_{1,1}$ or $I_{1,2}$. Since T_μ monotonic and the Cantor Set Λ is totally disconnected, x and y are equal (If not, there exists totally disconnected intervals $I_{k,n} \ni x$ and $I_{k,n+1} \ni y$ for some $k, n \in N$ such that $T_\mu(x) \in I_{1,1}$ and $T_\mu(y) \in I_{1,2}$). Hence, H is injective.

(ii) H is surjective:

Let $J \subset I$ be a closed interval. Denote $T_\mu^{-n}(J) = \left\{ x \in I : T_\mu^n(x) \in J \right\}$ where $T_\mu^{-n}(J)$ denotes the preimage of J. The first important result is that if $J \subset I$ is a closed interval, then $T_\mu^{-1}(J)$ consists of two subintervals.

Now, let $s = (s_0 s_1 s_2 \dots)$ be given and define

$$I_{s_0 s_1 s_2 \dots s_n} = \left\{ x \in I : x \in I_{s_0}, \dots, T_\mu^n(x) \in I_{s_n} \right\} = I_{s_0} \cap T_\mu^{-1}(I_{s_1}) \cap \dots \cap T_\mu^{-n}(I_{s_n}).$$

This is the set of all the points in I, whose first n iterations have the same place, and it is easily observed that I_{s_0} is either $I_{1,1}$ or $I_{1,2}$ which is closed. Suppose that $I_{s_1 s_2 \dots s_n}$ is a nonempty closed interval so that $T_\mu^{-1}(I_{s_1 s_2 \dots s_n})$ consists of two closed intervals: one in $I_{1,1}$ and the other in $I_{1,2}$. Hence,

$$I_{s_0 s_1 s_2 \dots s_n} = I_{s_0} \cap T_\mu^{-1}(I_{s_1 s_2 \dots s_n})$$

is closed. Since,

$$I_{s_0 s_1 s_2 \dots s_n} = I_{s_0 s_1 s_2 \dots s_{n-1}} \cap T_\mu^{-1}(I_{s_n}) \subset I_{s_0 s_1 s_2 \dots s_{n-1}}$$

these sets are nested. This implies that

$$I_{s_0 s_1 s_2 \dots} = \bigcap_{n \geq 0} I_{s_0 s_1 s_2 \dots s_n}$$

is non-empty. Hence, there is an $x \in I_{s_0 s_1 s_2 \dots}$ such that $H(x) = s$ where $s = (s_0 s_1 s_2 \dots)$.

(iii) H is continuous:

Let $x \in \Lambda$ such that $H(x) = (s_0 s_1 s_2 \dots)$ and $\epsilon > 0$ be given. Choose a value n such that $\frac{1}{2^n} < \epsilon$. Consider the closed interval $I_{s_0 s_1 s_2 \dots s_n}$. Since $I_{s_0 s_1 s_2 \dots s_n}$ has a finite length, we can take the length of $I_{s_0 s_1 s_2 \dots s_n}$ as $\delta > 0$. For such $\delta > 0$, if $|x - y| < \delta$ and $y \in \Lambda$, then $y \in I_{s_0 s_1 s_2 \dots s_n}$. This implies that $H(y) = (s_0 s_1 \dots s_n t_{n+1} t_{n+2} \dots)$ for some sequence $(s_0 s_1 \dots s_n t_{n+1} t_{n+2} \dots) \in \Sigma_2$.

By Proposition 2.1,

$$d\left(H(x), H(y)\right) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \leq \frac{1}{2^n} < \epsilon.$$

(iv) H^{-1} is continuous:

Let $s = (s_0 s_1 s_2 \dots) \in \Sigma_2$ such that $H^{-1}(s) = x$ where $x \in \Lambda$ and $\epsilon > 0$ be given. Choose a value $n \in N$ so that $\frac{1}{2^n} < \epsilon$. Take $\delta = \frac{1}{2^n} < \epsilon$ and define $B_\delta(s) = \{t \in \Sigma_2 : t = (s_0 s_1 \dots s_n t_{n+1} t_{n+2} \dots), d\,(t,s) \leq \delta\}$. Take $t \in B_\delta(s)$ where $H^{-1}(t) = y$. Since the first n terms of s and t are equal, $x, y \in I_{s_0 s_1 s_2 \dots s_n}$. By construction, $\{I_{s_0 s_1 s_2 \dots s_n}\}_{n \geq 0}$ is a decreasing sequence. Moreover, the length of $I_{s_0 s_1 s_2 \dots s_n}$ for any $n \in N$ is: $I_{s_0 s_1 s_2 \dots s_n} < \frac{1}{2^{n+1}}$ for any n. So for any $x, y \in I_{s_0 s_1 s_2 \dots s_n}$, $|x - y| < \frac{1}{2^{n+1}}$.

Hence, for any $\epsilon > 0$, choose $n \in N$ such that $\frac{1}{2^{n+1}} < \epsilon$ and take $\delta = \frac{1}{2^{n+1}} < \epsilon$ such that for all $s, t \in \Sigma_2$ where $d\,(s,t) < \delta$ implies $|H^{-1}(s) - H^{-1}(t)| = |x - y| < \frac{1}{2^{n+1}} < \epsilon$.

The homeomorphism H connects the shift map σ and the tent map T_μ with the parameter $\mu > 2$. By Theorem 2.1, the tent map T_μ with the parameter $\mu > 2$ is a chaotic dynamical map.

2.3. Logistic Map

Definition 2.7 *Let denote* $I := [0,1]$ *and define* Logistic Map*:* $L_\nu : I \to R$ *where* $0 < \nu < \infty$

$$L_\nu(x) = \nu x(1 - x).$$

We are related with the logistic map with parameter $\nu = 4$. It is a polynomial map of degree 2 on the interval $[0,1]$ exhibiting chaotic dynamics. There is a homeomorphism between the tent map T_μ and the logistic map L_ν for $\mu = 2$ and $\nu = 4$ respectively. A homeomorphism $h : \Lambda_\mu \to \Lambda_\nu$ is defined by $h(x) = \sin^2(x)$ if $x \in \left[0, \frac{1}{2}\right]$

and $h(x) = \sin^2(1-x)$ if $x \in \left[\frac{1}{2}, 1\right]$. Consider $L_4 \circ h$, for $x \in \left[0, \frac{1}{2}\right]$:

$$\begin{aligned}(L_4 \circ h)(x) &= 4\sin^2(x)(1-\sin^2(x)) \\ &= (2\sin^2(x)\cos^2(x))^2 = \sin(2x) = (h \circ T_2)(x).\end{aligned}$$

Similarly for $x \in \left[\frac{1}{2}, 1\right]$, $L_4 \circ h$ becomes:

$$\begin{aligned}(L_4 \circ h)(x) &= 4\sin^2(1-x)(1-\sin^2(1-x)) \\ &= (2\sin^2(1-x)\cos^2(1-x))^2 = \sin(2-2x) = (h \circ T_2)(x).\end{aligned}$$

We also directly prove the chaotic behavior of the logistic map with parameter $\nu = 4$ by using Theorem 2.2:

When $\nu = 4$, L_ν maps $[0,1]$ onto $[0,1]$. Since the logistic map L_ν is a symmetric one-hump map with negative Schwarzian ($S(L_4)(x) = -3(-8)^2 = -192 < 0$).

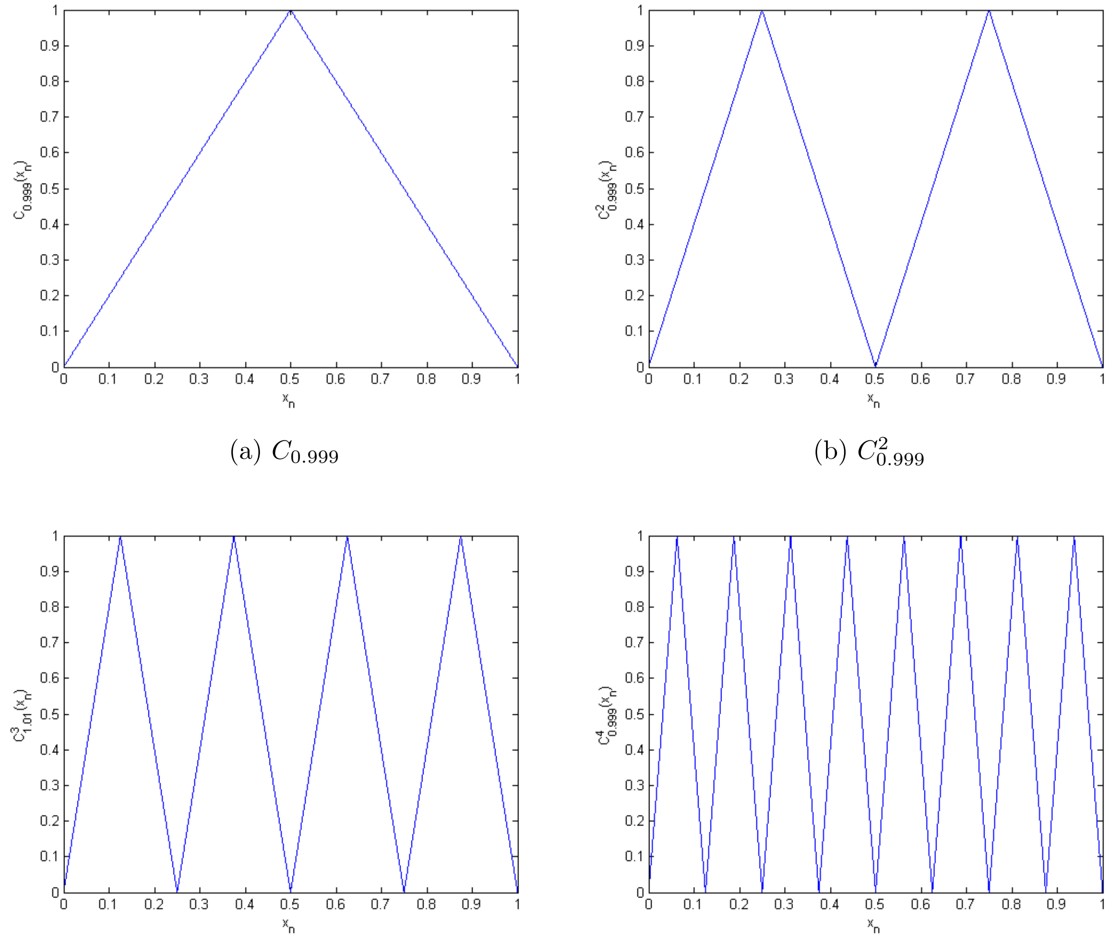
Hence the logistic map with parameter $\nu = 4$ has chaotic behaviour by Theorem 2.1, and Theorem 2.2.

2.4. Connecting Maps

Definition 2.8 *Let denote* $I := [0,1]$ *and define* Connecting Map*:* $C_\alpha : I \to I$ *where* $0 < \alpha < \infty$

$$C_\alpha(x) = 1 - |2x-1|^\alpha.$$

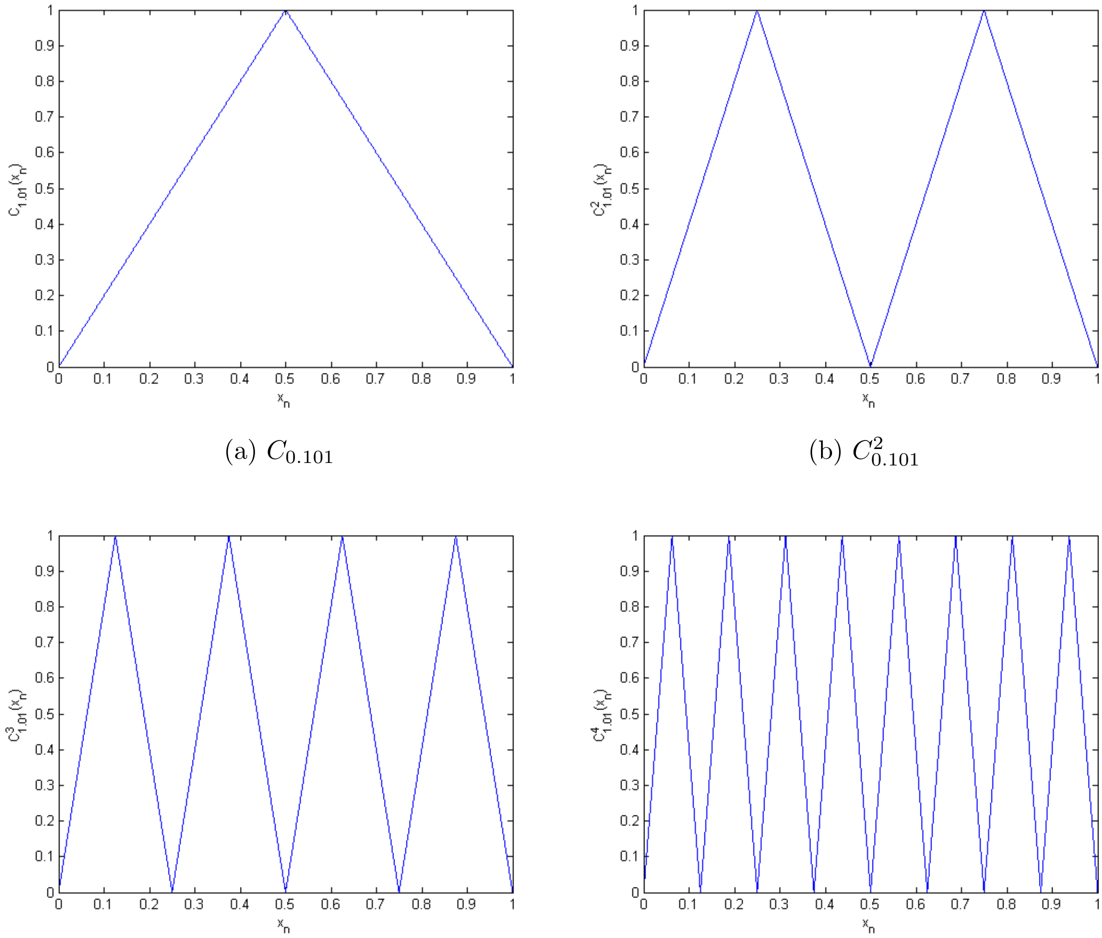
In this study, we consider two connecting maps with parameters $\alpha = 0.999$, and $\alpha = 1.01$. They are both symmetric one-hump maps from $[0,1]$ to $[0,1]$, and their iterations doubles humps, as seen in Figure 2.8 and Figure 2.9. The n^{th} iterates of them consist of 2^{n-1} humps. The largest bases, length of the humps, get smaller and smaller. As n tends to infinity, the largest bases lengths of the humps tend to zero. Since two conditions are satisfied, they have wiggly iterates. So they are chaotic maps by Lemma 2.2.



(a) $C_{0.999}$ (b) $C^2_{0.999}$

(c) $C^3_{0.999}$ (d) $C^4_{0.999}$

Figure 2.8. 4^{th} iteration of connecting map ($\alpha = 0.999$).



(a) $C_{0.101}$

(b) $C^2_{0.101}$

(c) $C^3_{0.101}$

(d) $C^4_{0.101}$

Figure 2.9. 4^{th} iteration of connecting map ($\alpha = 1.01$).

We have shown the chaotic behaviour of maps: tent map, logistic map, and the family of connecting maps. In the next part of study, the nature of chaotic maps will be exhibited in the generation of PRNs. We will be interested in exploring the random like behaviors of the chaotic maps itemized below. We claim that these five chaotic maps are applied to generate PRNs as RNGs. These five chaotic maps have three equilibrium points: $0, 0.5, 1$. When one of the maps generates one of them, it stabilizes immediately. Here, we subtract these points from the domains of maps since we do not want to choose one of them as a seed.

- $T_2(x) = 2\left(\frac{1}{2} - \left|\frac{1}{2} - x\right|\right)$ where $x \in [0, 1]$,
- $T_{2.00005}(x) = 2.00005\left(\frac{1}{2} - \left|\frac{1}{2} - x\right|\right)$ where $x \in [0, 1]$,
- $L_4(x) = 4x(1 - x)$ where $x \in [0, 1]$,
- $C_{0.999}(x) = 1 - |2x - 1|^{0.999}$ where $x \in [0, 1]$,
- $C_{0.01}(x) = 1 - |2x - 1|^{0.01}$ where $x \in [0, 1]$.

3. ANALYSIS OF CHAOTIC MAPS AS RNGs

We choose five types of chaotic maps as RNGs. These maps need to be examined and tested to detect any problem with randomness and uniformity. That's why, we generate a variety of sequences of numbers by using these maps, and then uniformity tests and dependence tests are applied.

We run Matlab software in order to produce the sequences of numbers. The program starts with a seed, which is chosen randomly. In this process there exist two termination criteria. Firstly, the program stops when it comes up with the same three successive iterated numbers that are already produced before. For instance, suppose that the group of numbers $(1, 5, 6)$ is produced before. The program runs until it produces the group of same numbers $(1, 5, 6)$. i.e. The sequence of generated numbers can be $(1, 5, 6, 10, 12, 1, 9, 6, ..., 7, 11, 102, 201, 20, 130, 1, 5, 6)$. In fact, producing one number that is generated before is enough to stop the iteration. But all computer calculations have numerical errors approximation, and we almost guarantee that it generates exactly equal successive numbers by checking the equality of not just one, but three numbers.

Secondly, termination occurs when the iteration number reaches the iteration limit set by the user. In other words, users can determine maximum number of iterations. In this study, we set the iteration numbers as 1000 and 10000 so we call the sequences as 1000-number-sequence, and 10000-number-sequence respectively. As a result, the program stops either when it generates the sequence of numbers that contains two groups of same three consecutive numbers or when it generates the specified number of numbers.

In the testing procedure, we firstly observe histograms and serial plots for the sequences of generated numbers. Because it is well known that human eye is the most powerful tester to detect patterns of non-uniformity on a sequence of numbers. Histograms illustrate the frequencies of numbers in the subintervals of maps' domain.

The domain of five maps is the interval $[0, 1]$ except three points $(0, 0.5, 1)$. Here, the interval is divided into 50 equal-subintervals. The number of subinterval is randomly determined as 50. We observe the histograms of samples for four different numbers of subintervals: 30, 50, 80, and 100. The histograms only show the general behavior of the generated numbers for the uniformity test. For each case, there is no significant information found about the distribution of the generated numbers, (see Appendix L). In Serial plots, we detect the relation between the iteration number and the generated numbers (n, x_n) as a point in xy-plane.

After eye control, we apply Kolmogorov-Smirnov test, and chi-square test to generated numbers. Both of the tests are conventional for testing the uniformity of a data sample, provided that the sample size is large. Kolmogorov-Smirnov test is a nonparametric test to compare a sample, that is the set of generated numbers, with a reference probability distribution, that is uniform distribution between 0 and 1 in our case. The null hypothesis is that the sample has a uniform distribution between 0 and 1, and the test rejects the null hypothesis at the 5% significance level.

The second uniformity test is chi-square test. This test allows us to observe whether the frequency distribution of generated numbers that are observed is consistent with uniform distribution between 0 and 1 or not. The null hypothesis is that the set of generated numbers is a random sample from uniform distribution between 0 and 1, and the test rejects the null hypothesis at the 5% significance level. Chi-square test requires setting the number of classes. This number is taken close to square root of the sample size. Since the sample sizes we consider are 1000 and 10000, the number of classes for chi-square test are 30 and 100. In this study, we will also use the number of classes 50 and 80 in addition to the number of classes 30 and 100 for chi-square test in order to examine uniformity property of the samples. Matlab software computes p-values of the sequences of generated numbers for two uniformity tests. The generated numbers pass the uniformity tests if p-values are larger than 0.05.

Dependency of the generated numbers is tested by *runs above and below mean test*, *runs up and down test*, and *tests for autocorrelation*. The runs tests examine the arrangement of the generated numbers in a sequence to test the hypothesis of independence. The runs above and below mean test is based on the number of runs of consecutive values above or below the mean of generated numbers. The null hypothesis is that generated numbers come in random order, and the test rejects the null hypothesis at the 5% significance level. The runs up and down test are based on a comparison of the expected and actual numbers of runs of various lengths. The null hypothesis and the significance level are same as used in the runs above and below mean test. Matlab program also computes p-values of the sequences of generated numbers for the runs tests. They pass the runs tests if p-values are larger than 0.05.

The tests for autocorrelation examine the dependence between numbers in a sequence. The test requires the computation of autocorrelation between every m numbers starting with the i^{th} number in the sequence of generated numbers. The program computes autocorrelation coefficients between every m numbers. It also computes the confidence bounds in which the autocorrelation coefficients are statistically zero. These confidence bounds are ± 0.0632 and ± 0.02 at approximate 95% confidence level for the sample sizes 1000 and 10000 respectively. (They are ± 0.0948 and ± 0.03 at approximate 99% confidence level for the sample sizes 1000 and 10000 respectively.) If all coefficients of the autocorrelation test are between the boundaries of the related sample size, then map passes the test.

3.1. Tent Map ($\mu = 2$)

The tent map with parameter $\mu = 2$ is defined as an operation that multiplies any number by 2. If we do not consider the numbers whose last digits are 5, as 0.125, the range of the tent map consists of the numbers whose last digits are even except zero. The only case that the last digit becomes zero is the case where the last digit of the previous number is 5. For the case that last digit is 5, tent map generates numbers whose last digits become any numbers.

Furthermore, tent map generates a number whose number of significant figures is same as those of the previously generated numbers. We deduce that the number of significant digits of generated numbers does not increase. Because of this fact the number of iterations cannot be infinite for tent map. Now we try to presume the lower and upper values of the period of tent map.

Let us take seed as 0.273 that has three significant figures. After the first iteration, the last digits of iterated numbers always become even numbers except zero. All generated numbers have three significant digits since the last digits of them are not 5. The sequence of the iterated numbers as follows (0.273, 0.546, 0.908, 0.184, 0.368, 0.736, 0.528, 0.944, 0.112,...).

If the last digit of seed or a generated number is 5, then the number of significance figures of the iterated numbers decreases. This decline depends on the power of 5 in the prime multiples of the iterated numbers. For instance, let seed be taken as 0.195 whose significant figures are 195. The prime multiples of 195 are 2, 3, 5^2, 13. Tent map reduces the significant digits of the seed as follows (0.195, 0.39, 0.78, ...). For another example 0.015625, the prime multiples of 15625 are 5^6. The significant digits of the seed decrease six times as follows (0.015625, 0.03125, 0.0625, 0.125, 0.25, 0.5, 1, 0,...).

We have an upper bound for the period of the tent map. Tent map only generates even numbers if the last digit of a seed is not taken as 5, and it does not increase the number of digits of generated numbers. When a seed is given, where the number of its significant digits is a, then tent map generates $\frac{4}{10} \times 10^a + 1$ numbers at most. +1 represents the given seed. For a lower value of the period of tent map, we cover the sequences of generated numbers.

We analyze the relationship between the number of elements in the sequence of generated numbers, and the number of significant figures of seeds. First of all, we take a seed that has one significant figure such as 0.3. Tent map generates four different numbers: $(0.3, 0.6, 0.8, 0.4, 0.8)$. For seed 0.4, the sequence has two generated numbers: $(0.4, 0.8, 0.4)$. For the other seeds that have one significant figure, the period of generated numbers cannot be smaller than 2.

Secondly, we are interested in seeds that have two significant figures such as 0.12. There is an irreducible part in the sequence that starts from 0.24 as follows $(0.12, 0.24, 0.48, 0.96, 0.08, 0.16, 0.32, 0.64, 0.72, 0.56, 0.88, 0.24)$. For all 2-significant-figure seeds, tent map generate this irreducible part of the sequence. So the minimum period of the iterations that are started with 2-significant-figure seeds is 10.

Thirdly, we are interested in seeds that have three significant figures such as 0.124. The number of elements in the sequence is 52. The irreducible part starts from 0.496 and tent map generate this part for all 3-significant-figure seeds. The minimum period of the iterations that are started with 3-significant-figure seeds is 50 as follows (0.124, 0.248, 0.496, 0.992, 0.016, 0.032, 0.064, 0.128, 0.256, 0.512, 0.976, 0.048, 0.096, 0.192, 0.384, 0.768, 0.464, 0.928, 0.144, 0.288, 0.576, 0.848, 0.304, 0.608, 0.784, 0.432, 0.864, 0.272, 0.544, 0.912, 0.176, 0.352, 0.704, 0.592, 0.816, 0.368, 0.736, 0.528, 0.944, 0.112, 0.448, 0.896, 0.208, 0.416, 0.832, 0.336, 0.672, 0.656, 0.688, 0.624, 0.752, 0.496).

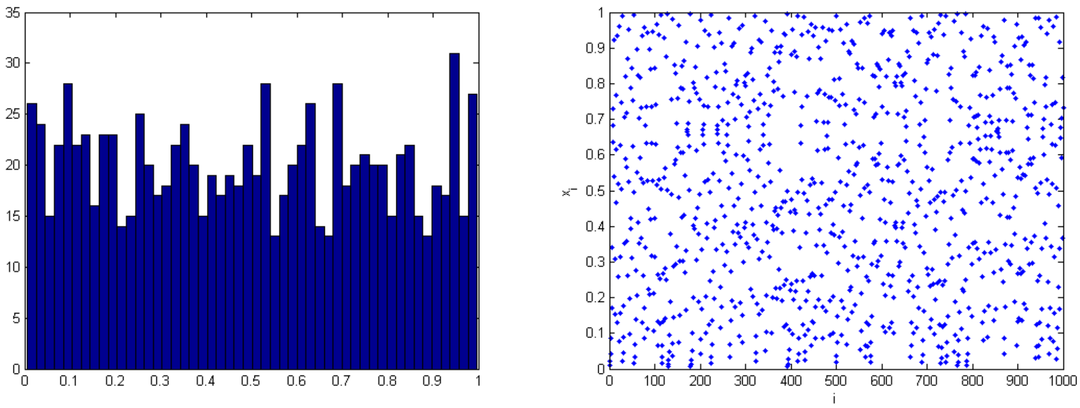
Inductively, we reach a fact that the minimum period of iterations depends on the number of significant figures of seeds. The number of elements in the irreducible part of the sequences determines the minimum period of tent map. When a seed is given, where the number of its significant digits is a, then the minimum period of tent map is $2 \times 5^{a-1}$.

Tent map with parameter $\mu = 2$ has chaotic nature. Although it is expected that the period of chaotic map is high, the period of tent map with the parameter $\mu = 2$ can be small depending on seeds. In order to use tent map as a RNG, the period of it becomes 1000000 at least. It occurs when 8-significant-digit seeds are chosen at least. In the thesis, we consider seeds that have 8 significant digits at least for tent map with parameter $\mu = 2$. Also the last digits of them are not equal to 5.

We examine tent map with randomly chosen seeds from the set $[0, 1]$ - $\{0, 0.5, 1\}$ that have 8 significant digits at least. For each seed, we generate 1000-number-sequences and 10000-number-sequences to test tent map. In general, the histograms for them are acceptable for a uniformly distributed map, and the serial plots for all sequences show that they are neatly spread on the xy-plane without any dependence.

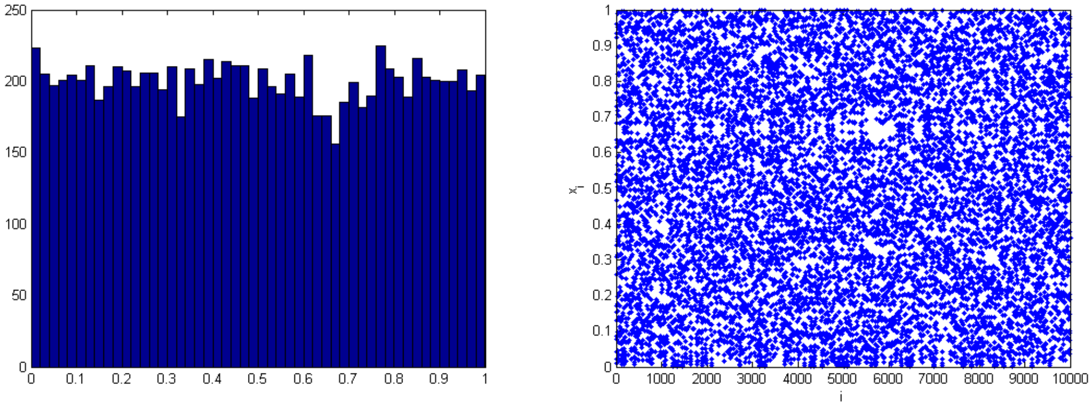
The results of the histograms of the numbers are also confirmed by Kolmogorov Smirnov and chi-square statistical tests. Almost all tests reject the null hypothesis at the 5% significance level. In the dependence case, nearly all p-values of runs tests are not less than 0.05. All results are nearly acceptable.

We randomly choose 10 different seeds in order to represent the performance of tent map in randomness tests. We illustrate histograms, and serial plots, and statistical test results. 1000-number-sequences and 10000-number-sequences are generated for 10 different seeds separately. The histograms and the serial plots for two different seeds under the tent map are shown in Figure 3.1-4. The rest of histograms and serial plots can be found in Appendix A.



(a) Histogram with 50 subintervals (b) Serial plot

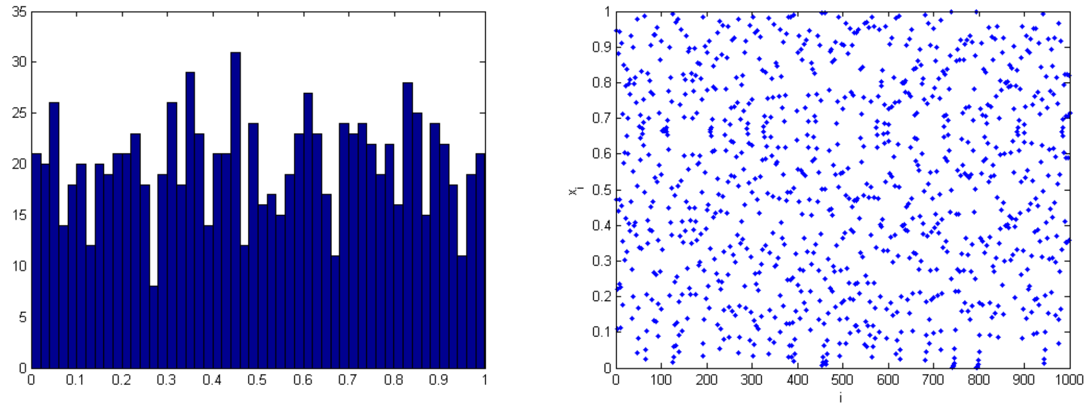
Figure 3.1. Tent map ($\mu = 2$): Seed = 0.010666884564, Iteration no = 1000.



(a) Histogram with 50 subintervals

(b) Serial plot

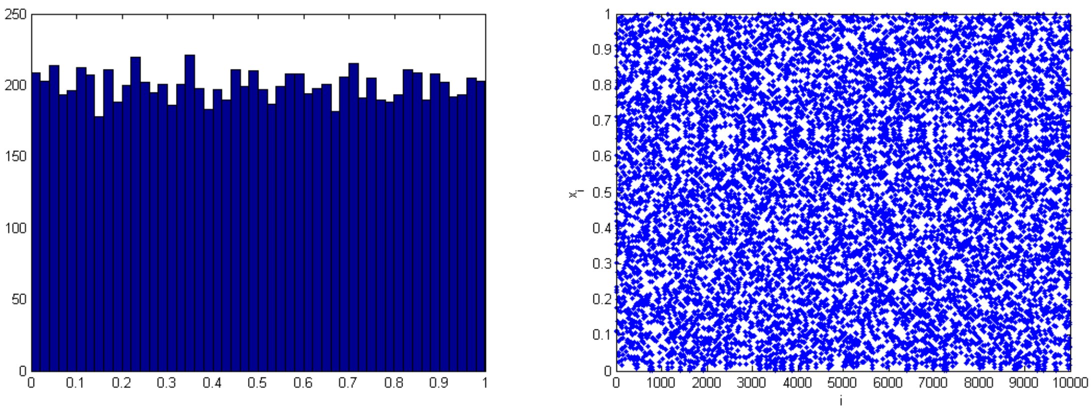
Figure 3.2. Tent map ($\mu = 2$): Seed = 0.010666884564, Iteration no = 10000.



(a) Histogram with 50 subintervals

(b) Serial plot

Figure 3.3. Tent map ($\mu = 2$): Seed = 0.4724338444224, Iteration no = 1000.



(a) Histogram with 50 subintervals

(b) Serial plot

Figure 3.4. Tent map ($\mu = 2$): Seed = 0.4724338444224, Iteration no = 10000.

The results of the statistical tests for 10 different seeds under the tent map with parameter $\mu = 2$ can be seen in the Table 3.1, and Table 3.2. The first two columns of the tables are the statistical test results for two different seeds whose histograms, and serial plots are seen above. As observed in the tables, almost all p-values of tests are larger than 0.05. All correlation coefficients are in the interval $[-0.635, 0.635]$ for iteration no $= 1000$, (see Table 3.1), and the half of the correlation coefficients are in the interval $[-0.2, 0.2]$ for iteration no $= 10000$, (see Table 3.2).

Table 3.1. The statistical test results for 10 different seeds under tent map ($\mu = 2$): Iteration no $= 1000$.

		Seeds				
	Statistical Tests	**0.010666884564**	**0.472433844224**	**0.197600506499**	**0.231069988461**	**0.983880834168782**
P-Values	**Kolmogorov Smirnov**	0.8497	0.4412	0.4305	0.6245	0.9859
	Chisqure 30	0.1887	0.8715	0.6802	0.4157	0.6431
	Chisqure 50	0.5962	0.7657	0.8214	0.1886	0.4005
	Chisqure 80	0.4968	0.9721	0.2603	0.3417	0.1413
	Chisquare 100	0.3254	0.9248	0.4827	0.4866	0.0991
	Runs Up - Down	0.7167	0.1301	0.4453	0.7167	0.9302
	Runs Above - Below	0.1952	0.2398	0.3572	0.2268	0.5093
	Max Autocorrelation coeff.	0.059438816	0.062058037	0.056830823	0.044625542	0.055569503

		Seeds				
	Statistical Tests	**0.389913876011**	**0.455019664777**	**0.490444465331**	**0.086948882143**	**0.686808244224**
P-Values	**Kolmogorov Smirnov**	0.909	0.7367	0.6102	0.7746	0.8676
	Chisqure 30	0.9998	0.9509	0.3539	0.2771	0.9719
	Chisqure 50	0.9854	0.9796	0.4274	0.1708	0.8372
	Chisqure 80	0.8247	0.9497	0.0785	0.4559	0.0353
	Chisquare 100	0.9726	0.9157	0.4105	0.144	0.5802
	Runs Up - Down	0.6981	0.323	0.608	0.2876	0.9501
	Runs Above - Below	0.987	0.217	0.9122	0.635	0.0897
	Max Autocorrelation coeff.	0.050154144	0.045369139	0.051782737	0.05609398	0.050358633

Table 3.2. The statistical test results for 10 different seeds under tent map ($\mu = 2$): Iteration no = 10000.

		Seeds				
	Statistical Tests	0.010666884564	0.472433844224	0.197600506499	0.231069988461	0.983880834169
P-Values	Kolmogorov Smirnov	0.354	0.9855	0.8496	0.617	0.8785
	Chisqure 30	0.6378	0.502	0.2848	0.5544	0.8785
	Chisqure 50	0.7653	0.5034	0.8266	0.1063	0.1395
	Chisqure 80	0.7767	0.3233	0.7811	0.0817	0.1534
	Chisquare 100	0.9198	0.2566	0.8532	0.0691	0.0471
	Runs Up - Down	0.5091	0.9213	0.6324	0.2279	0.3653
	Runs Above - Below	0.2977	0.8328	0.9923	0.5165	0.2419
	Max Autocorrelation coeff.	0.014279405	0.02165682	0.024787165	0.022381273	0.029123478

		Seeds				
	Statistical Tests	0.389913876011	0.455019664777	0.490444465331	0.086948882143	0.686808244224
P-Values	Kolmogorov Smirnov	0.8658	0.8209	0.5953	0.9951	0.6973
	Chisqure 30	0.5958	0.4654	0.0414	0.9999	0.8664
	Chisqure 50	0.2209	0.6102	0.542	0.9989	0.8279
	Chisqure 80	0.1111	0.6738	0.0075	0.9999	0.9149
	Chisquare 100	0.2796	0.3612	0.4938	1	0.9032
	Runs Up - Down	0.4455	0.3954	0.2666	0.725	0.6723
	Runs Above - Below	0.7749	0.8118	0.0144	0.4414	0.016
	Max Autocorrelation coeff.	0.019026983	0.019897044	0.022368968	0.010507252	0.013412629

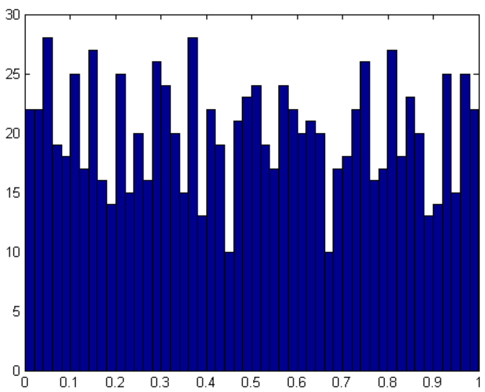
3.2. Tent Map ($\mu = 2.00005$)

The second suggested RNG, tent map with parameter $\mu = 2.00005$, generates more than 1000000 numbers for any given seed. But there is a problem about the range of the map. The tent map with the parameter μ goes from its domain $[0, 1]$ to the range $\left[0, \frac{\mu}{2}\right]$. Rarely, taking the parameter $\mu = 2.00005$ results in the generated numbers that are larger than 1 whereas they have to be in the interval $[0, 1]$. In order to overcome this problem, we offer IF method. It is a simple IF statement that asks if tent map produces a number either larger than 1 or not. When tent map generates a number that is larger than 1, IF statement converts it to a number which is the symmetric to the generated number based on 1. For instance, if tent map produce $x_n = 1.000021$, then it is converted to $x_n^* = 0.999979$. This operation does not harm uniformity since the probability of the generation of numbers larger than 1 is equal to $\frac{0.000025}{1.000025}$.

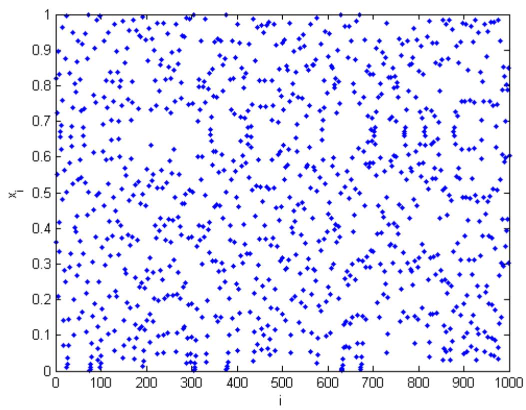
The period of the map does not depend on seeds, and it is 1000000 at least. Since there is no constraint on seeds, any seed works for the generation of numbers.

We use many seeds to test the map statistically. For each seed, we generate 1000-number-sequences and 10000-number-sequences. The histograms and serial plots of them are observed, and the acceptance of them are confirmed. Then we consider the statistical tests. The performance of the sequences in the statistical tests is considerably high. Almost all p-values are in the acceptance region of the tests, and the correlation coefficients are sufficient.

We represent the test performance of tent map with parameter $\mu = 2.00005$ by using 10 randomly selected different seeds. For each 10 different seeds, 1000-number-sequences and 10000-number-sequences are generated. The histograms and the serial plots of 2 out of 10 sequences from each group are shown below in Figure 3.5-8. Histograms and serial plots of other sequences of generated numbers can be found in Appendix B.

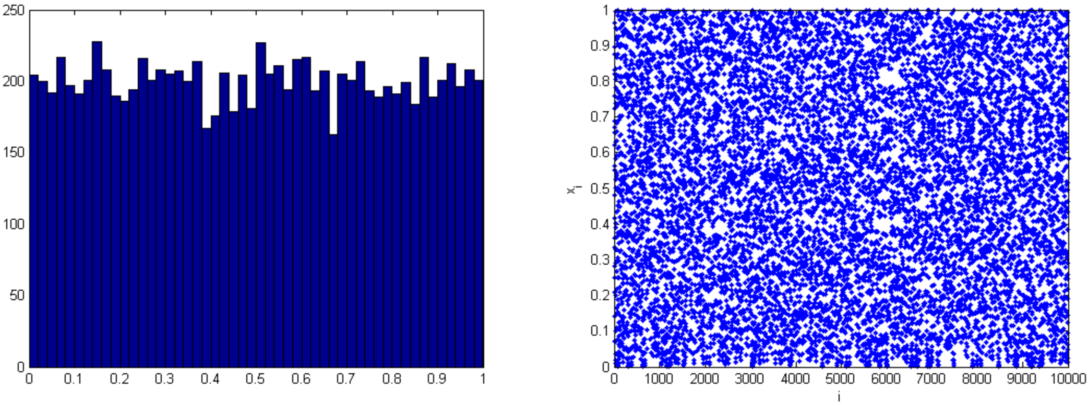


(a) Histogram with 50 subintervals



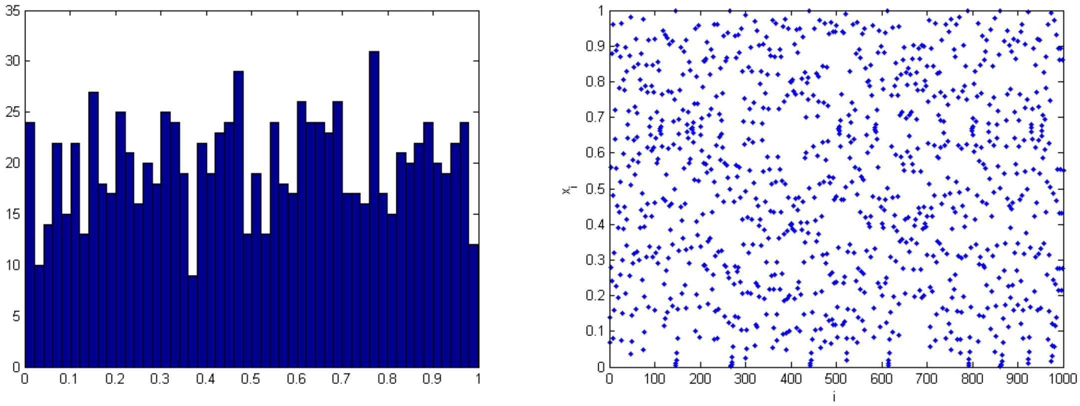
(b) Serial plot

Figure 3.5. Tent map ($\mu = 2.00005$): Seed = 0.819, Iteration no = 1000.



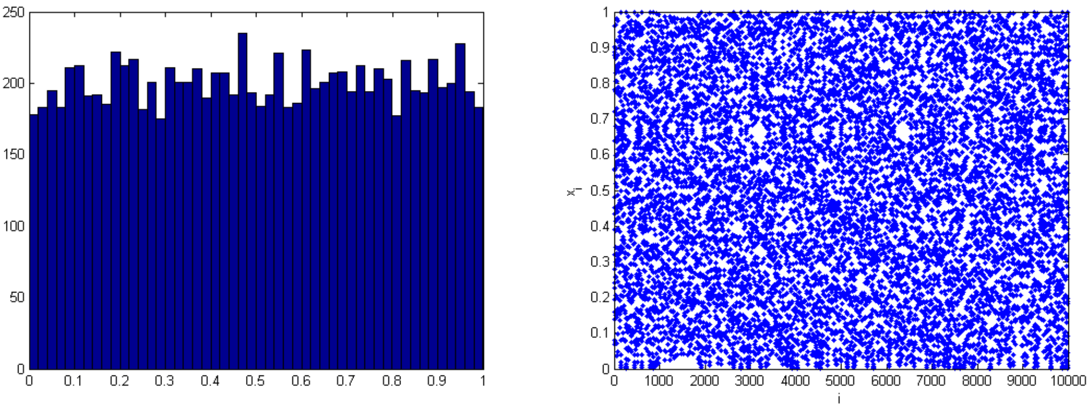
(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.6. Tent map ($\mu = 2.00005$): Seed = 0.819, Iteration no = 10000.



(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.7. Tent map ($\mu = 2.00005$): Seed = 0.07, Iteration no = 1000.



(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.8. Tent map ($\mu = 2.00005$): Seed = 0.07, Iteration no = 10000.

The results of the statistical tests of 20 sequences of numbers can be seen in the Table 3.3 and Table 3.4. The first two columns of the tables are the statistical test results of the sequences of numbers which have histograms, and serial plots above. As shown in the tables, almost all p-values of tests are larger than 0.05. More than half of the correlation coefficients are in the acceptable region for autocorrelation test.

Table 3.3. The statistical test results for 10 different seeds under tent map (μ = 2.00005): Iteration no = 1000.

		Seeds				
	Statistical Tests	**0.819**	**0.07**	**0.945195357852**	**0.005382078129**	**0.385874582044**
P-Values	**Kolmogorov Smirnov**	0.8871	0.5761	0.4461	0.6072	0.0423
	Chisqure 30	0.6636	0.3578	0.8094	0.8126	0.2739
	Chisqure 50	0.3756	0.183	0.8859	0.2626	0.7141
	Chisqure 80	0.9629	0.0433	0.9676	0.4682	0.6983
	Chisquare 100	0.2294	0.0967	0.8952	0.1721	0.6793
	Runs Up - Down	0.2991	0.3611	0.6615	0.5398	0.0083
	Runs Above - Below	0.8247	0.3572	0.2965	0.4527	0.0122
	Max Autocorrelation coeff.	0.06941559	0.058586651	0.02429429	0.035747446	0.083919106

		Seeds				
	Statistical Tests	**0.991**	**0.517**	**0.197600506499**	**0.267781590398**	**0.897496645632**
P-Values	**Kolmogorov Smirnov**	0.8299	0.7048	0.8164	0.1384	0.9998
	Chisqure 30	0.8419	0.9058	0.9824	0.2398	0.976
	Chisqure 50	0.8239	0.7954	0.8593	0.052	0.8713
	Chisqure 80	0.734	0.9263	0.9315	0.4406	0.9522
	Chisquare 100	0.7753	0.9869	0.7819	0.0347	0.6379
	Runs Up - Down	0.2064	0.2249	0.3611	0.1178	0.8708
	Runs Above - Below	0.1246	0.8911	0.5064	0.1551	0.8734
	Max Autocorrelation coeff.	0.04867941	0.062237957	0.03521395	0.077975175	0.0735589

Table 3.4. The statistical test results for 10 different seeds under tent map ($\mu = 2.00005$): Iteration no = 10000.

		Seeds				
	Statistical Tests	**0.819**	**0.07**	**0.945195357852**	**0.005382078128663**	**0.385874582044063**
P-Values	**Kolmogorov Smirnov**	0.7941	0.6542	0.9056	0.9611	0.5656
	Chisqure 30	0.2914	0.8475	0.9982	0.6043	0.9881
	Chisqure 50	0.6322	0.4585	0.9899	0.8755	0.775
	Chisqure 80	0.0464	0.2366	0.9996	0.372	0.9708
	Chisquare 100	0.1903	0.1759	0.9964	0.6861	0.9482
	Runs Up - Down	0.2372	0.2533	0.7428	0.9025	0.1989
	Runs Above - Below	0.2207	0.1359	0.9578	0.5966	0.3123
	Max Autocorrelation coeff.	0.008310828	0.018133114	0.029705719	0.024152498	0.02792109

		Seeds				
	Statistical Tests	**0.991**	**0.517**	**0.197600506499**	**0.267781590398**	**0.897496645632**
P-Values	**Kolmogorov Smirnov**	0.4908	0.4098	0.9991	0.8024	0.9968
	Chisqure 30	0.5818	0.6609	0.9999	0.7277	0.6261
	Chisqure 50	0.7991	0.8149	0.9959	0.7686	0.3289
	Chisqure 80	0.9887	0.9522	0.988	0.6835	0.6048
	Chisquare 100	0.9473	0.9721	0.999	0.8892	0.1818
	Runs Up - Down	0.2985	0.2101	0.9087	0.3529	0.8526
	Runs Above - Below	0.1585	0.5694	0.8808	0.5322	0.8489
	Max Autocorrelation coeff.	0.023193961	0.013029741	0.013943226	0.020615924	0.019938516

3.3. Logistic Map ($\nu = 4$)

Logistic map with parameter $\nu = 4$ is the third suggested RNG. It is not only a chaotic dynamical system but also a high-periodic map. It generates 1000000 numbers and more. We generate hundreds of sequences of 1000 and 10000 numbers to understand the properties of logistic map. An interesting point for the histograms of them is that they look like U-shape. In other words, the sequences of generated numbers are not uniformly distributed, (see Figure 3.9). We need to transform the sequences of generated numbers to the sequences of uniformly distributed numbers.

We obtain the natural invariant density of the logistic map by the derivation of the transformation [0]. For x in the interval $\Lambda_\mu = [0, 1]$, define y, in $\Lambda_\nu = [0, 1]$, by

$$x = sin^2(\frac{\pi y}{2}) = \frac{1}{2}[1 - cos(\pi y)]. \tag{3.1}$$

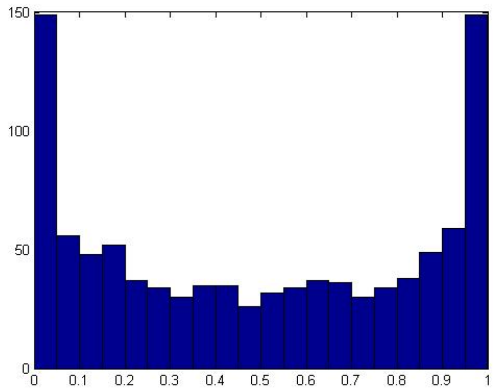


Figure 3.9. Histogram for logistic map.

The natural invariant density of the logistic map is derived easily

$$\rho(x) = \left|\frac{dy}{dx}\right| \rho(y)$$

where $\rho(y) = 1$. Since under Equation 3.1 y is shown to have uniform invariant density [0]. One can easily compute that

$$\rho(x) = \frac{\pi^{-1}}{[x(1-x)]^{\frac{1}{2}}}. \tag{3.2}$$

The graph of Equation 3.2 can be seen in Figure 3.10. This graph and Figure 3.9 are very similar. This result implies that the logistic map does not uniformly distributed on the interval $[0, 1]$, and the transformation is necessary to have a uniformly distributed function.

After the transformation, we get the sequences of numbers that are uniformly distributed. The sequence of numbers used in Figure 3.9 transforms to the sequence of numbers that is uniformly distributed, as seen in Figure 3.11. Now, we can take into consideration the transformed sequences of numbers for logistic map. In the thesis, we apply the transformation on the sequences of the numbers that are generated by logistic map, and we call transformed logistic map as T-logistic map.

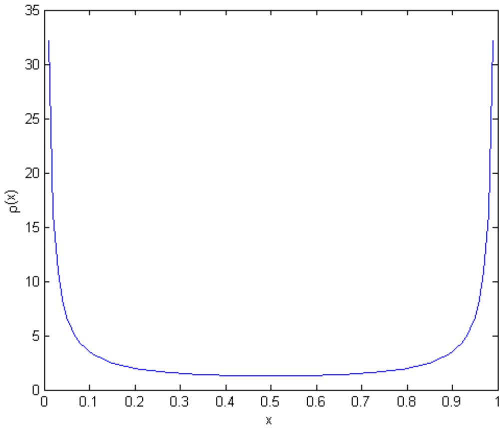


Figure 3.10. The graph of invariant density of logistic map

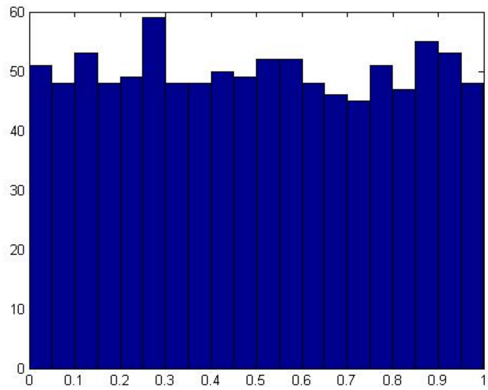
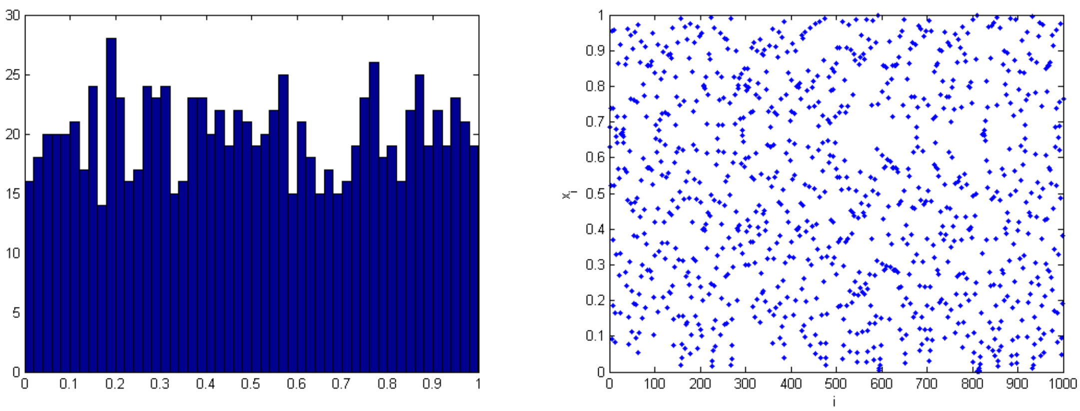


Figure 3.11. Histogram for transformed logistic map (T-logistic map).

The period of T-logistic map does not depend on seeds, and it can be more than 1000000 numbers. The hundreds of seeds from $[0, 1]$ are used in the randomness tests of T-logistic map. For each seed, the sequences of 1000 and 10000 numbers are produced, and transformed. The histograms of the sequences of the numbers are acceptable to pass the uniformity tests. The serial plots of them imply the independence of the numbers. They are also tested statistically. According to the tests results, T-logistic map produces the sequences of random numbers.

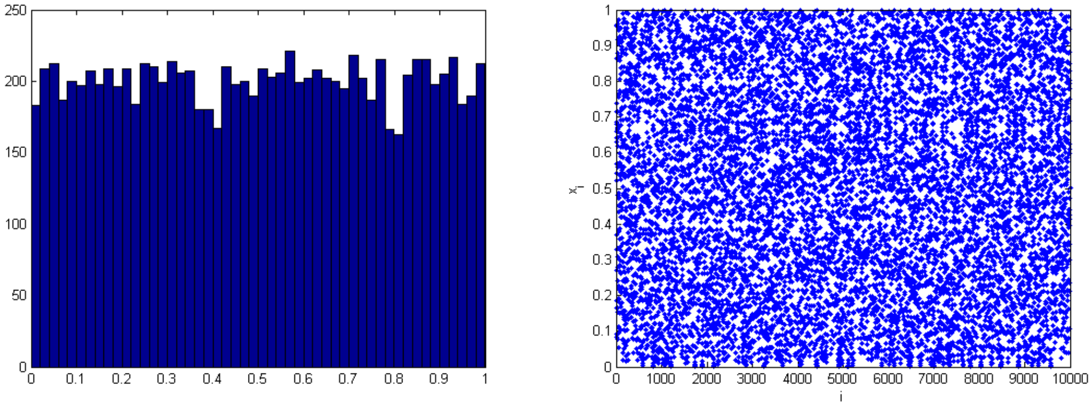
We randomly select 10 different seeds. Ten of 1000-number-sequences and ten of 10000-number-sequences are computed and transformed. The histograms and the serial plots of 2 out of 10 sequences from each group are shown below in Figure 3.12-15. The histograms and the serial plots of 16 sequences can be found in Appendix C.



(a) Histogram with 50 subintervals (b) Serial plot

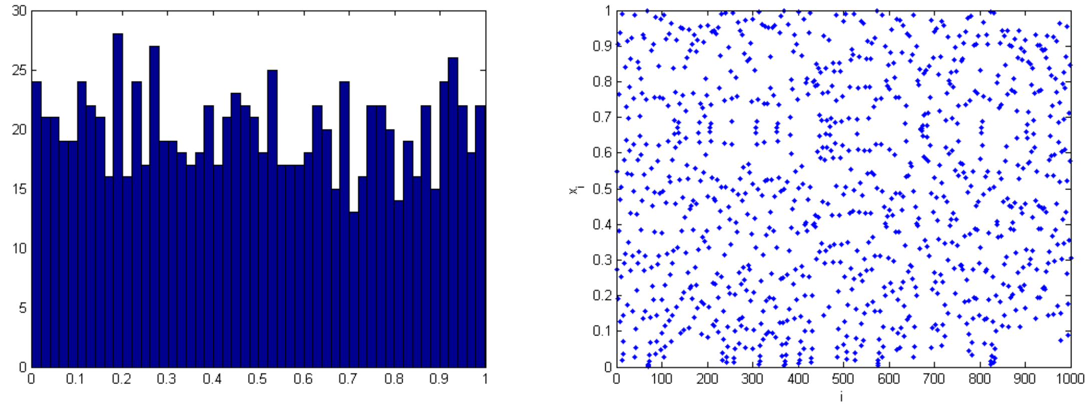
Figure 3.12. T-logistic map ($\nu = 4$): Seed = 0.774, Iteration no = 1000.

The results of eye test are also confirmed by statistical tests. We apply the statistical tests into 20 sequences of numbers. Almost in all tests the null hypothesis is rejected at the 5% significance level. The test results are affirmative. The results of the statistical tests of 20 sequences can be seen in the Table 3.5, and Table 3.6. The first two columns of the tables are the statistical test results of the sequences of numbers which have histograms, and serial plots above. The test results show that T-logistic map with parameter $\nu = 4$ can be used as a RNG.



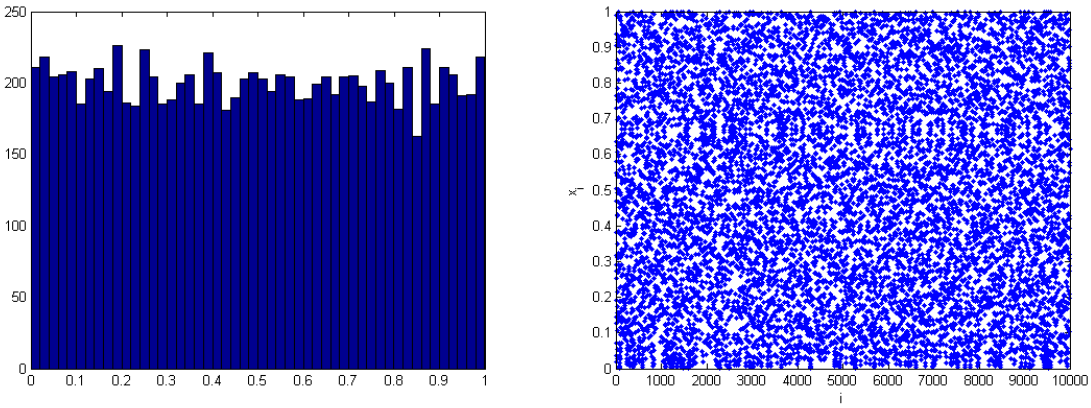
(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.13. T-logistic map ($\nu = 4$): Seed = 0.774, Iteration no = 10000.



(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.14. T-logistic map ($\nu = 4$): Seed = 0.174, Iteration no = 1000.



(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.15. T-logistic map ($\nu = 4$): Seed = 0.174, Iteration no = 10000.

Table 3.5. The statistical test results for 10 different seeds under T-logistic map ($\nu = 4$): Iteration no = 1000.

		Seeds				
	Statistical Tests	**0.774**	**0.174**	**0.973**	**0.912091805655299**	**0.68**
P-Values	**Kolmogorov Smirnov**	0.8893	0.6776	0.0597	0.8095	0.4972
	Chisqure 30	0.9405	0.8795	0.433	0.8767	0.9549
	Chisqure 50	0.9954	0.992	0.8198	0.8267	0.9802
	Chisqure 80	0.916	0.9619	0.8072	0.5398	0.986
	Chisquare 100	0.9834	0.9901	0.9193	0.1641	0.9847
	Runs Up - Down	0.6615	0.2991	0.0103	0.7167	0.3354
	Runs Above - Below	0.3904	0.9122	0.0288	0.5464	0.1546
	Max Autocorrelation coeff.	0.052851976	0.073149259	0.056233972	0.047851908	0.055770758

		Seeds				
	Statistical Tests	**0.58**	**0.282**	**0.4**	**0.758167338031333**	**0.809**
P-Values	**Kolmogorov Smirnov**	0.0179	0.853	0.9744	0.5588	0.7817
	Chisqure 30	0.076	0.8232	0.9127	0.9135	0.6207
	Chisqure 50	0.1899	0.3101	0.2057	0.8975	0.7005
	Chisqure 80	0.2173	0.4307	0.8575	0.936	0.8091
	Chisquare 100	0.2243	0.3075	0.0524	0.9289	0.8319
	Runs Up - Down	0.0865	0.3611	0.9501	0.4453	0.2346
	Runs Above - Below	0.9183	0.8256	0.9744	0.6749	0.6868
	Max Autocorrelation coeff.	0.055371363	0.067400378	0.080307197	0.036576894	0.065886361

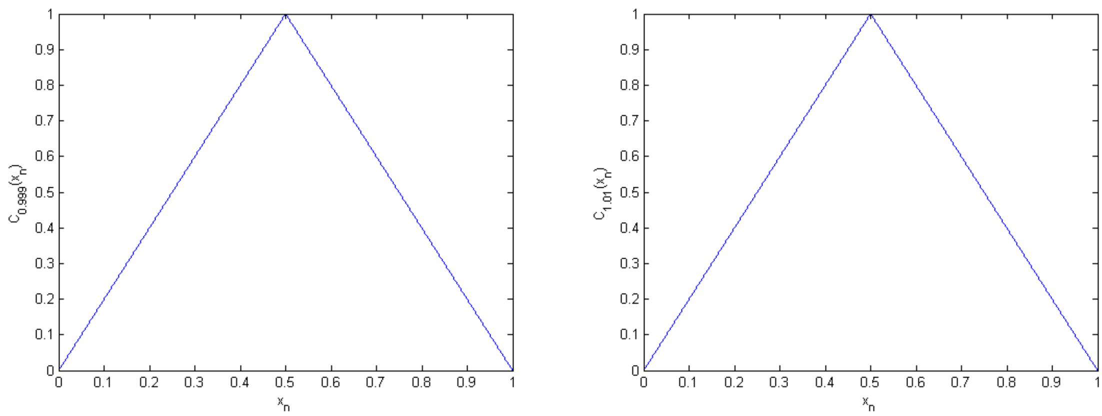
Table 3.6. The statistical test results for 10 different seeds under T-logistic map ($\nu = 4$): Iteration no = 10000.

		Seeds				
	Statistical Tests	**0.774**	**0.174**	**0.973**	**0.912091805655299**	**0.68**
P-Values	**Kolmogorov Smirnov**	0.9033	0.6841	0.0707	0.8948	0.4631
	Chisqure 30	0.1599	0.9293	0.4854	0.014	0.6709
	Chisqure 50	0.6208	0.7858	0.7869	0.0241	0.6352
	Chisqure 80	0.572	0.9347	0.346	0.0274	0.304
	Chisquare 100	0.159	0.7644	0.7759	0.0243	0.6807
	Runs Up - Down	0.2876	0.4087	0.024	0.865	0.3823
	Runs Above - Below	0.1916	0.242	0.1206	0.2842	0.242
	Max Autocorrelation coeff.	0.017941045	0.014693134	0.017313217	0.026406264	0.019955813

		Seeds				
	Statistical Tests	**0.58**	**0.282**	**0.4**	**0.758167338031333**	**0.809**
P-Values	**Kolmogorov Smirnov**	0.68	0.0903	0.682	0.0278	0.9725
	Chisqure 30	0.905	0.0349	0.4658	0.4679	0.8779
	Chisqure 50	0.34	0.0144	0.9983	0.4305	0.9691
	Chisqure 80	0.8463	0.0347	0.9979	0.5302	0.9789
	Chisquare 100	0.2525	0.0148	0.9957	0.7887	0.9476
	Runs Up - Down	0.6551	0.0929	0.13	0.0035	0.7608
	Runs Above - Below	0.6714	0.4881	0.2983	0.2019	0.4403
	Max Autocorrelation coeff.	0.027581553	0.013427137	0.013061025	0.016267296	0.0287351

3.4. Connecting Family of Maps ($\alpha = 0.999$ and $\alpha = 1.01$)

The last suggested RNGs are connecting family of mappings with parameters $\alpha = 0.999$ and $\alpha = 1.01$. The first observation about the maps is the graphs of them. They are similar to the graphs of the tent map with parameter $\mu = 2$, as shown in Figure 3.16. They have same properties as the tent map, except the relationship between seed and period of the tent map. The periods of connecting maps are not sensitive on seeds. Moreover, the periods of them become more than 1000000.

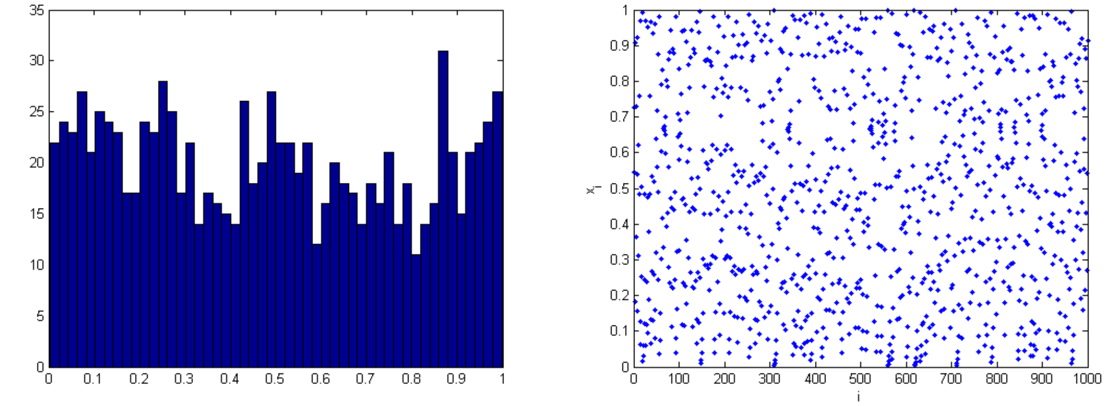


(a) $C_{0.999}$ (b) $C_{1.01}$

Figure 3.16. The graph of connecting maps ($\alpha = 0.999$ and $\alpha = 1.01$).

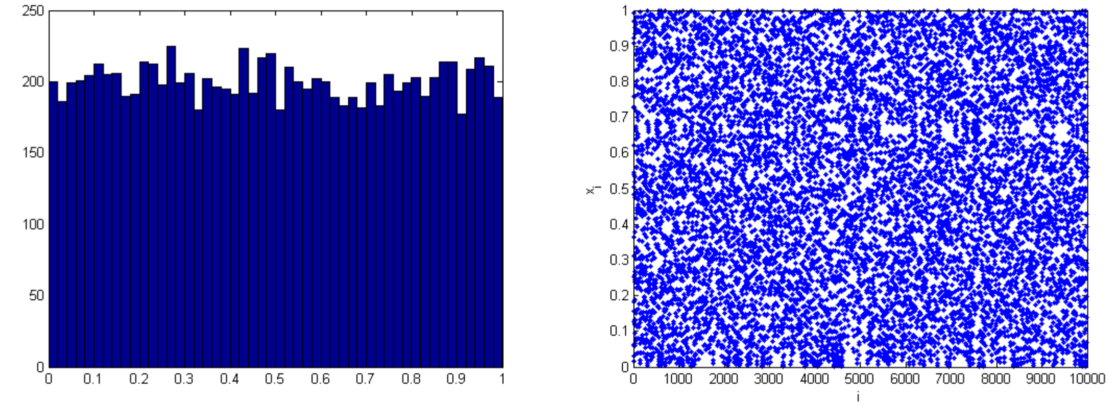
We generate hundreds of sequences of numbers, and observe histograms and serial plots. All histograms confirm uniformity of the sequences of the numbers. Serial plots illustrate that the numbers are independently spread on the xy-plane. The sequences of numbers also pass the statistical tests. Almost all p-values of uniformity tests and dependence tests are adequate that the connecting maps have random behavior.

We select 10 different seeds randomly and generate 20 sequences of numbers for each parameter $\alpha = 0.999$ and $\alpha = 1.01$. The histograms and the serial plots of 2 out of 10 sequences from each group for $\alpha = 0.999$ and $\alpha = 1.01$ are shown in Figure 3.17-20 and Figure 3.21-24 respectively. The rest of histograms and serial plots of two maps can be found in Appendix D and Appendix E.



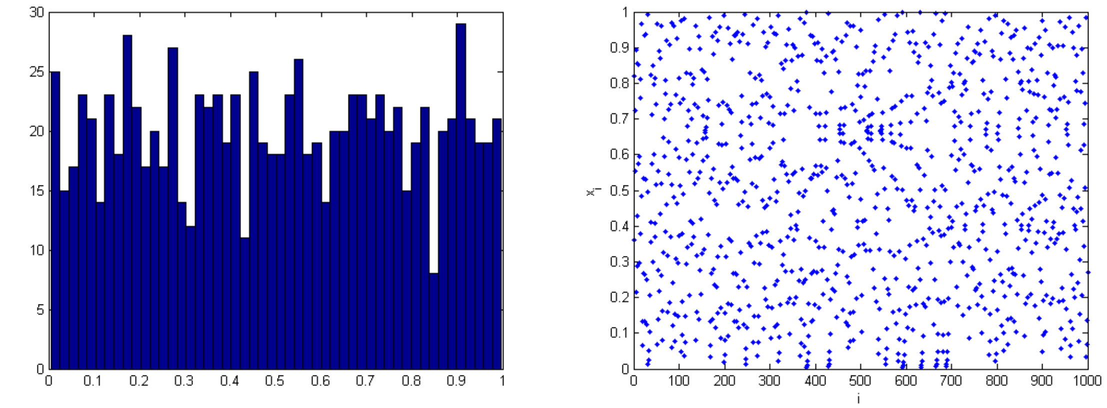
(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.17. Connecting map ($\alpha = 0.999$): Seed = 0.727, Iteration no = 1000.



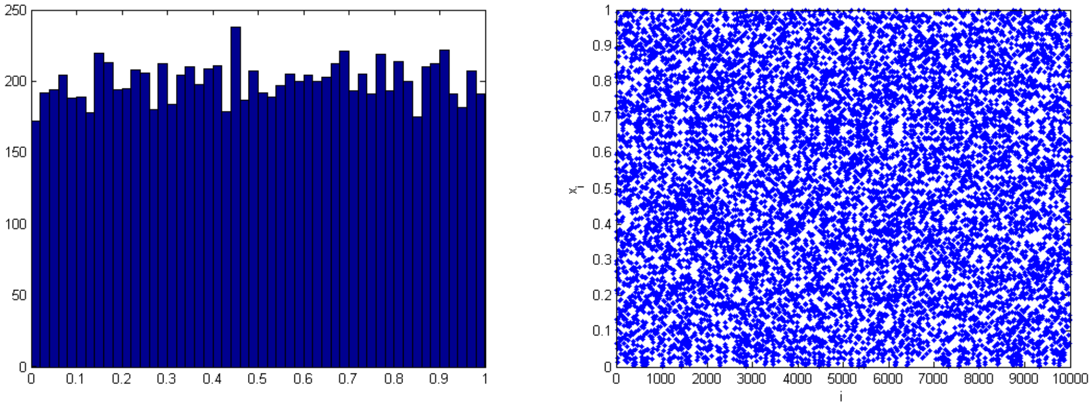
(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.18. Connecting map ($\alpha = 0.999$): Seed = 0.727, Iteration no = 10000.



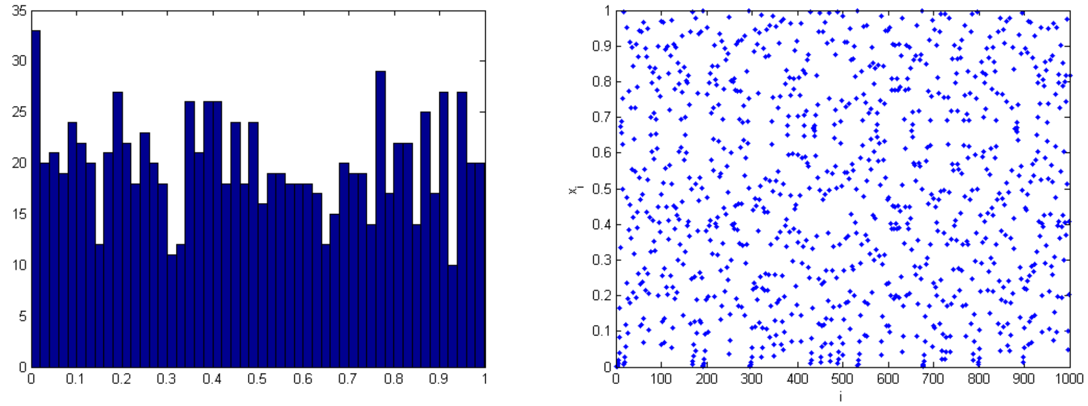
(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.19. Connecting map ($\alpha = 0.999$): Seed = 0.819, Iteration no = 1000.



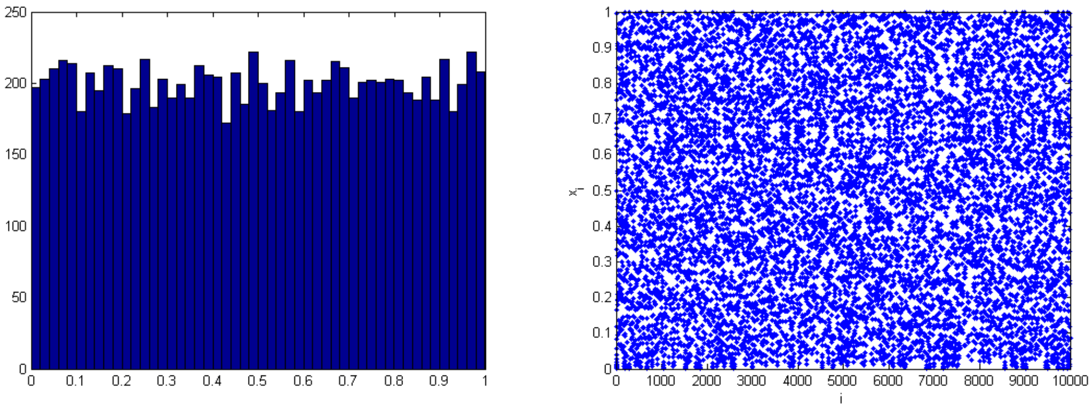
(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.20. Connecting map ($\alpha = 0.999$): Seed = 0.819, Iteration no = 10000.



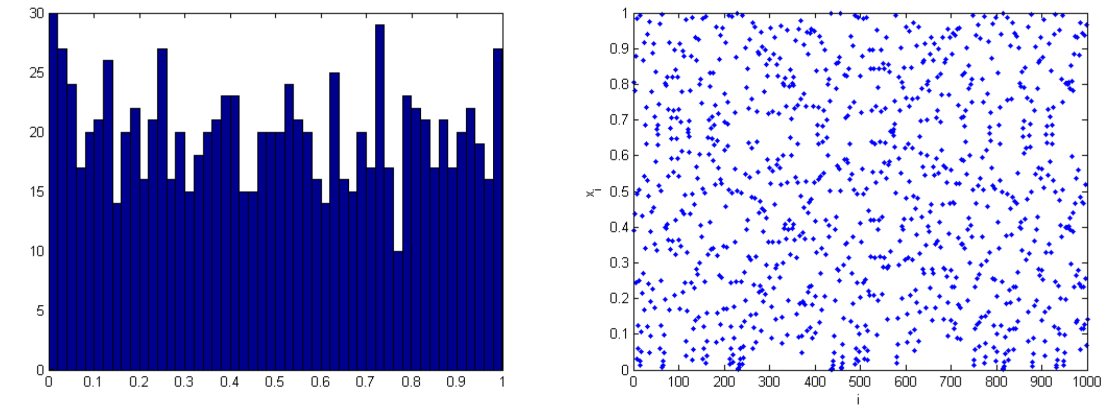
(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.21. Connecting map ($\alpha = 1.01$): Seed = 0.0006, Iteration no = 1000.



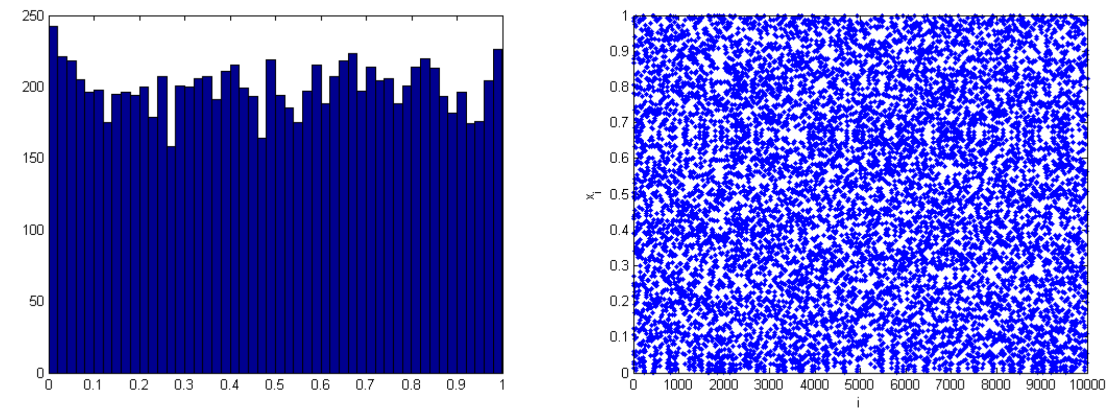
(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.22. Connecting map ($\alpha = 1.01$): Seed = 0.0006, Iteration no = 10000.



(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.23. Connecting map ($\alpha = 1.01$): Seed = 0.8066, Iteration no = 1000.



(a) Histogram with 50 subintervals (b) Serial plot

Figure 3.24. Connecting map ($\alpha = 1.01$): Seed = 0.8066, Iteration no = 10000.

We apply the statistical tests to 20 sequences of generated numbers. The results of the statistical tests of 20 sequences can be seen in the Table 3.7, Table 3.8, Table 3.9, and Table 3.10. The first two columns of the tables are the statistical test results of the sequences of numbers which have histograms, and serial plots above. The connecting family of mappings with parameters $\alpha = 0.999$ and $\alpha = 1.01$ pass randomness tests, and uniformity tests.

Table 3.7. The statistical test results for 10 different seeds under connecting map ($\alpha = 0.999$): Iteration no = 1000.

		Seeds				
	Statistical Tests	**0.727**	**0.819**	**0.1**	**0.98**	**0.6811**
P-Values	**Kolmogorov Smirnov**	0.038	0.9987	0.5695	0.8204	0.6199
	Chisqure 30	0.0163	0.9431	0.8497	0.0374	0.6634
	Chisqure 50	0.3995	0.7156	0.2832	0.0004	0.305
	Chisqure 80	0.4005	0.7565	0.8428	0.0002	0.6021
	Chisquare 100	0.6447	0.7956	0.3792	0.0003	0.4978
	Runs Up - Down	0.0045	0.4019	0.5398	0.5398	0.5565
	Runs Above - Below	0.1082	0.5065	0.955	0.1031	0.0209
	Max Autocorrelation coeff.	0.059235358	0.048950492	0.03572489	0.098787684	0.044833969

		Seeds				
	Statistical Tests	**0.002**	**0.7**	**0.298698545648568**	**0.525620161505902**	**0.333**
P-Values	**Kolmogorov Smirnov**	0.8595	0.7166	0.3603	0.8885	0.5507
	Chisqure 30	0.7502	0.975	0.9809	0.8938	0.9277
	Chisqure 50	0.9609	0.7788	0.8805	0.363	0.8264
	Chisqure 80	0.988	0.853	0.9416	0.6383	0.8428
	Chisquare 100	0.9685	0.7448	0.9789	0.276	0.9753
	Runs Up - Down	0.2655	0.2249	0.1807	0.8316	0.2655
	Runs Above - Below	0.0357	0.4253	0.6695	0.3264	0.2425
	Max Autocorrelation coeff.	0.047901021	0.047827382	0.045113752	0.104493299	0.034917029

Table 3.8. The statistical test results for 10 different seeds under connecting map ($\alpha = 0.999$): Iteration no = 10000.

	Statistical Tests	Seeds				
		0.727	**0.819**	**0.1**	**0.98**	**0.6811**
P-Values	**Kolmogorov Smirnov**	0.63	0.4453	0.9044	0.8625	0.5858
	Chisqure 30	0.9778	0.7672	0.7245	0.1655	0.7167
	Chisqure 50	0.9464	0.6171	0.9125	0.0483	0.9895
	Chisqure 80	0.3834	0.6378	0.5297	0.0382	0.9527
	Chisquare 100	0.8292	0.6394	0.9883	0.0204	0.9938
	Runs Up - Down	0.4502	0.0672	0.4223	0.4598	0.4743
	Runs Above - Below	0.1729	0.1901	0.847	0.0556	0.9795
	Max Autocorrelation coeff.	0.017671249	0.013714806	0.021939378	0.028739863	0.014280245

	Statistical Tests	Seeds				
		0.002	**0.7**	**0.298698545648568**	**0.525620161505902**	**0.333**
P-Values	**Kolmogorov Smirnov**	0.4611	0.9661	0.5896	0.9754	0.0365
	Chisqure 30	0.78	0.8665	0.8531	9818	0.0028
	Chisqure 50	0.8955	0.9135	0.5972	0.9548	0.0443
	Chisqure 80	0.4257	0.7216	0.9229	0.9775	0.1141
	Chisquare 100	0.7897	0.912	0.6352	0.7653	0.0216
	Runs Up - Down	0.13	0.834	0.6045	0.3328	0.0468
	Runs Above - Below	0.0221	0.142	0.3698	0.3628	0.0067
	Max Autocorrelation coeff.	0.024157563	0.019605673	0.021330712	0.008664005	0.025587361

Table 3.9. The statistical test results for 10 different seeds under connecting map ($\alpha = 1.01$): Iteration no = 1000.

	Statistical Tests	Seeds				
		0.0006	**0.80669**	**0.672551918078352**	**0.189**	**0.747**
P-Values	**Kolmogorov Smirnov**	0.4359	0.4726	0.1151	0.9235	0.9667
	Chisqure 30	0.3034	0.9199	0.8078	0.9405	0.5423
	Chisqure 50	0.2093	0.7381	0.7178	0.891	0.5642
	Chisqure 80	0.4051	0.5054	0.5965	0.9366	0.3573
	Chisquare 100	0.43	0.3766	0.6648	0.8238	0.6194
	Runs Up - Down	0.7167	0.3744	0.0495	0.6615	0.7735
	Runs Above - Below	0.8567	0.1545	0.8419	0.6351	0.4477
	Max Autocorrelation coeff.	0.059697053	0.07437212	0.061974722	0.06346677	0.063672062

	Statistical Tests	Seeds				
		0.880223477310122	**0.9**	**0.361436078541452**	**0.6001**	**0.24**
P-Values	**Kolmogorov Smirnov**	0.4914	0.2944	0.2397	0.9978	0.968
	Chisqure 30	0.6395	0.1619	0.3633	0.9975	0.35
	Chisqure 50	0.539	0.0234	0.5172	0.9954	0.4284
	Chisqure 80	0.9397	0.0095	0.7743	0.988	0.6111
	Chisquare 100	0.679	0.0172	0.4045	0.9947	0.7383
	Runs Up - Down	0.082	0.323	0.1502	0.9302	0.7544
	Runs Above - Below	0.6292	0.046	0.7994	0.6809	0.9683
	Max Autocorrelation coeff.	0.053683726	0.079586322	0.076766905	0.049357745	0.054377972

Table 3.10. The statistical test results for 10 different seeds under connecting map ($\alpha = 1.01$): Iteration no = 10000.

		Seeds				
	Statistical Tests	**0.0006**	**0.80669**	**0.672551918078352**	**0.189**	**0.747**
P-Values	**Kolmogorov Smirnov**	0.9627	0.3363	0.847	0.5652	0.9447
	Chisqure 30	0.9214	0.0847	0.7223	0.9943	0.9931
	Chisqure 50	0.8904	0.0447	0.3161	0.9632	1
	Chisqure 80	0.7922	0.0694	0.8108	0.924	0.9857
	Chisquare 100	0.8774	0.2579	0.5019	0.9042	0.9761
	Runs Up - Down	0.865	0.5193	0.6494	0.3328	0.4042
	Runs Above - Below	0.7872	0.3038	0.4065	0.6245	0.2184
	Max Autocorrelation coeff.	0.019349793	0.031863549	0.01651682	0.007327072	0.01902768

		Seeds				
	Statistical Tests	**0.880223477310122**	**0.9**	**0.361436078541452**	**0.6001**	**0.24**
P-Values	**Kolmogorov Smirnov**	0.5296	0.6261	0.6549	0.3971	0.7475
	Chisqure 30	0.6758	0.6059	0.7595	0.4191	0.6573
	Chisqure 50	0.9557	0.4187	0.8866	0.0083	0.316
	Chisqure 80	0.9928	0.0812	0.6588	0.065	0.3785
	Chisquare 100	0.9798	0.2043	0.8251	0.0449	0.1873
	Runs Up - Down	0.1111	0.8279	0.1222	0.6839	0.3288
	Runs Above - Below	0.2184	0.1027	0.4295	0.4198	0.0908
	Max Autocorrelation coeff.	0.024816983	0.015855071	0.024378107	0.029451123	0.019392261

3.5. Lyapunov Exponents Of The Suggested Maps

The Lyapunov exponent measures the rate of exponential divergence of nearby points. In practice, it gives an idea about how fast a chaotic map becomes unpredictable. If the Lyapunov exponent is $ln(c)$ then we can expect an error between two corresponding final points $c^n\epsilon$ after n iterations where ϵ was the initial error between two starting points.

The computation of the Lyapunov exponent varies depending on the type of the map: discrete maps and continuous maps. Since the suggested chaotic maps are continuous functions on the interval $[0, 1]$, Equation 3.3 is used in order to compute the Lyapunov exponents (Λ) of the maps. The integral equation consists of two parts, invariant density of map ($\rho(x)$) and logarithm of the derivative of map.

$$\Lambda = \int_0^1 \rho(x) ln \left| f^{'}(x) \right| dx \tag{3.3}$$

For the Tent map $f(x) = 1 - |1 - 2x|$, the natural invariant density ($\rho(x)$) of the map is known to be 1, [0]. The derivative of tent map ($f^{'}(x)$) is equal to 2 if $x < \frac{1}{2}$ and -2 otherwise. Therefore, the Lyapunov exponent of the tent map with parameter $\mu = 2$ is $ln2$ ($\cong 0.69314$).

For the tent map $f(x) = 1 - |1 - \mu x|$ with $\mu = 2.00005$, we need to find the invariant density of the map after applied IF method. This tent map consists of four subintervals after IF method was applied, which are listed below.

$$f(x) = \begin{cases} 2.00005x & , 0 < x < \frac{1}{2.00005} \\ -2.00005x + 2 & , \frac{1}{2.00005} < x < \frac{1}{2} \\ 2.00005x - 0.00005 & , \frac{1}{2} < x < \frac{1.00005}{2.00005} \\ -2.00005x + 2.00005 & , \frac{1.00005}{2.00005} < x < 1 \end{cases}$$

The stationary probabilities of each part are calculated in order to compute the invariant density of the tent map. They are 0.499975001, 0.000012499, 0.000012499, 0.5 respectively for each interval. The natural invariant density of the tent map is:

$$\rho(x) = \begin{cases} \frac{0.499975001}{0.4999875} & , 0 < x < \frac{1}{2.00005} \\ \frac{0.000012499}{0.0000125} & , \frac{1}{2.00005} < x < \frac{1}{2} \\ \frac{0.000012499}{0.0000125} & , \frac{1}{2} < x < \frac{1.00005}{2.00005} \\ \frac{0.5}{0.4999875} & , \frac{1.00005}{2.00005} < x < 1 \end{cases}$$

The above function is thus very close to 1.

The derivative of this tent map is equal to 2.00005 if $x < \frac{1}{2}$ and -2.00005 otherwise. So the integral equation becomes the multiplication of $ln(2.00005)$ and the integral of the invariant density. Since the integral of the density function is 1, the Lyapunov exponent of the tent map with parameter $\mu = 2.00005$ is $ln(2.00005)$ ($\cong 0.69317$).

For transformed logistic map (T-logistic map) $f(x) = 1 - |1 - 2x|$, it is equivalent to the tent map with parameter $\mu = 2$. That's why the natural invariant density of the T-logistic map is 1, and the Lyapunov exponent of the T-logistic map is $ln2$ ($\cong 0.69314$).

For the connecting maps $f(x) = 1 - |1 - 2x|^{\alpha}$, the Lyapunov exponents of the connecting maps with parameters $\alpha = 0.999$ and $\alpha = 1.01$ are not easily computed due to the fact that there is no explicit function to represent the invariant densities of the connecting maps. To determine the invariant densities of the connecting maps we consider the histograms of the numbers that are generated by the connecting maps, since the invariant density of a map is a function that fits to the histograms of the numbers generated by map.

Firstly, we observe histograms of the generated numbers for two connecting maps. As can be seen in the frequency histograms in Section 3.4, the invariant densities of two connecting maps have uniformly distributed shapes, (also see Appendix D and E).

Secondly, we apply the statistical chi-square tests to the numbers generated by these connecting maps. As can be seen in Section 3.4, ten 1000-number-sequences and ten 10000-number-sequences for 10 different seeds are generated and chi-square tests are applied to analyze the property of uniformity of the connecting maps. The performances of the maps for chi-square tests are summarized in Table 3.11. The table shows the number of chi-square tests passed with 10 different seeds under two connecting maps where p-values of 0.05 is used as the test criteria.

Table 3.11. Number of chi-square tests passed with 10 different seeds for two connecting maps (p-values of 0.05 is used as the test criteria).

	Iteration no = 1000		Iteration no = 10000	
Statistical Tests	$\alpha = 0.999$	$\alpha = 1.01$	$\alpha = 0.999$	$\alpha = 1.01$
Total # of passes in 10 Chisquare Tests	8	10	8	9
Passing Success	**80.0%**	**100.0%**	**80.0%**	**90.0%**

In Section 4.2, more test results comparing the uniformities of connecting maps versus linear congruential generators are provided. In conclusion, we can assume that invariant density of these connecting maps is approximately 1.

The Lyapunov exponent of the connecting maps with parameter $\alpha = 1.01$ is next computed:

$$
\begin{aligned}
\Lambda &= \int_0^1 \rho(x) ln \left| f^{'}(x) \right| dx \\
&= \int_0^1 ln \left| 2(1.01)(1-2x)^{0.01} \right| dx \\
&= ln(2.02) + \int_0^1 ln \left| (1-2x)^{0.01} \right| dx \\
&= ln(2.02) + \int_0^{\frac{1}{2}} ln((1-2x)^{0.01}) dx + \int_{\frac{1}{2}}^1 ln((2x-1)^{0.01}) dx \\
&= ln(2.02) + (0.01) \int_0^{\frac{1}{2}} ln(1-2x) dx + (0.01) \int_{\frac{1}{2}}^1 ln(2x-1) dx \\
&= ln(2.02) + (0.01) \frac{1}{2} \int_0^1 ln(u) du + (0.01) \frac{1}{2} \int_0^1 ln(u) du \\
&= ln(2.02) + (0.01) \left[u ln(u) - u \right]_0^1 \\
&= ln(2.02) - (0.01) \cong 0.69309.
\end{aligned}
$$

The Lyapunov exponent of the connecting maps with parameter $\alpha = 0.999$ is computed:

$$\begin{aligned}
\Lambda &= \int_0^1 \rho(x) ln \left| f^{'}(x) \right| dx \\
&= \int_0^1 ln \left| 2(0.999)(1-2x)^{-0.001} \right| dx \\
&= ln(1.998) + \int_0^1 ln \left| (1-2x)^{-0.001} \right| dx \\
&= ln(1.998) + \int_0^{\frac{1}{2}} ln((1-2x)^{-0.001}) dx + \int_{\frac{1}{2}}^1 ln((2x-1)^{-0.001}) dx \\
&= ln(1.998) - (0.001) \int_0^{\frac{1}{2}} ln(1-2x) dx - (0.001) \int_{\frac{1}{2}}^1 ln(2x-1) dx \\
&= ln(1.998) - (0.001) \frac{1}{2} \int_0^1 ln(u) du - (0.001) \frac{1}{2} \int_0^1 ln(u) du \\
&= ln(1.998) - (0.001) \left[u ln(u) - u \right]_0^1 \\
&= ln(1.998) + (0.001) \cong 0.69314.
\end{aligned}$$

To conclude, the Lyapunov exponents of all suggested maps are exactly or approximately 0.6931 ($\cong ln2$). If the initial error between two points is $\epsilon = 10^{-16}$, then the error $2^n \epsilon$ is approximately equal to 1 after $n = 53$ iterations.

4. COMPARISON BETWEEN FIVE SUGGESTED CHAOTIC MAPS AND SOME WELL-KNOWN LINEAR CONGRUENTIAL GENERATORS (LCGs)

4.1. Linear Congruential Generators (LCGs)

In this part of the thesis, we will compare our five suggested chaotic maps with some well-known linear congruential generators (LCGs). We would like to choose LCGs that are conventionally used in many areas. Five different LCGs are selected for the comparison. Two of them are multiplicative linear congruential generators (MLCGs), and the other ones are linear congruential generators (LCGs), as listed in Table 4.1. These generators are actively used in Borland C/C++, CodeWarrior, IBM VisualAge C/C++, Apple CarbonLib, Borland Delphi, and Virtual Pascal. It is necessary to compare five suggested chaotic maps with the well-known LCGs to understand whether they are successful in generating of random numbers or not. If they are as successful as LCGs, then we can approve the actual use of them as RNGs.

LCGs produces a sequence of integers, $X_1, X_2, ...$ between 0 and $m-1$ by following a recursive relation:

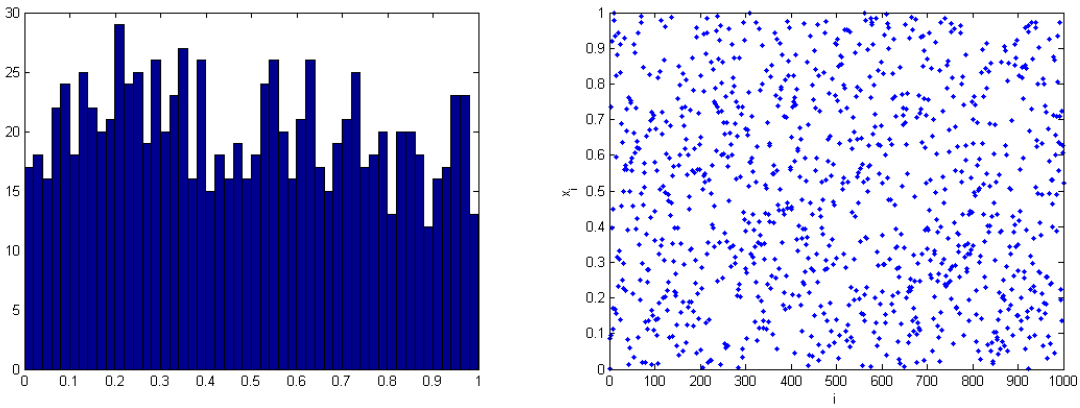
$$X_{n+1} = (aX_n + c)\ mod\, m,\ i = 0, 1, 2, ...$$

They are defined from the set of integers $\{0, 1, 2, ..., m-1\}$ to the set of integers $\{0, 1, 2, ..., m-1\}$, but the five suggested RNGs are defined from $[0, 1]$ to $[0, 1]$. For meaningful results, we convert the chosen integer seeds for LCGs to real numbers in $[0, 1]$. For the conversion, the chosen integer seeds for LCGs are divided by multiples of 10. For instance, $X_0 = 123, 457$ converts to $X_0 = 0.00123457$ for tent map with parameter $\mu = 2$, and $X_0 = 0.123457$ for the other four maps. (The reason of keeping tent map with parameter $\mu = 2$ separate are explained in the previous chapter.) The largest possible periods (P) of LCGs can be calculated by the choice of a, c, m and X_0, [2].

Table 4.1. List of some well-known linear congruential generators (LCGs).

	Source	a	m	c	P
MLCG 1	-	742938285	$2^{31}-1$	0	$2^{31}-2$
MLCG 2	Apple CarbonLib	16807	$2^{31}-1$	0	$2^{31}-2$
LCG 3	Borland C/C++	22695477	2^{32}	1	2^{32}
LCG 4	CodeWarrior, IBM VisualAge C/C++	1103515245	2^{32}	12345	2^{32}
LCG 5	Microsoft Visual, Quick C/C++	214013	2^{32}	2531011	2^{32}

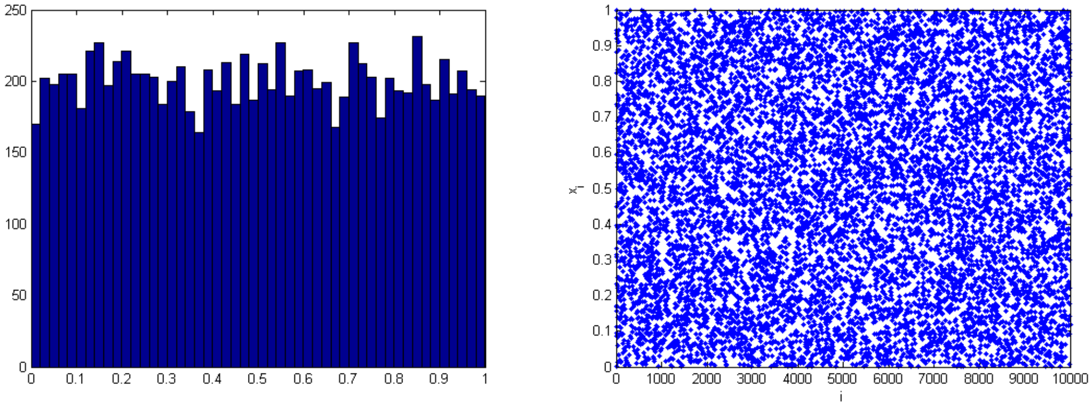
Before starting the comparison, we observe the histograms and serial plots of the sequences of numbers generated by LCGs. We take two randomly selected seeds for each LCG, and generate ten of 1000-number-sequences and ten of 10000-number-sequences. The histograms and serial plots for LCGs are shown in Figure 4.1-20.



(a) Histogram with 50 subintervals

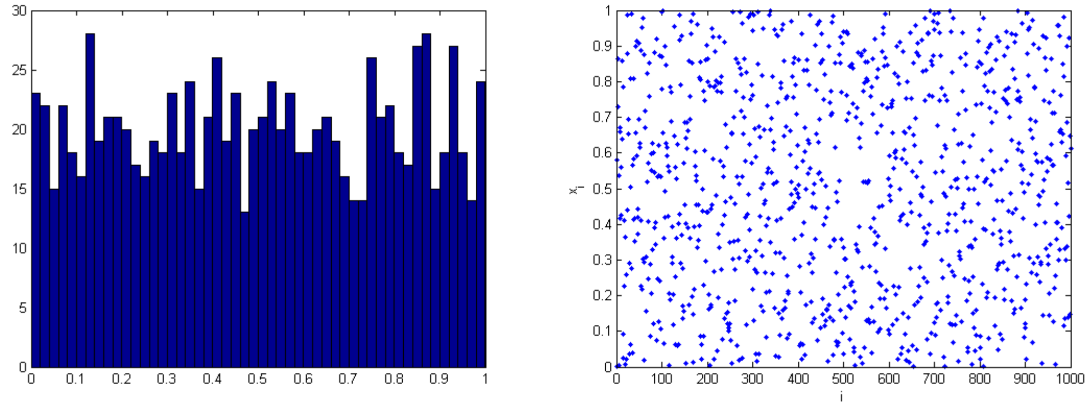
(b) Serial plot

Figure 4.1. MLCG 1: Seed = 931, Iteration no = 1000.



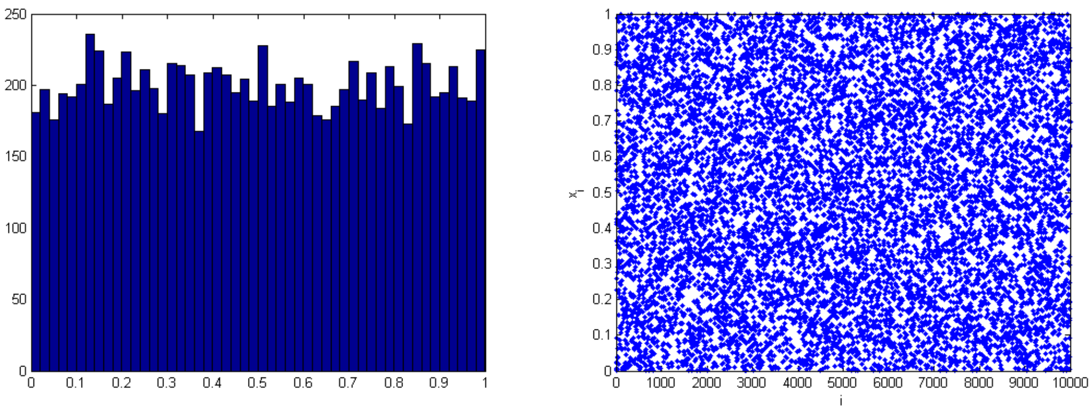
(a) Histogram with 50 subintervals (b) Serial plot

Figure 4.2. MLCG 1: Seed = 931, Iteration no = 10000.



(a) Histogram with 50 subintervals (b) Serial plot

Figure 4.3. MLCG 1: Seed = 6647, Iteration no = 1000.

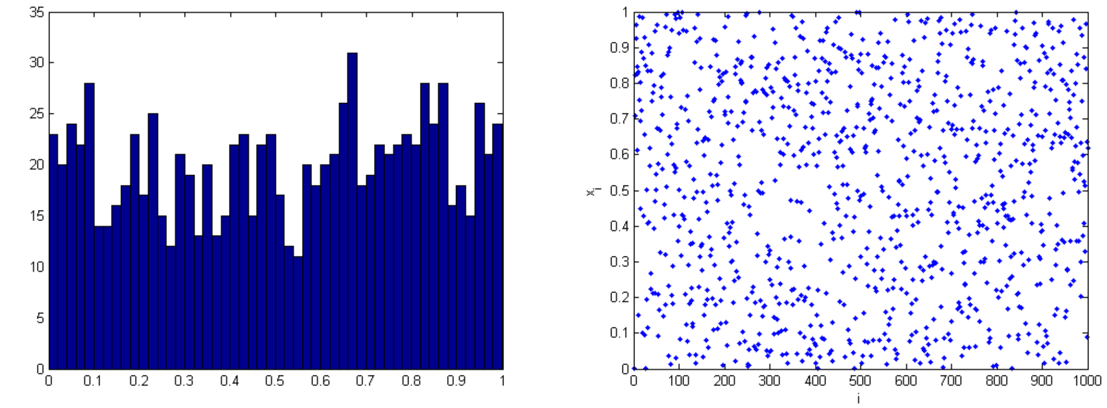


(a) Histogram with 50 subintervals (b) Serial plot

Figure 4.4. MLCG 1: Seed = 6647, Iteration no = 10000.

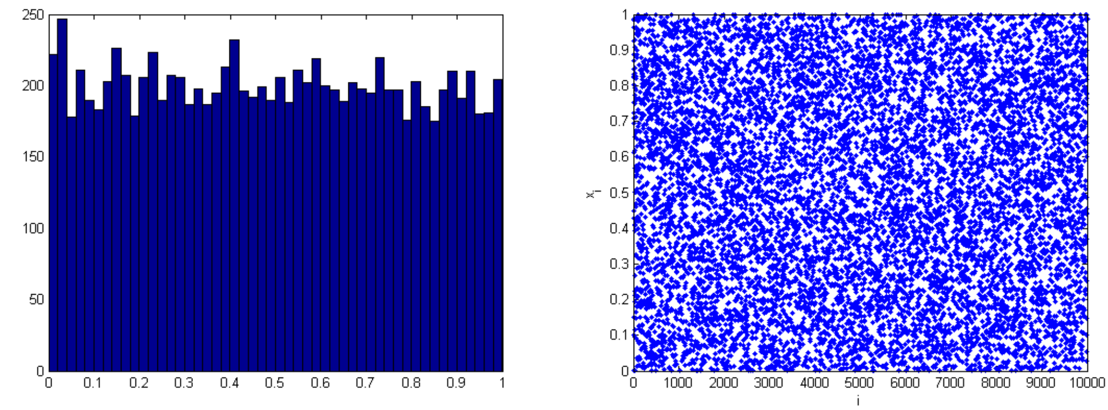
Table 4.2. The statistical test results for two different seeds under MLCG 1.

		Iteration No = 1,000		Iteration No = 10,000	
	Statistical Tests	**931**	**6647**	**931**	**6647**
P-Values	**Kolmogorov Smirnov**	0.0792	0.9276	0.8389	0.7919
	Chisqure 30	0.3631	0.8775	0.4869	0.4213
	Chisqure 50	0.8025	0.9032	0.1392	0.1082
	Chisqure 80	0.6502	0.8086	0.3899	0.1465
	Chisqure 100	0.945	0.9527	0.1643	0.0816
	Runs Up - Down	0.323	0.3354	0.8156	0.3096
	Runs Above - Below	0.9711	0.5911	0.6097	0.8044
	Auto Corr Func	0.044670445	0.032876603	0.015130087	0.016594983



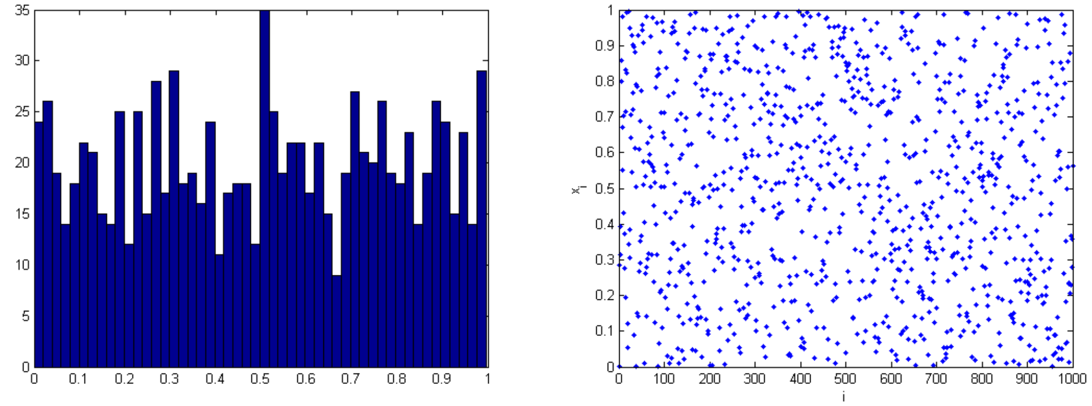
(a) Histogram with 50 subintervals (b) Serial plot

Figure 4.5. MLCG 2: Seed = 13, Iteration no = 1000.



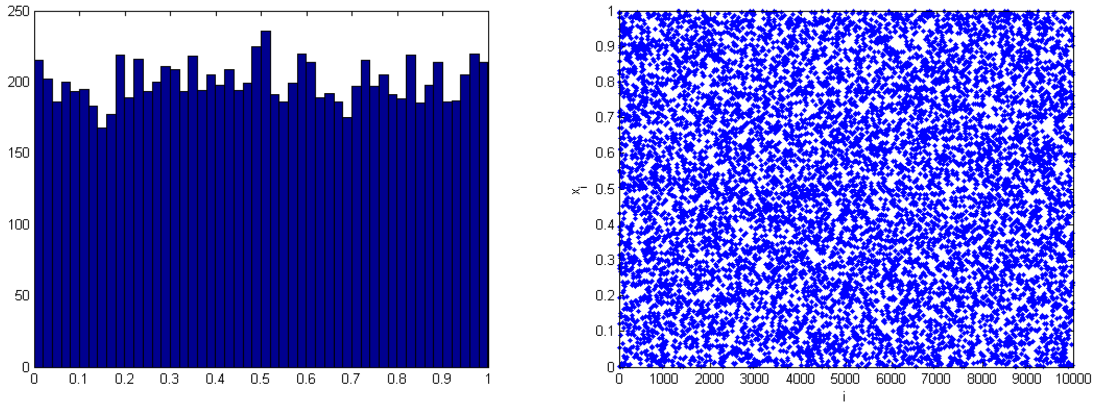
(a) Histogram with 50 subintervals (b) Serial plot

Figure 4.6. MLCG 2: Seed = 13, Iteration no = 10000.



(a) Histogram with 50 subintervals (b) Serial plot

Figure 4.7. MLCG 2: Seed = 36509, Iteration no = 1000.

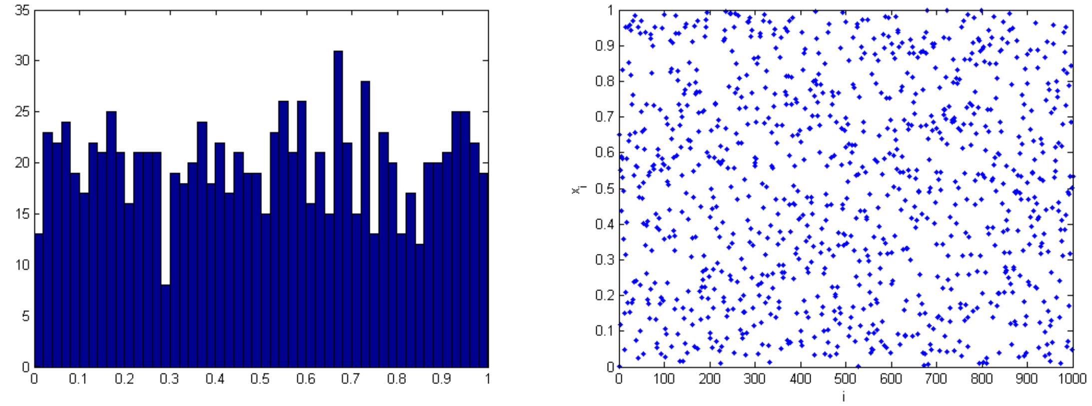


(a) Histogram with 50 subintervals (b) Serial plot

Figure 4.8. MLCG 2: Seed = 36509, Iteration no = 10000.

Table 4.3. The statistical test results for two different seeds under MLCG 2.

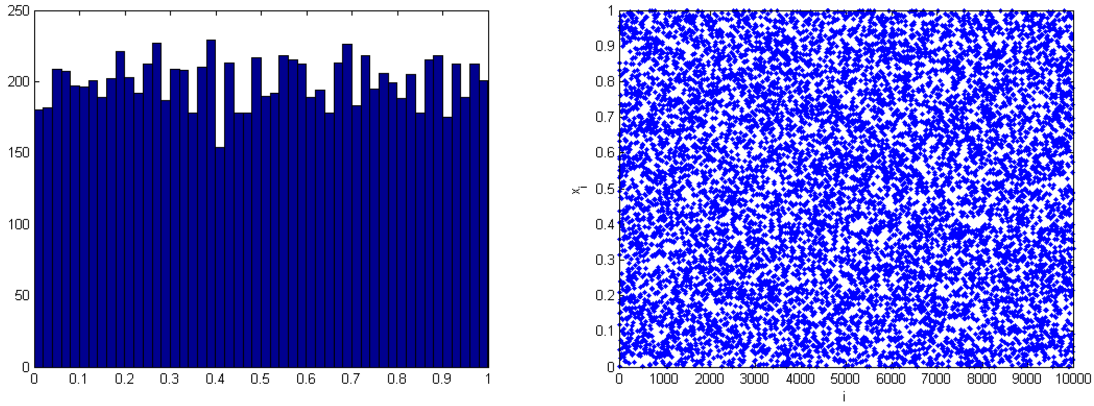
		Iteration No = 1,000		Iteration No = 10,000	
	Statistical Tests	**13**	**36509**	**13**	**36509**
P-Values	**Kolmogorov Smirnov**	0.0201	0.6978	0.8057	0.4504
	Chisqure 30	0.0278	0.1314	0.2569	0.2197
	Chisqure 50	0.3054	0.0395	0.7327	0.451
	Chisqure 80	0.0895	0.3798	0.7678	0.7664
	Chisqure 100	0.5359	0.159	0.6585	0.7762
	Runs Up - Down	0.3354	0.4161	0.7608	0.725
	Runs Above - Below	0.692	1	0.57	0.6234
	Auto Corr Func	0.04073585	0.032263478	0.015607527	0.011541214



(a) Histogram with 50 subintervals

(b) Serial plot

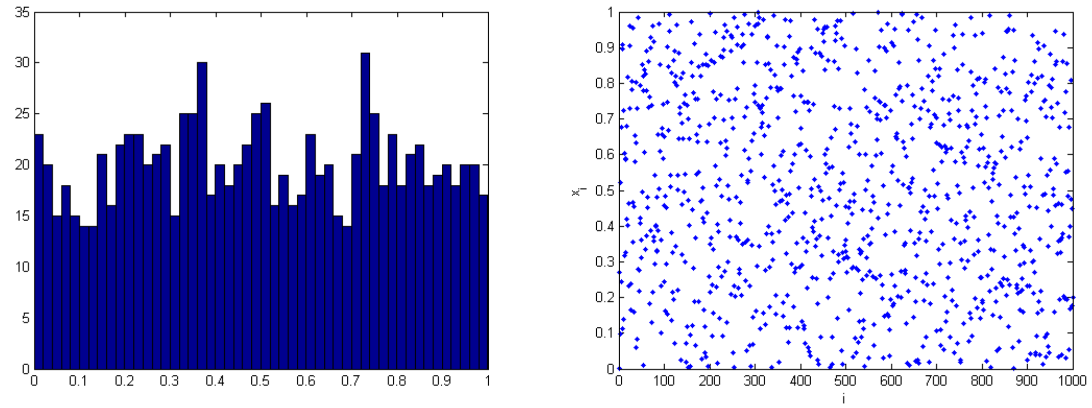
Figure 4.9. LCG 3: Seed = 1637, Iteration no = 1000.



(a) Histogram with 50 subintervals

(b) Serial plot

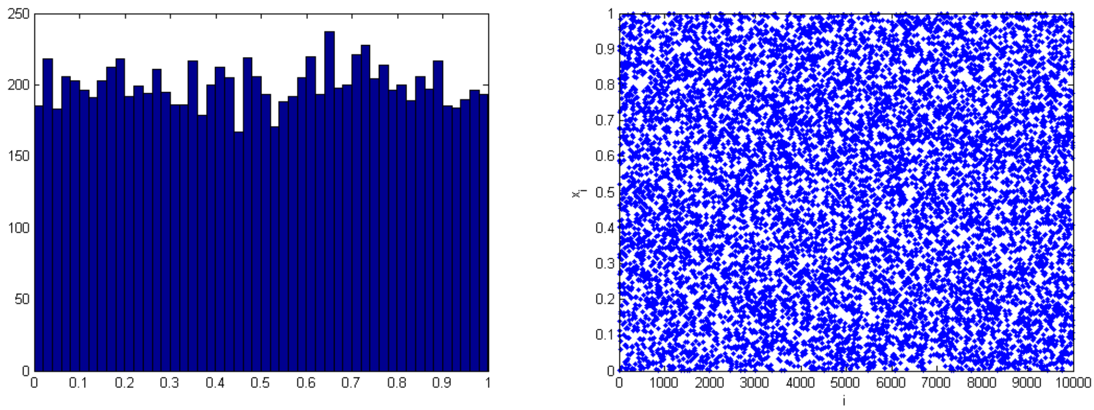
Figure 4.10. LCG 3: Seed = 1637, Iteration no = 10000.



(a) Histogram with 50 subintervals

(b) Serial plot

Figure 4.11. LCG 3: Seed = 5161, Iteration no = 1000.



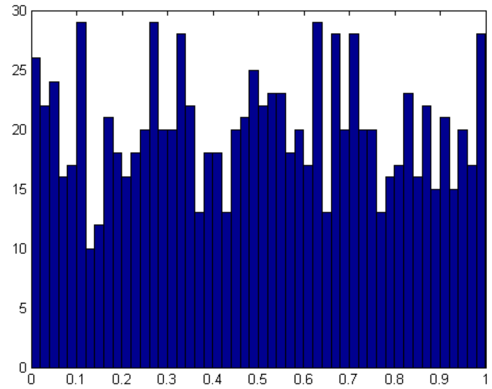
(a) Histogram with 50 subintervals (b) Serial plot

Figure 4.12. LCG 3: Seed = 5161, Iteration no = 10000.

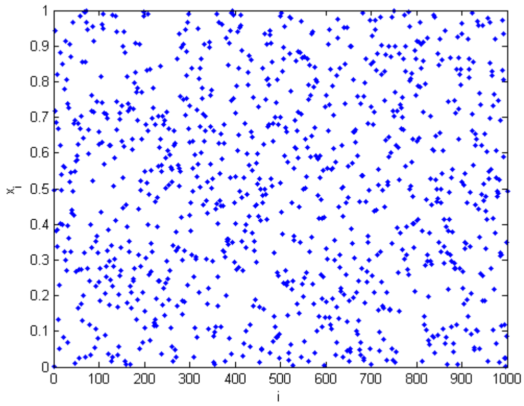
Table 4.4. The statistical test results for two different seeds under LCG 3.

		Iteration No = 1,000		Iteration No = 10,000	
	Statistical Tests	**1637**	**5161**	**1637**	**5161**
P-Values	**Kolmogorov Smirnov**	0.9225	0.4718	0.9567	0.5725
	Chisqure 30	0.2481	0.3352	0.0608	0.1388
	Chisqure 50	0.5893	0.9011	0.0664	0.4357
	Chisqure 80	0.4014	0.2194	0.1856	0.7166
	Chisqure 100	0.5369	0.27	0.136	0.654
	Runs Up - Down	0.2876	0.0414	0.865	0.391
	Runs Above - Below	0.8694	0.9162	0.407	0.569
	Auto Corr Func	0.061035185	0.057628314	0.012726317	0.015810861

The histograms and serial plots seem to be acceptable to pass the eye randomness test. The test results of LCGs are similar with those of five suggested chaotic maps as seen in Table 4.2-6.

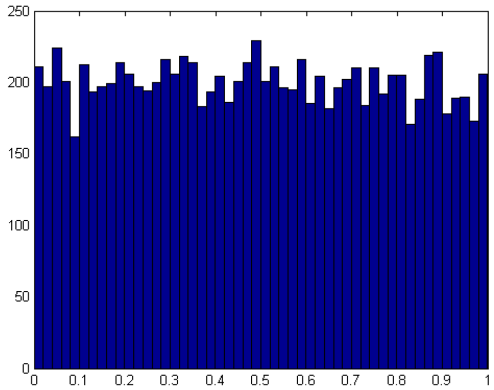


(a) Histogram with 50 subintervals

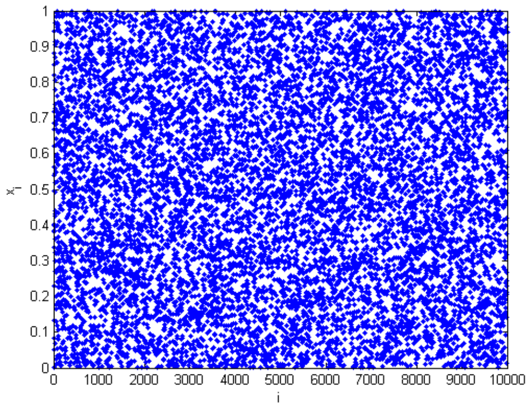


(b) Serial plot

Figure 4.13. LCG 4: Seed = 42807, Iteration no = 1000.

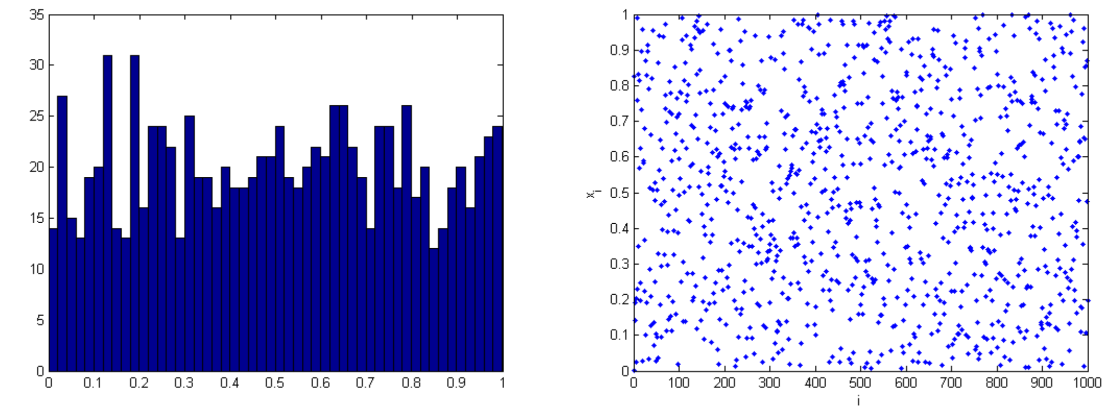


(a) Histogram with 50 subintervals



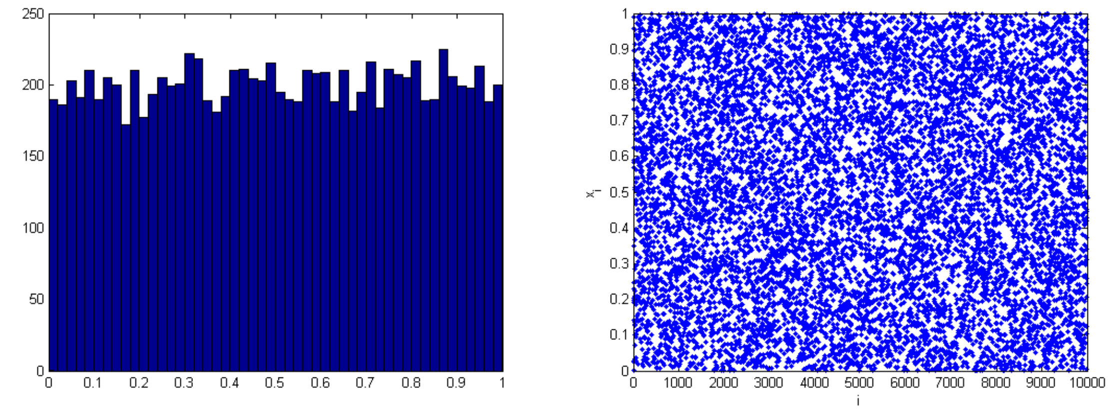
(b) Serial plot

Figure 4.14. LCG 4: Seed = 42807, Iteration no = 10000.



(a) Histogram with 50 subintervals (b) Serial plot

Figure 4.15. LCG 4: Seed = 11, Iteration no = 1000.

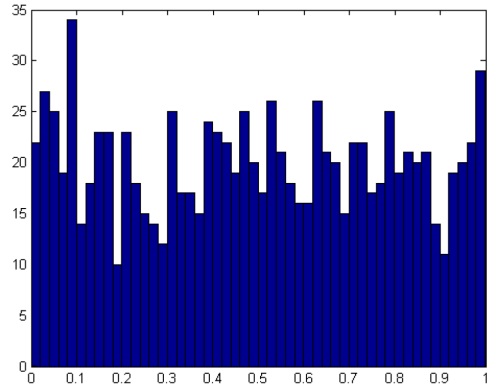


(a) Histogram with 50 subintervals (b) Serial plot

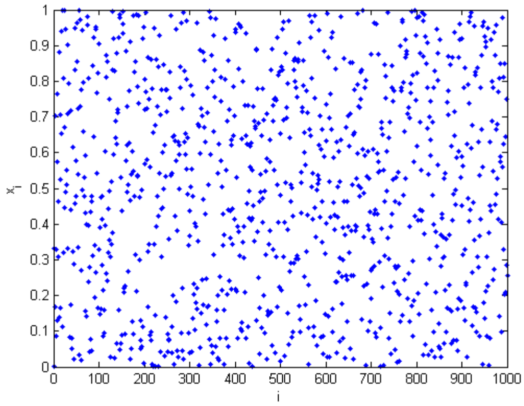
Figure 4.16. LCG 4: Seed = 11, Iteration no = 10000.

Table 4.5. The statistical test results for two different seeds under LCG 4.

		Iteration No = 1,000		Iteration No = 10,000	
	Statistical Tests	**42807**	**11**	**42807**	**11**
P-Values	**Kolmogorov Smirnov**	0.8513	0.8982	0.2875	0.5208
	Chisqure 30	0.6618	0.7866	0.3166	0.9404
	Chisqure 50	0.1895	0.4338	0.4633	0.9137
	Chisqure 80	0.6781	0.8275	0.9116	0.9355
	Chisqure 100	0.2851	0.6602	0.6469	0.6211
	Runs Up - Down	0.7544	0.5565	0.4361	0.4646
	Runs Above - Below	0.3973	0.0607	0.0991	0.2668
	Auto Corr Func	0.072887885	0.063600942	0.018742471	0.019221117

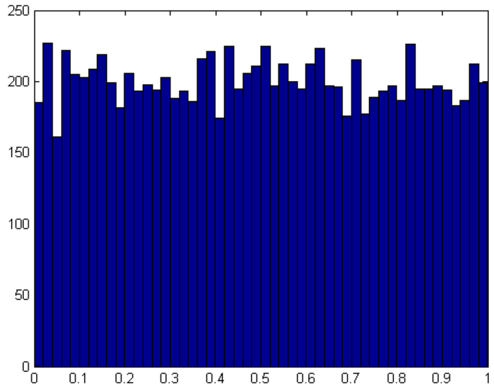


(a) Histogram with 50 subintervals

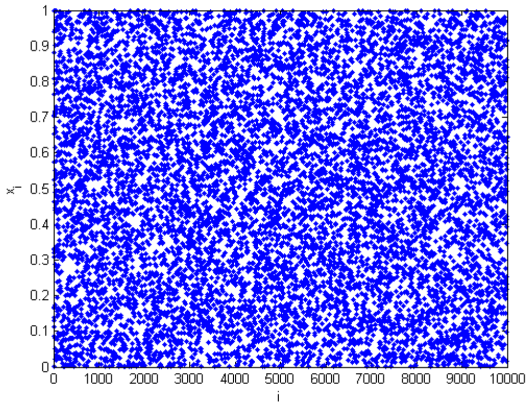


(b) Serial plot

Figure 4.17. LCG 5: Seed = 6625, Iteration no = 1000.

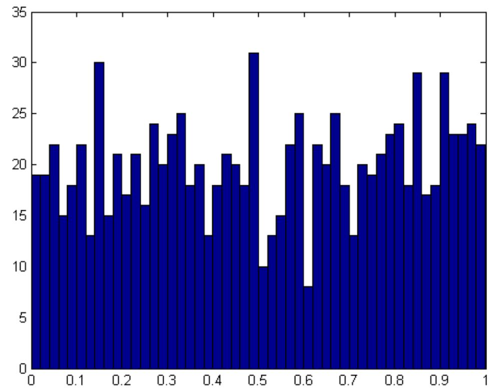


(a) Histogram with 50 subintervals

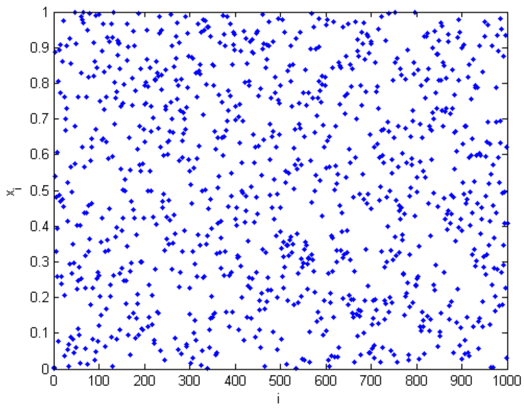


(b) Serial plot

Figure 4.18. LCG 5: Seed = 6625, Iteration no = 10000.

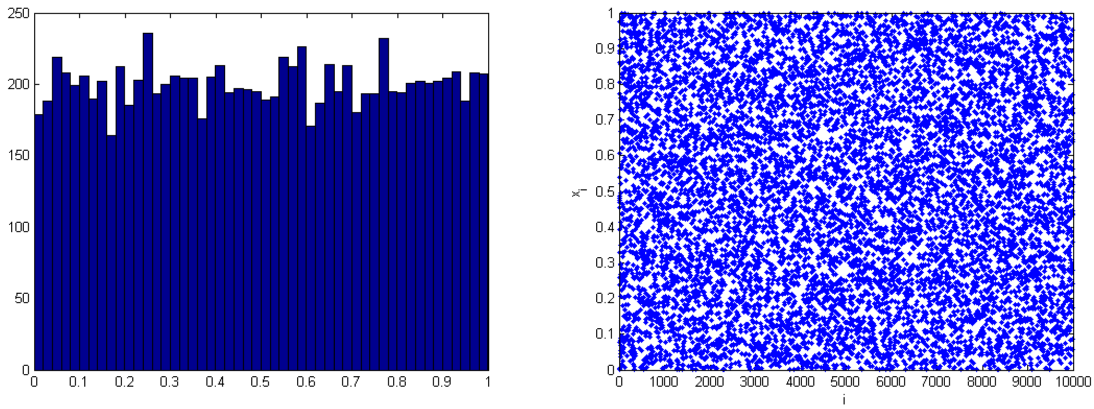


(a) Histogram with 50 subintervals



(b) Serial plot

Figure 4.19. LCG 5: Seed = 87, Iteration no = 1000.



(a) Histogram with 50 subintervals

(b) Serial plot

Figure 4.20. LCG 5: Seed = 87, Iteration no = 10000.

Table 4.6. The statistical test results for two different seeds under LCG 5.

		Iteration No = 1,000		Iteration No = 10,000	
	Statistical Tests	**6625**	**87**	**6625**	**87**
P-Values	**Kolmogorov Smirnov**	0.322	0.2167	0.2541	0.9304
	Chisqure 30	0.1156	0.221	0.4775	0.6343
	Chisqure 50	0.3056	0.1966	0.2542	0.4695
	Chisqure 80	0.1302	0.1357	0.013	0.2796
	Chisqure 100	0.2691	0.7043	0.0666	0.1779
	Runs Up - Down	0.0286	0.8316	0.9213	0.54
	Runs Above - Below	0.3267	0.3894	0.7112	0.2261
	Auto Corr Func	0.05100644	0.047151764	0.014500821	0.02872687

4.2. Basic Comparison of Five Suggested Chaotic Maps and LCGs

We test the performance of five suggested chaotic maps and LCGs. We use the seeds which are suggested by developers and actual users of LCGs. 20 suggested different seeds are selected. Using these seeds, we run LCGs and the five suggested maps to generate 1000-number-sequences and 10000-number-sequences. For each seed, there are 20 different 1000-number-sequences and 10000-number-sequences individually, generated by 10 different maps.

The first step is to observe histograms and serial plots of the sequences of numbers. We scan 400 different histograms and serial plots, and all of them are good enough to pass the eye test. Furthermore there is no significant difference between histograms and serial plots of the sequences of numbers. Twenty of 400 figures are provided in Appendix F and G.

The second step is the statistical tests. Kolmogorov Smirnov test and chi square test are applied to compare the property of uniformity of the maps with each other. Runs tests and autocorrelation test are taken into account to examine independence property in each sequence of generated numbers. The test results of the sequences of numbers for one of 20 seeds is listed below, (see Table 4.7 and Table 4.8). Although the numbers of tests passed by LCGs are larger than ones by the five suggested chaotic maps, there is no significant difference between the test results. When these results are taken into account for the usability of the five chaotic maps as RNGs, they are as successful as LCGs. The test results of the sequences of numbers for 19 seeds can be found in Appendix H, I, J and K.

The overall performances of the tests are summarized in Table 4.9. The table shows the number of tests passed with 20 different seeds under LGCs and five suggested maps where p-values of 0.05 and the related confidence bounds, which are ± 0.0632 and ± 0.02 for the sample sizes 1000 and 10000 respectively, are used as the test criteria. We consider Kolmogorov-Smirnov test and chi-square test for uniformity tests, and autocorrelation test for serial test for each map. Test results are over 20 for each test,

Table 4.7. The statistical test results for LGCs and five suggested maps: Seed = 3316779, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**3316779**	**3316779**	**3316779**	**3316779**	**3316779**
P-Values	**Kolmogorov Smirnov**	0.2645	0.6882	0.3255	0.9182	0.5212
	Chisqure 30	0.2115	0.7098	0.5857	0.9681	0.5212
	Chisqure 50	0.7628	0.844	0.3478	0.657	0.7441
	Chisqure 80	0.6679	0.5818	0.5983	0.6802	0.891
	Chisqure 100	0.4358	0.7589	0.4592	0.2794	0.8494
	Runs Up - Down	0.0089	0.5906	0.0345	0.9501	0.0089
	Runs Above - Below	0.3405	0.4278	0.0878	0.4271	1
	Auto Corr Func	0.078007416	0.045626096	0.045626096	0.026621392	0.07701126

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.03316779**	**0.3316779**	**0.3316779**	**0.3316779**	**0.3316779**
P-Values	**Kolmogorov Smirnov**	0.4591	0.5211	0.0014	0.7766	0.9248
	Chisqure 30	0.0102	0.9021	0.00001	0.5078	0.9686
	Chisqure 50	0.2217	0.9132	0.0001	0.43	0.9268
	Chisqure 80	0.3055	0.9298	0.0153	0.0092	0.8986
	Chisqure 100	0.0385	0.9034	0.0022	0.1743	0.9955
	Runs Up - Down	0.2655	0.1727	0.0157	0.7544	0.7167
	Runs Above - Below	0.2985	0.8518	1	0.7016	1
	Auto Corr Func	0.091024076	0.044200226	0.052844041	0.101173377	0.039105022

Table 4.8. The statistical test results for LGCs and five suggested maps: Seed = 3316779, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**3316779**	**3316779**	**3316779**	**3316779**	**3316779**
P-Values	**Kolmogorov Smirnov**	0.2841	0.1042	0.2211	0.777	0.4723
	Chisqure 30	0.6883	0.2585	0.7049	0.8705	0.5833
	Chisqure 50	0.4586	0.7274	0.4759	0.7504	0.3042
	Chisqure 80	0.3627	0.8502	0.6159	0.7854	0.4345
	Chisqure 100	0.3573	0.8051	0.2624	0.7884	0.5287
	Runs Up - Down	0.1076	0.9276	0.1907	0.2279	0.89
	Runs Above - Below	0.7058	0.5572	0.12	0.2422	0.0117
	Auto Corr Func	0.017870699	0.014595199	0.014595199	0.014415561	0.021320281

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.03316779**	**0.3316779**	**0.3316779**	**0.3316779**	**0.3316779**
P-Values	**Kolmogorov Smirnov**	0.3415	0.6251	0.6022	0.4978	0.8481
	Chisqure 30	0.6044	0.3102	0.7806	0.7327	0.5337
	Chisqure 50	0.687	0.8911	0.41	0.9452	0.4106
	Chisqure 80	0.3702	0.864	0.5733	0.7712	0.3995
	Chisqure 100	0.7772	0.9445	0.1141	0.9266	0.1431
	Runs Up - Down	0.0672	0.1751	0.599	0.0961	0.7369
	Runs Above - Below	0.1526	0.6033	0.7443	0.5677	0.5302
	Auto Corr Func	0.018933564	0.020643347	0.017049909	0.01868399	0.020478566

and the sum of these results can be 60 at most.

LCGs seem to be somewhat more successful than the chaotic RNGs in statistical tests. Tent map with parameter $\mu = 2.00005$ and MLCG 2 reach success ratios of 90% and 95% respectively, which are the highest ratios of the tests, when the number of iterations is 1000. The minimum ratios are observed for T-logistic map and MLCG 1, which are 75% and 85%. Overall average success rate of five suggested maps in tests is 81%, and it is observed for LCGs as 90.3%.

When the number of iterations is 10000, the five suggested chaotic maps reach the highest ratio at 85%. The highest ratio for LCGs is 86.7% in this case. The minimum ratios of five suggested maps and LCGs are seen as 80% and 83.3% for connecting map with the parameter $\alpha = 1.01$ and LCG 5 respectively. Overall average success rate of five suggested chaotic maps in tests is 83.3%, whereas it is 87.7% for LCGs, (see Table 4.9).

Table 4.9. Number of tests passed with 20 different seeds under LGCs and five suggested maps (p-values of 0.05 is used as the test criteria).

Iteration no = 1,000	**LCGs**					**FIVE MAPS**				
Statistical Tests	**MLCG 1**	**MLCG 2**	**LCG 3**	**LCG 4**	**LCG 5**	**Tent Map μ = 2**	**Tent Map μ = 2.00005**	**T-Logistic Map**	**Connecting Map α = 0.999**	**Connecting Map α = 1.01**
Kolmogorov Smirnov	16	20	20	20	20	19	20	19	19	20
Chisqure 30	20	20	19	18	20	18	20	15	19	17
Auto Corr Func	20	20	20	20	20	20	20	20	20	20
Total passes (over 60)	**56**	**60**	**59**	**58**	**60**	**57**	**60**	**54**	**58**	**57**
Passing Success	**93.3%**	**100.0%**	**98.3%**	**96.7%**	**100.0%**	**95.0%**	**100.0%**	**90.0%**	**96.7%**	**95.0%**

Iteration no = 10,000	**LCGs**					**FIVE MAPS**				
Statistical Tests	**MLCG 1**	**MLCG 2**	**LCG 3**	**LCG 4**	**LCG 5**	**Tent Map μ = 2**	**Tent Map μ = 2.00005**	**T-Logistic Map**	**Connecting Map α = 0.999**	**Connecting Map α = 1.01**
Kolmogorov Smirnov	20	20	20	20	20	20	19	19	18	18
Chisquare 100	15	19	19	20	19	18	15	19	17	18
Auto Corr Func	18	12	12	18	11	13	15	13	16	12
Total passes (over 60)	**53**	**51**	**51**	**58**	**50**	**51**	**49**	**51**	**51**	**48**
Passing Success	**88.3%**	**85.0%**	**85.0%**	**96.7%**	**83.3%**	**85.0%**	**81.7%**	**85.0%**	**85.0%**	**80.0%**

These test ratios increase when p-values of 0.01 and the related confidence bounds, which are ± 0.0948 and ± 0.03 for the sample sizes 1000 and 10000 respectively, are used as the test criteria. The highest success ratios of 100% and of 98.3% are observed for five suggested maps and LCGs respectively when the number of iterations is taken as 1000. The minimum ratios become 90% for five suggested maps, and 96.7% for LCGs.

Overall average success rate of five suggested chaotic maps in tests is 96.7%, and it is observed for LCGs as 97.7%.

When iteration number is 10000, a decrement is observed in the highest ratio for five suggested chaotic maps. It decreases to 98.3%. However there is an increment in the lowest ratio for them. It is observed as 93.3% for them. The highest ratio of LCGs increases to 100%, and the lowest ratio of them reach to 98.3% in this situation. Overall average success rate of five suggested chaotic maps and LCGs in tests become 95.7% and 99.7% respectively, (see Table 4.10).

Table 4.10. Number of tests passed with 20 different seeds under LGCs and five suggested maps (p-values of 0.01 is used as the test criteria).

Iteration no = 1,000	LCGs					FIVE MAPS				
Statistical Tests	MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5	Tent Map $\mu = 2$	Tent Map $\mu = 2.00005$	T-Logistic Map	Connecting Map $\alpha = 0.999$	Connecting Map $\alpha = 1.01$
Kolmogorov Smirnov	19	20	20	20	20	19	20	19	20	20
Chisqure 30	20	20	19	20	20	19	20	17	20	20
Auto Corr Func	19	19	19	19	19	20	20	18	18	20
Total passes (over 60)	**58**	**59**	**58**	**59**	**59**	**58**	**60**	**54**	**58**	**60**
Passing Success	**96.7%**	**98.3%**	**96.7%**	**98.3%**	**98.3%**	**96.7%**	**100.0%**	**90.0%**	**96.7%**	**100.0%**

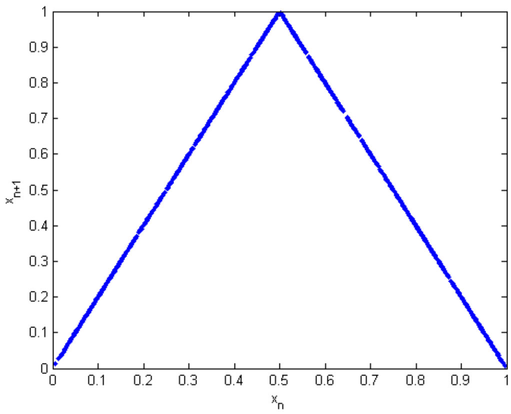
Iteration no = 10,000	LCGs					FIVE MAPS				
Statistical Tests	MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5	Tent Map $\mu = 2$	Tent Map $\mu = 2.00005$	T-Logistic Map	Connecting Map $\alpha = 0.999$	Connecting Map $\alpha = 1.01$
Kolmogorov Smirnov	20	20	20	20	20	20	20	19	20	18
Chisquare 100	19	20	20	20	20	19	19	19	18	19
Auto Corr Func	20	20	20	20	20	18	20	20	19	19
Total passes (over 60)	**59**	**60**	**60**	**60**	**60**	**57**	**59**	**58**	**57**	**56**
Passing Success	**98.3%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**95.0%**	**98.3%**	**96.7%**	**95.0%**	**93.3%**

4.3. Further Analysis and Comparisons of the Suggested Chaotic Maps

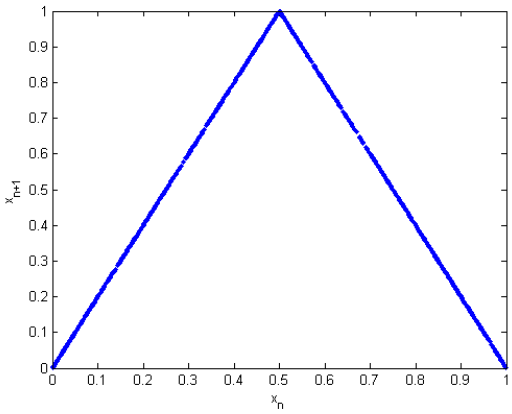
We consider five suggested chaotic maps and LCGs. The five chaotic maps pass statistical tests satisfactorily, (see Table 4.9 and Table 4.10). These results support the idea that the chaotic maps can generate random numbers like LCGs.

In the empirical tests, we consider histograms and serial plots for the generated numbers for each map. Histograms show the frequencies of the generated numbers, and serial plots illustrate the relations between the iteration number and the generated numbers (n, x_n).

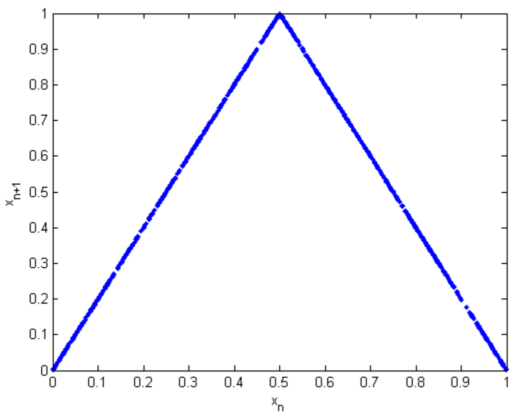
For further analysis of the generated random numbers, we plot the graphs of two consecutive iterated numbers (x_n, x_{n+1}). We take a seed, which is used in the comparison section 4.2, and generate 1000 numbers to plot (x_n, x_{n+1}) graph of each map. The graphs can be seen in Figure 4.21 and Figure 4.22 for chaotic maps and LCGs respectively, and the statistical test results of the generated numbers can be seen in Table 4.11.



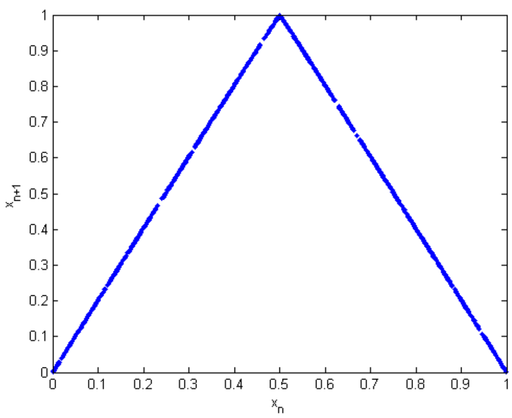
(a) Tent Map ($\mu = 2$)



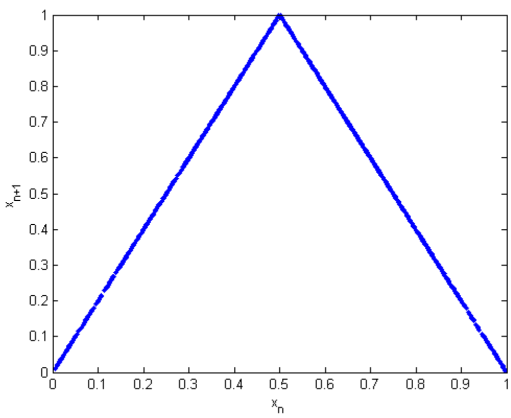
(b) Tent Map ($\mu = 2.00005$)



(c) T-logistic Map ($\nu = 4$)

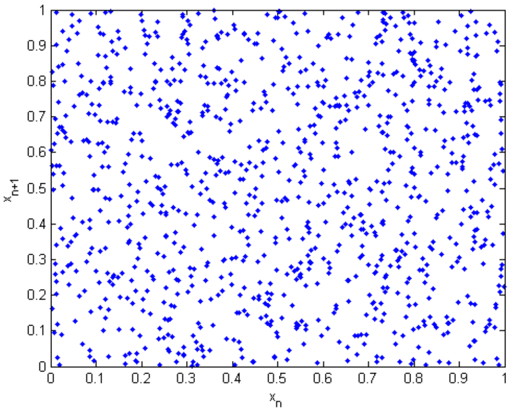


(d) Connecting Map ($\alpha = 0.999$)

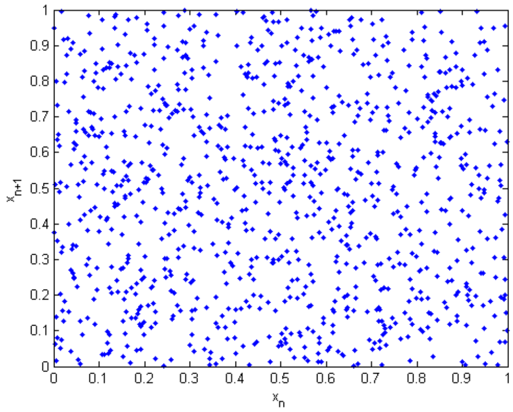


(e) Connecting Map ($\alpha = 1.01$)

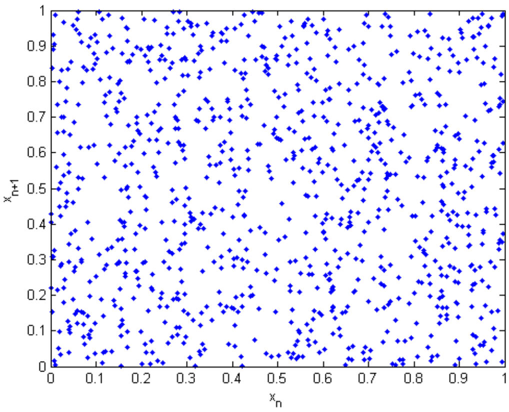
Figure 4.21. The (x_n, x_{n+1}) graphs of chaotic maps.



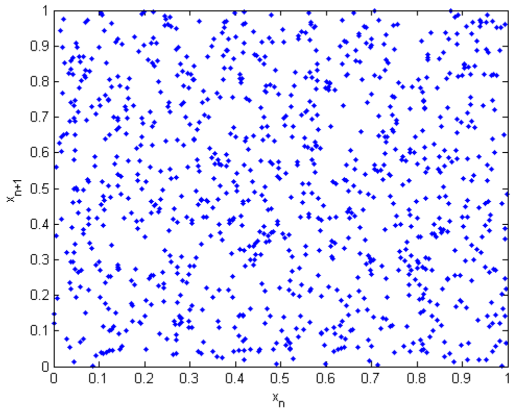
(a) MLCM1



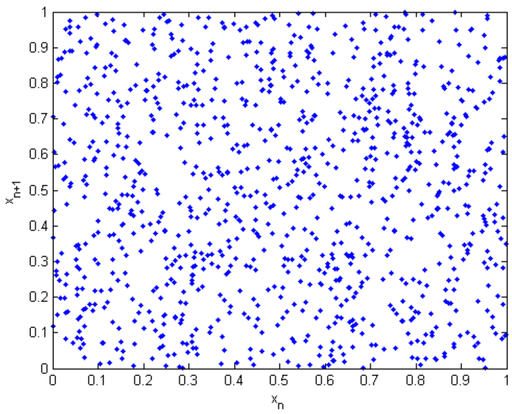
(b) MLCM2



(c) LCM3



(d) LCM4



(e) LCM5

Figure 4.22. The (x_n, x_{n+1}) graph of LCGs.

The figures show that the numbers generated by LCGs diffuse randomly in (x_n, x_{n+1}) plane. This situation results from the modulus operation inside LCGs. When chaotic maps are examined, the generated numbers draw the graphs of related equations, which are obviously not random. Although the generated numbers are not randomly distributed in (x_n, x_{n+1}) plane, the nature of chaotic dynamical maps do contain strong unpredictability. Due to the property of chaotic maps, the generated numbers are randomly distributed in (n, x_n) plane and pass the related randomness tests (as already seen in Section 4.2).

Table 4.11. The statistical test results for LCGs and five suggested maps: Seed = 915612027, Iteration no = 1000.

	Statistical Tests	MLCG 1 915612027	MLCG 2 915612027	LCG 3 915612027	LCG 4 915612027	LCG 5 915612027
P-Values	Kolmogorov Smirnov	0.0154	0.0698	0.0601	0.9929	0.9647
	Chisqure 30	0.7861	0.1085	0.6298	0.8828	0.4165
	Chisqure 50	0.723	0.0092	0.2625	0.7926	0.4206
	Chisqure 80	0.9051	0.0441	0.2258	0.9023	0.5242
	Chisqure 100	0.6507	0.0317	0.2482	0.8767	0.2373
	Runs Up - Down	0.1012	0.99	0.0623	0.2991	0.323
	Runs Above - Below	0.2802	0.9226	0.394	0.9797	0.6775
	Auto Corr Func	0.059462529	0.057422749	0.057422749	0.07011717	0.065801652

	Statistical Tests	Tent map $\mu = 2$ 0.915612027	Tent map $\mu = 2.00005$ 0.915612027	T-Logistic map 0.915612027	Connecting map $\alpha = 0.999$ 0.915612027	Connecting map $\alpha = 1.01$ 0.915612027
P-Values	Kolmogorov Smirnov	0.0751	0.2069	0.3533	0.492	0.0659
	Chisqure 30	0.4515	0.055	0.0289	0.9026	0.039
	Chisqure 50	0.1901	0.1234	0.5296	0.4521	0.0774
	Chisqure 80	0.7253	0.2414	0.2418	0.7738	0.3364
	Chisqure 100	0.1461	0.105	0.2902	0.5294	0.4168
	Runs Up - Down	0.0128	0.0589	0.2249	0.3744	0.0495
	Runs Above - Below	0.0173	0.0246	0.0711	0.1562	1
	Auto Corr Func	0.084997272	0.052317922	0.051399705	0.060246897	0.070297064

The unpredictability property of chaotic maps is so strong that they generate randomly distributed numbers in (n, x_n) plane. In order to obtain randomness in (x_n, x_{n+1}) plane as well, using modulus operation with chaotic maps to generate random numbers will probably produce better results. Such generated numbers may produce independence test results as those of LCGs. The use of such chaotic maps with modulus operations as RNGs can be a further research topic.

5. CONCLUSION

This thesis deals with the generation of pseudo random numbers (PRNs) by means of chaotic dynamical maps. The nature of a chaotic dynamical map contains unpredictability itself, a motivation behind our thesis study. The research question is: to what extent some chaotic dynamical maps can be used in the generation of random numbers.

We develop five different random number generators (RNGs) that are based on five different chaotic dynamical maps. In three steps, we check the performances of five suggested chaotic maps as RNGs.

Firstly, the periods of five suggested maps are controlled. Conditions that guarantee the generation of more than 1000000 numbers without any cycles are determined for each chaotic map. After it is confirmed that each map can generate 1000000 numbers at least, the sequences of 1000 and 10000 numbers are generated as samples in order to test the uniformity and randomness of each map.

Secondly, the histograms and the serial plots of the sequences of generated numbers are examined visually as a first test. Such graphs give an idea about the randomness of the maps. We check uniformity of the generated numbers visually with their histograms, and independence of the sequences of these numbers with serial plots. It is observed that the suggested RNGs perform well according to visual tests.

Thirdly, the generated numbers are tested by Kolmogorov Smirnov test, and chi-square test to assess their properties of uniformity. (Chi-square tests are run for different numbers of classes: 30, 50, 80, and 100). Runs tests (runs up and down test, and runs above and below mean test), and autocorrelation tests are next applied to the sequences of generated numbers to test their independence property. The five suggested maps performed well in all these tests, and uniformity and independence of the sequences of the generated numbers are confirmed.

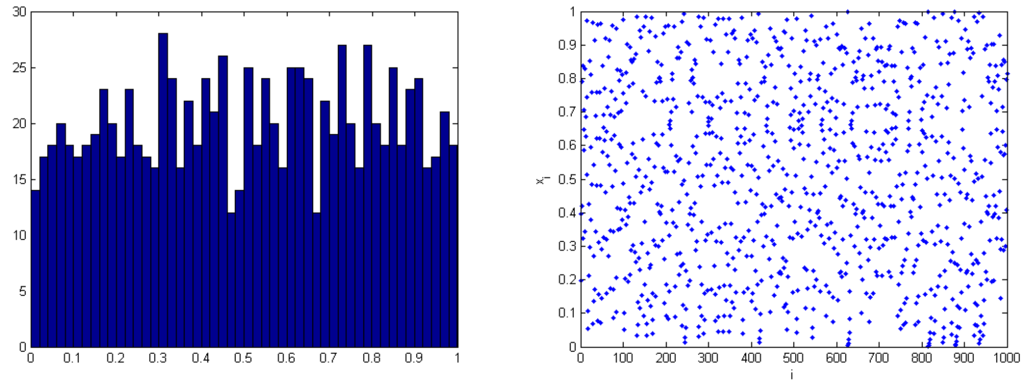
We finally compare our suggested maps with some best known linear congruential generators (LCGs). We select 20 different seeds that conform the desired properties of LCGs for the comparison of LCGs and five suggested chaotic RNGs. The visual tests and statistical tests are both taken into account. No undesirable patterns are observed in visual tests for any of the generators. In statistical tests, we compare the test results of LCGs and five suggested maps.

LCGs perform well since they are already suggested as some of the best generators in the literature and the seeds satisfy some suggested specific conditions. (ie. the seeds are all prime or odd numbers.) However, the five suggested chaotic dynamical maps perform almost as well as LCGs. Similarly the performance of five suggested chaotic RNGs can be improved through defining seeds that perform well for these five RNGs.

To sum up, we propose that our five chaotic dynamical maps can be potentially used as random number generators. These suggested random number generators have only one simple chaotic function each, so they are practical and efficient.

The use of such chaotic maps as RNGs can be further studied by investigating the relation between seeds used and the resulting random number properties. More detailed mathematical, statistical and numerical properties of the suggested generators also constitute useful further research topics. As another further research topic in order to improve performance of suggested chaotic generators, the use of modulus operations with suggested chaotic generators can be analyzed.

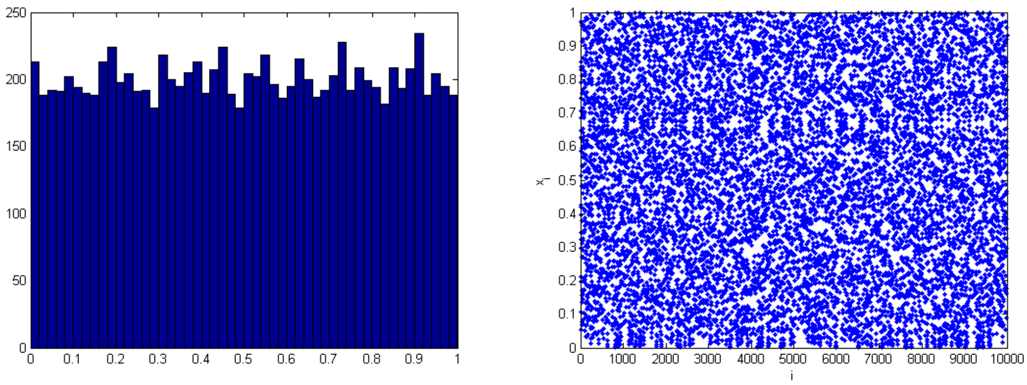
APPENDIX A: HISTOGRAMS FOR TENT MAP ($\mu = 2$)



(a) Histogram with 50 subintervals

(b) Serial plot

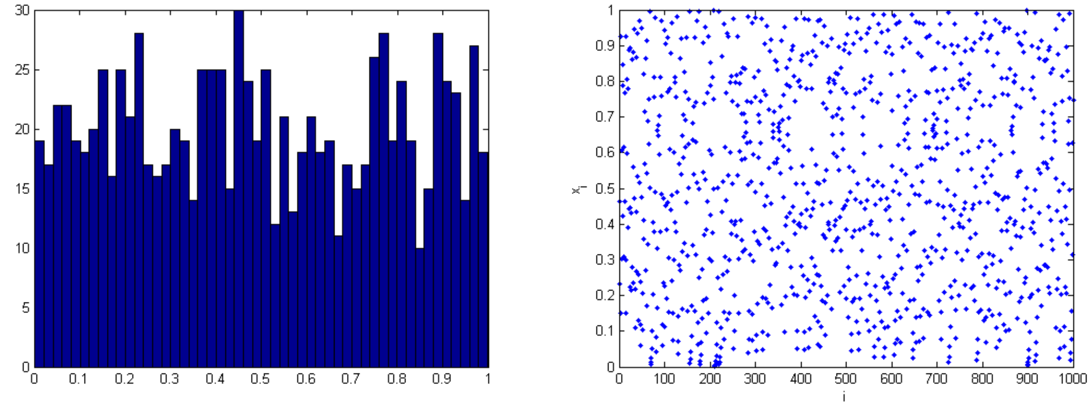
Figure A.1. Tent map ($\mu = 2$): Seed = 0.197600506499, Iteration no = 1000.



(a) Histogram with 50 subintervals

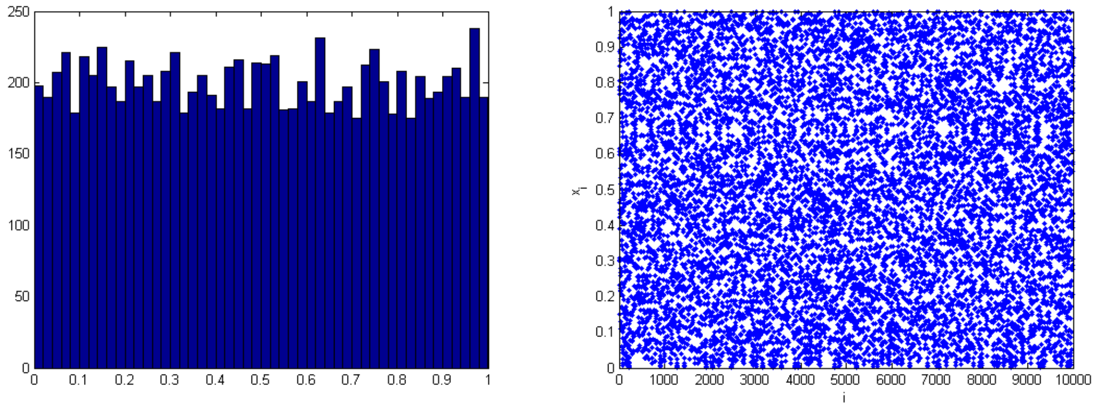
(b) Serial plot

Figure A.2. Tent map ($\mu = 2$): Seed = 0.197600506499, Iteration no = 10000.



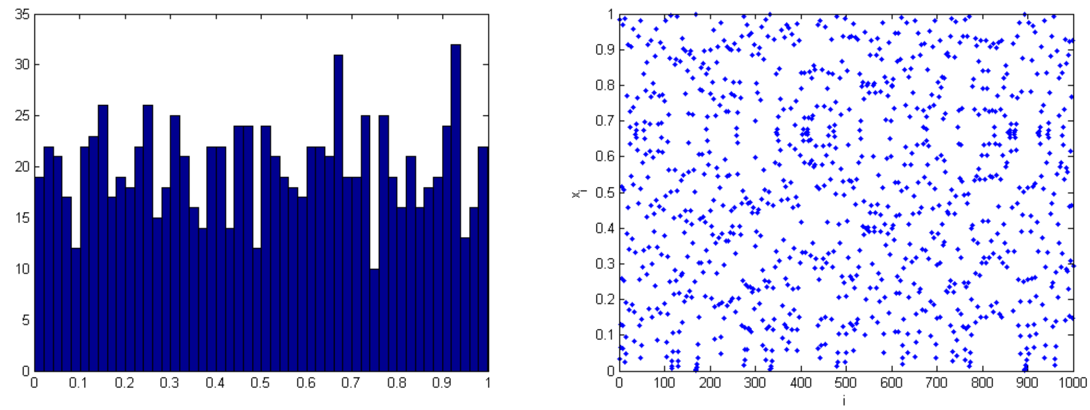
(a) Histogram with 50 subintervals (b) Serial plot

Figure A.3. Tent map ($\mu = 2$): Seed = 0.231069988461, Iteration no = 1000.



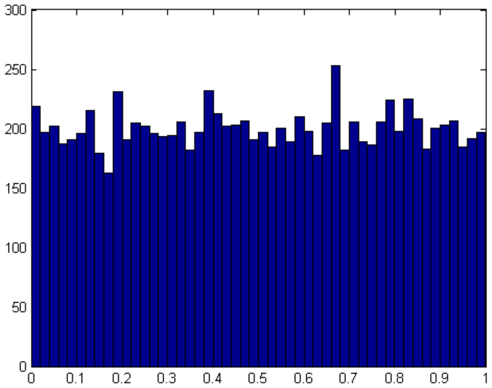
(a) Histogram with 50 subintervals (b) Serial plot

Figure A.4. Tent map ($\mu = 2$): Seed = 0.231069988461, Iteration no = 10000.

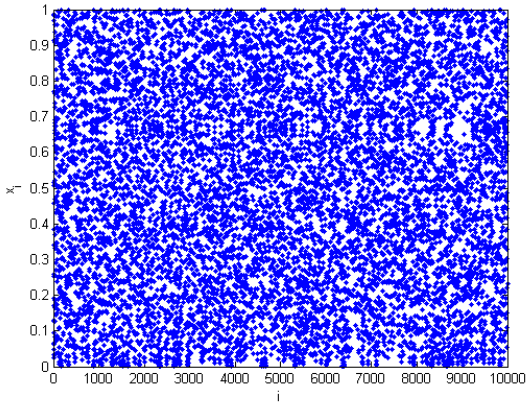


(a) Histogram with 50 subintervals (b) Serial plot

Figure A.5. Tent map ($\mu = 2$): Seed = 0.98388083416, Iteration no = 1000.

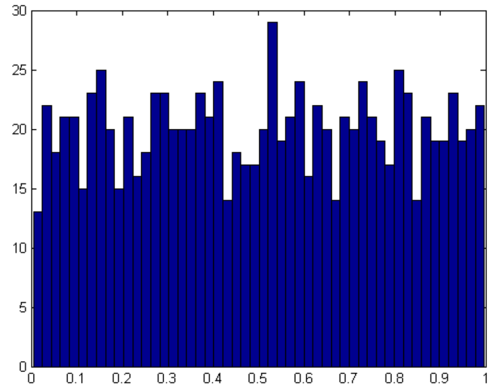


(a) Histogram with 50 subintervals

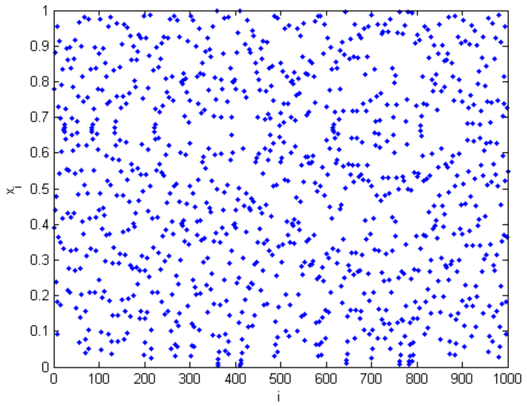


(b) Serial plot

Figure A.6. Tent map ($\mu = 2$): Seed = 0.98388083416, Iteration no = 10000.

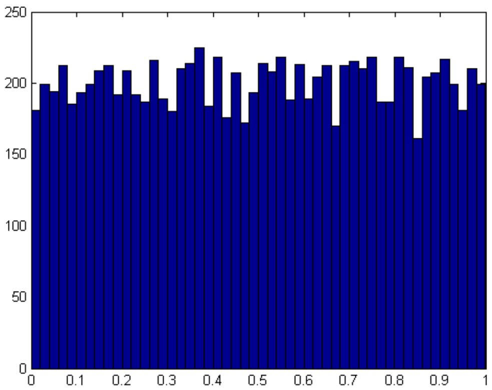


(a) Histogram with 50 subintervals

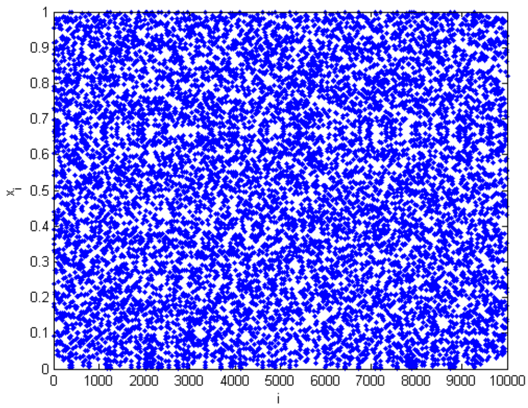


(b) Serial plot

Figure A.7. Tent map ($\mu = 2$): Seed = 0.389913876011, Iteration no = 1000.

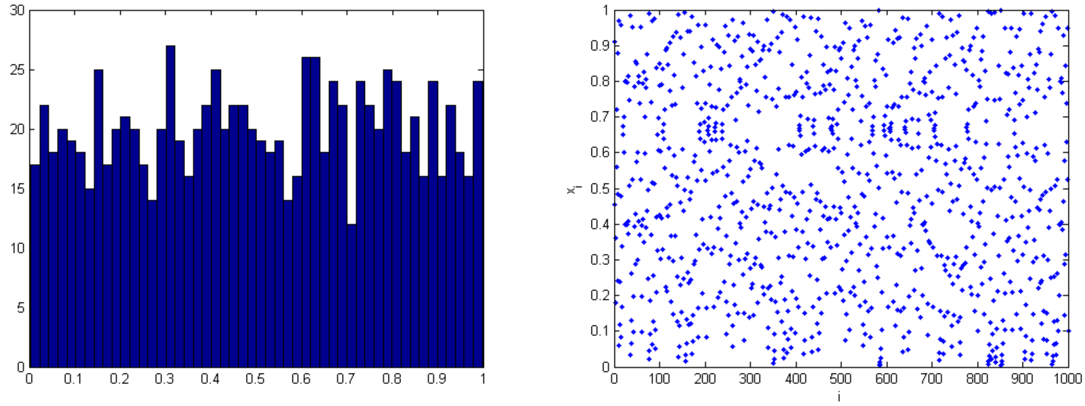


(a) Histogram with 50 subintervals



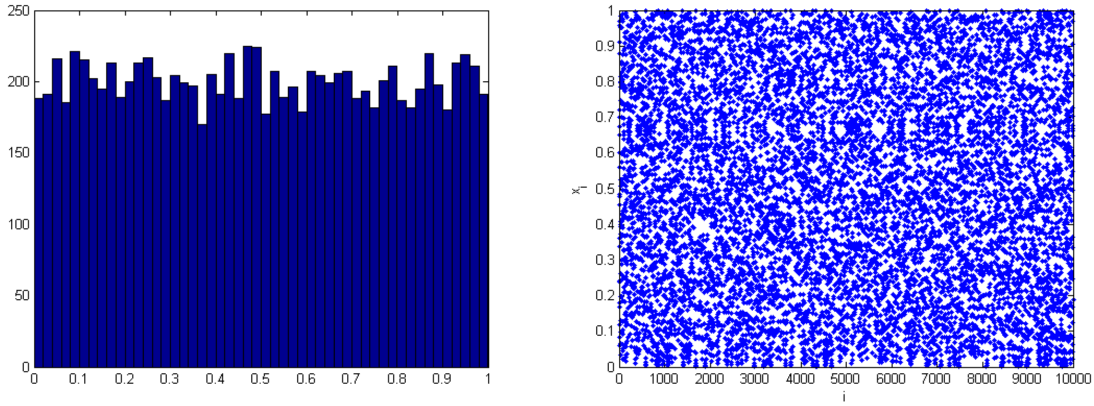
(b) Serial plot

Figure A.8. Tent map ($\mu = 2$): Seed = 0.389913876011, Iteration no = 10000.



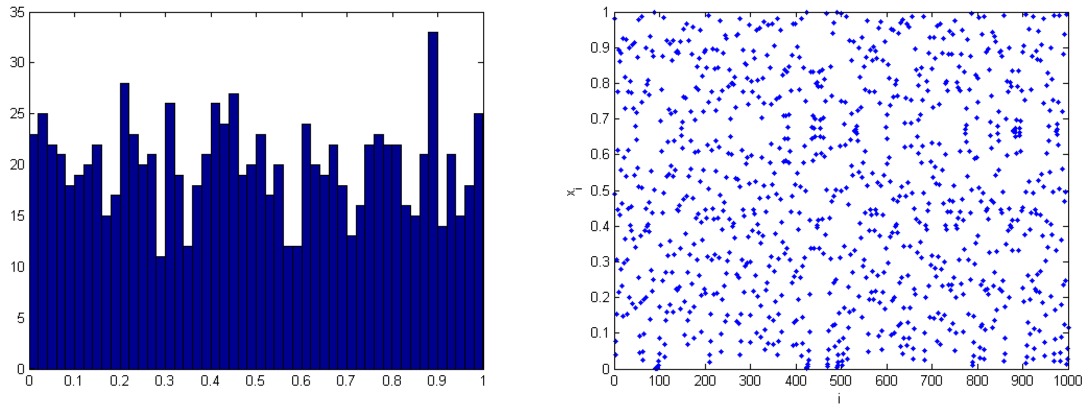
(a) Histogram with 50 subintervals (b) Serial plot

Figure A.9. Tent map ($\mu = 2$): Seed = 0.455019664777, Iteration no = 1000.



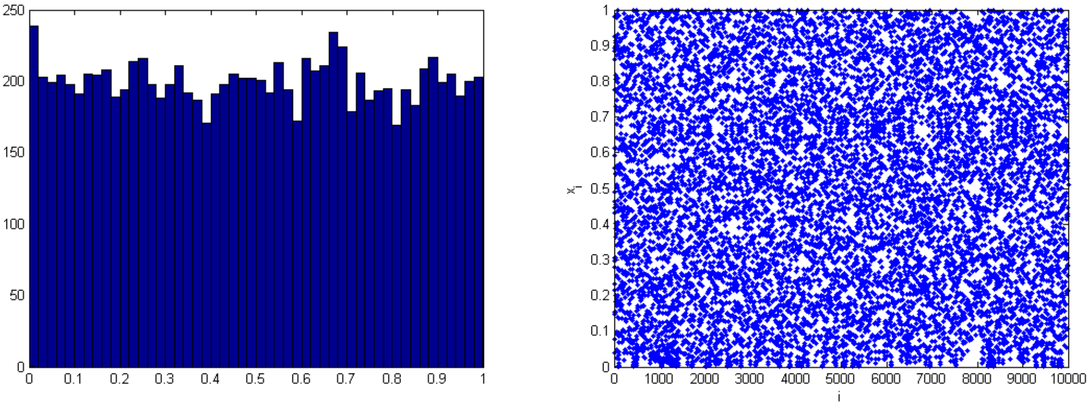
(a) Histogram with 50 subintervals (b) Serial plot

Figure A.10. Tent map ($\mu = 2$): Seed = 0.455019664777, Iteration no = 10000.



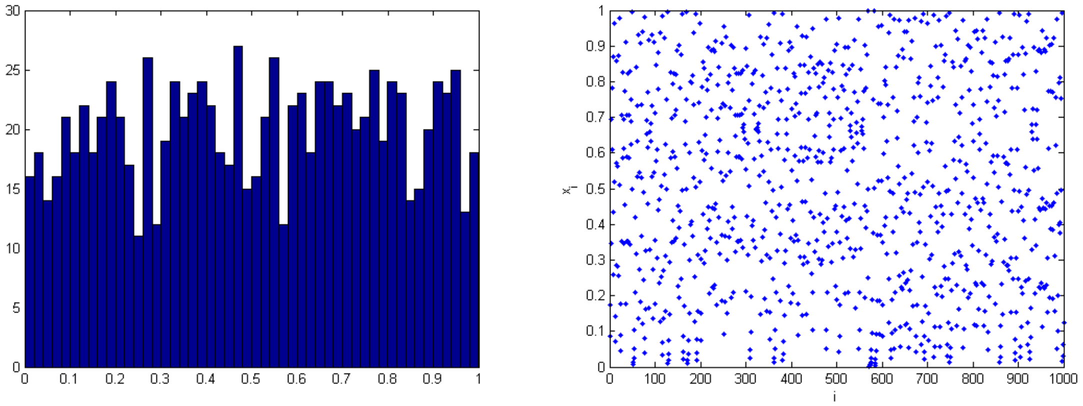
(a) Histogram with 50 subintervals (b) Serial plot

Figure A.11. Tent map ($\mu = 2$): Seed = 0.490444465331, Iteration no = 1000.



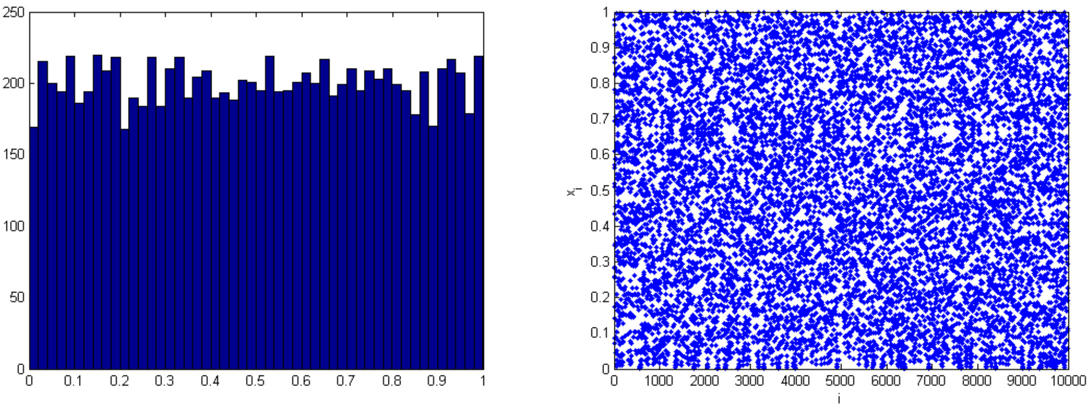
(a) Histogram with 50 subintervals (b) Serial plot

Figure A.12. Tent map ($\mu = 2$): Seed = 0.490444465331, Iteration no = 10000.



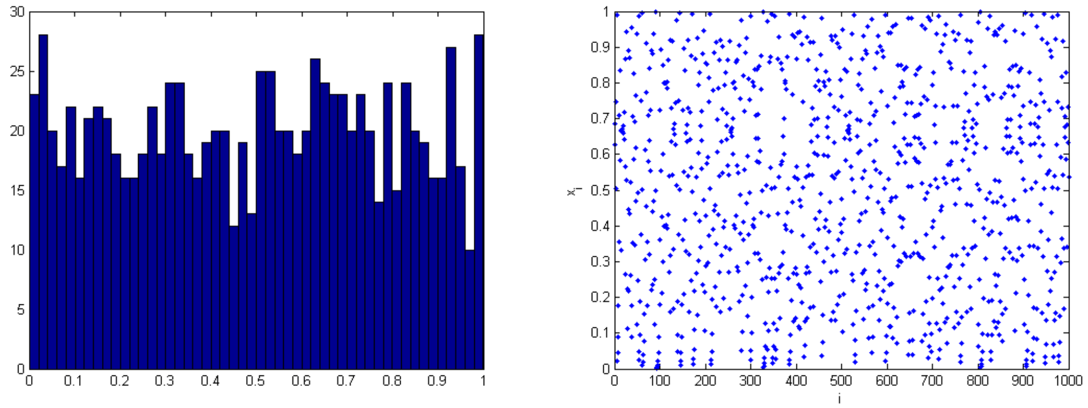
(a) Histogram with 50 subintervals (b) Serial plot

Figure A.13. Tent map ($\mu = 2$): Seed = 0.086948882143, Iteration no = 1000.



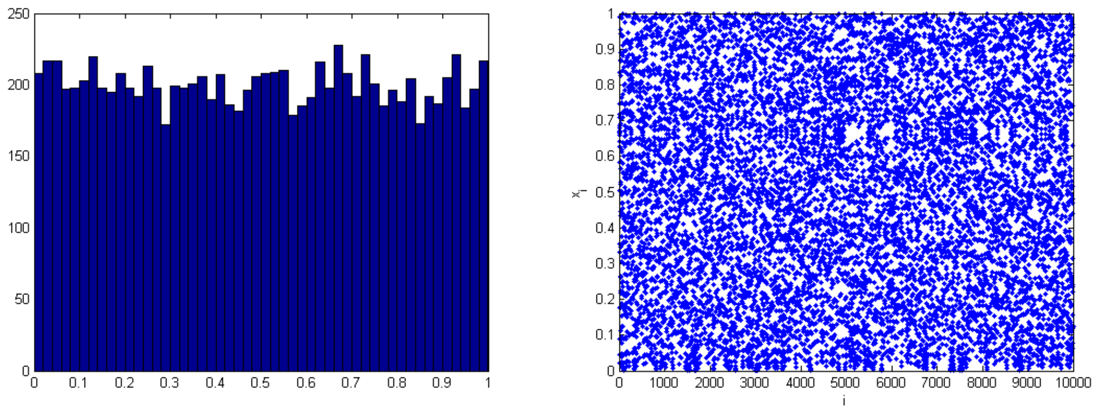
(a) Histogram with 50 subintervals (b) Serial plot

Figure A.14. Tent map ($\mu = 2$): Seed = 0.086948882143, Iteration no = 10000.



(a) Histogram with 50 subintervals (b) Serial plot

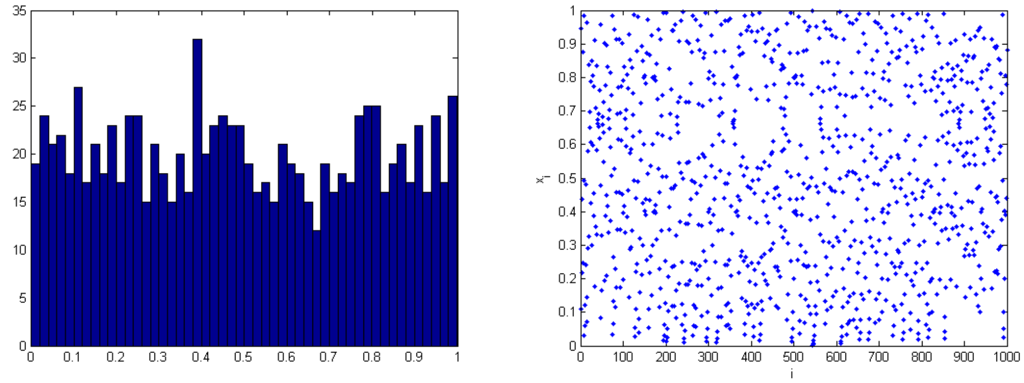
Figure A.15. Tent map ($\mu = 2$): Seed = 0.686808244224, Iteration no = 1000.



(a) Histogram with 50 subintervals (b) Serial plot

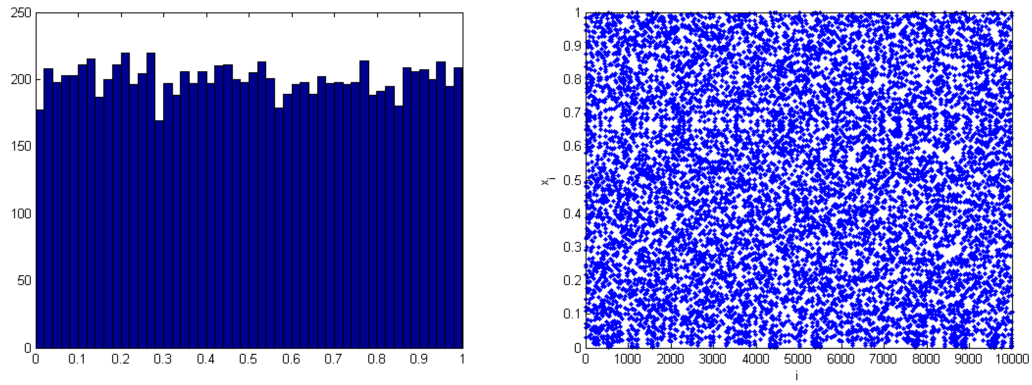
Figure A.16. Tent map ($\mu = 2$): Seed = 0.686808244224, Iteration no = 10000.

APPENDIX B: HISTOGRAMS FOR TENT MAP ($\mu = 2.00005$)



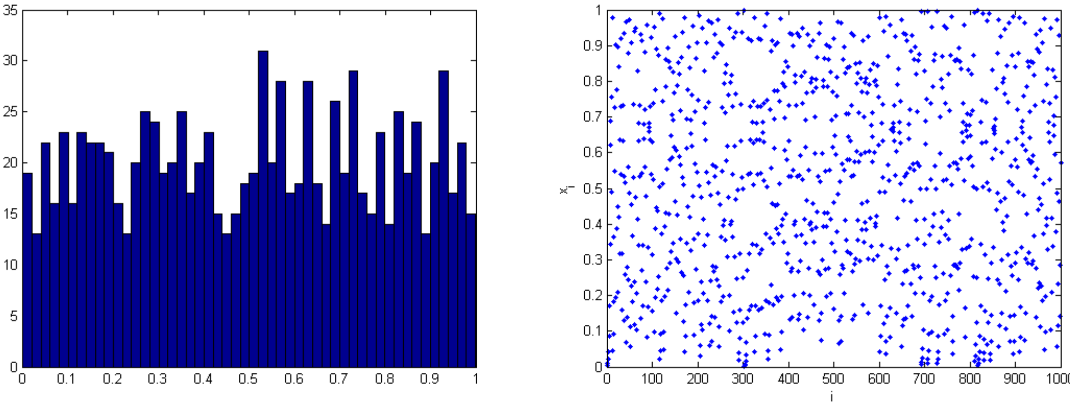
(a) Histogram with 50 subintervals (b) Serial plot

Figure B.1. Tent map ($\mu = 2.00005$): Seed $= 0.945195357852$, Iteration no $= 1000$.



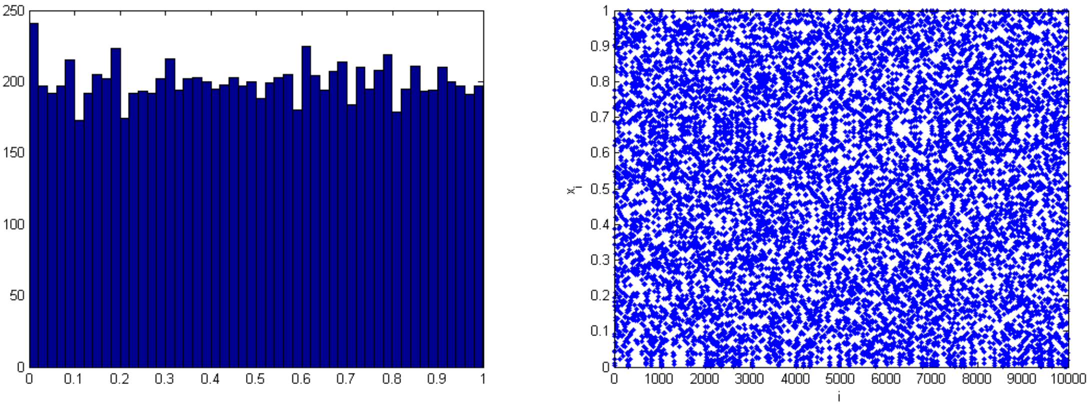
(a) Histogram with 50 subintervals (b) Serial plot

Figure B.2. Tent map ($\mu = 2.00005$): Seed $= 0.945195357852$, Iteration no $= 10000$.



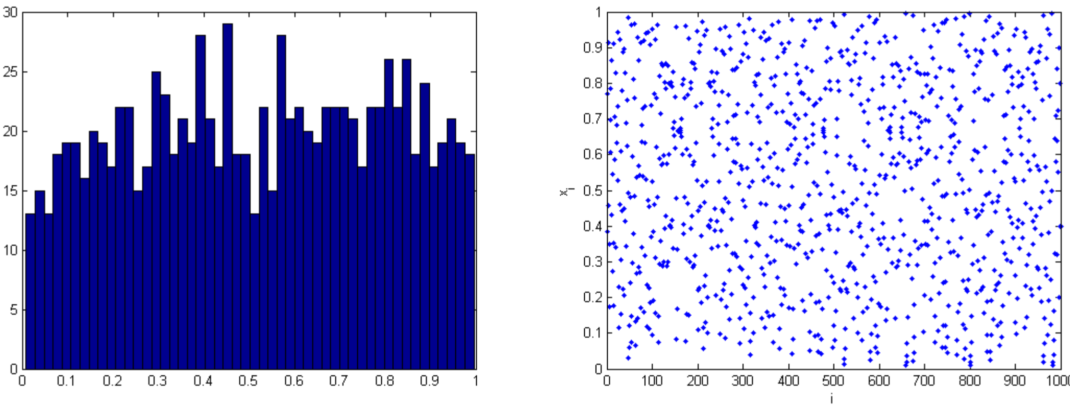
(a) Histogram with 50 subintervals (b) Serial plot

Figure B.3. Tent map ($\mu = 2.00005$): Seed = 0.005382078128663, Iteration no = 1000.



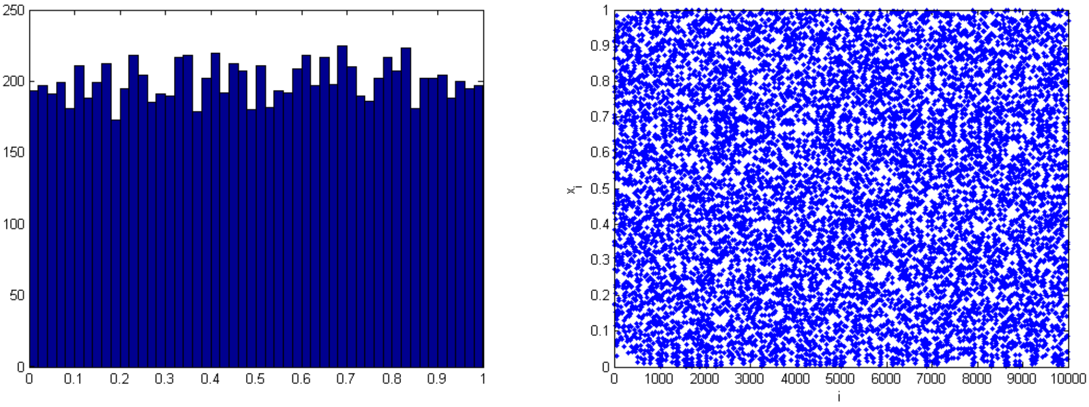
(a) Histogram with 50 subintervals (b) Serial plot

Figure B.4. Tent map ($\mu = 2.00005$): Seed = 0.005382078128663, Iteration no = 10000.



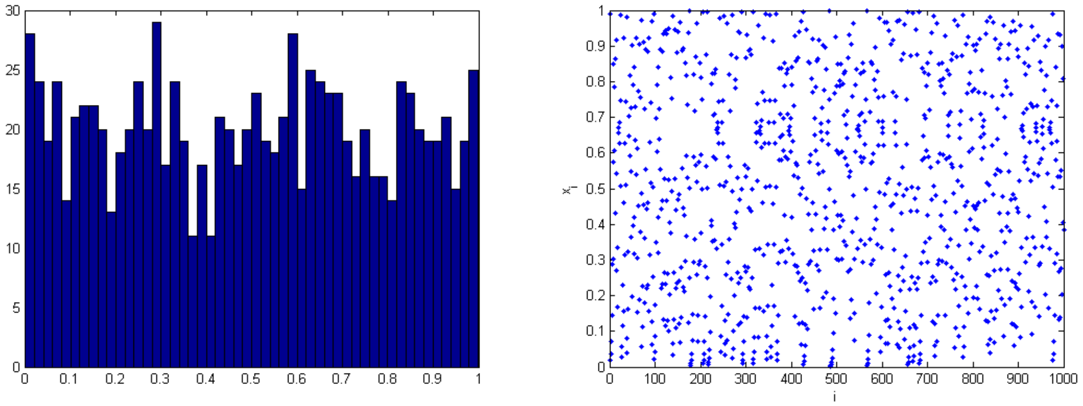
(a) Histogram with 50 subintervals (b) Serial plot

Figure B.5. Tent map ($\mu = 2.00005$): Seed = 0.385874582044063, Iteration no = 1000.



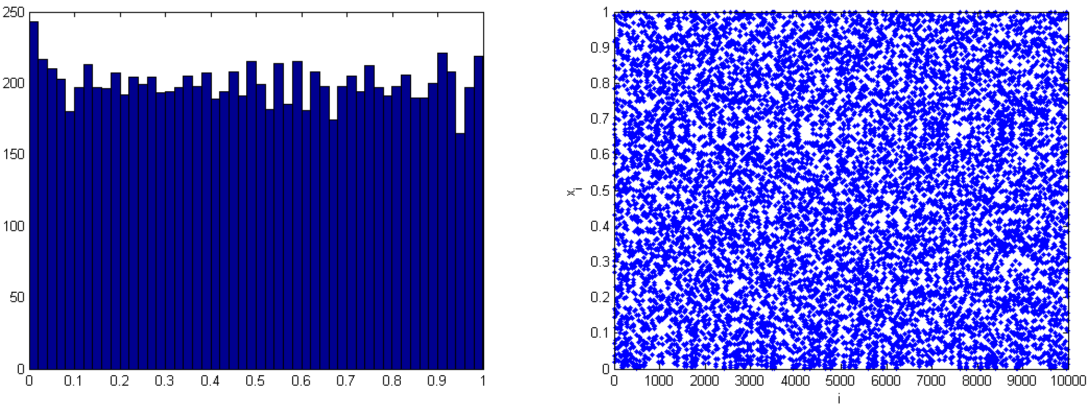
(a) Histogram with 50 subintervals (b) Serial plot

Figure B.6. Tent map ($\mu = 2.00005$): Seed $= 0.385874582044063$, Iteration no $= 10000$.



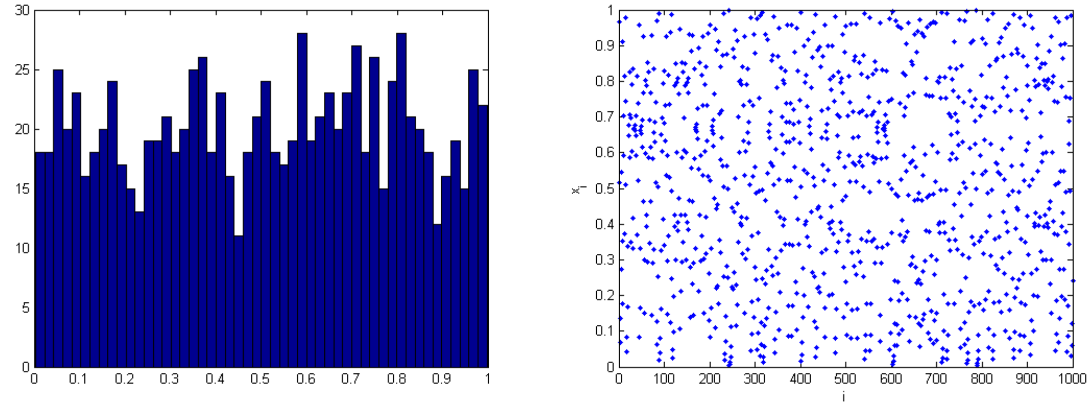
(a) Histogram with 50 subintervals (b) Serial plot

Figure B.7. Tent map ($\mu = 2.00005$): Seed $= 0.991$, Iteration no $= 1000$.



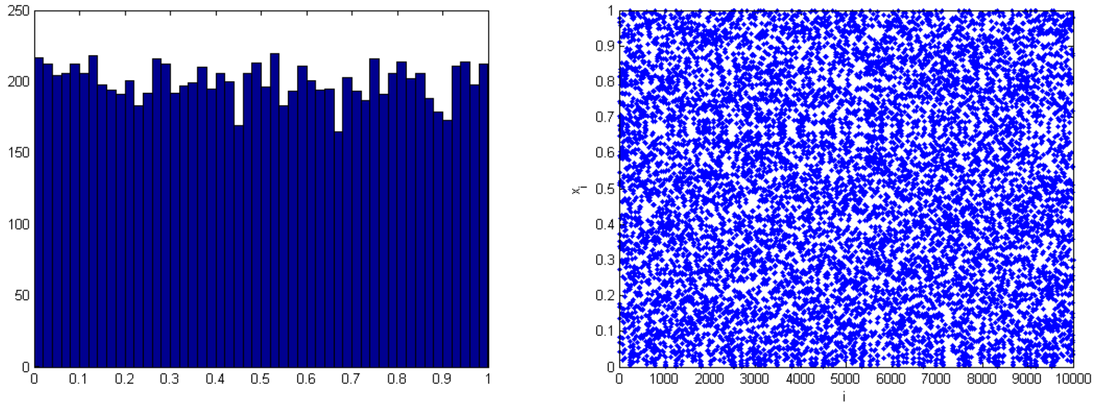
(a) Histogram with 50 subintervals (b) Serial plot

Figure B.8. Tent map ($\mu = 2.00005$): Seed $= 0.991$, Iteration no $= 10000$.



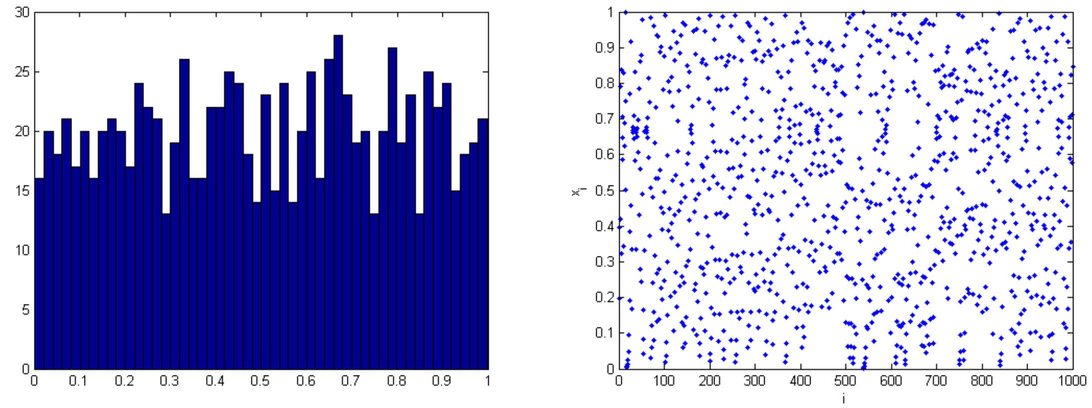
(a) Histogram with 50 subintervals (b) Serial plot

Figure B.9. Tent map ($\mu = 2.00005$): Seed = 0.517, Iteration no = 1000.



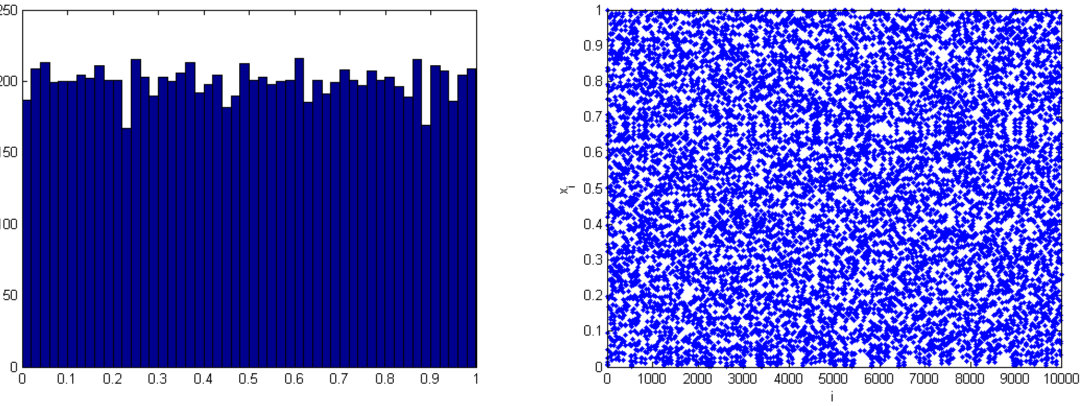
(a) Histogram with 50 subintervals (b) Serial plot

Figure B.10. Tent map ($\mu = 2.00005$): Seed = 0.517, Iteration no = 10000.



(a) Histogram with 50 subintervals (b) Serial plot

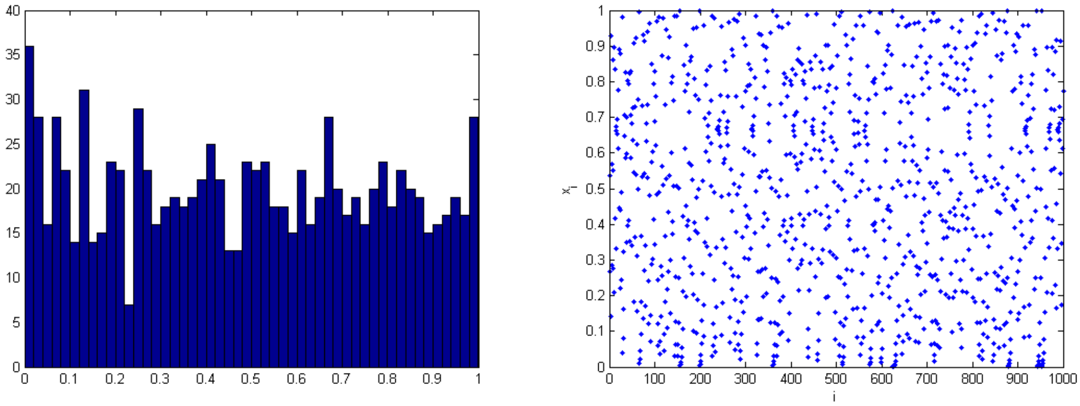
Figure B.11. Tent map ($\mu = 2.00005$): Seed = 0.197600506499, Iteration no = 1000.



(a) Histogram with 50 subintervals

(b) Serial plot

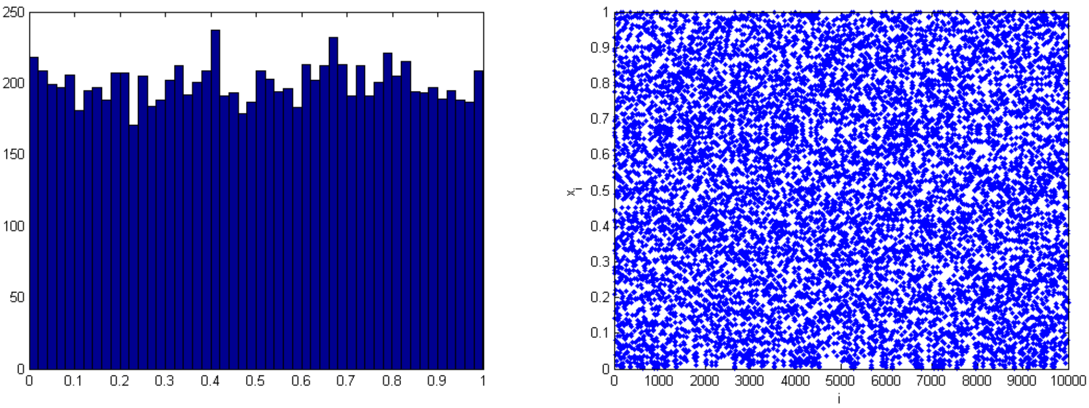
Figure B.12. Tent map ($\mu = 2.00005$): Seed = 0.197600506499, Iteration no = 10000.



(a) Histogram with 50 subintervals

(b) Serial plot

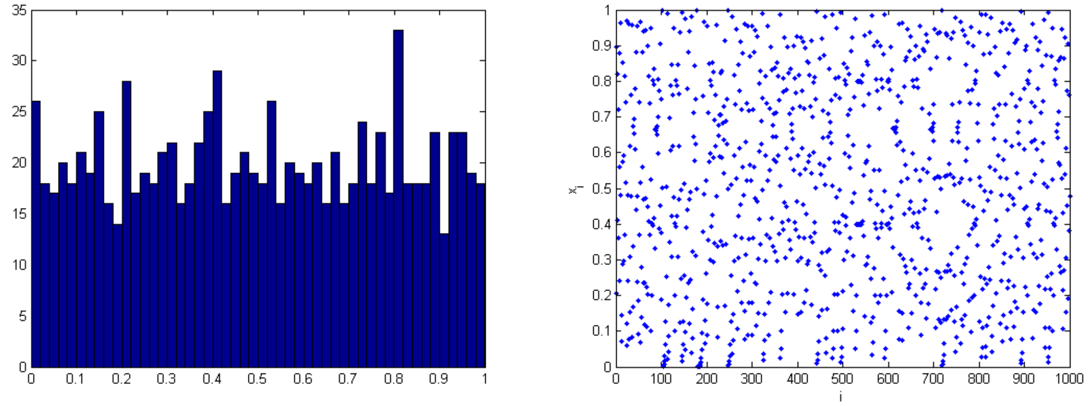
Figure B.13. Tent map ($\mu = 2.00005$): Seed = 0.267781590398, Iteration no = 1000.



(a) Histogram with 50 subintervals

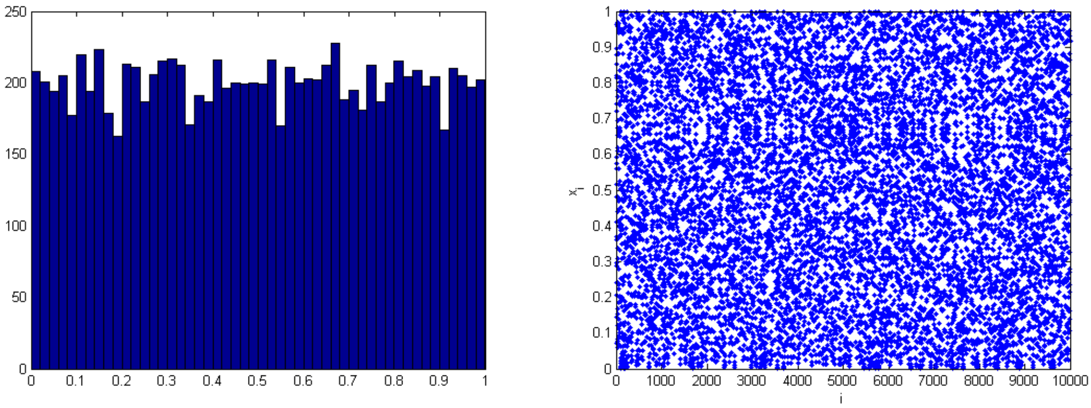
(b) Serial plot

Figure B.14. Tent map ($\mu = 2.00005$): Seed = 0.267781590398, Iteration no = 10000.



(a) Histogram with 50 subintervals (b) Serial plot

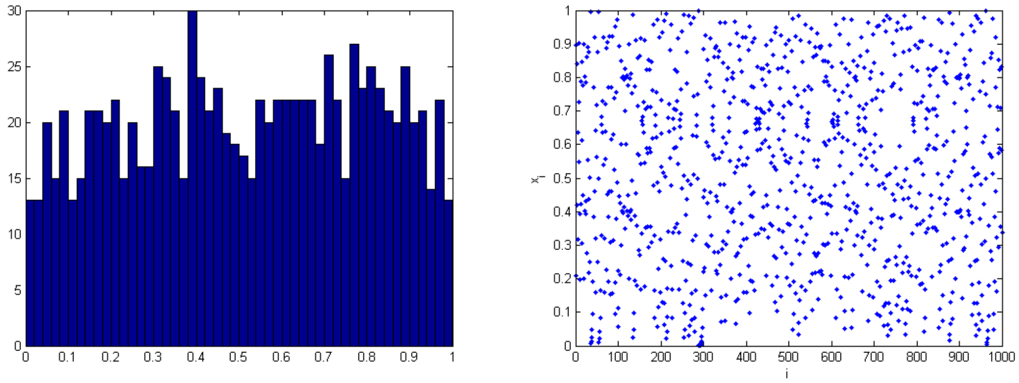
Figure B.15. Tent map ($\mu = 2.00005$): Seed = 0.897496645632, Iteration no = 1000.



(a) Histogram with 50 subintervals (b) Serial plot

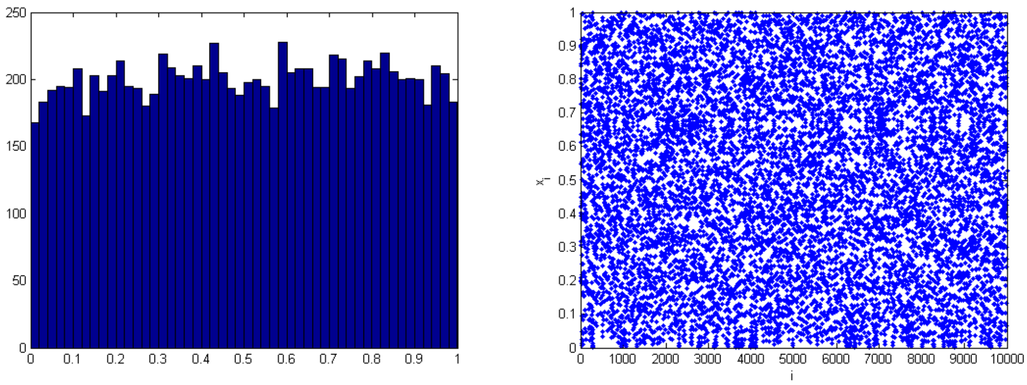
Figure B.16. Tent map ($\mu = 2.00005$): Seed = 0.897496645632, Iteration no = 10000.

APPENDIX C: HISTOGRAMS FOR T-LOGISTIC MAP ($\nu = 4$)



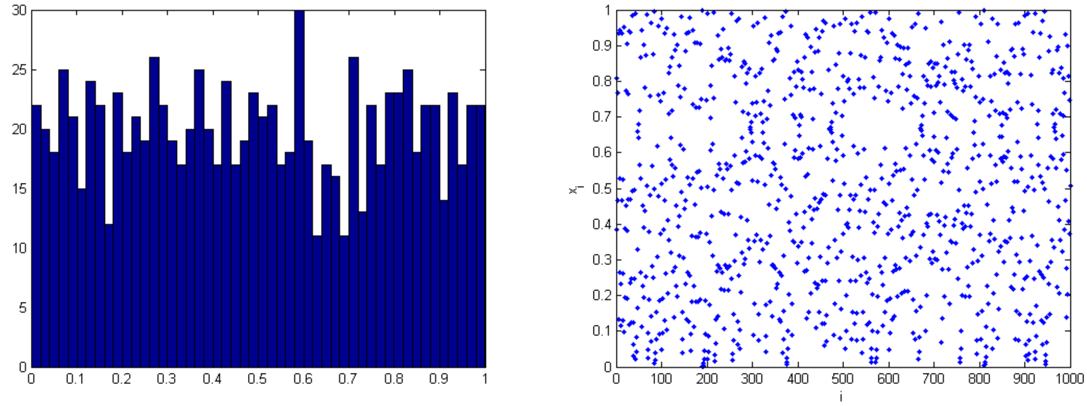
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.1. T-logistic map ($\nu = 4$): Seed = 0.973, Iteration no = 1000.



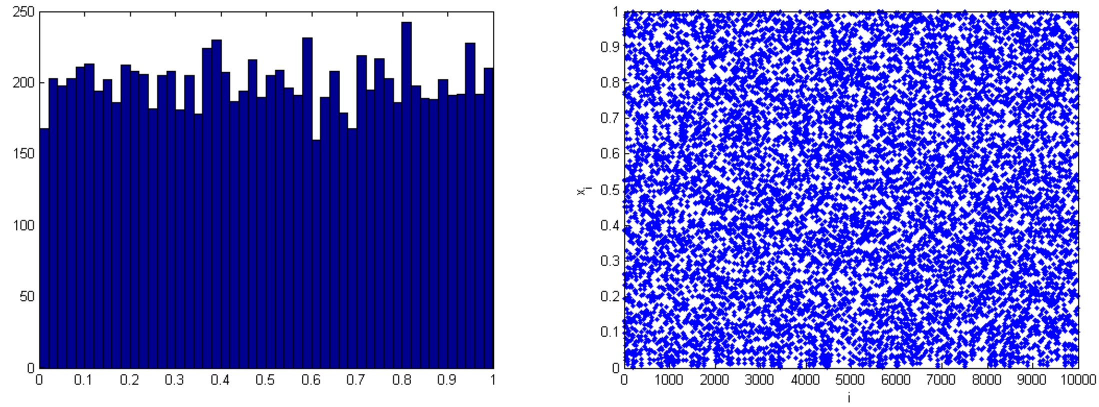
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.2. T-logistic map ($\nu = 4$): Seed = 0.973, Iteration no = 10000.



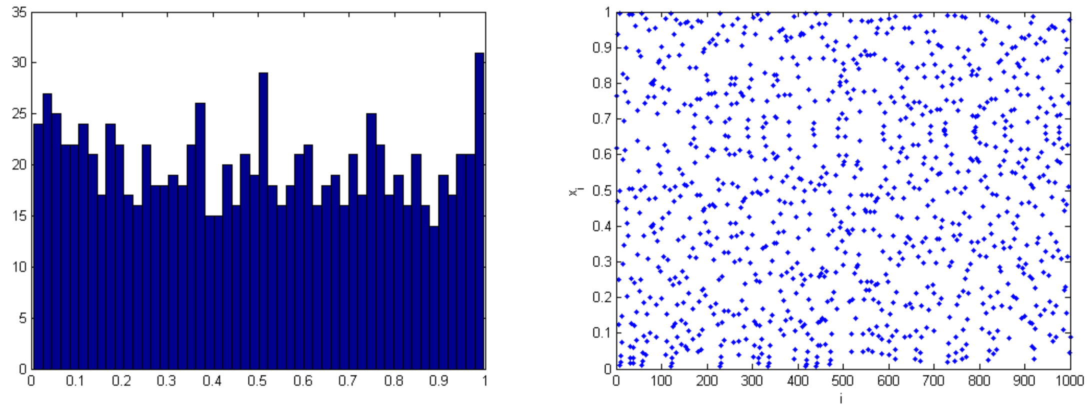
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.3. T-logistic map ($\nu = 4$): Seed = 0.912091805655299, Iteration no = 1000.



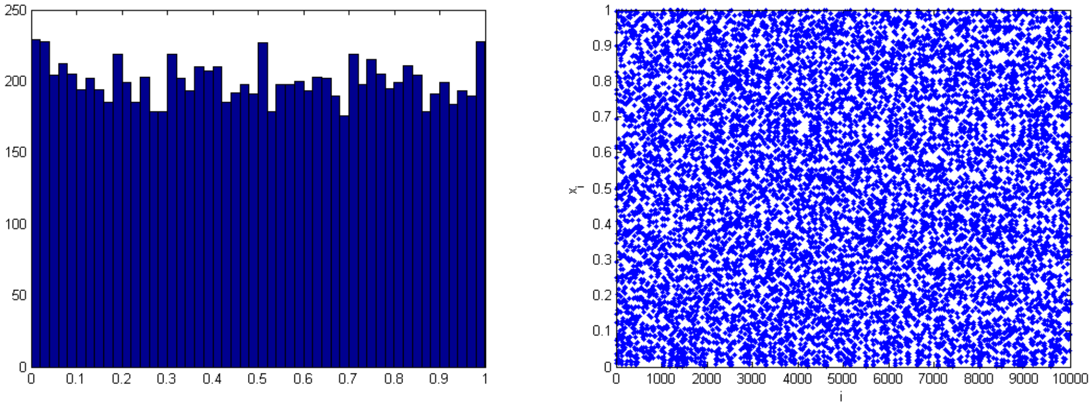
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.4. T-logistic map ($\nu = 4$): Seed = 0.912091805655299, Iteration no = 10000.



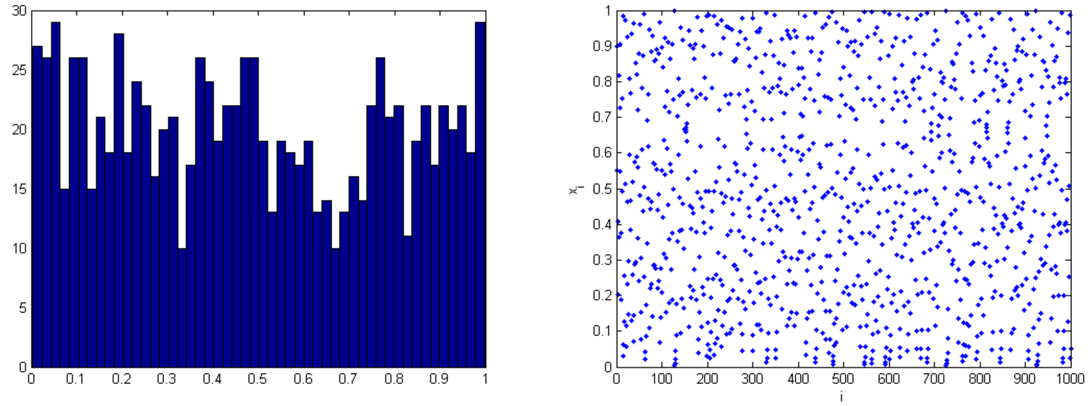
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.5. T-logistic map ($\nu = 4$): Seed = 0.68, Iteration no = 1000.



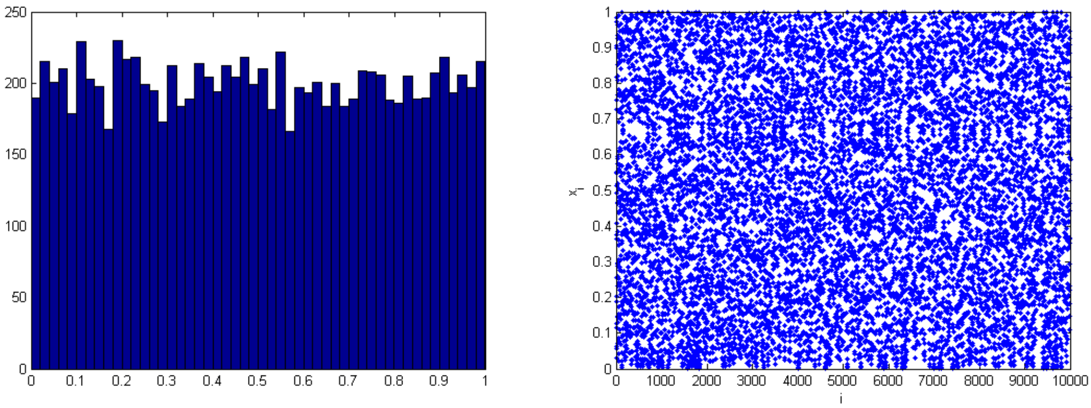
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.6. T-logistic map ($\nu = 4$): Seed = 0.68, Iteration no = 10000.



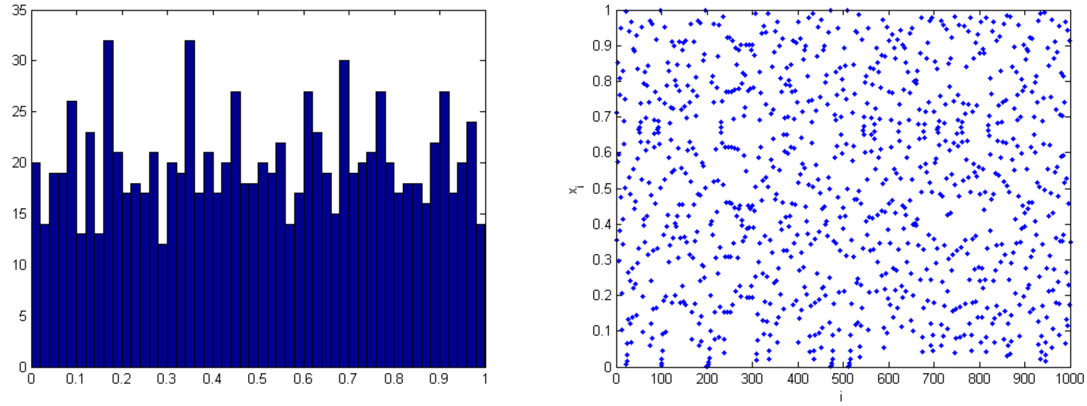
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.7. T-logistic map ($\nu = 4$): Seed = 0.58, Iteration no = 1000.



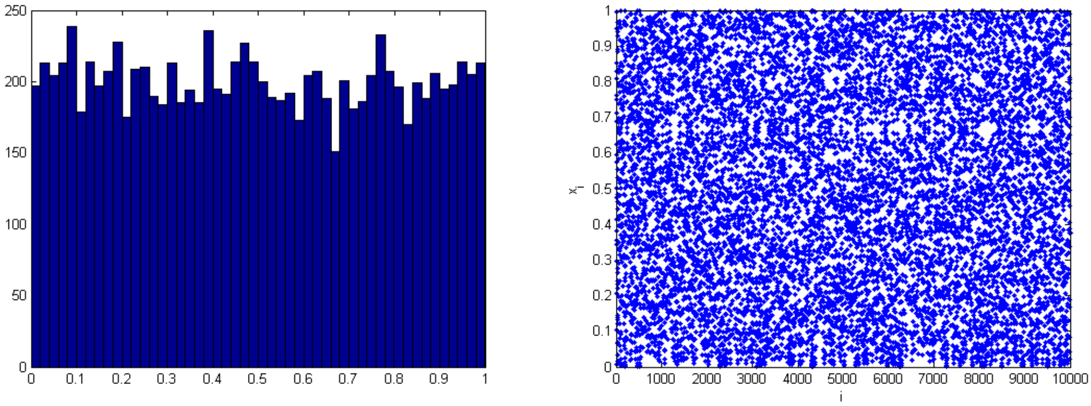
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.8. T-logistic map ($\nu = 4$): Seed = 0.58, Iteration no = 10000.



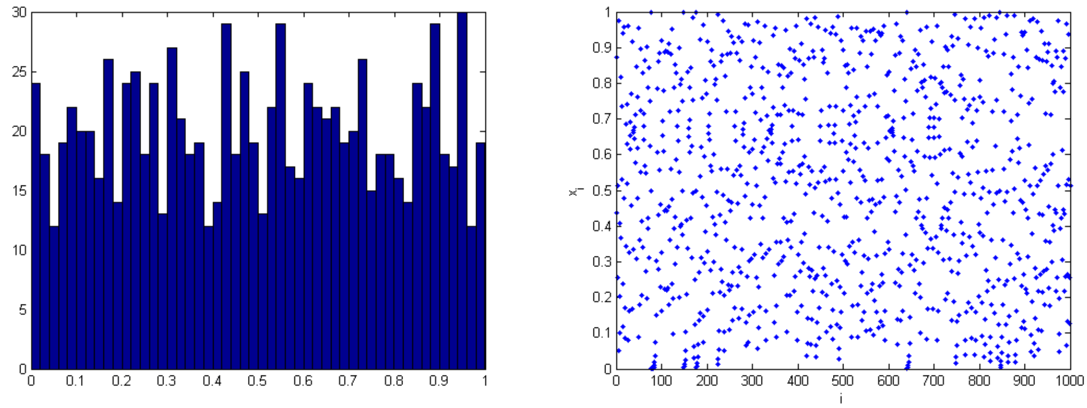
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.9. T-logistic map ($\nu = 4$): Seed = 0.282, Iteration no = 1000.



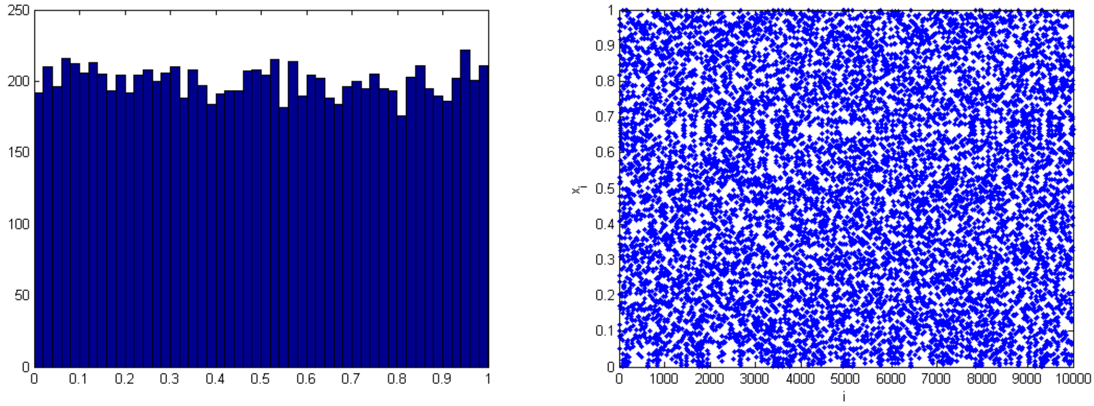
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.10. T-logistic map ($\nu = 4$): Seed = 0.282, Iteration no = 10000.



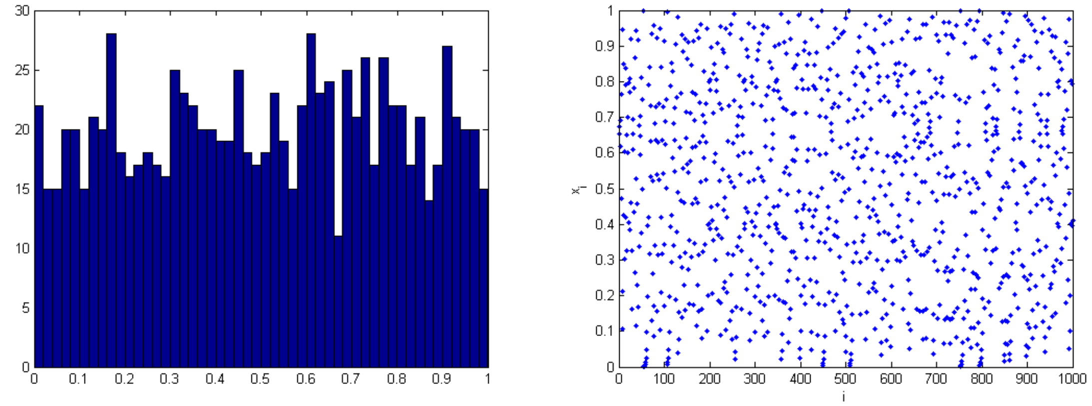
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.11. T-logistic map ($\nu = 4$): Seed = 0.4, Iteration no = 1000.



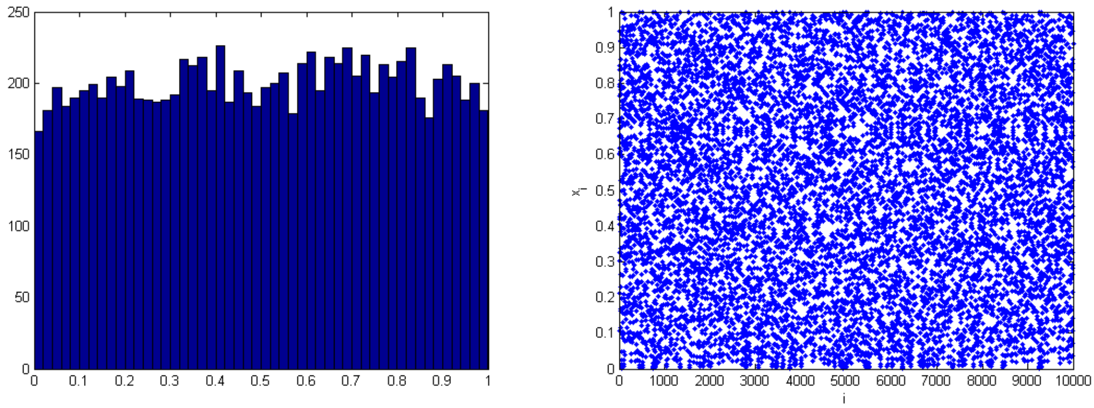
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.12. T-logistic map ($\nu = 4$): Seed = 0.4, Iteration no = 10000.



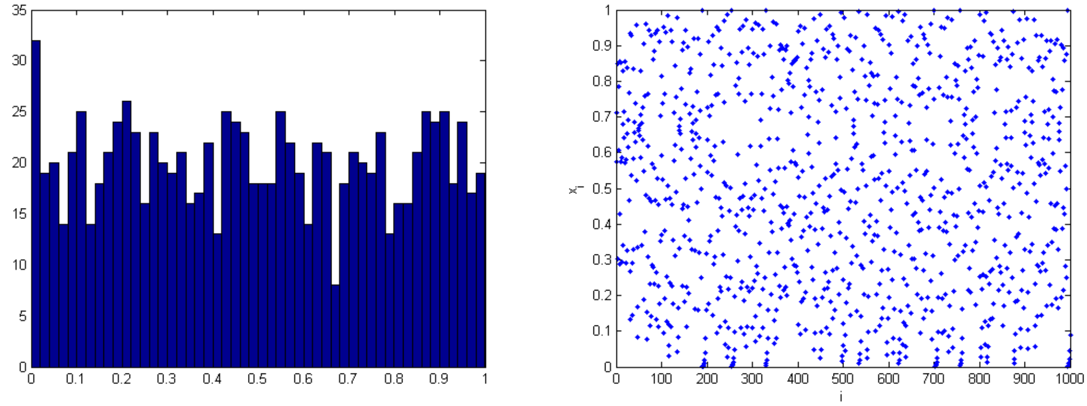
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.13. T-logistic map ($\nu = 4$): Seed = 0.758167338031333, Iteration no = 1000.



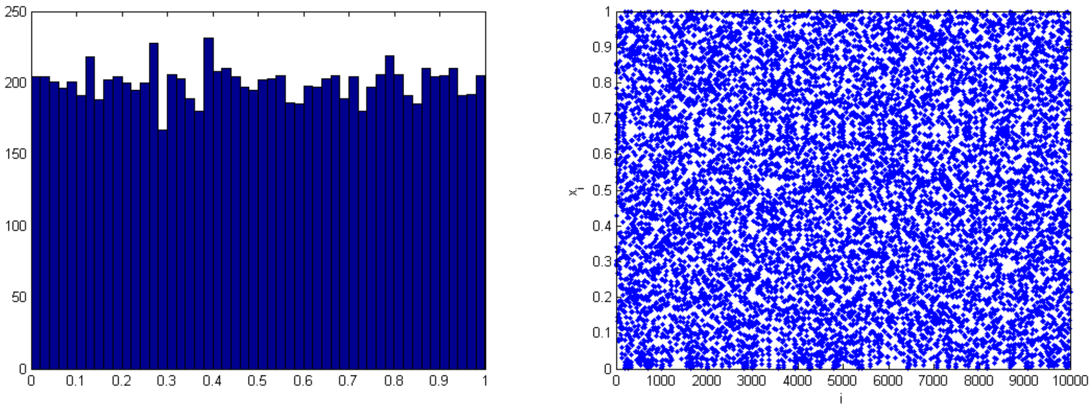
(a) Histogram with 50 subintervals (b) Serial plot

Figure C.14. T-logistic map ($\nu = 4$): Seed = 0.758167338031333, Iteration no = 10000.



(a) Histogram with 50 subintervals (b) Serial plot

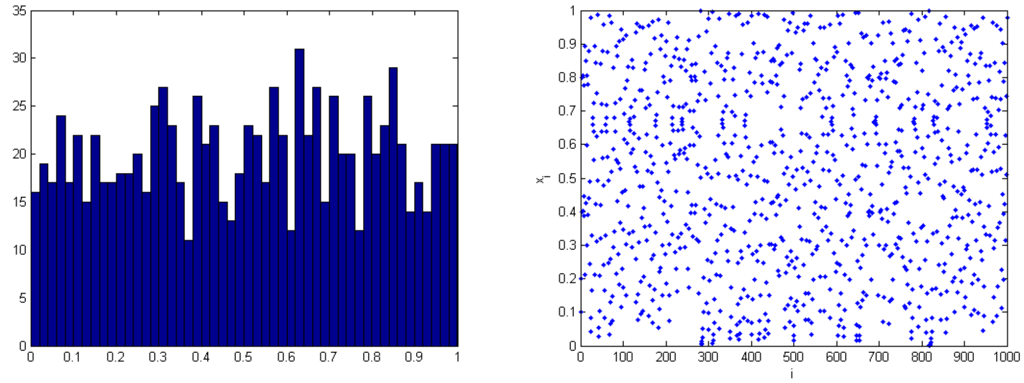
Figure C.15. T-logistic map ($\nu = 4$): Seed = 0.809, Iteration no = 1000.



(a) Histogram with 50 subintervals (b) Serial plot

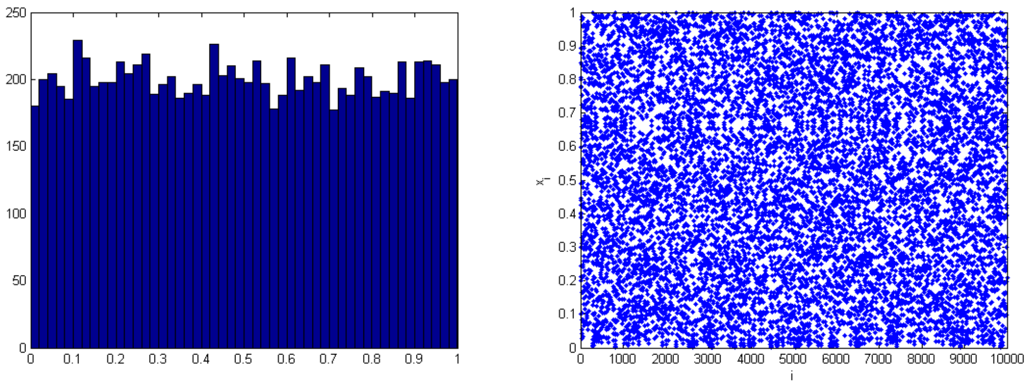
Figure C.16. T-logistic map ($\nu = 4$): Seed = 0.809, Iteration no = 10000.

APPENDIX D: HISTOGRAMS FOR CONNECTING MAP ($\alpha = 0.999$)



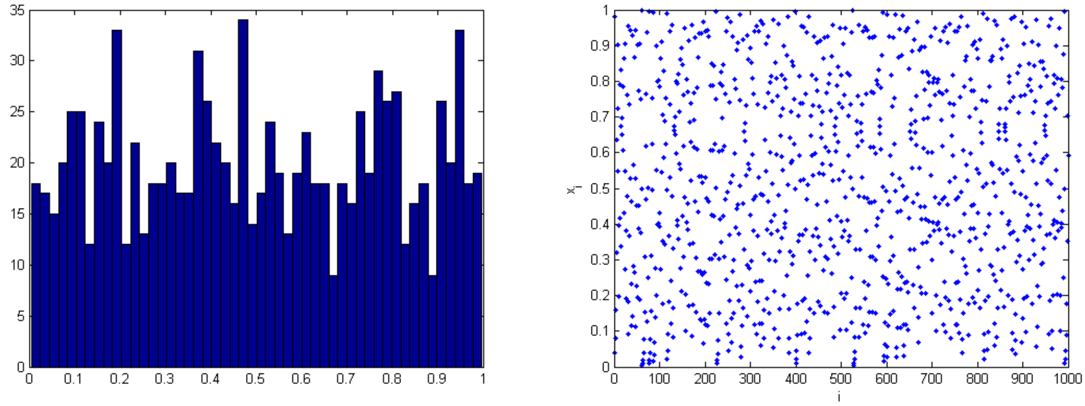
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.1. Connecting map ($\alpha = 0.999$): Seed $= 0.1$, Iteration no $= 1000$.



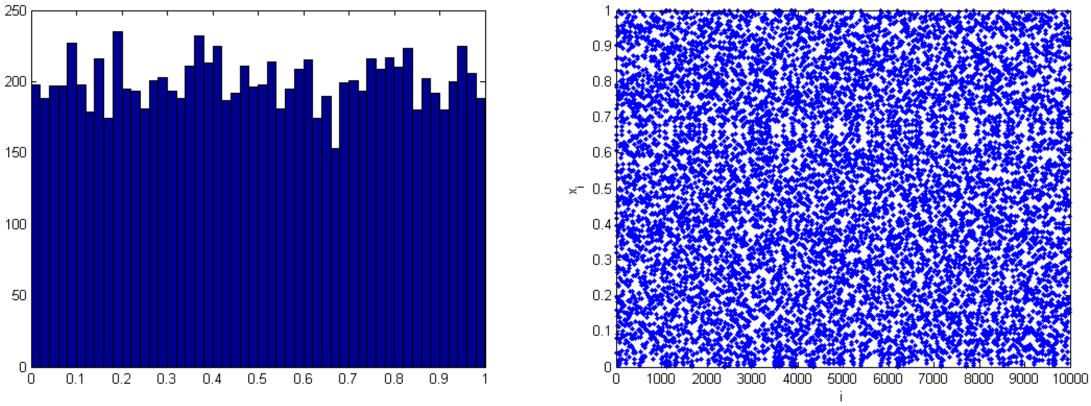
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.2. Connecting map ($\alpha = 0.999$): Seed $= 0.1$, Iteration no $= 10000$.



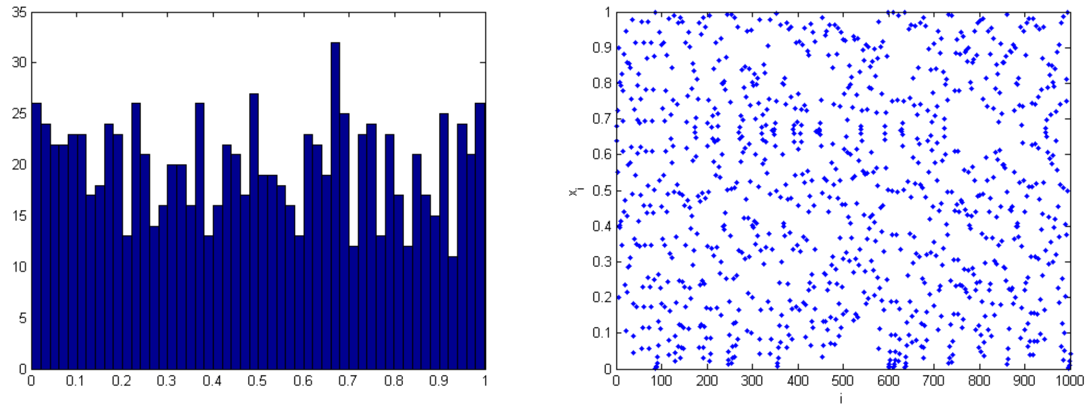
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.3. Connecting map ($\alpha = 0.999$): Seed $= 0.98$, Iteration no $= 1000$.



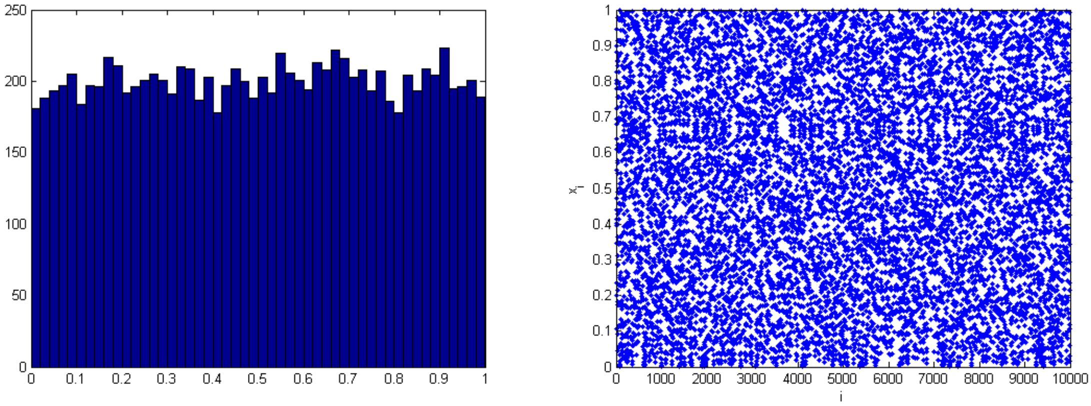
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.4. Connecting map ($\alpha = 0.999$): Seed $= 0.98$, Iteration no $= 10000$.



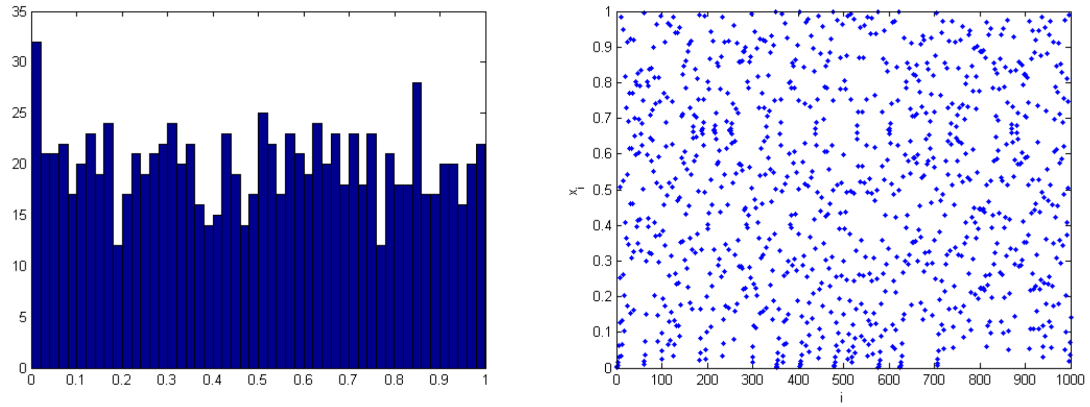
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.5. Connecting map ($\alpha = 0.999$): Seed $= 0.6811$, Iteration no $= 1000$.



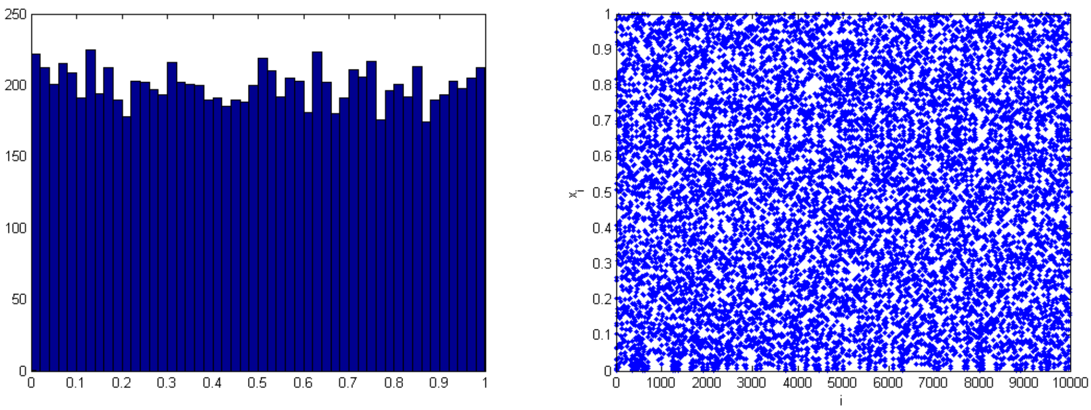
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.6. Connecting map ($\alpha = 0.999$): Seed = 0.6811, Iteration no = 10000.



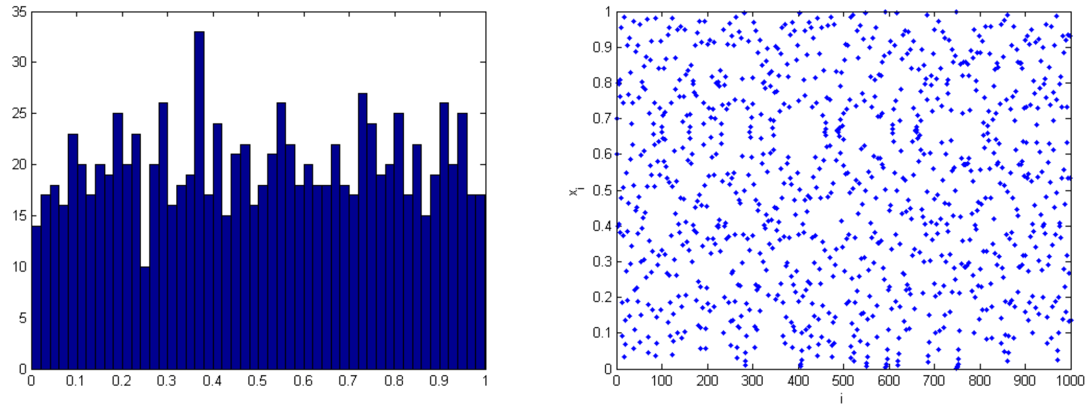
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.7. Connecting map ($\alpha = 0.999$): Seed = 0.002, Iteration no = 1000.



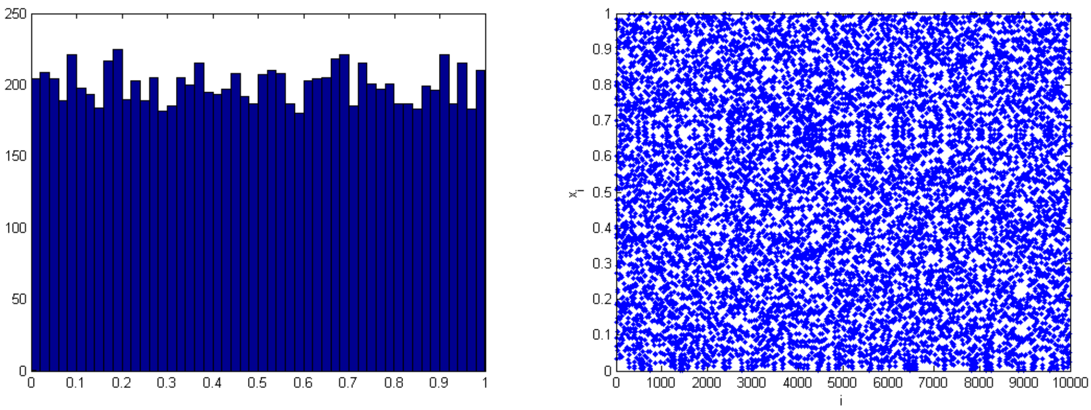
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.8. Connecting map ($\alpha = 0.999$): Seed = 0.002, Iteration no = 10000.



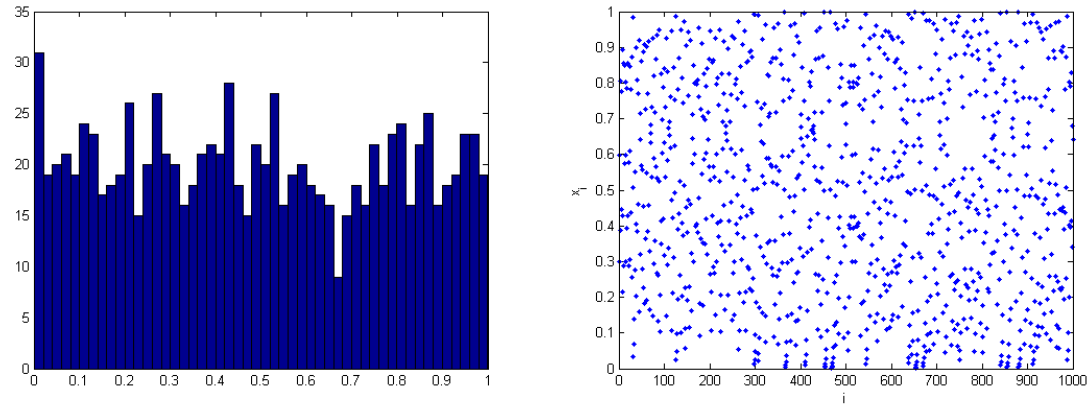
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.9. Connecting map ($\alpha = 0.999$): Seed = 0.7, Iteration no = 1000.



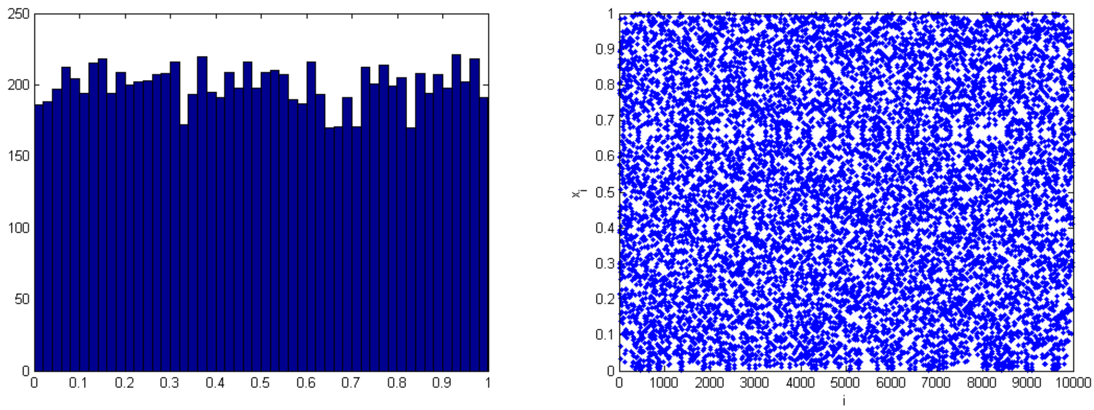
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.10. Connecting map ($\alpha = 0.999$): Seed = 0.7, Iteration no = 10000.



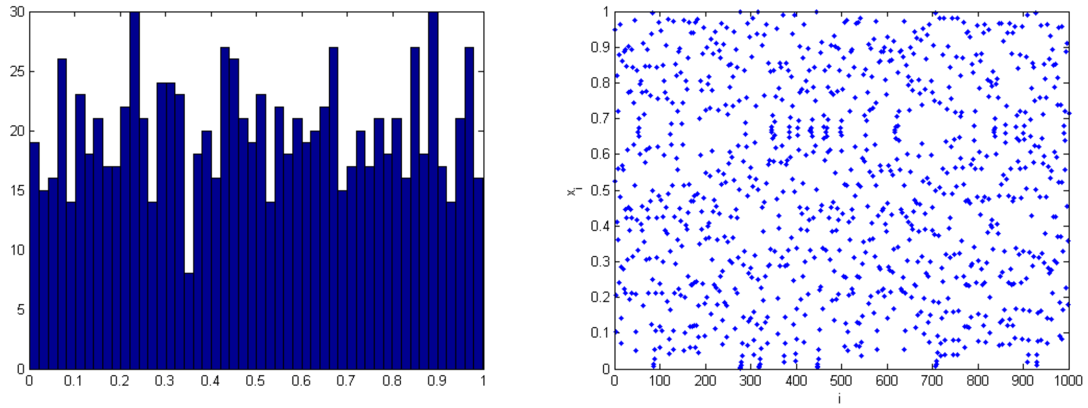
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.11. Connecting map ($\alpha = 0.999$): Seed = 0.298698545648568, Iteration no = 1000.



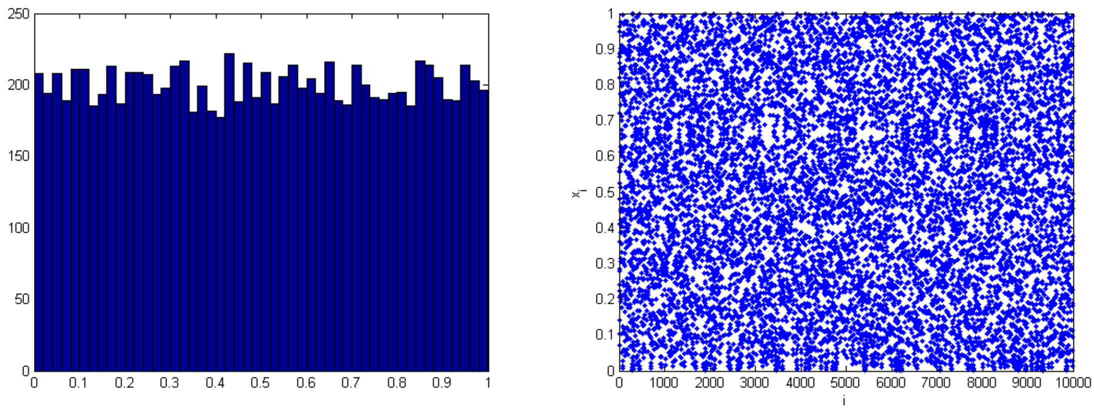
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.12. Connecting map ($\alpha = 0.999$): Seed = 0.298698545648568, Iteration no = 10000.



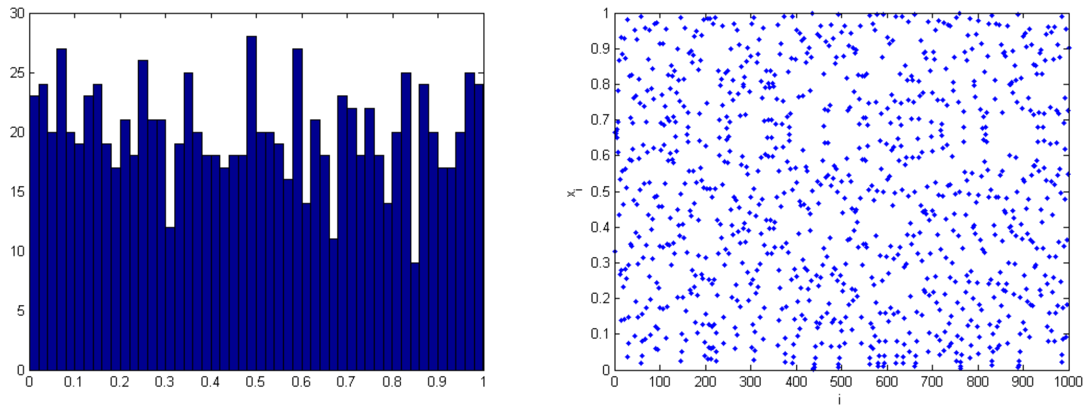
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.13. Connecting map ($\alpha = 0.999$): Seed = 0.525620161505902, Iteration no = 1000.



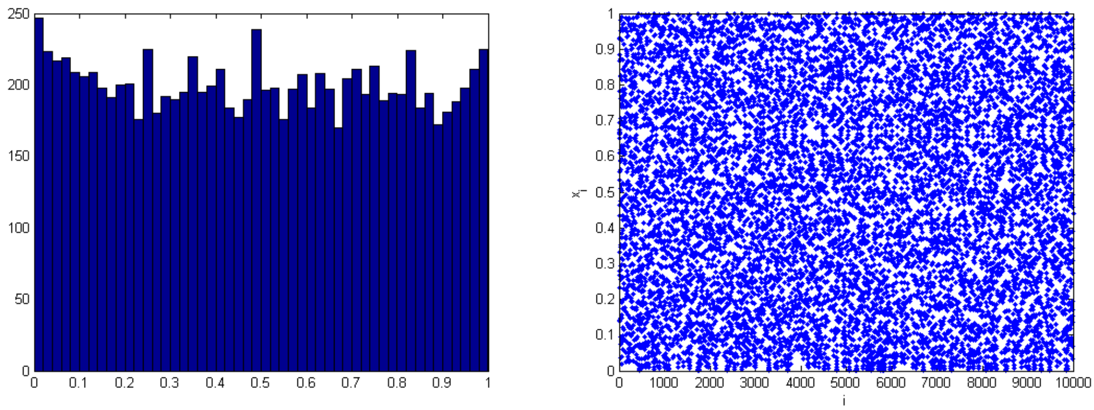
(a) Histogram with 50 subintervals (b) Serial plot

Figure D.14. Connecting map ($\alpha = 0.999$): Seed = 0.525620161505902, Iteration no = 10000.



(a) Histogram with 50 subintervals (b) Serial plot

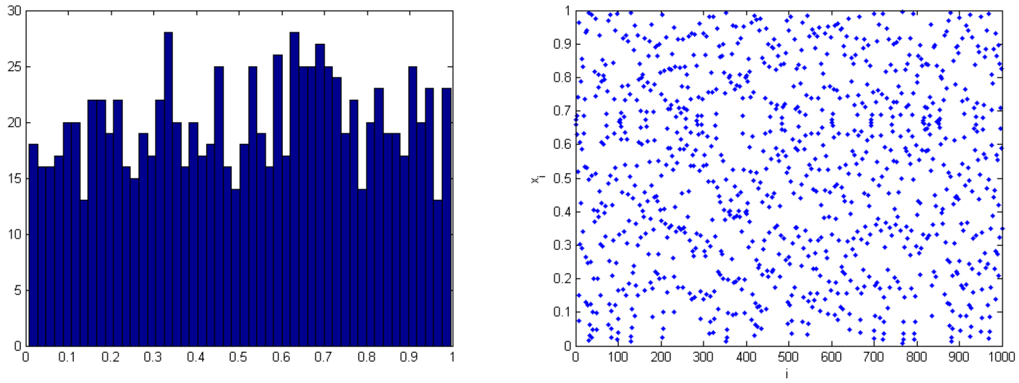
Figure D.15. Connecting map ($\alpha = 0.999$): Seed $= 0.333$, Iteration no $= 1000$.



(a) Histogram with 50 subintervals (b) Serial plot

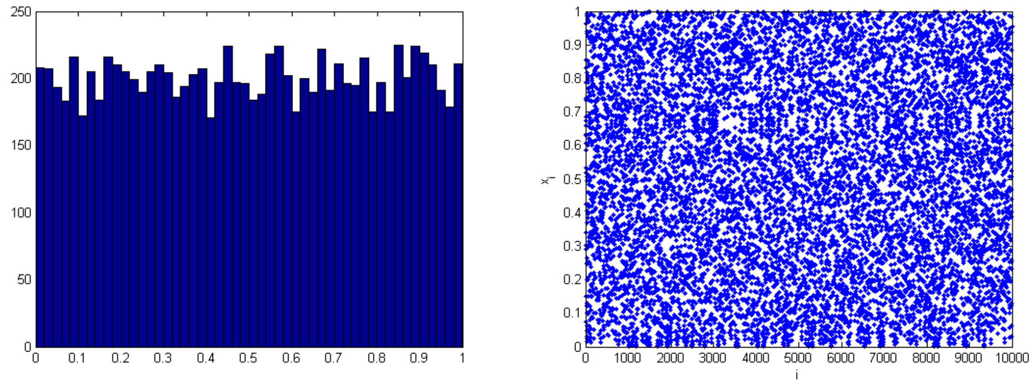
Figure D.16. Connecting map ($\alpha = 0.999$): Seed $= 0.333$, Iteration no $= 10000$.

APPENDIX E: HISTOGRAMS FOR CONNECTING MAP ($\alpha = 1.01$)



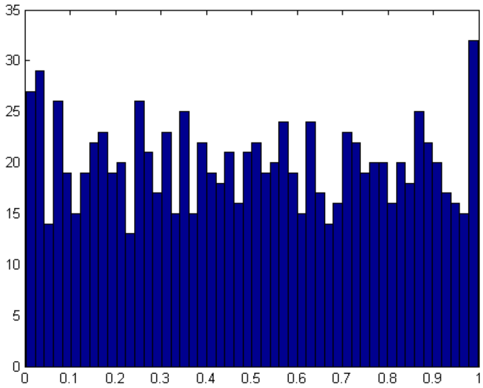
(a) Histogram with 50 subintervals (b) Serial plot

Figure E.1. Connecting map ($\alpha = 1.01$): Seed = 0.672551918078352, Iteration no = 1000.

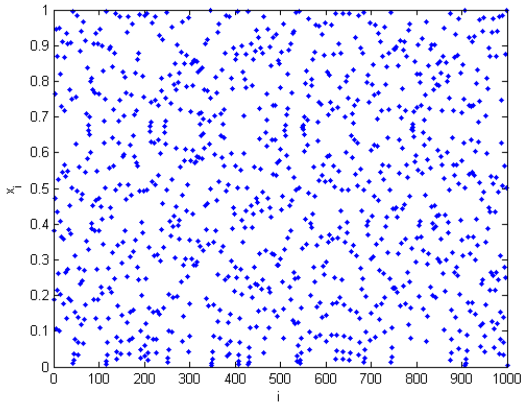


(a) Histogram with 50 subintervals (b) Serial plot

Figure E.2. Connecting map ($\alpha = 1.01$): Seed = 0.672551918078352, Iteration no = 10000.

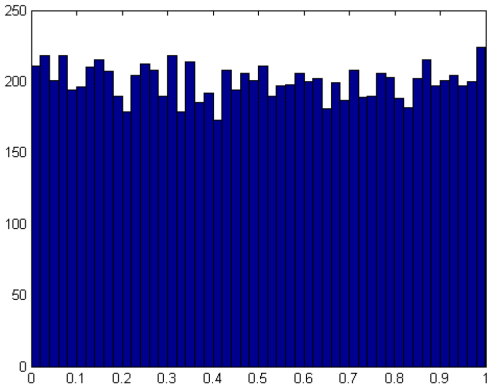


(a) Histogram with 50 subintervals

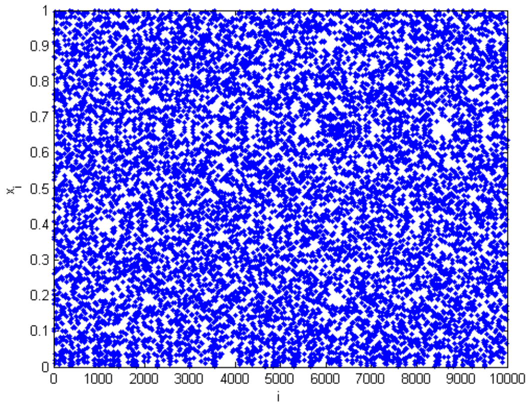


(b) Serial plot

Figure E.3. Connecting map ($\alpha = 1.01$): Seed = 0.189, Iteration no = 1000.

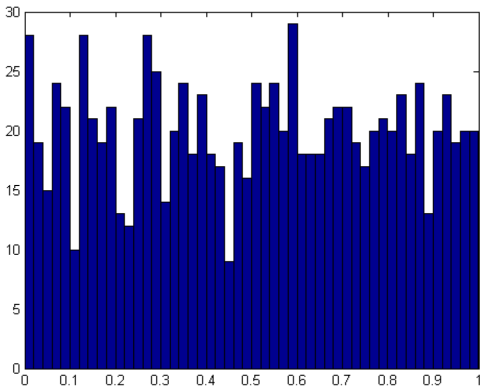


(a) Histogram with 50 subintervals

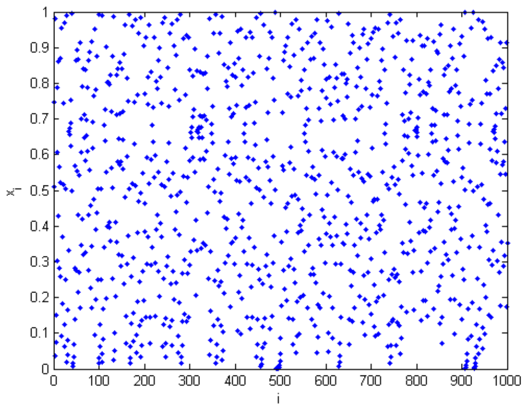


(b) Serial plot

Figure E.4. Connecting map ($\alpha = 1.01$): Seed = 0.189, Iteration no = 10000.

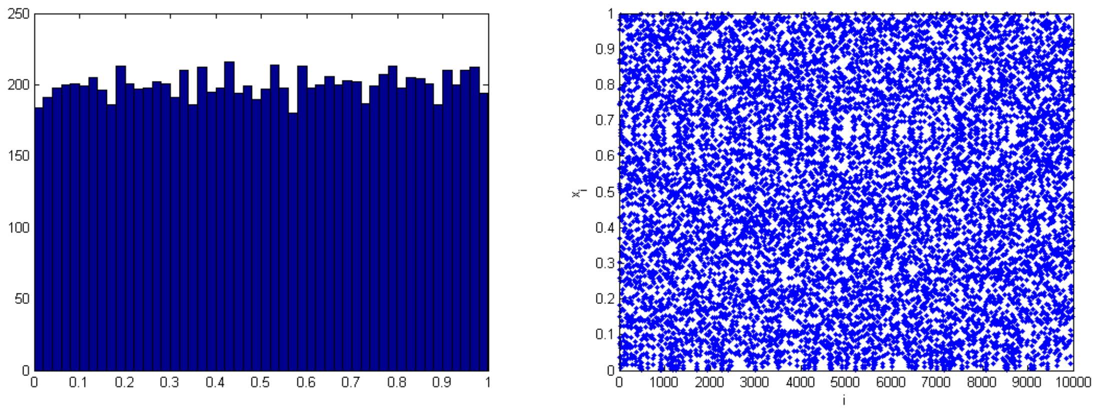


(a) Histogram with 50 subintervals



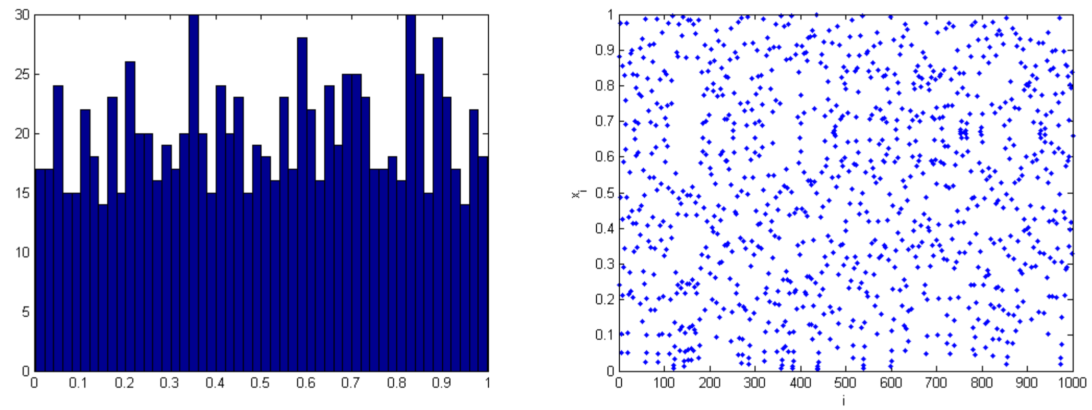
(b) Serial plot

Figure E.5. Connecting map ($\alpha = 1.01$): Seed = 0.747, Iteration no = 1000.



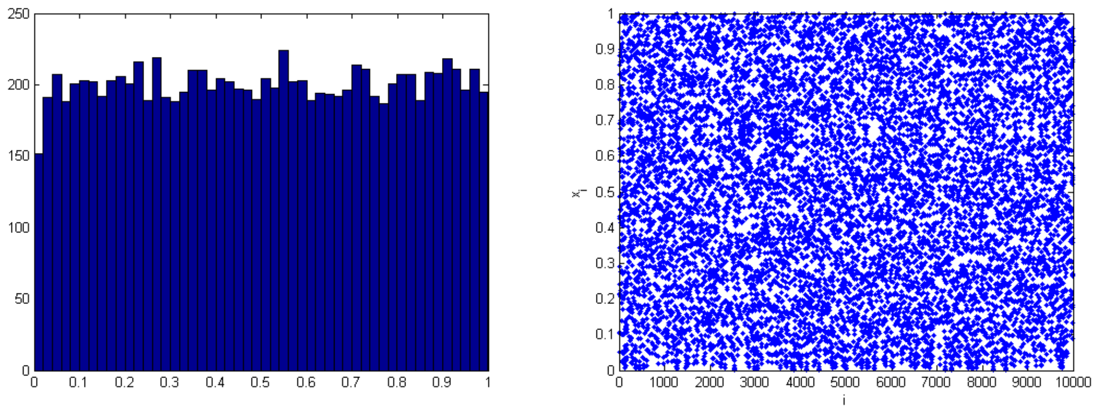
(a) Histogram with 50 subintervals (b) Serial plot

Figure E.6. Connecting map ($\alpha = 1.01$): Seed = 0.747, Iteration no = 10000.



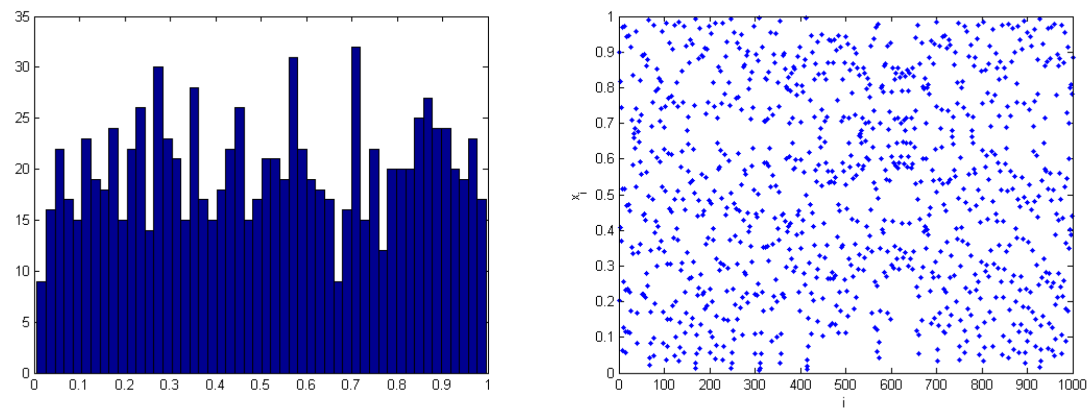
(a) Histogram with 50 subintervals (b) Serial plot

Figure E.7. Connecting map ($\alpha = 1.01$): Seed = 0.880223477310122, Iteration no = 1000.



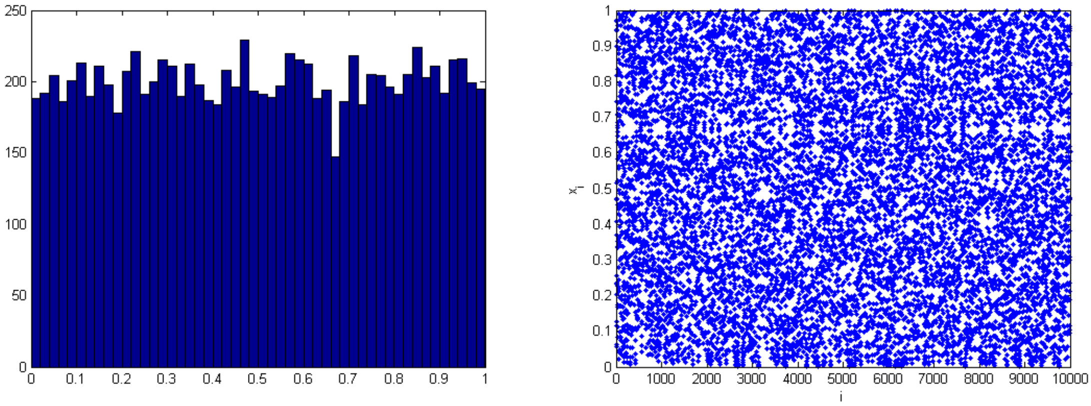
(a) Histogram with 50 subintervals (b) Serial plot

Figure E.8. Connecting map (α = 1.01): Seed = 0.880223477310122, Iteration no = 10000.



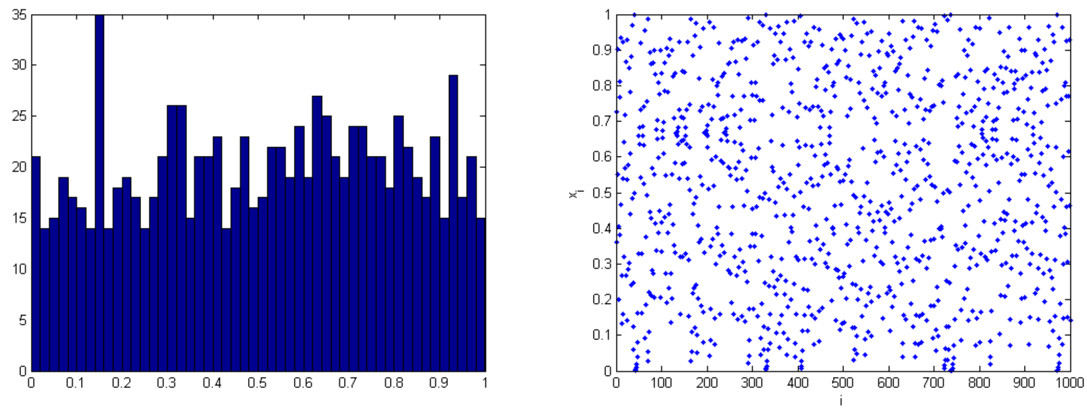
(a) Histogram with 50 subintervals (b) Serial plot

Figure E.9. Connecting map (α = 1.01): Seed = 0.9, Iteration no = 1000.



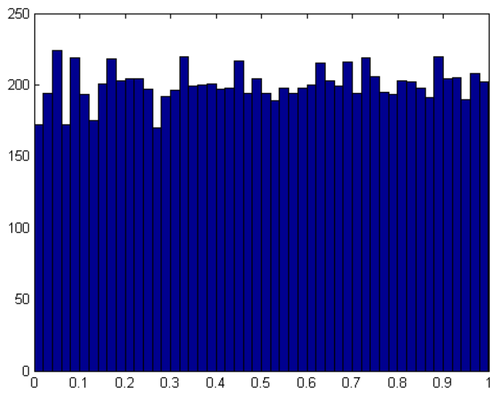
(a) Histogram with 50 subintervals (b) Serial plot

Figure E.10. Connecting map ($\alpha = 1.01$): Seed = 0.9, Iteration no = 10000.

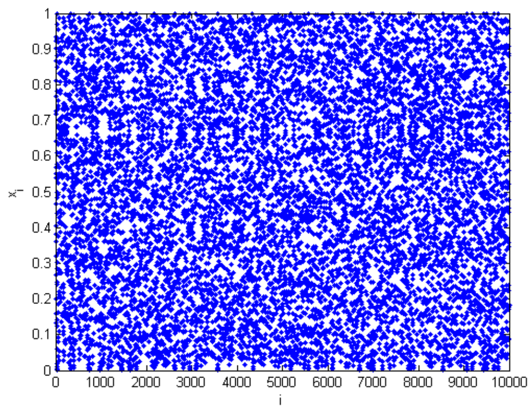


(a) Histogram with 50 subintervals (b) Serial plot

Figure E.11. Connecting map ($\alpha = 1.01$): Seed = 0.361436078541452, Iteration no = 1000.

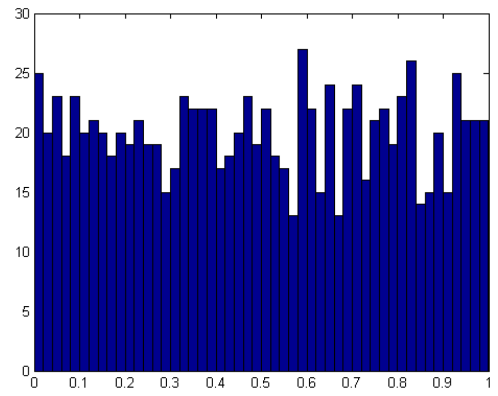


(a) Histogram with 50 subintervals

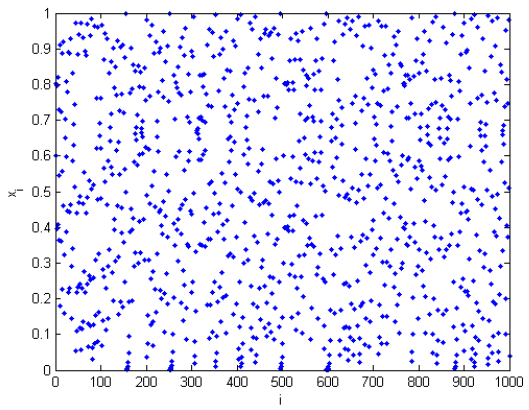


(b) Serial plot

Figure E.12. Connecting map ($\alpha = 1.01$): Seed = 0.361436078541452, Iteration no = 10000.

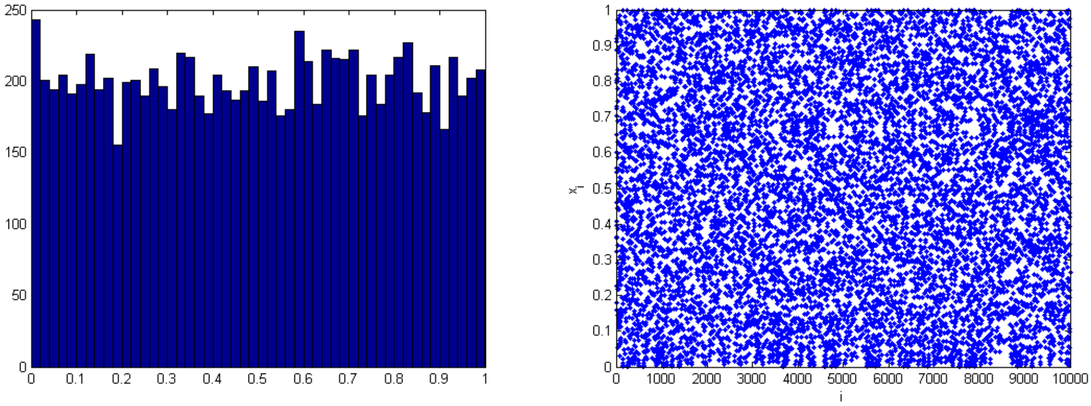


(a) Histogram with 50 subintervals



(b) Serial plot

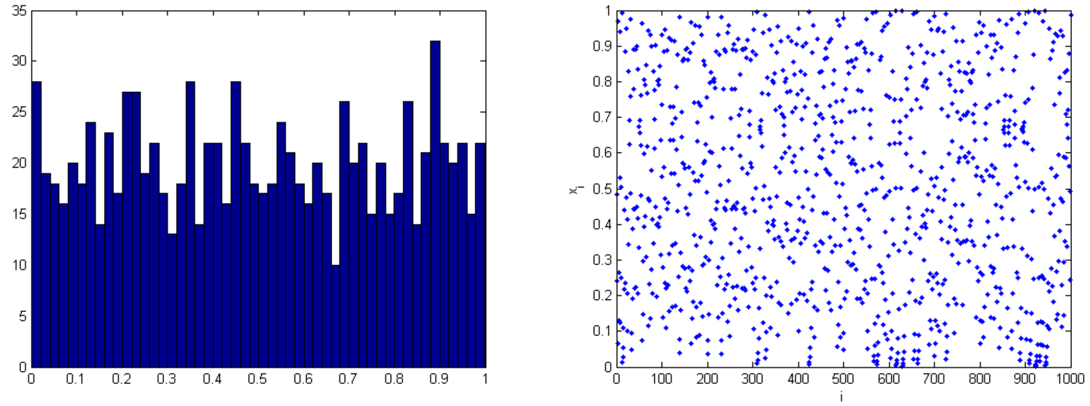
Figure E.13. Connecting map ($\alpha = 1.01$): Seed = 0.6001, Iteration no = 1000.



(a) Histogram with 50 subintervals

(b) Serial plot

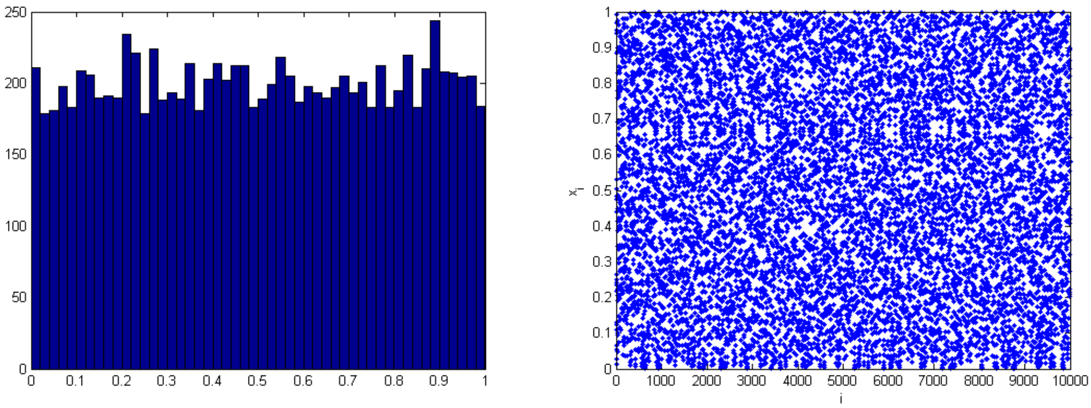
Figure E.14. Connecting map ($\alpha = 1.01$): Seed = 0.6001, Iteration no = 10000.



(a) Histogram with 50 subintervals

(b) Serial plot

Figure E.15. Connecting map ($\alpha = 1.01$): Seed = 0.24, Iteration no = 1000.

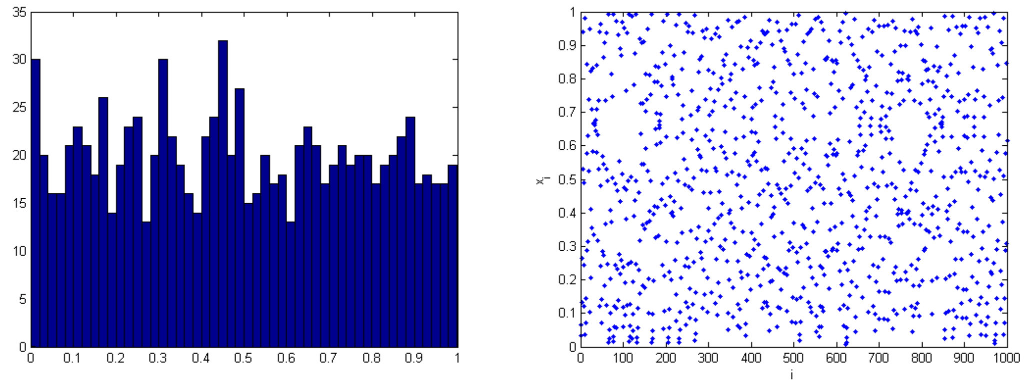


(a) Histogram with 50 subintervals

(b) Serial plot

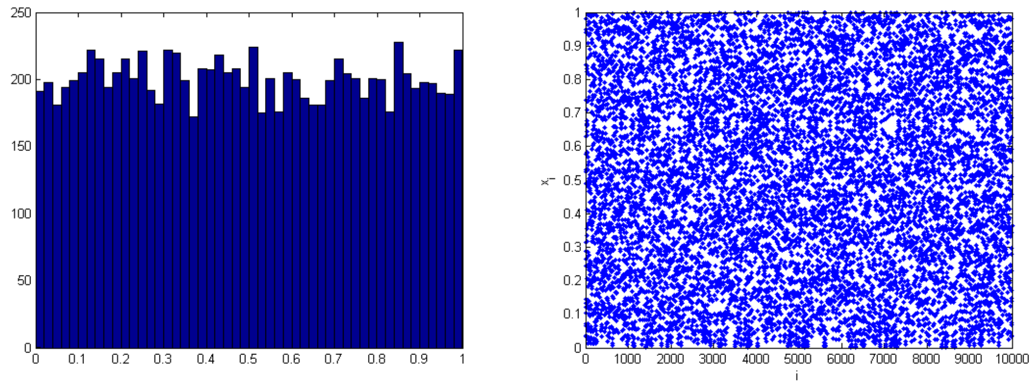
Figure E.16: Connecting map ($\alpha = 1.01$): Seed = 0.24, Iteration no = 10000.

APPENDIX F: HISTOGRAMS FOR LCGs



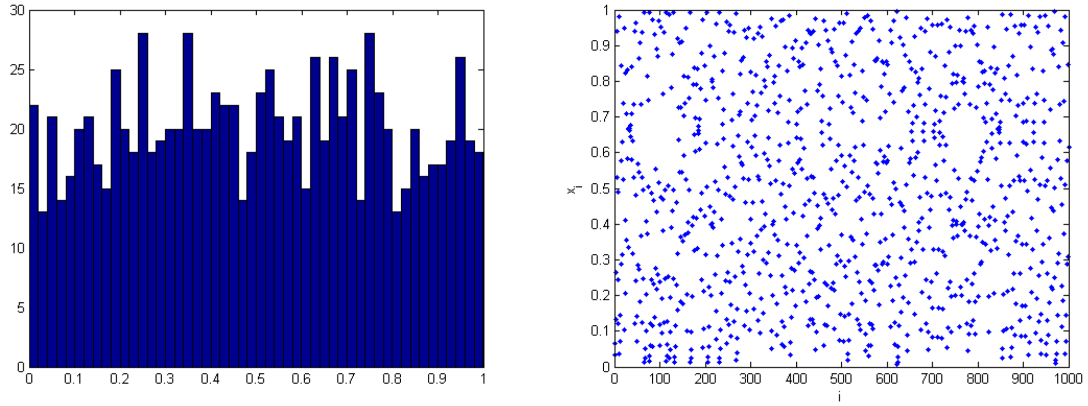
(a) Histogram with 50 subintervals (b) Serial plot

Figure F.1. MLCG 1: Seed = 3316779, Iteration no = 1000.



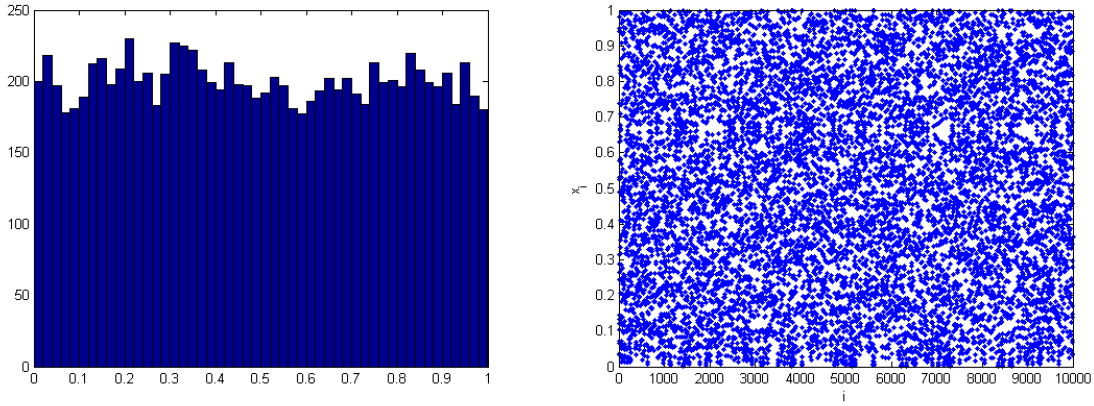
(a) Histogram with 50 subintervals (b) Serial plot

Figure F.2. MLCG 1: Seed = 3316779, Iteration no = 10000.



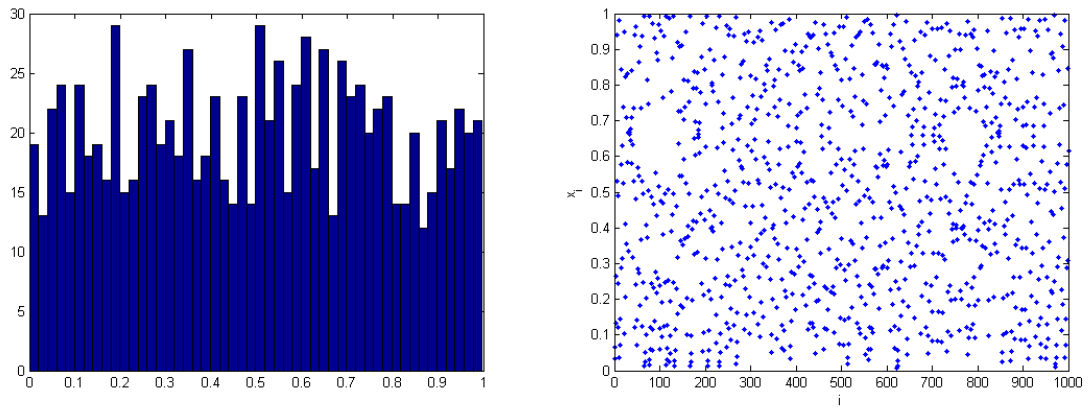
(a) Histogram with 50 subintervals (b) Serial plot

Figure F.3. MLCG 2: Seed = 3316779, Iteration no = 1000.



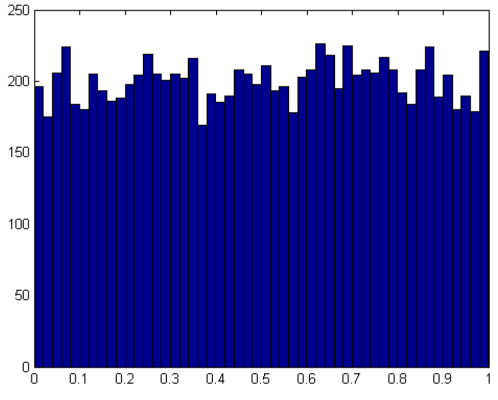
(a) Histogram with 50 subintervals (b) Serial plot

Figure F.4. MLCG 2: Seed = 3316779, Iteration no = 10000.

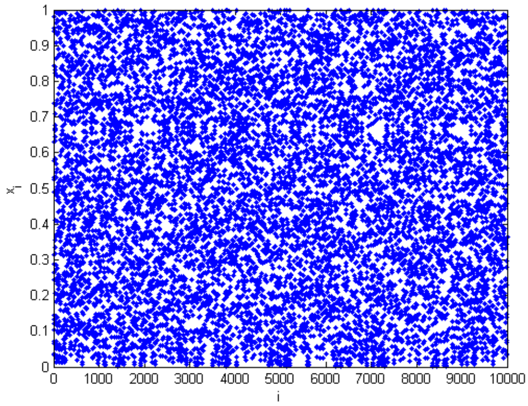


(a) Histogram with 50 subintervals (b) Serial plot

Figure F.5. LCG 3: Seed = 3316779, Iteration no = 1000.

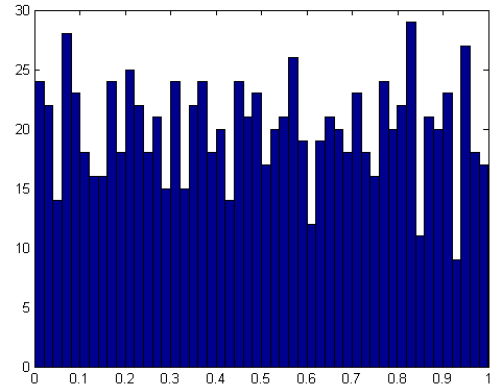


(a) Histogram with 50 subintervals

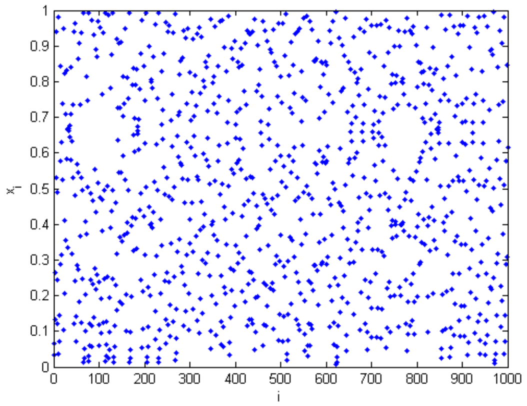


(b) Serial plot

Figure F.6. LCG 3: Seed = 3316779, Iteration no = 10000.

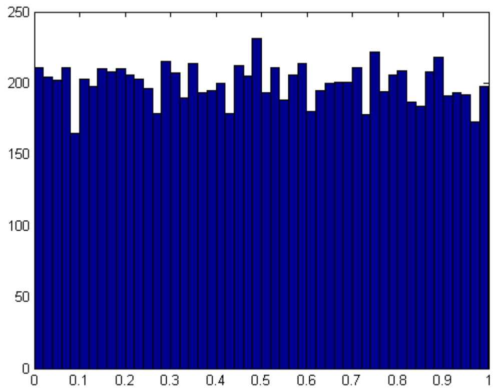


(a) Histogram with 50 subintervals

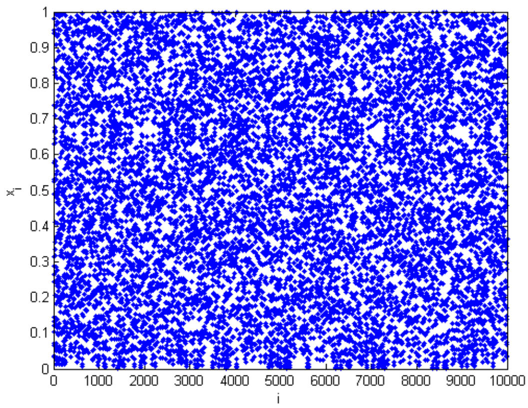


(b) Serial plot

Figure F.7. LCG 4: Seed = 3316779, Iteration no = 1000.

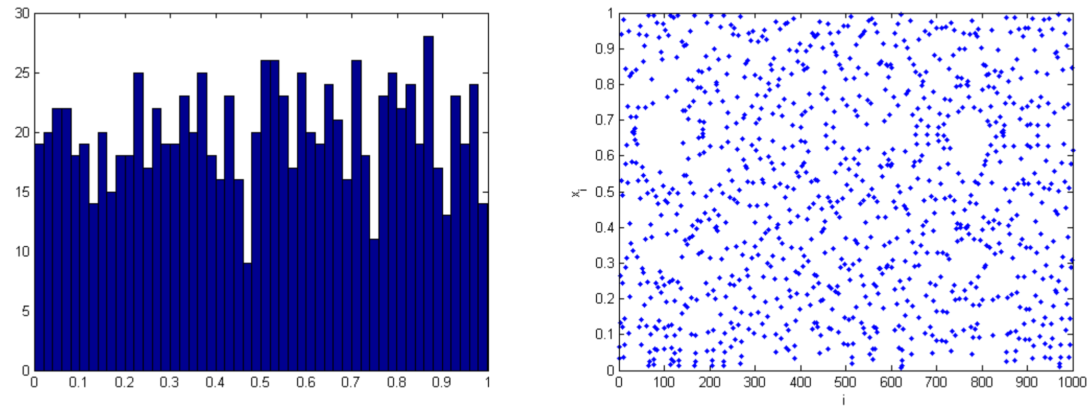


(a) Histogram with 50 subintervals



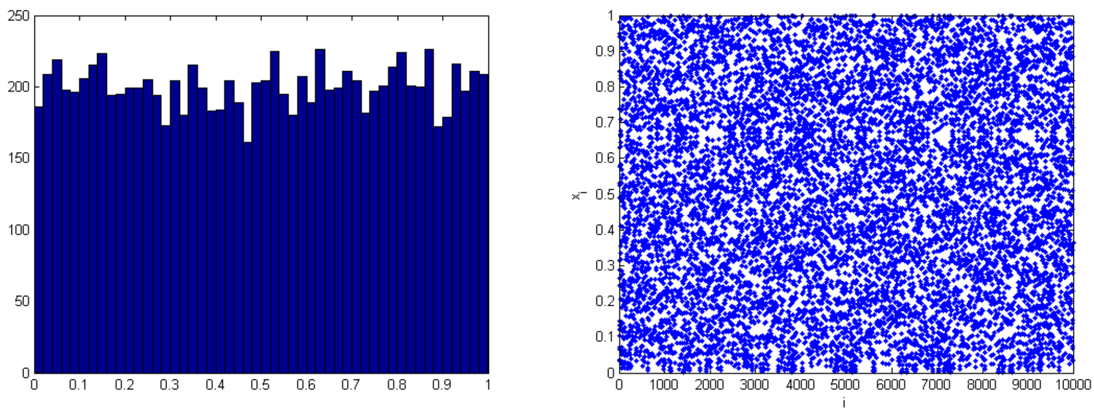
(b) Serial plot

Figure F.8. LCG 4: Seed = 3316779, Iteration no = 10000.



(a) Histogram with 50 subintervals (b) Serial plot

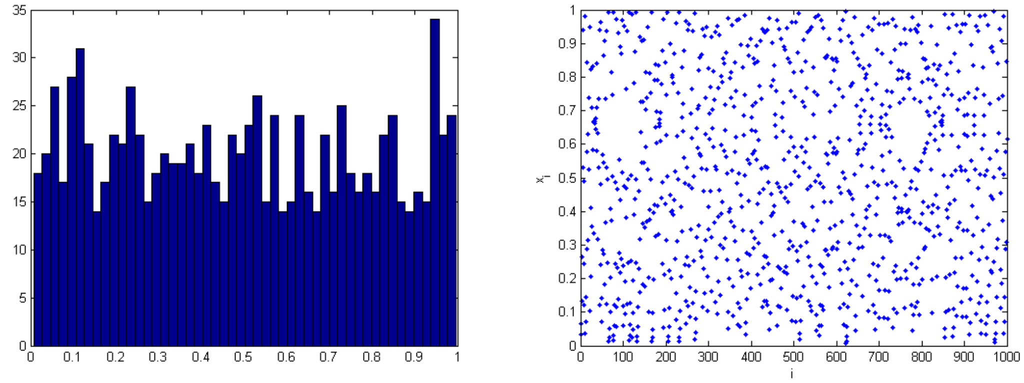
Figure F.9. LCG 5: Seed = 3316779, Iteration no = 1000.



(a) Histogram with 50 subintervals (b) Serial plot

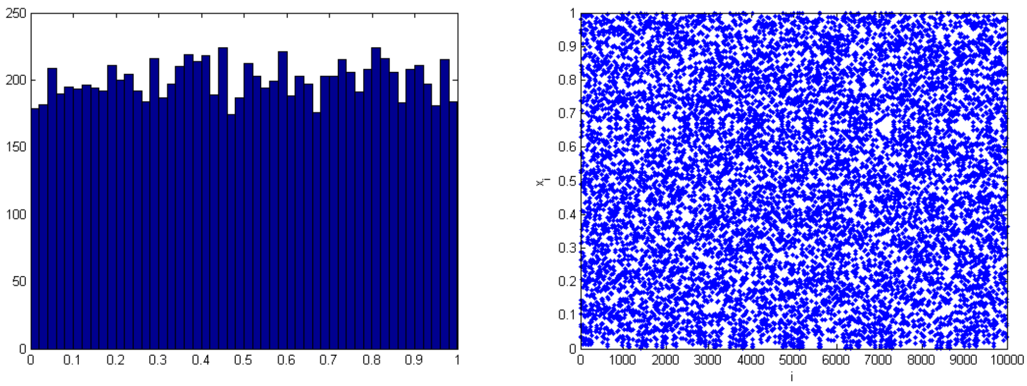
Figure F.10. LCG 5: Seed = 3316779, Iteration no = 10000.

APPENDIX G: HISTOGRAMS FOR FIVE SUGGESTED MAPS



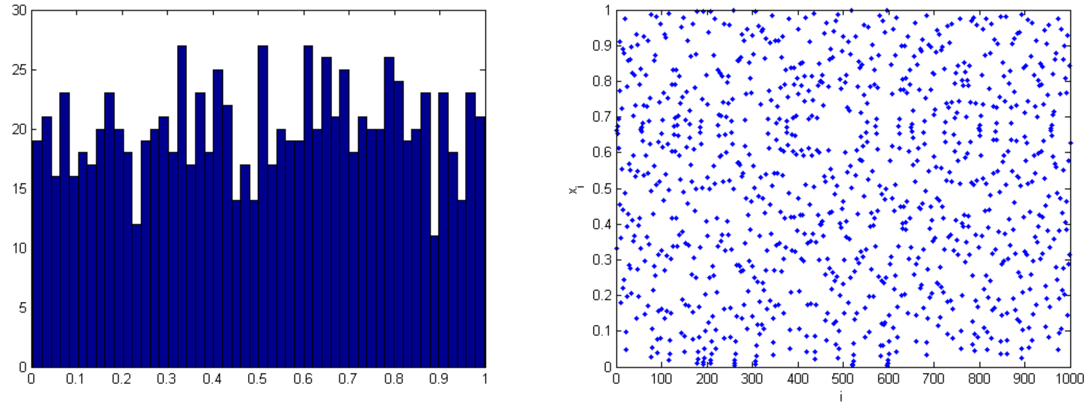
(a) Histogram with 50 subintervals (b) Serial plot

Figure G.1. Tent map ($\mu = 2$): Seed = 0.03316779, Iteration no = 1000.



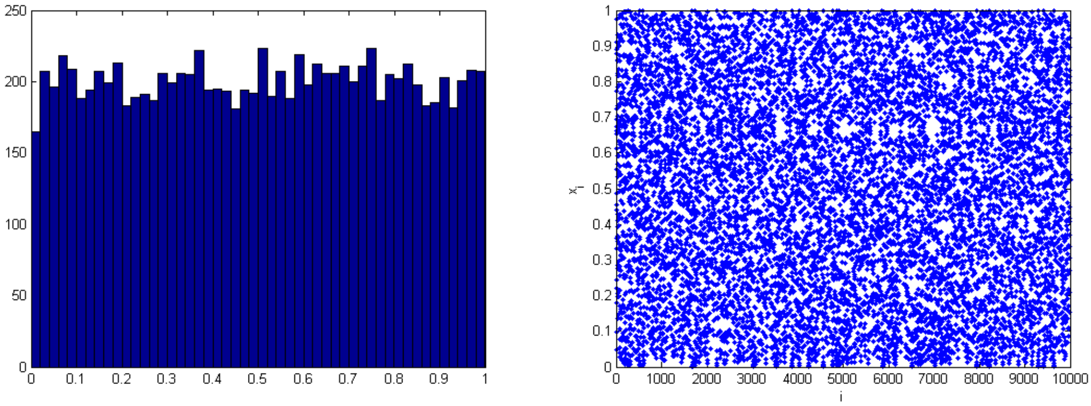
(a) Histogram with 50 subintervals (b) Serial plot

Figure G.2. Tent map ($\mu = 2$): Seed = 0.03316779, Iteration no = 10000.



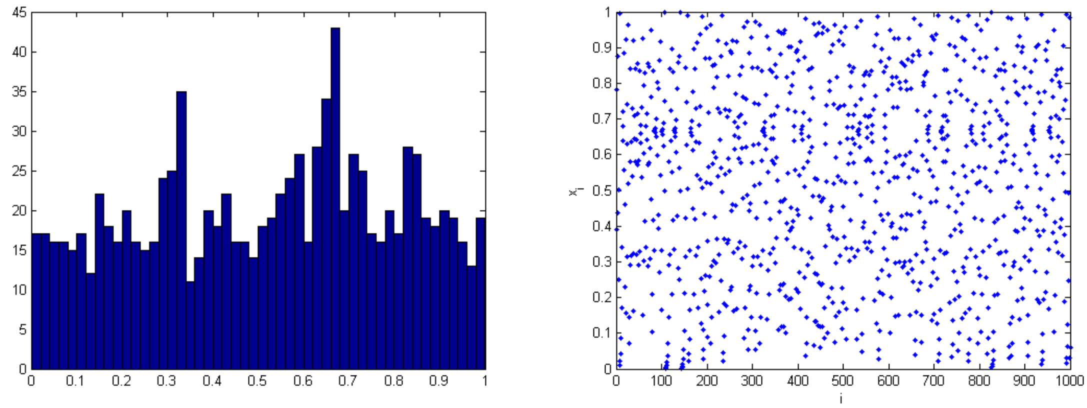
(a) Histogram with 50 subintervals (b) Serial plot

Figure G.3. Tent map ($\mu = 2.00005$): Seed = 0.3316779, Iteration no = 1000.



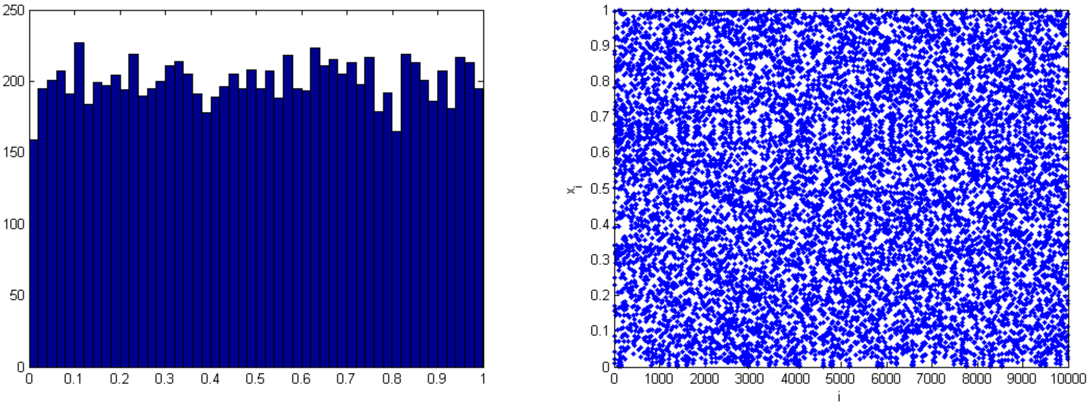
(a) Histogram with 50 subintervals (b) Serial plot

Figure G.4. Tent map ($\mu = 2.00005$): Seed = 0.33167, Iteration no = 10000.



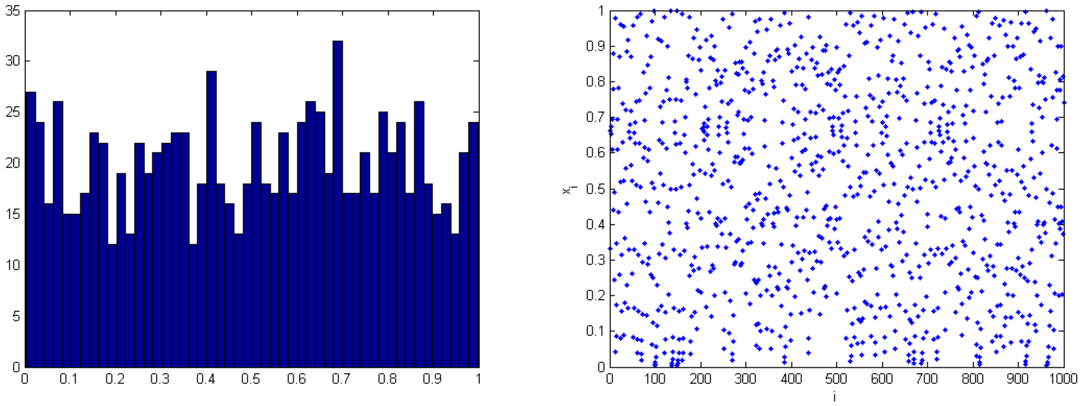
(a) Histogram with 50 subintervals (b) Serial plot

Figure G.5. T-logistic map ($\nu = 4$): Seed = 0.33167, Iteration no = 1000.



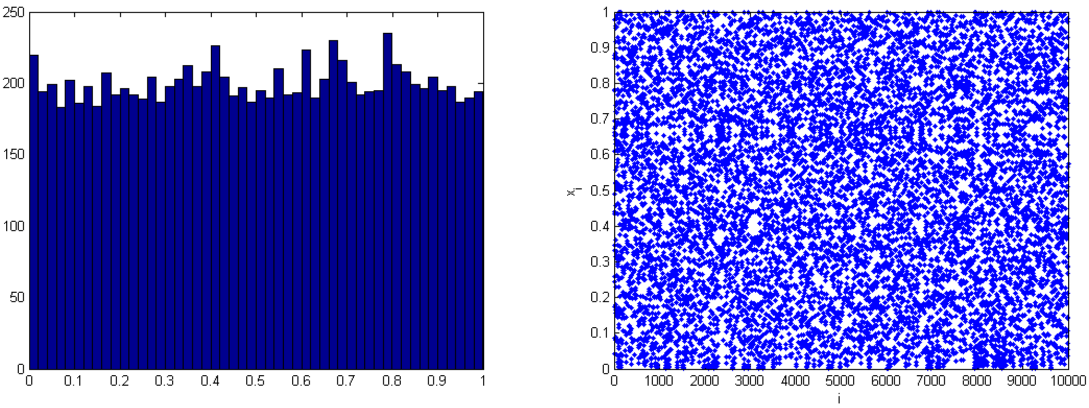
(a) Histogram with 50 subintervals (b) Serial plot

Figure G.6. T-logistic map ($\nu = 4$): Seed = 0.3316779, Iteration no = 10000.



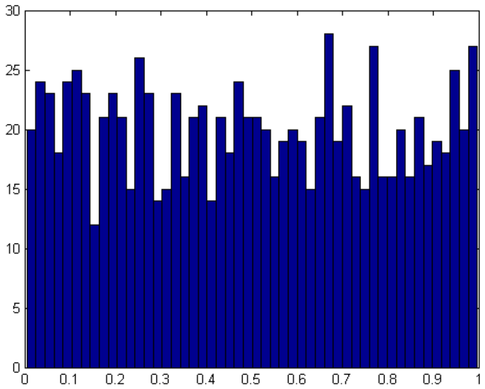
(a) Histogram with 50 subintervals (b) Serial plot

Figure G.7. Connecting map ($\alpha = 0.999$): Seed = 0.3316779, Iteration no = 1000.

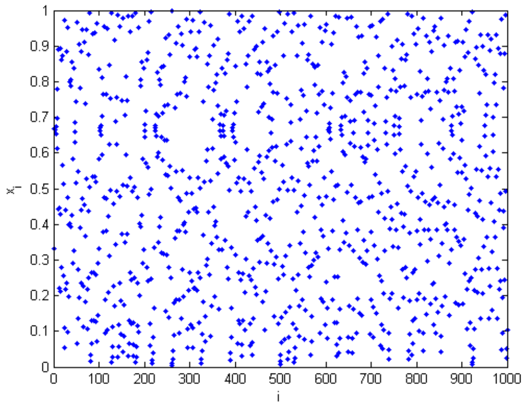


(a) Histogram with 50 subintervals (b) Serial plot

Figure G.8. Connecting map ($\alpha = 0.999$): Seed = 0.3316779, Iteration no = 10000.

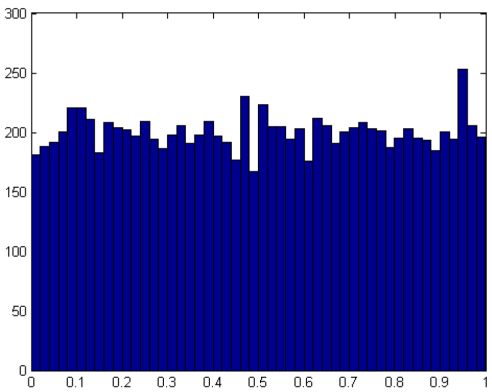


(a) Histogram with 50 subintervals

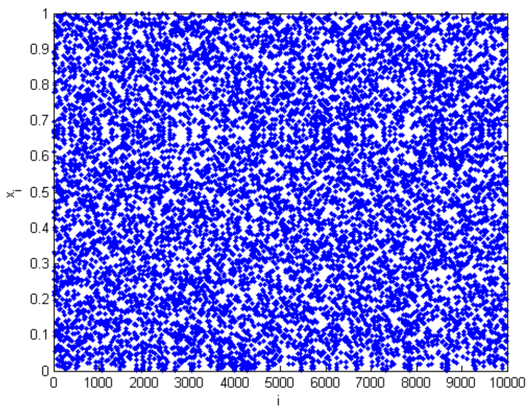


(b) Serial plot

Figure G.9. Connecting map ($\alpha = 1.01$): Seed = 0.3316779, Iteration no = 1000.



(a) Histogram with 50 subintervals



(b) Serial plot

Figure G.10. Connecting map ($\alpha = 1.01$): Seed = 0.3316779, Iteration no = 10000.

APPENDIX H: THE STATISTICAL TEST RESULTS - 1

Table H.1. The statistical test results: Seed = 449, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**449**	**449**	**449**	**449**	**449**
P-Values	**Kolmogorov Smirnov**	0.0376	0.93	0.3995	0.3117	0.742
	Chisqure 30	0.2001	0.6603	0.4472	0.2282	0.4236
	Chisqure 50	0.6272	0.3714	0.3945	0.1895	0.6504
	Chisqure 80	0.5454	0.2357	0.8642	0.3315	0.3313
	Chisqure 100	0.5503	0.7414	0.1812	0.2458	0.5821
	Runs Up - Down	0.2655	0.8708	0.5398	0.2876	0.5398
	Runs Above - Below	0.3047	0.7774	0.4143	0.8742	0.8768
	Auto Corr Func	0.047556043	0.045087869	0.045087869	0.057598787	0.055220955

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00000449**	**0.449**	**0.449**	**0.449**	**0.449**
P-Values	**Kolmogorov Smirnov**	0.478	0.4094	0.9541	0.9939	0.6557
	Chisqure 30	0.5346	0.9811	0.4315	0.9551	0.3778
	Chisqure 50	0.6525	0.8186	0.9119	0.921	0.2939
	Chisqure 80	0.8095	0.7351	0.6656	0.9552	0.5669
	Chisqure 100	0.513	0.6257	0.683	0.8732	0.3983
	Runs Up - Down	0.608	0.2064	0.3611	0.8316	0.8121
	Runs Above - Below	0.0151	0.0823	0.7274	0.27	0.8531
	Auto Corr Func	0.069741843	0.0485342	0.08827836	0.060945058	0.052280719

Table H.2. The statistical test results: Seed = 181081, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**181081**	**181081**	**181081**	**181081**	**181081**
P-Values	**Kolmogorov Smirnov**	0.6033	0.4058	0.5773	0.6335	0.6914
	Chisqure 30	0.8977	0.8181	0.1882	0.8328	0.6551
	Chisqure 50	0.5387	0.8269	0.1362	0.6287	0.2827
	Chisqure 80	0.6591	0.6742	0.1216	0.6329	0.0281
	Chisqure 100	0.1927	0.6374	0.1507	0.3821	0.1294
	Runs Up - Down	0.608	0.4604	0.608	0.5073	0.6981
	Runs Above - Below	0.8747	0.7354	0.8739	0.5907	0.5144
	Auto Corr Func	0.076842388	0.047334459	0.047334459	0.018374855	0.095790153

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00181081**	**0.181081**	**0.181081**	**0.181081**	**0.181081**
P-Values	**Kolmogorov Smirnov**	0.6725	0.5932	0.0586	0.9303	0.8512
	Chisqure 30	0.5068	0.2376	0.0027	0.9241	0.7274
	Chisqure 50	0.2716	0.3106	0.0002	0.9457	0.126
	Chisqure 80	0.7494	0.0396	0.0009	0.466	0.1069
	Chisqure 100	0.4847	0.1484	0.00001	0.5396	0.0263
	Runs Up - Down	0.5906	0.5398	0.0495	0.5073	0.8121
	Runs Above - Below	0.4797	0.3564	0.2766	0.8251	0.2758
	Auto Corr Func	0.062191667	0.090899974	0.080056131	0.075404215	0.078319979

Table H.3. The statistical test results: Seed = 890123, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**890123621**	**890123621**	**890123621**	**890123621**	**890123621**
P-Values	**Kolmogorov Smirnov**	0.7691	0.2361	0.7175	0.9494	0.8394
	Chisqure 30	0.2624	0.8676	0.2165	0.3497	0.9014
	Chisqure 50	0.3689	0.9222	0.6043	0.5378	0.804
	Chisqure 80	0.3601	0.7434	0.2604	0.2863	0.3455
	Chisqure 100	0.716	0.8241	0.1311	0.1744	0.39
	Runs Up - Down	0.8121	0.7544	0.6981	0.6981	0.0865
	Runs Above - Below	0.8759	0.5423	0.7231	0.775	0.5915
	Auto Corr Func	0.045852733	0.095784997	0.095784997	0.039215282	0.033972788

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.890123621**	**0.890123621**	**0.890123621**	**0.890123621**	**0.890123621**
P-Values	**Kolmogorov Smirnov**	0.9354	0.8199	0.1988	0.0132	0.0965
	Chisqure 30	0.7447	0.3474	0.0019	0.0745	0.017
	Chisqure 50	0.6529	0.2817	0.1253	0.0499	0.0778
	Chisqure 80	0.8445	0.0305	0.0585	0.2305	0.0007
	Chisqure 100	0.7944	0.1014	0.0171	0.0552	0.0311
	Runs Up - Down	0.4161	0.5565	0.2346	0.0066	0.082
	Runs Above - Below	0.9244	0.7312	0.5993	0.0494	0.489
	Auto Corr Func	0.075594375	0.074092724	0.078960882	0.048316732	0.077622527

Table H.4. The statistical test results: Seed = 1, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**1**	**1**	**1**	**1**	**1**
P-Values	**Kolmogorov Smirnov**	0.0013	0.364	0.1099	0.5349	0.8909
	Chisqure 30	0.3725	0.1228	0.8079	0.7649	0.2767
	Chisqure 50	0.3603	0.4817	0.723	0.9227	0.5568
	Chisqure 80	0.2261	0.0803	0.1467	0.7745	0.1736
	Chisqure 100	0.606	0.2919	0.2169	0.739	0.1602
	Runs Up - Down	0.7167	0.0071	0.7544	0.4019	0.4604
	Runs Above - Below	0.8072	0.1554	0.6883	0.1075	1
	Auto Corr Func	0.049307906	0.03906343	0.03906343	0.101495898	0.053192559

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.0000001**	**0.1**	**0.1**	**0.1**	**0.1**
P-Values	**Kolmogorov Smirnov**	0.8845	0.9714	0.2511	0.5695	0.332
	Chisqure 30	0.8945	0.7987	0.0158	0.8497	0.1775
	Chisqure 50	0.5613	0.9857	0.0141	0.2832	0.0272
	Chisqure 80	0.4281	0.991	0.0023	0.8428	0.0092
	Chisqure 100	0.4545	0.9624	0.0159	0.3792	0.0172
	Runs Up - Down	0.99	0.5906	0.0961	0.5398	0.3611
	Runs Above - Below	0.047	0.6365	0.0039	0.955	0.0534
	Auto Corr Func	0.09151591	0.060073549	0.067779824	0.03572489	0.076475716

Table H.5. The statistical test results: Seed = 9301, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**9301**	**9301**	**9301**	**9301**	**9301**
P-Values	**Kolmogorov Smirnov**	0.1355	0.3397	0.2935	0.1396	0.6486
	Chisqure 30	0.9148	0.7882	0.6772	0.8339	0.5793
	Chisqure 50	0.4966	0.0718	0.8062	0.4534	0.3991
	Chisqure 80	0.7362	0.5628	0.9184	0.8959	0.3909
	Chisqure 100	0.879	0.0421	0.9781	0.3547	0.6536
	Runs Up - Down	0.1975	0.9302	0.8708	0.4019	0.1502
	Runs Above - Below	0.526	0.5929	0.0909	0.3257	0.1172
	Auto Corr Func	0.052164587	0.061685049	0.061685049	0.050099914	0.063276358

		Tent map $\mu = 2$	Tent map $\mu = 2.00005$	T-Logistic map	Connecting map $\alpha = 0.999$	Connecting map $\alpha = 1.01$
	Statistical Tests	**0.00009301**	**0.9301**	**0.9301**	**0.9301**	**0.9301**
P-Values	**Kolmogorov Smirnov**	0.8165	0.4812	0.9943	0.8183	0.9096
	Chisqure 30	0.5336	0.7824	0.9998	0.8487	0.7754
	Chisqure 50	0.135	0.9294	0.9989	0.7967	0.7372
	Chisqure 80	0.0463	0.9641	0.9999	0.1206	0.9065
	Chisqure 100	0.3761	0.8874	0.9993	0.4298	0.9237
	Runs Up - Down	0.2549	0.3611	0.6981	0.5565	0.3744
	Runs Above - Below	0.0597	0.9194	0.4655	0.8282	0.5012
	Auto Corr Func	0.082869823	0.048837575	0.040104555	0.068965176	0.057901614

Table H.6. The statistical test results: Seed = 56623185, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**566231857**	**566231857**	**566231857**	**566231857**	**566231857**
P-Values	**Kolmogorov Smirnov**	0.5124	0.68	0.9282	0.743	0.6372
	Chisqure 30	0.7924	0.7299	0.8142	0.6575	0.2066
	Chisqure 50	0.5395	0.6219	0.4267	0.2254	0.4664
	Chisqure 80	0.6382	0.5205	0.962	0.8618	0.4604
	Chisqure 100	0.411	0.6958	0.7073	0.3648	0.4398
	Runs Up - Down	0.5906	0.9501	0.7544	0.7544	0.8905
	Runs Above - Below	0.5012	0.253	0.9245	0.5115	0.9162
	Auto Corr Func	0.07242455	0.060914871	0.060914871	0.048139405	0.058427481

		Tent map $\mu = 2$	Tent map $\mu = 2.00005$	T-Logistic map	Connecting map $\alpha = 0.999$	Connecting map $\alpha = 1.01$
	Statistical Tests	**0.566231857**	**0.566231857**	**0.566231857**	**0.566231857**	**0.566231857**
P-Values	**Kolmogorov Smirnov**	0.6114	0.7395	0.7832	0.63	0.8757
	Chisqure 30	0.0773	0.2973	0.9495	0.9881	0.9931
	Chisqure 50	0.5848	0.0541	0.989	0.8121	0.4298
	Chisqure 80	0.2955	0.0382	0.9984	0.6836	0.5142
	Chisqure 100	0.5899	0.0077	0.9954	0.47	0.219
	Runs Up - Down	0.5073	0.7544	0.2876	0.6434	0.5906
	Runs Above - Below	0.488	0.1118	0.9365	0.7594	0.5077
	Auto Corr Func	0.044585591	0.045348958	0.061796187	0.072647904	0.034808142

Table H.7. The statistical test results: Seed = 27683, Iteration no = 1000.

	MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
Statistical Tests	**27683**	**27683**	**27683**	**27683**	**27683**
Kolmogorov Smirnov	0.9726	0.5321	0.7959	0.3688	0.3258
Chisqure 30	0.7065	0.1914	0.6808	0.3852	0.2152
Chisqure 50	0.9817	0.43	0.9339	0.4996	0.0589
Chisqure 80	0.7012	0.6803	0.509	0.3064	0.1176
Chisqure 100	0.8432	0.3826	0.8201	0.4119	0.1299
Runs Up - Down	0.7167	0.2249	0.0961	0.9302	0.0206
Runs Above - Below	0.9253	0.5072	0.4514	0.3437	0.0398
Auto Corr Func	0.046164143	0.04709321	0.04709321	0.025309565	0.079654078

	Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
Statistical Tests	**0.00027683**	**0.27683**	**0.27683**	**0.27683**	**0.27683**
Kolmogorov Smirnov	0.9856	0.5634	0.9965	0.5007	0.671
Chisqure 30	0.9857	0.2439	0.9627	0.1704	0.543
Chisqure 50	0.9826	0.1799	0.929	0.125	0.1851
Chisqure 80	0.9254	0.1655	0.5355	0.0024	0.0896
Chisqure 100	0.8922	0.2509	0.7273	0.0362	0.021
Runs Up - Down	0.6981	0.608	0.5073	0.5906	0.5398
Runs Above - Below	0.8345	0.0212	0.5064	0.0264	0.121
Auto Corr Func	0.054509906	0.034047475	0.044060032	0.075014771	0.057864743

Table H.8. The statistical test results: Seed = 77, Iteration no = 1000.

	MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
Statistical Tests	**77**	**77**	**77**	**77**	**77**
Kolmogorov Smirnov	0.1791	0.6366	0.8947	0.9001	0.9873
Chisqure 30	0.8873	0.6839	0.355	0.552	0.3068
Chisqure 50	0.9837	0.8734	0.2857	0.0763	0.588
Chisqure 80	0.8333	0.8508	0.0579	0.3379	0.4078
Chisqure 100	0.9558	0.8571	0.2309	0.1277	0.7413
Runs Up - Down	0.2064	0.1807	0.99	0.4161	0.7735
Runs Above - Below	0.4348	0.6526	0.0299	0.7763	0.27
Auto Corr Func	0.062412062	0.030946895	0.030946895	0.037639745	0.049394469

	Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
Statistical Tests	**0.00000077**	**0.77**	**0.77**	**0.77**	**0.77**
Kolmogorov Smirnov	0.691	0.3115	0.6479	0.153	0.7991
Chisqure 30	0.051	0.3983	0.2678	0.0339	0.7502
Chisqure 50	0.0031	0.8483	0.4609	0.0776	0.9453
Chisqure 80	0.0156	0.8865	0.2362	0.0878	0.8835
Chisqure 100	0.0002	0.6226	0.4202	0.2609	0.7683
Runs Up - Down	0.6615	0.4161	0.2655	0.0367	0.6615
Runs Above - Below	0.0506	0.3963	0.8622	0.1066	0.3269
Auto Corr Func	0.081271172	0.06474141	0.052172116	0.060074263	0.052296231

Table H.9. The statistical test results: Seed = 74601189, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**746011899**	**746011899**	**746011899**	**746011899**	**746011899**
P-Values	**Kolmogorov Smirnov**	0.2832	0.6264	0.1308	0.3553	0.2401
	Chisqure 30	0.3071	0.3643	0.7256	0.7083	0.2464
	Chisqure 50	0.8634	0.3591	0.8311	0.384	0.2585
	Chisqure 80	0.9881	0.5814	0.8284	0.3464	0.6213
	Chisqure 100	0.9669	0.671	0.6011	0.2793	0.8509
	Runs Up - Down	0.99	0.4914	0.6434	0.4604	0.7544
	Runs Above - Below	0.2435	0.3918	0.5977	0.4291	0.0803
	Auto Corr Func	0.060209218	0.052427678	0.052427678	0.02325975	0.038612762

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.746011899**	**0.746011899**	**0.746011899**	**0.746011899**	**0.746011899**
P-Values	**Kolmogorov Smirnov**	0.0013	0.554	0.8106	0.47	0.8403
	Chisqure 30	0.0001	0.8612	0.8846	0.9476	0.4671
	Chisqure 50	0.0002	0.5896	0.8205	0.3591	0.428
	Chisqure 80	0.00001	0.0711	0.9674	0.6861	0.6495
	Chisqure 100	0.00001	0.2045	0.9038	0.4295	0.8819
	Runs Up - Down	0.0028	0.4453	0.6981	0.1807	0.5906
	Runs Above - Below	0.0007	0.6011	0.7767	0.5507	0.4926
	Auto Corr Func	0.052718056	0.045422488	0.03870393	0.059067713	0.060941231

APPENDIX I: THE STATISTICAL TEST RESULTS - 2

Table I.1. The statistical test results: Seed = 89, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**89**	**89**	**89**	**89**	**89**
P-Values	**Kolmogorov Smirnov**	0.2122	0.3793	0.0792	0.3075	0.7566
	Chisqure 30	0.7812	0.4088	0.1166	0.9722	0.3442
	Chisqure 50	0.9155	0.1503	0.1301	0.6307	0.4922
	Chisqure 80	0.5311	0.6268	0.0112	0.7088	0.5862
	Chisqure 100	0.9456	0.1793	0.0358	0.8877	0.6331
	Runs Up - Down	0.5906	0.5398	0.1301	0.9501	0.7167
	Runs Above - Below	0.7063	0.5065	0.4904	0.3985	0.9602
	Auto Corr Func	0.04282828	0.061425047	0.061425047	0.032351294	0.056207467

		Tent map $\mu = 2$	Tent map $\mu = 2.00005$	T-Logistic map	Connecting map $\alpha = 0.999$	Connecting map $\alpha = 1.01$
	Statistical Tests	**0.00000089**	**0.89**	**0.89**	**0.89**	**0.89**
P-Values	**Kolmogorov Smirnov**	0.3991	0.4322	0.9947	0.995	0.9737
	Chisqure 30	0.7353	0.6397	0.9845	0.9779	0.9992
	Chisqure 50	0.8372	0.2596	0.9451	0.9993	0.4772
	Chisqure 80	0.967	0.3127	0.9435	0.9998	0.9124
	Chisqure 100	0.7028	0.2421	0.9793	0.9993	0.4299
	Runs Up - Down	0.0736	0.3354	0.5565	0.608	0.608
	Runs Above - Below	0.0087	0.3269	0.1802	0.3932	0.2174
	Auto Corr Func	0.073994131	0.060680078	0.054923486	0.058159962	0.078978695

Table I.2. The statistical test results: Seed = 590223, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**590223**	**590223**	**590223**	**590223**	**590223**
P-Values	**Kolmogorov Smirnov**	0.0868	0.7076	0.3013	0.6596	0.8401
	Chisqure 30	0.4887	0.5916	0.2932	0.0231	0.2047
	Chisqure 50	0.5318	0.7237	0.4924	0.0049	0.3402
	Chisqure 80	0.0326	0.2738	0.8335	0.0732	0.1151
	Chisqure 100	0.1757	0.0703	0.2844	0.0315	0.3884
	Runs Up - Down	0.7735	0.1727	0.6615	0.6981	0.4453
	Runs Above - Below	0.7324	0.0504	0.7302	0.526	0.9744
	Auto Corr Func	0.078492259	0.065748521	0.065748521	0.046377118	0.045607639

		Tent map $\mu = 2$	Tent map $\mu = 2.00005$	T-Logistic map	Connecting map $\alpha = 0.999$	Connecting map $\alpha = 1.01$
	Statistical Tests	**0.00590223**	**0.590223**	**0.590223**	**0.590223**	**0.590223**
P-Values	**Kolmogorov Smirnov**	0.8313	0.9227	0.6355	0.9827	0.3486
	Chisqure 30	0.6032	0.2941	0.1394	0.8406	0.8988
	Chisqure 50	0.9443	0.1729	0.0565	0.9987	0.6785
	Chisqure 80	0.7747	0.1992	0.0083	0.996	0.8866
	Chisqure 100	0.8235	0.2331	0.0537	0.9998	0.8264
	Runs Up - Down	0.2876	0.5398	0.3354	0.9501	0.2876
	Runs Above - Below	0.782	0.5915	0.0266	0.4705	0.3801
	Auto Corr Func	0.077642041	0.069530951	0.078744227	0.032226744	0.066968828

Table I.3. The statistical test results: Seed = 39818, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**39818317**	**39818317**	**39818317**	**39818317**	**39818317**
P-Values	**Kolmogorov Smirnov**	0.0366	0.2868	0.7681	0.2779	0.1318
	Chisqure 30	0.122	0.1843	0.8833	0.3476	0.3545
	Chisqure 50	0.26	0.1711	0.9508	0.2663	0.2136
	Chisqure 80	0.569	0.1859	0.9922	0.345	0.2351
	Chisqure 100	0.2323	0.0347	0.9815	0.6502	0.2935
	Runs Up - Down	0.3611	0.1727	0.5073	0.7544	0.1807
	Runs Above - Below	0.3595	0.5012	0.768	0.2979	0.8212
	Auto Corr Func	0.057665144	0.043983037	0.043983037	0.043602274	0.083676217

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.39818317**	**0.39818317**	**0.39818317**	**0.39818317**	**0.39818317**
P-Values	**Kolmogorov Smirnov**	0.9985	0.9451	0.7561	0.9995	0.9067
	Chisqure 30	0.6739	0.9963	0.9965	0.9761	0.2577
	Chisqure 50	0.7773	1	0.8398	0.9878	0.0843
	Chisqure 80	0.5118	0.9989	0.9702	0.988	0.0506
	Chisqure 100	0.4988	0.9999	0.7586	0.9925	0.1311
	Runs Up - Down	0.6434	0.9501	0.4453	0.7167	0.8316
	Runs Above - Below	0.8256	0.5906	0.2404	0.359	0.4278
	Auto Corr Func	0.051615347	0.040935273	0.054644522	0.053316373	0.054755869

Table I.4. The statistical test results: Seed = 123, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**123**	**123**	**123**	**123**	**123**
P-Values	**Kolmogorov Smirnov**	0.8649	0.424	0.7317	0.5414	0.8847
	Chisqure 30	0.6519	0.9765	0.1976	0.8888	0.931
	Chisqure 50	0.1261	0.9968	0.4968	0.9663	0.9806
	Chisqure 80	0.7639	0.777	0.3852	0.7894	0.9224
	Chisqure 100	0.3634	0.9945	0.676	0.8908	0.6897
	Runs Up - Down	0.3354	0.7544	0.6981	0.608	0.3354
	Runs Above - Below	0.5481	0.9752	0.1209	0.3578	0.9975
	Auto Corr Func	0.054357109	0.041662485	0.041662485	0.06630245	0.066449165

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00000123**	**0.123**	**0.123**	**0.123**	**0.123**
P-Values	**Kolmogorov Smirnov**	0.1289	0.9627	0.2255	0.3295	0.2717
	Chisqure 30	0.1598	0.6327	0.2974	0.2808	0.3136
	Chisqure 50	0.2975	0.7458	0.7225	0.357	0.5307
	Chisqure 80	0.7075	0.7622	0.2494	0.728	0.7139
	Chisqure 100	0.5289	0.3678	0.5185	0.5722	0.431
	Runs Up - Down	0.1975	0.6981	0.1178	0.3354	0.1727
	Runs Above - Below	0.0324	0.9784	0.0247	0.0026	0.1881
	Auto Corr Func	0.058645225	0.073996154	0.064102881	0.045843661	0.062965321

Table I.5. The statistical test results: Seed = 6979, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**6979**	**6979**	**6979**	**6979**	**6979**
P-Values	**Kolmogorov Smirnov**	0.5415	0.6464	0.0758	0.6891	0.3491
	Chisqure 30	0.4802	0.2621	0.0067	0.8847	0.3925
	Chisqure 50	0.2206	0.9465	0.0237	0.6145	0.5756
	Chisqure 80	0.2038	0.012	0.0796	0.2717	0.2229
	Chisqure 100	0.0704	0.1501	0.0211	0.3627	0.2117
	Runs Up - Down	0.4914	0.0589	0.99	0.4019	0.4914
	Runs Above - Below	0.27	0.3809	0.7806	0.1632	0.111
	Auto Corr Func	0.094986885	0.061999049	0.061999049	0.063253432	0.041412525

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00006979**	**0.6979**	**0.6979**	**0.6979**	**0.6979**
P-Values	**Kolmogorov Smirnov**	0.9559	0.3213	0.0777	0.9805	0.6432
	Chisqure 30	0.5551	0.6588	0.1065	0.9944	0.4722
	Chisqure 50	0.6405	0.4969	0.0054	0.9913	0.1874
	Chisqure 80	0.7585	0.2092	0.0396	0.9594	0.3807
	Chisqure 100	0.7193	0.4485	0.0063	0.9474	0.4327
	Runs Up - Down	0.7167	0.1975	0.0066	0.2991	0.7167
	Runs Above - Below	0.6805	0.0458	0.5622	0.2965	0.6625
	Auto Corr Func	0.042778599	0.048448651	0.1185108	0.044664927	0.069337217

Table I.6. The statistical test results: Seed = 821129, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**821129**	**821129**	**821129**	**821129**	**821129**
P-Values	**Kolmogorov Smirnov**	0.5623	0.0663	0.6348	0.9789	0.9586
	Chisqure 30	0.8294	0.3442	0.3194	0.8355	0.5731
	Chisqure 50	0.3008	0.5206	0.5411	0.5509	0.247
	Chisqure 80	0.5943	0.3315	0.381	0.9459	0.3826
	Chisqure 100	0.5941	0.1989	0.7708	0.6684	0.4178
	Runs Up - Down	0.7167	0.2991	0.1727	0.4914	0.99
	Runs Above - Below	0.5012	0.5758	0.644	1	0.4629
	Auto Corr Func	0.041730028	0.072120044	0.072120044	0.053042563	0.052647811

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00821129**	**0.821129**	**0.821129**	**0.821129**	**0.821129**
P-Values	**Kolmogorov Smirnov**	0.969	0.7908	0.9984	0.982	0.6557
	Chisqure 30	0.8377	0.9883	0.9322	0.9971	0.8422
	Chisqure 50	0.5851	0.9763	0.9678	0.823	0.6638
	Chisqure 80	0.6233	0.9875	0.9235	0.9087	0.8406
	Chisqure 100	0.4635	0.9759	0.9632	0.6604	0.8815
	Runs Up - Down	0.9501	0.7167	0.5906	0.8905	0.1975
	Runs Above - Below	0.6794	0.9308	0.3973	0.6383	0.8886
	Auto Corr Func	0.069861839	0.069861839	0.062599061	0.043226991	0.066413134

Table I.7. The statistical test results: Seed = 91561202, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**915612027**	**915612027**	**915612027**	**915612027**	**915612027**
P-Values	**Kolmogorov Smirnov**	0.0154	0.0698	0.0601	0.9929	0.9647
	Chisqure 30	0.7861	0.1085	0.6298	0.8828	0.4165
	Chisqure 50	0.723	0.0092	0.2625	0.7926	0.4206
	Chisqure 80	0.9051	0.0441	0.2258	0.9023	0.5242
	Chisqure 100	0.6507	0.0317	0.2482	0.8767	0.2373
	Runs Up - Down	0.1012	0.99	0.0623	0.2991	0.323
	Runs Above - Below	0.2802	0.9226	0.394	0.9797	0.6775
	Auto Corr Func	0.059462529	0.057422749	0.057422749	0.07011717	0.065801652

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.915612027**	**0.915612027**	**0.915612027**	**0.915612027**	**0.915612027**
P-Values	**Kolmogorov Smirnov**	0.0751	0.2069	0.3533	0.492	0.0659
	Chisqure 30	0.4515	0.055	0.0289	0.9026	0.039
	Chisqure 50	0.1901	0.1234	0.5296	0.4521	0.0774
	Chisqure 80	0.7253	0.2414	0.2418	0.7738	0.3364
	Chisqure 100	0.1461	0.105	0.2902	0.5294	0.4168
	Runs Up - Down	0.0128	0.0589	0.2249	0.3744	0.0495
	Runs Above - Below	0.0173	0.0246	0.0711	0.1562	1
	Auto Corr Func	0.084997272	0.052317922	0.051399705	0.060246897	0.070297064

Table I.8. The statistical test results: Seed = 112319, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**112319**	**112319**	**112319**	**112319**	**112319**
P-Values	**Kolmogorov Smirnov**	0.6442	0.7944	0.3427	0.4552	0.4094
	Chisqure 30	0.6366	0.3327	0.4707	0.8791	0.2772
	Chisqure 50	0.2562	0.5603	0.2493	0.7178	0.4318
	Chisqure 80	0.8022	0.3505	0.4338	0.9399	0.251
	Chisqure 100	0.338	0.4275	0.2501	0.7144	0.3648
	Runs Up - Down	0.5906	0.7167	0.7735	0.1301	0.5398
	Runs Above - Below	0.9698	0.9218	0.175	0.68	0.7724
	Auto Corr Func	0.061878599	0.043714351	0.043714351	0.038436559	0.039492864

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00112319**	**0.112319**	**0.112319**	**0.112319**	**0.112319**
P-Values	**Kolmogorov Smirnov**	0.9232	0.7599	0.8736	0.8411	0.5873
	Chisqure 30	0.9315	0.9356	0.0863	0.9955	0.2144
	Chisqure 50	0.8764	0.9936	0.0193	0.9974	0.4215
	Chisqure 80	0.9265	0.9777	0.0032	0.8447	0.0524
	Chisqure 100	0.8566	0.2876	0.0057	0.7807	0.3029
	Runs Up - Down	0.5073	0.2876	0.4453	0.6981	0.99
	Runs Above - Below	0.68	0.9744	0.3457	0.5893	0.7324
	Auto Corr Func	0.038210798	0.061486134	0.046110423	0.0595149	0.088853209

Table I.9. The statistical test results: Seed = 277, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**277**	**277**	**277**	**277**	**277**
P-Values	**Kolmogorov Smirnov**	0.6414	0.2322	0.816	0.8735	0.4043
	Chisqure 30	0.9989	0.1088	0.8353	0.6842	0.2552
	Chisqure 50	0.8946	0.1341	0.8975	0.7512	0.0783
	Chisqure 80	0.9192	0.103	0.9232	0.9762	0.3722
	Chisqure 100	0.8931	0.0639	0.9613	0.8992	0.265
	Runs Up - Down	0.8121	0.8708	0.4604	0.8708	0.99
	Runs Above - Below	0.1066	0.5104	0.4271	0.8547	0.7277
	Auto Corr Func	0.056743439	0.042435139	0.042435139	0.033698601	0.043130747

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00000277**	**0.277**	**0.277**	**0.277**	**0.277**
P-Values	**Kolmogorov Smirnov**	0.4999	0.9803	0.967	0.4478	0.672
	Chisqure 30	0.7284	0.9835	0.2492	0.5859	0.7342
	Chisqure 50	0.6104	0.999	0.7584	0.5048	0.2048
	Chisqure 80	0.8784	0.8216	0.5546	0.8525	0.5864
	Chisqure 100	0.1012	0.9867	0.7806	0.8954	0.6063
	Runs Up - Down	0.4604	0.7167	0.2991	0.2064	0.9302
	Runs Above - Below	0.1373	0.7293	0.5578	0.0294	0.1098
	Auto Corr Func	0.079287835	0.025937683	0.065660781	0.096450451	0.05729885

Table I.10. The statistical test results: Seed = 4612967, Iteration no = 1000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**4612967**	**4612967**	**4612967**	**4612967**	**4612967**
P-Values	**Kolmogorov Smirnov**	0.8001	0.5005	0.2396	0.6732	0.7768
	Chisqure 30	0.3457	0.1223	0.2407	0.0495	0.5195
	Chisqure 50	0.6553	0.0463	0.0602	0.3468	0.8726
	Chisqure 80	0.3384	0.1235	0.0638	0.6984	0.8661
	Chisqure 100	0.6164	0.0248	0.0638	0.3658	0.9803
	Runs Up - Down	0.1121	0.608	0.9501	0.8905	0.8121
	Runs Above - Below	0.6552	0.8743	0.559	0.4661	0.1328
	Auto Corr Func	0.055841251	0.0552054	0.0552054	0.05628132	0.050851706

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.04612967**	**0.4612967**	**0.4612967**	**0.4612967**	**0.4612967**
P-Values	**Kolmogorov Smirnov**	0.5624	0.2822	0.8229	0.6198	0.3824
	Chisqure 30	0.3215	0.82	0.1532	0.3968	0.0472
	Chisqure 50	0.2352	0.6407	0.0871	0.7562	0.2278
	Chisqure 80	0.1809	0.671	0.0098	0.8004	0.2027
	Chisqure 100	0.2308	0.8727	0.0212	0.9242	0.2957
	Runs Up - Down	0.3354	0.1365	0.99	0.1807	0.1121
	Runs Above - Below	0.1099	0.062	0.5578	0.1381	0.0619
	Auto Corr Func	0.05630209	0.046407432	0.100544412	0.075782829	0.068564728

APPENDIX J: THE STATISTICAL TEST RESULTS - 3

Table J.1. The statistical test results: Seed = 449, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**449**	**449**	**449**	**449**	**449**
P-Values	**Kolmogorov Smirnov**	0.6465	0.2998	0.519	0.1186	0.9744
	Chisqure 30	0.4894	0.4456	0.7261	0.3321	0.9783
	Chisqure 50	0.3298	0.8096	0.7358	0.3599	0.9107
	Chisqure 80	0.3034	0.8446	0.8266	0.3828	0.7429
	Chisqure 100	0.078	0.9781	0.8337	0.5864	0.8955
	Runs Up - Down	0.8712	0.5041	0.5881	0.7972	0.6156
	Runs Above - Below	0.8323	0.726	0.3734	0.7108	0.4295
	Auto Corr Func	0.017293476	0.023008912	0.023008912	0.014440308	0.017418483

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00000449**	**0.449**	**0.449**	**0.449**	**0.449**
P-Values	**Kolmogorov Smirnov**	0.8925	0.7593	0.2913	0.8334	0.0625
	Chisqure 30	0.9916	0.0475	0.3676	0.9489	0.5223
	Chisqure 50	0.9519	0.0043	0.4823	0.4997	0.2197
	Chisqure 80	0.9119	0.0174	0.5491	0.9766	0.3876
	Chisqure 100	0.6883	0.0005	0.5796	0.5246	0.1577
	Runs Up - Down	0.3328	0.6551	0.2279	0.3096	0.0615
	Runs Above - Below	0.6965	0.3635	0.1585	0.3954	0.5981
	Auto Corr Func	0.014646416	0.022027571	0.019776283	0.018477826	0.018992281

Table J.2. The statistical test results: Seed = 181081, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**181081**	**181081**	**181081**	**181081**	**181081**
P-Values	**Kolmogorov Smirnov**	0.4499	0.5931	0.0994	0.8459	0.067
	Chisqure 30	0.4876	0.0966	0.2259	0.7973	0.6906
	Chisqure 50	0.335	0.5626	0.267	0.8485	0.8091
	Chisqure 80	0.161	0.2047	0.5786	0.8989	0.5191
	Chisqure 100	0.1333	0.5087	0.207	0.9599	0.7428
	Runs Up - Down	0.1025	0.4087	0.959	0.1629	0.7072
	Runs Above - Below	0.9943	0.2645	0.1717	0.8335	0.0308
	Auto Corr Func	0.019454107	0.028839553	0.028839553	0.014827983	0.023399574

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00181081**	**0.181081**	**0.181081**	**0.181081**	**0.181081**
P-Values	**Kolmogorov Smirnov**	0.1621	0.1603	0.0043	0.7501	0.5182
	Chisqure 30	0.0438	0.1026	0.00001	0.9153	0.17
	Chisqure 50	0.0045	0.0703	0.0228	0.8549	0.2528
	Chisqure 80	0.0012	0.1661	0.0417	0.9383	0.215
	Chisqure 100	0.0053	0.0217	0.0802	0.9701	0.3327
	Runs Up - Down	0.1989	0.0572	0.0044	0.5558	0.494
	Runs Above - Below	1	0.858	0.0079	0.7264	0.0086
	Auto Corr Func	0.021608613	0.015353979	0.025427231	0.011357471	0.021626626

Table J.3. The statistical test results: Seed = 890123, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**890123621**	**890123621**	**890123621**	**890123621**	**890123621**
P-Values	**Kolmogorov Smirnov**	0.3722	0.9998	0.2569	0.6036	0.9715
	Chisqure 30	0.5245	0.9986	0.2469	0.6887	0.7792
	Chisqure 50	0.243	0.9664	0.781	0.775	0.8471
	Chisqure 80	0.6551	0.9908	0.4101	0.7059	0.4813
	Chisqure 100	0.2251	0.9017	0.5796	0.7547	0.6079
	Runs Up - Down	0.2985	0.391	0.1828	0.1629	0.0615
	Runs Above - Below	0.6816	0.0365	0.0461	0.277	0.1369
	Auto Corr Func	0.01701497	0.028076533	0.028076533	0.020285054	0.008662253

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.890123621**	**0.890123621**	**0.890123621**	**0.890123621**	**0.890123621**
P-Values	**Kolmogorov Smirnov**	0.9614	0.9785	0.882	0.991	0.0078
	Chisqure 30	0.5321	0.9082	0.3365	0.7884	0.018
	Chisqure 50	0.179	0.883	0.1236	0.9304	0.3262
	Chisqure 80	0.2543	0.9062	0.6705	0.4754	0.3164
	Chisqure 100	0.1251	0.5858	0.2237	0.7594	0.7322
	Runs Up - Down	0.8279	0.7548	0.9968	0.5719	0.0019
	Runs Above - Below	0.6677	0.8015	0.1262	0.4667	0.0615
	Auto Corr Func	0.031997261	0.020981078	0.016416721	0.019948318	0.016935344

Table J.4. The statistical test results: Seed = 1, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**1**	**1**	**1**	**1**	**1**
P-Values	**Kolmogorov Smirnov**	0.2144	0.7086	0.4354	0.3475	0.0739
	Chisqure 30	0.379	0.753	0.6076	0.3991	0.1484
	Chisqure 50	0.0341	0.1972	0.4929	0.7444	0.3702
	Chisqure 80	0.0188	0.6235	0.156	0.9388	0.2852
	Chisqure 100	0.0041	0.5518	0.5076	0.6937	0.2356
	Runs Up - Down	0.4646	0.1776	0.5506	0.638	0.5881
	Runs Above - Below	0.6263	0.2127	0.9271	0.0803	0.2244
	Auto Corr Func	0.020463228	0.015119696	0.015119696	0.015201131	0.018853987

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.0000001**	**0.1**	**0.1**	**0.1**	**0.1**
P-Values	**Kolmogorov Smirnov**	0.4484	0.0337	0.4884	0.9044	0.643
	Chisqure 30	0.1395	0.0194	0.3368	0.7245	0.6095
	Chisqure 50	0.1077	0.0628	0.7652	0.9125	0.4148
	Chisqure 80	0.494	0.1212	0.3391	0.5297	0.0804
	Chisqure 100	0.1338	0.0159	0.9081	0.9883	0.2013
	Runs Up - Down	0.3823	0.0225	0.2101	0.4223	0.8464
	Runs Above - Below	0.9959	0.9123	0.4085	0.847	0.1161
	Auto Corr Func	0.033430041	0.027263319	0.014372213	0.021939378	0.015567442

Table J.5. The statistical test results: Seed = 9301, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**9301**	**9301**	**9301**	**9301**	**9301**
P-Values	**Kolmogorov Smirnov**	0.8631	0.5784	0.8129	0.66	0.514
	Chisqure 30	0.8658	0.3019	0.2196	0.3756	0.1395
	Chisqure 50	0.1519	0.2196	0.6525	0.5222	0.0544
	Chisqure 80	0.4872	0.6598	0.9019	0.5116	0.1541
	Chisqure 100	0.0957	0.2499	0.9085	0.5415	0.1127
	Runs Up - Down	0.8156	0.3172	0.5665	0.277	0.2073
	Runs Above - Below	0.5152	0.7113	0.0907	0.2851	0.3736
	Auto Corr Func	0.017429212	0.014230102	0.014230102	0.014246685	0.018586687

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00009301**	**0.9301**	**0.9301**	**0.9301**	**0.9301**
P-Values	**Kolmogorov Smirnov**	0.8283	0.4981	0.2132	0.9994	0.5501
	Chisqure 30	0.981	0.7709	0.7251	0.7549	0.9611
	Chisqure 50	0.9888	0.399	0.3452	0.8292	0.7781
	Chisqure 80	0.9992	0.8703	0.728	0.8076	0.9751
	Chisqure 100	0.9948	0.64	0.2008	0.7879	0.8178
	Runs Up - Down	0.6156	0.2876	0.1605	0.6839	0.3288
	Runs Above - Below	0.3027	0.016	0.1074	0.5282	0.0314
	Auto Corr Func	0.017268807	0.019760127	0.020379339	0.03040788	0.020844143

Table J.6. The statistical test results: Seed = 566231, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**566231857**	**566231857**	**566231857**	**566231857**	**566231857**
P-Values	**Kolmogorov Smirnov**	0.1708	0.1722	0.9606	0.8479	0.6853
	Chisqure 30	0.1342	0.2476	0.2287	0.6894	0.5966
	Chisqure 50	0.1075	0.1629	0.8127	0.4597	0.7661
	Chisqure 80	0.0324	0.4412	0.9437	0.6297	0.3946
	Chisqure 100	0.0408	0.1224	0.6311	0.3766	0.5427
	Runs Up - Down	0.6723	0.2533	0.5719	0.2189	0.5193
	Runs Above - Below	0.9555	0.0886	0.3142	0.0877	0.131
	Auto Corr Func	0.014291193	0.013761149	0.013761149	0.016228158	0.012517501

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.566231857**	**0.566231857**	**0.566231857**	**0.566231857**	**0.566231857**
P-Values	**Kolmogorov Smirnov**	0.8315	0.9646	0.5064	0.1352	0.221
	Chisqure 30	0.5156	0.4778	0.7141	0.0313	0.0321
	Chisqure 50	0.0708	0.2848	0.9968	0.0346	0.0055
	Chisqure 80	0.0573	0.7118	0.9373	0.0957	0.0003
	Chisqure 100	0.0225	0.1542	0.1629	0.0162	0.0071
	Runs Up - Down	0.8837	0.4891	0.1629	0.2735	0.7548
	Runs Above - Below	0.1094	0.9435	0.1772	0.146	0.0095
	Auto Corr Func	0.017257342	0.01958584	0.010897262	0.021038092	0.019383905

Table J.7. The statistical test results: Seed = 27683, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**27683**	**27683**	**27683**	**27683**	**27683**
P-Values	**Kolmogorov Smirnov**	0.4544	0.2681	0.6632	0.2995	0.4299
	Chisqure 30	0.433	0.9893	0.225	0.9481	0.0603
	Chisqure 50	0.2742	0.4396	0.8758	0.9265	0.1651
	Chisqure 80	0.4493	0.4165	0.7805	0.7063	0.156
	Chisqure 100	0.185	0.2703	0.6122	0.6282	0.0414
	Runs Up - Down	0.3448	0.2735	0.9213	0.638	0.834
	Runs Above - Below	0.6656	0.9438	0.1013	0.3317	0.5421
	Auto Corr Func	0.018287319	0.01551624	0.01551624	0.016971693	0.017211663

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00027683**	**0.27683**	**0.27683**	**0.27683**	**0.27683**
P-Values	**Kolmogorov Smirnov**	0.6901	0.7935	0.2251	0.4468	0.9177
	Chisqure 30	0.8371	0.9048	0.9021	0.024	0.8645
	Chisqure 50	0.2532	0.9638	0.4442	0.0078	0.0696
	Chisqure 80	0.4436	0.9631	0.5275	0.0087	0.4887
	Chisqure 100	0.1425	0.924	0.6212	0.0059	0.2586
	Runs Up - Down	0.1425	0.2876	0.149	0.4743	0.5558
	Runs Above - Below	0.267	0.2188	0.6566	0.0314	0.5961
	Auto Corr Func	0.015073741	0.008966916	0.011319264	0.015869356	0.017443941

Table J.8. The statistical test results: Seed = 77, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**77**	**77**	**77**	**77**	**77**
P-Values	**Kolmogorov Smirnov**	0.8397	0.6465	0.3475	0.5737	0.9632
	Chisqure 30	0.9552	0.7872	0.5174	0.2582	0.8178
	Chisqure 50	0.2832	0.9083	0.5037	0.4721	0.4904
	Chisqure 80	0.6584	0.4014	0.4642	0.4642	0.612
	Chisqure 100	0.2692	0.647	0.2686	0.513	0.2378
	Runs Up - Down	0.9968	0.5558	0.834	0.1241	0.8279
	Runs Above - Below	0.1901	0.1523	0.4179	0.1709	0.9254
	Auto Corr Func	0.017478759	0.011501058	0.011501058	0.016441879	0.028216088

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00000077**	**0.77**	**0.77**	**0.77**	**0.77**
P-Values	**Kolmogorov Smirnov**	0.9999	0.2174	0.1689	0.9521	0.9709
	Chisqure 30	0.9966	0.1195	0.0885	0.9685	0.2334
	Chisqure 50	0.9846	0.4636	0.0989	0.7812	0.3962
	Chisqure 80	1	0.3861	0.0272	0.8483	0.3454
	Chisqure 100	0.9955	0.5268	0.1382	0.4589	0.2201
	Runs Up - Down	0.779	0.0649	0.1425	0.6045	0.9464
	Runs Above - Below	0.3854	0.0085	0.0047	0.8965	0.8807
	Auto Corr Func	0.010996194	0.015907782	0.024853115	0.015685976	0.029931379

Table J.9. The statistical test results: Seed = 746011, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**746011899**	**746011899**	**746011899**	**746011899**	**746011899**
P-Values	**Kolmogorov Smirnov**	0.4362	0.6578	0.4298	0.9558	0.5953
	Chisqure 30	0.4362	0.6273	0.7183	0.8183	0.9422
	Chisqure 50	0.216	0.327	0.6768	0.8601	0.9971
	Chisqure 80	0.328	0.7333	0.387	0.8958	0.9858
	Chisqure 100	0.1451	0.4501	0.5589	0.9412	0.9501
	Runs Up - Down	0.2016	0.0976	0.8464	0.0841	0.9401
	Runs Above - Below	0.897	0.0136	0.1586	0.977	0.0867
	Auto Corr Func	0.017568745	0.024075737	0.024075737	0.014605701	0.018608589

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.746011899**	**0.746011899**	**0.746011899**	**0.746011899**	**0.746011899**
P-Values	**Kolmogorov Smirnov**	0.4078	0.3733	0.9866	0.5638	0.6098
	Chisqure 30	0.6683	0.4836	0.9452	0.0715	0.6418
	Chisqure 50	0.3184	0.9495	0.7509	0.0754	0.8023
	Chisqure 80	0.3482	0.9848	0.9854	0.2438	0.8269
	Chisqure 100	0.2005	0.9715	0.8601	0.2169	0.4266
	Runs Up - Down	0.277	0.1751	0.725	0.725	0.2372
	Runs Above - Below	0.2853	0.009	0.9762	0.1471	0.2501
	Auto Corr Func	0.024937472	0.017395836	0.010613236	0.018117446	0.021837637

APPENDIX K: THE STATISTICAL TEST RESULTS - 4

Table K.1. The statistical test results: Seed = 89, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**89**	**89**	**89**	**89**	**89**
P-Values	**Kolmogorov Smirnov**	0.7951	0.6767	0.8798	0.5187	0.8943
	Chisqure 30	0.8738	0.2802	0.3002	0.5002	0.7962
	Chisqure 50	0.4784	0.5558	0.1057	0.7979	0.8802
	Chisqure 80	0.6656	0.5989	0.0634	0.7279	0.9112
	Chisqure 100	0.2456	0.5772	0.1914	0.5511	0.9494
	Runs Up - Down	0.9968	0.2735	0.1009	0.3448	0.4087
	Runs Above - Below	0.3507	0.9921	0.7264	0.4995	0.0993
	Auto Corr Func	0.023081921	0.024543547	0.024543547	0.019753245	0.019166929

		Tent map $\mu = 2$	Tent map $\mu = 2.00005$	T-Logistic map	Connecting map $\alpha = 0.999$	Connecting map $\alpha = 1.01$
	Statistical Tests	**0.00000089**	**0.89**	**0.89**	**0.89**	**0.89**
P-Values	**Kolmogorov Smirnov**	0.6311	0.171	0.3156	0.87	0.7954
	Chisqure 30	0.794	0.0492	0.9431	0.7813	0.8606
	Chisqure 50	0.7681	0.0768	0.8568	0.1814	0.5784
	Chisqure 80	0.5932	0.1844	0.9188	0.6867	0.9032
	Chisqure 100	0.8204	0.042	0.9303	0.2078	0.9329
	Runs Up - Down	0.719	0.1854	0.0442	0.5091	0.8094
	Runs Above - Below	0.3654	0.7921	0.2762	1	0.161
	Auto Corr Func	0.012229834	0.017521686	0.011678062	0.013353291	0.014822285

Table K.2. The statistical test results: Seed = 590223, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**590223**	**590223**	**590223**	**590223**	**590223**
P-Values	**Kolmogorov Smirnov**	0.8679	0.3157	0.8726	0.3319	0.1611
	Chisqure 30	0.6737	0.6027	0.7974	0.7942	0.5738
	Chisqure 50	0.2351	0.612	0.8724	0.9603	0.6065
	Chisqure 80	0.1797	0.37	0.8286	0.5623	0.6891
	Chisqure 100	0.1076	0.3286	0.6885	0.4142	0.713
	Runs Up - Down	0.1076	0.7972	0.1059	0.5881	0.0869
	Runs Above - Below	0.9289	0.9608	0.5263	0.4172	0.1362
	Auto Corr Func	0.017346572	0.017867838	0.017867838	0.015148013	0.020250812

		Tent map $\mu = 2$	Tent map $\mu = 2.00005$	T-Logistic map	Connecting map $\alpha = 0.999$	Connecting map $\alpha = 1.01$
	Statistical Tests	**0.00590223**	**0.590223**	**0.590223**	**0.590223**	**0.590223**
P-Values	**Kolmogorov Smirnov**	0.883	0.6312	0.0566	0.0367	0.9821
	Chisqure 30	0.9491	0.998	0.2024	0.1912	0.7335
	Chisqure 50	0.9842	1	0.6225	0.0912	0.8758
	Chisqure 80	0.725	0.9965	0.4422	0.1543	0.5624
	Chisqure 100	0.9707	1	0.4776	0.2221	0.8215
	Runs Up - Down	0.5827	0.2467	0.0468	0.0096	0.4891
	Runs Above - Below	0.557	0.3031	0.0446	0.0043	0.8808
	Auto Corr Func	0.009641777	0.01469662	0.02390522	0.015736155	0.013656965

Table K.3. The statistical test results: Seed = 39818317, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**39818317**	**39818317**	**39818317**	**39818317**	**39818317**
P-Values	Kolmogorov Smirnov	0.3309	0.7338	0.9013	0.5376	0.664
	Chisqure 30	0.5655	0.26	0.9484	0.541	0.1402
	Chisqure 50	0.3343	0.64	0.9605	0.6877	0.8115
	Chisqure 80	0.2734	0.2854	0.755	0.9313	0.7149
	Chisqure 100	0.0745	0.0347	0.9913	0.8849	0.799
	Runs Up - Down	0.1701	0.8156	0.2101	0.4792	0.4223
	Runs Above - Below	0.8463	0.8788	0.7703	0.1036	0.8501
	Auto Corr Func	0.016139271	0.014408079	0.014408079	0.015280884	0.019894086

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.39818317**	**0.39818317**	**0.39818317**	**0.39818317**	**0.39818317**
P-Values	Kolmogorov Smirnov	0.7636	0.8576	0.4666	0.9974	0.0044
	Chisqure 30	0.9817	0.6428	0.8374	0.9977	0.00001
	Chisqure 50	0.9878	0.3511	0.9674	0.9981	0.004
	Chisqure 80	0.9998	0.4451	0.9918	0.9857	0.0021
	Chisqure 100	0.9998	0.1427	0.9837	0.9925	0.0185
	Runs Up - Down	0.3823	0.5506	0.0684	0.5665	0.0045
	Runs Above - Below	0.7423	0.1564	0.1164	0.9922	0.001
	Auto Corr Func	0.012690263	0.017917491	0.024418786	0.017263698	0.025017469

Table K.4. The statistical test results: Seed = 123, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**123**	**123**	**123**	**123**	**123**
P-Values	Kolmogorov Smirnov	0.8115	0.159	0.5473	0.1459	0.2323
	Chisqure 30	0.7823	0.0095	0.6475	0.3092	0.4241
	Chisqure 50	0.4925	0.0572	0.3979	0.4746	0.321
	Chisqure 80	0.3083	0.1009	0.3062	0.9462	0.2103
	Chisqure 100	0.1613	0.3832	0.4786	0.7311	0.6238
	Runs Up - Down	0.1241	0.9276	0.6897	0.3328	0.2948
	Runs Above - Below	0.8473	0.8523	0.4272	0.1841	0.7716
	Auto Corr Func	0.018011841	0.014004948	0.014004948	0.019106092	0.020413666

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00000123**	**0.123**	**0.123**	**0.123**	**0.123**
P-Values	Kolmogorov Smirnov	0.9651	0.2861	0.6084	0.2033	0.7774
	Chisqure 30	0.8673	0.1649	0.2805	0.4291	0.7681
	Chisqure 50	0.3736	0.0467	0.045	0.1553	0.4949
	Chisqure 80	0.153	0.2224	0.5633	0.3143	0.2101
	Chisqure 100	0.1096	0.0235	0.0612	0.3594	0.3488
	Runs Up - Down	0.7369	0.0785	0.3328	0.1059	0.3172
	Runs Above - Below	0.4902	0.2112	0.8324	0.3319	0.0536
	Auto Corr Func	0.024356075	0.017623321	0.008755097	0.019272507	0.012761038

Table K.5. The statistical test results: Seed = 6979, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	***6979***	***6979***	***6979***	***6979***	***6979***
P-Values	**Kolmogorov Smirnov**	0.5486	0.7959	0.6508	0.0778	0.0787
	Chisqure 30	0.5759	0.6707	0.9404	0.5419	0.0649
	Chisqure 50	0.3073	0.3932	0.8641	0.3894	0.1668
	Chisqure 80	0.1571	0.1655	0.7991	0.8667	0.3266
	Chisqure 100	0.0474	0.328	0.825	0.8567	0.3951
	Runs Up - Down	0.3096	0.9968	0.1907	0.9087	0.4743
	Runs Above - Below	0.864	0.3415	0.6081	0.2597	0.2062
	Auto Corr Func	0.015628322	0.021856282	0.021856282	0.013742474	0.023893731

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00006979**	**0.6979**	**0.6979**	**0.6979**	**0.6979**
P-Values	**Kolmogorov Smirnov**	0.6618	0.9159	0.6776	0.4709	0.8948
	Chisqure 30	0.7943	0.5383	0.509	0.9089	0.497
	Chisqure 50	0.7171	0.7035	0.3024	0.4589	0.6302
	Chisqure 80	0.9331	0.1649	0.0367	0.1153	0.1253
	Chisqure 100	0.8597	0.7946	0.0967	0.2063	0.4638
	Runs Up - Down	0.8526	0.89	0.1828	0.5348	0.5193
	Runs Above - Below	0.9405	0.051	0.865	0.2262	0.36
	Auto Corr Func	0.022594979	0.01142359	0.023518492	0.016739042	0.026456539

Table K.6. The statistical test results: Seed = 821129, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**821129**	**821129**	**821129**	**821129**	**821129**
P-Values	**Kolmogorov Smirnov**	0.5064	0.8152	0.707	0.3403	0.1042
	Chisqure 30	0.7782	0.6503	0.4326	0.3724	0.5126
	Chisqure 50	0.3829	0.8138	0.3643	0.7182	0.5955
	Chisqure 80	0.5122	0.5134	0.7596	0.7304	0.6657
	Chisqure 100	0.1736	0.4875	0.8844	0.6737	0.8751
	Runs Up - Down	0.9464	0.89	0.5348	0.5091	0.959
	Runs Above - Below	0.0868	0.3121	0.4414	0.7234	0.652
	Auto Corr Func	0.011455198	0.013206686	0.013206686	0.017424533	0.022524389

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00821129**	**0.821129**	**0.821129**	**0.821129**	**0.821129**
P-Values	**Kolmogorov Smirnov**	0.6104	0.4286	0.1436	0.9509	0.7552
	Chisqure 30	0.8832	0.8234	0.0448	0.9542	0.7235
	Chisqure 50	0.9668	0.8012	0.0875	0.6717	0.915
	Chisqure 80	0.9863	0.7924	0.0127	0.9714	0.8675
	Chisqure 100	0.9874	0.8604	0.084	0.7814	0.9842
	Runs Up - Down	0.5665	0.2566	0.1468	0.865	0.6839
	Runs Above - Below	0.1706	0.006	0.0483	0.8493	0.2041
	Auto Corr Func	0.016603891	0.019641184	0.02086094	0.018567155	0.012422952

Table K.7. The statistical test results: Seed = 915612, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**915612027**	**915612027**	**915612027**	**915612027**	**915612027**
P-Values	**Kolmogorov Smirnov**	0.6728	0.0862	0.4123	0.3707	0.4097
	Chisqure 30	0.7988	0.0551	0.9342	0.4306	0.0255
	Chisqure 50	0.2048	0.0245	0.5854	0.6046	0.197
	Chisqure 80	0.4081	0.239	0.4929	0.5736	0.452
	Chisqure 100	0.1866	0.0892	0.5377	0.7272	0.0619
	Runs Up - Down	0.8156	0.4792	0.494	0.2189	0.8837
	Runs Above - Below	0.1828	0.371	0.0938	0.3963	0.9442
	Auto Corr Func	0.016135728	0.022779501	0.022779501	0.018049773	0.017053264

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.915612027**	**0.915612027**	**0.915612027**	**0.915612027**	**0.915612027**
P-Values	**Kolmogorov Smirnov**	0.2923	0.0704	0.9061	0.8149	0.8293
	Chisqure 30	0.8421	0.004	0.8615	0.9474	0.9253
	Chisqure 50	0.9821	0.0626	0.8488	0.6533	0.482
	Chisqure 80	0.9919	0.387	0.0341	0.7516	0.8798
	Chisqure 100	0.1009	0.2047	0.6776	0.6865	0.7633
	Runs Up - Down	0.1009	0.0976	0.5665	0.4598	0.2249
	Runs Above - Below	0.2327	0.2448	0.555	0.0082	0.7025
	Auto Corr Func	0.018141104	0.019937024	0.017284718	0.018777548	0.013383938

Table K.8. The statistical test results: Seed = 112319, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**112319**	**112319**	**112319**	**112319**	**112319**
P-Values	**Kolmogorov Smirnov**	0.3385	0.9998	0.1939	0.6223	0.2664
	Chisqure 30	0.6978	0.9901	0.1721	0.7911	0.8786
	Chisqure 50	0.0995	0.9802	0.1727	0.8201	0.5733
	Chisqure 80	0.0687	0.9801	0.0917	0.8266	0.1831
	Chisqure 100	0.034	0.9833	0.0597	0.3998	0.3386
	Runs Up - Down	0.9276	0.1751	0.5091	0.6551	0.1989
	Runs Above - Below	0.4065	0.9759	0.4053	0.6223	0.4902
	Auto Corr Func	0.013526284	0.014261944	0.014261944	0.022642897	0.020438572

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00112319**	**0.112319**	**0.112319**	**0.112319**	**0.112319**
P-Values	**Kolmogorov Smirnov**	0.9997	0.2837	0.4061	0.869	0.0869
	Chisqure 30	1	0.8526	0.4187	0.9043	0.5179
	Chisqure 50	0.9999	0.9834	0.0006	0.8474	0.6733
	Chisqure 80	0.9995	0.9221	0.0003	0.6754	0.7676
	Chisqure 100	0.9987	0.999	0.00001	0.5217	0.9944
	Runs Up - Down	0.725	0.1629	0.1468	0.7548	0.0325
	Runs Above - Below	0.9923	0.4325	0.2846	0.9094	0.3886
	Auto Corr Func	0.015093751	0.018520993	0.016589174	0.019081744	0.017155772

Table K.9. The statistical test results: Seed = 277, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**277**	**277**	**277**	**277**	**277**
P-Values	**Kolmogorov Smirnov**	0.3091	0.1889	0.0521	0.6709	0.443
	Chisqure 30	0.7679	0.4718	0.6059	0.5388	0.2219
	Chisqure 50	0.3102	0.6714	0.0635	0.5798	0.1283
	Chisqure 80	0.4644	0.9186	0.6092	0.4779	0.0522
	Chisqure 100	0.1924	0.7385	0.2867	0.4472	0.091
	Runs Up - Down	0.89	0.6324	0.7911	0.2985	0.2666
	Runs Above - Below	0.3162	0.3956	0.0798	0.2049	0.6974
	Auto Corr Func	0.016745862	0.014835294	0.014835294	0.012035639	0.009280854

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.00000277**	**0.277**	**0.277**	**0.277**	**0.277**
P-Values	**Kolmogorov Smirnov**	0.9999	0.9887	0.9546	0.0189	0.4078
	Chisqure 30	0.9885	0.8443	0.8985	0.0022	0.6876
	Chisqure 50	0.9998	0.7845	0.9971	0.0028	0.0877
	Chisqure 80	0.9962	0.9975	0.998	0.0002	0.7563
	Chisqure 100	0.9998	0.9465	0.9995	0.0002	0.1704
	Runs Up - Down	0.6839	0.9779	0.4743	0.0026	0.3823
	Runs Above - Below	0.5266	0.7885	0.9613	0.0949	0.5287
	Auto Corr Func	0.011195943	0.009091166	0.017671223	0.02834014	0.014440612

Table K.10. The statistical test results: Seed = 4612967, Iteration no = 10000.

		MLCG 1	MLCG 2	LCG 3	LCG 4	LCG 5
	Statistical Tests	**4612967**	**4612967**	**4612967**	**4612967**	**4612967**
P-Values	**Kolmogorov Smirnov**	0.8807	0.3945	0.2471	0.2521	0.5649
	Chisqure 30	0.5106	0.3945	0.051	0.8149	0.0407
	Chisqure 50	0.1574	0.6569	0.053	0.7608	0.0299
	Chisqure 80	0.1557	0.4638	0.0594	0.7482	0.1837
	Chisqure 100	0.038	0.536	0.0161	0.6169	0.1852
	Runs Up - Down	0.8156	0.8156	0.834	0.54	0.89
	Runs Above - Below	0.5258	0.7875	0.0911	0.6094	0.816
	Auto Corr Func	0.017877339	0.023409966	0.023409966	0.019794854	0.026743209

		Tent map μ = 2	Tent map μ = 2.00005	T-Logistic map	Connecting map α = 0.999	Connecting map α = 1.01
	Statistical Tests	**0.04612967**	**0.4612967**	**0.4612967**	**0.4612967**	**0.4612967**
P-Values	**Kolmogorov Smirnov**	0.5526	0.3131	0.4579	1	0.0653
	Chisqure 30	0.175	0.3419	0.4993	0.9994	0.0454
	Chisqure 50	0.6942	0.2282	0.5262	0.9811	0.1536
	Chisqure 80	0.4358	0.2869	0.3253	0.995	0.0784
	Chisqure 100	0.3662	0.4433	0.3241	0.8944	0.0658
	Runs Up - Down	0.4177	0.1934	0.4891	0.9276	0.041
	Runs Above - Below	0.2752	0.1654	0.0315	0.8339	0.0331
	Auto Corr Func	0.027844795	0.023696674	0.019143394	0.018851738	0.031878973

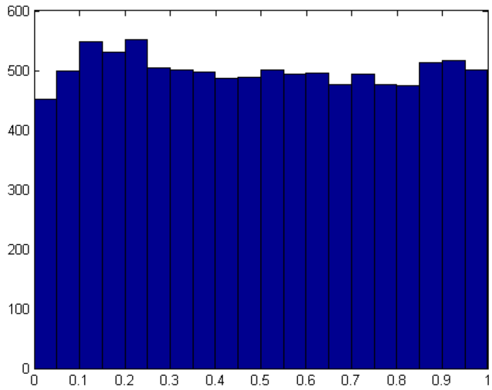
APPENDIX L: HISTOGRAMS WITH DIFFERENT SUBINTERVALS

In this part, we show that the change in the number of subintervals does not improve to determine the uniformity of the numbers. Three maps are selected to represent the relations between the histograms of samples for four different numbers of subintervals: 30, 50, 80, and 100. As seen in Table L.1, the first map passes the half of the chi-square tests, the second one is successful for all tests, and the last one is unsuccessful for all tests. Note that p-values of 0.05 is used as the test criteria.

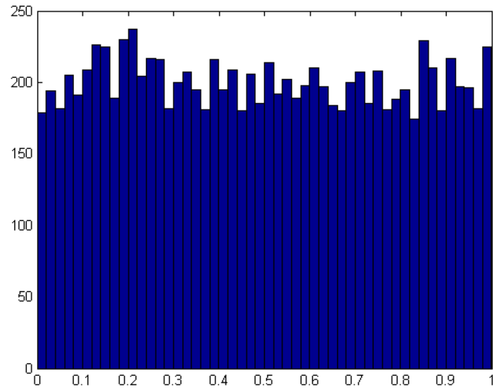
Table L.1. The chi-square test results for MLCG 1, LCG 3, and connecting map ($\alpha = 1.01$): Seed = 566231857, Iteration no = 10000.

		MLCG 1	LCG 3	Connecting map α = 1.01
	Statistical Tests	566231857	566231857	0.566231857
P-Values	Chisqure 30	0.1342	0.2287	0.0321
	Chisqure 50	0.1075	0.8127	0.0055
	Chisqure 80	0.0324	0.9437	0.0003
	Chisqure 100	0.0408	0.6311	0.0071

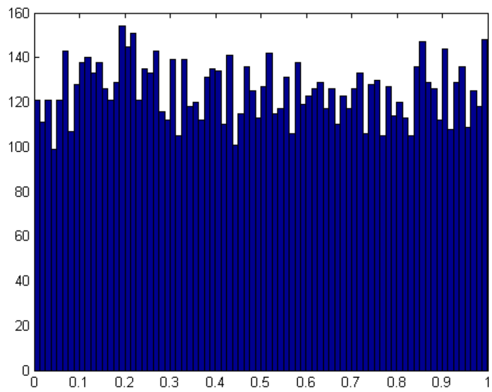
As can be observed in Figure L.1, Figure L.2, and Figure L.3, there is no relation between the test results and the histograms of the numbers. The difference between the histogram of the numbers that are passed chi-square test and the histogram of the number that are not passed chi-square test cannot be detected easily. So the number of subinterval is randomly selected as 50 to understand the general behavior of the numbers for the uniformity test.



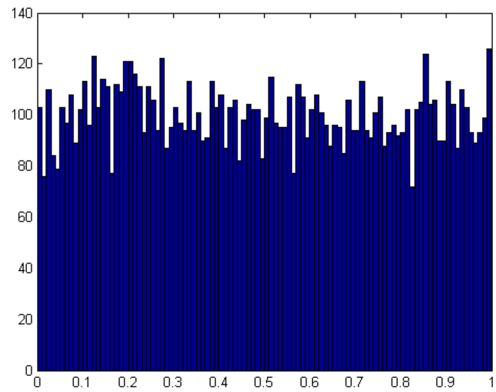
(a) Histogram with 30 subintervals



(b) Histogram with 50 subintervals

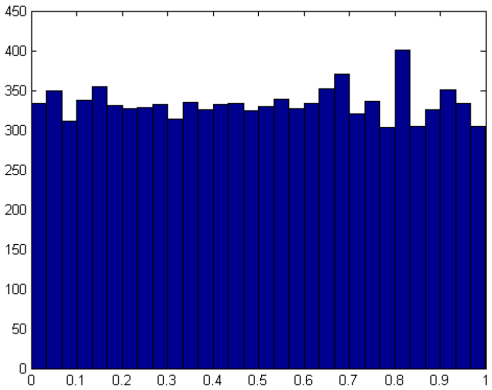


(c) Histogram with 80 subintervals

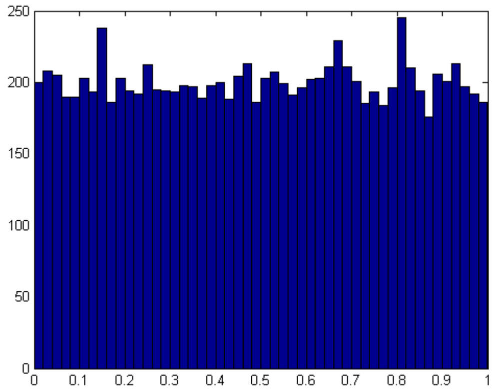


(d) Histogram with 100 subintervals

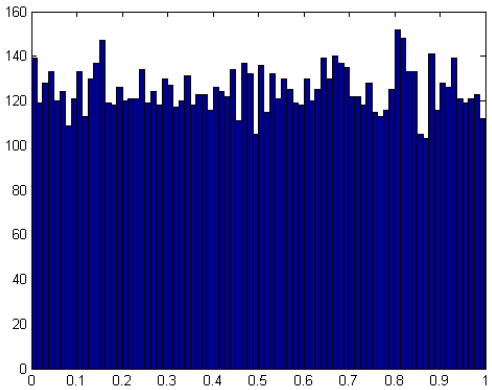
Figure L.1. Histograms for MLCG 1: Seed = 566231857, Iteration no = 10000.



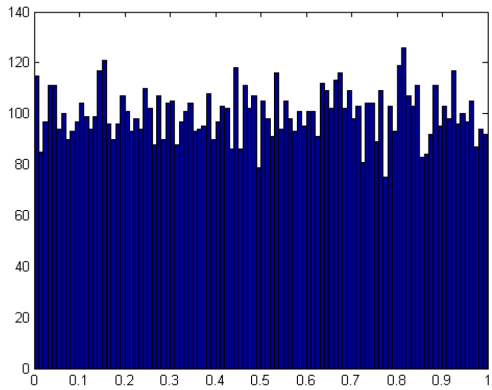
(a) Histogram with 30 subintervals



(b) Histogram with 50 subintervals

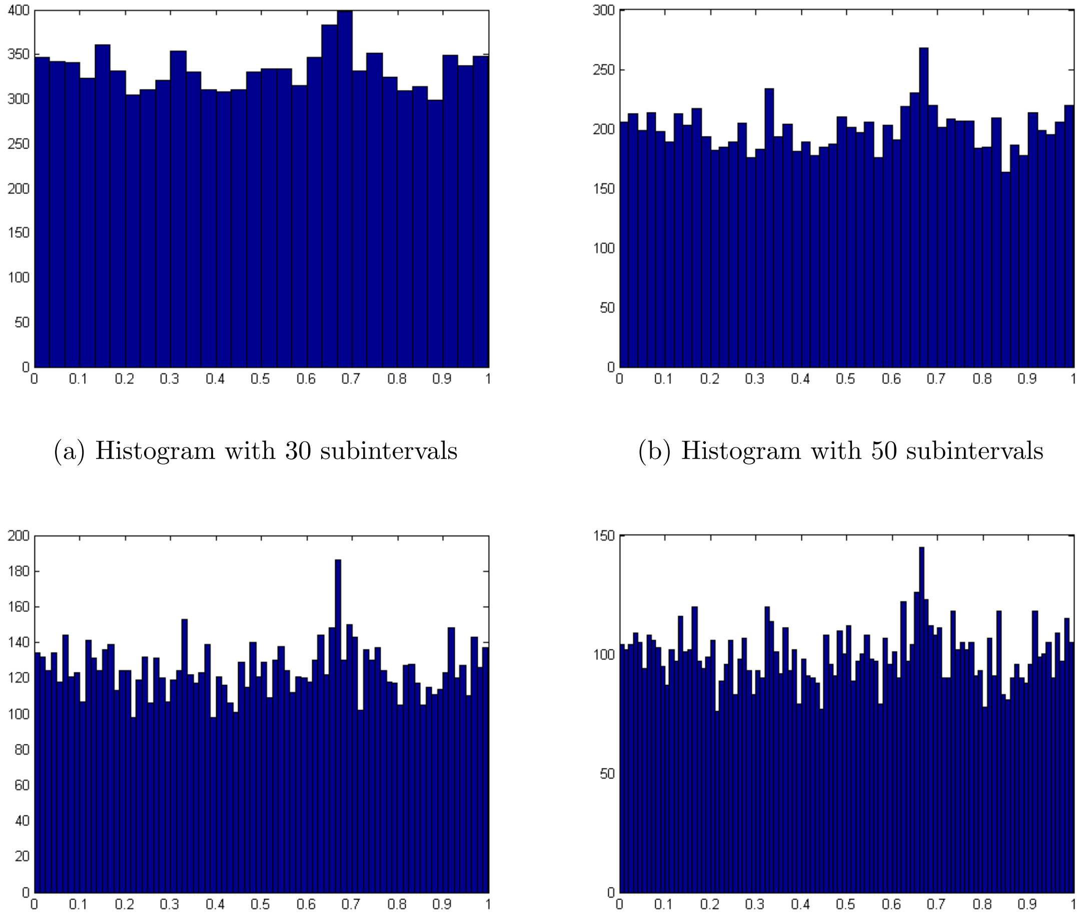


(c) Histogram with 80 subintervals



(d) Histogram with 100 subintervals

Figure L.2. Histograms for LCG 3: Seed = 566231857, Iteration no = 10000.



(a) Histogram with 30 subintervals

(b) Histogram with 50 subintervals

(c) Histogram with 80 subintervals

(d) Histogram with 100 subintervals

Figure L.3. Histogram for connecting map ($\alpha = 1.01$): Seed $= 0.566231857$, Iteration no $= 10000$.

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