TWO ESSAYS IN MODELING AND ANALYSIS OF EXCHANGE RATE DYNAMICS

by

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ABSTRACT

TWO ESSAYS IN MODELING AND ANALYSIS OF EXCHANGE RATE DYNAMICS

This thesis has two main chapters that are focusing on the different aspects of foreign exchange modeling. In the first chapter, a dynamic programming approach is used to model the problem that; an investor has to decide on the foreign currency levels at the beginning of each period in order to meet an uncertain demand. The investor tries to minimize his cost with the constraint of meeting the uncertain demand. The problem is formulated with dynamic programming approach and solved, the structure of the optimal decision is given. In the second part of the thesis, the problem of pricing FX options is tackled. FX options market is getting very popular among emerging market economies. Since traders in emerging markets are more prone to FX risk instead of interest-rate, FX derivatives market has been developing faster than Interest-Rate derivatives market. The importance of FX options market and the its unique conventions are described firstly in the second part of the thesis. FX options market has its own market conventions, and it is necessary to understand those conventions before going deep in modeling. After the FX conventions are described, the problem of options pricing is handled through stochastic modeling. The model used is commonly known as Heston's stochastic volatility model. Theoretical background of the model and the interpretation of the parameters are discussed and the relation between implied-distribution and option smile are discussed. The calibration procedure is described and practical use of the Heston's model is given. Heston's model is commonly used as a tool to price exotic options through Monte Carlo simulation or finite difference method. In this thesis, the model will be used for creating trading signals in the market.

ÖZET

DÖVİZ KURU MODELLEMESİ VE ANALİZİ ÜZERİNE ÇALIŞMA

Bu çalışma iki ana bölümden oluşmaktadır. İlk bölümde dinamik programlama yaklaşımıyla; miktarı önceden bilinmeyen ve yabancı para cinsinden olan talebi karşılamaya çalışan problem yapısı ele alınmıştır. Karar verici her periyodta elinde ne kadar yabancı para tutması gerektiğine karar verirken, risksiz faiz oranları da modelde girdiyi oluşturmaktadır. Karar verici talebi karşılarken, toplam maliyetleri de enküçüklemeye çalışmaktadır. Bu problem yapısı altında, dinamik programlama ile ideal çözüm bulunmuştur. İkinci ana bölümde döviz kuruna yazılan opsiyonların fiyatlanması problemi ele alınmıştır. Gelişmekte olan ülkelerde dövize dayalı opsiyonlar son yıllarda popülerlik kazanmıştır. Gelişmekte olan ülke ekonomilerindeki yatırımcıların döviz kuru riski, faiz riskine göre daha fazla olduğu için, dövize dayalı türev araçlar piyasası, faize dayalı türev araçlar piyasasından daha hızlı gelişmektedir. FX opsiyon piyasasında modelleme yapmadan önce, bu piyasanın kendine has dinamiklerini anlamak gerekir. FX opsiyon piyasası anlatıldıktan sonra Heston'un stokastik volatilite modeli tanıtılmıştır. Bu modelin teorik bileşenleri ve diğer stokastik modellerden üstünlüğü anlatıldıktan sonra piyasa verilerine kalibrasyonuna geçilmiştir. Kalibrasyon problemi anlatılıp, örnek sonuçlar ve grafikler verilmiştir. Heston modelinin literatürde ve pratikte, çoğunlukla egzotik opsiyon fiyatlamasında kullanıldığını bilerek, bu tezde farklı bir yol izlenip piyasada alım-satım stratejileri geliştirilmeye çalışılmıştır.

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LIST OF SYMBOLS

D_{t_n}	Demand observed after the given decision at time t_n
$\mathbb{E}[.]$	Expectation Operator
$\mathbb{F}_n^{-1}(.)$	Inverse cumulative distribution function of D_{t_n} at time t_n
r^d	Risk free interest rate for domestic currency
r^f	Risk free interest rate for foreign currency
t_n	Time n
V_{t_n}	Value function at time t_n
x_{t_n}	Initial foreign currency at time t_n
y_{t_n}	Decision variable at time t_n
Γ_{t_n}	Random variable representing the inverse of the exchange rate process for the period t_n
γ_{t_n}	Realization of the inverse of the exchange rate process for the period t_n
$\phi_n(.)$	Density function of D_{t_n} at time t_n
Ξ_{t_n}	Random variable representing the exchange rate for the period t_n
ξ_{t_n}	Realization of the exchange rate for the period t_n

LIST OF ACRONYMS/ABBREVIATIONS

ATM	At-the-money
BIS	Bank for International Settlements
BS	Black and Scholes
CHF	Swiss Frank
DOM	Domestic Currency
EME	Emerging Market Economy
EUR	Euro Currency
FOR	Foreign Currency
FX	Foreign Exchange
GBP	Great Britain Pound
JPY	Japanese Yen
MCMC	Markov Chain Monte Carlo
OTC	Over-the-counter
RR	Risk Reversal
SABR	Stocahstic Alpha Beta Rho Model
STR	Strangle
TRY	Turkish Lira
USD	United States Dollar

1. INTRODUCTION

Foreign exchange market has long been one of the largest, liquid and attractive market for speculators. Exchange rate is essential in today's economy because of the huge trading volume between countries. In today's economy most of the main currencies are floating, meaning that the level is determined in the spot market. In 1960s, Bretton Woods system was in use. This system simply fixes the exchange rate to a level and this level remains fixed as the time passes. However today most of the major currencies are in floating rate regime, except the Chinese Yuan. Chinese Yuan is in pegged exchange rate regime, meaning that conversion to US Dollar is allowed between some upper and lower barriers.

Average daily transaction of the market is estimated to be around 4 trillion US Dollars, which indicates how big the market is. Approximately the 90% of this turnover is composed of spot transactions and foreign exchange swaps. Largest participants of the market are banks, central-banks, commercial companies and large funds. Since foreign exchange market has participants those making large transactions and those making small transactions, the market is divided into levels of access unlike the stock market. As an inevitable result of this large volume trading, most of the transactions are made in over-the-counter (OTC) market. Inter-bank market is at the top of this level; largest commercial banks, security dealers make their transactions in this market. Since larger volumes are traded at the top of this level of access, ask-bid spreads of the currencies are smaller compared to the lower level of access. Ask-bid spread of currencies is increasing as one goes down from this level. Most traded currency is US Dollar as one may expect, then Euro, Great Britain Pound, Swiss Frank and Japanese Yen can be said as the major currencies that are traded.

Foreign exchange market is getting more and more important especially in Emerging Market Economies. After the global financial crisis in 2008 and 2009, developed countries increased regulation on the financial speculative instruments and as a result of this situation large FX trading desks has started shifting to Emerging Markets, where regulations are looser.

Exchange rate parity of one major currency with one emerging market currency behaves differently when compared to a parity that is composed of two major currencies. Interest rate differential is large and positive in these exchanges, meaning that risk-free rate of emerging market currency is larger than that of the major currency. This situation makes emerging market currencies attractive because of the high carry and the dynamics of the exchange rate becomes highly dependent on the size of the carry and the uncertainty, or volatility, in the market.

This thesis starts with modeling the exchange rate through the dynamic programming approach. A simple decision making problem is constructed and solved. The decision maker has to satisfy uncertain demand and he can convert his money to foreign currency. Interest rate differential becomes highly dominant in the decision. The main difference of this problem setting from classical portfolio management is that demand is uncertain and independent of the exchange rate process. This problem setting enables the demand risk hedging impossible, since they are driven by independent sources of uncertainties. In the second part of the thesis, FX options market will be investigated. Theoretical background of the Heston's stochastic volatility model will be introduced. In this problem setting, option price will be a function of spot price and volatility. Because of the previous reason, it is possible to hedge the risk of the option position, by trading in money market, underlying security and an option with same underlying. This problem setting is different from the one that will be handled in the first chapter. So, two different types of problem settings will be handled in this thesis in the sense of hedging the risk of the obligation.

Options market enables one to trade the volatility of the underlying security. Information about the future level of the uncertainty could be extracted in this market, and more accurate trading strategies could be constructed. FX options market is different from the classical equity options market. It has its own conventions. Before starting any type of modeling, one needs to understand the options market dynamics. For instance, level of moneyness is measured with Black-Scholes Delta instead of strike price. In equity options, strike price of the option is quoted along with its price, however in FX options market risk-reversals, butterflies and at-the-money option volatilities are quoted. From this information, delta-volatility pairs are extracted and the smile or surface can be drawn. After understanding the FX specific option smile convention, one may start modeling the exchange rate. There exists several number of studies in stochastic modeling and option pricing in literature. In this thesis Heston's Stochastic Volatility Model is selected and used. Simplicity and ability to handle the option smile are the main reasons for selecting Heston's model.

This thesis models the exchange rate in two different aspects. First chapter analysis foreign exchange consumption model through dynamic programming and the second chapter tackles the option pricing. Before pricing options, FX options market is introduced since it has unique conventions. In the next section exchange rate basics will be explained. Motivation, contributions of the thesis and literature review will follow respectively. Then Dynamic programming model for foreign exchange consumption will be explained and after that the chapter of pricing FX options in emerging markets begin.

1.1. Exchange Rate Basics

In finance, exchange rate is the rate between two currencies that defines how one currency will be converted to another currency. For example, an exchange rate of 1.5100 for USDTRY parity means that 1 USD will be exchanged for 1.5100 TRY. The actual rate quoted by money dealers in spot will be different for buying or selling the currency, because of the fact that there is allowance for the dealer's profit (or commission). Spot exchange rate refers to the current exchange rate, and forward exchange rate refers to an exchange rate that is quoted and traded today but the delivery will be made on the specific future time. Note that in theory forward is never unique, forward price of a security can be calculated according to any security whose future payoff is known by today, e.g. bonds, bank account. In foreign exchange (FX) market, forward rate is calculated according to the risk-free interest rate differences of the two currencies.

FX rate is quoted in terms of the number of domestic (DOM) currency is needed to exchange 1 unit of foreign (FOR) currency. Foreign and domestic does not refer to any geographical region, rather they refer to underlying and numeraire currencies respectively. Quotation is always made in FOR-DOM, that is first currency is always foreign and the second one is always considered as domestic currency. For the example above, 1.5100 USDTRY rate means; 1.5100 domestic currency (Turkish Lira) is needed to buy 1 foreign currency (US Dollar).

Today most of the currencies are floating, that is the rate is allowed to vary against other currencies by the market forces of supply and demand. Exchange rates for such currencies are changing almost instantenously in financial markets mainly by the intervention of banks. As a result of this floating rate situation, stochastic modelling of the exchange rates becomes crucial in understanding the internal dynamics of the market. On the other side of the floating rate there is fixed exchange rate system. However there are no major economic players that uses fixed exchange rate system.

FX market is a very large financial market that, according to the findings of BIS, total daily average turnover is nearly 2300 million USD. Nearly 2000 million USD daily turnover belongs to advanced economies, such as United States, Japan and United Kingdom. Although the daily average turnover is relatively small in Emerging Market Economies (EME), there is significant increase for the FX market over the past decade. Mostly traded currencies are U.S. Dollar, Euro, Hong Kong Dollar, Japanese Yen, Singapore Dollar, Australian Dollar, Korean Won, Chinese Renminbi and G.B. Pound. As it can easily be seen that, these currencies used mainly in international trade and it is an expected result that they constitute the nearly 90% of the total turnover.

The participant of the FX market is almost everyone. Any company engaging in international trade has to deal with foreign exchange. Other than that hedge funds and banks are the major players in the FX market, both in spot and derivatives. Hedge funds try to generate a return of 10% - 15% annually regardless of the market direction, since there is no restriction for hedge funds in shorting assets. They try to achieve this return by taking low risk, as the name of the fund indicates, by hedging the risk. Banks are trading in FX market for variety of reasons.

Drivers of the FX market are mainly buyers and sellers. These drivers are also driven by another factors, such as international trade. Foreign direct investment is another factor, not just buying stocks of foreign company, that affects the FX rate. Interest rate differential, inflation expectations, monetary policy and even rumors are the major drivers of the buyers and sellers in FX market. Unlike the equity market, FX market incorporates all the macro economic factors into the pricing and trading. Moreover FX rates are sensitive to political news and events. Also central banks intervention is a major driver of the FX spot rate.

1.2. Motivation

The main motivation behind the study of the FX market is its liquidity. When an asset is sold without causing any price change, or with the minimum loss of value, the market is called liquid. Unlike in equity markets, FX market is highly liquid that investors can easily buy and sell their currencies with very little spread. This means that, at almost every price level there exits a buyer and a seller in the market.

Another motivation is that, FX market is getting popular in EMEs and Turkey is one the biggest EME. With the increasing importance of FX market, understanding the dynamics of FX rate through stochastic modelling and pricing the instruments, such as options, is crucial in trading. Big financial institutions started to shift their trading desks to EMEs, because of the over-regularization in advanced economies. Because of this reason, they tend to seek equipped FX experts in those countries.

1.3. Contributions

Main contribution of this thesis is, it provides complete guide to foreign exchange market. The market specific quotation mechanism and the relation between the smile and implied distribution is explained in detail. Besides the theoretical knowledge provided about the options market, practical use of Heston's stochastic volatility model is explained in detail also.

Another main contribution of this thesis is to show simple trading strategies work even in FX market, which is a highly volatile market. However these simple trading strategies are not any simpler than the existing ones. It shows that, extracting information from option data enables successful trades in spot price. The trading strategy, two standard deviation rule, is simple but the signals generated from the data series, rho, contains correct mixture of information about the market. That is the reason for the successful trades, even if simple trading strategies are used.

This thesis can be used as practical and theoretical guide for beginners, it contains both theoretical and practical knowledge about the FX options market. It illustrates trading strategies both in spot and option market. Trading in FX spot market is not easy, since it is highly volatile and has many drivers. However a successful trading signal is generated in USDTRY currency by the use of Heston's model parameters. Main idea behind this strategy is emerging market currencies are driven by rate differential, later it will be changed to skewness.

To sum up, this thesis provides theoretical information in FX options market and illustrates how to use models in practice. Then shows that model based trading systems outperforms the ones that are simpler. Finally, it emphasizes simple trading rules with model based inputs works fine even in highly volatile FX market.

2. LITERATURE REVIEW

This thesis is composed of two chapters and consequently the literature review will contain the previous researches that are related with the two chapters. First chapter is modelling dynamic model for the consumption of foreign exchange demand. Second chapter discusses the FX options market. The related literature for the first chapter are mainly discusses the foreign exchange uncertainty and foreign direct investment of the firms.

In the pioneering work of Campa [1], the relationship of real exchange rate fluctuations and foreign direct investment is investigated in United States during 1980s. The aim of this paper is to guide foreign direct investments under the volatile exchange rate regime, which is observed after the crash of Bretton Woods system of fixed exchange rates. In this model, the investor enters the market as long as the future dividends is greater than the initial sunk cost. The paper also handles the other foreign direct investments and concludes that the exchange rate volatility to be negatively correlated with the number of foreign investments. The research is done in the wholesale industry of the United States.

In the masterwork paper of Sung and Lapan [2], the relationship between the exchange rate volatility and foreign direct investment decisions of the firm is investigated. The firm is risk neutral multinational firm, that can open plants in both countries. This paper emphasized that exchange rate volatility can create opportunities of lower cost production plants by shifting from one country to another. Their decision model is in spirit of real options, and high volatility increases the option value and deters the decision of foreign direct investment. They conclude that exchange rate volatility has relationship with the domestic competetive market through the idea of high volatility deters the foreign direct investment and the multi national firms starts to invest in domestic plants.

Similar work in the first chapter of the thesis has been handled in the paper of Gurnani and Tang [3]. They tackled a problem of two stage in which the retailer orders a seasonal product prior to a single selling horizon. This problem is similar to one handled in the first part of the thesis in some ways; first of all exchange rate can be thought of unit production cost and in this paper the cost and demand is uncertain. Their problem structure allows two stage decision making and the uncertain cost is allowed to have two different states in the future period. Their model is nested newsvendor for determining the optimal ordering quantity.

The related literature with the first chapter mainly discusses the exchange rate volatility and foreign direct investment. One similar problem setting has been observed in a different problem context. Uncertain production cost can be thought as uncertain exchange rate in the model of foreign exchange consumption model of this thesis. Second similarity was the uncertain demand, which is also the case in this thesis.

The literature review continues for the second part of the thesis. Second chapter mainly discusses FX options market dynamics and modelling the exchange rate. Theoretical background of Heston's stochastic volatility model is discussed and practical usage of the model is represented. The related literature handles mainly option pricing and FX conventions.

In the study of Reiswich and Wystup [4], FX options market conventions described first and then smile construction problem is tackled. Market conventions are different from any classical equity options market. In FX options market, moneyness level is represented by the delta of the option, instead of its strike price. At-the-money and delta conventions are described in this paper, and it is very important to understand the market conventions before getting deeper in modelling.

After the market specific conventions are understood, the need of modelling the exchange rate is necessary in order to understand the underlying dynamics of the market and also to price derivatives. In mathematical finance Black & Scholes published a benchmark paper that gave shape to derivatives market. However Black & Scholes options pricing formula (BS) was not enough to explain the market observed implied volatilities. Black & Scholes assumes in their paper that stock price process has constant mean and variance (or volatility), and in the risk neutral probability measure, the drift is replaced with the risk-free interest rate. When the market prices of options are inverted in the BS formula, different volatilities (or implied volatilities) are observed for different strike prices and time-to-maturities. This problem is called smile problem. The selected model should handle smile, and obviously this cannot be Black & Scholes model, because of the constant volatility assumption. So the literature continues in order to handle this problem.

Next step in the literature is local volatility models. Merton [5] suggested that making the volatility a function of time. Although this approach explains the different implied volatilities across time-to-maturities, it fails to explain the smile shape across strike prices.
Dupire [6], Derman and Kani [7], and Rubinstein [8] came up with the idea that making the volatility not only a function of time, but a function of state variables, i.e. volatility is a function of stock price and time. This approach was successful in fitting the volatility surface, however it failed to explain the persistent smile shape which does not vanish as the time passes.

Failure of local volatility models would be the success of stochastic volatility models. In the study of Scott [9], it is assumed that the variance of the stock price changes randomly and the risk of random variance is diversifiable which leads to uncorrelated variance with stock returns. Same type of modelling approach is conducted by Hull and White [10] and Wiggins [11] and all of these models have one common flaw, that they do not have closed form solution for european type of options.

However the Heston's stochastic volatility model [12], has the closed form solution for european type of options and it can be considered as the second milestone in option theory, after Black & Scholes. The solution technique is based on characteristic functions and it can be extended to even stochastic interest rates. Heston's stochastic volatility model is loved by the practitioners by two main reasons; first one is that it has a closed form solution for european options which results in fast calibration of the model to the market data, second the variance process is mean-reverting which in reality what they observe. Heston's option pricing formula is semi-analytical because it requires numeric integration of characteristic functions. Nevertheless, it is one of most common model in practice.

Another very common model in stochastic volatility model pool is Hagan's Stochastic Alpha Beta Rho model (SABR) [13]. The main purpose of this paper is to show that local volatility models would yield to very wrong results and delta and vega hedges derived from local volatility models may perform worse than naive Black & Scholes. The authors use singular perturbation techniques to obtain the prices of european options and from those prices the closed form algebraic formulas for the implied volatilities. SABR model can be considered as the simplest stochastic volatility model in the literature, it is also very commonly used in practice.

3. A DYNAMIC MODEL FOR PLANNING FOREIGN EXCHANGE CONSUMPTION

3.1. Problem Definition

An investor with an infinite amount of domestic currency, has to meet an uncertain demand, whose unit is foreign currency. The planning horizon, $(t_1, t_2 \ldots, t_n, \ldots, t_N)$, is fixed and finite. At the beginning of each period, the decision of new foreign currency level will be given. If there is excess amount of foreign currency, it will be carried to next period and during that time it will grow with foreign interest rate. On the other hand, the investor loses the opportunity of investing in domestic interest rate. Every period, demand must be satisfied and the unmet demand, if there is any, will be satisfied with the next period's exchange rate.The problem will be formulated in Dynamic Programming approach and under some assumptions, the structure of the optimal decision will be given.

3.2. Sequence of Events

(i) At the beginning of the period, the investor realizes the state of the system, which are the exchange rate and on hand foreign currency.

(ii) The investor decides on the new level of foreign currency.

(iii) After the given decision, the uncertain demand is realized and at the same time we move to the next period.

(iv) The unmet demand must be satisfied with the new exchange rate, if there is any.

(v) The cost of excess foreign currency will incur, note that this cost can actually be profit if the foreign interest rate is higher than the domestic interest rate.

3.3. Model and Dynamic Programming Formulation

Where x_{t_n} is the initial foreign currency at the beginning of period t_n and Ξ_{t_n} is the random variable representing the exchange rate for the period t_n. ξ_{t_n} is the realization of the exchange rate for the period t_n. Γ_{t_n} is the random variable representing the inverse of the exchange rate process for the period t_n and γ_{t_n} is the realization of the inverse of the exchange rate process for the period t_n. Where D_{t_n} is the demand observed after the given decision at time t_n and r^d and r^f are the risk free interest rates for domestic and foreign currency respectly.

$$\begin{aligned} G_{t_n}(y_{t_n}, \xi_{t_n}) &= \frac{y_{t_n}}{\xi_{t_n}} + e^{-r^d(t_{n+1}-t_n)}\mathbb{E}\left[\frac{(D_{t_n} - y_{t_n})^+}{\Xi_{t_{n+1}}}\right] \\ &+ \frac{e^{r^d(t_{n+1}-t_n)} - 1}{\xi_{t_n} e^{r^d(t_{n+1}-t_n)}}\mathbb{E}[(y_{t_n} - D_{t_n})^+] \\ &+ e^{-r^d(t_{n+1}-t_n)}\mathbb{E}[V_{t_{n+1}}((y_{t_n} - D_{t_n})^+ e^{r^f(t_{n+1}-t_n)}, \Xi_{t_{n+1}})] \quad (3.1) \\ V_{t_n}(x_{t_n}, \xi_{t_n}) &= \min_{y_{t_n} \geq 0}\{G_{t_n}(y_{t_n}, \xi_{t_n})\} - \frac{x_{t_n}}{\xi_{t_n}} \quad (3.2) \\ V_{t_{N+1}}(x_{t_{N+1}}, \xi_{t_{N+1}}) &= -\frac{x_{t_{N+1}}}{\xi_{t_{N+1}}} \quad (3.3) \end{aligned}$$

The cost of changing initial foreign currency level is, $\frac{y_{t_n} - x_{t_n}}{\xi_{t_n}}$, and this part could be seen from the combination of the first term in Equation 3.1 and the second term in Equation 3.2. Cost of unsatisfied demand is represented as the second term in Equation 3.1. Excess demand, $(D_{t_n} - y_{t_n})^+$, is satisfied with the next period's exchange rate, $\Xi_{t_{n+1}}$. Since next period's exchange rate is uncertain, expectation is taken over $\Xi_{t_{n+1}}$ and this cost is discounted to time t_n. When demand becomes less than the new level of foreign currency, the excess amount is carried with foreign risk-free rate for one period and this is represented as the third term in Equation 3.1. Last term in Equation 3.1 is the next period's discounted value function. Expectation operator is both over D_{t_n} and Ξ_{t_n}.

The value function in (3.2), is the minimization of $G_{t_n}(y_{t_n}, \xi_{t_n})$ over y_{t_n}, and subtracting the initial domestic currency. Note that y_{t_n} cannot be negative, by the assumption that the investor cannot borrow money.

Equation (3.3) is the boundary condition. When the planning period ends, the excess amount of foreign currency is changed back to domestic currency with that period's exchange rate. That means a cash inflow, so it is multiplied with -1.

3.4. Assumptions

(i) Demand is a stationary continuous positive random variable, and it must be satisfied in each period

(ii) $y_{t_n} \geq 0$, meaning that increasing or decreasing the initial foreign currency position is possible but borrowing is not

(iii) There is no ask - bid spread. (But can be incorporated)

(iv) r^d and r^f are domestic and foreign risk-free interest rates respectively and they are deterministic

(v) Demand in each period is independent of anything else

(vi) Exchange rate model is either Geometric Brownian Motion or Merton's Jump Diffusion Model

(vii) Jump sizes are iid log-normal random variables and they are independent of other random variables

3.5. Structure of the Optimal Decision

Without loss of generality, let the planning period be equally spaced, $t_{n+1} - t_n = \Delta t$; $\forall n = 1, 2, \ldots, N$ and let $\phi_n(.)$ be the density function of D_{t_n} and $d = e^{r^d \Delta t}$

$$
\begin{aligned}
\frac{\partial G_{t_n}(y_{t_n}, \xi_{t_n})}{\partial y_{t_n}} &= \gamma_{t_n} + \frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d} \frac{\partial}{\partial y_{t_n}} \int_{y_{t_n}}^{\infty} (u - y_{t_n}) \phi_n(u) du \\
&+ \gamma_{t_n} \frac{d-1}{d} \frac{\partial}{\partial y_{t_n}} \int_0^{y_{t_n}} (y_{t_n} - u) \phi_n(u) du \\
&+ \frac{1}{d} \frac{\partial}{\partial y_{t_n}} \mathbb{E}\left[\min_{y_{t_{n+1}} \geq 0} \{ G_{t_{n+1}}(y_{t_{n+1}}, \xi_{t_{n+1}}) \} - (y_{t_n} - D_{t_n})^+ e^{r^f \Delta t} \Gamma_{t_{n+1}} \right]
\end{aligned}
$$

Let $\mathbb{F}_n(.)$ be the cumulative distribution function of D_{t_n}

$$\begin{aligned}
\frac{\partial G_{t_n}(y_{t_n},\xi_{t_n})}{\partial y_{t_n}} &= \gamma_{t_n} + \frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d}\int_{y_{t_n}}^{\infty}(-1)\phi_n(u)du + \gamma_{t_n}\frac{d-1}{d}\int_0^{y_{t_n}}(1)\phi_n(u)du \\
&- \frac{e^{r^f\Delta t}}{d}\mathbb{E}[\Gamma_{t_{n+1}}]\int_0^{y_{t_n}}(1)\phi_n(u)du \\
&= \gamma_{t_n} - \frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d} + \frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d}\mathbb{F}_n(y_{t_n}) + \gamma_{t_n}\frac{d-1}{d}\mathbb{F}_n(y_{t_n}) \\
&- \frac{e^{r^f\Delta t}}{d}\mathbb{E}[\Gamma_{t_{n+1}}]\mathbb{F}_n(y_{t_n}) \\
y^*_{t_n} &= \mathbb{F}_n^{-1}\left(\frac{\frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d} - \gamma_{t_n}}{\gamma_{t_n}\frac{e^{r^d\Delta t}-1}{d} - \frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d}(e^{r^f\Delta t}-1)}\right) \qquad (3.4)
\end{aligned}$$

The first order condition has closed form solution for every y_{t_n}. Since the function

$$\mathbb{E}\left[\min_{y_{t_{n+1}}\geq 0}\{G_{t_{n+1}}(y_{t_{n+1}},\xi_{t_{n+1}})\}\right]$$

is independent of y_{t_n}, its first derivative with respect to y_{t_n} becomes zero. The partial derivatives of the rest of the terms are straight-forward. It is assumed that the investor can drop down the level of foreign currency and this assumption enables the minimization of $G_{t_{n+1}}(.)$ to be independent of y_{t_n}.

The optimality of the above formula depends on the second derivative of $G_{t_n}(y_{t_n},\xi_{t_n})$ with respect to y_{t_n}, it must be positive in order Equation 3.4 to be valid. Required condition on the convexity of $G_{t_n}(y_{t_n},\xi_{t_n})$, will be given in Section 3.7. The expression inside the parenthesis of the above formula will be called fractile from now on.

3.6. Exchange Rate Models

Two different exchange rate processes are considered. The first one is Geometric Brownian Motion (GBM) [14] and the second one is Merton's Jump Diffusion (MJD) Model [15]. In the following two subsections, general characteristics of these processes will be given.

3.6.1. Geometric Brownian Motion

GBM is the standard model used in classical Black-Scholes. It also makes sure that inverse process is in the same model class.

$$\begin{aligned}
d\xi_t &= \xi_t \mu dt + \xi_t \sigma dW_t \\
d\gamma_t &= (\sigma^2 - \mu)\gamma_t dt - \sigma \gamma_t dW_t \\
\Gamma_{t_{n+1}} &= \gamma_{t_n} exp\left\{(\sigma^2 - \mu - \frac{1}{2}\sigma^2)\Delta t - \sigma(W_{t_{n+1}} - W_{t_n})\right\} \\
\mathbb{E}_{t_n}[\Gamma_{t_{n+1}}] &= \gamma_{t_n} exp\left\{(\sigma^2 - \mu)\Delta t\right\} \\
Let & \\
c_1 &= exp\left\{(\sigma^2 - \mu)\Delta t\right\} \\
\mathbb{E}_{t_n}[\Gamma_{t_n+1}] &= \gamma_{t_n} c_1 \qquad (3.5)
\end{aligned}$$

When $\mathbb{E}[\Gamma_{t_{n+1}}]$ in Equation 3.4, is replaced by the expression in Equation 3.5, the following line will give the explicit formula for the optimal decision when Geometric Brownian Motion Model is used for exchange rate process.

$$y^*_{t_n} = \mathbb{F}_n^{-1}\left(\frac{\frac{c_1}{d} - 1}{\frac{e^{r^d \Delta t} - 1}{d} - \frac{c_1}{d}(e^{r^f \Delta t} - 1)}\right) \qquad (3.6)$$

Note that, in risk-neutral probability measure, μ should be replaced with $r^f - r^d$.

3.6.2. Merton's Jump Model

The following process is the standard jump diffusion model of the stock price. Jump size, Y, is a log-normally distributed random variable with parameters μ^{Jump} and σ^{Jump}, and λ is the Jump intensity. In the risk-neutral probability measure, μ will be replaced with $r^f - r^d$.

$$
\begin{aligned}
d\xi_t &= \xi_t(\mu - \lambda K)dt + \xi_t \sigma dW_t + \xi_t(Y-1)dq_t \\
K &= E[Y-1] \\
dq_t &= \begin{cases} 1 & \lambda dt \\ 0 & (1-\lambda dt) \end{cases} \\
d\gamma_t &= (\sigma^2 - \mu + \lambda K)\gamma_t dt - \sigma \gamma_t dW_t + \frac{1-Y}{Y}\gamma_t dq_t \\
\Gamma_{t_{n+1}} - \gamma_{t_n} &= (\sigma^2 - \mu + \lambda K)\gamma_{t_n}\Delta t - \sigma\gamma_{t_n}[W_{t_{n+1}} - W_{t_n}] - \frac{Y-1}{Y}\gamma_{t_n}[q_{t_{n+1}} - q_{t_n}] \\
\mathbb{E}_{t_n}[\Gamma_{t_{n+1}}] &= \gamma_{t_n}[1 + (\sigma^2 - \mu + \lambda K)\Delta t - E\left[\frac{Y-1}{Y}\right]\lambda \Delta t] \\
Let & \\
c_2 &= 1 + (\sigma^2 - \mu + \lambda K)\Delta - \lambda E\left[\frac{Y-1}{Y}\right]\Delta t \\
\mathbb{E}_{t_n}[\Gamma_{t_{n+1}}] &= \gamma_{t_n} c_2
\end{aligned} \tag{3.7}
$$

When $\mathbb{E}[\Gamma_{t_{n+1}}]$ in Equation 3.4 is replaced by the expression in Equation 3.7, the following line will give the explicit formula for the optimal decision when Merton's Jump Diffusion model is used for exchange rate process.

$$
y^*_{t_n} = \mathbb{F}_n^{-1}\left(\frac{\frac{c_2}{d} - 1}{\frac{e^{r^d\Delta t}-1}{d} - \frac{c_2}{d}(e^{r^f\Delta t} - 1)}\right) \tag{3.8}
$$

3.7. Conditions on the Convexity and the Boundedness of the Decision

In order the formula for optimal decision in Equation 3.4 to hold, the second derivative of the function $G_{t_n}(y_{t_n}, e_{t_n})$ must be positive.

$$
\frac{\partial^2 G_{t_n}(y_{t_n}, \gamma_{t_n})}{\partial y^2_{t_n}} = \left(\gamma_{t_n}\frac{e^{r^d\Delta t} - 1}{d} - \frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d}(e^{r^f\Delta t} - 1)\right)\phi_n(y_{t_n})
$$

When $\mathbb{E}[\Gamma_{t_{n+1}}]$ is replaced with $\gamma_{t_n} c_i$, $i = 1$ denotes GBM Model and $i = 2$ denotes MJD Model, the convexity condition simplifies as the following.

$$\frac{\partial^2 G_{t_n}(y_{t_n}, \gamma_{t_n})}{\partial y_{t_n}^2} = \left(\frac{e^{r^d \Delta t} - 1}{d} - \frac{c_1}{d}(e^{r^f \Delta t} - 1) \right) \phi_n(y_{t_n}) \gamma_{t_n}$$

Since $\phi_n(y_{t_n})$, γ_{t_n} and $\frac{1}{d}$ are always positive numbers, the convexity condition would become:

$$\left(e^{r^d \Delta t} - 1\right) \geq c_i \left(e^{r^f \Delta t} - 1\right) \tag{3.9}$$

If the investor carries one foreign currency for one period, then the expected profit would be $c_i \gamma_{t_n} \left(e^{r^f \Delta t} - 1\right)$ amount of domestic currency (Remember that 1 foreign = γ_{t_n} domestic, at time t_n). On the other hand, if that 1 foreign currency is first changed into γ_{t_n} amount of domestic currency and then carried for one period, the profit would be $\gamma_{t_n} \left(e^{r^d \Delta t} - 1\right)$. So if the condition in Equation 3.9 is not satisfied, it would be profitable, on the expected, to change all of the domestic currency into foreign currency. This situation makes the goal of the investor irrelevant, which was satisfying demand while cost minimizing, because it is possible to make infinite amount of profits by just changing all of the domestic currency into foreign currency.

Assuming that the convexity condition holds; the fractile in Equation 3.4 should be between 0 and 1, in order to have bounded decisions.

$$\begin{aligned} 0 &\leq \left(\frac{\frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d} - \gamma_{t_n}}{\gamma_{t_n} \frac{e^{r^d \Delta t} - 1}{d} - \frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d}(e^{r^f \Delta t} - 1)} \right) \\ \gamma_{t_n} &\leq \frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d} \end{aligned} \tag{3.10}$$

If there is unmet demand, in any period, then it should be satisfied with $\frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d}$ instead of γ_{t_n}. So the decision maker should be penalized for not satisfying demand, otherwise it would be unreasonable to hold foreign currency.

$$\left(\frac{\frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d} - \gamma_{t_n}}{\gamma_{t_n} \frac{e^{r^d \Delta t} - 1}{d} - \frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d}(e^{r^f \Delta t} - 1)} \right) \leq 1$$

$$\frac{\mathbb{E}[\Gamma_{t_{n+1}}]}{d} - \gamma_{t_n} \leq \frac{1}{d}\left(\gamma_{t_n}(e^{r^d \Delta t} - 1) - c_i \gamma_{t_n}(e^{r^f \Delta t} - 1)\right) \tag{3.11}$$

Assuming that Equation 3.9 and 3.10 holds, the right and the left hand sides of the above equation are positive. Left hand side is the expected cost of not satisfying one unit of demand, and the right hand side is the expected discounted cost of carrying 1 unit of foreign currency for one period. So if the above condition is not satisfied, then the investor would try to satisfy the demand as much as possible, because not satisfying is very expensive.

3.8. Structure of the Value Functions

Let,

$$B(y^*_{t_n}) = y^*_{t_n} + \frac{c_i}{d}\mathbb{E}\left[(D_{t_n} - y^*_{t_n})^+\right] + \frac{d-1}{d}\mathbb{E}\left[(y^*_{t_n} - D_{t_n})^+\right] - \frac{c_i e^{r^f \Delta t}}{d}\mathbb{E}\left[(y^*_{t_n} - D_{t_n})^+\right]$$

Since it is possible to find $y^*_{t_n}$ $\forall n = 1, 2, \ldots, N$ explicitly, it is also possible to calculate $B(y^*_{t_n})$ $\forall n = 1, 2, \ldots, N$.

$$\begin{aligned}
G_{t_N}(y^*_{t_N}, \xi_{t_N}) &= \frac{B(y^*_{t_N})}{\xi_{t_N}} \\
G_{t_n}(y^*_{t_n}, \xi_{t_n}) &= \frac{B(y^*_{t_n})}{\xi_{t_n}} + \mathbb{E}\left[G_{t_{n+1}}(y^*_{t_{n+1}}, \Xi_{t_n+1})\right]\frac{1}{d} \\
G_{t_n}(y^*_{t_n}, \xi_{t_n}) &= \frac{1}{\xi_{t_n}} B(y^*_{t_n}) \sum_{j=0}^{N-n} \left(\frac{c_i}{d}\right)^j \\
V^*_{t_n}(x_{t_n}, \xi_{t_n}) &= \frac{1}{\xi_{t_n}}\left(B(y^*_{t_n}) \sum_{j=0}^{N-n} \left(\frac{c_i}{d}\right)^j - x_{t_n}\right)
\end{aligned} \tag{3.12}$$

3.9. Conclusion

Part one shows that dynamic programming approach in FX market has some unrealisticities, for instance the convexity condition. When the expected return of foreign currency is greater than the domestic one, model suggests to convert all of the money to higher yielding currency. Because the demand and exchange rate processes are driven by two independent uncertainties, it is impossible to hedge the demand risk by trading in foreign currency. These independent uncertainties may lead to unbounded solution in the dynamic programming, in the case where convexity condition does not hold. This problem structure is different from the classical portfolio management problem in the sense of hedging the risk of the obligation. First chapter has provided solution to the consumption problem when the demand and exchange rate processes are independent.

In Chapter 2, FX OTC options market will be studied. This time, option price will be a function of the spot price and the volatility of the spot price, which enables perfect hedging for options. Hedging the risks of option price, such as delta, gamma and vega, will be possible in this problem setting because it is possible to trade in money market, underlying security and an option with the same underlying. So the first chapter has provided solution to the problem where the demand risk is impossible to hedge and on the other hand the second chapter will be discussing the problem of option pricing where the risks of option position is possible to be hedged in the market.

4. PRICING FX-OPTIONS IN EMERGING MARKET ECONOMIES

4.1. FX Derivatives in Emerging Market Economies

There are many differences between emerging market economies (EME) and advanced economies, such as daily turnover, traded instruments and percent share of exchange trading and OTC trading. All the information contained in this section is referenced from the 2010 December report of BIS [16]. Average daily turnover of FX derivatives market in EMEs has grown four times over the past decade. Unlike advanced economies, FX derivatives are the most traded in EMEs with 50% share in the total turnover. Exchange traded and OTC traded FX derivatives have the same share in EMEs, whereas OTC has one third and exchange traded has two third of the turnover in advanced economies. See figure 4.1 for the graphs that summarizes derivatives turnover information between advanced and emerging markets.

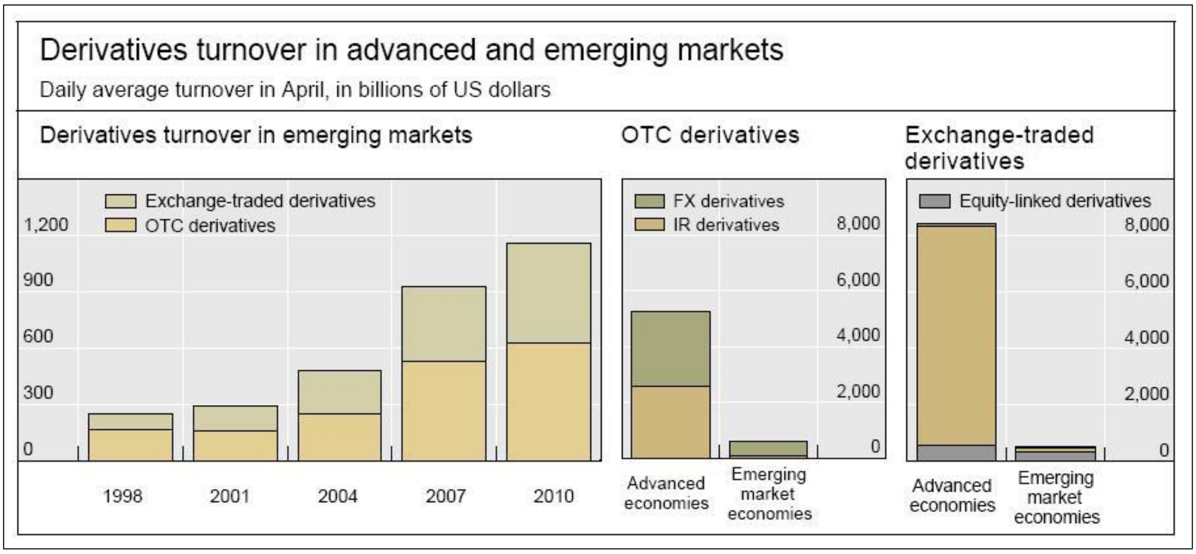


Figure 4.1. Comparison of Derivatives Turnover in Advanced and Emerging Markets.

Largest FX derivatives market among EMEs are Brazil, Korea, Hong Kong SAR and Singapore. Brazil and Korea constitutes the 90% of the turnover among EMEs.

The main reasons of usage are hedging and speculating. Figure 4.2 shows the derivatives turnover by country and also the ratio of FX and interest rate derivatives. Mostly financial institutions (30%), such as pension funds or hedge funds, commercial and investment banks (58%) are trading in FX derivatives, rest (12%) is the non-financial customers.

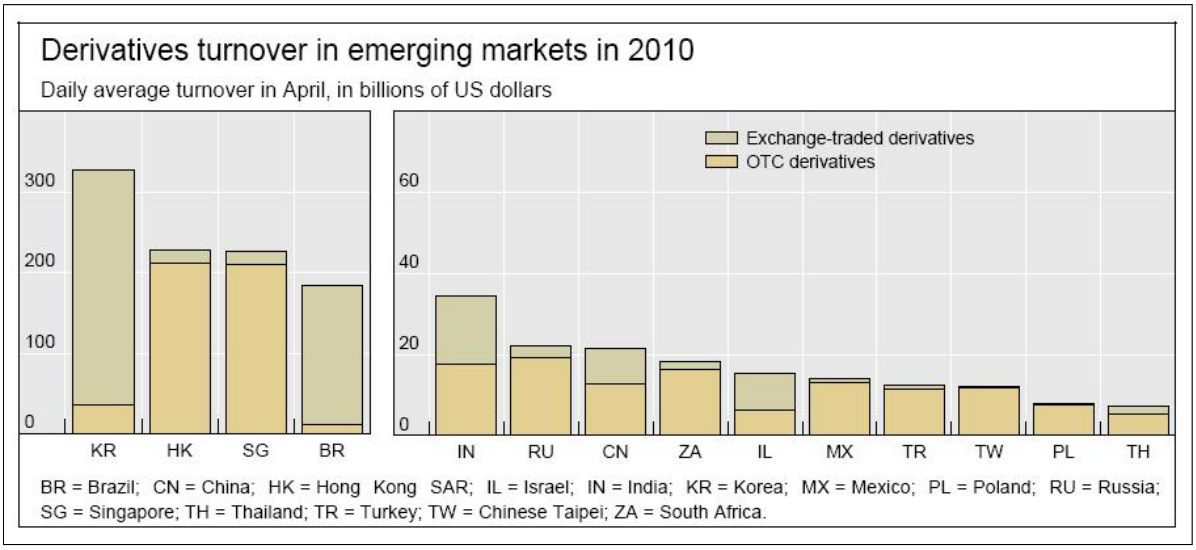


Figure 4.2. Derivatives Turnover in Emerging Markets in 2010.

Figure 4.3 shows the FX derivatives turnover by instrument. FX swaps comprise the lion's share of turnover with 70% and the FX options are less than 10%. However trading in options also involve trading outright forward contracts and/or FX swaps in order the trader to hedge his delta risk.

Cross border transactions have increased to 67% in 2010, which is the same growth size as advanced economies. Importance of FX derivatives market is more prominent when the growth rates of interest-rate derivatives and FX derivatives markets are compared in EMEs. The growth rate of interest-rate derivatives market is -8% since 2007. The main reason to this situation is a mojar dealer, which is accounted for 40-50% of interest-rate derivatives, has shifted its trading desk during crisis. Rest of the interest-rate derivatives dealers in EMEs increased their turnover however there is still huge gap between these two derivatives market in EMEs. In advanced economies, the growth rate of interest-rate derivatives market is 24% from 2007 to 2010. The reason of controversy between advanced economies and EMEs is liquidity of bond and

Foreign exchange derivatives turnover by instrument, counterparty and location Daily averages in April, in billions of US dollars and percentages						
	Emerging market economies				Advanced economies	
	2004	2007	2010	% share	2010	% share
OTC FX derivatives	**159**	**299**	**380**	**100**	***2,110***	***100***
Outright forwards	21	47	73	19	*402*	*19*
FX swaps	125	231	277	73	*1,488*	*71*
Currency swaps	3	4	7	2	*36*	*2*
Currency options and others	10	18	24	6	*184*	*9*

Figure 4.3. FX Derivatives Turnover by Instrument.

money markets in EMEs. With an increasing interest for the FX derivatives market in EMEs, the transaction sizes of currencies has changed with this growth. Turnover of EUR, JPY, GBP and CHF has decreased significantly from 2004 to 2010. Local currency and US dollar are more preferred in transactions. The reasons for such growth in FX derivatives market can be related to strong growth of internal trade, import and export, financial globalization and increase of per capita income. The FX derivatives market is fast growing and one of the largest market in EMEs, which requires accurate pricing of derivatives and calculating extreme event probabilities.

4.2. FX Options Market Dynamics

4.2.1. Introduction

In options market it is common to summarize the information of the vanilla options in volatility smile (surface). When the market prices of vanilla options are inverted in Black-Scholes option pricing formula, the implied volatilities are obtained. These volatilities and moneyness levels (and different time-to-maturities) are represented in volatility smile (surface). Moneyness level of an option can be represented by its strike price or by any linear or non-linear transformation of strike, like log-moneyness, or delta. Market participants can not generally observe the smile directly in FX OTC derivatives market. This situation is opposite to what happens in equity market, where strike versus price or strike versus volatility pairs can be observed. In FX OTC derivatives market, delta specific risk-reversals (RR), strangles (STR) and at-the-money (ATM) volatilities are directly observed for a given time-to-maturity and currency pair. The reason for that is they are the most traded portfolios. Sample quotation is represented in Table 4.1. These quotes can be used to construct a volatility smile, which then one can extract the volatility for any delta or strike price.

Table 4.1. Sample Quotation in FX market

Date	ATM	25ΔRR	10ΔRR	25ΔSTR	10ΔSTR
04.01.2011	11.595	2.2775	3.93	0.51	1.4775
03.01.2011	11.535	2.3025	4.0025	0.505	1.5125

Rest of this chapter are organized as follows: Definitions of spot, forward and options, delta conventions, and ATM conventions. Most of the definitions and formulas in the rest of this chapter are obtained from the paper Reiswich *et al.* [4].

4.2.2. Spot, Forward and Options Price

<u>4.2.2.1. Spot Rate.</u> The FX spot rate S_t = foreign-domestic represents the amount of domestic currency needed to buy one unit of foreign currency at time t. For example, USD-TRY= 1.5100 means that 1.5100 TRY can buy 1 USD. In this example USD is the foreign currency (FOR) and TRY is the domestic currency (DOM). The term foreign does not refer to any geographical region, rather it means underlying just like in equity market. The term domestic also doesn't refer to any geographical region, it refers to numeraire currency.

<u>4.2.2.2. FX Outright Forward Rate.</u> Outright forward contract is the most liquid hedge contract that trades at time t at a zero cost and at time T there is exchange of notionals at pre-specified are $f(t,T)$. Long leg of the contract will give N units of domestic currency and short leg will give $N \times f(t,T)$ units of foreign currency. The outright

forward rate is related to spot via the formula:

$$f(t,T) = S_t e^{(r_d - r_f)\tau} \tag{4.1}$$

where r_d is the continuously compounded risk-free interest rate of the domestic currency, r_f is the continuously compounded risk-free interest rate of the foreign currency, τ is the time-to-maturity, which is also equal to $T - t$

4.2.2.3. FX Forward Value. Value of the forward contract is zero at the time it is traded but as the markets move, the value of the contract does not stay at zero rather it worths;

$$v_f(t,T) = e^{-r_d\tau}(f(t,T) - K) = S_t e^{-r_f\tau} - K e^{-r_d\tau} \tag{4.2}$$

where K is the pre-specified exchange rate. The value of the forward contract is in terms of domestic currency.

4.2.2.4. FX Vanilla Option Price. In FX markets, options are usually physically settled, i.e. buyer of the vanilla call option will receive a FOR currency with notional amount of N and gives $N \times K$ units of DOM currency. Price of such a vanilla option is computed by the Black-Scholes formula

$$\begin{aligned} v(S_t, K, \sigma, \phi) &= \phi \left[e^{-r_f\tau} S_t N(\phi d_+) - e^{-r_d\tau} K N(\phi d_-) \right] && (4.3) \\ &= \phi e^{-r_d\tau} \left[f(t,T) N(\phi d_+) - K N(\phi d_-) \right] && (4.4) \end{aligned}$$

where $d_{\pm} = \frac{\ell n\left(\frac{f(t,T)}{K}\right) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$, ϕ is $+1$ for call options and -1 for put options, K is the strike price of the option, σ is Black-Scholes volatility and $N(x)$ is the cdf of standard normal distribution.

4.2.3. Delta Conventions

In FX OTC options market, moneyness level of options are represented by its delta as previously mentioned, however there are two characteristics of the delta type. One characteristic is being spot or forward delta, meaning that one can hedge his option position either with the spot or with the forward contract. Second characteristic is premium-adjusted or unadjusted delta. In FX options market, the premium of the some currency pairs are foreign currency and sometimes deltas are adjusted with the option premium. This situation is like, shorting a stock option in equity market and receiving stock as premium instead of money. However not all the deltas are adjusted even the premium is paid in foreign currency. For example USDTRY currency pair, the premium currency is USD but the quoted delta volatility pair information are not premium-adjusted. Analogously when an option is shorted in USDTRY currency, short position will receive underlying (USD) as premium, although this kind of situation never happens in equity markets.

So with these two characteristics, there are four different delta types that are used in practice. Those are "Spot Delta", "Forward Delta", "Premium Adjusted Spot Delta" and "Premium Adjusted Forward Delta".

4.2.3.1. Spot Delta. Spot delta is the first derivative of Black-Scholes price (4.4) with respect to S_t.

$$\begin{aligned} \Delta_S(K,\sigma,\phi) &= \frac{\partial v}{\partial S} \\ \Delta_S(K,\sigma,\phi) &= \phi e^{-r_f \tau} N(\phi d_+) \end{aligned} \tag{4.5}$$

And the put-call delta parity is:

$$\Delta_S(K,\sigma,+1) - \Delta_S(K,\sigma,-1) = e^{-r_f \tau} \tag{4.6}$$

In FX market, one needs to buy $\Delta_S \times N$ foreign currency in order to hedge a short vanilla position, or equivalently needs to sell $\Delta_S \times N \times S_t$ units of domestic currency.

When calibrating a model to the market data, one will need to extract the strike volatility pairs from delta volatility pairs. So the strike price must be needed to backed-out from the above formula (Equation 4.5) and after simple algebra the formula for the strike price is given below when the spot delta type is used.

$$\frac{K}{f} = exp\left\{ -\phi N^{-1}(\phi e^{r_f \tau} \Delta_S)\sigma\sqrt{\tau} + \frac{1}{2}\sigma^2\tau \right\} \tag{4.7}$$

4.2.3.2. Forward Delta. Forward delta is the first derivative of Black-Scholes price (4.4) with respect to v_f. This time the position is hedged with forward contract instead of spot.

$$\begin{aligned} \Delta_f(K, \sigma, \phi) &= \frac{\partial v}{\partial v_f} = \frac{\partial v}{\partial S}\frac{\partial S}{\partial v_f} = \frac{\partial v}{\partial S}\left(\frac{\partial v_f}{\partial S}\right)^{-1} \\ \Delta_f(K, \sigma, \phi) &= \phi N(\phi d_+) \end{aligned} \tag{4.8}$$

And the put-call delta parity is:

$$\Delta_f(K, \sigma, +1) - \Delta_f(K, \sigma, -1) = 1 \tag{4.9}$$

In FX market, one needs to enter $\Delta_f \times N$ amount of forward contracts in order to hedge a short vanilla position. Forward delta type is very commonly used convention in most of the currency pairs, because of the fact that delta of a call and absolute value of the delta of a put adds up to 1. i.e 10ΔP and 90ΔC has the same volatility.

Again in the calibration phase, extraction of the strike volatility pairs from delta volatility are needed. So the formula for the strike price is given below when the forward delta type is used,

$$\frac{K}{f} = exp\left\{ -\phi N^{-1}(\phi \Delta_f)\sigma\sqrt{\tau} + \frac{1}{2}\sigma^2\tau \right\} \tag{4.10}$$

4.2.3.3. Premium-Adjusted Spot Delta. The premium-adjusted spot delta makes the correction induced by the option premium. Since the option premium is paid in foreign currency, the actual amount needed to hedge the short position must decrease by the amount of option value. So it can be represented as,

$$\Delta_{S,pa} = \Delta_S - \frac{v}{S}$$

In this delta convention, one needs to buy $N \times \left(\Delta_S - \frac{v}{S}\right)$ amount of foreign currency in order to hedge the short option position. Equivalently, one needs to sell $N \times (S_t \Delta_S - v)$ units of domestic currency in order to hedge the short position. Another way of finding the delta in domestic currency units is flipping around the quotation and computing delta, meaning that taking the partial derivative of $\frac{v}{S}$ with respect to $\frac{1}{S}$, because the value of the option, $\frac{v}{S}$, is in domestic currency now and $\frac{1}{S}$ is $\frac{DOM}{FOR}$. By flipping the quotation domestic currency now becomes the underlying and foreign currency becomes numeraire, and since the premium is paid in foreign currency now the formula for delta should be $S_t \Delta_S - v$

$$\begin{aligned} \frac{\partial \frac{v}{S}}{\partial \frac{1}{S}} &= \frac{\partial \frac{v}{S}}{\partial S} \frac{\partial S}{\partial \frac{1}{S}} \\ &= \frac{S v_S - v}{S^2} \left(\frac{\partial \frac{1}{S}}{\partial S}\right)^{-1} = \frac{S v_S - v}{S^2} \left(-\frac{1}{S^2}\right)^{-1} \\ &= -(S v_S - v) \quad \text{DOM to buy} = S v_S - v \quad \text{DOM to sell} = v_S - \frac{v}{S} \quad \text{FOR to buy} \end{aligned}$$

which confirms the definition of premium-adjusted spot delta and the formula is as follows:

$$\Delta_{S,pa}(K, \sigma, \phi) = \phi e^{-r_f \tau} \frac{K}{f} N(\phi d_-) \tag{4.11}$$

Put-call delta parity relation becomes:

$$\Delta_{S,pa}(K, \sigma, +1) - \Delta_{S,pa}(K, \sigma, -1) = e^{-r_f \tau} \frac{K}{f} \tag{4.12}$$

The strike versus delta relation is not injective this time, because there is variable K in d_- and also as a multiplier of $N(.)$. So there might exist more than one strike per delta. The relationship between delta and strike is as follows,

$$\Delta_{S,pa}(K,\sigma,\phi) = \phi\frac{K}{f}e^{-r_f\tau}N\left(\phi\frac{\ell n\left(\frac{f}{K}\right)-\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right) \tag{4.13}$$

In order to find the strike, the authors of the reference paper of this chapter [4] suggest to search for $K \in [K_{min}, K_{max}]$. Where K_{max} is simply the strike of spot delta case and K_{min} is the solution of the following equation,

$$\sigma\sqrt{\tau}N(d_-) = n(d_-) \tag{4.14}$$

with $n(.)$ being the probability density function of standard normal distribution. After finding the bounds, solve the Equation 4.13 in terms of K, and that strike would be the corresponding strike for the premium-adjusted spot delta.

4.2.3.4. Premium-Adjusted Forward Delta. Just like in the premium-adjusted spot delta case, the hedge quantity needs to be adjusted because of the premium paid in foreign currency, and the resulting formula for the premium-adjusted forward delta is,

$$\Delta_{f,pa}(K,\sigma,\phi) = \phi\frac{K}{f}N(\phi d_-) \tag{4.15}$$

The Put-call delta parity relation becomes,

$$\Delta_{f,pa}(K,\sigma,+1) - \Delta_{f,pa}(K,\sigma,-1) = \frac{K}{f} \tag{4.16}$$

Again the strike delta relation is not injective like in Spot-pa delta case. However the technique is the same as Spot-pa case, search for $K \in [K_{min}, K_{max}]$. This time K_{max} is the strike of forward delta, however K_{min} is the same as in Spot-pa case, it is the solution of K in the Equation 4.14.

4.2.4. At-the-Money Convention

Since it is an attempt to specify the middle of the spot distribution, at-the-money (ATM) definition may not be obvious. One can think of many definitions for ATM, such as

ATM-spot	$K = S_0$
ATM-forward	$K = f$
ATM-Δ-neutral	K such that call delta $= -$ put delta

The delta-neutral ATM definition has sub-categories depending on which delta type is used. This ATM convention is default for short-dated FX options. The strike prices and deltas for every combination of ATM definition are given in the following Table 4.2.

Table 4.2. Strike and Delta Values for Different Type of Delta Conventions.

	Δneutral strike	Forw strike	Spot strike	Δneutral Delta	ATM Forward Delta	ATM Spot Delta
Spot Delta	$fe^{\frac{1}{2}\sigma^2\tau}$	f	S_0	$\frac{1}{2}\phi e^{-r_f\tau}$	$\phi e^{-r_f\tau}N(\phi\frac{1}{2}\sigma\sqrt{\tau})$	$\phi e^{-r_f\tau}\cdot N\left(\phi(\frac{r_d-r_f}{\sigma}\sqrt{\tau}+\frac{1}{2}\sigma\sqrt{\tau})\right)$
Forward Delta	$fe^{\frac{1}{2}\sigma^2\tau}$	f	S_0	$\frac{1}{2}\phi$	$\phi N(\phi\frac{1}{2}\sigma\sqrt{\tau})$	$\phi N\left(\phi(\frac{r_d-r_f}{\sigma}\sqrt{\tau}+\frac{1}{2}\sigma\sqrt{\tau})\right)$
Spot Delta p.a.	$fe^{-\frac{1}{2}\sigma^2\tau}$	f	S_0	$\frac{1}{2}\phi e^{-r_f\tau-\frac{1}{2}\sigma^2\tau}$	$\phi e^{-r_f\tau}N(-\phi\frac{1}{2}\sigma\sqrt{\tau})$	$\phi e^{-r_d\tau}\cdot N\left(\phi(\frac{r_d-r_f}{\sigma}\sqrt{\tau}-\frac{1}{2}\sigma\sqrt{\tau})\right)$
Forward Delta p.a.	$fe^{-\frac{1}{2}\sigma^2\tau}$	f	S_0	$\frac{1}{2}\phi e^{-\frac{1}{2}\sigma^2\tau}$	$\phi N(-\phi\frac{1}{2}\sigma\sqrt{\tau})$	$\phi e^{(r_f-r_d)\tau}\cdot N\left(\phi(\frac{r_d-r_f}{\sigma}\sqrt{\tau}-\frac{1}{2}\sigma\sqrt{\tau})\right)$

4.3. The Heston Model

4.3.1. Introduction

In option pricing theory, Black&Scholes [14] published a benchmark paper that option prices were related with the distribution of spot returns. Although the Black-Scholes european option pricing formula is successful in relating distribution of spot and option prices, it does not explain the smile effect. Meaning that options with different strike prices and time-to-maturities have different implied volatilities. When the market option prices are inverted with the Black-Scholes formula to obtain volatility, with known strike, time-to-maturity, risk-free-interest rate and spot price, it is called BS-implied-volatility. This situation is conflicting with the model assumption that the spot price process has constant volatility, since it has been observed that options written on the same underlying asset with varying strikes and time-to-maturities actually have different volatilities. One simple way of handling this situation is using different models for every different strike and time-to-maturity pairs, so that these models would capture the different volatilities across strike and time-to-maturity. However using different models would result inconsistency in the management of option books that contains several levels of strikes and time-to-maturities. So the literature continued with the relaxation of this assumption.

Next step in the literature is local volatility models. Merton [5] suggested that making the volatility a function of time. Although this approach explains the different implied volatilities across time-to-maturities, it fails to explain the smile shape across strike prices. Dupire [6], Derman and Kani [7], and Rubinstein [8] came up with the idea that making the volatility not only a function of time, but a function of state variables. This approach was successful in fitting the volatility surface, however it failed to explain the persistent smile shape which does not vanish as the time passes.

Failure of local volatility models is followed by the idea of making the volatility a stochastic process. With the pioneering work of Scott [9], Hull and White [10], and Wiggins [11] the idea of stochastic volatility is further developed. These models have the disadvantage of not having closed form solution for european type of options and also they require extensive use of numerical techniques.

The Heston's stochastic volatility model [12] is different from the other models for two reasons. First one is the stochastic process for the volatility is non-negative and mean-reverting, which is observed in the market. Second one is there exists a semi-analytical formula for european type of options, which one can easily implement. The second advantage is very important in calibrating the model to the market data. So these two advantages made Heston's model very popular and many practitioners uses this model in front office implementation.

4.3.2. The Model Definition

Heston [12] proposed the following model:

$$dS_t = \mu S_t dt + \sqrt{V}_t S_t dW_t^1 \tag{4.17}$$

$$dV_t = \kappa \left(\theta - V_t\right) dt + \sigma \sqrt{V}_t dW_t^2 \tag{4.18}$$

$$dW_t^1 dW_t^2 = \rho dt \tag{4.19}$$

where S_t and V_t are the spot price and variance processes respectively. W_t^1 and W_t^2 are correlated brownian motions with correlation parameter ρ. The intuition of the parameters of the mean-reverting variance process are: long term variance level θ, rate of mean reversion κ and volatility of volatility, sometimes referred as volofvol, σ. All the parameters are independent of time and the current state of the system. Note that parameter μ will be replaced with $r^d - r^f$ in risk-neutral-world, so the parameters that will be calibrated to market data would be: κ, θ, σ, ρ and initial variance v_0.

4.3.3. Option Pricing Formula

Consider a contingent claim, whose value is $G(t, v, S)$ at time t, paying $G(T, v, S)$ at time T. Since the Heston model has two sources of uncertainty, the self-financing portfolio must include the possibilities of trading in the money market, underlying and another derivative whose value function is $V(t, v, S)$. The differential of the process X is as follows:

$$dX = \Delta dS + \Gamma dV + r_d \left(X - \Gamma V - \Delta S \right) dt + r_f \Delta S dt$$

where Δ is the number of underlyings and Γ is the number of derivative securities V, held at time t. The aim is constructing a portfolio that has initial wealth of X_0, and finding Δ and Γ so that $X_t = G(t, v, S)$ for all $t = [0, T]$. The standard approach is to compare the differentials of the processes X and G. After some algebra the partial differential equation which G must satisfy, in order the market to be arbitrage free, is as follows:

$$\begin{aligned} &\frac{1}{2} v S^2 \frac{\partial^2 G}{\partial S^2} + \rho \sigma v S \frac{\partial^2 G}{\partial S \partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 G}{\partial v^2} + \left(r_d - r_f \right) S \frac{\partial G}{\partial S} \\ &\qquad + \left\{ \kappa \left(\theta - v \right) - \lambda(t, v, S) \right\} \frac{\partial G}{\partial v} + \frac{\partial G}{\partial t} - r_d G = 0 \end{aligned} \tag{4.20}$$

Note that $\lambda(t, v, S)$ is the market price of volatility risk and every different level of $\lambda(t, v, S)$ would lead to different risk-neutral measure, meaning that the risk-neutral measure is not unique and that the market is incomplete. However this is an expected result for the Heston model, because there is another brownian motion for the variance process. The above partial differential equation can be solved with the appropriate boundary conditions. For an European type of option, these conditions are:

$$G\left(T, v, S\right) = max\left\{ \phi(S - K), 0 \right\} \tag{4.21}$$

$$G\left(t, v, 0\right) = \frac{1 - \phi}{2} K e^{-r_d \tau} \tag{4.22}$$

$$\frac{\partial G}{\partial S}\left(t, v, \infty\right) = \frac{1 + \phi}{2} e^{-r_f \tau} \tag{4.23}$$

$$r_d G\left(t, 0, S\right) = \left(r_d - r_f \right) S \frac{\partial G}{\partial S}\left(t, 0, S\right) + \kappa \theta \frac{\partial G}{\partial v}\left(t, 0, S\right) + \frac{\partial G}{\partial t}\left(t, 0, S\right) \tag{4.24}$$

$$G\left(t, \infty, S\right) = \begin{cases} S e^{-r_f \tau} & \phi = +1 \\ K e^{-r_d \tau} & \phi = -1 \end{cases} \tag{4.25}$$

where K is the strike price, $\phi = \pm 1$ for call and put options respectively and $\tau = T - t$ is the time-to-maturity. Heston solved the partial differential equation analytically and the european FX option price is given by:

$$H(\kappa,\theta,\sigma,\rho,\lambda,r_d,r_f,v_t,S_t,K,\tau,\phi) = \phi\left\{S_t e^{-r_f\tau}P_+(\phi) - Ke^{-r_d\tau}P_-(\phi)\right\} \tag{4.26}$$

where $u_{1,2} = \pm\frac{1}{2}$, $b_1 = \kappa+\lambda-\sigma\rho$, $b_2 = \kappa+\lambda$,

$$d_j = \sqrt{(\rho\sigma\psi i - b_j)^2 - \sigma^2(2u_j\psi i - \psi^2)} \tag{4.27}$$

$$g_j = \frac{b_j - \rho\sigma\psi i + d_j}{b_j - \rho\sigma\psi i - d_j} \tag{4.28}$$

$$C_j(\tau,\psi) = (r_d - r_f)\psi i\tau + \frac{\kappa\theta}{\sigma^2}\left\{(b_j - \rho\sigma\psi i + d_j)\tau - 2log\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right\} \tag{4.29}$$

$$D_j(\tau,\psi) = \frac{b_j - \rho\sigma\psi i + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right) \tag{4.30}$$

$$f_j(logS_t,v_t,\tau,\psi) = exp\left\{C_j(\tau,\psi) + D_j(\tau,\psi)v_t + i\psi logS_t\right\} \tag{4.31}$$

$$P_j(logS_t,v_t,\tau,logK) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty R\left\{\frac{e^{-i\psi logK}f_j(logS_t,v_t,\tau,\psi)}{i\psi}\right\}d\psi \tag{4.32}$$

$$P_+(\phi) = \frac{1-\phi}{2} + \phi P_1(logS_t,v_t,\tau,logK) \tag{4.33}$$

$$P_-(\phi) = \frac{1-\phi}{2} + \phi P_2(logS_t,v_t,\tau,logK) \tag{4.34}$$

The above formulas require integration of complex logarithms and this situation causes numerical instability. Efficient transformation of g_j, C_j and D_j are proposed in the paper of Janek et. al. [17], so when coding in Matlab the Equations 4.28, 4.29 and 4.30 are changed with the following ones,

$$\tilde{g}_j = \frac{1}{g_j} = \frac{b_j - \rho\sigma\psi i - d_j}{b_j - \rho\sigma\psi i + d_j} \tag{4.35}$$

$$C_j(\tau,\psi) = (r_d - r_f)\psi i\tau + \frac{\kappa\theta}{\sigma^2}\left\{(b_j - \rho\sigma\psi i - d_j)\tau - 2log\left(\frac{1 - \tilde{g}_j e^{-d_j\tau}}{1 - \tilde{g}_j}\right)\right\} \tag{4.36}$$

$$D_j(\tau,\psi) = \frac{b_j - \rho\sigma\psi i - d_j}{\sigma^2}\left(\frac{1 - e^{-d_j\tau}}{1 - \tilde{g}_j e^{-d_j\tau}}\right) \tag{4.37}$$

Above formulas are coded in Matlab and details of the code are given in Appendix A.

4.4. Calibration

4.4.1. Effects of Parameters in the Smile and in Implied Distribution

Before starting the calibration of model to the market data, it is preferred to examine firstly the qualitative effects of parameters on the smile and also on the implied distribution (ImpDist). Knowing which parameter effects which characteristic of the smile (or ImpDist) will help to better understand the calibration results and possibly give clues about which parameters to fix before calibration.

Empirical studies show that the distribution of asset's log-return is non-Gaussian, which conflicts with the Black-Scholes model assumption. Classical BS model assumes Gaussian log-return distribution. The log-return distribution with high peaks and heavy tails is referred as leptokurtic and Heston's model can generate these kinds of distributions.

The parameter ρ, correlation between the log-returns and volatility, affects the heaviness of the tails, meaning that ρ being positive will make the right tail fatter and squeeze the left tail, and being negative will do the reverse. Therefore ρ affects the skewness of the distribution. In particular positive ρ makes calls more expensive, while negative ρ makes puts more expensive. Figure 4.4 shows the effect of different values of ρ on the distribution and smile. From the figure one can also imply the relation between the smile and ImpDist, such as; if the right end of the smile is higher than the left end side in the smile, it means the ImpDist is right-skewed, and vise-versa.
Volatility of the volatility, σ, effects the kurtosis (peakness) of the distribution. Increasing σ will result in peaker distributions and that will cause heavier tails on both sides. In the smile, increasing sigma increases the convexity of the fit. Higher volofvol means market has greater potential to make extreme movements, therefore the price of both call and put options should go up. Figure 4.5 shows the effect of different values of σ on the distribution and smile. Again from the figure one can realize that convexity of the smile increases as σ increases. In other words, probability of extreme event increases in the market, which also means that ImpDist has heavier tails.

Changing initial variance, v_t affects the height of the smile curve. Moreover the long term variance θ has similar effect on the smile. That is why in the paper "FX

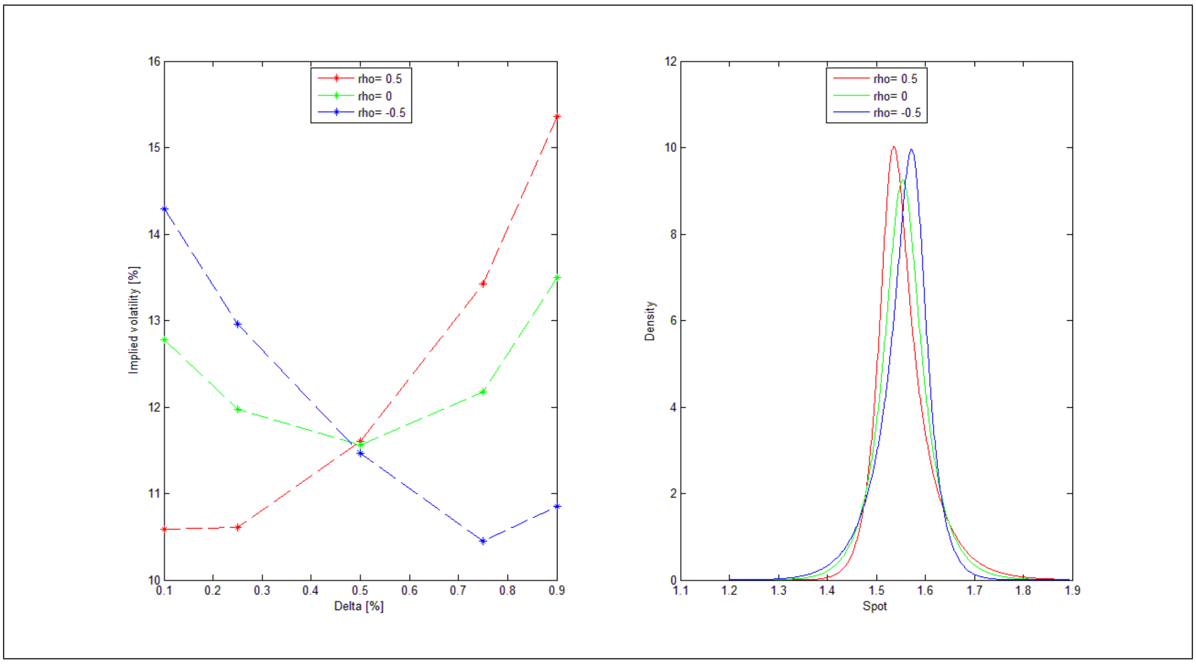


Figure 4.4. The Effect of Changing Parameter ρ.

Smile in the Heston Model" [17], the authors suggest fixing the initial variance at ATM volatility and let θ vary. Figure 4.6a and Figure 4.6b show the effects of v_t and θ on the distribution and smile respectively.

The mean-reversion speed parameter, κ affects the ATM part and the wings of the smile, however its effect on the level of the curve is more prominent. When the current volatility is far away from the long term volatility level, higher κ makes current volatility approach to long term volatility faster. This situation can be referred as volatility clustering, because higher κ makes large price variations followed by large price variations. In the paper, "FX Smile in the Heston Model" [17], authors suggest to fix κ at some level and calibrate the rest of the three parameters. Figure 4.6c shows the effect of different values of κ on the distribution and smile.

4.4.2. Calibration Scheme

Calibration of stochastic volatility models are done basically in two conceptually different ways. One is estimation from the historical time seires data such as; generalized method of moments, efficient method of moments and bayesian MCMC. Second way of calibration is fitting the empirical distributions of returns to the marginal dis-

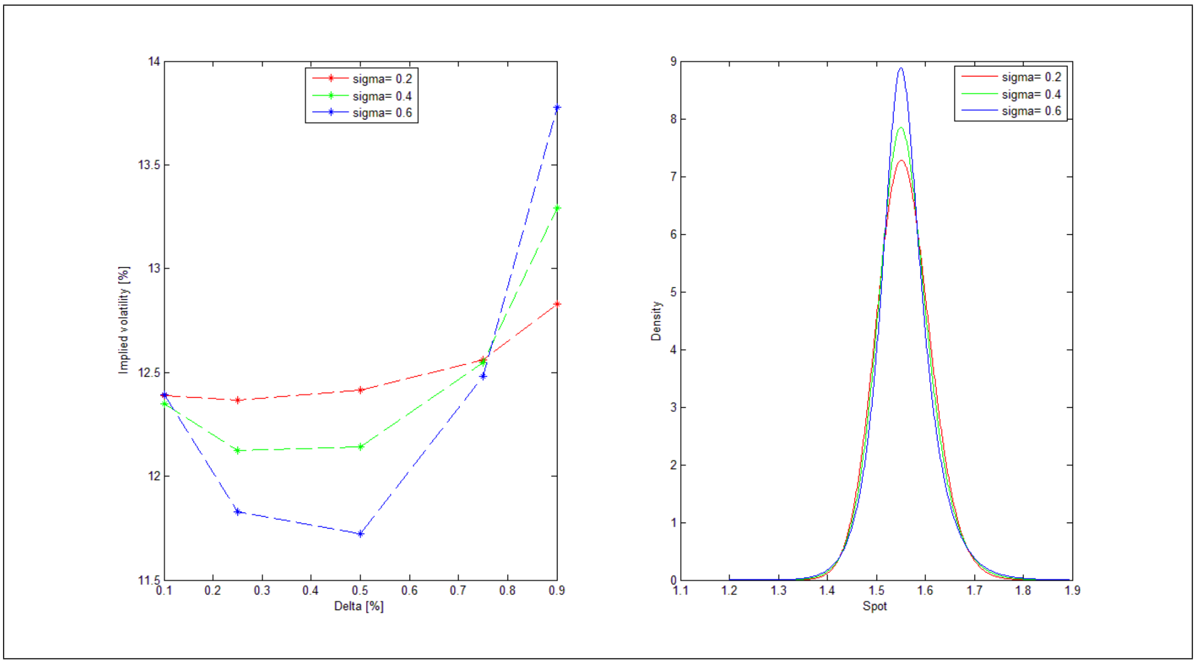
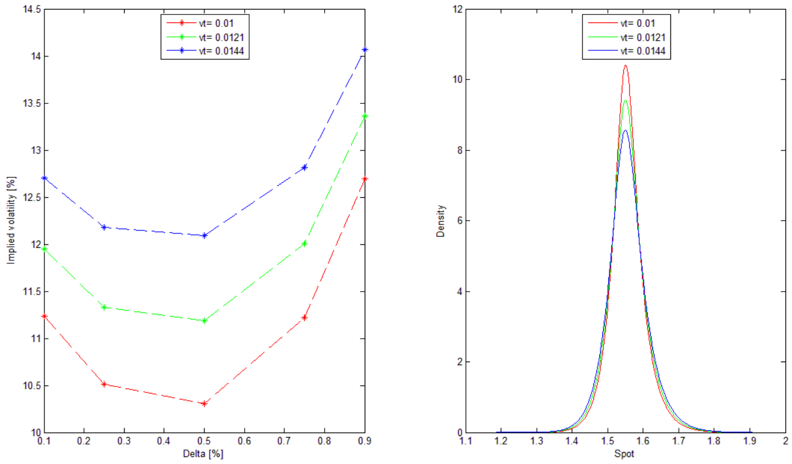
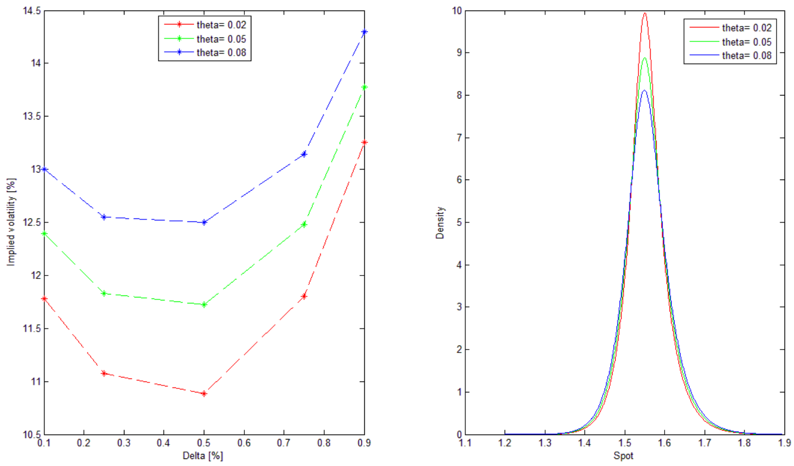


Figure 4.5. The Effect of Changing Parameter σ.

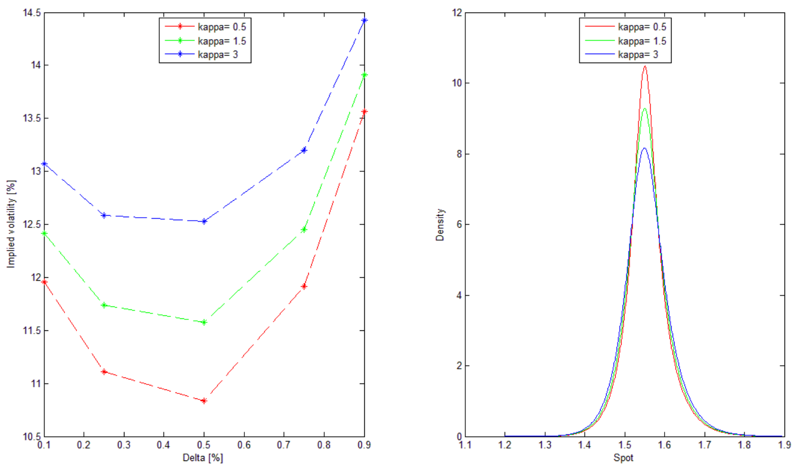
tributions via a minimization scheme. This thesis chooses to use the second way, because the historical approaches fail to estimate the market price of volatility risk, namely $\lambda(t, v, S)$. When second way of calibration scheme is used one need not worry about $\lambda(t, v, S)$, because that information is embedded in the market smile. Simply set $\lambda(t, v, S) = 0$ and continue with the calibration process.



(a) The Effect of Changing Parameter v_t.



(b) The Effect of Changing Parameter θ.



(c) The Effect of Changing Parameter κ.

Figure 4.6. The Effects of Changing Parameters v_t, θ and κ.

More specifically this approach tries to minimize the discrepancy between the model implied volatilities and market implied volatilities. i.e.

$$min \sum_{\forall i} \left[\sigma^{BS}(\Omega, K_i) - \sigma^{Mrkt}(K_i)\right]^2 \tag{4.38}$$

$\sigma^{BS}(\Omega, K_i)$ is the model implied volatility, and Ω is the parameter vector, for the i^{th} calibration object. Here note that the model implied volatility is Black-Scholes volatility, so the price argument of the Black-Scholes-implied-volatility function should be parametric Heston price. More specifically, after finding the heston price of the i^{th} calibration object (price is a function of Ω), the Black-Scholes volatility needed to be backed-out from that price. After computing the parametric volatility, the i^{th} element of the objective function is constructed by taking the squared differences of model volatility and market volatility.

As a preliminary step, one needs to retrieve the strike versus volatility pairs from the delta versus volatility pair information. This step can be done by using the formulas given in Section 4.2.3. Next thing is to fix the parameter, v_0 to the σ^2_{ATM}. As it is argued previously, v_0 affects the level of the smile curve and fixing it to σ^2_{ATM} is not insensible. This step is suggested in the paper Janek [17], and the authors of that paper also suggest to fix the parameter κ to some level, say 1.5, and optimize the rest of the parameters, θ, σ and ρ. The Matlab function of calibration is provided in Appendix B.

To summarize the calibration step by step:

(i) Retrieve the $\sigma(K_i)$ from the data $\sigma(\Delta_i)$

(ii) Decide which parameters to fix, such as $v_0 = \sigma^2_{ATM}$ and/or $\kappa = 1.5$

(iii) Find the heston price of the i^{th} object, $h_i(\theta, \sigma, \rho)$ assume v_0 and κ are fixed

(iv) Find the Black-Scholes implied volatility of the i^{th} object using its heston price, $h_i(\theta, \sigma, \rho)$. Matlab function for this step is “blsimpv.m”

(v) Construct the objective function by taking squared differences of model implied volatilities and market volatilities, and then summing them up. The Matlab function for this step is provided in Appendix B

(vi) Minimize the objective function using "fmincon.m" in Matlab by defining sensible bounds for the parameters, such as $-1 \leq \rho \leq 1$.

4.4.3. Sample Calibration Results

Now one can easily calibrate heston model to market data. Sample experiments are made in the USDTRY 1 month options. Table 4.3 shows the market quoted volatilities. Note that this information has to be converted to delta volatility pairs by the use of simplified formula (4.39), which is given below

$$\begin{aligned} \sigma_{\Delta C} &= \sigma_{ATM} + \frac{1}{2}\sigma_{RR} + \sigma_{STR} \\ \sigma_{\Delta P} &= \sigma_{ATM} - \frac{1}{2}\sigma_{RR} + \sigma_{STR} \end{aligned} \tag{4.39}$$

After converting quotations to delta-volatility pairs, it is now ready to be given to the calibration function as input. Note that, this delta-volatility information will be converted to strike-volatility pairs inside the calibration function. Table 4.4 shows the delta-volatility pairs of sample quotations represented in Table 4.3.

Table 4.3. Four Days Sample Quote for USDTRY 1 Month Options.

Date	Spot	r_d	r_f	σ_{ATM}	25 ΔRR	10 ΔRR	25 ΔSTR	10 ΔSTR
04.01.2011	1.5476	0.07	0.02	11.595	2.2725	3.93	0.51	1.4775
03.01.2011	1.5376	0.07	0.02	11.535	2.3025	4.0025	0.505	1.5125
31.12.2010	1.5460	0.07	0.02	11.44	2.355	4.13	0.51	1.535
30.12.2010	1.5567	0.07	0.02	11.57	2.355	4.09	0.4925	1.5525

In USDTRY OTC options market, the delta convention used is "forward-delta" and the ATM type is "forward-delta-neutral". Sample calibration results, parameter values, are represented in the Table 4.5. It can be seen from the results that parameter

Table 4.4. Delta Volatility Pairs Given in Table 4.3.

Date	Spot	rd	rf	10ΔP	25ΔP	ATM	25ΔC	10ΔC
04.01.2011	1.5476	0.07	0.02	11.1075	10.96875	11.595	13.24125	15.0375
03.01.2011	1.5376	0.07	0.02	11.04625	10.88875	11.535	13.19125	15.04875
31.12.2010	1.546	0.07	0.02	10.91	10.7725	11.44	13.1275	15.04
30.12.2010	1.5567	0.07	0.02	11.0775	10.885	11.57	13.24	15.1675

κ is more steady compared to other variables as the time passes. Fixing κ to level 1.5 does not make much difference in the estimates of the other variables, so a model with one less parameter is always better, so fixing κ seems not a bad idea. In this table, v_0 is set to σ^2_{ATM}. Note that when calibrating to volatility surface, fixing $v_0 = \sigma^2_{ATM}$ is not suggested, because then for every different time-to-maturity there will be an initial variance estimate, which is unreasonable and will result in inconsistency in the model.

In figure 4.7, sample smile results for four days are represented. In these figures, the model is calibrated with 3 parameters, θ, σ and ρ, the initial variance is set to square of ATM-volatility, $v_0 = \sigma^2_{ATM}$, and $\kappa = 1.5$. It can be seen that heston model has fitted the market data almost perfectly.

There are mainly three parameters in the model, when calibrating to smile, those are θ, σ and ρ. The long-run average variance level, θ, is related to the level of the smile, which corresponds to ATM level; vol-of-vol, σ, is related with the convexity of the smile, which corresponds to quoted strangle volatilities; and correlation parameter, ρ, is related with the skewness of the smile, which also corresponds to quoted risk-reversals. These direct relationship of the parameters with the quoted volatilities makes the heston model more attractive.

4.5. Trading Algorithm

After calibrating the Heston's model to the market data, it is now time to construct trading strategies. The most basic trading strategy in the literature is two

Table 4.5. Calibration Results for 1 Month USDTRY Option Data for 10 Days.

	When κ is free				When $\kappa = 1.5$		
Date	κ	θ	σ	ρ	θ	σ	ρ
04.01.2011	1.517	0.0502	0.6621	0.4019	0.0506	0.6617	0.4019
03.01.2011	1.5169	0.0507	0.6712	0.4054	0.0511	0.6708	0.4055
31.12.2010	1.5171	0.0509	0.6769	0.4159	0.0513	0.6765	0.4159
30.12.2010	1.5166	0.0516	0.684	0.4087	0.052	0.6836	0.4087
29.12.2010	1.517	0.0516	0.6842	0.4201	0.052	0.6838	0.4201
28.12.2010	1.5175	0.0499	0.6682	0.4176	0.0503	0.6678	0.4176
27.12.2010	1.5178	0.0496	0.6601	0.4207	0.05	0.6597	0.4208
24.12.2010	1.5177	0.0501	0.6658	0.4229	0.0505	0.6654	0.4229
23.12.2010	1.5171	0.0506	0.6765	0.4123	0.051	0.6761	0.4123
22.12.2010	1.5181	0.0494	0.6516	0.4205	0.0498	0.6512	0.4206

standard deviation rule, and in this study two standard deviation rule is used to construct trading signals.

The rule is simple, when the time series hits its $\mu + 2\sigma$ level the sell signal is generated and when it hits the $\mu - 2\sigma$ level the buy signal is generated. According to the Law of One Price, time series is mean-reverting and when it deviates too much from its long term mean, it will start to close that deviation by returning to its mean. The rule is simple, straight-forward and widely used in practice specifically in pairs trading. Figure 4.8 demonstrates the strategy on the line chart in order to understand better.

4.5.1. Two Standard Deviation Rule in Trading Exchange Rate

In this section, the previously defined trading strategy will be constructed on the exchange rate (spot price). Differently from the literature, signal generating process will be different from the one that is traded, in our example it is spot price. The

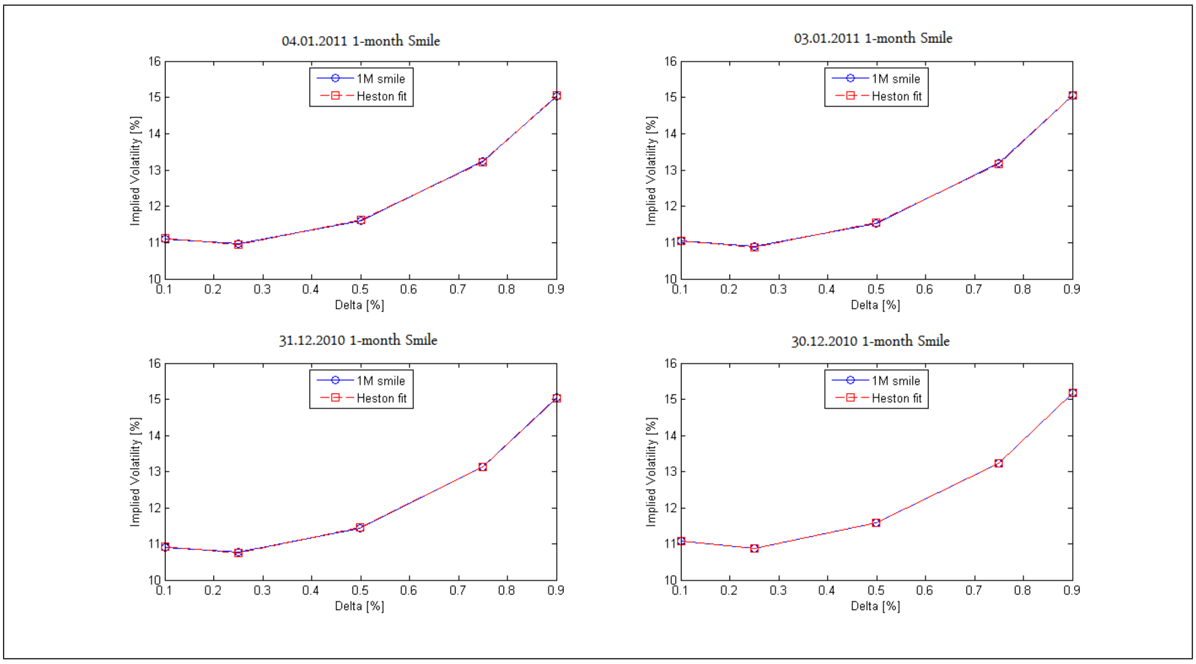


Figure 4.7. Sample Smile Fits for 4 Days.

candidates for the signal generating process are: 10 Delta Risk-Reversal, 25-Delta Risk-Reversal, Interest Rate Differential ($r^d - r^f$), Heston's model parameter rho (ρ), and the skewness parameter of the log-normal distribution. Note that all of these candidates are kind of measure of skewness in the implied distribution. The idea is, interest rate differential and spot price is highly correlated in emerging market currencies, so the candidates are all belong to same class in a sense. Before starting to construct the trading signal, a simple regression analysis is made in order to have an intuition among the candidates. The dependent variable in the regression model is spot price, and the independent ones are all of the candidates listed above. Results of the Linear Regression is given in Figure 4.9.

The P-values of the covariates 10 Delta and 25 Delta Risk-Reversal are too high to be in the model, so the type one error will be very small when we assume that the coefficients of those are zero and continue with the regression model without them. Results of the second regression model is given in Figure 4.10. As it can be seen easily, the covariates rho and skewness are very significant compared to Rate Differential. Trading signal will be constructed individually by using these covariates and decide on which one will perform best, however for the sake of completeness the results of

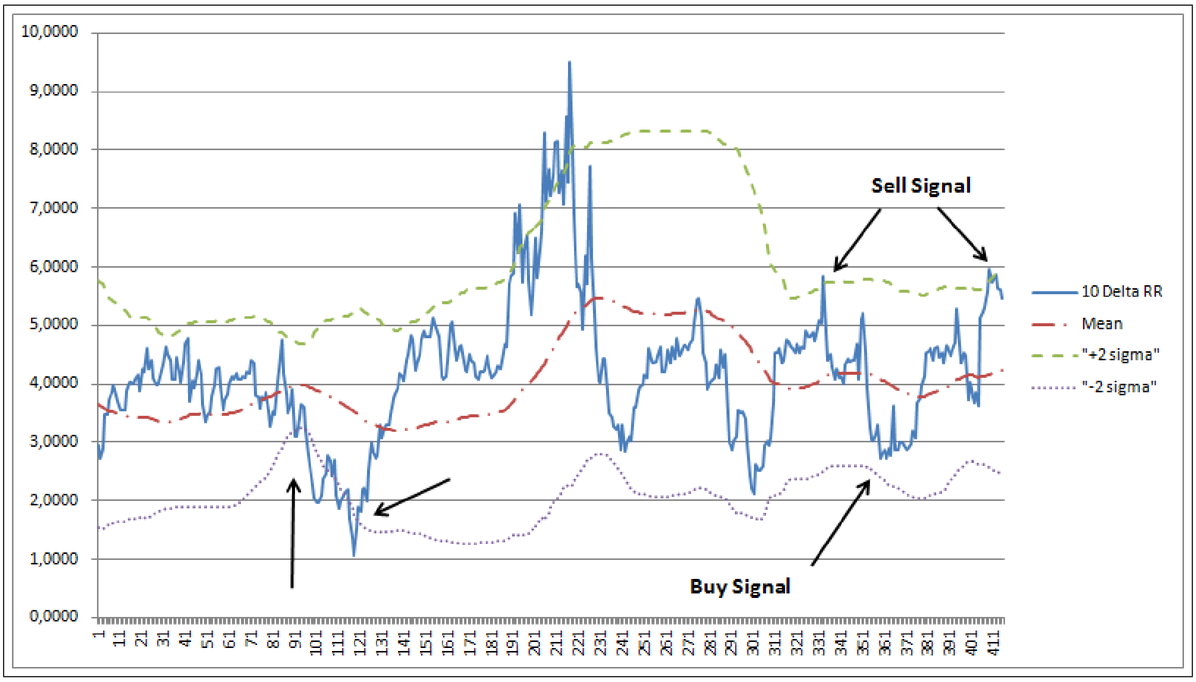


Figure 4.8. Two Standard Deviation Rule.

the third regression is provided in Figure 4.11. This last regression has two covariates; those are rho and skewness.

Now the signal generating process will be selected from the covariates and according to the signal coming from that, long or short position will be taken in the spot price. Specifically when rho is selected as the signal generating one, the 90-days moving average and standard deviation will be calculated. After that the sigma position ($\frac{x-\mu}{\sigma}$) will be calculated and if it is above 2, sell signal will be generated and if it is below -2 then the buy signal will be generated.

The results of the back-test is provided in the Table 4.6. To make the results more realistic, a take profit (3%) and stop loss limits (2%) are applied in the back-test, since FX market is too volatile to trade without take-profit and stop-loss limits in practice.

The results are promising in the case of rho, since the strategy results in an annualized return of 26%. The success probability of the signal is 0.67, meaning that the signal gives correct direction, nearly two third of the time. The back-test results of skewness and rate differential are given in Table 4.7 and 4.8.

When skewness and interest-rate differential is used as the sigma position input,

Regression Statistics	
Multiple R	0.746062206
R Square	0.556608815
Adjusted R Square	0.5511751
Standard Error	0.031867438
Observations	414

ANOVA

	df	SS	MS	F	Significance F
Regression	5	0.520136693	0.104027339	102.4361352	8.13207E-70
Residual	408	0.414337714	0.001015534		
Total	413	0.934474407			

	Coefficients	Standard Error	t Stat	P-value
Intercept	1.471275735	0.018553604	79.298649	5.1832E-250
10D_RR_Quote	-0.008253915	0.01051206	-0.785185278	0.432800643
25D_RR_Quote	0.027931948	0.018720414	1.492058203	0.136456682
Rate Diff	-0.828353774	0.237116683	-3.493443664	0.000529019
Rho	0.261566615	0.040361315	6.480626599	2.63754E-10
Skewness	-0.155081801	0.035304919	-4.392640027	1.42908E-05

Figure 4.9. Regression Results-1.

the annualized return of 1.50% and −2.48% are resulted, which shows that rho significantly outperforms the rest of the two input type.

The reason why skewness has failed to perform well in trade is because of the underlying distribution is not log-normal. In fact the implied distribution is leptokurtic, meaning that it has heavy tails and large kurtosis compared to log-normal distribution. In calculation of the skewness of the log-normal distribution, historic volatility is used for the variance parameter, that is why the observations in regression model are 414. 90 days moving standard deviation is used for the calculation of skewness, with this 90 days a total of 504 days (2 years, 252 in a year) are used in the analysis.

As our trading idea suggests, interest-rate differential should be one of the most important covariate and the back-test results should be promising. Since most of the skewness is coming from the positivity of the rate differential, it should perform well in trading also. The reason why it underperformed in the trade is that, it contains very less information about the spot price. Although the rate differential is the main

Regression Statistics	
Multiple R	0.741573783
R Square	0.549931675
Adjusted R Square	0.546638492
Standard Error	0.032028087
Observations	414

ANOVA

	df	SS	MS	F	Significance F
Regression	3	0.513897076	0.171299025	166.9909318	1.00513E-70
Residual	410	0.420577331	0.001025798		
Total	413	0.934474407			

	Coefficients	Standard Error	t Stat	P-value
Intercept	1.47068002	0.018156142	81.00178957	2.2381E-254
Rate Diff	-0.614252934	0.212252105	-2.893978054	0.004007024
Rho	0.335723847	0.017761402	18.90187764	9.22755E-58
Skewness	-0.208514869	0.02686239	-7.762334934	6.71608E-14

Figure 4.10. Regression Results-2.

reason of positive skewness, trading in exchange rate by just looking to rate differential is not enough, since FX market is highly volatile and it has many drivers besides rate differential. This reasoning can be justified with the results of the regression analysis, the p-value was 0.004 in the last regression (See Figure 4.11), which means that interest rate differential has information in explaining the behavior of the spot, however this information is obviously not enough to trade in exchange rate, at least with two standard deviation rule.

The Heston parameter rho has significantly outperformed its rivals both in regression analysis and in trading. The reason is that it contains the information at every delta risk reversal, i.e. 10D RR, 15D RR, 25D RR etc. To be more specific, rho is the output of calibration procedure and obviously it contains all the information coming from risk-reversals, butterflies, rate differentials and spot. So this right mixture of information makes rho out-performer in the trade. Still, FX market has many drivers and this situation is specific to emerging market currencies, which the underlying trade idea is: skewness is the main driver of the exchange rate. At this point, it is crucial to

Regression Statistics	
Multiple R	0.735348956
R Square	0.540738087
Adjusted R Square	0.538503236
Standard Error	0.03231417
Observations	414

ANOVA

	df	SS	MS	F	Significance F
Regression	2	0.505305903	0.252652952	241.9570916	3.57617E-70
Residual	411	0.429168504	0.001044206		
Total	413	0.934474407			

	Coefficients	Standard Error	t Stat	P-value
Intercept	1.426973062	0.010167373	140.3482597	0
Rho	0.34160516	0.017802361	19.188756	4.63498E-59
Skewness	-0.211340495	0.027084423	-7.803027425	5.05031E-14

Figure 4.11. Regression Results-3.

emphasize that simple trading strategies work even in FX market, with the input(s) having the right mixture of information about the market.

Table 4.6. Back-Testing Results When Heston Parameter Rho is Used as Signal Generating Series.

Enter	Direction	Exit	Return	# of days in trade
1.4706	long	1.4395	-2.12 %	9
1.4395	long	1.4092	-2.11 %	4
1.4092	long	1.4565	3.22 %	25
1.5771	short	1.5226	3.58 %	6
1.4825	long	1.5300	3.20 %	17
1.4871	long	1.4563	-2.07 %	9
1.4563	long	1.5079	3.54 %	12
1.5234	short	1.4760	3.21 %	7
1.4633	long	1.5351	4.91 %	60

4.5.2. Conclusion

In the second part of the thesis, FX OTC options market is introduced and quotation mechanism is explained in detail. FX options market is different in various ways compared to equity options market; level of moneyness is measured with the options delta in FX market, however in classical option markets strike price of the option is used to measure the level of moneyness. The reason for this difference was not arbitrary, communicating in terms of delta makes the system efficient. Traders can directly understand the spot risk when they trade options with the use of delta mechanism instead of strike price. Second difference from the classical option market is quotation mechanism. Calls and puts are not quoted in the market, instead risk-reversals and butterflies (strangles) are quoted. The reason for this situation is, those are the mostly traded instruments in the market. Traders prefer to finance their call option premium by shorting same delta put option for instance. The premium obtained from shorting put option is paid to long call option. By this strategy initial investment is zero, but the spot risk (delta) is doubled. This situation is same for the butterflies.

After the definition of the market, Heston's stochastic volatility model is in-

Table 4.7. Back-Testing Results When Skewness of Log-normal Distribution is Used as Signal Generating Series.

Enter	Direction	Exit	Return	# of days in trade
1.4545	long	1.4162	-2.63 %	8
1.4162	short	1.4546	-2.64 %	26
1.5644	long	1.6128	3.09 %	18
1.6128	long	1.5763	-2.26 %	3
1.5763	long	1.5430	-2.11 %	10
1.5235	short	1.4743	3.34 %	19
1.4767	long	1.5351	3.96 %	40

troduced and calibrated to market data. Heston's model is selected because of this simplicity, realisticity and ability to handle the smile implied by options. One to one correspondence between Heston parameters and market quoted instruments is another reason for the popularity of this model in FX options market. The correlation parameter ρ and risk-reversals are affecting the skewness of the smile, and butterflies and volofvol parameter, σ are affecting the curvature of the smile, which is an indication of extreme event movement.

FX options market is getting increasingly popular in emerging market economies, and trading strategies should be generated in this market. The underlying trade idea was, emerging market currencies are mostly driven by the interest rate differential. Interest rate differential was found to be under-performer in trade and not successful in regression analysis also. However the reason is that, rate differential affects the skewness of the implied distribution and this skewness is the main driver of the spot price. Following this idea, risk-reversals, skewness parameter of log-normal distribution and the Heston model parameter rho are tested and rho has outperformed its rivals in two sigma trading rule. The reason of this huge out-performance is the right mixture of information in parameter rho, since it is the output of the calibration procedure. Rho has the right mixture of skewness information and other market drivers, so that the signals generated from rho has resulted in an annual return of 26%. In all of the

Table 4.8. Back-Testing Results When Interest-Rate Differential is Used as Signal Generating Series.

Enter	Direction	Exit	Return	# of days in trade
1.5439	long	1.5058	-2.47 %	8
1.4889	long	1.4545	-2.31 %	26
1.4161	short	1.4546	-2.65 %	18
1.4643	long	1.5103	3.14 %	3
1.5155	long	1.5205	0.33 %	10
1.5771	long	1.5430	-2.16 %	19
1.5942	short	1.5358	3.80 %	40
1.4852	short	1.5042	-1.26 %	10
1.4871	short	1.4739	0.90 %	19
1.5234	long	1.5351	0.77 %	40

regression models, rho has been the most significant covariate. Since the parameter rho affects the skewness of the implied distribution, the initial idea was proven to be successful with the back-test results, with a slight change, emerging market currencies are driven by the skewness of the implied distribution. This thesis also showed that simple trading strategies, the two standard deviation rule, can be successful even in highly volatile FX market if the correct input is used. In our example this input is Heston's model parameter rho.

5. CONCLUSION

This thesis has started with dynamic programming approach to model the exchange rate dynamics and finalized with generating trading signals. Spot market and options market of exchange rate has explained in detail. In both markets, it is observed that interest rate differential is highly dominant in the direction of the spot price, especially in emerging market currencies. Attractive carry in emerging market currencies results in fast depreciation of the EM currency. In the first part, the decision maker was observing just the spot price and interest rates of the two currencies and the convexity condition was dependent on the difference of interest rates. The difference from classical portfolio management problem was demand and exchange rate were independent, and this assumption lead to unbounded solutions in DP when the convexity condition did not hold. The reason for this situation was independence assumption made demand risk hedging impossible. In the second chapter FX option pricing is tackled, which now hedging the risks of option position is possible, since option price is a function of spot and volatility. However FX options market convention is different from the classical options market, and before going deep in modelling one needs to understand the market conventions. After explaining the FX specific quotation mechanism and delta and ATM conventions, Heston's stochastic volatility model is introduced. Its advantages were; simplicity, semi-analytic solution for european type of options and its ability to capture the option smile. Calibration of the model to the market data has been explained and sample results were given. Relationship between the option smile and implied distribution has been described. Parameters of the Heston model are closely related with the market quoted risk-reversals and butterflies and with the shape of the smile. Correlation coefficient rho and risk-reversals both affect the skewness of the implied distribution, for instance.

In the second chapter, FX basics, spot market, options market, calibration of option smile were discussed. After learning FX market in-depth, simple trading rules were constructed using the output of the calibration. Following the trading idea of rate differential and spot price movement in EMs, two standard deviation rule is used with Heston's model parameter rho, in trading the spot price. Regression analysis have been made and rho had been found to be the most significant covariate in predicting the spot price. The success of the parameter rho was it is being the output of the calibration procedure, meaning that it contains more information about the expectations of the spot price. This thesis has shown that simple trading strategies works even in highly volatile FX market, if the correct information mixture is used. Rho is a very good indicator in trading the spot price, since it extracts the information coming from option smile.

To sum up, this thesis discussed two different problems in FX market. In the first chapter, demand consumption was handled, where hedging the demand risk was impossible. In the second chapter, FX option pricing problem was tackled and this time it was possible to hedge the risks of option position. So this thesis tackled two different problem settings in the sense of hedging the risk of the obligation is possible and impossible. It will be good starting point for those who want to study in foreign exchange. One another purpose was to provide theoretical and practical information together. Matlab codes for both Heston pricer and calibration procedure was provided, and detailed explanation of the model is also provided, effects of parameters in the smile and implied distribution.

APPENDIX A: Vanilla Option Pricer

```
function [price]=HestonVanilla(k,teta,sigma,ro,lambda,...
rd,rf,vt,St,K,tao,fi)
y=log(K);
x=log(St);
u1=0.5;
u2=-0.5;
b1=k+lambda-sigma.*ro;
b2=k+lambda;
d1=@(psi) sqrt( (ro.*sigma.*psi.*1i-b1).^2-sigma^2.*...
(2.*u1.*psi.*1i-psi.^2) );
d2=@(psi) sqrt( (ro.*sigma.*psi.*1i-b2).^2-sigma^2.*...
(2.*u2.*psi.*1i-psi.^2) );
g1=@(psi) (b1-ro.*sigma.*psi.*1i-d1(psi))./(b1-ro.*...
sigma.*psi.*1i+d1(psi));
g2=@(psi) (b2-ro.*sigma.*psi.*1i-d2(psi))./(b2-ro.*...
sigma.*psi.*1i+d2(psi));
C1= @(tao,psi) (rd-rf).*psi.*1i.*tao+k.*teta./sigma.^2.*...
((b1-ro.*sigma.*psi.*1i-d1(psi)).*tao-2.*log((1-g1(psi).*...
exp(-d1(psi).*tao))./(1-g1(psi))));
C2= @(tao,psi) (rd-rf).*psi.*1i.*tao+k.*teta./sigma.^2.*...
((b2-ro.*sigma.*psi.*1i-d2(psi)).*tao-2.*log((1-g2(psi).*...
exp(-d2(psi).*tao))./(1-g2(psi))));
D1= @(tao,psi) (b1-ro.*sigma.*psi.*1i-d1(psi))./sigma.^2.*...
(1-exp(-d1(psi).*tao))./(1-g1(psi).*exp(-d1(psi).*tao));
D2= @(tao,psi) (b2-ro.*sigma.*psi.*1i-d2(psi))./sigma.^2.*...
(1-exp(-d2(psi).*tao))./(1-g2(psi).*exp(-d2(psi).*tao));
f1= @(x,vt,tao,psi) exp(C1(tao,psi)+D1(tao,psi).*vt+1i.*psi.*x);
f2= @(x,vt,tao,psi) exp(C2(tao,psi)+D2(tao,psi).*vt+1i.*psi.*x);
P1= @(x,vt,tao,y) 0.5+1/pi.*quadgk(@(psi) real((exp(-1i.*psi.*y).*...
```

```
f1(x,vt,tao,psi))./(1i.*psi)),0,inf);
P2= @(x,vt,tao,y) 0.5+1/pi.*quadgk(@(psi) real((exp(-1i.*psi.*y).*...
f2(x,vt,tao,psi))./(1i.*psi)),0,inf);
Pplus= @(fi) (1-fi)./2+fi.*P1(x,vt,tao,y);
Pminus= @(fi) (1-fi)./2+fi.*P2(x,vt,tao,y);
price=fi.*(exp(-rf.*tao).*St.*Pplus(fi)-K.*exp(-rd.*tao).*Pminus(fi));
end
```

APPENDIX B: Calibration Scheme

```
function [output] =HestonCalibration(kappa,parity)
data=xlsread('option data[Bloomberg].xls',...
'1Month_Matlab_format','B2:I1232');
tstart=tic;
numberofdays=length(data(:,1));
output=zeros(numberofdays,4);
yaxis2=zeros(1,length(data(1,4:end)));
K=zeros(1,length(data(1,4:end)));
tao=1/12;
if strcmp(parity,'USDTRY')
putcallseq=[-1 -1 1 1 1];
deltaseq=[-10 -25 0 25 10]./100;
modelimpvol=zeros(numberofdays,length(deltaseq));
delta_type='forward';
ATM_type='delta-neutral-forward';
end

if strcmp(delta_type,'spot')
strikef=@(f,tao,sigma,phi,rf,delta) f.*...
exp( -phi.*sigma.*sqrt(tao).*...
icdf('normal',phi.*exp(rf.*tao).*delta,0,1)+...
0.5.*sigma.^2.*tao );
putcallparityf=@(rf,tao,K,f) exp(-rf.*tao)-...
deltaseq(round(length(deltaseq)/2)+1:end);
deltaf=@(phi,rf,tao,f,K,sigma) phi.*exp(-rf.*tao).*...
cdf('normal',phi./...
sigma./sqrt(tao).*(log(f./K)+0.5.*sigma.^2.*tao),0,1);
elseif strcmp(delta_type,'forward')
strikef=@(f,tao,sigma,phi,rf,delta) f.*exp( -phi.*...
```

```
sigma.*sqrt(tao).*...
icdf('normal',phi.*delta,0,1)+0.5.*sigma.^2.*tao );
putcallparityf=@(rf,tao,K,f) 1-...
deltaseq(round(length(deltaseq)/2)+1:end);
deltaf=@(phi,rf,tao,f,K,sigma) phi.*cdf('normal',phi./sigma./...
sqrt(tao).*(log(f./K)+0.5.*sigma.^2.*tao),0,1);
elseif strcmp(delta_type,'spot-pa')
dminus=@(f,K,sigma,tao) (log(f./K)-0.5.*sigma.^2.*tao)./...
(sigma.*sqrt(tao));
Kmax=@(f,tao,sigma,phi,rf,delta) f.*exp( -phi.*sigma.*sqrt(tao).*...
icdf('normal',phi.*exp(rf.*tao).*delta,0,1)+0.5.*sigma.^2.*tao );
Kminf=@(sigma,tao,f,K) sigma.*sqrt(tao).*cdf('normal',...
dminus(f,K,sigma,tao),0,1)-pdf('normal',dminus(f,K,sigma,tao),0,1);
Kmin=@(sigma,tao,f) fzero(@(K)Kminf(sigma,tao,f,K),1);
strikef=@(f,tao,sigma,phi,rf,delta) fzero(@(K)phi.*exp(-rf.*tao).*...
K./f.*cdf('normal',phi.*dminus(f,K,sigma,tao),0,1)-delta,...
[Kmin(sigma,tao,f) Kmax(f,tao,sigma,phi,rf,delta)]);
putcallparityf=@(rf,tao,K,f) exp(-rf.*tao).*...
K(round(length(deltaseq)/2)+1:end)./...
f-deltaseq(round(length(deltaseq)/2)+1:end);
deltaf=@(phi,rf,tao,f,K,sigma) phi.*exp(-rf.*tao).*K./f.*...
cdf('normal',phi./sigma./sqrt(tao).*...
(log(f./K)-0.5.*sigma.^2.*tao),0,1);
elseif strcmp(delta_type,'forward-pa')
dminus=@(f,K,sigma,tao) (log(f./K)-0.5.*sigma.^2.*tao)./...
(sigma.*sqrt(tao));
Kmax=@(f,tao,sigma,phi,rf,delta) f.*exp( -phi.*sigma.*sqrt(tao).*...
icdf('normal',phi.*delta,0,1)+0.5.*sigma.^2.*tao );
Kminf=@(sigma,tao,f,K) sigma.*sqrt(tao).*cdf('normal',...
dminus(f,K,sigma,tao),0,1)-pdf('normal',dminus(f,K,sigma,tao),0,1);
Kmin=@(sigma,tao,f) fzero(@(K)Kminf(sigma,tao,f,K),1);
```

```
strikef=@(f,tao,sigma,phi,rf,delta) fzero(@(K)phi.*exp(-rf.*tao).*...
K./f.*cdf('normal',phi.*dminus(f,K,sigma,tao),0,1)-delta,...
[Kmin(sigma,tao,f), Kmax(f,tao,sigma,phi,rf,delta)]);
putcallparityf=@(rf,tao,K,f) K(round(length(deltaseq)/2)+1:end)./...
f-deltaseq(round(length(deltaseq)/2)+1:end);
deltaf=@(phi,rf,tao,f,K,sigma) phi.*K./f.*...
cdf('normal',phi./sigma./sqrt(tao).*...
(log(f./K)-0.5.*sigma.^2.*tao),0,1);
end

if strcmp(ATM_type,'spot')
atmstrikef=@(f,rd,rf,sigma,tao) f./exp((rd-rf).*tao);
elseif strcmp(ATM_type,'forward')
atmstrikef=@(f,rd,rf,sigma,tao) f;
elseif strcmp(ATM_type,'delta-neutral-spot')
atmstrikef=@(f,rd,rf,sigma,tao) f.*exp(0.5.*sigma.^2.*tao);
elseif strcmp(ATM_type,'delta-neutral-forward')
atmstrikef=@(f,rd,rf,sigma,tao) f.*exp(0.5.*sigma.^2.*tao);
elseif strcmp(ATM_type,'delta-neutral-spotpa')
atmstrikef=@(f,rd,rf,sigma,tao) f.*exp(-0.5.*sigma.^2.*tao);
elseif strcmp(ATM_type,'delta-neutral-forwardpa')
atmstrikef=@(f,rd,rf,sigma,tao) f.*exp(-0.5.*sigma.^2.*tao);
end

elapsedtime(40,1)=0;
for i=1:5
looptime=tic;
vols=data(i,4:end)./100;
st=data(i,1);
rd=data(i,2);
rf=data(i,3);
```

```
mid=round(length(vols)/2);
f=st.*exp(rd.*tao-rf.*tao);

for m=1:length(vols)
if m~=mid
K(m)=strikef(f,tao,vols(m),putcallseq(m),rf,deltaseq(m));
end
end
K(mid)=atmstrikef(f,rd,rf,vols(mid),tao);

vt=vols(mid).^2;
if isempty(kappa)
objf=@(teta,sigma,ro,kappa) 0;
else
objf=@(teta,sigma,ro) 0;

end

for j=1:length(vols)
if j~=mid
pricevanilla=@(teta,sigma,ro) HestonVanilla(kappa,teta,sigma,ro,...
0,rd,rf,vt,st,K(j),tao,putcallseq(j));

if putcallseq(j)==1; class='call';else class='put'; end

impsigma=@(teta,sigma,ro) blsimpv(st,K(j),rd,tao,...
max(0,pricevanilla(teta,sigma,ro)),[],rf,[],{class});

objf=@(teta,sigma,ro) objf(teta,sigma,ro)+...
(impsigma(teta,sigma,ro)-vols(j)).^2;
elseif j==mid
```

```
ATMpricecall=@(teta,sigma,ro) HestonVanilla(kappa,teta,sigma,ro,...
0,rd,rf,vt,st,K(j),tao,1);
ATMpriceput=@(teta,sigma,ro) HestonVanilla(kappa,teta,sigma,ro,...
0,rd,rf,vt,st,K(j),tao,-1);
ATMimpsigmacall=@(teta,sigma,ro) blsimpv(st,K(j),rd,tao,...
max(0,ATMpricecall(teta,sigma,ro)),[],rf,[],{'call'});
ATMimpsigmaput=@(teta,sigma,ro) blsimpv(st,K(j),rd,tao,...
max(0,ATMpriceput(teta,sigma,ro)),[],rf,[],{'put'});
objf=@(teta,sigma,ro) objf(teta,sigma,ro)+...
(ATMimpsigmacall(teta,sigma,ro)-vols(j)).^2+...
(ATMimpsigmaput(teta,sigma,ro)-vols(j)).^2;
end
end

options=optimset('Algorithm','interior-point');
problem.objective=@(x)objf(x(1),x(2),x(3));
problem.x0=[0.5,1,0.5];
problem.lb=[0 0 -1];
problem.ub=[1 1 1];
problem.solver='fmincon';
problem.options=options;
[output(i,1:3) output(i,4)]=fmincon(problem);
output(i,1:3)

voltemp=zeros(1,length(vols));
for m=1:length(vols)
if putcallseq(m)==1; class='call';else class='put'; end
pricetemp=HestonVanilla(kappa,output(i,1),...
output(i,2),output(i,3),0,rd,rf,...
vt,st,K(m),tao,putcallseq(m));
```

```
voltemp(m)=blsimpv(st,K(m),rd,tao,...
max(0,pricetemp),[],rf,[],{class}).*100;
end
modelimpvol(i,:)=voltemp;

if 1==1
xaxis=[-deltaseq(1:mid-1),...
deltaf(1,rf,tao,f,atmstrikef(f,rd,rf,vols(mid),tao),...
vols(mid)),putcallparityf(rf,tao,K,f)];
yaxis1=vols(1:end).*100;
for m=1:length(vols)

if putcallseq(m)==1; class='call';else class='put'; end
pricetemp=HestonVanilla(kappa,output(i,1),...
output(i,2),output(i,3),0,rd,rf,vt,st,...
K(m),tao,putcallseq(m));
yaxis2(m)=blsimpv(st,K(m),rd,tao,...
max(0,pricetemp),[],rf,[],{class}).*100;
end

figure(1);subplot(2,2,1);
plot(xaxis,yaxis1,'-o',xaxis,yaxis2,'--sr');
title('Delta vs Volatility Plot')
xlabel('Delta [\%]');ylabel('Implied Volatility [\%]');
legend('1M smile','Heston fit','location','north');
subplot(2,2,2);plot(K,yaxis1,'-o',K,yaxis2,'--sr');
title('Strike vs Volatility Plot')
xlabel('Strike');ylabel('Implied Volatility [\%]');
legend('1M smile','Heston fit','location','north');

end
```

```
elapsedtime(i,1)=toc(looptime);
end
xlswrite('HestonResults.xlsx',modelimpvol,'Sheet1','J3');
xlswrite('estimationresults.xlsx',output(:,1:3),'Sheet1','B3');
xlswrite('estimationresults.xlsx',output(:,4)*1e7,'Sheet1','E3');
xlswrite('estimationresults.xlsx',elapsedtime,'Sheet1','F3');
toc(tstart);
end
```

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