QUOTIENTS OF HOM-FUNCTORS

by

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ABSTRACT

QUOTIENTS OF HOM-FUNCTORS

Quotients of Hom-functors are functors of the form $\mathrm{Hom}_R(P,-)/\mathrm{Hom}_R(P,-)J$ where P is a projective R-module and J is a certain ideal of the endomorphism ring of P. These functors were used by R. Dipper in the articles *On Quotients of Hom-Functors and Representations of Finite General Linear Groups I-II*, to obtain a classification of the irreducible l-modular representations of $GL_n(q)$ for primes l not dividing q. In this thesis, the general properties of these functors are examined following Dipper's articles [6] and [7]. Besides, the relation between the quotients of Hom-functors and the Harish-Chandra theory is investigated.

ÖZET

HOM-İZLEÇLERİN BÖLÜMLERİ

Hom-izleçlerin bölümleri, projektif bir R-modülü P ve P'nin endomorfizma halkasının bir ideali J için $\mathrm{Hom}_R(P,-)/\mathrm{Hom}_R(P,-)J$ şeklinde tanımlanan izleçlerdir. Bu izleçler R. Dipper'ın *On Quotients of Hom-Functors and Representations of Finite General Linear Groups I-II* adlı makalelerinde, q'yu bölmeyen l asal sayıları için $GL_n(q)$'nun indirgenemez l-modüler temsillerinin sınıflandırılmasında kullanılmıştır. Bu tezde, Dipper'ın makaleleri ([6] ve [7]) kullanılarak, bu izleçlerin genel özellikleri incelenmiştir. Ayrıca, Hom-izleçlerin bölümleri ile Harish-Chandra kuramı arasındaki ilişki çalışılmıştır.

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LIST OF SYMBOLS

$\mathrm{c}^f_{K,G}$	The (K,G)-biset K for groups G and K and a group isomorphism $f : G \to K$ with the left action of K by multiplication, and right action of G by taking image by f, and then multiplying on the right in K
$\mathrm{C}^f_{K,G}$	Transport of structure functor from the category of FG-modules to the category of FK-modules
$\mathrm{def}^G_{G/N}$	The $(G/N,G)$-biset G/N for a group G and a normal subgroup of G with the left action of G/N by multiplication, and the right action of G by projection to G/N, and then right multiplication in G/N
$\mathrm{Def}^G_{G/N}$	Deflation functor from the category of FG-modules to the category of $F[G/N]$-modules
$\mathrm{End}_T(M)$	The set of T-module endomorphisms of M
$(\mathrm{End}_T(P))_\beta$	The set of T-module endomorphisms of P under which $\ker\beta$ is invariant
FX	Permutation FG-module with permutation basis X where X is a G-set
${}^xG$	Conjugate group $x^{-1}Gx$ for a group G
$hd(V)$	Head of V
$(H,K)_x$	Stabilizer of x in $H \times K$
$\mathrm{Hom}_R(A,B)$	The set of R-linear maps from A to B
$\mathrm{im}f$	Image of f
ind^G_H	The (G,H)-biset G for a group G and a subgroup H of G with actions left and right multiplications in G
Ind^G_H	Induction functor from the category of FH-modules to the category of FG-modules
$\mathrm{inf}^G_{G/N}$	The $(G,G/N)$-biset G/N for a group G and a normal subgroup N of G with the left action of G by projection to G/N, and then left multiplication in G/N, and the right action of G/N by multiplication

$\mathrm{Inf}_{G/N}^{G}$	Inflation functor from the category of $F[G/N]$-modules to the category of FG-modules
$\mathrm{Irr}T$	The complete set of non-isomorphic irreducible T-modules
J_{β}	The set of T-module homomorphisms of P whose images contained in $\ker\beta$
$Jac(V)$	Jacobson radical of V
$\ker f$	Kernel of f
$\ker_P$	Kernel of P
$\mathbf{M}$	Mackey system
M^x	Conjugate module for the conjugate group xGx^{-1} for an FG-module M
mod_R	Category of finitely generated right R-modules
${}_R\mathrm{mod}$	Category of finitely generated left R-modules
$P\backslash G/Q$	P-Q-double coset of representatives in G
res_H^G	The (H,G)-biset G for a group G and a subgroup H of G with actions given by left and right multiplications in G
Res_H^G	Restriction functor from the category of FG-modules to the category of FH-modules
$R_{P/U}^{G}$	Harish-Chandra induction from $F[P/U]$-modules to FG-modules for a field F
$soc(V)$	Socle of V
$T_{P/U}^{G}$	Harish-Chandra truncation from FG-modules to $F[P/U]$-modules for a field F
$\mathrm{tor}_P(V)$	P-torsion submodule of V
$\mathrm{tr}_{V_1}(V_2)$	Trace of V_1 in V_2
YM	The submodule of M generated by the images of homorphisms in Y for a subset Y of $\mathrm{End}_T(M)$
$V \times_H U$	the composition of V and U for an (H,G)-biset U and (K,H)-biset V
1_M	Identity map on M

1. INTRODUCTION

Quotients of Hom-functors are functors of the form $\text{Hom}(P, -)/\text{Hom}(P, -)J$ where P is projective and J is a certain ideal of the endomorphism ring of P. Their terminology and properties were developed by R. Dipper in the articles [6] and [7], and they were used to obtain a classification of the irreducible l-modular representations of $GL_n(q)$ for primes l not dividing q, and to obtain information on decomposition numbers in terms of Hecke algebras and q-Schur algebras, in [7].

For a Noetherian commutative ring R, semiperfect R-algebra T with a multiplicative identity, and a projective presentation $\beta : P \to M$ where P and M are T-modules, the map

$$H = \text{Hom}_T(P, -)/\text{Hom}_T(P, -)J_\beta$$

where J_β is the ideal of $\text{End}_T(P)$ consists of endomoprhisms of P under which $\ker\beta$ is invariant, is a functor from the category of T-modules to the category of $\text{End}_T(M)$-modules. After studying the properties of that functor in [6] and [7], Dipper considered a more specialized situation; taking a discrete complete valuation ring O with quotient field K and residue field F, he replaced the algebra T with the R-algebra T_R where $R = K,\ O,\ F$, and constructed H using this T_R and obtained results similar to the general case.

It was stated in Dipper [7] that, for a finite reductive group G and $R = F,\ K$, the irreducible RG-modules are determined using the following method: For any Levi subgroup L of G, the irreducible RL-modules are found. Then, for any Levi subgroup L and a cuspidal irreducible RL-module C, the irreducible $\text{End}_{RG}(R_L^G(C))$-modules are found where R_L^G is the Harish-Chandra induction. Then using the bijection between the isomoprhism classes of the irreducible RG-modules occuring in the head of $R_L^G(C)$ and a set of representatives of the isomorphism classes of the irreducible $\text{End}_{RG}(R_L^G(C))$-modules, the classification of the irreducible RG-modules is achieved.

As an aplication to the general theory, it was proved in Dipper [7] that, in the case $G = GL_n(q)$, the endomorphism ring $\mathrm{End}_{RG}(R_L^G(C))$ is isomorphic to a product of some Hecke algebras associated with symmetric groups. Therefore, the representation theory of $GL_n(q)$ is related to Hecke algebras associated with symmetric groups through the functor

$$H = \mathrm{Hom}_{RG}(P_R, -)/\mathrm{Hom}_{RG}(P_R, -)J_{\beta_R}.$$

Using this method, the classification of non-isomorphic irreducible $RGL_n(q)$-modules was achieved in [7], and also, a complete set of non-isomorphic cuspidal irreducible $FGL_n(q)$-modules was given.

The aim of this thesis is to examine the properties of quotients of Hom-functors and their connection with the Harish-Chandra theory, and to understand the application of the theory of Hom-functors to the classification of representations of general linear groups, using Dipper [6] and [7]. The thesis is organized as follows:

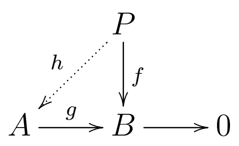
In Chapter 2, some preliminary definitions and results which are required to construct quotients of Hom-functors are stated.

In Chapter 3, the theory of quotients of Hom-functors is introduced and the properties of those functors are examined in a detailed way.

In Chapter 4, the connection between quotients of Hom-functors and the Harish-Chandra theory is studied. Besides, the notion of bisets is introduced and Mackey Decomposition Theorem (Dipper [7, 2.2.1]) is proved using biset functors.

2. PRELIMINARIES

We start with defining what a semiperfect ring is. Firstly, we need some preliminary definitions. A module P over a ring R is said to be *projective* if given any diagram of R-module homomorphisms f and g



with bottom row exact (that is, g is an epimorphism), there exists an R-module homomorphism $h : P \to A$ such that the diagram commutes, that is $gh = f$. A submodule S of a module M is *superfluous* if, whenever L is a submodule of M with $L + S = M$, then $L = M$. A *projective cover* of a module M is an ordered pair (P, φ), where P is a projective module and $\varphi : P \to M$ is a surjective map with ker φ a superfluous submodule of P.

A ring R is *semiperfect* if every finitely generated right R-module has a projective cover.

For a ring R, the category of finitely generated right R-modules is denoted by mod_R and the category of finitely generated left R-modules is denoted by ${}_R\text{mod}$. Let $M \in \text{mod}_R$, $P \in \text{mod}_R$ and P be projective. Let $\beta : P \to M$ be an epimorphism of right R-modules. Then β is called a *projective presentation* of M.

An R-module M is said to satisfy the *ascending chain condition on submodules* (or is *Noetherian*) if for every chain

$$M_1 \subseteq M_2 \subseteq M_2 \subseteq M_3 \subseteq ...$$

of submodules of M, there is an integer n such that $M_i = M_n$ for all $i \geq n$.

An R-module N is said to satisfy the *descending chain condition on submodules* (or is *Artinian*) if for every chain

$$N_1 \supseteq N_2 \supseteq N_2 \supseteq N_3 \supseteq ...$$

of submodules of N, there is an integer m such that $N_i = N_m$ for all $i \geq m$.

A ring R is said to be *left* [resp.*right*] *Noetherian* if R satisfies the ascending chain condition on left [resp. right] ideals. R is said to be *Noetherian* if R is both left and right Noetherian.

A ring R is said to be *left* [resp.*right*] *Artinian* if R satisfies the descending chain condition on left [resp. right] ideals. R is said to be *Artinian* if R is both left and right Artinian.

Definiton 2.1. *Let V be an R-module. The* Jacobson radical *of V is defined as the intersection of all maximal submodules of V, denoted by $Jac(V)$.*

The head *of V is the factor module $V/Jac(V)$, denoted by $hd(V)$. Therefore $hd(V)$ is the largest semisimple factor module of V.*

The socle *of V is the largest semisimple submodule of V, denoted by $soc(V)$.*

Definiton 2.2. *Let R be a ring.*

(i) *A nonzero element e of R is called an* idempotent *if $e^2 = e$.*
(ii) *Two idempotents e_1 and e_2 of R are said to be* orthogonal *if $e_1e_2 = e_2e_1 = 0$.*
(iii) *An idempotent is called* primitive *if it is not the sum of two orthogonal idempotents.*
(iv) *An* idempotent decomposition *of 1 in R is a set of pairwise orthogonal idempotents $e_1, ..., e_r$ such that $1 = \sum_{i=1}^{r} e_i$. An idempotent decomposition is called* primitive *if all the involved idempotents are primitive.*

Lemma 2.3. *(Fitting's Lemma) Let R be a ring and M be an R-module. Then there* *is a one to one correspondence between idempotent decompositions of* $1 = \sum_{i\in I} e_i$ *in* $\mathrm{End}_R(M)$*, where* I *is finite, and decompositions* $M = \sum_{i\in I} M_i$*, characterized by the fact that* e_j *is the projection of* M *onto* M_j *with kernel* $\sum_{i\neq j} M_i$*.*

Proof. See [9, I.1.4]. □

Proposition 2.4. *Let* R *be a ring.*

(i) Let P *be a projective* R*-module and* ϕ *be in* $\mathrm{End}_R(P)$*. Then* ϕ *is in* $Jac(\mathrm{End}_R(P))$ *if and only if* $\mathrm{im}\phi$ *is superfluous in* P*.*

(ii) If R *is left Artinian, then* $Jac(R)$ *is nilpotent.*

Proof. (i) See [1, 17.11].

(ii) See [9, I.3.6(i)]

□

Proposition 2.5. *Let* R *be a right Artinian ring and let* $\{e_i\}$ *be a set of primitive idempotents of* R*. Set* $P_i = e_iR$*. Then,* P_i *contains a unique maximal submodule, namely* $e_iJac(R)$*.*

Proof. See [9, I.3.14]. □

Definiton 2.6. *(i) A ring* R *is called* self-injective *if the regular* R*-module* R *is injective.*

(ii) A ring R *is called* quasi-Frobenius *if it is Noetherian and injective as an* R*-module.*

(iii) If a ring R *is a direct sum of indecomposable modules, say* $R = \bigoplus_i L_i$*, then any module* M *isomorphic to some* L_i *is called a* principal indecomposable module.

Proposition 2.7. *If* R *is quasi-Frobenius, then there is a bijection between its minimal left ideals and its principal indecomposable modules.*

Proof. See [11, 4.48]. □

Proposition 2.8. *Let R be a ring, and let M and N be R-modules.*

(i) We have

$$Jac(M) = \sum\{L \leq M | L \text{ is superfluous in } M\}.$$

(ii) If $f : M \to N$ is an epimorphism and $\ker f$ is a submodule of $Jac(M)$, then $Jac(N) = f(Jac(M))$.

Proof. (i) See [1, 9.13].

(ii) See [1, 9.15].

□

Proposition 2.9. *Let S and T be rings, U be an $S-T$-bimodule, N be a left T-module and P be a projective left T-module. Then, there is a natural homomorphism*

$$\eta : \mathrm{Hom}_S(P, U) \otimes_T N \to \mathrm{Hom}_S(P, (U \otimes_T N))$$

defined by

$$\eta(\gamma \otimes_T n) : p \mapsto \gamma(p) \otimes_T n$$

where $\gamma \in \mathrm{Hom}_S(P, U)$, n in N and p in P. If P is finitely generated and projective, then η is an isomorphism.

Proof. See [1, 20.10]. □

Proposition 2.10. *A finitely generated left module over a Noetherian ring is Noetherian.*

Proof. See [5, 3.3]. ☐

Proposition 2.11. *Let R be a semiperfect ring and consider only finitely generated R-modules. Let $N = Jac(R)$. Let $f : P \to X$ be a surjection with P projective. Then f gives a projective cover if and only if* $\ker f \subseteq NP$.

Proof. See [5, 6.25(i)]. ☐

Lemma 2.12. *(Nakayama's Lemma) Let R be a commutative ring. Let I be an ideal of R which is contained in every maximal ideal of R. If M is a finitely generated R-module and $MI = M$, then $M = (0)$.*

Proof. See [10, X.4.1]. ☐

3. THE QUOTIENTS OF HOM-FUNCTORS

3.1. The Ideal J_β

Let R be a commutative Noetherian ring and T be a semiperfect R-algebra which is finitely generated as an R-module. Assume that both T and R have multiplicative identities, and that T is unital as R-module. Let M be a finitely generated left T-module. Since T is semiperfect, there exists a projective presentation (β, P) of M. In this work, all modules are finitely generated unless stated otherwise. The set of R-module endomorphisms of M is denoted by $\mathrm{End}_R(M)$.

Notation. $(\mathrm{End}_T(P))_\beta = \{\phi \in \mathrm{End}_T(P) \mid \phi(\ker\beta) \subseteq \ker\beta\}$

$$J_\beta = \{\psi \in \mathrm{End}_T(P) \mid \mathrm{im}\psi \leq \ker\beta\}$$

In [6], it was stated that $(\mathrm{End}_T(P))_\beta/J_\beta$ and $\mathrm{End}_T(M)$ are isomorphic as R-algebras. Now, we prove this statement.

Proposition 3.1. *J_β is an ideal of $(\mathrm{End}_T(P))_\beta$ and $(\mathrm{End}_T(P))_\beta/J_\beta \cong \mathrm{End}_T(M)$ as R-algebra canonically .*

Proof. Clearly, the set J_β is a subset of $(\mathrm{End}_T(P))_\beta$. Also J_β is nonempty since 0 is an element of J_β. Let ψ_1 and ψ_2 be in J_β. For any p in P, we have

$$(\psi_1 - \psi_2)(p) = \psi_1(p) - \psi_2(p) \in \ker\beta$$

since $\mathrm{im}\psi_1$ and $\mathrm{im}\psi_2$ are submodules of $\ker\beta$. So, the set $\mathrm{im}(\psi_1 - \psi_2)$ is also a submodule of $\ker\beta$. Hence the element $\psi_1 - \psi_2$ is in J_β. Let ψ be in J_β and ϕ be in $(\mathrm{End}_T(P))_\beta$. For p in P, we have $\phi(\psi(p))$ is in $\ker\beta$ since $\mathrm{im}\psi$ is a submodule of $\ker\beta$ and $\phi(\ker\beta)$ is a subset of $\ker\beta$. Hence J_β is an ideal of $(\mathrm{End}_T(P))_\beta$.

Now, define $\widetilde{\beta} : (\mathrm{End}_T(P))_\beta \to \mathrm{End}_T(M)$ as

$$\widetilde{\beta}(\phi)(m) := \beta(\phi(p))$$

for ϕ in $(\mathrm{End}_T(P))_\beta$, the element m in M and p in P such that $\beta(p) = m$. Such a p always exists since β is surjective.

For each ϕ in $(\mathrm{End}_T(P))_\beta$, the map $\widetilde{\beta}(\phi)$ is well-defined since for p_1 and p_2 in P such that $p_1 \neq p_2$, if $\beta(p_1) = \beta(p_2)$ then $p_1 - p_2$ is in $\ker\beta$. This implies $\phi(p_1 - p_2)$ is in $\ker\beta$ since ϕ is in $(\mathrm{End}_T(P))_\beta$. That means $\beta(\phi(p_1 - p_2)) = 0$. Then, we have $\beta(\phi(p_1)) = \beta(\phi(p_2))$, that is $\widetilde{\beta}(\phi)(\beta(p_1)) = \widetilde{\beta}(\phi)(\beta(p_1))$. Also $\widetilde{\beta}$ is well-defined as β is well-defined.

Now, we are to show that $\widetilde{\beta}$ is an R-algebra homomorphism. Let ϕ, ϕ_1 and ϕ_2 be in $(\mathrm{End}_T(P))_\beta$, the element r be in R, the element m be in M and p be in P such that $\beta(p) = m$. Then

$$\begin{aligned}\widetilde{\beta}(\phi_1 + \phi_2)(m) = \beta((\phi_1 + \phi_2)(p)) = \beta(\phi_1(p) + \phi_2(p)) &= \beta(\phi_1(p)) + \beta(\phi_2(p)) \\ &= \widetilde{\beta}(\phi_1)(m) + \widetilde{\beta}(\phi_2)(m)\end{aligned}$$

and

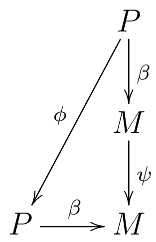
$$\begin{aligned}\widetilde{\beta}(\phi_1\phi_2)(m) = \beta((\phi_1\phi_2)(p)) &= \beta(\phi_1(\phi_2(p))) = \widetilde{\beta}(\phi_1)(\beta(\phi_2(p))) \\ &= \widetilde{\beta}(\phi_1)(\widetilde{\beta}(\phi_2)(m)) = \widetilde{\beta}(\phi_1)\widetilde{\beta}(\phi_2)(m)\end{aligned}$$

and

$$\widetilde{\beta}(r\phi)(m) = \beta(r\phi(p)) = r\beta(\phi(p)) = r\widetilde{\beta}(\phi)(m)$$

since β is an R-module homomorphism. That proves $\widetilde{\beta}$ is an R-algebra homomorphism.

Now, we are to prove that $\widetilde{\beta}$ is surjective. To this end, let ψ be in $\text{End}_T(M)$. Since $\psi\beta$ is an R-module homomorphism, the map β is surjective and P is projective, there exists a ϕ in $\text{End}_T(P)$ such that the diagram



commutes. That is, we have $\beta\phi = \psi\beta$. Then,

$$\beta(\phi(\ker\beta)) = \psi(\beta(\ker\beta)) = \psi(0) = 0.$$

So, the set $\phi(\ker\beta)$ is a subset of $\ker\beta$. Hence, the map ϕ is in $(\text{End}_T(P))_\beta$, and it is mapped to ψ under $\widetilde{\beta}$ since for m in M and p in P such that $\beta(p) = m$, we have

$$\widetilde{\beta}(\phi)(m) = \beta(\phi(p)) = \psi(\beta(p)) = \psi(m).$$

Therefore, the map $\widetilde{\beta}$ is surjective.

Finally, we are to show that $J_\beta = \ker\widetilde{\beta}$. Let ϕ be in $(\text{End}_T(P))_\beta$. By definition, ϕ is in J_β means $\text{im}\phi$ is a submodule of $\ker\beta$, and that means $\beta(\phi(p)) = 0$ for all p in P, and so $\widetilde{\beta}(\phi)(m) = 0$ for all m in M, or equivalently, the map ϕ is an element of $\ker\widetilde{\beta}$.

Therefore, using the First Isomorphism Theorem, we conclude that

$$\text{End}_T(P)/J_\beta \cong \text{End}_T(M).$$

That proves the proposition. □

3.2. Definition of the Functor H

Using a projective presentation (β, P) of M, a functor H from mod_T to $\text{mod}_{\text{End}_T(M)}$ was defined in [6]. We state this definition and prove that H is a covariant functor.

Proposition 3.2. *Assume* $(\text{End}_T(P))_\beta = \text{End}_T(P)$. *The mapping*

$$H := H^\beta := H^\beta_M : \text{mod}_T \to \text{mod}_{\text{End}_T(M)}$$

defined for $V \in \text{mod}_T$ *by*

$$H(V) = \text{Hom}_T(P, V)/\text{Hom}_T(P, V)J_\beta$$

is a covariant functor.

Proof. $\text{Hom}_T(P, V)$ is an $\text{End}_T(P)$-module via the action $\delta\theta(p) = \delta(\theta(p))$ for θ in $\text{End}_T(P)$ and δ in $\text{Hom}_T(P, V)$ and p in P. Thus, there is an induced $\text{End}_T(P)$-action on $H(V) = \text{Hom}_T(P, V)/\text{Hom}_T(P, V)J_\beta$. Also, we have $J_\beta(H(V)) = 0$ since for $\delta + \text{Hom}_T(P, V)J_\beta$ in $H(V) = \text{Hom}_T(P, V)/\text{Hom}_T(P, V)J_\beta$ and θ in J_β, we have

$$(\delta + \text{Hom}_T(P, V)J_\beta)\theta = \delta\theta + \text{Hom}_T(P, V)J_\beta = \text{Hom}_T(P, V)J_\beta.$$

Then $\text{End}_T(P)/J_\beta$ acts on $H(V)$ via the action $(\theta + J_\beta)\delta = \theta\delta$ for θ in $\text{End}_T(P)$ and δ in $\text{Hom}_T(P, V)$. Hence $H(V)$ is an $\text{End}_T(P)/J_\beta$-module. By Proposition 3.1, we have $\text{End}_T(P)/J_\beta \cong \text{End}_T(M)$, then $H(V)$ is also an $\text{End}_T(M)$-module.

Let V and V' be in mod_T, and $f : V \to V'$ be a T-module homomorphism. Then

$$f_* = \text{Hom}_T(P, f) : \text{Hom}_T(P, V) \to \text{Hom}_T(P, V'), \phi \mapsto f\phi$$

is an $\text{End}_T(P)$-homomorphism, so an $(\text{End}_T(P))_\beta$-homomorphism.
Also, the set $f_*(\text{Hom}_T(P, V)J_\beta)$ is a subset of $\text{Hom}_T(P, V')J_\beta$ since if ϕ is an element

of $\mathrm{Hom}_T(P,V)J_\beta$, then $\phi = \alpha\gamma$ for some α in $\mathrm{Hom}_T(P,V)$, some γ in J_β, and so

$$f_*(\phi) = f\phi = f(\alpha\gamma) = (f\alpha)\gamma \in \mathrm{Hom}_T(P,V')J_\beta.$$

Then f_* induces an $\mathrm{End}_T(M)$-homomorphism $H(f) : H(V) \to H(V')$ defined for $\psi + \mathrm{Hom}_T(P,V)J_\beta$ in $H(V)$ as

$$H(f)(\psi + \mathrm{Hom}_T(P,V)J_\beta) = f_*(\psi) + \mathrm{Hom}_T(P,V')J_\beta.$$

Also, we have $H(1_V) = 1_{H(V)}$ since for $\psi + \mathrm{Hom}_T(P,V)J_\beta$ in $H(V)$ we have

$$\begin{aligned} H(1_V)((\psi + \mathrm{Hom}_T(P,V)J_\beta) \in H(V)) &= 1_V\psi + \mathrm{Hom}_T(P,V)J_\beta \\ &= 1_{H(V)}(\psi + \mathrm{Hom}_T(P,V)J_\beta) \end{aligned}$$

and for elements V, V' and V'' in mod_T, and morphisms $f : V \to V'$ and $g : V' \to V''$, we have $H(gf) = H(g)H(f)$ since for $\psi + \mathrm{Hom}_T(P,V)J_\beta$ in $H(V)$ we have

$$\begin{aligned} H(gf)(\psi + \mathrm{Hom}_T(P,V)J_\beta) &= (gf)\psi + \mathrm{Hom}_T(P,V'')J_\beta \\ &= g(f\psi) + \mathrm{Hom}_T(P,V'')J_\beta \\ &= H(g)(f\psi + \mathrm{Hom}_T(P,V')J_\beta) \\ &= H(g)(H(f)(\psi + \mathrm{Hom}_T(P,V)J_\beta)) \\ &= H(g)H(f)(\psi + \mathrm{Hom}_T(P,V)J_\beta) \end{aligned}$$

Therefore H is a covariant functor from mod_T to $\mathrm{mod}_{\mathrm{End}_T(M)}$. □

3.3. The Functors for Different Projective Presentations

The functor H depends on the projective presention (β, P) we choose. We are to investigate what happens if we change (β, P) with the minimal projective cover (β_1, P_1) of M. In [7], a necessary and sufficient condition for the equivalence of H^β and H^{β_1} was stated and a sketch of a proof for that statement was given. Here, we give a detailed proof following the sketch in [7].

Definiton 3.3. *Let S be a ring. For S-modules V_1 and V_2 and for a submodule U of V_1, the S-module* $\mathrm{Hom}_S(V_1, V_2)U$ *defined as the submodule of V_2 spanned by all images* $\mathrm{im}\phi$ *for restrictions ϕ of U of homomorphisms from V_1 to V_2. In the case U is equal to V_1, the T-module* $\mathrm{Hom}_S(V_1, V_2)V_1$ *is called* trace of V_1 in V_2 *and denoted as* $\mathrm{tr}_{V_1}(V_2)$.

Since P and P_1 are both projective, we have $P = P_1 \oplus P_2$ where $P_2 = \ker\beta/\ker\beta_1$ and $\beta_2 : P_2 \to M$ is the zero map. Then $\ker\beta = \ker\beta_1 \oplus P_2$ and we may express β as

$$\beta = \beta_1 \oplus 0.$$

Hence, we have the following short exact sequence:

$$0 \to \ker\beta_1 \oplus P_2 \to P_1 \oplus P_2 \xrightarrow{\beta_1 \oplus 0} M \to 0 \tag{3.1}$$

Proposition 3.4. *Let $\beta = (\beta_1, 0) : P_1 \oplus P_2 \to M$ be given as in the exact sequence in* (3.1)*. Then* $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$ *if and only if* $\mathrm{tr}_{P_2}(P_1)$ *is a submodule of* $\ker\beta_1$*. Moreover, for a T-module V we have*

$$H^\beta(V) \cong H^{\beta_1}(V)/\mathrm{Hom}_T(P_2, V)\mathrm{Hom}(P_1, P_2).$$

Thus $H^\beta = H^{\beta_1}$ if and only if every homomorphism from P_1 to P_2 factors through a linear combination of endomorphisms of P_1 whose image is contained in $\ker\beta_1$*, that is,* $H^{\beta_1}(P_2) = (0)$.
If this condition does not hold, then H^β is a proper quotient of H^{β_1}.

Proof. Since $P = P_1 \oplus P_2$, we can write the elements of P as column vectors with two components, the first one from P_1 and the second one from P_2. Consequently, we can represent the endomorphisms of P as 2×2 matrices with entries in the appropriate

Hom-spaces, hence we have

$$\mathrm{End}_T(P) = \begin{pmatrix} \mathrm{Hom}_T(P_1, P_1) & \mathrm{Hom}_T(P_2, P_1) \\ \mathrm{Hom}_T(P_1, P_2) & \mathrm{Hom}_T(P_2, P_2) \end{pmatrix}.$$

Then, since $\mathrm{End}_T(P_1) = (\mathrm{End}_T(P_1))_{\beta_1}$ and

$$\begin{aligned} (\mathrm{End}_T(P_2))_{\beta_2} &= \{\phi \in \mathrm{End}_T(P_2) \mid \phi(\ker\beta_2) \subseteq \ker\beta_2\} \\ &= \{\phi \in (\mathrm{End}_T(P_2))_{\beta_2} \mid \phi(P_2) \subseteq P_2\} \\ &= \mathrm{End}_T(P_2) \end{aligned}$$

we have

$$\mathrm{End}_T(P) = \begin{pmatrix} (\mathrm{End}_T(P_1))_{\beta_1} & \mathrm{Hom}_T(P_2, P_1) \\ \mathrm{Hom}_T(P_1, P_2) & (\mathrm{End}_T(P_2))_{\beta_2} \end{pmatrix}.$$

Then we have

$$\begin{aligned} \mathrm{End}_T(P)\ker\beta &= \begin{pmatrix} (\mathrm{End}_T(P_1))_{\beta_1} & \mathrm{Hom}_T(P_2, P_1) \\ \mathrm{Hom}_T(P_1, P_2) & (\mathrm{End}_T(P_2))_{\beta_2} \end{pmatrix} \begin{pmatrix} \ker\beta_1 \\ P_2 \end{pmatrix} \\ &= \begin{pmatrix} (\mathrm{End}_T(P_1))_{\beta_1}\ker\beta_1 + \mathrm{Hom}_T(P_2, P_1)P_2 \\ \mathrm{Hom}_T(P_1, P_2)\ker\beta_1 + (\mathrm{End}_T(P_2))_{\beta_2}P_2 \end{pmatrix} \\ &\subseteq \begin{pmatrix} \ker\beta_1 + \mathrm{tr}_{P_2}(P_1) \\ P_2 \end{pmatrix} \end{aligned} \tag{3.2}$$

Now, we are to prove that $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$ if and only if $\mathrm{tr}_{P_2}(P_1)$ is a submodule of $\ker\beta_1$. If $\mathrm{tr}_{P_2}(P_1)$ is a submodule of $\ker\beta_1$, then $\ker\beta_1 + \mathrm{tr}_{P_2}(P_1) = \ker\beta_1$, thus, by the inclusion in (3.2), we have $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$. Conversely, assume that $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$. Let ψ be in $\mathrm{Hom}_T(P_2, P_1)$. Define

$$\phi : P_1 \oplus P_2 \to P_1 \oplus P_2,\ (\alpha_1, \alpha_2) \mapsto (\psi(\alpha_2), 0).$$

Clearly, ϕ is well-defined since ψ is. Since $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$, the set $\phi(\ker\beta)$ is a submodule of $\ker\beta$. Then $\phi(\ker\beta_1 \oplus P_2)$ is a submodule of $\ker\beta_1 \oplus P_2$. That means for any α_1 in $\ker\beta_1$ and α_2 in P_2 we have $\phi(\alpha_1, \alpha_2)$ in $\ker\beta_1 \oplus P_2$, that is, $(\psi(\alpha_2), 0)$ in $\ker\beta_1 \oplus P_2$. Then we have $\psi(\alpha_2)$ in $\ker\beta_1$. Since α_2 is arbitrary, we conclude that $\mathrm{im}\psi$ is a submodule of $\ker\beta_1$, and consequently $\mathrm{tr}_{P_2}(P_1)$ is a submodule of $\ker\beta_1$. Therefore $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$ if and only if $\mathrm{tr}_{P_2}(P_1)$ is a submodule of $\ker\beta_1$, or equivalently, $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$ if and only if $\mathrm{Hom}_T(P_2, P_1)$ is a subset of $\mathrm{Hom}_T(P_2, \ker\beta_1)$. The first part of the proposition is proved.

Now, we are to prove the second part. Assume $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$. Let $\theta = \left(\begin{smallmatrix}\theta_1 & \theta_2\\ \theta_3 & \theta_4\end{smallmatrix}\right)$ be in J_β. Then for any $\gamma = \left(\begin{smallmatrix}\gamma_1\\ \gamma_2\end{smallmatrix}\right)$ in $\ker\beta$ we have

$$\begin{pmatrix}\theta_1 & \theta_2\\ \theta_3 & \theta_4\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} \subseteq \ker\beta,$$

that is,

$$\begin{pmatrix}\theta_1(\gamma_1) + \theta_2(\gamma_2)\\ \theta_3(\gamma_1) + \theta_4(\gamma_2)\end{pmatrix} \subseteq \ker\beta = \ker\beta_1 \oplus P_2.$$

Then $\theta_1(\gamma_1) + \theta_2(\gamma_2)$ is in $\ker\beta_1$. Since θ_2 is in $\mathrm{Hom}_T(P_2, \ker\beta_1)$, we have $\theta_2(\gamma_2)$ in $\ker\beta_1$. So $\theta_1(\gamma_1)$ is in $\ker\beta_1$. As γ_1 is arbitrary, we have $\mathrm{im}\theta_1$ a submodule of $\ker\beta_1$. Hence we obtain

$$J_\beta = \begin{pmatrix} J_{\beta_1} & \mathrm{Hom}_T(P_2, \ker\beta_1)\\ \mathrm{Hom}_T(P_1, P_2) & (\mathrm{End}_T(P_2))_{\beta_2}\end{pmatrix}.$$

We represent homomorphisms from P into a T-module V as row vectors $\delta = (\delta_1, \delta_2)$, where δ_i is in $\mathrm{Hom}_T(P_i, V)$ for $i = 1, 2$. Then, we have

$$\mathrm{Hom}_T(P, V)J_\beta = \Big(\mathrm{Hom}_T(P_1, V),\quad \mathrm{Hom}_T(P_2, V)\Big)\begin{pmatrix} J_{\beta_1} & \mathrm{Hom}_T(P_2, P_1)\\ \mathrm{Hom}_T(P_1, P_2) & (\mathrm{End}_T(P_2))_{\beta_2}\end{pmatrix}$$

$$= \big(\mathrm{Hom}_T(P_1,V)J_{\beta_1} + \mathrm{Hom}_T(P_1,V)\mathrm{Hom}_T(P_1,P_2),\ \mathrm{Hom}_T(P_1,V)\mathrm{Hom}_T(P_2,P_1) + \mathrm{Hom}_T(P_2,V)(\mathrm{End}_T(P_2))_{\beta_2} \big)$$

Clearly, the T-module $\mathrm{Hom}_T(P_1, V)\mathrm{Hom}_T(P_2, P_1) + \mathrm{Hom}_T(P_2, V)(\mathrm{End}_T(P_2))_{\beta_2}$ is a submodule of $\mathrm{Hom}_T(P_2, V)$. Also, any ξ in $\mathrm{Hom}_T(P_2, V)$ can be written as $\xi = \xi \circ \mathrm{id}_{(\mathrm{End}_T(P_2))_{\beta_2}}$, hence it is an element of $\mathrm{Hom}_T(P_2, V)(\mathrm{End}_T(P_2))_{\beta_2}$. Thus, the set $\mathrm{Hom}_T(P_2, V)$ is a submodule of $\mathrm{Hom}_T(P_2, V)(\mathrm{End}_T(P_2))_{\beta_2}$. Then we have

$$\mathrm{Hom}_T(P, V)J_\beta = \Big(\mathrm{Hom}_T(P_1, V)J_{\beta_1} + \mathrm{Hom}_T(P_1, V)\mathrm{Hom}_T(P_1, P_2), \quad \mathrm{Hom}_T(P_2, V)\Big).$$

Now, we can write $H^\beta(V)$ as

$$\begin{aligned} H^\beta(V) &= \frac{\mathrm{Hom}_T(P, V)}{\mathrm{Hom}_T(P, V)J_\beta} \\ &= \frac{(\mathrm{Hom}_T(P_1, V), \mathrm{Hom}_T(P_2, V))}{(\mathrm{Hom}_T(P_1, V)J_{\beta_1} + \mathrm{Hom}_T(P_2, V)\mathrm{Hom}_T(P_1, P_2), \mathrm{Hom}_T(P_2, V))} \\ &\cong \frac{\mathrm{Hom}_T(P_1, V)}{(\mathrm{Hom}_T(P_1, V)J_{\beta_1} + \mathrm{Hom}_T(P_2, V)\mathrm{Hom}_T(P_1, P_2))} \\ &\cong \frac{\mathrm{Hom}_T(P_1, V)/\mathrm{Hom}_T(P_1, V)J_{\beta_1}}{\mathrm{Hom}_T(P_2, V)\mathrm{Hom}_T(P_1, P_2)} \\ &\cong \frac{H^{\beta_1}(V)}{\mathrm{Hom}_T(P_2, V)\mathrm{Hom}_T(P_1, P_2)} \end{aligned} \tag{3.3}$$

Finally, we are to show that $H^\beta = H^{\beta_1}$ if and only if $H^{\beta_1}(P_2) = (0)$. First, assume $H^{\beta_1}(P_2) = (0)$. Then $\mathrm{Hom}_T(P_1, P_2) = \mathrm{Hom}_T(P_1, P_2)J_{\beta_1}$. Hence, we have

$$\begin{aligned} \mathrm{Hom}_T(P_2, V)\mathrm{Hom}_T(P_1, P_2) &= \mathrm{Hom}_T(P_2, V)\mathrm{Hom}_T(P_1, P_2)J_{\beta_1} \\ &\subseteq \mathrm{Hom}_T(P_1, V)J_{\beta_1} \end{aligned}$$

Then, by Equation 3.3, we have

$$H^\beta(V) \cong \mathrm{Hom}_T(P_1, V)/\mathrm{Hom}_T(P_1, V)J_{\beta_1} = H_1^\beta(V).$$

Conversely, assume $H^\beta = H^{\beta_1}$. Then, we have

$$H^{\beta_1}(V) = \frac{H^{\beta_1}(V)}{\mathrm{Hom}_T(P_2, V)\mathrm{Hom}_T(P_1, P_2)}$$

for any T-module V. Then, we obtain $\mathrm{Hom}_T(P_2, V)\mathrm{Hom}_T(P_1, P_2) = (0)$. For $V = P_2$, we have $\mathrm{Hom}_T(P_2, V) = \mathrm{Hom}_T(P_2, P_2) \neq (0)$. Hence $\mathrm{Hom}_T(P_1, P_2) = (0)$. Therefore, we have $H^{\beta_1}(P_2) = (0)$. □

Now, we prove a relevant lemma. First we need a definition:

Definiton 3.5. *Let P and V be in* mod_T *and assume that P is projective.*

(i) The P-torsion submodule $\mathrm{tor}_P(V)$ *is the sum of all submodules X of V with respect to the property* $\mathrm{Hom}_T(P, X) = (0)$. *If* $\mathrm{tor}_P(V) = (0)$, *then V is called P-torsionless.*

(ii) The kernel ker_P *is the full subcategory of* mod_T *whose objects are the T-modules V with* $\mathrm{Hom}_T(P, V) = (0)$. *Therefore, the T-module V is in* ker_P *if and only if* $\mathrm{tor}_P(V) = (0)$.

Lemma 3.6. *Let $\beta = (\beta_1, 0)$ be as in Lemma 3.4. Then* $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$ *if and only if* $\mathrm{Hom}_T(P_2, M) = (0)$. *In this case, M is P-torsionless if and only if it is P_1-torsionless.*

Proof. Assume $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$. Then by Lemma 3.4, we have $\mathrm{tr}_{P_2}(P_1)$ a submodule of $\ker\beta_1$. Let ϕ be in $\mathrm{Hom}_T(P_2, M)$. Since P_2 is projective and β is surjective, there exists a homomorphism ψ in $\mathrm{Hom}_T(P_2, P_1)$ such that $\phi = \beta\psi$. Then we have $\phi = 0$ as

$$\mathrm{im}\psi \subseteq \mathrm{tr}_{P_2}(P_1) \subseteq \ker\beta_1.$$

Conversely, assume that $\mathrm{End}_T(P) \neq (\mathrm{End}_T(P))_\beta$. Then $\mathrm{tr}_{P_2}(P_1)$ is not a submodule of $\ker\beta_1$. That means there exists a homomorphism θ in $\mathrm{Hom}_T(P_2, P_1)$ whose image is not contained in $\ker\beta_1$. Then, the map $\beta_1\theta$ in $\mathrm{Hom}_T(P_2, M)$ is nonzero. Hence $\mathrm{Hom}_T(P_2, M) = (0)$.

Now, assume that $\mathrm{Hom}_T(P_2, M) = (0)$. Let X be a submodule of M. Then $\mathrm{Hom}_T(P_2, X) = (0)$. Hence, we have

$$\mathrm{Hom}_T(P, X) = \mathrm{Hom}_T(P_1 \oplus P_2, X) = \mathrm{Hom}_T(P_1, X) \oplus \mathrm{Hom}_T(P_2, X) = \mathrm{Hom}_T(P_1, X).$$

Therefore, we conclude that M is P-torsionless if and only if it is P_1-torsionless, as claimed. □

Corollary 3.7. *Suppose P_2 is a projective T-module such that $\mathrm{Hom}_T(P_1, P_2) = (0)$ and that $\mathrm{tr}_{P_2}(P_1)$ is a submodule of $\ker\beta_1$. Then, for*

$$\beta = \beta_1 \oplus 0 : P_1 \oplus P_2 \to M$$

we have $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$, and $H^\beta = H^{\beta_1}$.

Proof. If $\mathrm{Hom}_T(P_1, P_2) = (0)$ then $H^{\beta_1}(P_2) = (0)$, hence the result follows by Lemma 3.4.

□

3.4. Right Inverse of H

The functor H has a right inverse. Before proving this statement, we need some definitions and lemma which were stated and proved in [6].

Lemma 3.8. *Let V and V' be in mod_T and let P be projective T-module. Then $\mathrm{tor}_P(V)$ is the unique maximal submodule X of V such that $\mathrm{Hom}_T(P, X) = (0)$. Moreover, $\mathrm{tor}_P(V/\mathrm{tor}_P(V)) = (0)$ and for a T-module homomorphism $f : V \to V'$ we have $f(\mathrm{tor}_P(V))$ is a subset of $\mathrm{tor}_P(V')$.*

Proof. First, we are to prove that $\mathrm{tor}_P(V)$ is the unique maximal P-torsion submodule of V with respect to the property $\mathrm{Hom}_T(P, X) = (0)$. We need only to show the uniqueness part. Assume there exists another maximal submodule N of V satisfying the condition $\mathrm{Hom}_T(P, N) = (0)$. Then, we have $\mathrm{Hom}_T(P, N/(\mathrm{tor}_P(V) \cap N)) = (0)$ since, if there would exist a nonzero morphism in $\mathrm{Hom}_T(P, N/(\mathrm{tor}_P(V) \cap N))$, then, by projectivity of P and surjectivity of the natural projection from N onto $N/(\mathrm{tor}_P(V) \cap N)$, there would exist a nonzero morphism in $\mathrm{Hom}_T(P, N)$, which is not the case. Hence, by the Second Isomorphism Theorem, we conclude that $\mathrm{Hom}_T(P, (\mathrm{tor}_P(V) + N)/\mathrm{tor}_P(V)) = (0)$. Then, for any homomorphism ϕ in $\mathrm{Hom}_T(P, \mathrm{tor}_P(V) + N)$, we have the T-module $\mathrm{im}\phi$ is a submodule of $\mathrm{tor}_P(V)$. Then, since $\mathrm{Hom}_T(P, \mathrm{tor}_P(V)) = (0)$, the map ϕ must be the zero map. Hence, we obtain $\mathrm{Hom}_T(P, \mathrm{tor}_P(V) + N) = (0)$. However, this result is contradicting the maximality of $\mathrm{tor}_P(V)$ since $\mathrm{tor}_P(V)$ is a submodule of $(\mathrm{tor}_P(V) + N)$. Therefore, we must have $\mathrm{tor}_P(V)$ as the unique maximal P-torsion submodule of V.

Next, we are to show that $\mathrm{tor}_P(V/\mathrm{tor}_P(V)) = (0)$. To this end, we have to prove that for any submodule $W/\mathrm{tor}_P(V)$ of $V/\mathrm{tor}_P(V)$, we have $\mathrm{Hom}_T(P, W/\mathrm{tor}_P(V)) \neq (0)$. We prove by contradiction; assume that there exists a submodule $W_0/\mathrm{tor}_P(V)$ of $V/\mathrm{tor}_P(V)$ such that $\mathrm{Hom}_T(P, W_0/\mathrm{tor}_P(V)) = (0)$. Then we have $\mathrm{Hom}_T(P, W_0) = (0)$ since $\mathrm{Hom}_T(P, \mathrm{tor}_P(V)) = (0)$. But, that contradicts maximality of $\mathrm{tor}_P(V)$ since $\mathrm{tor}_P(V)$ is a subset of W_0. Hence, we obtain $\mathrm{tor}_P(V/\mathrm{tor}_P(V)) = (0)$.

Finally, we are to establish the last statement; for any T-module homomorphism $f : V \to V'$, we have $f(\mathrm{tor}_P(V))$ as a submodule of $\mathrm{tor}_P(V')$. Our goal is to prove the equality $\mathrm{Hom}_T(P, f(\mathrm{tor}_P(V))) = (0)$, then, since $\mathrm{tor}_P(V')$ is maximal, the result follows. If there would be a nonzero homomorphism in $\mathrm{Hom}_T(P, f(\mathrm{tor}_P(V)))$, then by projectivity of P and surjectivity of the map $f|_{\mathrm{tor}_P(V)} : \mathrm{tor}_P(V) \to f(\mathrm{tor}_P(V))$ which is obtained by restricting f to $\mathrm{tor}_P(V)$, there would exist a nonzero homomorphism in $\mathrm{Hom}_T(P, \mathrm{tor}_P(V))$, which is not the case. Therefore $\mathrm{Hom}_T(P, f(\mathrm{tor}_P(V))) = (0)$. $\square$

Using the above result $f(\mathrm{tor}_P(V)) \subseteq \mathrm{tor}_P(V')$, we can conclude that for V and V' in mod_T, any T-module homomorphism $f : V \to V'$ induces a T-module homomorphism from $V/\mathrm{tor}_P(V)$ to $V'/\mathrm{tor}_P(V')$. Now, we define a functor A_P which has an intermediate role in the definition of right inverse of H:

Definiton 3.9. *Define the functor*

$$A_P : \mathrm{mod}_T \to \mathrm{mod}_T, V \mapsto V/\mathrm{tor}_P(V)$$

for V in mod_T and define $A_P(f)$ as the induced morphism from $V/\mathrm{tor}_P(V)$ to $V'/\mathrm{tor}_P(V')$ for any T-module homomorphism $f : V \to V'$.

Now, we are ready to define inverses of H:

Definiton 3.10. *Assume $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$. We define four functors from $\mathrm{mod}_{\mathrm{End}_T(M)}$ to mod_T as*

$$F_M = \_ \otimes_{\mathrm{End}_T(M)} M$$
$$\tilde{F}_M = A_P \circ (\_ \otimes_{\mathrm{End}_T(M)} M)$$
$$G_M = \_ \otimes_{\mathrm{End}_T(P)} P$$
$$\tilde{G}_M = A_P \circ (\_ \otimes_{\mathrm{End}_T(P)} P)$$

Before stating the proposition on inverses of H, we state a lemma which shall be used in the proof of that proposition. The sketch of the proof was given in [6]. Here, we give a detailed proof.

Lemma 3.11. *Assume $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$. Then $\mathrm{End}_T(M) \cong \mathrm{Hom}_T(P, M)$ as $\mathrm{End}_T(M) - \mathrm{End}_T(M)$ bimodules.*

Proof. Firstly, we observe that, given a morphism α in $\mathrm{Hom}_T(P, M)$, by projectivity of P and surjectivity of β, we have a morphism ϕ in $\mathrm{End}_T(P)$ such that $\alpha = \beta\phi$, Then, since we assume that $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$, by Proposition 3.1, there exists a morphism ψ in $\mathrm{End}_T(M)$ such that for m in M $\psi(m) = \beta(\phi(p))$ where $\beta(p) = m$. Combining these two results, we obtain for any α in $\mathrm{Hom}_T(P, M)$, a ψ in $\mathrm{End}_T(M)$ given by $\psi(m) = \alpha(p)$ for m in M where p is in P such that $\beta(p) = m$.

Now, we define

$$\Phi : \mathrm{Hom}_T(P, M) \to \mathrm{End}_T(M)$$

mapping α in $\mathrm{Hom}_T(P, M)$ to ψ in $\mathrm{End}_T(M)$ where ψ is defined as above. Well-definedness of Φ is clear since even if different morphisms ϕ_1 and ϕ_2 satisfy the property $\alpha = \beta\phi$, the resulting morphims ψ_1 and ψ_2 are the same, as we have

$$\psi_1(m) = \beta(\phi_1(p)) = \alpha(p) = \beta(\phi_2(p)) = \psi_2(m)$$

for any m in M and p in P such that $\beta(p) = m$. The map Φ is an $\mathrm{End}_T(M)$-module homomorphism since, for $\psi \in \mathrm{End}_T(M)$ and $\alpha \in \mathrm{Hom}_T(P, M)$,

$$\Phi(\psi\alpha)(m) = \psi\alpha(p) = \psi(\alpha(p)) = \psi\Phi(\alpha)(m).$$

Also Φ is surjective since, by Proposition 3.1, for ψ in $\mathrm{End}_T(M)$, there exists a ϕ in $\mathrm{End}_T(P)$ such that $\widetilde{\beta}(\phi) = \psi$ and we have $\beta\phi$ in $\mathrm{Hom}_T(P, M)$, and for all m in M and p in P such that $\beta(p) = m$ we have $\Phi(\beta\phi)(m) = \beta\phi(p) = \psi(m)$. Finally, Φ is injective since if α is in $\ker\Phi$, then $\Phi(\alpha) = 0$, and as $\widetilde{\beta}$ in the proof of Proposition 3.1 is an isomorphism, the corresponding ϕ in $(\mathrm{End}_T(P))_\beta$ is in J_β, that is $\mathrm{im}\phi$ is a subset of $\ker\beta$, hence $\alpha = \beta\phi = 0$. Therefore Φ is an isomorphism and the lemma follows. □

The following proposition gives right inverses for H. The proof is taken from [6].

Proposition 3.12. *Assume* $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$. *Let* $\hat{H}$ *be one of the four functors defined in Definition 3.10. Then* $\hat{H}$ *is a right inverse of the functor* H.

Proof. Let X be an $\mathrm{End}_T(M)$-module. Firstly, we observe that, by Proposition 3.1, $\mathrm{End}_T(M)$-module M is also an $\mathrm{End}_T(P)/J_\beta$-module. Besides, the ideal J_β acts on M trivially, that is $J_\beta \cdot M = (0)$ since the action of any element of $\mathrm{End}_T(P)/J_\beta$ on M is defined as the action of the corresponding element in $\mathrm{End}_T(M)$ and J_β is mapped to the zero element of $\mathrm{End}_T(M)$, therefore

$$J_\beta \cdot M = 0 \cdot M = 0.$$

Hence, we have

$$X \otimes_{\mathrm{End}_T(P)} M \cong X \otimes_{\mathrm{End}_T(M)} M.$$

Also, by Proposition 2.9, we know that

$$\mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(P)} M) \cong X \otimes_{\mathrm{End}_T(P)} \mathrm{Hom}_T(P, M).$$

Thus, using Lemma 3.11, we obtain

$$\begin{aligned} \mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(M)} M) &\cong \mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(P)} M) \\ &\cong X \otimes_{\mathrm{End}_T(P)} \mathrm{Hom}_T(P, M) \\ &\cong X \otimes_{\mathrm{End}_T(P)} \mathrm{End}_T(M) \\ &\cong X \otimes_{\mathrm{End}_T(M)} \mathrm{End}_T(M) \\ &\cong X \end{aligned}$$

Besides, by Proposition 2.9, any morphism ϕ in $\mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(M)} M)$ can be written of the form $\phi_{x,\beta}$ for some x in X such that $\phi_{x,\beta}(p) = x \otimes \beta(p)$ for p in P. Thus we have

$$\mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(M)} M) J_\beta = 0$$

since for ψ in J_β and $\phi_{x,\beta}$ in $\mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(M)} M)$ and p in P,

$$\phi_{x,\beta}\psi(p) = x \otimes \beta(\psi(p)) = x \otimes 0$$

as imϕ is a submodule of kerβ.

Therefore we have

$$\begin{aligned} H_M(F_M(X)) &= \mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(M)} M)/\mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(M)} M)J_\beta \\ &\cong \mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(M)} M) \\ &\cong X \end{aligned}$$

and hence F_M is a right inverse of H.

Obviously, for V in mod$_T$, we have $\mathrm{Hom}_T(P, V) \cong \mathrm{Hom}_T(P, A_P(V))$. Thus $H(V) = H(A_P(V))$. Then, using the above result we get

$$H(\tilde{F}_M(X)) = H(A_P(F_M(X))) = H(F_M(X)) = X.$$

Therefore $\tilde{F}_M$ is also a right inverse of H. The statement can be proved similarly also for G_M and $\tilde{G}_M$. □

3.5. Correspondence between $(\mathrm{Irr}T)_H$ and $\mathrm{IrrEnd}_T(M)$

Our next aim is to maintain a correspondence between certain irreducible T-modules and non-isomorphic irreducible $\mathrm{End}_T(M)$-modules. First, we need some lemma. Proofs of those lemma are taken from [6].

Lemma 3.13. *Assume* $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$. *Let* V *be an irreducible* T*-module. Then* $H(V) = (0)$ *or* $\mathrm{Hom}_T(P, V)J_\beta = (0)$, *and* $H(V) = \mathrm{Hom}_T(P, V) \neq (0)$ *is an irreducible* $\mathrm{End}_T(M)$*-module.*

Proof. First note that if $\mathrm{Hom}_T(P, V) \neq (0)$ then $\mathrm{Hom}_T(P, V)$ is an irreducible $\mathrm{End}_T(P)$-module. For proof, see [2, 6.3].
Assume $H(V) \neq 0$. Then $\mathrm{Hom}_T(P, V) \neq 0$ and $\mathrm{Hom}_T(P, V)J_\beta \neq \mathrm{Hom}_T(P, V)$. Then

$\mathrm{Hom}_T(P,V)J_\beta$ is a proper submodule of $\mathrm{Hom}_T(P,V)$. But, since $\mathrm{Hom}_T(P,V)$ is an irreducible $\mathrm{End}_T(P)$-module, we have $\mathrm{Hom}_T(P,V)J_\beta = (0)$. So $H(V) = \mathrm{Hom}_T(P,V)$ is an irreducible $\mathrm{End}_T(P)$-, hence $\mathrm{End}_T(M)$-module. □

Lemma 3.14. *Let X be an $\mathrm{End}_T(M)$-module. Then $\mathrm{tr}_P(X \otimes_{\mathrm{End}_T(M)} M) = X \otimes_{\mathrm{End}_T(M)} M$ and $\mathrm{tr}_P(X \otimes_{\mathrm{End}_T(P)} P) = X \otimes_{\mathrm{End}_T(P)} P$.*

Proof. We know by Lemma 3.11 and Proposition 2.9 that $X \cong \mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(M)} M)$ via the map $x \mapsto \phi_{x,\beta}$ where x is in X and $\phi_{x,\beta}$ is in $\mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(M)} M)$ defined as $\phi_{x,\beta}(p) = x \otimes \beta(p)$ for p in P. Let $x \otimes m$ be an arbitrary generator of $X \otimes_{\mathrm{End}_T(M)} M$. As β is surjective, there exists a p in P such that $\beta(p) = m$, hence $\phi_{x,\beta}(p) = x \otimes m$. Thus $x \otimes m$ is in $\mathrm{tr}_P(X \otimes_{\mathrm{End}_T(M)} M)$. Then, since $x \otimes m$ is arbitrary, we obtain $X \otimes_{\mathrm{End}_T(M)} M \subseteq \mathrm{tr}_P(X \otimes_{\mathrm{End}_T(M)} M)$. Also we have, by definition, $\mathrm{tr}_P(X \otimes_{\mathrm{End}_T(M)} M) \subseteq X \otimes_{\mathrm{End}_T(M)} M$. Therefore we proved

$$\mathrm{tr}_P(X \otimes_{\mathrm{End}_T(M)} M) = X \otimes_{\mathrm{End}_T(M)} M.$$

Since X is also an $\mathrm{End}_T(P)$-module, the same proof provided M replaced by P gives us the second statement, $\mathrm{tr}_P(X \otimes_{\mathrm{End}_T(P)} P) = X \otimes_{\mathrm{End}_T(P)} P$. □

Lemma 3.15. *Let X be an irreducible $\mathrm{End}_T(M)$-module. Then $\tilde{F}_M(X) \neq (0)$, and $\tilde{F}_M(X)$ is an irreducible T-module, and we have $\tilde{F}_M(X) = \tilde{G}_M(X)$.*

Proof. By Proposition 3.12 we know that $H_M(\tilde{F}_M(X)) \cong X \neq (0)$. Hence, we have $\tilde{F}_M(X) \neq (0)$.

Now, we are to show that $\tilde{F}_M(X)$ is irreducible. Let U be a submodule of $F_M(X)$ which is equal to $X \otimes_{\mathrm{End}_T(M)} M$. Then, the $\mathrm{End}_T(P)$-module $\mathrm{Hom}_T(P,U)$ is canonically embedded into $\mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(M)} M) \cong X$. Since X is an irreducible $\mathrm{End}_T(M)$-module, hence irreducible $\mathrm{End}_T(P)$-module, we have either $\mathrm{Hom}_T(P,U) = (0)$ or $\mathrm{Hom}_T(P,U) = X \cong \mathrm{Hom}_T(P, X \otimes_{\mathrm{End}_T(M)} M)$. If the latter holds, then the image of every homomorphism from P to $X \otimes_{\mathrm{End}_T(M)} M$ is contained in U. That means

$\mathrm{tr}_P(X \otimes_{\mathrm{End}_T(M)} M)$ is a submodule of U. Then, by Lemma 3.14 $X \otimes_{\mathrm{End}_T(M)} M$ is a submodule of U. Hence $U = X \otimes_{\mathrm{End}_T(M)} M$. This shows that if U is a proper submodule of $X \otimes_{\mathrm{End}_T(M)} M$, then $\mathrm{Hom}_T(P, U) = (0)$. Then, by definition of $\mathrm{tor}_P(X \otimes_{\mathrm{End}_T(M)} M)$, for all proper submodules U of $X \otimes_{\mathrm{End}_T(M)} M$, U is a submodule of $\mathrm{tor}_P(X \otimes_{\mathrm{End}_T(M)} M)$. So $\mathrm{tor}_P(X \otimes_{\mathrm{End}_T(M)} M)$ is the unique maximal submodule of $X \otimes_{\mathrm{End}_T(M)} M$, hence, we obtain that T-module $\tilde{F}_M(X) = X \otimes_{\mathrm{End}_T(M)} M/\mathrm{tor}_P(X \otimes_{\mathrm{End}_T(M)} M)$ is irreducible.

Since X is also irreducible as $\mathrm{End}_T(P)$-module, substituting M with P in the above argument gives that $\tilde{G}_M(X) = X \otimes_{\mathrm{End}_T(P)} P/\mathrm{tr}_P(X \otimes_{\mathrm{End}_T(P)} P)$ is also irreducible.

Finally we are to show that $\tilde{F}_M(X) = \tilde{G}_M(X)$. Since the functor $X \otimes_{\mathrm{End}_T(P)} -$ is right exact, the morphism

$$1 \otimes \beta : X \otimes_{\mathrm{End}_T(P)} P \to X \otimes_{\mathrm{End}_T(P)} M \cong X \otimes_{\mathrm{End}_T(M)} M$$

induced by β is an epimorphism. So the induced mapping

$$A_P(1 \otimes \beta) : \tilde{G}_M(X) \to \tilde{F}_M(X)$$

is also an epimorphism. As $A_P(1 \otimes \beta)$ is nonzero and $\ker A_P(1 \otimes \beta)$ is a submodule of the irreducible module $\tilde{G}_M(X)$, we have $\ker A_P(1 \otimes \beta) = (0)$. Therefore, by the First Isomorphism Theorem, $\tilde{F}_M(X) \cong \tilde{G}_M(X)$.

□

Now we state the correspondence theorem mentioned before. The proof is partly taken from [6].

Notation. *Let R be a ring. The complete set of non-isomorphic irreducible R-modules is denoted by* $\mathrm{Irr}R$.

Theorem 3.16. *Assume* $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$ *Define the set*

$$(\mathrm{Irr}T)_H = \{V \in \mathrm{Irr}T \mid H_M(V) \neq (0)\}.$$

Then H_M *induces a bijective correspondence*

$$H_M : (\mathrm{Irr}T)_H \to \mathrm{Irr}(\mathrm{End}_T(M))$$

and the inverse of H_M *is*

$$\tilde{F}_M : \mathrm{Irr}(\mathrm{End}_T(M)) \to (\mathrm{Irr}T)_H.$$

On $\mathrm{Irr}(\mathrm{End}_T(M))$*, the functors* $\tilde{F}_M$ *and* $\tilde{G}_M$ *coincide.*

Proof. We proved before in Proposition 3.12 that $\tilde{F}_M$ is a right inverse for H_M. So if we show that $\tilde{F}_M$ is also a left inverse for H_M we obtain the required correspondence. To this end, let V be in $(\mathrm{Irr}T)_H$. Then $X := H_M(V) \neq (0)$ and so, by Lemma 3.13, we have $H_M(V) = \mathrm{Hom}_T(P, V)$ and $H_M(V)$ is irreducible. Now we define the map $\phi : \mathrm{Hom}_T(P, V) \otimes_{\mathrm{End}_T(P)} P \to V$, for f in $\mathrm{Hom}_T(P, V)$ and p in P, as $\phi(f \otimes p) = f(p)$. Clearly, ϕ is well-defined. Now, we are to show that ϕ is surjective. First, observe that any nonzero f in $\mathrm{Hom}_T(P, V)$ is surjective since otherwise $\mathrm{im}f$ is a proper submodule of V and that contradicts the irreducibility of V. Now let v be in V. As $\mathrm{Hom}_T(P, V) \neq (0)$ there exists a nonzero f in $\mathrm{Hom}_T(P, V)$ and since f is surjective there exists p in P such that $f(p) = v$. Then $\phi(f \otimes p) = f(p) = v$. Since v is arbitrary, we have ϕ surjective.

Observe that, by Lemma 3.13, we have $\mathrm{tor}_P(V) = (0)$ as $\mathrm{Hom}_T(P, V) \neq (0)$ and V is irreducible. Also, by Lemma 3.8, $\phi(\mathrm{tor}_P(X \otimes_{\mathrm{End}_T(P)} P)) \subseteq \mathrm{tor}_P(V) = (0)$. Then $\mathrm{tor}_P(X \otimes_{\mathrm{End}_T(P)} P) = (0)$. Then, since $\tilde{G}_M(X)$ is irreducible, ϕ induces an isomorphism, and by Lemma 3.15, we have

$$\tilde{F}_M(X) \cong \tilde{G}_M(X) = X \otimes_{\mathrm{End}_T(P)} P/\mathrm{tor}_P(X \otimes_{\mathrm{End}_T(P)} P) \cong V/\mathrm{tor}_P(V) \cong V. \quad \square$$

Definiton 3.17. *Let* $Y \subseteq \mathrm{End}_T(M)$*. Define* YM *as the ideal of* M *generated by the images of homomorphisms in* Y*, that is,* $YM = \langle \mathrm{im}\phi : \phi \in Y \rangle$*. The below theorem was stated and partly proved in [6]. We give a full proof here.*

Theorem 3.18. *Let* Y *be a right ideal of* $\mathrm{End}_T(M)$*. Then* $\tilde{F}_M(Y) = A_P(YM)$ *and* $H_M(YM) = Y$*. In particular, if* M *is* P*-torsionless, then* $\tilde{F}_M(Y) = YM$*.*

Proof. First, observe that, by general theory and Lemma 3.11, we have the isomorphisms

$$Y \cong Y \otimes_{\mathrm{End}_T(M)} \mathrm{End}_T(M) \cong Y \otimes_{\mathrm{End}_T(M)} \mathrm{Hom}_T(P, M)$$

via maps $y \mapsto y \otimes 1 \mapsto y \otimes \beta$, and by Proposition 2.9 the isomorphism

$$Y \otimes_{\mathrm{End}_T(M)} \mathrm{Hom}_T(P, M) \cong \mathrm{Hom}_T(P, Y \otimes_{\mathrm{End}_T(M)} M)$$

via the map $y \otimes \beta \mapsto \phi_{y,\beta}$ where $\phi_{y,\beta}(p) = y \otimes \beta(p)$ for p in P. Hence the elements of $\mathrm{Hom}_T(P, Y \otimes_{\mathrm{End}_T(M)} M)$ are of the form $\phi_{y,\beta}$. By the definition of YM the map $\gamma : Y \otimes_{\mathrm{End}_T(M)} M \to YM$ defined by $\gamma(y \otimes m) = y(m)$ is surjective.
So the induced map

$$\gamma_* : \mathrm{Hom}_T(P, Y \otimes_{\mathrm{End}_T(M)} M) \to \mathrm{Hom}_T(P, YM)$$

is also surjective. Now we are to prove the injectivity of γ_*: For ϕ in $\ker\gamma_*$ we have $\gamma_*(\phi) = 0$, that is $\gamma\phi(p) = 0$ for all p in P. Then, since any ϕ in $\mathrm{Hom}_T(P, Y \otimes_{\mathrm{End}_T(M)} M)$ can be written as $y \otimes \beta$ for some y in Y, we have $\gamma(y \otimes \beta(p)) = 0$ for some y in Y, for all p in P. Then, by surjectivity β, we have $\gamma(y \otimes m) = 0$ for all m in M. By definition of γ, that means $y(m) = 0$ for all m in M. Then $y = 0$, hence $\phi = y \otimes \beta = 0$. Therefore $\ker\gamma_* = (0)$. So γ_* is an isomorphism and we obtain $Y \cong \mathrm{Hom}_T(P, YM)$.

In particular, $\mathrm{Hom}_T(P, YM)J_\beta = (0)$ since for ϕ in J_β, ψ in $\mathrm{Hom}_T(P, YM)$, p in P, using the isomorphism γ_*, ψ can be written as $y \otimes \beta$ for some y in Y and hence

$\psi\phi(p) = \psi(\phi(p)) = y \otimes \beta(\phi(p)) = y \otimes 0 = 0.$

Thus,

$$H_M(YM) = \mathrm{Hom}_T(P, YM)/\mathrm{Hom}_T(P, YM)J_\beta = \mathrm{Hom}_T(P, YM) \cong Y.$$

Second, we are to prove $\tilde{F}_M(Y) = A_P(YM)$. Since $\gamma : Y \otimes_{\mathrm{End}_T(M)} M \to YM$ is surjective, it is enough to show that $\ker\gamma$ is a submodule of $\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M)$ and $\gamma(\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M)) = \mathrm{tor}_P(YM)$. Then, by general theory, we can conclude that

$$\tilde{F}_M(Y) = Y \otimes_{\mathrm{End}_T(M)} M/\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M) \cong YM/\mathrm{tor}_P(YM) = A_P(YM).$$

Now, applying the functor $\mathrm{Hom}_T(P, -)$ to the exact sequence

$$0 \longrightarrow \ker\gamma \xrightarrow{\iota} Y \otimes_{\mathrm{End}_T(M)} M \xrightarrow{\gamma} YM \longrightarrow 0$$

we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_T(P, \ker\gamma) \xrightarrow{\iota_*} \mathrm{Hom}_T(P, Y \otimes_{\mathrm{End}_T(M)} M) \xrightarrow{\gamma_*} \mathrm{Hom}_T(P, YM) \longrightarrow 0$$

Then, since γ_* is an isomorphism, ι_* is the zero map. So $\mathrm{Hom}_T(P, \ker\gamma) = (0)$. Hence $\ker\gamma \subseteq \mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M)$.

Since the restriction $\gamma|_{\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M)}$ of γ to $\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M)$ is surjective and P is projective, the induced morphism

$$(\gamma|_{\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M)})_* : \mathrm{Hom}_T(P, \mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M)) \to \mathrm{Hom}_T(P, \gamma(\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M)))$$

is also surjective. Then, we have $\mathrm{Hom}_T(P, \gamma(\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M))) = (0)$. Now, assume there is a submodule W of YM such that $\gamma(\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M))$ is a submodule of W

and $\mathrm{Hom}_T(P, W) = (0)$. Then there exists a submodule Z of $Y \otimes_{\mathrm{End}_T(M)} M$ such that $\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)})M$ is a submodule of Z and $\gamma(Z) = W$ since γ is surjective. For any η in $\mathrm{Hom}_T(P, Z)$, we have η in $\mathrm{Hom}_T(P, \ker\gamma)$ since $\gamma\eta$ is in $\mathrm{Hom}_T(P, \mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)})M)$ and $\mathrm{Hom}_T(P, \mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)})M) = (0)$. However, we know that $\mathrm{Hom}_T(P, \ker\gamma) = (0)$, hence we obtain $\mathrm{Hom}_T(P, Z) = (0)$, but this contradicts the maximality of the P-torsion submodule $\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M)$. Hence we conclude that

$$\gamma(\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M)) = \mathrm{tor}_P(YM).$$

Therefore we have $\tilde{F}_M(Y) \cong A_P(YM)$.

Finally, if M is P-torsionless, so is YM since otherwise, if there would exist a submodule W of YM such that $\mathrm{Hom}_T(P, W) = (0)$, then as YM is a submodule of M, W is also a submodule of M and that would contradict the assumption that M is P-torsionless. Thus, we have

$$\tilde{F}_M(Y) \cong A_P(YM) = YM/\mathrm{tor}_P(YM) = YM.$$

□

The proof of the following theorem was given in [6].

Theorem 3.19. *Assume* $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$. *Let X and Y be right ideals of* $\mathrm{End}_T(M)$. *Suppose M is P-torsionless. Then H induces an isomorphism, also denoted by H, from* $\mathrm{Hom}_T(XM, YM)$ *onto* $\mathrm{Hom}_{\mathrm{End}_T(M)}(X, Y)$.

Proof. Using functoriality of H_M we define

$$H : \mathrm{Hom}_T(XM, YM) \to \mathrm{Hom}_{\mathrm{End}_T(M)}(X, Y),\ \phi \mapsto H(\phi)$$

where $H(\phi) : H(XM) \to H(YM)$. Since M is P-torsionless, we have $H(XM) = X$ and $H(YM) = Y$ by Theorem 3.18. Then $H(\phi)$ is a map from X to Y. Since H is a functor, for ϕ and ψ in $\mathrm{Hom}_T(XM, YM)$, we have $H(\phi\psi) = H(\phi)H(\psi)$. Hence H is a homomorphism. Clearly H is R-linear.

Now, we are to show that H is surjective. Let α be an element of $\mathrm{Hom}_{\mathrm{End}_T(M)}(X, Y)$. Consider the map $\alpha \otimes 1_M : X \otimes_{\mathrm{End}_T(M)} M \to Y \otimes_{\mathrm{End}_T(M)} M$. This map induces a T-linear map

$$\gamma : \frac{X \otimes_{\mathrm{End}_T(M)} M}{\mathrm{tor}_P(X \otimes_{\mathrm{End}_T(M)} M)} \to \frac{Y \otimes_{\mathrm{End}_T(M)} M}{\mathrm{tor}_P(Y \otimes_{\mathrm{End}_T(M)} M)}$$

By Theorem 3.18, we have $A_P(X \otimes_{\mathrm{End}_T(M)} M) = XM$ and $A_P(Y \otimes_{\mathrm{End}_T(M)} M) = YM$. Hence γ is a T-linear map from XM to YM. Also, we have

$$H(\gamma) = H(A_P(\alpha \otimes 1_M)) = H(A_P(F_M(\alpha))) = H(\hat{H}(\alpha)) = \alpha.$$

Therefore H is surjective.

H is also injective: Let $f : XM \to YM$ be a nonzero T-linear map. We are to show that $H(f)$ is also nonzero. Set $U := \mathrm{im} f$. U is a submodule of M. Then since M is P-torsionless, so is U. Then $\mathrm{Hom}_T(P, U) \neq (0)$. By Lemma 3.11 we have $\mathrm{Hom}_T(P, M) \cong \mathrm{End}_T(M)$. Then, by projectivity of P and Proposition 3.1, we conclude that $\mathrm{Hom}_T(P, M)J_\beta = (0)$. So, since $\mathrm{Hom}_T(P, U)$ is a submodule of $\mathrm{Hom}_T(P, M)$, we have $\mathrm{Hom}_T(P, U)J_\beta = (0)$ as well. Hence $H(U) = \mathrm{Hom}_T(P, U)$. Since P is projective and $f : XM \to U$ is surjective, the map $f_* : \mathrm{Hom}_T(P, XM) \to \mathrm{Hom}_T(P, U)$ is also surjective. Then, for each nonzero element ρ of $\mathrm{Hom}_T(P, U)$, there exists an element τ of $\mathrm{Hom}_T(P, XM) = H(XM)$ such that $\rho = f\tau$. Then, we have

$$H(f)(\tau) = \mathrm{Hom}_T(P, f)(\tau) = f\tau = \rho \neq 0.$$

So $H(f) : H(XM) \to H(YM)$ is not the zero map. Therefore H is injective.

Thus, we have shown that H is bijective, hence an isomorphism. □

3.6. Correspondence between $(\mathrm{Irr}T)_H$ and Constituents of $hd(M)$

Now we are to give the exposition of the proof for one of the main theorems of this thesis. Firstly, we need a lemma which were proved in [6]:

Lemma 3.20. *Assume* $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$*. Let* X *be an* $\mathrm{End}_T(M)$*-module and* Y *be a maximal submodule of* X*. Let* $i : Y \to X$ *be the canonical embedding. Let* V *denote* $F_M(X) = X \otimes_{\mathrm{End}_T(M)} M$*, and* U *be the image of the* T*-linear map*

$$i \otimes 1_M : Y \otimes_{\mathrm{End}_T(M)} M \to X \otimes_{\mathrm{End}_T(M)} M.$$

Then $\mathrm{tor}_P(V/U)$ *is the unique maximal submodule of* V/U *and the factor module* $A_P(V/U)$ *is canonically isomorphic to the irreducible* T*-module* $\tilde{F}_M(X/Y)$*.*

Proof. Applying the right exact functor $- \otimes_{\mathrm{End}_T(M)} M$ to the exact sequence

$$0 \to Y \xrightarrow{i} X \to X/Y \to 0$$

we obtain the exact sequence

$$Y \otimes_{\mathrm{End}_T(M)} M \xrightarrow{i \otimes 1_M} X \otimes_{\mathrm{End}_T(M)} M \to X/Y \otimes_{\mathrm{End}_T(M)} M \to 0,$$

hence the exact sequence

$$0 \to U \to V \to X/Y \otimes_{\mathrm{End}_T(M)} M \to 0.$$

Then we have $V/U \cong X/Y \otimes_{\mathrm{End}_T(M)} M$ and therefore

$$(V/U)/\mathrm{tor}_P(V/U) \cong X/Y \otimes_{\mathrm{End}_T(M)} M/\mathrm{tor}_P((X/Y) \otimes_{\mathrm{End}_T(M)} M).$$

By definition, the left hand side is equal to $A_P(V/U)$ and the right hand side is equal to $\tilde{F}_M(X/Y)$. Hence, we obtain $A_P(V/U) = \tilde{F}_M(X/Y)$. Since Y is maximal, the quotient module X/Y is irreducible. Then, by Lemma 3.15, we have $\tilde{F}_M(X/Y)$ irreducible and hence $\mathrm{tor}_P(V/U)$ is maximal. $\square$

We are to state a corollary of Lemma 3.20. Let X be an $\mathrm{End}_T(M)$-module and $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq ...$ be a filtration of X such that $Y_i := X_{i-1}/X_i$ is an irreducible $\mathrm{End}_T(M)$-module for $i \geq 1$. Let $V := F_M(X) = X \otimes_{\mathrm{End}_T(M)} M$, and let V_i be the canonical image of $X_i \otimes_{\mathrm{End}_T(M)} M$ in V for $i \geq 0$.

Corollary 3.21. *Assume* $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$ *and let* X_i *be an* $\mathrm{End}_T(M)$*-module for* $i \geq 0$ *and* $X = X_0$*. For the induced filtration*

$$V = V_0 \supseteq V_1 \supseteq V_2 \supseteq ...$$

where $V_i = F_M(X_i)$*, let* U_i *be the factor module* V_{i-1}/V_i*. Then* $\mathrm{tor}_P(U_i)$ *is the unique maximal submodule of* U_i *and the irreducibe* T*-module* $A_P(U_i) = U_i/\mathrm{tor}_P(U_i)$ *is canonically isomorphic to* $\tilde{F}_M(Y_i)$ *for all* $i \geq 0$*.*

The next theorem gives us a correspondence between $(\mathrm{Irr}T)_H$ and constituents of $\mathrm{hd}(M)$. In the proof, we follow the sketch given in [6].

Theorem 3.22. *Let* R *be a field. Assume* $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$*. Then,* $(\mathrm{Irr}T)_H$ *is a complete set of non-isomorphic irreducible constituents of head of* M $hd(M)$*. Every indecomposable direct summand of* M *has a simple head and factoring out the Jacobson radical induces a bijection between the isomorphism classes of indecomposable direct summands of* M *and the elements of* $(\mathrm{Irr}T)_H$*.*

Proof. Let $\{e_1, ..., e_k\}$ be a complete set of non-conjugate, primitive idempotents of $\mathrm{End}_T(M)$, that is $e_i\mathrm{End}_T(M) \neq e_j\mathrm{End}_T(M)$ for $i \neq j$. Then, by Lemma 2.3, the set $\{e_iM | 1 \leq i \leq k\}$ is a complete set of non-isomorphic, indecomposable direct summands of M. Hence, there is a bijective correspondence between the indecomposable direct summands of M and projective indecomposable $\mathrm{End}_T(M)$-modules.

We may consider $\beta : P \to M$ as the projective cover of M. Then $\ker\beta$ is superfluous in P. Clearly, any submodule of $\ker\beta$ is also superfluous. Then, since the module $\mathrm{im}\phi$ is a submodule of $\ker\beta$ for any ϕ in J_β, by Proposition 2.4(i), we have J_β is an ideal of $Jac(\mathrm{End}_T(P))$. As $\mathrm{End}_T(P)$ is finitely generated, it is right Artinian. Then, by Proposition 2.4(ii), the Jacobson radical $Jac(\mathrm{End}_T(P))$ is nilpotent. Then, J_β is also nilpotent as it is an ideal of $Jac(\mathrm{End}_T(P))$.

As $\mathrm{End}_T(P)/J_\beta \cong \mathrm{End}_T(M)$ by Proposition 3.1, and J_β is nilpotent, the non-conjugate primitive idempotents of $\mathrm{End}_T(P)$ are in one-to-one correspondence with the non-conjugate primitive idempotents of $\mathrm{End}_T(M)$. Hence, we can lift idempotents from $\mathrm{End}_T(M)$ to $\mathrm{End}_T(P)$. Therefore, we have a one-to-one correspondence between the indecomposable direct summands of P and those of M, given by restricting β to indecomposable direct summands of P. In particular, the projective cover P_N of any indecomposable direct summand N of M is an indecomposable projective T-module. Then, by Proposition 2.5, P_N has a unique maximal submodule, namely $Jac(P_N)$.

As indecomposable direct summands of P are in one-to-one correspondence with those of M, hence with projective indecomposable $\mathrm{End}_T(M)$-modules, the $\mathrm{End}_T(M)$-modules

$$hd(e_i\mathrm{End}_T(M)) = e_i\mathrm{End}_T(M)/Jac(e_i\mathrm{End}_T(M))$$

are simple for each $i \in 1, ..., k$. Then the set

$$\{hd(e_i\mathrm{End}_T(M))|\ 1 \le i \le k\}$$

is a complete set of non-isomorphic irreducible $\mathrm{End}_T(M)$-modules. Since this set is equal to $\mathrm{Irr}(\mathrm{End}_T(M))$, by Theorem 3.16, the set $(\mathrm{Irr}T)_H$ can be written as

$$(\mathrm{Irr}T)_H = \{\tilde{F}_M(e_i\mathrm{End}_T(M)/Jac(e_i\mathrm{End}_T(M)))|\ 1 \le i \le k\}.$$

By the assumption in the theorem, R is a field, T a finite dimensional algebra, so, all

T-modules and $\mathrm{End}_T(M)$-modules being finitely generated, have composition series, and the multiplicities of $Y \in \mathrm{IrrEnd}_T(M)$ a a composition factor of $e_i\mathrm{End}_T(M)$ equals to the multiplicity of $\tilde{F}_M(Y)$ as a composition factor of $\tilde{F}_M(X)$. Applying the functor F_M to the filtration

$$e_i\mathrm{End}_T(M) \supseteq Jac(e_i\mathrm{End}_T(M)) \supseteq ...$$

we obtain

$$e_iM \supseteq Jac(e_iM) \supseteq ...$$

Then, by Corollary 3.21, the irreducible T-module $e_iM/Jac(e_iM)$ is canonically isomorphic to $\tilde{F}_M(e_i\mathrm{End}_T(M)/Jac(e_i\mathrm{End}_T(M))$. Therefore, the set $(\mathrm{Irr}T)_H$ consists of precisely the direct summands of the head $M/Jac(M)$ of M. □

Now, we state a more specific version of the previous theorem. In the proof, we use the sketch given in [7].

Theorem 3.23. *Let R be a field. Assume $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$, the T-module M is P-torsionless, and $\mathrm{End}_T(M)$ is self-injective. Then*

(i) Every element of $\mathrm{Irr}(\mathrm{End}_T(M))$ is isomorphic to XM for some minimal ideal X of $\mathrm{End}_T(M)$.

(ii) The set $(\mathrm{Irr}T)_H$ is up to isomorphism a complete set of irreducible constituents of $soc(M)$ as well as $hd(M)$.

(iii) Every indecomposable direct summand of M has a simple socle and a simple head, and taking socles, respectively heads, induce bijections between the isomorphism classes of indecomposable direct summands of M and the elements of $(\mathrm{Irr}T)_H$.

(iv) Socle and heads of the indecomposable direct summands of M are isomorphic if in addition $\mathrm{End}_T(M)$ is a symmetric algebra.

Proof. As $\mathrm{End}_T(M)$ is self-injective, by Proposition 2.7, minimal ideals of $\mathrm{End}_T(M)$

correspond to principal indecomposable direct summands of the module $\mathrm{End}_T(M)$. Thus, part (i) of the theorem follows from Theorem 3.18.

The parts (ii) and (iii) are proved for $hd(M)$ in Theorem 3.22, so, it is enough to prove the statements for soc(M). Consider the projective cover (β_1, P_1) of M. By definition of the projective cover, $\ker\beta_1$ is superfluous. Then, by Proposition 2.8(i), $\ker\beta_1$ is a submodule of $Jac(P_1)$. Then, since β_1 is an epimorphism, by Proposition 2.8(ii), we have

$$\beta_1(Jac(P_1)) = Jac(M).$$

Then, we obtain

$$P_1/Jac(P_1) \cong M/Jac(M).$$

Thus, using Theorem 3.22, we conclude that the set $(\mathrm{Irr}T)_H$ is, up to isomorphism, precisely the set of the irreducible constituents of $hd(P_1)$.

Let S be a simple submodule of M. Then there exists a surjective map ϕ from P onto S. Since ϕ is surjective, $\ker\phi$ is maximal. Then $Jac(P)$ is a submodule of $\ker\phi$. Thus, we have

$$S \cong \frac{P}{\ker\phi} \subset \frac{P}{Jac(P)} = hd(P).$$

Therefore, every simple submodule of M, or equivalently, every constituent of $soc(M)$ is isomorphic to an irreducible constituent of $hd(P)$.

We use the notation of (3.1). By assumption, M is P-torsionless. Then, by Lemma 3.6, it is P_1-torsionless, that is, we have $\mathrm{Hom}_T(P_2, M) = (0)$. Since R is a field, T is a finite dimensional algebra. So, every T-module has a composition series. Since $\mathrm{Hom}_T(P_2, M) = (0)$, no composition factor of $hd(P_2)$ occurs as a composition factor of M. In particular, no simple submodule of $hd(P_2)$ occurs as a simple submodule of M.

Then, we conclude that, every constituent of $soc(M)$ is isomorphic to an irreducible constituent of $hd(P_1)$, hence to an element of $(\mathrm{Irr}T)_H$.

Now, let S be an element of $(\mathrm{Irr}T)_H$. Then, by Lemma 3.15 and Theorem 3.16 we have $S = \hat{H}(X)$ for some irreducible $\mathrm{End}_T(M)$-module X. By the part (i), we may assume X to be a minimal right ideal of $\mathrm{End}_T(M)$. M is P-torsionless, hence by Theorem 3.18, we have $S = XM$. Then, the T-module S is contained in the socle of M. Therefore, the set $(\mathrm{Irr}T)_H$ is, up to isomorphism, a complete set of irreducible constituents of $soc(M)$.

Finally, for part (iv), we refer to [9, I.8.6] which enables us to obtain a correspondence between heads and socles of $\mathrm{End}_T(M)$-modules in the case that $\mathrm{End}_T(M)$ is a symmetric algebra. □

Recall that the functor H depends on P. The functor $\tilde{F}_M$ also depends on the choice of P since it is the composition of the functor $\_\otimes_{\mathrm{End}_T(M)} M$ and the functor A_P that is determined by P. As in the case of H we are to compare functors $\tilde{F}_M^{\beta}$ and $\tilde{F}_M^{\beta_1}$ where $\beta_1 : P_1 \to M$ is the minimal projective cover of M. The following lemma was stated in [7]. Here, we give a proof.

Lemma 3.24. *Let X be an $\mathrm{End}_T(M)$-module. Then there is a natural epimorphism from $\tilde{F}_M^{\beta_1}(X)$ onto $\tilde{F}_M^{\beta}(X)$.*

Proof. Since $\tilde{F}_M^{\beta}(X) = A_P(X \otimes_{\mathrm{End}_T(M)} M)$, it is enough to show that for any T-module V, $A_{P_1}(V)$ is an epimorphic image of $A_P(V)$.

Firstly observe that if $\mathrm{Hom}_T(P, \mathrm{tor}_P(V)) = (0)$, then $\mathrm{Hom}_T(P_1, \mathrm{tor}_P(V)) = (0)$, hence $\mathrm{tor}_P(V) \leq \mathrm{tor}_{P_1}(V)$. Now, since $1_V : V \to V$ is surjective and

$$1_V(\mathrm{tor}_P(V)) = \mathrm{tor}_P(V) \leq \mathrm{tor}_{P_1}(V),$$

1_V induces an epimorphism

$$\bar{1_V} : A_P(V) \to A_{P_1}(V) \ v + \mathrm{tor}_P(V)) \mapsto v + \mathrm{tor}_{P_1}(V)$$

where v is in V. Hence $A_{P_1}(V)$ is an epimorphic image of $A_P(V)$ and lemma follows. □

For later use, we are to show that the ideal J_{β_1} of the endomorphism ring $(\mathrm{End}_T(P_1))_{\beta_1}$ is contained in the Jacobson radical $Jac((\mathrm{End}_T(P_1))_{\beta_1})$ of $(\mathrm{End}_T(P_1))_{\beta_1}$, under the assumption that T is Noetherian. The proof is taken from [7].

Lemma 3.25. *Suppose that T is Noetherian. Then J_{β_1} is a subset of $Jac((\mathrm{End}_T(P_1))_{\beta_1})$.*

Proof. Since P_1 is a finitely generated module over the Noetherian ring T, it is Noetherian. Then every surjective endomorphism of P_1 is actually an isomorphism. For details see [5, 3.3] and [5, 5.8]. Using this fact, we observe that for a maximal submodule V of P_1, the set

$$\{\phi \in (\mathrm{End}_T(P_1))_{\beta_1} \mid \mathrm{im}\phi \subseteq V\}$$

is a maximal right ideal of $(\mathrm{End}_T(P_1))_{\beta_1}$, and every maximal right ideal of V is obtained in this way.

Since $\beta_1 : P \to M$ is a minimal projective cover of M and T is semiperfect, by [5, 6.25(i)], we conclude that $\ker\beta_1$ is a submodule of $Jac(P_1)$. Then, we have

$$\begin{aligned} J_{\beta_1} &= \{\phi \in (\mathrm{End}_T(P_1))_{\beta_1} \mid \mathrm{im}\phi \leq \ker\beta_1\} \\ &\subseteq \{\phi \in (\mathrm{End}_T(P_1))_{\beta_1} \mid \mathrm{im}\phi \leq Jac(P_1)\} \\ &= Jac((\mathrm{End}_T(P_1))_{\beta_1}) \end{aligned}$$

□

Since $\hat{H}$ has a left inverse, it is injective on objects. Moreover, decomposable $\mathrm{End}_T(M)$-modules are taken to decomposable T-modules by $\hat{H}$. However, it does not preserve indecomposability in general. Next lemma concerns with these facts. In the proof , we use the sketch given in [7].

Lemma 3.26. *Let X be an indecomposable $\mathrm{End}_T(M)$-module, and let $\hat{H}(X) = V$. Let $V = V_1 \oplus V_2 \oplus ... \oplus V_k$ be a decomposition of V into a direct sum of indecomposable T-modules. Then there is an index i in $\{1, ..., k\}$ such that the following holds (viewing X if needed as an $\mathrm{End}_T(P)$-module via the epimorphism $\widetilde{\beta} : \mathrm{End}_T(P) \to \mathrm{End}_T(M)$ in the proof of Proposition 3.1):*

(i) $H(V_i) = X$ *and* $H(V_j) = (0)$ *for* $j \neq i$ *in* $\{1, ..., k\}$

(ii) $V_i \ncong V_j$ *for* $i \neq j$ *in* $\{1, ...k\}$

(iii) $\mathrm{Hom}_T(P_1, V_i) \neq (0)$ *and* $\mathrm{Hom}_T(P_1, V_j) = (0)$ *for* $i \neq j$ *in* $\{1, ..., k\}$.

Proof. Since H is a left inverse for $\hat{H}(X) = V$, we have $H(V) = H(\hat{H}(X)) = X$. Then, since X is indecomposable by assumption, so is $H(V)$. Observe that H preserves direct sums since the functor $\mathrm{Hom}_T(P, \_)$ does. Then

$$X = H(V) = H(V_1 \oplus V_2 \oplus ... \oplus V_k) = H(V_1) \oplus H(V_2) \oplus ... \oplus H(V_k)$$

and since X is indecomposable, we have $H(V_i) = X$ for some i and $H(V_j) = (0)$ for all $j \neq i$. That proves part (i).

Part (ii) follows from part (i) as H is well-defined, so if it would be the case $V_i \cong V_j$ for some j in $\{1, ...k\}$ and $j \neq i$, then we must have had $H(V_i) = H(V_j)$ which is not the case.

Part (iii) is proved first for the two choices of V obtained using the functors F_M and G_M stated in Definition 3.10, since they are mapped to X under the functor H^{β_1}. We have $H^{\beta}(V) = H^{\beta_1}(V)$. Then, by part (i), we have $H^{\beta_1}(V_i) \neq (0)$, and hence $\mathrm{Hom}_T(P_1, V_i) \neq (0)$. Also, we obtain by part (i) that $H^{\beta_1}(V_j) = (0)$ for all $j \neq i$. That means

$$\mathrm{Hom}_T(P_1, V_j)/\mathrm{Hom}_T(P_1, V_j)J_{\beta_1} = (0).$$

Then by Lemma 3.25 and Lemma 2.12, we have $\mathrm{Hom}_T(P_1, V_j) = (0)$. Hence, we have $\mathrm{Hom}_T(P_1, V_j) = (0)$ for all $j \neq i$.

Before proving part (iii) for the remaining two choices of V which obtained using the functors $\tilde{F}_M$ and $\tilde{G}_M$, we need to show that the functor A_P preserves direct sums. Now, let W_1 and W_2 be T-modules. Assume that $\mathrm{Hom}_T(P, W_1 \oplus W_2) \neq (0)$. Since $\mathrm{Hom}_T(P, \mathrm{tor}_P(W_1)) = (0)$ and $\mathrm{Hom}_T(P, \mathrm{tor}_P(W_2)) = (0)$, we have

$$\mathrm{Hom}_T(P, \mathrm{tor}_P(W_1) \oplus \mathrm{tor}_P(W_2)) = \mathrm{Hom}_T(P, \mathrm{tor}_P(W_1)) \oplus \mathrm{Hom}_T(P, \mathrm{tor}_P(W_2)) = (0).$$

Then, we have $\mathrm{tor}_P(W_1) \oplus \mathrm{tor}_P(W_2)$ is a submodule of $\mathrm{tor}_P(W_1 \oplus W_2)$. Since we assume that $\mathrm{Hom}_T(P, W_1 \oplus W_2) \neq (0)$, we have one of the statements $\mathrm{Hom}_T(P, W_1) \neq (0)$ and $\mathrm{Hom}_T(P, W_2) \neq (0)$ true. Then, we have either $\mathrm{Hom}_T(P, W_1/\mathrm{tor}_P(W_1)) \neq (0)$ or $\mathrm{Hom}_T(P, W_2/\mathrm{tor}_P(W_2)) \neq (0)$. Then, we have

$$\begin{aligned} (0) \neq \mathrm{Hom}_T &\left(P, \frac{W_1}{\mathrm{tor}_P(W_1)} \oplus \frac{W_2}{\mathrm{tor}_P(W_2)}\right) \\ = \mathrm{Hom}_T &\left(P, \frac{W_1 \oplus W_2}{\mathrm{tor}_P(W_1) \oplus \mathrm{tor}_P(W_2)}\right) \end{aligned}$$

Hence, we conclude that $\mathrm{tor}_P(W_1) \oplus \mathrm{tor}_P(W_2) = \mathrm{tor}_P(W_1 \oplus W_2)$. Then, we obtain

$$\begin{aligned} A_P(W_1 \oplus W_2) &= (W_1 \oplus W_2)/\mathrm{tor}_P(W_1 \oplus W_2) \\ &= (W_1 \oplus W_2)/\mathrm{tor}_P(W_1) \oplus \mathrm{tor}_P(W_2) \\ &= W_1/\mathrm{tor}_P(W_1) \oplus W_2/\mathrm{tor}_P(W_2) = A_P(W_1) \oplus A_P(W_2). \end{aligned}$$

For $V = \tilde{F}_M(X)$, we have $V = A_P(F_M(X)) = A_P(V_1) \oplus A_P(V_2) \oplus ... \oplus A_P(V_k)$ for some decomposition of $F_M(X) = V_1 \oplus V_2 \oplus ... \oplus V_k$ of $F_M(X)$ into a direct sum of indecomposable T-modules. Set $V_l' = A_P(V_l)$ for all $l \in \{1, ..., k\}$. Since there does not exist any T-module homomorphism from P to $\text{tor}_P(V_s)$ for all $s \in \{1, ..., k\}$, we have

$$\text{Hom}_T(P, V_s) = \text{Hom}_T(P, V_s/\text{tor}_P(V_s)) = \text{Hom}_T(P, V_s').$$

In particular, we have $H(V_s') = H(V_s)$.

Since part (iii) is proved for the functor F_M, we have $\text{Hom}_T(P_1, V_i) \neq (0)$ and $\text{Hom}_T(P_1, V_j) = (0)$ for $i \neq j$ in $\{1, ..., k\}$. Then we obtain $\text{Hom}_T(P_1, V_j') = (0)$ for $j \neq i$ in $\{1, ..., k\}$. Even if V_j' for $j \neq i$ is decomposable, for each indecomposable constituent W' of V_j', we have $\text{Hom}_T(P_1, W') = (0)$.

Since $\text{Hom}_T(P_1, V_i) \neq (0)$ for some $i \in \{1, ..., k\}$ and there does not exist any T-homomorphism from P_1 to $\text{tor}_P(V_i)$, we have

$$\text{Hom}_T(P_1, V_i') = \text{Hom}_T(P_1, A_P(V_i)) = \text{Hom}_T(P_1, V_i/\text{tor}_P(V_i)) \neq (0).$$

Now, assume V_i' is decomposable, say $V_i' = W_1' \oplus W_2'$. Since H is a left inverse for $\tilde{F}_M$, and $H(V_j') = (0)$ for all $j \neq i$, we have

$$\begin{aligned} X = H(\tilde{F}_M(X)) &= H(V_1' \oplus V_2' \oplus ...V_k') = H(V_1') \oplus H(V_2') \oplus ...H(V_k') \\ &= H(V_i') = H(W_1') \oplus H(W_2'). \end{aligned}$$

Since X is indecomposable, we have either $H(W_1') = (0)$ or $H(W_2') = (0)$. Assume $H(W_2') = (0)$. Then we have

$$\frac{\text{Hom}_T(P, W_2')}{\text{Hom}_T(P, W_2')J_\beta} = (0).$$

We have shown in the proof of Theorem 3.22 that J_β is contained in $Jac(\text{End}_T(P))$. Then, using Lemma 2.12 we conclude that $\text{Hom}_T(P, W_2') = (0)$. Hence we have $\text{Hom}_T(P_1, W_2') = (0)$.

Therefore, for a decomposition of $\tilde{F}_M(X) = V'_1 \oplus V'_2 ... \oplus V'_{k'}$ into indecomposable T-modules, $\mathrm{Hom}_T(P, V'_i) \neq (0)$ for some $i \in \{1, ...k'\}$ and $\mathrm{Hom}_T(P, V'_j) = (0)$ for all $j \neq i$. The proof for $V = \tilde{G}_M(X)$ is exactly the same as the proof for $\tilde{F}_M(X)$, therefore is omitted. □

The previous lemma has a corollary stated in [7]. Here, we give a proof.

Corollary 3.27. *Under the assumption of Lemma 3.26 suppose that $H^\beta = H^{\beta_1}$, where T is a finite-dimensional algebra over some field. Then no composition factor of the head of P_1, hence of M, occurs as a composition factor of V_j for $i \neq j\{1, ..., k\}$.*

Proof. We have shown in the proof of Theorem 3.23 that P_1 and M have the same head. By Theorem 3.22, we know that the set of constituents of $hd(M)$ isomorphic to the set $\mathrm{Irr}T_H$. However, for any $i \neq j \in \{1, ..., k\}$ $H^{\beta_1}(V) = (0)$. Hence, V_j is not an element of $\mathrm{Irr}T_H$. The result follows. □

Now we state an application of Theorem 3.19. The proof is taken from [7].

Corollary 3.28. *Asssume that M is P-torsionless. Let X be an indecomposable right ideal of* $\mathrm{End}_T(M)$. *Then $\hat{H}(X) = XMisindecomposable$.*

Proof. Since M is P-torsionless, by Theorem 3.18, we have $\hat{H}(X) = XM$. Also, by Theorem 3.19, there exists an isomorphism between the endomorphism rings of X and XM. For a Noetherian R-module M, the endomorphism ring of M is local if and only if M is indecomposable, for details see [3, VII.1.27]. Then, since X is indecomposable, its endomorphism ring is local, hence the endomorphism ring of XM is local as well. Then, we conclude that XM is indecomposable. □

4. APPLICATION OF H TO HARISH-CHANDRA THEORY

4.1. Harish-Chandra Induction and Truncation

Let G be a finite group and F be a field. For subgroups P and U of G with U normal in P, we define *Harish-Chandra induction* from $F[P/U]$-modules to FG-modules, denoted by $R^G_{P/U}$, as the functor that lifts an $F[P/U]$-module to an FP-module by letting U act trivially and then inducing it from P to G. The right adjoint functor of $R^G_{P/U}$, *Harish-Chandra truncation*, denoted by $T^G_{P/U}$, is defined as the functor that restricts an FG-module to FP-module and takes U-fixed points to yield an $F[P/U]$-module.

Theorem 4.1. *(Mackey Decomposition Theorem) Let P, Q, U, V be subgroups of G with U normal in P and V normal in Q. Suppose that the orders of U and V are invertible in F. Let M be and $F[P/U]$-module. Then*

$$T^G_{Q/V} \circ R^G_{P/U}(M) \cong \bigoplus_{x \in P\backslash G/Q} R^{Q/V}_{\frac{(P^x\cap Q)V}{(U^x\cap Q)V}} \mathrm{C}^{\phi}_{\frac{(Q\cap P^x)U^x}{(V\cap P^x)U^x}, \frac{(P^x\cap Q)V}{(U^x\cap Q)V}} T^{P^x/U^x}_{\frac{(Q\cap P^x)U^x}{(V\cap P^x)U^x}}(M^x)$$

where

$$\mathrm{C}^{\phi}_{\frac{(Q\cap P^x)U^x}{(V\cap P^x)U^x}, \frac{(P^x\cap Q)V}{(U^x\cap Q)V}} : \frac{(Q\cap P^x)U^x}{(V\cap P^x)U^x} \to \frac{(P^x\cap Q)V}{(U^x\cap Q)V}$$

is an isomorphism, and M^x denotes the conjugate module for the conjugate factor group $x(P/U)x^{-1}$, and $P\backslash G/Q$ is a set of $P-Q$-double coset of representatives in G.

We are to prove Theorem 4.1 using biset functors. At this section, we are to introduce the notion of bisets and prove some facts about bisets.

4.2. Biset Functors

Definiton 4.2. *Let H and K be groups.*

*(i) An (H,K)-*biset *X is both a left H-set and a right K-set such that the H-action and the K-action commute, that is, for any $x \in X$, for all $h \in H$ and $k \in K$, we have*

$$(h \cdot x) \cdot k = h \cdot (x \cdot k).$$

(ii) An (H,K)-biset X is called transitive *if for any elements x, y in X there exists (h,k) in $H \times K$ such that*

$$h \cdot x \cdot k = y.$$

(iii) The stabilizer $(H,K)_x$ of x in $(H \times K)$ is the subgroup of $H \times K$ defined by

$$(H,K)_x = \{(h,k) \in H \times K \mid h \cdot x = x \cdot k\}.$$

Lemma 4.3. *Let H and K be groups, and X be an (H,K) biset. Choose a set $H\backslash X/K$ of representatives of (H,K)-orbits of X. Then there is an isomorphism of (H,K)-bisets*

$$X \cong \bigsqcup_{x \in H\backslash X/K} \frac{(H \times K)}{(H,K)_x}.$$

In particular, any transitive (H,K)-biset is isomorphic to $(H \times K)/L$, for some subgroup L of $H \times K$.

Proof. See [4, 2.3.4]. □

Composition of bisets is defined as follows:

Definiton 4.4. *Let G, H and K be groups. If U is an (H, G)-biset, and V is a (K, H)-biset,* the composition of V and U *is the set of H-orbits on the cartesian product $V \times U$, where the right action of H is defined by*

$$(v, u) \cdot h = (v \cdot h, h^{-1} \cdot u)$$

for all (v, u) in $V \times U$. It is denoted by $V \times_H U$.

Now, we are to state a lemma which provides us a useful formula for the composition of bisets:

Lemma 4.5. *(Mackey Formula for Bisets) Let G, H and K be groups. If L is a subgroup of $H \times G$, and if M is a subgroup of $K \times H$, then there is an isomorphism of (K, G)-bisets*

$$\frac{K \times H}{M} \times_H \frac{H \times G}{L} \cong \bigsqcup_{x \in p_2(M)\backslash H/p_1(L)} \frac{K \times G}{M *^{(x,1)} L}$$

where $p_2(M)\backslash H/p_1(L)$ is a set of representatives of double cosets and

$$M *^{(x,1)} L = \{(k, g) \in K \times G \mid (k, h) \in M \text{ and } (h, g) \in {}^{(x,1)}L \text{ for some } h \in H\}.$$

Proof. See [4, 2.3.24]. □

Let X be a G-set. We define the permutation FG-module with permutation basis X as FX. That is,

$$FX = \bigoplus_{x \in X} F \cdot x.$$

Then, G acts on FX, for $g \in G$, $x \in X$, and $\lambda_x \in F$, as

$$g \left(\sum_{x \in X} \lambda_x x \right) = \sum_{x \in X} \lambda_x g x.$$

Similarly, for a (G,H)-biset X, FX is an $FG-FH$-bimodule.

Lemma 4.6. *For a (G,H)-biset X and a (H,K)-biset Y*

$$F(X \times_H Y) = FX \otimes_{FH} FY.$$

Proof. Firstly, we observe that,

$$\begin{aligned} FX \otimes_{FH} FY &= \left(\bigoplus_{x\in X} Fx\right) \otimes_{FH} \left(\bigoplus_{y\in Y} Fy\right) \\ &= \bigoplus_{(x,y)\in X\times Y} Fx \otimes_{FH} Fy \\ &= \bigoplus_{\substack{(x,y)\in X\times Y \\ h\in H}} \frac{Fx \times Fy}{\sim} \end{aligned}$$

where $\sim$ is an equivalence relation on $X \times Y$ relating the elements (xh, y) and (x, hy) for every h in H. Also, we have $(xh, y) \sim (x, hy)$ if and only if $(x, y) \sim (xh, yh^{-1})$. Then we obtain

$$FX \otimes_{FH} FY = \bigoplus_{(x,y)\in X\times_H Y} F(x,y) = F(X \times_H Y).$$

□

Definiton 4.7. *Let G and K be groups. Let H be a subgroup of G and N be a normal subgroup of G.*

(i) The set G is a (H,G)-biset for the actions given by left and right multiplications in G. It is denoted by res^G_H.

(ii) The set G is a (G,H)-biset for the actions given by left and right multiplications in G. It is denoted by ind^G_H.

(iii) The set G/N is a $(G,G/N)$-biset for the left action of G by projection to G/N, and then left multiplication in G/N, and the right action of G/N by multiplication. It is denoted by $\mathrm{inf}^G_{G/N}$.

(iv) The set G/N is a $(G/N, G)$-biset for the left action of G/N by multiplication, and the right action of G by projection to G/N, and then right multiplication in G/N. It is denoted by $\mathrm{def}^G_{G/N}$.

(v) If $f : G \to K$ is a group isomorphism, then the set K is an (K, G)-biset for the left action of K by multiplication, and the right action of G given by taking image by f, and then multiplying on the right in K. It is denoted by $\mathrm{c}^f_{K,G}$.

These five bisets defined in Definition 4.7 are transitive, therefore their orbit sets have cardinality 1. Then, using Lemma 4.3 we can rewrite those elementary bisets as follows:

$$\mathrm{res}^G_H = (H \times G)/R \text{ where } R = \{(h, h) \mid h \in H\}$$
$$\mathrm{ind}^G_H = (G \times H)/T \text{ where } T = \{(h, h) \mid h \in H\}$$
$$\mathrm{inf}^G_{G/N} = (G \times G/N)/I \text{ where } I = \{(g, gN) \mid g \in G\}$$
$$\mathrm{def}^G_{G/N} = (G/N \times G)/D \text{ where } D = \{(gN, g) \mid g \in G\}$$
$$\mathrm{c}^f_{K,G} = (K \times G)/C^f \text{ where } C^f = \{(f(g), g) \mid g \in G\}$$

Now, using Lemma 4.6, we define five elementary biset functors:

Definiton 4.8. *Let G and K be groups. Let H be a subgroup of G and N be a normal subgroup of G.*

(i) For an FG-module V, the restriction *functor is defined as*

$$\mathrm{Res}^G_H V := F(\mathrm{res}^G_H \times_G V) =_{FH} FG_{FG} \otimes_{FG} V.$$

(ii) For an FH-module V, the induction *functor is defined as*

$$\mathrm{Ind}^G_H(V) := F(\mathrm{ind}^G_H \times_H V) =_{FG} FG_{FH} \otimes_{FH} V.$$

(iii) For an $F[G/N]$*-module* V*, the* inflation *functor is defined as*

$$\mathrm{Inf}_{G/N}^{G}(V) := F(\mathrm{inf}_{G/N}^{G} \times_{G/N} V) =_{FG} FG_{F[G/N]} \otimes_{F[G/N]} V.$$

(iv) For an FG*-module* V*, the* deflation *functor is defined as*

$$\mathrm{Def}_{G/N}^{G}(V) := F(\mathrm{def}_{G/N}^{G} \times_{G} V) =_{F[G/N]} F[G/N]_{FG} \otimes_{FG} V.$$

(v) For an FG*-module* V *and an isomorphism* $f : G \to K$*, the* transport of structure *functor is defined as*

$$\mathrm{C}_{K,G}^{f}(V) := F(\mathrm{c}_{K,G}^{f} \times_{G} V) =_{FK} FG_{FG} \otimes_{FG} V.$$

4.3. Mackey Decomposition Theorem

At this section, first, we are to prove Mackey Decomposition Theorem using the results of the previous section. Then, we are to show the adjointness of $T_{P/U}^{G}$ and $R_{P/U}^{G}$ on both sides.

Proof of Theorem 4.1. The equality in the statement of Theorem 4.1 can be rewritten as

$$\begin{aligned} &\mathrm{Def}_{Q/V}^{Q}\mathrm{Res}_{Q}^{G}\mathrm{Ind}_{P}^{G}\mathrm{Inf}_{P/U}^{P}(M) \\ &\qquad = \bigsqcup_{x \in P\backslash G/Q} \mathrm{Ind}_{(P^x\cap Q)V}^{Q/V}\mathrm{Inf}_{\frac{(P^x\cap Q)V}{(U^x\cap Q)V}}^{(P^x\cap Q)V}\mathrm{C}^{\phi}_{\frac{(P^x\cap Q)V}{(U^x\cap Q)V},\frac{(Q\cap P^x)U^x}{(V\cap P^x)U^x}}\mathrm{Def}_{\frac{(Q\cap P^x)U^x}{(V\cap P^x)U^x}}^{(Q\cap P^x)U^x}\mathrm{Res}_{(Q\cap P^x)U^x}^{P^x/U^x}(M^x) \end{aligned}$$

We can write

$$\mathrm{Def}_{Q/V}^{Q}\mathrm{Res}_{Q}^{G}\mathrm{Ind}_{P}^{G}\mathrm{Inf}_{P/U}^{P} = \frac{Q/V \times Q}{D} \times_{Q} \frac{Q \times G}{R} \times_{G} \frac{G \times P}{T} \times_{P} \frac{P \times P/U}{I}$$

where

$$D = \{(qV, q)|\ q \in Q\},\ R = \{(q, q)|\ q \in Q\},\ T = \{(p, p)|\ p \in P\},\ I = \{(p, pU)|\ p \in P\}.$$

By Lemma 4.5, we know

$$\frac{Q/V \times Q}{D} \times_Q \frac{Q \times G}{R} = \bigsqcup_{q \in p_2(D) \backslash Q / p_1(R)} \frac{Q/V \times G}{D *^{(q,1)} R}$$

where

$$\begin{aligned} p_1(R) &= \{q \in Q|\ (q, g) \in R \text{ for some } g \in G\}, \\ p_2(D) &= \{q \in Q|\ (q'V, q) \in D \text{ for some } q'V \in Q/V\}, \\ D *^{(q,1)} R &= \{(qV, g) \in Q/V \times G|\ (qV, q') \in D \text{ and } (q', g) \in {}^{(q,1)}R \text{ for some } q' \in Q\}. \end{aligned}$$

For any q in Q, the element (q, q) is in R. Thus, we have $p_1(R) = Q$. Also, for any q in Q, the element (q, qV) is in D. So, we have $p_2(D) = Q$. Then, the set $p_2(D)\backslash Q/p_1(R)$ contains only one coset and the union consists of only one biset. We can take the identity element 1 of Q as the coset representative q. Then we have

$$\frac{Q/V \times Q}{D} \times_Q \frac{Q \times G}{R} = \frac{Q/V \times G}{D *^{(1,1)} R}.$$

Clearly the set ${}^{(1,1)}R$ is equal to R. Therefore

$$\begin{aligned} D *^{(q,1)} R &= D * R \\ &= \{(qV, g) \in Q/V \times G|\ (qV, q') \in D \text{ and } (q', g) \in R \text{ for some } q' \in Q\} \\ &= \{(qV, q) \in Q/V \times Q\} \\ &= \{(qV, q)|\ q \in Q\} \\ &= D \end{aligned}$$

Hence, we obtain

$$\mathrm{Def}^{Q}_{Q/V}\mathrm{Res}^{G}_{Q} = \frac{Q/V \times Q}{D} \times_{Q} \frac{Q \times G}{R} = \frac{Q/V \times G}{D}.$$

Similarly, we have

$$\frac{G \times P}{T} \times_{P} \frac{P \times P/U}{I} = \bigsqcup_{p \in p_2(T)\backslash P/p_1(I)} \frac{G \times P/U}{T * {}^{(p,1)}I}$$

where

$$\begin{aligned} p_1(I) &= \{p \in P|\ (p, p'U) \in I \text{ for some } p'U \in P/U\}, \\ p_2(T) &= \{p \in P|\ (p, g) \in T \text{ for some } g \in G\}, \\ T * {}^{(p,1)}R &= \{(g, pU) \in G \times P/U|\ (g, p') \in T \text{ and } (p', pU) \in {}^{(p,1)}I \text{ for some } p' \in P\}. \end{aligned}$$

For any p in P, the element (p, pU) is in I. Thus, we have $p_1(I) = P$. Also, for any p in P, the element (p, p) is in T. So, we have $p_2(T) = P$. Then, the set $p_2(T)\backslash P/p_1(I)$ contains only one coset and the union consists of only one biset. We can take the identity element 1 of P as the coset representative p. Then we have

$$\frac{G \times P}{T} \times_{P} \frac{P \times P/U}{I} = \frac{G \times P/U}{T * {}^{(1,1)}I}.$$

Clearly the set ${}^{(1,1)}I$ is equal to I. Therefore

$$\begin{aligned} T * {}^{(p,1)}I &= T * I \\ &= \{(g, pU) \in G \times P/U|\ (g, p') \in T \text{ and } (p', pU) \in I \text{ for some } p' \in P\} \\ &= \{(p, pU) \in P \times P/U\} \\ &= \{(p, pU)|\ p \in P\} \\ &= I \end{aligned}$$

Hence, we obtain

$$\mathrm{Ind}_P^G \mathrm{Inf}_{P/U}^P = \frac{G \times P}{T} \times_P \frac{P \times P/U}{I} = \frac{G \times P/U}{I}.$$

Now, by Lemma 4.5, we obtain

$$\frac{Q/V \times G}{D} \times_G \frac{G \times P/U}{I} = \bigsqcup_{g \in [p_2(D) \backslash G / p_1(I)]} \frac{Q/V \times P/U}{D *^{(g,1)} I}$$

where

$$p_1(I) = \{p \in P \mid (p, p'U) \in I \text{ for some } p'U \in P/U\}$$
$$p_2(D) = \{q \in Q \mid (q'V, q) \in D \text{ for some } q'V \in Q/V\}$$

For any p in P, we have (p, pU) in I, and hence p in $p_1(I)$. Thus, we have $p_1(I) = P$. Also, for any q in Q, we have (qV, q) in D, and hence q in $p_2(D)$. Thus, we have $p_2(D) = Q$. Then, we obtain

$$p_2(D) \backslash G / p_1(I) = Q \backslash G / P.$$

Also, for any g in $p_2(D) \backslash G / p_1(I)$, we have

$$\begin{aligned} {}^{(g,1)}I &= (g,1)I(g^{-1},1) = \{(g,1)(p,pU)(g^{-1},1) \mid p \in P\} \\ &= \{(gpg^{-1}, pU) \mid p \in P\} \end{aligned}$$

and

$$\begin{aligned} L := D *^{(g,1)} I &= \{(qV, pU) \in Q/V \times P/U \mid (qV, g) \in D \ (g, pU) \in {}^{(g,1)}I \text{ for some } g \in G\} \\ &= \{(qV, pU) \in Q/V \times P/U \mid q = gpg^{-1}\} \end{aligned}$$

Now, by [4, 2.3.25] and [4, 2.3.26] we have

$$\frac{Q/V \times P/U}{L} \cong \mathrm{Ind}_{p_1(L)}^{Q/V}\mathrm{Inf}_{p_1(L)/k_1(L)}^{p_1(L)}\mathrm{C}_{p_1(L)/k_1(L),p_2(L)/k_2(L)}^{f}\mathrm{Def}_{p_2(L)/k_2(L)}^{p_2(L)}\mathrm{Res}_{p_2(L)}^{P/U}$$

where

$$\begin{aligned} p_1(L) &= \{qV \in Q/V \mid (qV, pU) \in L \text{ for some } pU \in P/U\} \\ &= \{qV \in Q/V \mid q = gpg^{-1} \text{ for some } p \in P\} \\ k_1(L) &= \{qV \in Q/V \mid (qV, U) \in L\} = \{qV \in Q/V \mid q = gug^{-1} \text{ for some } u \in U\} \\ p_2(L) &= \{pU \in P/U \mid (qV, pU) \in L \text{ for some } qV \in Q/V\} \\ &= \{pU \in P/U \mid q = gpg^{-1} \text{ for some } q \in Q\} \\ k_2(L) &= \{pU \in P/U \mid (V, pU) \in L\} = \{pU \in P/U \mid v = gpg^{-1} \text{ for some } v \in V\} \end{aligned}$$

and

$$f : p_2(L)/k_2(L) \to p_1(L)/k_1(L),\ (pU)k_2(L) \mapsto (gpg^{-1}V)k_1(L)$$

On the other hand,

$$\begin{aligned} (P^g \cap Q)V &= \{qV \in Q/V \mid q = g^{-1}pg \text{ for some } p \in P\} = p_1(L) \\ (U^g \cap Q)V &= \{qV \in Q/V \mid q = g^{-1}ug \text{ for some } u \in U\} = k_1(L) \\ (Q \cap P^g)U^g &= \{(pU)^g \mid g^{-1}pg = q \text{ for some } q \in Q\} = (p_2(L))^g \\ (V \cap P^g)U^g &= \{(pU)^g \mid g^{-1}pg = v \text{ for some } v \in V\} = (k_2(L))^g \end{aligned}$$

and, by Butterfly Lemma [10, 3.3], there is an isomorphism

$$\phi : (Q \cap P^g)U^g/(V \cap P^g)U^g \to (P^g \cap Q)V/(U^g \cap Q)V$$

Therefore, we obtain

$$
\begin{aligned}
T^G_{Q/V} R^G_{P/U}(M) &= \mathrm{Def}^G_{Q/V}\mathrm{Res}^G_Q\mathrm{Ind}^G_P\mathrm{Inf}^P_{P/U}(M)\\
&= \bigsqcup_{g\in p_2(D)\backslash G/p_1(I)} \mathrm{Ind}^{Q/V}_{p_1(L)}\mathrm{Inf}^{p_1(L)}_{\frac{p_1(L)}{k_1(L)}}\mathrm{C}^f_{\frac{p_2(L)}{k_2(L)},\frac{p_1(L)}{k_1(L)}}\mathrm{Def}^{p_2(L)}_{\frac{p_2(L)}{k_2(L)}}\mathrm{Res}^{P/U}_{p_2(L)}(M)\\
&= \bigsqcup_{g\in p_2(D)\backslash G/p_1(I)} \mathrm{Ind}^{Q/V}_{p_1(L)}\mathrm{Inf}^{p_1(L)}_{\frac{p_1(L)}{k_1(L)}}\mathrm{C}^f_{(\frac{p_2(L)}{k_2(L)})^g,\frac{p_1(L)}{k_1(L)}}\mathrm{Def}^{(p_2(L))^g}_{(\frac{p_2(L)}{k_2(L)})^g}\mathrm{Res}^{(P/U)^g}_{(p_2(L))^g}(M^g)\\
&= \bigsqcup_{g\in P\backslash G/Q} \mathrm{Ind}^{Q/V}_{(P^g\cap Q)V}\mathrm{Inf}^{(P^g\cap Q)V}_{\frac{(P^g\cap Q)V}{(U^g\cap Q)V}}\mathrm{C}^\phi_{\frac{(Q\cap P^g)U^g}{(V\cap P^g)U^g},\frac{(P^g\cap Q)V}{(U^g\cap Q)V}}\mathrm{Def}^{(p_2(L))^g}_{\frac{(Q\cap P^g)U^g}{(V\cap P^g)U^g}}\mathrm{Res}^{(P/U)^g}_{(V\cap P^g)U^g}(M^g)\\
&= \bigoplus_{g\in P\backslash G/Q} R^{Q/V}_{\frac{(P^g\cap Q)V}{(U^g\cap Q)V}} T^{(P/U)^g}_{\frac{(Q\cap P^g)U^g}{(V\cap P^g)U^g}}
\end{aligned}
$$

□

Let P be a subgroup of G and U be a normal subgroup of P. The quotient P/U is called a *subquotient* of G.

Lemma 4.9. *For a subquotient P/U of G, if the order of U is invertible in F, the functor $T^G_{P/U}$ is adjoint on both sides of $R^G_{P/U}$.*

Proof. By Definition 4.8, for an FG-module A, we have

$$T^G_{P/U}(A) = \mathrm{Def}^P_{P/U}\mathrm{Res}^G_P(A)$$

and, for an $F[P/U]$-module B, we have

$$R^G_{P/U}(B) = \mathrm{Ind}^G_P\mathrm{Inf}^P_{P/U}(B).$$

Clearly, Ind^G_P is adjoint on both sides of Res^G_P. So, to prove the statement, it is enough to examine left and right adjoints of $\mathrm{Inf}^P_{P/U}$.

The left and right adjoints of $\mathrm{Inf}^P_{P/U}$ is not necessarily equal. However, in our case, we are to prove that they are isomorphic. Firstly, we are to show that the functor

$$\mathrm{Def}^P_{P/U} : \mathrm{mod}_{FP} \to \mathrm{mod}_{F[P/U]},\ M \mapsto M^U$$

where $M^U = \{m \in M \mid Um = m\}$, is the left adjoint of $\mathrm{Inf}^P_{P/U}$.

Let M be an FP-module and N be an $F[P/U]$-module. We define a map

$$\Phi : \mathrm{Hom}_{FP}(M, \mathrm{Inf}^P_{P/U}(N)) \to \mathrm{Hom}_{F[P/U]}(\mathrm{Def}^P_{P/U}(M), N),\ \phi \mapsto \tilde{\phi}$$

where $\tilde{\phi}$ is defined as

$$\tilde{\phi} : \mathrm{Def}^P_{P/U}(M) \to N,\ m \mapsto \phi(m).$$

$\tilde{\phi}$ is an $F[P/U]$-module homomorphism since

$$\tilde{\phi}(pUm) = \tilde{\phi}(pm) = \phi(pm) = p\phi(m) = pU\phi(m) = pU\tilde{\phi}(m)$$

for p in P and m in M^U. Now, we define a second map

$$\Psi : \mathrm{Hom}_{F[P/U]}(\mathrm{Def}^P_{P/U}(M), N) \to \mathrm{Hom}_{FP}(M, \mathrm{Inf}^P_{P/U}(N)),\ \psi \mapsto \hat{\psi}$$

where $\hat{\psi}$ is defined as

$$\hat{\psi} : M \to \mathrm{Inf}^P_{P/U}(N),\ m \mapsto \psi\left(\frac{1}{|U|}\sum_{u \in U} um\right).$$

The map $\hat{\psi}$ is an FP-module homomorphism since, for p in P and m in M,

$$\begin{aligned}\hat{\psi}(pm) &= \psi\left(\frac{1}{|U|}\sum_{u\in U} upm\right) = \psi\left(\frac{1}{|U|}\sum_{u\in U} pum\right) = \psi\left(p\frac{1}{|U|}\sum_{u\in U} um\right)\\ &= p\psi\left(\frac{1}{|U|}\sum_{u\in U} um\right) = p\hat{\psi}(m).\end{aligned}$$

Now, we are to show that Ψ is the inverse of Φ. For ψ in $\mathrm{Hom}_{F[P/U]}(\mathrm{Def}^P_{P/U}(M), N)$ and m in M^U, we have

$$\Phi\Psi(\psi)(m) = \Phi(\hat{\psi}(m)) = \tilde{\hat{\psi}}(m) = \hat{\psi}(m) = \psi\left(\frac{1}{|U|}\sum_{u\in U} um\right) = \psi\left(\frac{1}{|U|}\sum_{u\in U} m\right) = \psi(m)$$

Also, for ϕ in $\mathrm{Hom}_{FP}(M, \mathrm{Inf}^P_{P/U}(N))$ and m in M, we have

$$\begin{aligned}\Psi\Phi(\phi)(m) &= \Psi(\tilde{\phi}(m)) = \hat{\tilde{\phi}}(m) = \tilde{\phi}\left(\frac{1}{|U|}\sum_{u\in U} um\right) = \phi\left(\frac{1}{|U|}\sum_{u\in U} um\right)\\ &= \frac{1}{|U|}\sum_{u\in U} u\phi(m) = \phi(m)\end{aligned}$$

Therefore, we obtain

$$\mathrm{Hom}_{FP}(M, \mathrm{Inf}^P_{P/U}(N)) \cong \mathrm{Hom}_{F[P/U]}(\mathrm{Def}^P_{P/U}(M), N),$$

that is, the functor $\mathrm{Def}^P_{P/U}$ is the left adjoint of $\mathrm{Inf}^P_{P/U}$.

Secondly, we are to show that the functor

$$\mathrm{Codef}^P_{P/U} : \mathrm{mod}_{FP} \to \mathrm{mod}_{F[P/U]},\ M \mapsto M_U$$

where $M_U = \{\sum_{u\in U} um \mid m \in M\}$, is the right adjoint of $\mathrm{Inf}^P_{P/U}$.

We define a map

$$\Theta : \mathrm{Hom}_{FP}(\mathrm{Inf}_{P/U}^{P}(N), M) \to \mathrm{Hom}_{F[P/U]}(N, \mathrm{Codef}_{P/U}^{P}(M)),\ \theta \mapsto \tilde{\theta}$$

where $\tilde{\theta}$ is defined as

$$\tilde{\theta} : N \to \mathrm{Codef}_{P/U}^{P}(M),\ n \mapsto \sum_{u \in U} u\theta(n).$$

The map $\tilde{\theta}$ is an $F[P/U]$-module homomorphism since

$$\tilde{\theta}(pUn) = \sum_{u \in U} u\theta(pn) = \sum_{u \in U} pu\theta(n) = p\sum_{u \in U} u\theta(n) = p\tilde{\theta}(n) = pU\tilde{\theta}(n)$$

for p in P and n in N. Now, we define a second map

$$\Gamma : \mathrm{Hom}_{F[P/U]}(N, \mathrm{Codef}_{P/U}^{P}(M)) \to \mathrm{Hom}_{FP}(\mathrm{Inf}_{P/U}^{P}(N), M),\ \gamma \mapsto \hat{\gamma}$$

where $\hat{\gamma}$ is defined as

$$\hat{\gamma} : \mathrm{Inf}_{P/U}^{P} N \to M,\ n \mapsto \frac{1}{|U|}\gamma(n).$$

The map $\hat{\gamma}$ is an $F[P/U]$-module homomorphism since

$$\hat{\gamma}(pn) = \frac{1}{|U|}\gamma(pn) = p\frac{1}{|U|}\gamma(n) = p\hat{\gamma}(n)$$

for p in P and n in N.

Now, we prove that Γ is the inverse of Θ. For γ in $\mathrm{Hom}_{F[P/U]}(N, \mathrm{Codef}_{P/U}^{P}(M))$

and n in N, we have

$$\begin{aligned}\Theta\Gamma(\gamma)(n) &= \Theta(\hat{\gamma}(n)) = \tilde{\hat{\gamma}}(n) = \sum_{u\in U} u\hat{\gamma}(n) = \sum_{u\in U} u\frac{1}{|U|}\gamma(n)\\ &= \frac{1}{|U|}\sum_{u\in U} u\gamma(n) = \frac{1}{|U|}\sum_{u\in U}\gamma(n)\\ &= \gamma(n).\end{aligned}$$

Also, for θ in $\mathrm{Hom}_{FP}(\mathrm{Inf}^P_{P/U}(N), M)$ and n in $\mathrm{Inf}^P_{P/U}(N)$, we have

$$\begin{aligned}\Gamma\Theta(\theta)(n) &= \Gamma(\tilde{\theta}(n)) = \hat{\tilde{\theta}}(n) = \frac{1}{|U|}\tilde{\theta}(n) = \frac{1}{|U|}\sum_{u\in U} u\theta(n)\\ &= \frac{1}{|U|}\sum_{u\in U}\theta(un) = \frac{1}{|U|}\sum_{u\in U}\theta(n) = \theta(n)\end{aligned}$$

Therefore we obtain

$$\mathrm{Hom}_{FP}(\mathrm{Inf}^P_{P/U}(N), M) \cong \mathrm{Hom}_{F[P/U]}(N, \mathrm{Codef}^P_{P/U}(M)),$$

that is, $\mathrm{Codef}^P_{P/U}$ is the right adjoint of $\mathrm{Inf}^P_{P/U}$.

Now, we are to show that, for an $F[P/U]$-module M, the FP-modules $\mathrm{Def}^P_{P/U}(M)$ and $\mathrm{Codef}^P_{P/U}(M)$ are isomorphic. To this end, we define two maps

$$\zeta : \mathrm{Def}^P_{P/U}(M) \to \mathrm{Codef}^P_{P/U}(M),\ m \mapsto \sum_{u\in U} um$$

and

$$\xi : \mathrm{Codef}^P_{P/U}(M) \to \mathrm{Def}^P_{P/U}(M),\ \sum_{u\in U} um \mapsto \frac{1}{|U|}\sum_{u\in U} um.$$

ζ and ξ are $F[P/U]$-module homomorphisms since for m in M^U and $\sum_{u\in U} un$ in M_U, we have

$$\zeta(pUm) = \zeta(pm) = \sum_{u\in U} upm = p\sum_{u\in U} um = pU\sum_{u\in U} um = pU\zeta(m)$$

and

$$\begin{aligned}\xi(pU\sum_{u\in U} un = \xi(p\sum_{u\in U} un) = \xi(\sum_{u\in U} upn) &= \frac{1}{|U|}\sum_{u\in U} u(pn)\\ &= p\left(\frac{1}{|U|}\sum_{u\in U} un\right) = pU\left(\frac{1}{|U|}\sum_{u\in U} un\right) = pU\xi(\sum_{u\in U} un)\end{aligned}$$

Also, we have

$$\begin{aligned}\zeta\xi(\sum_{u\in U} um) &= \zeta\left(\frac{1}{|U|}\sum_{u\in U} um\right) = \frac{1}{|U|}\sum_{u\in U} u\xi(m) = \frac{1}{|U|}\sum_{u\in U} u\sum_{u\in U} um\\ &= \frac{1}{|U|}\sum_{u\in U}\sum_{u\in U} um = \sum_{u\in U} um\end{aligned}$$

for $m + UM$ in M_U, and

$$\xi\zeta(n) = \xi(n + UM) = \frac{1}{|U|}\sum_{u\in U} un = n$$

for n in M^U. Therefore $\mathrm{Def}^P_{P/U}(M)$ and $\mathrm{Codef}^P_{P/U}(M)$ are isomorphic.

Now, for an FG-module N and FP-module M, we have

$$\begin{aligned}\mathrm{Hom}_{FP}(T^G_{P/U}(N), M) &= \mathrm{Hom}_{FP}(\mathrm{Def}^P_{P/U}\mathrm{Res}^G_P(N), M)\\ &\cong \mathrm{Hom}_{FP}(\mathrm{Res}^G_P(N), \mathrm{Inf}^P_{P/U}(M))\\ &\cong \mathrm{Hom}_{FP}(N, \mathrm{Ind}^G_P\mathrm{Inf}^P_{P/U}(M))\\ &= \mathrm{Hom}_{FP}(N, R^G_{P/U}(M))\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_{FG}(N, T^G_{P/U}(M)) &= \mathrm{Hom}_{FG}(N, \mathrm{Def}^P_{P/U}\mathrm{Res}^G_P(M)) \\ &\cong \mathrm{Hom}_{FG}(N, \mathrm{Codef}^P_{P/U}\mathrm{Res}^G_P(M)) \\ &\cong \mathrm{Hom}_{FG}(\mathrm{Inf}^P_{P/U}(N), \mathrm{Res}^G_P(M)) \\ &\cong \mathrm{Hom}_{FG}(\mathrm{Ind}^G_P\mathrm{Inf}^P_{P/U}(N), M) \\ &= \mathrm{Hom}_{FG}(R^G_{P/U}(N), M)\end{aligned}$$

Thus, $T^G_{P/U}$ is adjoint on both sides of $R^G_{P/U}$. □

4.4. Mackey System

Recall that, for a subgroup P of G and a normal subgroup U of P, a subquotient of G is the quotient P/U. A system $\mathbf{M}$ of subquotients of G is called a *Mackey system*, if it contains G, is closed under conjugation and the operation

$$P/U \sqcap Q/V = (P \cap Q)U/(P \cap V)U$$

for P/U and Q/V in $\mathbf{M}$.

For a prime p, the system $\mathbf{M}$ is called *p-modular*, if for all P/U in $\mathbf{M}$, U is p-regular, that is, the order of U is not divisible by p. Thus, if F has characteristic p and $\mathbf{M}$ is p-modular, we may apply Theorem 4.1 to the elements of $\mathbf{M}$.

For P/U in $\mathbf{M}$, the set

$$\mathbf{M}_{P/U} = \{P/U \sqcap Q/V \mid Q/V \in \mathbf{M}\}$$

defines a Mackey system in P/U. We give a proof of this fact. Let $P/U \sqcap Q/V$ and $P/U \sqcap R/Y$ be two subquotients in $\mathbf{M}_{P/U}$. Then, since $U \leq P$ and $P \cap VU \leq P \cap QU$, we have

$$\begin{aligned}(P/U \sqcap Q/V) \sqcap (P/U \sqcap R/Y) &= \frac{(P\cap Q)U}{(P\cap V)U} \sqcap \frac{(P\cap R)U}{(P\cap Y)U} \\ &= \frac{[(P\cap Q)U \cap (P\cap R)U](P\cap V)U}{[(P\cap Q)U \cap (P\cap Y)U](P\cap V)U} \\ &= \frac{(P\cap QU \cap P\cap RU)(P\cap VU)}{(P\cap QU \cap P\cap YU)(P\cap VU)} \\ &= \frac{(P\cap QU \cap RU)(P\cap VU)}{(P\cap QU\cap YU)(P\cap VU)} \\ &= \frac{P\cap RU(P\cap VU)\cap QU}{P\cap YU(P\cap VU)\cap QU} \\ &= \frac{[P\cap RU(P\cap VU)\cap Q]U}{[P\cap YU(P\cap VU)\cap Q]U} \\ &= \frac{P}{U} \sqcap \frac{RU(P\cap VU)\cap Q}{YU(P\cap VU)\cap Q}\end{aligned}$$

Hence, $\mathbf{M}_{P/U}$ is closed under the operation $\sqcap$. Clearly, it is closed under conjugation. Also, we have

$$P/U = PU/U = \frac{(P\cap P)U}{(P\cap U)U} = P/U \sqcap P/U.$$

So, P/U is an element of $\mathbf{M}_{P/U}$.

If $\mathbf{M}$ is p-modular, so is $\mathbf{M}_{P/U}$. To show this statement, assume $\mathbf{M}$ is p-modular. Let $Q/V \sqcap P/U = (Q\cap P)V/(Q\cap U)V$ be an element of $\mathbf{M}_{P/U}$. If $\mathbf{M}$ is p-modular, then U and V are p-regular. Then, the order of $Q\cap U$ is not divisible by p since $Q\cap U$ is a submodule of U and the order of U is not divisible by p. Then p does not divide the order of the module $(Q\cap U)V$. So, the Mackey system $\mathbf{M}_{P/U}$ is also p-modular.

Now, assume $\mathbf{M}$ is p-modular, where p is the characteristic of F. An FG-module M is called *cuspidal* with respect to $\mathbf{M}$, if $T^G_{P/U}(M) = (0)$ for all subquotients P/U of G different from G. For a subquotient P/U in $\mathbf{M}$, an $F[P/U]$-module N is called *cuspidal* with respect to $\mathbf{M}$ if it is cuspidal with respect to $\mathbf{M}_{P/U}$. If $\mathbf{M}$ contains a proper subgroup $P/1$, then FG does not have any cuspidal modules since for any nonzero FG-module M, we have

$$T^G_{P/1}(M) = \mathrm{Res}^G_P(M) \neq (0).$$

Even if $\mathbf{M}$ contains a proper subgroup and hence FG does not have any cuspidal modules with respect to $\mathbf{M}$, the same might not be true for $\mathbf{M}_{P/U}$, so $F[P/U]$ might have cuspidal modules with respect to $\mathbf{M}_{P/U}$.

The following theorem establishes a relation between Harish-Chandra theory and the results of the first chapter. In [7], a sketch for the proof was given. Here, we give a full proof using this sketch.

Theorem 4.10. *Let F be of characteristic p where $p > 0$. Let $\boldsymbol{M}$ be a p-modular Mackey system for G. For P/U in $\boldsymbol{M}$, let M be an irreducible cuspidal $F[P/U]$-module, and $\beta : X \to M$ be a minimal projective cover of M. Then we have*

$$\mathrm{End}_{FG}(R^G_{P/U}(X)) = (\mathrm{End}_{FG}(R^G_{P/U}(X))_{R^G_{P/U}(\beta)}$$

where $R^G_{P/U}(\beta)$ is the map

$$R^G_{P/U}(\beta) : R^G_{P/U}(X) \to R^G_{P/U}(M)$$

induced from the map $\beta : X \to M$.

Proof. We apply the functor $\mathrm{Hom}_{FG}(R^G_{P/U}(X), -)$ to the map

$$R^G_{P/U}(\beta) : R^G_{P/U}(X) \to R^G_{P/U}(M)$$

to obtain the map

$$(R^G_{P/U}(\beta))_* : \mathrm{End}_{FG}(R^G_{P/U}(X)) \to \mathrm{Hom}_{FG}(R^G_{P/U}(X), R^G_{P/U}(M)),\ \phi \mapsto \beta\phi$$

Since X is projective it can be written as $X = \bigoplus_{n\in\mathbb{N}} F[P/U]$. Then we have

$$\begin{aligned} R^G_{P/U}(X) = \mathrm{Ind}^G_P\mathrm{Inf}^P_{P/U}F[P/U](X) &= FG\otimes_P (FP\otimes_{F[P/U]} X) \\ &= FG\otimes_P (FP\otimes_{F[P/U]} \bigoplus_{n\in\mathbb{N}} F[P/U]) \\ &= FG\otimes_P \bigoplus_{n\in\mathbb{N}}(FP\otimes_{F[P/U]} F[P/U]) \\ &= \bigoplus_{n\in\mathbb{N}}(FG\otimes_P (FP\otimes_{F[P/U]} F[P/U])) \\ &= \bigoplus_{n\in\mathbb{N}}(\mathrm{Ind}^G_P\mathrm{Inf}^P_{P/U}F[P/U](F[P/U])) \\ &= \bigoplus_{n\in\mathbb{N}} R^G_{P/U}(F[P/U]). \end{aligned}$$

Also we have

$$\mathrm{Inf}^P_{P/U}F[P/U] = \mathrm{Inf}^P_{P/U}\mathrm{Ind}^{P/U}_{U/U}F = \mathrm{Ind}^P_U\mathrm{Inf}^U_{U/U}F = \mathrm{Ind}^P_U F.$$

Therefore we obtain

$$R^G_{P/U}(F[P/U]) = \mathrm{Ind}^G_P\mathrm{Inf}^P_{P/U}F[P/U] = \mathrm{Ind}^G_P\mathrm{Ind}^P_U F.$$

Since $|U|$ is invertible in F, the field F is a projective FU-module. Also, since induction preserves projectivity, we have $R^G_{P/U}(F[P/U])$ projective, and hence, being the direct sum of projective modules, $R^G_{P/U}(X)$ is projective.

Then, using projectivity of $R^G_{P/U}(X)$ and surjectivity of β, we conclude that, the map $(R^G_{P/U}(\beta))_*$ is surjective. Also $\ker(R^G_{P/U}(\beta))_* = J_{R^G_{P/U}(\beta)}$ where

$$J_{R^G_{P/U}(\beta)} = \{\psi \in \mathrm{End}_{FG}(R^G_{P/U}(X)) \mid \mathrm{im}\psi \text{ is a submodule of } \ker R^G_{P/U}(\beta)\}.$$

Then, we have

$$\dim_F\mathrm{Hom}_{FG}(R^G_{P/U}(X), R^G_{P/U}(M)) = \dim_F\mathrm{End}_{FG}(R^G_{P/U}(X)) - \dim_F J_{R^G_{P/U}(\beta)}.$$

By Proposition 3.1, we have the isomorphism

$$\mathrm{End}_{FG}(R^G_{P/U}(M)) \cong (\mathrm{End}_{FG}(R^G_{P/U}(X)))_{R^G_{P/U}(\beta)}/J_{R^G_{P/U}(\beta)}.$$

Therefore, we obtain

$$\dim_F \mathrm{End}_{FG}(R^G_{P/U}(M)) = \dim_F (\mathrm{End}_{FG}(R^G_{P/U}(X)))_{R^G_{P/U}(\beta)} - \dim_F J_{R^G_{P/U}(\beta)}.$$

Then, these two equations imply that

$$\mathrm{End}_{FG}(R^G_{P/U}(X)) = \mathrm{End}_{FG}(R^G_{P/U}(X)))_{R^G_{P/U}(\beta)}$$

if and only if

$$\dim_F \mathrm{Hom}_{FG}(R^G_{P/U}(X), R^G_{P/U}(M)) = \dim_F \mathrm{End}_{FG}(R^G_{P/U}(M)).$$

By Theorem 4.1, we have

$$T^G_{P/U} \circ R^G_{P/U}(M) = \bigoplus_{x \in P\backslash G/P} R^{P/U}_{\frac{(P^x \cap P)U}{(U^g \cap P)U}} \circ T^{P^x/U^x}_{\frac{(p \cap P^x)U^x}{(U \cap P^x)U^x}}(M^x).$$

Applying the functor $\mathrm{Hom}_{FG}(M, -)$ to this equation, we obtain

$$\mathrm{Hom}_{FG}(M, T^G_{P/U} \circ R^G_{P/U}(M)) = \bigoplus_{x \in P\backslash G/P} \mathrm{Hom}_{F[P/U]}(M, R^{P/U}_{\frac{(P^x \cap P)U}{(U^g \cap P)U}} \circ T^{P^x/U^x}_{\frac{(p \cap P^x)U^x}{(U \cap P^x)U^x}}(M^x)).$$

Now, using adjointness of the functors $R^G_{P/U}$ and $T^G_{P/U}$ we get

$$\mathrm{Hom}_{FG}(R^G_{P/U}(M), R^G_{P/U}(M)) = \bigoplus_{x \in P\backslash G/P} \mathrm{Hom}_{F[P/U]}(T^{P/U}_{\frac{(P^x \cap P)U}{(U^x \cap P)U}}(M), T^{P^x/U^x}_{\frac{(P \cap P^x)U^x}{(U \cap P^x)U^x}}(M^x)).$$

Since M is cuspidal, $T^G_{P/U}(M) = (0)$ for any proper subquotient of P/U in $\mathbf{M}_{P/U}$.

Then, we have

$$T^{P/U}_{\frac{(P^x\cap P)U}{(U^x\cap P)U}}(M) \neq (0)$$

if and only if $\frac{(P^x\cap P)U}{(U^x\cap P)U}$ is equal to P/U, and

$$T^{P^x/U^x}_{\frac{(P\cap P^x)U^x}{(U\cap P^x)U^x}}(M^x)) \neq (0)$$

if and only if $\frac{(P\cap P^x)U^x}{(U\cap P^x)U^x}$ is equal to P^x/U^x. Therefore we have

$$\mathrm{Hom}_{FG}(R^G_{P/U}(M), R^G_{P/U}(M)) = \bigoplus_{x\in N_G(P,U)\cap(P\backslash G/P)} \mathrm{Hom}_{F[P/U]}(M, M^x)$$

where $N_G(P,U) := \{x \in G \mid (P^x \cap P)U/(U^x \cap P)U = P/U\}$.

Similarly, we have

$$\mathrm{Hom}_{FG}(R^G_{P/U}(X), R^G_{P/U}(M)) = \bigoplus_{x\in N_G(P,U)\cap(P\backslash G/P)} \mathrm{Hom}_{F[P/U]}(X, M^x)$$

Since M is irreducible, $\dim_F\mathrm{Hom}_{FG}(M, M^x) = (0)$ unless $M \cong M^x$ in which case that dimension equals to 1. Also since X is the minimal projective cover of M, similarly we have $\dim_F\mathrm{Hom}_{FG}(X, M^x) = (0)$, unless $M \cong M^x$, and it equals to 1 in that case. Therefore, we have

$$\dim_F\mathrm{Hom}_{FG}(M, M^x) = \dim_F\mathrm{Hom}_{FG}(X, M^x)$$

and hence

$$\mathrm{End}_{FG}(R^G_{P/U}(X)) = \mathrm{End}_{FG}(R^G_{P/U}(X)))_{R^G_{P/U}(\beta)}$$

□

Corollary 4.11. *Let F be of characteristic p where $p > 0$. Let $\boldsymbol{M}$ be a p-modular Mackey system for G. For P/U in $\boldsymbol{M}$, let M be a cuspidal $F[P/U]$-module, and $\beta : X \to M$ be a minimal projective cover of M. The functors H^β and $\hat{H}^\beta$ provide a bijection between the isomorphism classes of the irreducible FG-modules occuring in the head of $R^G_{P/U}(M)$ and a set of representatives of the isomorphism classes of irreducible $\mathrm{End}_{FG}(R^G_{P/U}(M))$-modules.*

APPENDIX A: SUMMARY OF RESULTS

In this appendix, we restate some definitions and main theorems of the text to help the reader to understand the notation and terminology easily.

- $(\mathrm{End}_T(P))_\beta = \{\phi \in \mathrm{End}_T(P) \mid \phi(\ker\beta) \subseteq \ker\beta\}$
  $J_\beta = \{\psi \in \mathrm{End}_T(P) \mid \mathrm{im}\psi \leq \ker\beta\}$
- J_β is an ideal of $(\mathrm{End}_T(P))_\beta$ and $(\mathrm{End}_T(P))_\beta/J_\beta \cong \mathrm{End}_T(M)$ as R-algebra canonically, (Proposition 3.1).
- Assume $(\mathrm{End}_T(P))_\beta = \mathrm{End}_T(P)$. The mapping

$$H := H^\beta := H^\beta_M : \mathrm{mod}_T \to \mathrm{mod}_{\mathrm{End}_T(M)}$$

  defined for $V \in \mathrm{mod}_T$ by

$$H(V) = \mathrm{Hom}_T(P,V)/\mathrm{Hom}_T(P,V)J_\beta$$

  is a covariant functor, (Proposition 3.2).
- Let S be a ring. For S-modules V_1 and V_2, *trace of* V_1 *in* V_2, $\mathrm{tr}_{V_1}(V_2)$, is defined as the submodule of V_2 spanned by images of all homomorphisms from V_1 to V_2, (Definition 3.3).
- Let P and V be in mod_T and assume that P is projective. The *P-torsion submodule* $\mathrm{tor}_P(V)$ is the sum of all submodules X of V with respect to the property $\mathrm{Hom}_T(P,X) = (0)$. The *kernel* $\ker_P$ is the full subcategory of mod_T whose objects are the T-modules V with $\mathrm{Hom}_T(P,V) = (0)$. Therefore, the T-module V is in $\ker_P$ if and only if $\mathrm{tor}_P(V) = (0)$, (Proposition 3.5).
- Define the functor

$$A_P : \mathrm{mod}_T \to \mathrm{mod}_T, V \mapsto V/\mathrm{tor}_P(V)$$

  for V in mod_T and define $A_P(f)$ as the induced morphism from $V/\mathrm{tor}_P(V)$ to

$V'/\text{tor}_P(V')$ for any T-module homomorphism $f : V \to V'$, (Definition 3.9).

- Assume $(\text{End}_T(P))_\beta = \text{End}_T(P)$. We define four functors from $\text{mod}_{\text{End}_T(M)}$ to mod_T as

$$F_M = \_ \otimes_{\text{End}_T(M)} M$$
$$\tilde{F}_M = A_P \circ (\_ \otimes_{\text{End}_T(M)} M)$$
$$G_M = \_ \otimes_{\text{End}_T(P)} P$$
$$\tilde{G}_M = A_P \circ (\_ \otimes_{\text{End}_T(P)} P)$$

Let $\hat{H}$ be one of the four functors defined above. Then $\hat{H}$ is a right inverse of the functor H, (Definition 3.10 and Proposition 3.12).

- Assume $(\text{End}_T(P))_\beta = \text{End}_T(P)$ Define the set

$$(\text{Irr}T)_H = \{V \in \text{Irr}T \mid H_M(V) \neq (0)\}.$$

Then H_M induces a bijective correspondence

$$H_M : (\text{Irr}T)_H \to \text{Irr}(\text{End}_T(M))$$

and the inverse of H_M is

$$\tilde{F}_M : \text{Irr}(\text{End}_T(M)) \to (\text{Irr}T)_H.$$

On $\text{Irr}(\text{End}_T(M))$, the functors $\tilde{F}_M$ and $\tilde{G}_M$ coincide, (Theorem 3.16).

- Let R be a field. Assume $(\text{End}_T(P))_\beta = \text{End}_T(P)$. Then, $(\text{Irr}T)_H$ is a complete set of non-isomorphic irreducible constituents of head of M $hd(M)$. Every indecomposable direct summand of M has a simple head and factoring out the Jacobson radical induces a bijection between the isomorphism classes of indecomposable direct summands of M and the elements of $(\text{Irr}T)_H$, (Theorem 3.22).

- Let R be a field. Assume $(\text{End}_T(P))_\beta = \text{End}_T(P)$, the T-module M is P-

torsionless, and $\mathrm{End}_T(M)$ is self-injective. Then

(i) Every element of $\mathrm{Irr}(\mathrm{End}_T(M))$ is isomorphic to XM for some minimal ideal X of $\mathrm{End}_T(M)$.

(ii) The set $(\mathrm{Irr}T)_H$ is up to isomorphism a complete set of irreducible constituents of $soc(M)$ as well as $hd(M)$.

(iii) Every indecomposable direct summand of M has a simple socle and a simple head, and taking socles, respectively heads, induce bijections between the isomorphism classes of indecomposable direct summands of M and the elements of $(\mathrm{Irr}T)_H$.

(iv) Socle and heads of the indecomposable direct summands of M are isomorphic if in addition $\mathrm{End}_T(M)$ is a symmetric algebra.

(Theorem 3.23).

- Let P, Q, U, V be subgroups of G with U normal in P and V normal in Q. Suppose that the orders of U and V are invertible in F. Let M be and $F(P/U)$-module. Then

$$T^G_{Q/V} \circ R^G_{P/U}(M) \cong \bigoplus_{x \in P\backslash G/Q} R^{Q/V}_{\frac{(P^x\cap Q)V}{(U^x\cap Q)V}} \mathrm{C}^{\phi}_{\frac{(Q\cap P^x)U^x}{(V\cap P^x)U^x}, \frac{(P^x\cap Q)V}{(U^x\cap Q)V}} T^{P^x/U^x}_{\frac{(Q\cap P^x)U^x}{(V\cap P^x)U^x}}(M^x)$$

where

$$\mathrm{C}^{\phi}_{\frac{(Q\cap P^x)U^x}{(V\cap P^x)U^x}, \frac{(P^x\cap Q)V}{(U^x\cap Q)V}} : \frac{(Q\cap P^x)U^x}{(V\cap P^x)U^x} \to \frac{(P^x\cap Q)V}{(U^x\cap Q)V}$$

is an isomorphism, and M^x denotes the conjugate module for the conjugate factor group $x(P/U)x^{-1}$, and $P\backslash G/Q$ is a set of $P-Q$-double coset of representatives in G, (Theorem 4.1).

- A system $\mathbf{M}$ of subquotients of G is called a *Mackey system*, if it contains G, is closed under conjugation and the operation

$$P/U \sqcap Q/V = (P\cap Q)U/(P\cap V)U, \text{ for } P/U \text{ and } Q/V \text{ in } \mathbf{M}.$$

- For P/U in $\mathbf{M}$, the set

$$\mathbf{M}_{P/U} = \{P/U \sqcap Q/V \mid Q/V \in \mathbf{M}\}$$

defines a Mackey system in P/U.

- Let F be of characteristic p where $p > 0$. Let $\mathbf{M}$ be a p-modular Mackey system for G. For P/U in $\mathbf{M}$, let M be a cuspidal $F[P/U]$-module, and $\beta : X \to M$ be a minimal projective cover of M. Then we have

$$\mathrm{End}_{FG}(R^G_{P/U}(X)) = (\mathrm{End}_{FG}(R^G_{P/U}(X))_{R^G_{P/U}(\beta)}$$

where $R^G_{P/U}(\beta)$ is the map

$$R^G_{P/U}(\beta) : R^G_{P/U}(X) \to R^G_{P/U}(M)$$

induced from the map $\beta : X \to M$, (Theorem 4.10).

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