EFFECT OF A PRICE DRIVEN SECONDARY MARKET ON A SYSTEM WITH RANDOM DEMAND AND UNCERTAIN COSTS

by

Yücel Gürel

B.S., Industrial Engineering, Boğaziçi University, 2013

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Industrial Engineering
Boğaziçi University
2016

EFFECT OF A PRICE DRIVEN SECONDARY MARKET ON A SYSTEM WITH RANDOM DEMAND AND UNCERTAIN COSTS

APPROVED BY:

Prof. Refik Güllü (Thesis Supervisor)	....
Assoc. Prof. Mehmet Güray Güler	....
Assoc. Prof. Gönenç Yücel	....

DATE OF APPROVAL: 29.08.2016

ACKNOWLEDGEMENTS

First of all, I would like to thank Prof. Refik Güllü for his endless support, patience and guidance throughout my entire graduate study. This work would not be completed without his wise advices and suggestions. Secondly, I would like to express my gratitudes to Assoc. Prof. Mehmet Güray Güler and Assoc. Prof. Gönenç Yücel for taking part in my thesis jury and providing valuable comments.

I would like to thank all the graduate assistants, especially Gökalp Erbeyoğlu for answering my endless questions on every little detail of everything with his vast knowledge and enthusiasm.

I will never be able to find correct words to express my gratitudes to my family. I would like to thank my brothers for trusting and encouraging me at times when I feel doubtful about my decisions. I thank my mother for her unlimited encouragement and my father for giving me perspective whenever I needed. I am sure that things would be much harder if I did not have their support and love.

ABSTRACT

EFFECT OF A PRICE DRIVEN SECONDARY MARKET ON A SYSTEM WITH RANDOM DEMAND AND UNCERTAIN COSTS

In this thesis, we consider the problem of purchasing a commodity with a fluctuating market price so as to maximize its profit from selling it over two selling season. Over the primary season, the market price evolution is described by the Geometric Brownian Motion and we describe the demand process as a Poisson Process. Moreover demand and price is independent during this season. Over the secondary selling season, demand depends on the price the firm assigns. Considering the salvage value in place of the secondary market, we develop a mathematical model and find a closed form of solution. For the general model, we model the secondary market demand as a linear model and first solve optimization problem of maximizing the revenue from the secondary market. Then, we combine two selling seasons and model the problem for maximizing the total expected profit. In addition, we perform a numerical study to investigate how the optimal quantity and the optimal profit values depend on the price process parameters. We also test the performance of the optimal policy against the policy that ignores the volatility of the price.

ÖZET

FİYATA DAYALI İKİNCİL PAZARIN RASSAL TALEP VE BELİRSİZ MALİYETLER ALTINDAKİ SİSTEME OLAN ETKİSİ

Bu tez çalışmasında, dalgalı piyasa fiyatlandırması altında ürün alım problemi incelendi. Tezin amacı; iki satış sezonu boyunca elde edilecek karı maksimuma çıkaran başlangıç ürün alış miktarını belirlemektir. Birinci satış sezonunda, piyasa fiyatı sürecini Geometric Brown Hareket Süreci ve talep süresini ise Poisson Süreci olarak ele alındı. Ayrıca bu sezonda talep ve alış fiyatı birbirinden bağımsız olarak düşünüldü. İkinci satış sezonunda ise, talebin; firmanın belirlediği fiyata göre şekilleneceği kabul edildi. İkinci market yerine hurda değerini kabul edilerek, matematiksel model geliştirildi ve kapalı bir çözüm yöntemi bulundu. Genel model için, ikincil piyasa modellendi ve öncelikle bu dönemden en iyi gelir elde edilmesini sağlıyacak şekilde optimizasyon problemi çözüldü. Daha sonra bu iki marketin gelirleri birleştirildi ve toplam geliri eniyileyecek şekilde problem modellendi. Bunlara ek olarak, fiyatı süreci değişkenlerinin optimal miktarı ve optimal karı nasıl etklediğini incelemek için bazı sayısal örnekleme çalışmaları yapıldı. Ayrıca fiyat değişikliğini dikkata alan ve almayan iki optimal planın performansları karşılaştırıldı.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

$D_1(t)$	Primary market demand over $[0, t]$
$D_2(p)$	Secondary market demand under price p
h	Holding cost multiplier
$HC(Q)$	Present value of expected holding cost with initial purchase quantity Q
$J(Q)$	Total expected profit from the initial order quantity Q
$P(0)$	Initial purchase price
$P(0)Q$	Purchase cost with initial order quantity Q
$P(t)$	Price at time t
Q	Initial purchase quantity
r	Yearly interest rate
s	Salvage multiplier
$SR(Q)$	Present value of expected sales revenue with initial purchase quantity Q
T_j	Arrival time of the jth demand
$V_1(Q)$	Expected profit over the primary season when initial order quantity is Q
$V_2(x, y)$	Expected value of secondary market revenue where x is the inventory on hand at the end of primary selling season and y is the price at the end of primary selling season
α	Mark-up for the sales price
λ	Demand rate
μ	Drift of the price process
σ_p	Variance of the price process
τ	Length of the primary selling season

LIST OF ACRONYMS/ABBREVIATIONS

GBM	Geometric Brownian Motion
OPEC	Organization of the Petroleum Exporting Countries

1. INTRODUCTION

Price fluctuation is a common challenge for the companies purchasing from spot markets and/or abroad. The prices of raw materials, precious metals, agricultural products, minerals and energy resources and electronic components, among many others, can fluctuate substantially over short periods. Variations in market conditions, the presentation of the new technology or change in supply and demand may cause these fluctuations. On the one hand, some firms depend on spot markets for procurement and this dependence brings on input price variability. For example, the spot price of the OPEC basket yearly price varies considerably between 2003 to 2016, as shown in Figure 1.1. It started from \$28.10 in 2003 and reached \$94.45 in 2008. Then it declined sharply to \$61.06 in a year. After an increasing trend, it was peak in 2012 by \$109.45. Afterwards it fell heavily again to \$37.17 in 2016 and still displays much fluctuations. On the other hand, fluctuations in exchange rates bring about many challenges for the companies in Turkey that import goods from abroad. For instance, the daily spot rate of the dollar against the Turkish lira fluctuates significantly over the last ten years as shown in Figure 1.2. These fluctuations has been, followed by several similar ups and downs.

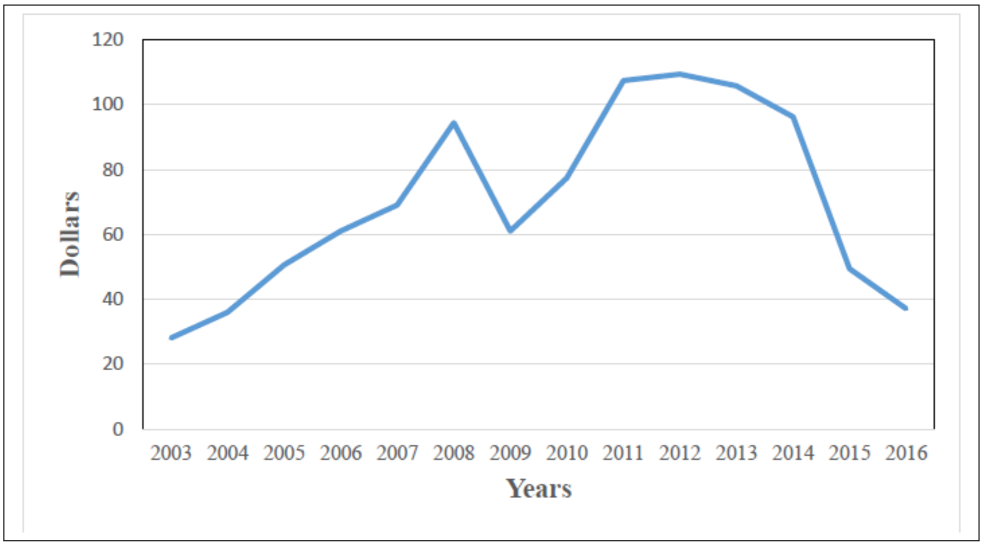

Figure 1.1. Yearly Spot Price of OPEC Reference Basket

Figure 1.2. Daily Spot Rate of the Dollar against the Turkish Lira

Some companies use derivatives to hedge against the fluctuations. However, in a stochastic demand environment, the time that the future sales to be made cannot be known in advance and this makes it difficult to find the right derivatives to deal with the price variations. Other firms choose to pass the fluctuations to their customers. But this requires to assign the selling price according to the price that the firms paid at the beginning rather than the current market price and it is not reasonable to expect that the customers would accept such a selling price. As a result, purchase decisions should include the price process as well in contrary to the classical inventory control models which assume constant or known purchase price.

This study considers the problem of purchasing a commodity with a fluctuating market price so as to maximize its profit from selling it over two selling seasons. Over the primary season, inventory levels are depleted by the demand which is a random process following a Poisson Process and the demand does not depend on the current price. Over the secondary selling season, the firm decides the price and face with the demand that depends on that price. The firm buys the commodity only at the beginning of the primary selling season and it sells the remaining inventory after the primary season over the secondary selling season. There are two expense items in our model. Firstly, the firm pays the initial purchase costs which is just multiplication of purchase price and purchase quantity. Secondly, holding cost is uncured directly proportional to shelf life of the product. Figure 1.3 depicts the framework of our model.

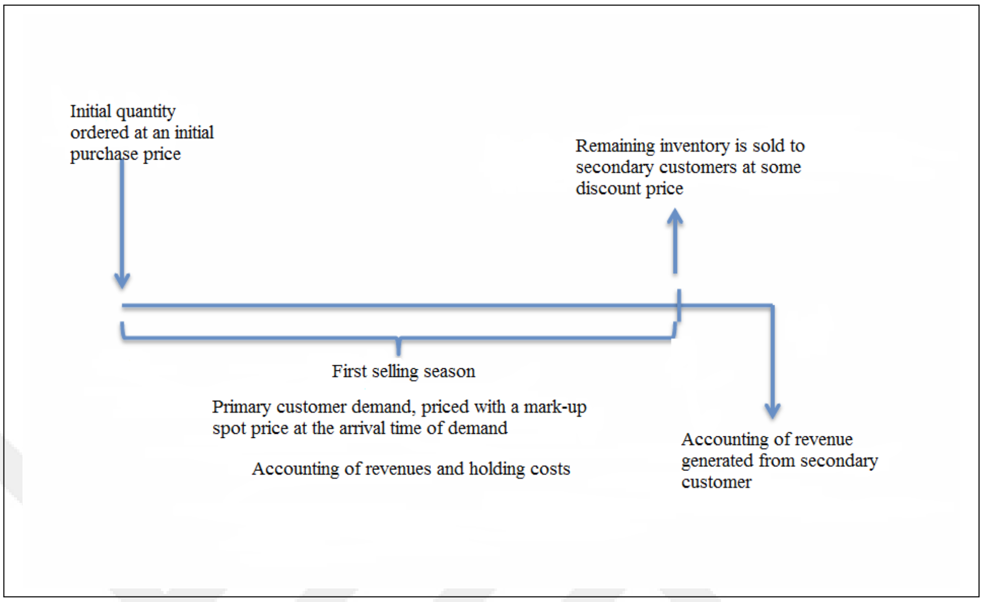

Figure 1.3. The Framework of the Model

This model can be applied for the firms that buy technology intensive products from abroad. The value of these products depreciates after the release of upper version of them. Hence the companies should also consider the life-cycle of that commodity and after certain time they should sell the remaining products in their inventories and order the new segment of it. Thus, the time from beginning to the introduction of the upper segment of that commodity can be seen as a primary selling season and the time before the new segment of orders arrive can be seen as a secondary selling season. For example, let's assume a company that imports apple products of iPhone 6s and sells it in Turkey. After a certain time, iPhone 7 which is a better segment of iPhone 6s will be introduced and the value of the iPhone 6s will be fallen. Hence the firm should consider this depreciation and should include this to the buying decisions. It is worth noting that the problem is typically applicable in the context of consumer-packaged goods (e.g. grocery items) which are bought by foreign currency as well. For instance, a shopping mall in Turkey imports some fruits from abroad. After some time of the order, the manager of the firm should decide how much of the remaining inventories is disposed of at a reduced sale price since these products are perishable.

The essential decision problem in buying on a fluctuating market is the timing of the purchase and the decision of how much to purchase when the time arises. Since the input price varies, the firm can buy the commodity cheaper or more expensive according to the time of ordering. Ordering more quantity at the beginning may result in too much remaining inventory quantity and ruinous sale. Ordering less quantity may not satisfy all the demand and cause potential profit losses. Hence, it is important to decide optimal purchase quantity.

This study contributes to the increasing analysis on the integration of finance and operations management, since we combine in our model different types of price processes and inventory management. For a novel approach, we expand the classical inventory management model by adding the analysis of the remaining inventories at the end of the cycle. This allows us to develop new understanding on the structure of the optimal policy and to examine in more detail how price evolution parameters effect it. Also in our model, we find innovative ways of thinking to calculate the holding costs.

The study is presented in two parts. In the first part, we assume that the remaining inventories after first selling season will be salvaged with a reduced price and there is no dependence between price and demand for the remaining inventories. In simple terms, all the remaining inventories will be sold for certain with a reduced price. With these assumption, we develop a mathematical model and find a closed form of solution. The second part develops the general model where the secondary market revenue is a variable which depends on the price that the firm assigns, the demand quantity for this price and the remaining inventories.

In addition, we perform an extensive numerical study to investigate how the optimal quantity and the optimal profit values depend on the expected price evolution and its volatility. We also test the performance of the optimal policy against the policy that ignore the volatility of the price.

The rest of this thesis is organized as follows: we review the related literature in Chapter 2. In Chapter 3 gives the notation, assumptions and formulation of the model and presents a solution method. Chapter 4 covers the numerical experiments and sensitivity of the optimal policy to the price process parameters. We conclude the thesis in Chapter 5 with a discussion on possible extensions of the model.

2. LITERATURE REVIEW

This chapter provides a brief literature survey on the random price in the inventory management with different model types in Section 2.1. Then, Section 2.2 outline analysis of price driven demand in procurement management.

2.1. Inventory Management with Price Uncertainty

There have been many studies about optimal inventory policies with random prices since 1950s. Fabian *et al.* (1959) studied the case of purchasing a commodity when the price of this commodity varies between different periods. In this model, it is assumed that price and demand functions are normally distributed. The objective of the study is to minimize average sum of purchasing, holding and shortage costs per unit.

Scarf (1960) analyzed inventory management similar to the study of Fabian et al. with cost preferences and different from Fabian's study, Scarf included reorder cost in the model as well. Order price is assumed constant and order costs has a type of nonlinear cost, ie. cost is zero if order is zero and K + cz otherwise where K is reorder cost, c is purchase price and z is quantity ordered. The aim of the study is to determine the ordering preferences which minimize the expectation of the discounted value of costs above. It is shown that (S, s) type policy is optimal under some linear cost assumptions to ensure the convexity of objective function with respect to ordering quantity.

Kalymon (1971) extended the Scarf's inventory model by considering the ordering prices determined by a Markovian stochastic process. Also the model assumes the convexity of holding, shortage and set-up costs. The purpose of this study is to derive the structure of optimal policies for both finite and infinite planning horizon. The optimality of state-dependent (S, s) policy where S and s depend on the order price is shown.

Gavirneni (2004) studied the periodic review inventory control problem with fluctuating purchasing costs. He showed that "an order up to policy is optimal and determine conditions under which the optimal up to levels are monotonically ordered".

Gaur and Seshadri (2005) examined mitigating the risk in carrying inventory for a short life cycle or seasonal item when there is a correlation between its demand and the price of a financial asset. They show "how to construct static hedging strategies in both the mean-variance framework and the more general utility-maximization framework".

Haksöz and Seshadri (2007) survey the literature about managing procurement in supply chains by using spot market operations. They classify the results in two categories; "the works that deal with optimal procurement strategies and work related to the valuation of the procurement contacts".

Goel and Gutierrez (2012) examined the procurement policies of the stochastic inventory system by integrating commodity markets in their model. They showed that incorporating spot and futures price information in the procurement decision making process assists notably to reduce the inventory related costs.

Secomandi and Kekre (2014) conducted research about the effective use of spot and forward markets in the energy procurement management. They assumed partially procuring supply in forward and spot markets. Their study showed the value of the forward procurement option on realistic natural gas instances and proposed that procuring the demand forecast in the forward market is nearly optimal.

Xie *et al.* (2013) examined the procurement policies for Chinese oil refineries to aim reducing the cost associated with fluctuating oil prices in the international spot market. They used Bayesian learning to incorporate market information dynamics with the model and their model performs well empirically and can result in considerable cost savings.

Chen *et al.* (2014) conducted research to explore "the impact of input price variability and correlation in the context of an inventory system with stochastic demand and stochastic input prices". They showed the concavity of the expected cost function in the input price and higher input price variability leads to lower the expected cost for different price processes.

A work close to ours is Berling and Martínez-de-Albéniz (2011) because we assume, as they do, that the demand follows a Poisson process and the price follows a Markov process. Their study aims to minimize the expected discounted cost including the purchase costs, the out-of-pocket holding costs and the backordering costs. They identifies the optimal base-stock level by the single-unit decomposition approach. In the first part of our model is similar to their studies, but we tries to maximize our profit. Hence, we include the sales processes and do not let backlogging the order.

Berling and Xie (2014) extended the findings of Berling's and Martinez-de-Albeniz's study by proposing an approximation that allow them to derive simple heuristics for determining the optimal thresholds that are close to optimal.

Gaur *et al.* (2015) studied "the optimal timing of inventory ordering decisions" with price uncertainty. They developed a "continuous time inventory model where demand and price are realized at the horizon date T and the stocking decision can be made at any time in the interval $[0, T]$ given progressively more accurate forecasts of price and demand and a time dependent purchasing cost". They show that "the optimal timing of inventory ordering decision" is independent of the demand and follows a "simple threshold policy in the price variable with a possible option of non-purchasing".

2.2. Price dependent Demand/Excess Inventory

Researchers has investigated inventory management from the economical perspective where demand is a function of the price for years. Whitin (1955) improved inventory control models by including price theory in it. In his model, demand depends on the price. The study aims to optimize price and stock level to maximize the firm's profit.

Young (1978) studied the relation between price, inventory and uncertain demand. He assumed a firm whose uncertain demand is a function of the price. He analyzed "the form of the policy, existence and uniqueness, the impact of the initial inventory and the relationship of the optimal policy to that of a firm facing the expected demand curve with certainty."

Khouja (1999) collected the literature for the single period problem (SPP) and give ideas about extension of the SPP and suggests some future direction for research.

Petruzzi and Dada (1999) examined a developed version of the news-vendor problem in which inventory quantity and selling price are determined at the same time. They suggested that simultaneous price setting and stock quantity selection help the firms with regard to handling the effect of uncertainty.

Khouja (2000) studied the extension of the SPP problem where "demand is price-dependent and multiple discounts with prices under the control of the news-vendor are used to sell excess inventory".

Zhang *et al.* (2008) provides "an analytical model for obtaining optimal decisions jointly on pricing, promotion and inventory control". Their study includes "a single item, finite horizon, periodic review model in which the demand is influenced by price and promotion, and the objective is to maximize the total profit". They show that "the optimal promotion policy is a threshold policy under some reasonable assumptions and once the promotion is determined the optimal inventory/pricing policy is a base-stock-list-price policy."

Çetinkaya and Parlar (2010) studied the excess inventory before a policy starts. They categorize three different demand processes, which are unlimited demand, demand depends on the sales price and demand is a random variable. The first type of demand looks similar to our model special case of secondary market process, salvage revenue. Second type of demand processes, looks similar to our demand modeling for the secondary market. We need to note that we use this type of modeling after a period of stochastic inventory in our model, which is more complex than the study of excess inventory before the inventory policies. To be more explicit, in our model the excess inventory quantity is a random variable itself as well, not a known variable.

Zhu and Çetinkaya (2015) extend inventory liquidation problem by analyzing "four special cases where the demand during the liquidation period and regular demand are assumed to be either exponential or uniform random variables". In our model, we use one of the case where the demand of the primary market is exponential and secondary market demand is uniform price function.

3. DESCRIPTION OF THE MATHEMATICAL MODEL

This chapter provides description of the mathematical model and its analytical solutions. Table 3.1 gives the sets, parameters and decision variables for the model. Figure 3.1 plots the framework of the model with decision variables and parameters. This chapter consists of two subsections. Firstly, we describe the modeling processes in a comprehensive manner. Then we provide deeply the mathematical analysis on the model.

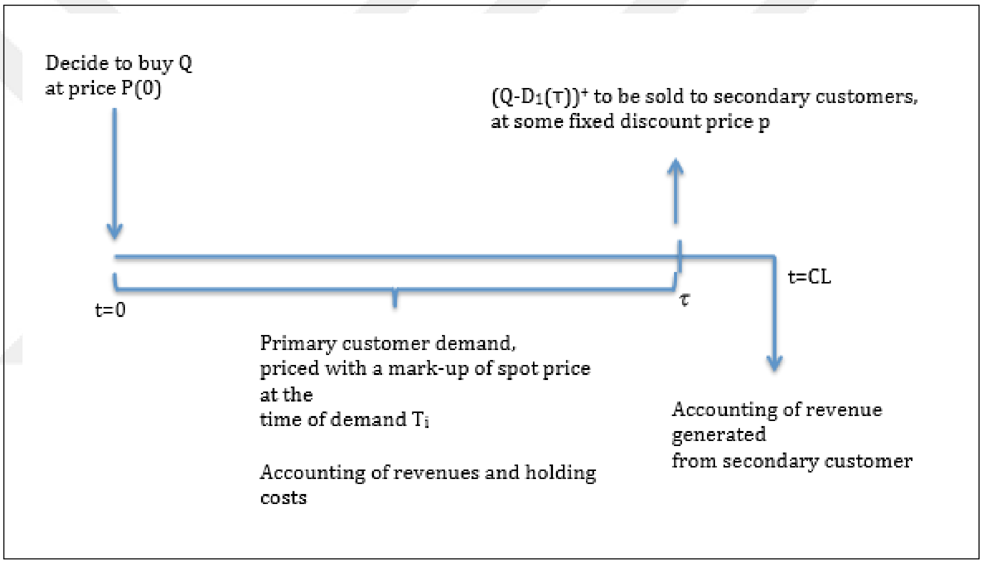

Figure 3.1. The Framework of the Model with Parameters

3.1. Modeling Processes

3.1.1. Demand Process

A great number of products are sold continually every year. While older products become obsolete and their demand decreases, the new and modern ones become popular as soon as they are launched. Most companies realize the importance of the different product life cycle stages and that their products have a limited lifespan (2011). Hence, they try to sell the commodities in their inventory as soon as possible to invest in new products to ensure that their business is sustainable.

Consider a firm sells a commodity in the primary market and if it still has commodities after the primary market, it sells the remaining commodities in the secondary market.

Table 3.1. Sets, Parameters, and Decision Variables for the Model.

$D_1(t)$	Primary market demand over $[0, t]$
$D_2(p)$	Secondary market demand under price p
h	Holding cost multiplier
$HC(Q)$	The present value of the holding cost with initial purchase quantity Q
$J(Q)$	Total profit from the initial order quantity Q
$P(0)$	Initial Price
$P(0)Q$	The purchase cost with initial purchase quantity Q
$P(t)$	The price at time t
Q	Initial purchase quantity
r	The yearly interest rate
s	Salvage multiplier
$SR(Q)$	The present value of the expected sales revenue with initial purchase quantity Q
T_j	The arrival of the jth demand
$V_1(Q)$	The expected profit in the primary season when initial order quantity is Q
$V_2(x, y)$	The expected value of the secondary market revenue where x is the inventory on hand at the end of the primary selling season and y is the price at the end of primary season
α	The mark-up for the sales price
λ	Demand rate
μ	The drift of the price process
σ_p	The variance of the price process
τ	the length of the primary selling season

In the primary market, customers arrive continuously in a stochastic pattern and demand one unit of the item. We assume that the arrival process is Poisson with rate λ . Moreover the primary market customers are assumed to be insensitive to the price, i.e. the primary market demand and price are independent. $D_1(t)$ denotes for the primary market demand over $[0, t]$.

$$D_1(t) \sim Poisson(\lambda t). \text{ That is, } \Pr(D_1(t) = k) = \frac{e^{-\lambda t}\lambda t^k}{k!} \tag{3.1}$$

In the secondary market, customers are responsive to the commodity price. The price that customers are willing to pay depends on the remaining life time of the product and the primary market price of the product. In a realistic approach, customers do not pay more than the primary market price of the product in a secondary market. Furthermore, customers are willing to pay less if the remaining product life is small. Thus, when using the model, the implication of these assumptions need to be carefully considered. In this context, we assume a linear relationship between the secondary market demand and price. $D_2(p)$ denotes for the secondary market demand under price p.

$$D_2(p) = A - Bp + \epsilon, \text{ where } \epsilon \sim (0, \sigma_\epsilon^2) \tag{3.2}$$

Note that regarding the use of the exponential and uniform demand distributions in our illustrations, we look at the related references of the Khouja's study(1999) and Wanke's paper(2008) and realize that they are very popular in inventory management literature and they offer an advantage of analytical simplicity for practical purposes.

3.1.2. The Unit Cost Process

The commodity which is seen as an input for a firm can be bought in a spot market. Hence its price changes continuously. We assume the price as a stochastic process. The Geometric Brownian Motion is used to model the price process that follows the coming Equation 3.3.

$$dP(t) = \mu P(t)dt + \sigma_P P(t)dW_P(t) \tag{3.3}$$

In this equation, μ defines the drift of the process and σ_P defines its variance. $W_P(t)$ is a Weiner process. It is a continuous-time stochastic process with zero drift and unit variance per unit time, i.e. $W_P(t + \Delta t) - W_P(t) \sim N(0, \Delta t)$.

Initial price and the price at time t are denoted $P(0)$ and $P(t)$, respectively. To sum up, the price at time t can be decided by the following Equation 3.4.

$$P(t) = P(0)e^{(\mu-(1/2)\sigma_P^2)t+\sigma_P W_P(t)} \tag{3.4}$$

For an arbitrary initial value P(0), the above Equation 3.3 has the analytic solution (under Ito's interpretation), as can be seen in Equation 3.4. To arrive this formula, we will divide the Equation 3.3 by $P(t)$ in order to have our choice random variable be on only one side. From there, we write the previous equation in Ito integral form:

$$\int_0^t \frac{dP(t)}{P(t)} = \mu t + \sigma_P W_P(t), \text{ assuming } W_P(0) = 0$$

Of course, $\frac{dPt}{Pt}$ looks related to the derivative of $\ln Pt$. However, $P(t)$ is an Ito process which requires the use of Ito calculus. Applying Ito's formula leads to

$$d \ln P(t) = \frac{dP(t)}{P(t)} - \frac{1}{2}\frac{1}{P(t)^2}\, dP(t)dP(t),$$

where $dP(t)dP(t)$ is the quadratic variation of the Equation 3.3. In this case we have:

$$dP(t)dP(t) = \sigma^2\, P(t)^2\, dt$$

Plugging the value of $dP(t)$ in the above equation and simplifying we obtain

$$\ln \frac{P(t)}{P(0)} = (\mu - \frac{\sigma_P^2}{2})t + \sigma_P W_P(t)$$

Taking the exponential and multiplying both sides by $P(0)$ gives the solution claimed above.

3.1.3. Accounting Costs and Revenues

There are two cost items in the model. The first one is initial order cost. The firm buys initial quantity Q and its cost is only quantity multiplied by the initial price. There is no setup cost in our model. The second one is holding cost. Every product we initially purchased spend time in the inventory until it is sold. The firm incurs a holding cost proportional to the time spent in the shelf with a constant holding cost coefficient h and the initial price $P(0)$.

3.2. The Expected Profit

The inventory manager seeks to maximize its profit when deciding to order quantity at the beginning. In our model, profit function consists of two parts; the profit from the primary market and the revenue from the secondary market. Over the primary selling season, there are three items: the initial purchase cost, the sales revenue and the holding costs. The expected profit in the primary season is denoted by $V_1(Q)$ when initial order quantity is Q. Equation 3.5 specifies the profit function in this season with a initial purchase quantity Q, where $P(0)Q$ is the purchase cost, $SR(Q)$ is the sales revenue and $HC(Q)$ is the holding cost, respectively.

$$V_1(Q) = -P(0)Q + SR(Q) - HC(Q) \tag{3.5}$$

Over the secondary selling season, the firm tries to maximize its revenue from the remaining commodities in its inventory at the end of the primary season. The revenue depends on the price and on hand inventory at the end of primary selling season. Equation 3.6 denotes the optimization problem we need to solve where x is the inventory on hand at the end of primary season, y is the price at the end of primary season and $D_2(p)$ is the demand in the secondary market with respect to the price p.

$$V_2(x, y) = \max_{p \leq y} E_{D_2}[p \min\{x^+, D_2(p)\}] \tag{3.6}$$

The sum of the primary season profit and discounted secondary season revenue gives us the total profit,$J(Q)$ the firm gets from the initial purchase quantity Q,as can be seen in Equation 3.7. In this study we aim to decide the initial order quantity Q maximizing this total profit, $J(Q)$.

$$\begin{aligned} J(Q) &= V_1(Q) + E[V_2((Q - D_1(\tau))^+, P(\tau)] \\ &= -P(0)Q + SR(Q) - HC(Q) + E[\max_{p \leq P(\tau)} E_{D_2}[p \min\{(Q - D_1(\tau))^+, D_2(p)\}]] \end{aligned} \tag{3.7}$$

3.2.1. Expected Sales Revenue over the Primary Selling Season

Customers arrive continuously and independently throughout the primary selling season. The firm earns revenue from the sales at the rate of the commodity price of arrival time with a mark-up α for sales price. Equation 3.8 gives the expected discounted sales revenue over the primary selling season,where T_j is the arrival of the jth demand and τ is the length of the primary selling season

$$SR(Q) = E[\sum_{j=1}^{Q} \alpha P(T_j) e^{-rT_j} 1_{\{T_j \leq \tau\}}] \tag{3.8}$$

Proposition 3.1. *$SR(Q) = \alpha P(0) \sum_{j=1}^{Q} (\frac{\lambda}{\lambda-\mu+r})^j \Pr\{T_j^* \leq \tau\}$,*
where $T_j^* \sim$ Erlang$(\lambda - \mu + r, j)$ provided that $\lambda - \mu + r \geq 0$

Proof. We know that if a stochastic process $P(t)$ follows a Geometric Brownian Motion, then its expected value is $P(0)e^{\mu t}$, where μ is the drift of the process. Since demand and price are independent, T_j is just a random variable like t above and $E[P(T_j)] = P(0)e^{-\mu T_j}$. Then, equation 3.8 becomes

$$SR(Q) = \alpha E[\sum_{j=1}^{Q} P(0)e^{-(\mu+r)T_j} 1_{\{T_j \leq \tau\}}]$$

The probability density function of T_j is Erlang distribution with shape parameter j and rate λ as the arrival process of customers is described by a Poisson process with rate λ. Taking expectation with respect to T_j we obtain the following equation.

$$\alpha P(0) \sum_{j=1}^{Q} \int_0^{\infty} e^{-(\mu+r)x} 1_{\{x \leq \tau\}} f_j(x)\, \mathrm{d}x \quad \text{where} \quad f_j(x) \text{ is the pdf of } T_j.$$

After writing open form of pdf of Erlang with shape parameter j and rate λ and getting rid of binary variable $1_{\{x \leq \tau\}}$, the equation turns into the following form:

$$SR(Q) = \alpha P(0) \sum_{j=1}^{Q} \int_0^{\tau} e^{-(\mu+r)x} \frac{\lambda^j x^{j-1} e^{-\lambda x}}{(j-1)!}\, \mathrm{d}x.$$

For the ease of interpretation, we translate the integral part as a pdf of the Erlang distribution. For this reason, we assume that $(\lambda - \mu + r)$ is a positive number. Then we multiply and divide the equation by $(\lambda - \mu + r)^j$ and do not change the value of the equation. After simplifying, we obtain:

$$SR(Q) = \alpha P(0) \sum_{j=1}^{Q} (\frac{\lambda}{\lambda - \mu + r})^j \int_0^{\tau} \frac{(\lambda - \mu + r)^j x^{j-1} e^{-(\lambda-\mu+r)x}}{(j-1)!}\, \mathrm{d}x$$

Since $\frac{(\lambda-\mu+r)^j x^{j-1} e^{-(\lambda-\mu+r)x}}{(j-1)!}$ is a probability density function of Erlang distribution with shape parameter j and rate $(\lambda-\mu+r)$, the equation is transformed into the following form.

$$SR(Q) = \alpha P(0) \sum_{j=1}^{Q} (\frac{\lambda}{\lambda-\mu+r})^j \Pr\{T_j^* \leq \tau\} \text{ where } T_j^* \sim Erlang(\lambda-\mu+r, j)$$

□

3.2.2. Expected Holding Costs over the Primary Selling Season

We define a recursive function to track every commodity the firm buys and calculate its holding time on shelves. In this model, we assume that there is a linear relation between holding cost and the shelf life of commodity. We calculate for every commodity the time between the beginning and the arrival of the demand for that commodity and multiply it with a holding cost coefficient h and the initial purchase price $P(0)$. To find the total holding cost, we sum the present values of the individual commodity holding cost. To illustrative, assume we purchase the commodity now and the demand for the commodity arrives after 3 months. Then, the present value for the holding cost for this commodity is $e^{-3r}3hP(0)$ where r is the yearly interest rate.

Let's assume that the firm orders initially one commodity and measures its present value holding cost. Denoted by $L(t)$ and its present value is found by taking expectation over the time of first customer arrives. Its value is in the following Proposition 3.2.

Proposition 3.2. $L(t) = E[\int_0^{min(t,T_1)} e^{-rx} \mathrm{d}x] = \frac{1}{\lambda+r}[1-e^{-(\lambda+r)t}]$

Proof. Expectation is taken by conditioning arrival time of the first customer, T_1.

$$E[\int_0^{min(t,T_1)} e^{-rx} \mathrm{d}x] = Pr(t \leq T_1) \int_0^t e^{-rx}\, \mathrm{d}x \, + Pr(t > T_1) \int_0^{T_1} e^{-rx}\, \mathrm{d}x$$

Since the pdf of T_1 is $\lambda e^{-\lambda k}$,

$$L(t) = \int_t^{\infty} \lambda e^{-\lambda k} \int_0^t e^{-rx}\,\mathrm{d}x\,\mathrm{d}k \; + \int_0^t \lambda e^{-\lambda k} \int_0^k e^{-rx}\,\mathrm{d}x\,\mathrm{d}k$$

After calculating the integrals and pairing up, we get the result $\frac{1}{\lambda+r}[1-e^{-(\lambda+r)t}]$. $\square$

Define $M(i,t)$ as total expected discounted inventories carried when there are i items in the stock and the length until the end of primary selling season is t. It can be written as a recursive function since when the first customer arrives before the end of primary selling season, we update the function with decreasing the order quantity by 1 and decreasing the time variable by the arrival time, T_1. Then it becomes similar function except changes in variables.

$$M(i,t) = \begin{cases} 0 & t \leq 0 \text{ or } i = 0; \\ iL(t) + E[e^{-rT_i} M(i-1, t-T_i)] & \text{otherwise.} \end{cases}$$

So,

$$\begin{aligned}
M(1,t) &= L(t) + E[e^{-rT_1} M(0, t-T_1)] = L(t) \\
M(2,t) &= 2L(t) + E[e^{-rT_1} M(1, t-T_1)] = 2L(t) + E[e^{-rT_1} L(t-T_1)] \\
M(3,t) &= 3L(t) + E[e^{-rT_1} M(2, t-T_1)] = 3L(t) + 2E[e^{-rT_1} L(t-T_1)] + E[e^{-rT_2} L(t-T_2)] \\
&\vdots \\
M(N,t) &= NL(t) + (N-1)E[e^{-rT_1} L(t-T_1)] + (N-2)E[e^{-rT_2} L(t-T_2)] + \ldots \\
&\quad + E[e^{-rT_{N-1}} L(t-T_{N-1})]
\end{aligned}$$

In this model, we want $M(Q,\tau)$, i.e. total expected discounted inventories carried when there are Q items in the stock and the length until the end of primary selling season is τ.

$$M(Q,\tau) = QL(\tau) + \sum_{i=1}^{Q-1}(Q-i)E[e^{-rT_i}L(\tau - T_i)]$$

Proposition 3.3. $E[e^{-rT_n}L(t-T_n)] = \frac{\lambda^n}{(\lambda+r)^{n+1}}\Pr(T_n^{**} < t) - \frac{e^{-rt}}{\lambda+r}\Pr(N(t)=n)$, *where* $T_n^{**} \sim Erlang(\lambda+r,n)$ *and* $N(t) \sim Poisson(\lambda t)$

Proof. Since T_n is distributed with Erlang, expectation is taken on the pdf of Erlang distribution.

$$\begin{aligned}
E[e^{-rT_n}L(t-T_n)] &= \int_0^t e^{-ry}\frac{1}{\lambda+r}(1-e^{-(\lambda+r)(t-y)})f_n(y)\,\mathrm{d}y \\
&= \frac{1}{\lambda+r}\int_0^t e^{-ry}f_n(y)\,\mathrm{d}y - \frac{1}{\lambda+r}\int_0^t e^{-(\lambda+r)t}e^{\lambda y}f_n(y)\,\mathrm{d}y \\
&= \frac{1}{\lambda+r}(\frac{\lambda}{\lambda+r})^n\int_0^t \frac{e^{-(\lambda+r)y}y^{n-1}(\lambda+r)^n}{(n-1)!}\,\mathrm{d}y \\
&- \frac{e^{-(\lambda+r)t}\lambda^n}{\lambda+r}\int_0^t \frac{y^{n-1}}{(n-1)!}\,\mathrm{d}y \\
&= (\frac{\lambda}{\lambda+r})^n\Pr(T_n^{**} < t) - \frac{e^{-(\lambda+r)t}\lambda^n}{\lambda+r}\frac{t^n}{n!} \\
&= (\frac{\lambda}{\lambda+r})^n\Pr(T_n^{**} < t) - \frac{e^{-rt}}{\lambda+r}\Pr(N(t)=n),
\end{aligned}$$

where $T_n^{**} \sim Erlang(\lambda+r,n),\ N(t) \sim Poisson(\lambda t)$. □

After using the findings of the proposition above, we reach the formula for the total expected discounted holding costs, denoted by $HC(Q)$, with a holding cost multiplier $hP(0)$.

$$HC(Q) = \frac{hP(0)}{\lambda+r}[Q + \sum_{i=1}^{Q-1}(Q-i)(\frac{\lambda}{\lambda+r})^i\Pr(T_i^{**}) - e^{-r\tau}\sum_{i=0}^{Q-1}(Q-i)Pr(N(\tau)=i)] \tag{3.9}$$

To sum up, the expected profit from the primary selling season becomes

$$
\begin{aligned}
E[V_1(Q)] &= -P(0)Q + SR(Q) - HC(Q) \\
&= -P(0)Q + \alpha P(0) \sum_{j=1}^{Q} (\frac{\lambda}{\lambda - \mu + r})^j \Pr\{T_j^* \leq \tau\} - \frac{hP(0)}{\lambda + r}[Q \\
&+ \sum_{i=1}^{Q-1} (Q-i)(\frac{\lambda}{\lambda + r})^i \Pr(T_i^{**}) - e^{-r\tau} \sum_{i=0}^{Q-1} (Q-i) Pr(N(\tau) = i)],
\end{aligned}
$$

where $T_j^* \sim Erlang(\lambda - \mu + r, j)$ and $T_i^{**} \sim Erlang(\lambda + r, i)$.

3.2.3. Special Case: Constant Salvage Value

In the second market, we assumed that every leftover item can be sold at a constant mark-up price. It is equivalent to externally imposed salvage "multiplier" $s \leq 1$. Hence there is no optimization over the discounted price. Then the revenue from the second selling season becomes

$$
V_2(Q - D_1(\tau), P(\tau)) = sP(\tau)(Q - D_1(\tau))^+
$$

Expectation is taken over the demand over the first selling season and the price at the end of first season. Since the price and demand is independent in the first season,

$$
\begin{aligned}
E[V_2(Q - D_1(\tau), P(\tau))] &= E[sP(\tau)e^{-r\tau}(Q - D_1(Q))^+] \\
&= se^{-r\tau} E[P(\tau)] E[(Q - D_1(Q))^+] \\
&= sP(0)e^{\mu\tau} e^{-r\tau} E[(Q - D_1(Q))^+] \\
&= sP(0)e^{(\mu - r)\tau} \sum_{i=0}^{Q-1} (Q-i) \Pr(D_1(\tau) = i),
\end{aligned}
$$

where $D_1(\tau) \sim Poisson(\lambda\tau)$.

Therefore, total expected profit with special case of constant salvage revenue is denoted by $J(Q)$ and its value is

$$\begin{aligned}
J(Q) &= E[V_1(Q)] + E[V_2(Q - D_1(\tau), P(\tau))] \\
&= -P(0)Q + \alpha P(0) \sum_{j=1}^{Q} (\frac{\lambda}{\lambda - \mu + r})^j \Pr\{T_j^* \leq \tau\} - \frac{hP(0)}{\lambda + r}[Q \\
&+ \sum_{i=1}^{Q-1} (Q - i)(\frac{\lambda}{\lambda + r})^i \Pr(T_i^{**} \leq \tau) - e^{-r\tau} \sum_{i=0}^{Q-1} (Q - i) Pr(N(\tau) = i)] \\
&+ sP(0)e^{(\mu - r)\tau} \sum_{j=1}^{Q-1} (Q - j) \Pr\{D_1(\tau) = j\},
\end{aligned}$$

where $T_j^* \sim Erlang(\lambda - \mu + r, j)$, $T_i^{**} \sim Erlang(\lambda + r, i)$ and $D_1(\tau) \sim Poisson(\lambda\tau)$.

Proposition 3.4. *$J(Q)$ is concave in Q for $\mu \leq r$.*

Proof. We need to check first and second increment of the profit function. First increment is

$$\begin{aligned}
\Delta(J(Q)) &= J(Q + 1) - J(Q) \\
&= -P(0) + \alpha P(0)(\frac{\lambda}{\lambda - \mu + r})^{Q+1} \Pr\{T_{Q+1}^* \leq \tau\} \\
&+ sP(0)e^{(\mu - r)\tau} \sum_{j=1}^{Q} \Pr\{D_1(\tau) = j\} \\
&- \frac{hP(0)}{\lambda + r}[1 + \sum_{i=1}^{Q} (\frac{\lambda}{\lambda + r})^i \Pr(T_i^{**} \leq \tau) - e^{-r\tau} \sum_{i=0}^{Q} Pr(N(\tau) = i)]
\end{aligned}$$

Second increment is

$$\begin{aligned}
\Delta^2(J(Q)) &= \Delta(J(Q+1)) - \Delta(J(Q)) \\
&= \alpha P(0)[(\frac{\lambda}{\lambda-\mu+r})^{Q+2}\Pr\{T^*_{Q+2}\leq\tau\} - (\frac{\lambda}{\lambda-\mu+r})^{Q+1}\Pr\{T^*_{Q+1}\leq\tau\}] \\
&+ sP(0)e^{(\mu-r)\tau}\Pr\{D_1(\tau)=Q+1\} + \frac{hP(0)}{\lambda+r}e^{-r\tau}Pr\{D_1(\tau)=Q+1\} \\
&- \frac{hP(0)}{\lambda+r}(\frac{\lambda}{\lambda+r})^{Q+1}\Pr\{T^{**}_{Q+1}\leq\tau\}
\end{aligned}$$

Since the cumulative distribution function of the Erlang distribution is $F(x;k,\lambda) = 1-\sum_{n=0}^{k-1}\frac{1}{n!}e^{-\lambda x}(\lambda x)^n = \sum_{n=k}^{\infty}\frac{1}{n!}e^{-\lambda x}(\lambda x)^n$ and the probability mass function of the Poisson distribution is $\frac{e^{-\lambda x}\lambda^k}{k!}$, after using these equalities in the second increment and bringing equation in a good shape, we get

$$\begin{aligned}
\Delta^2(J(Q)) = &-\alpha P(0)(\frac{r-\mu}{\lambda-\mu+r})(\frac{\lambda}{\lambda-\mu+r})^{Q+1}\sum_{n=Q+2}^{\infty}\frac{e^{-(\lambda-\mu+r)\tau}((\lambda-\mu+r)\tau)^n}{n!} \\
&- \alpha P(0)\frac{e^{-(\lambda-\mu+r)\tau}(\lambda\tau)^{Q+1}}{(Q+1)!} + sP(0)\frac{e^{-(\lambda-\mu+r)\tau}(\lambda\tau)^{Q+1}}{(Q+1)!} \\
&- \frac{hP(0)}{\lambda+r}(\frac{\lambda}{\lambda+r})^{Q+1}\sum_{n=Q+2}^{\infty}\frac{e^{-(\lambda+r)\tau}((\lambda+r)\tau)^n}{n!}
\end{aligned}$$

Therefore, for $\mu\leq r$ and $\alpha\geq s$, $\Delta^2(J(Q))\leq 0$. So the profit function is concave. $\square$

In this case, the smallest ordering quantity which makes the first increment negative is the optimal quantity. Hence, the first Q satisfying

$$\begin{aligned}
\frac{\lambda+h+r}{\lambda+r} &\geq \alpha(\frac{\lambda}{\lambda-\mu+r})^{Q+1}\Pr\{T^*_{Q+1}\leq\tau\} \\
&+ e^{-r\tau}(se^{\mu\tau}+\frac{h}{\lambda+r})\sum_{i=0}^{Q}\Pr\{D_1(\tau)=i\} \\
&- \frac{h}{\lambda+r}\sum_{i=1}^{Q}(\frac{\lambda}{\lambda+r})^i\Pr\{T^{**}_i\leq\tau\}
\end{aligned}$$

is optimal.

3.2.4. The Secondary Market Customers

The firm maximizes its revenue from the remaining inventory after the first selling season. The secondary market customers demand the firm's product with a negative correlation with price. We assume linear deterministic customer demand in the secondary market. $D_2(p) = A - Bp$ is the relation between price and demand in our model.

Unconstrained solution. The firm decides its product price according to its total revenue from the remaining inventory. There is no limitation for the price the firm assigns. Hence, the problem gets the form below where p denotes the price and x denotes the remaining inventory for the second market.

$$V_2(x, p) = \max_p p \min\{x^+, A - Bp\} \tag{3.10}$$

To solve the problem above, we need to analyze every possible case of the remaining inventory quantity.

Case 1. For $x^+ \geq A$, the problem becomes $\max_p p(A - Bp)$ since $min\{x^+, A - Bp\} = A - Bp$. It can be seen easily that it is a concave function and it gets its optimal value when the first order derivation equals to 0. Hence, $p^* = \frac{A}{2B}$ and $V_2(x, p^*) = \frac{A^2}{4B}$.

Case 2. For $A > x^+ \geq \frac{A}{2}$, there are two options:

For the first option, $x^+ \geq A - Bp$. Then $min\{x^+, A - Bp\} = A - Bp$. The problem is the same as for the Case 1. Since the optimal solution assures the constraints for these case, optimal solution is the same here as well. So, $p^* = \frac{A}{2B}$ and $V_2(x, p^*) = \frac{A^2}{4B}$.

For the second option, $x^+ < A-Bp$. Then, $min\{x^+, A-Bp\} = x^+$. The problem becomes $\max_p px^+$ and the optimal price is the greatest value the price can take. Since the price has a constraint $p < \frac{A-x^+}{B}$ from the option part and $x^+ \geq \frac{A}{2}$ from the case part, we get $p < \frac{A}{2B}$. So, if we take limit for the price, then $p^* = \frac{A}{2B}$ and $V_2(x, p^*) = \frac{A^2}{4B}$.

To sum up, for both options, the optimal solution is same and it is a general solution for this case.

Case 3. For $\frac{A}{2} > x^+ \geq 0$, there are two options similar to case 2:

For the first option, $x^+ \geq A - Bp$. Then, $min\{x^+, A - Bp\} = A - Bp$. The problem is the same as for the Case 1. Unfortunately, optimal solution for the case 1 is not feasible here because of the constraints $x^+ \geq A - Bp$ and $x^+ < \frac{A}{2}$. Since $Ap - Bp^2$ is a concave function, the optimal price is the least drift feasible solution from $p^* = \frac{A}{2B}$. Lets call this drift Δ. Then, $x^+ \geq A - B(\frac{A}{2B} + \Delta)$. So, we get the constraint $\Delta \geq \frac{A}{2B} - \frac{x^+}{B}$ for the drift and the least value it gets is $\Delta^* = \frac{A}{2B} - \frac{x^+}{B}$. Hence, $p^* = \frac{A}{2B} + \Delta^* = \frac{A}{2B} + \frac{A}{2B} - \frac{x^+}{B} = \frac{A-x^+}{B}$ and $V_2(x, p^*) = \frac{Ax^+ - {x^+}^2}{B}$.

For the second option, $x^+ < A-Bp$. Then, $min\{x^+, A-Bp\} = x^+$. The problem becomes $\max_p px^+$ and the optimal price is the greatest value the price can take similar to case 2 again. Since $p < \frac{A-x^+}{B}$ from the option part, when we take limit for the price, we get $p^* = \frac{A-x^+}{B}$ and $V_2(x, p^*) = \frac{Ax^+ - {x^+}^2}{B}$.

To sum up, for both options, the optimal solution is same and it is a general solution for this case.

In a nutshell, the unconstrained solution becomes

For $(Q - D_1(\tau))^+ \geq A/2$

$$p_u^* = \frac{A}{2B}, \quad V_2^* = \frac{A^2}{4B}$$

For $(Q - D_1(\tau))^+ < A/2$

$$p_u^* = \frac{A - (Q - D_1(\tau))^+}{B}, \quad V_2^* = \frac{A(Q - D_1(\tau))^+ - (Q - D_1(\tau))^2}{B}$$

Constrained solution. Due to the nature of the secondary market, for normal goods the price should be less and equal to the price of the same good in the first market. Therefore we need to add this constraint to the problem as well to be more realistic model. Then, we have four cases whether or not optimal price for unconstrained model is less than its normal price at that time and the remaining inventory is less than $\frac{A}{2}$.

		Unconstrained optimal	
		$\leq P(\tau)$	$> P(\tau)$
$(Q - D_1(\tau))^+$	$\geq A/2$	Case 1	Case 3
	$< A/2$	Case 2	Case 4

The total revenue from the secondary market is $\sum_{k=1}^{4} V_{2,k}(Q) = \sum_{k=1}^{4} E[V_2(Q - D_1(\tau), P(\tau))1_{\{\text{case k}\}}]$. In the following pages, every case of constrained solution is analyzed one by one.

Case 1: $P(\tau) \geq \frac{A}{2B}$ **and** $(Q - D_1(\tau))^+ \geq \frac{A}{2}$**.** In this case, since the price at the end of primary selling season is greater than the optimal value, the unconstrained solution is feasible and optimal here as well. After taking expectation over the primary selling season demand and the price at the end of this season, we get

$$\begin{aligned} V_{2,1}(Q) &= E[V_2(Q - D_1(\tau), P(\tau))1_{\{\text{case } 1\}}] \\ &= \sum_{i=0}^{Q-\frac{A}{2}} \frac{A^2}{4B} \Pr(D_1(\tau) = i) \Pr(P(\tau) \geq \frac{A}{2B}) \\ &= \sum_{i=0}^{Q-\frac{A}{2}} \frac{A^2}{4B} \Big\{ \int_{\frac{A}{2B}}^{\infty} f(y)\, dy \Big\} P(D_1(\tau) = i), \end{aligned}$$

where $f(y)$ is a probability density function of the price at the end of the primary season.

With the help of findings in Appendix A, we can take the integral. Then it gets

$$V_{2,1}(Q) = \frac{A^2}{4B}\Big\{1 - \Phi(\frac{\ln(\frac{A}{2B}) - In(P(0)) - (\mu - \frac{1}{2}\sigma^2)\tau}{\sigma_P\sqrt{\tau}})\Big\} \sum_{i=0}^{Q-\frac{A}{2}} P(D_1(\tau) = i) \quad (3.11)$$

Case 2: $P(\tau) \geq \frac{A-(Q-D_1(\tau))^+}{B}$ **and** $(Q - D_1(\tau))^+ < \frac{A}{2}$**.** Similar to case 1, unconstrained solution is feasible and optimal in this case as well. With similar steps to case 1, we get

$$\begin{aligned} V_{2,2}(Q) &= E[V_2(Q - D_1(\tau), P(\tau))1_{\{\text{case } 2\}}] \\ &= \sum_{i=Q+1-\frac{A}{2}}^{Q} \Big\{ \int_{\frac{A-(Q-i)}{B}}^{\infty} \frac{A - (Q - i)}{B}(Q - i) f(y)\, dy \Big\} P(D_1(\tau) = i) \\ &= \sum_{i=Q-\frac{A}{2}+1}^{Q} \frac{A - (Q - i)}{B}(Q - i)\Big\{1 - \Phi(\frac{\ln(\frac{A-(Q-i)}{B}) - \mu_\tau}{\sigma_P\sqrt{\tau}})\Big\} P(D_1(\tau) = i), \end{aligned}$$

where $\mu_\tau = In(P(0)) + (\mu - \frac{1}{2}\sigma^2)\tau$.

Case 3: $P(\tau) < \frac{A}{2B}$ **and** $(Q - D_1(\tau))^+ \geq \frac{A}{2}$**.** In this case, the unconstrained solution is not feasible since the optimal price is greater than the price at the end of the primary season. Then, the optimal price is the price at the end of the primary season because of the concavity of the revenue function.

$$
\begin{aligned}
V_{2,3}(Q) &= E[V_2(Q - D_1(\tau), P(\tau))1_{\{\text{case } 3\}}] \\
&= \sum_{i=0}^{Q-\frac{A}{2}} \Big\{ \int_0^{\frac{A}{2B}} y\, min(Q-i, A-By) f(y)\, dy \Big\} P(D_1(\tau)=i) \\
&= \sum_{i=0}^{Q-\frac{A}{2}} \Big\{ \int_0^{\frac{A-(Q-i)}{B}} y(Q-i) f(y)\, dy \\
&+ \int_{\frac{A-(Q-i)}{B}}^{\frac{A}{2B}} y(A - BP(\tau)) f(y)\, dy \Big\} P(D_1(\tau)=i) \\
&= \sum_{i=0}^{Q-\frac{A}{2}} (Q-i) P(D_1(\tau)=i) \int_0^{\frac{A-(Q-i)}{B}} y f(y)\, dy \\
&+ \sum_{i=0}^{Q-\frac{A}{2}} P(D_1(\tau)=i) \int_{\frac{A-(Q-i)}{B}}^{\frac{A}{2B}} (Ay - By^2) f(y)\, dy
\end{aligned}
$$

After taking integrals by the help of the findings in Appendix A, we get

$$
\begin{aligned}
V_{2,3}(Q) = \sum_{i=0}^{Q-\frac{A}{2}} P(D_1(\tau)=i) \Bigg[& (Q-i) e^{(\frac{\sigma_P^2 \tau}{2} + \mu_\tau)} \Phi\Big(\frac{\ln(\frac{A-(Q-i)}{B}) - \mu_\tau - \sigma_P^2 \tau}{\sigma_P \sqrt{\tau}} \Big) \\
& + A e^{(\frac{\sigma_P^2 \tau}{2} + \mu_\tau)} \Big[\Phi\Big(\frac{\ln(\frac{A}{2B}) - \mu_\tau - \sigma_P^2 \tau}{\sigma_P \sqrt{\tau}} \Big) - \Phi\Big(\frac{\ln(\frac{A-(Q-i)}{B}) - \mu_\tau - \sigma_P^2 \tau}{\sigma_P \sqrt{\tau}} \Big) \Big] \\
& - B e^{2(\sigma_P^2 \tau + \mu_\tau)} \Big[\Phi\Big(\frac{\ln(\frac{A}{2B}) - \mu_\tau - 2\sigma_P^2 \tau}{\sigma_P \sqrt{\tau}} \Big) - \Phi\Big(\frac{\ln(\frac{A-(Q-i)}{B}) - \mu_\tau - 2\sigma_P^2 \tau}{\sigma_P \sqrt{\tau}} \Big) \Big] \Bigg],
\end{aligned}
\tag{3.12}
$$

where $\mu_\tau = In(P(0)) + (\mu - \frac{1}{2}\sigma^2)\tau$.

Case 4: $P(\tau) < \frac{A-(Q-D_1(\tau))^+}{B}$ **and** $(Q - D_1(\tau))^+ < \frac{A}{2}$**.** Similar to Case 3, unconstrained solution is not feasible and the optimal price is the price at the end of the primary season. With similar steps, we get

$$
\begin{aligned}
V_{2,4}(Q) &= E[V_2(Q - D_1(\tau), P(\tau))1_{\{\text{case } 4\}}] \\
&= \sum_{i=Q+1-\frac{A}{2}}^{Q} \Big\{ \int_0^{\frac{A-(Q-i)}{B}} y\, min(Q-i, A-By) f(y)\, dy \Big\} P(D_1(\tau) = i) \\
&= \sum_{i=Q+1-\frac{A}{2}}^{Q} \Big\{ \int_0^{\frac{A-(Q-i)}{B}} y(Q-i) f(y)\, dy \Big\} P(D_1(\tau) = i) \\
&= \sum_{i=0}^{Q-\frac{A}{2}} P(D_1(\tau) = i) \Bigg[(Q-i) e^{(\frac{\sigma_P^2 \tau}{2} + \mu_\tau)} \Phi\Big(\frac{\ln(\frac{A-(Q-i)}{B}) - \mu_\tau - \sigma_P^2 \tau}{\sigma_P \sqrt{\tau}} \Big) \Bigg],
\end{aligned}
$$

where $\mu_\tau = In(P(0)) + (\mu - \frac{1}{2}\sigma^2)\tau$.

To sum up, total revenue from the primary and secondary market becomes

$$J(Q) = V_1(Q) + V_2(Q) = -P(0)Q + SR(Q) - HC(Q) + \sum_{k=1}^{4} V(2,k) \tag{3.13}$$

4. COMPUTATIONAL AND SENSITIVITY ANALYSIS

This chapter provides the aim of the computational and sensitivity analysis, parameter settings for the numerical analysis and observations from the results of numerical experiments.

4.1. Objectives

To validate our mathematical model and obtain information about the optimal policy, we perform some numerical experiments with different parameter settings. This numerical analysis also provide the answer for questions, such as how much to order with specific parameter setting, which parameter of the model is more effective in the optimal policy or how the price volatility information affects the optimal quantity and optimal policy.

4.2. Parameter Settings

Considering salvage value as a secondary market, in our analysis we fix $P(0)$ as 1 TL, and τ is selected as 1 year. The yearly interest rate is set as $r = 0.1$. Other parameters take at least two different values as $\alpha \in \{1.00, 1.01, 1.05\}$, $\lambda \in \{10, 100\}$, $\mu \in \{0.09, 0.10\}$, $h \in \{0.01, 0.10\}$ and $s \in \{0.95, 0.99, 1.00\}$.

For the general secondary market, before analyzing the model, we need to specify the parameters of the demand model of the secondary market. In our computational study, $D_2(p) = A - Bp + \epsilon$ is the relationship between price and the secondary market demand.

The parameter A can be interpreted as the potential demand over the secondary selling season. Assuming the total expected demand for a commodity is simply the multiplication of the product life and the rate of demand, the potential demand for the secondary market will be directly proportional to the remaining product life. Hence, $A = d_1(CL - \tau)\lambda$ is a reasonable choice, where d_1 is a coefficient depending on the commodity, CL is the commodity life cycle, τ is the length of the primary selling season and λ is the rate of demand over the primary selling season.

The parameter B is the elasticity of the price over the secondary selling season. Assuming a linear model for the total demand in the primary selling season, we can find the elasticity of the price over the primary selling season. Then, $\tilde{D}_1 = A_0 - B_0 p + \epsilon$. After taking expectation from both side of the equation, we get

$$E[\tilde{D}_1] = E[A_0 - B_0 p + \epsilon] = \lambda\tau = d_2 CL\lambda - B_0 \bar{p}.$$

To find the elasticity B_0, we also need to find $\bar{p}$, which is the average price over the primary selling season. It can be determined by taking expectation over the price throughout the primary selling season and divide by the length of the primary selling season. Then,

$$\bar{p} = \frac{E[\int_0^\tau P(t)dt]}{\tau} = \frac{\int_0^\tau E[P(t)]dt}{\tau} = \frac{\int_0^\tau P(0)e^{\mu t}dt}{\tau} = P(0)\frac{e^{\mu\tau}-1}{\mu\tau}.$$

Assuming that the elasticity of the price is the same for the primary and secondary selling seasons, we get

$$B = B_0 = \frac{(d_2 CL - \tau)\lambda}{\bar{p}}.$$

To test the secondary market demand model, we made experiments with many different parameter settings, as can be seen in Table 4.2. In that table, we assume that the commodity life cycle, CL is 5 years and the initial purchase price, $P(0)$ is 1. Min is defined as the minimum optimal price when there is no price constraint. In a similar manner, Max is defined as the maximum optimal price. To be remembered, if there is enough remaining inventory, the optimal price is $\frac{A}{2B}$ and if not, the optimal price depends on the remaining quantity. $(Q - D_1(\tau))^+$ and it is $\frac{A-(Q-D_1(\tau))^+}{B}$. Hence, the optimal price can take values between $\frac{A}{2B}$ and $\frac{A}{B}$. Min is $\frac{A}{2B}$ and Max is $\frac{A}{B}$ and as we can see in the Table 4.2, both is less than the expected price of the commodity at the end of the first selling season. This is a reasonable indicator of the modeling and makes us sure that we are on the right path.

Table 4.1. Parameter Analysis for the Secondary Market Demand Model

	τ	***1***		***2***		***3***		***4***	
	μ	***0.10***	***0.05***	***0.10***	***0.05***	***0.10***	***0.05***	***0.10***	***0.05***
Expected Price		1.105	1.105	1.221	1.105	1.350	1.162	1.492	1.221
$\bar{p}$		1.052	1.025	1.107	1.052	1.166	1.079	1.230	1.107
d_1=d_2=***1***	**Min**	0.553	0.553	0.611	0.553	0.675	0.581	0.746	0.611
	Max	1.105	1.105	1.221	1.105	1.350	1.162	1.492	1.221
d_1=d_2=***1.5***	**Min**	0.485	0.473	0.453	0.351	0.389	0.360	0.263	0.237
	Max	0.971	0.947	0.906	0.701	0.777	0.719	0.527	0.474
d_1=***1***, d_2=***1.5***	**Min**	0.789	0.769	0.830	0.789	0.875	0.809	0.922	0.830
	Max	1.578	1.538	1.661	1.578	1.749	1.618	1.844	1.661
d_1=***1.5***, d_2=***1***	**Min**	0.324	0.316	0.302	0.234	0.259	0.240	0.176	0.158
	Max	0.647	0.631	0.604	0.467	0.518	0.480	0.351	0.316
d_1=d_2=***2***	**Min**	0.467	0.456	0.415	0.300	0.333	0.308	0.205	0.185
	Max	0.935	0.911	0.830	0.601	0.666	0.617	0.410	0.369

After deciding the parameters of the secondary market demand models, we analyze the general model with numerical experiments. We assume that $d_1 = 1.5$ and $d_2 = 1.5$, the first market demand rate, λ is 10, the product life cycle, CL is 5 years, the yearly interest rate, r is 0.1, and the initial purchase price, $P(0)$ is 1. Other parameters are $\sigma_P \in \{0, 0.5, 0.8, 1\}$, $\tau \in \{1, 2, 3, 4, 4.5\}$, $\mu \in \{0.09, 0.1, 0.15\}$, $h \in \{0.01, 0.03, 0.05\}$ and $\alpha \in \{1, 1.01, 1.05\}$.

4.3. Observations

Considering salvage value as a secondary market, we perform numerical analysis with different parameter settings to validate the model and its solution. For example, in one case, we calculate the optimal quantity and optimal profit with $(\alpha, s, \lambda, \mu, h) = (1.01, 0.95, 10, 0.10, 0.01)$ and find that optimal quantity is 6 and the corresponding expected profit is 0.033 TL. Other numerical results can be seen in Table 4.3 for the optimal quantity and in Table 4.3 for the expected optimal profit values for the optimal quantities. Note that in the optimal quantity table, 0 means do not order anything. There are many observations from the results. Firstly, optimal quantity seems to be directly proportional to the demand rate. This can be expected since if there are many requests for the product, it is expected that the firm will order more product. Secondly, sales mark-up multiplier has obviously large impact on the optimal policy. When the firm is able to sell the commodity with a higher price, it orders more at the beginning of the season since the demand and price during the primary season is independent. Thirdly, even the expected present value of price is decreasing, the firm still orders quantity according to our model. This can be possible, since the revenue from sales of the product can outweigh the cost of buying higher initial prices. Moreover, the salvage multiplier becomes inactive when holding multiplier is higher. This is reasonable considering outbalance of holding costs to the salvage revenue. Lastly, holding cost multiplier is very effective on the optimal quantity.

In Table 4.3, the optimal quantity and in Table 4.5, optimal profit values can be seen when the volatility of the price process, σ_p is 1. Similarly, optimal quantity and profit values with different volatilities can be seen in the following tables: Table 4.6 and Table 4.7 where σ_p is 0.8, Table 4.8 and Table 4.9 where σ_p is 0.5 and Table 4.10 and Table 4.11 where the volatility information is ignored, i.e. $\sigma_p = 0$.

Let's summarize the observations from the computational analysis for the general model. Firstly, the optimal quantity and the optimal profit value increase as one of the following three parameters increases. These parameters are the length of the first selling season, τ, the drift of the price process, μ and the mark-up α for sales price. It is reasonable that if the price is expected to be increased, the firms tend to order more quantity at the beginning to get the benefit of increase in price, even they could not sell much. Similarly, if the firm sells with higher mark-up prices, they will gain more and sales revenue can outweigh the holding costs. The effect of the length of the first selling season on the optimal quantity is decreasing as the length of the primary market increases. This is also be reasonable since the product life cycle is limited and there will be too little revenue from the secondary market for the remaining inventories. Secondly, the optimal quantity and profit value decrease as holding cost coefficient, h increases. Lastly, the optimal quantity and profit values decrease as the variance of the price, σ_p increases. This is not expected outcome at the first thinking. There is a reasonable explanation for this result. The price constraint becomes more active as σ_p increases. This causes any excess stock can only be sold at a less favorable price. Hence, if there is more volatility for the price, less quantity of the product is ordered at the beginning.

To analyze the importance of taking volatility of the price into consideration when deciding the optimal inventory quantity, we look at the difference in the optimal quantity between the numerical results with $\sigma_p = 1$ and $\sigma_p = 0$, as can be seen in Table 4.12. The value of the information of the volatility becomes more valuable when the drift of the price is higher and the length of the first selling season is smaller. Moreover,in Table 4.13 we look at the percentage change of the optimal profit value when we pay attention to the variance of the price process. For instance, for the parameter setting, "$\tau = 2$, $\mu = 0.15$, $\alpha = 1.01$, $h = 0.03$", we can gain % 10.09 more when we considering the information of $\sigma = 1$.

Table 4.2. Optimal Quantity

	λ	***10***				***100***			
	μ	***0.09***		***0.10***		***0.09***		***0.10***	
	h	***0.01***	***0.10***	***0.01***	***0.10***	***0.01***	***0.10***	***0.01***	***0.10***
α=***1.00***	***s=0.95***	0	0	0	0	0	0	0	0
	s=0.99	0	0	0	0	0	0	0	0
	s=1.00	0	0	0	0	0	0	0	0
α=***1.01***	***s=0.95***	4	0	6	0	49	9	82	10
	s=0.99	4	0	7	0	50	9	86	10
	s=1.00	4	0	8	0	50	9	90	10
α=***1.05***	***s=0.95***	9	4	9	4	95	46	98	51
	s=0.99	10	4	11	5	100	46	104	51
	s=1.00	11	4	13	5	103	46	109	51

Table 4.3. Expected Optimal Profit Values

	λ	***10***				***100***			
	μ	***0.09***		***0.10***		***0.09***		***0.10***	
	h	***0.01***	***0.10***	***0.01***	***0.10***	***0.01***	***0.10***	***0.01***	***0.10***
α=***1.00***	***s=0.95***	0	0	0	0	0	0	0	0
	s=0.99	0	0	0	0	0	0	0	0
	s=1.00	0	0	0	0	0	0	0	0
α=***1.01***	***s=0.95***	0.019	0.000	0.033	0.000	0.246	0.041	0.482	0.045
	s=0.99	0.020	0.000	0.038	0.000	0.246	0.041	0.482	0.045
	s=1.00	0.020	0.000	0.041	0.000	0.246	0.041	0.490	0.045
α=***1.05***	***s=0.95***	0.284	0.090	0.329	0.101	3.641	1.123	4.127	1.247
	s=0.99	0.321	0.091	0.381	0.101	3.752	1.123	4.298	1.247
	s=1.00	0.336	0.091	0.407	0.102	3.799	1.123	4.378	1.247

Table 4.4. The Optimal Quantities where $\sigma_p = 1$

	μ	**0.09**			**0.1**			**0.15**		
	h	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**
τ=**1**	α=**1.00**	0	0	0	0	0	0	5	4	1
	α=**1.01**	3	2	1	4	2	1	6	5	4
	α=**1.05**	6	5	5	6	6	5	7	6	6
τ=**2**	α=**1.00**	0	0	0	0	0	0	14	13	10
	α=**1.01**	5	2	1	9	3	2	15	13	12
	α=**1.05**	12	11	8	13	12	9	16	15	14
τ=**3**	α=**1.00**	0	0	0	0	0	0	24	22	19
	α=**1.01**	5	2	1	10	3	2	24	22	20
	α=**1.05**	19	12	8	21	17	10	25	24	23
τ=**4**	α=**1.00**	0	0	0	0	0	0	33	31	28
	α=**1.01**	5	2	1	10	3	2	34	32	29
	α=**1.05**	24	13	8	28	18	10	34	33	31
τ=**4.5**	α=**1.00**	0	0	0	0	0	0	38	36	33
	α=**1.01**	5	2	1	10	3	2	38	37	34
	α=**1.05**	25	13	8	32	18	10	39	38	36

Table 4.5. The Optimal Profit Values where $\sigma_p = 1$

	μ	**0.09**			**0.1**			**0.15**		
	h	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**
τ=**1**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	0.041	0.015	0.000
	α=**1.01**	0.017	0.008	0.004	0.024	0.011	0.005	0.094	0.062	0.036
	α=**1.05**	0.205	0.170	0.141	0.226	0.186	0.156	0.345	0.293	0.252
τ=**2**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	0.339	0.152	0.011
	α=**1.01**	0.020	0.008	0.004	0.045	0.012	0.005	0.483	0.286	0.123
	α=**1.05**	0.418	0.279	0.187	0.511	0.350	0.231	1.101	0.871	0.660
τ=**3**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	0.995	0.493	0.073
	α=**1.01**	0.020	0.008	0.004	0.047	0.012	0.005	1.247	0.725	0.280
	α=**1.05**	0.554	0.295	0.188	0.769	0.413	0.236	2.272	1.704	1.184
τ=**4**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	2.059	1.088	0.242
	α=**1.01**	0.020	0.008	0.004	0.047	0.012	0.005	2.414	1.432	0.553
	α=**1.05**	0.607	0.295	0.188	0.987	0.419	0.236	3.874	2.833	1.867
τ=**4.5**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	2.748	1.497	0.384
	α=**1.01**	0.020	0.008	0.004	0.047	0.012	0.005	3.161	1.892	0.751
	α=**1.05**	0.613	0.295	0.188	1.081	0.419	0.236	4.847	3.515	2.282

Table 4.6. The Optimal Quantities where $\sigma_p = 0.8$

	μ	**0.09**			**0.1**			**0.15**		
	h	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**
τ=**1**	α=**1.00**	0	0	0	0	0	0	5	4	1
	α=**1.01**	3	2	1	4	3	1	6	5	4
	α=**1.05**	6	5	5	6	6	5	7	7	6
τ=**2**	α=**1.00**	0	0	0	0	0	0	14	13	10
	α=**1.01**	5	2	1	9	3	2	15	14	12
	α=**1.05**	13	11	8	13	12	9	16	15	14
τ=**3**	α=**1.00**	0	0	0	0	0	0	24	22	19
	α=**1.01**	5	2	1	10	3	2	24	23	20
	α=**1.05**	19	12	8	21	17	10	25	24	23
τ=**4**	α=**1.00**	0	0	0	0	0	0	33	32	29
	α=**1.01**	5	2	1	10	3	2	34	32	29
	α=**1.05**	24	13	8	28	18	10	35	33	32
τ=**4.5**	α=**1.00**	0	0	0	0	0	0	38	36	33
	α=**1.01**	5	2	1	10	3	2	38	37	34
	α=**1.05**	25	13	8	32	18	10	39	38	36

Table 4.7. The Optimal Profit Values where $\sigma_p = 0.8$

	μ	**0.09**			**0.1**			**0.15**		
	h	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**
τ=**1**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	0.044	0.016	0.000
	α=**1.01**	0.017	0.008	0.004	0.025	0.011	0.005	0.101	0.065	0.036
	α=**1.05**	0.212	0.173	0.144	0.234	0.193	0.159	0.362	0.308	0.260
τ=**2**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	0.350	0.158	0.011
	α=**1.01**	0.020	0.008	0.004	0.045	0.012	0.005	0.503	0.295	0.126
	α=**1.05**	0.423	0.281	0.188	0.517	0.353	0.231	1.132	0.890	0.671
τ=**3**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	1.019	0.503	0.075
	α=**1.01**	0.020	0.008	0.004	0.047	0.012	0.005	1.271	0.740	0.283
	α=**1.05**	0.556	0.295	0.188	0.775	0.413	0.236	2.307	1.728	1.200
τ=**4**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	2.075	1.099	0.245
	α=**1.01**	0.020	0.008	0.004	0.047	0.012	0.005	2.436	1.444	0.557
	α=**1.05**	0.608	0.295	0.188	0.989	0.419	0.236	3.899	2.849	1.877
τ=**4.5**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	2.755	1.501	0.385
	α=**1.01**	0.020	0.008	0.004	0.047	0.012	0.005	3.169	1.897	0.753
	α=**1.05**	0.613	0.295	0.188	1.082	0.419	0.236	4.857	3.522	2.286

Table 4.8. The Optimal Quantities where $\sigma_p = 0.5$

	μ	**0.09**			**0.1**			**0.15**		
	h	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**
τ=**1**	α=**1.00**	0	0	0	0	0	0	6	5	2
	α=**1.01**	3	2	1	4	3	2	6	6	5
	α=**1.05**	6	6	5	7	6	6	8	7	7
τ=**2**	α=**1.00**	0	0	0	0	0	0	15	14	10
	α=**1.01**	5	2	1	9	3	2	15	14	12
	α=**1.05**	13	11	8	14	12	10	17	16	15
τ=**3**	α=**1.00**	0	0	0	0	0	0	24	23	20
	α=**1.01**	5	2	1	10	3	2	25	23	21
	α=**1.05**	19	12	8	21	17	10	26	25	23
τ=**4**	α=**1.00**	0	0	0	0	0	0	34	32	29
	α=**1.01**	5	2	1	10	3	2	34	32	30
	α=**1.05**	24	13	8	29	18	10	35	33	32
τ=**4.5**	α=**1.00**	0	0	0	0	0	0	38	36	33
	α=**1.01**	5	2	1	10	3	2	38	37	34
	α=**1.05**	25	13	8	32	18	10	39	38	36

Table 4.9. The Optimal Profit Values where $\sigma_p = 0.5$

	μ	**0.09**			**0.1**			**0.15**		
	h	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**
τ=**1**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	0.053	0.019	0.000
	α=**1.01**	0.017	0.008	0.004	0.026	0.011	0.005	0.113	0.072	0.040
	α=**1.05**	0.223	0.183	0.148	0.246	0.204	0.163	0.391	0.333	0.279
τ=**2**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	0.380	0.169	0.012
	α=**1.01**	0.020	0.008	0.004	0.045	0.012	0.005	0.533	0.313	0.131
	α=**1.05**	0.433	0.283	0.188	0.532	0.358	0.232	1.183	0.926	0.694
τ=**3**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	1.055	0.523	0.079
	α=**1.01**	0.020	0.008	0.004	0.047	0.012	0.005	1.314	0.764	0.292
	α=**1.05**	0.558	0.295	0.188	0.784	0.414	0.236	2.368	1.766	1.224
τ=**4**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	2.099	1.114	0.250
	α=**1.01**	0.020	0.008	0.004	0.047	0.012	0.005	2.464	1.459	0.564
	α=**1.05**	0.608	0.295	0.188	0.993	0.419	0.236	3.935	2.870	1.892
τ=**4.5**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	2.762	1.505	0.387
	α=**1.01**	0.020	0.008	0.004	0.047	0.012	0.005	3.175	1.902	0.755
	α=**1.05**	0.613	0.295	0.188	1.082	0.419	0.236	4.865	3.529	2.289

Table 4.10. The Optimal Quantities where $\sigma_p = 0$

	μ	**0.09**			**0.1**			**0.15**		
	h	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**
τ=**1**	α=**1.00**	0	0	0	0	0	0	7	6	2
	α=**1.01**	4	2	1	5	3	2	7	6	5
	α=**1.05**	7	6	6	7	7	6	8	8	7
τ=**2**	α=**1.00**	0	0	0	0	0	0	16	14	11
	α=**1.01**	5	2	1	9	3	2	16	15	13
	α=**1.05**	13	11	8	14	12	10	17	16	15
τ=**3**	α=**1.00**	0	0	0	0	0	0	25	23	20
	α=**1.01**	5	2	1	10	3	2	25	23	21
	α=**1.05**	19	12	8	22	17	10	26	25	23
τ=**4**	α=**1.00**	0	0	0	0	0	0	34	32	29
	α=**1.01**	5	2	1	10	3	2	34	32	30
	α=**1.05**	24	13	8	29	18	10	35	33	32
τ=**4.5**	α=**1.00**	0	0	0	0	0	0	38	36	33
	α=**1.01**	5	2	1	10	3	2	38	37	34
	α=**1.05**	25	13	8	32	18	10	39	38	36

Table 4.11. The Optimal Profit Values where $\sigma_p = 0$

	μ	**0.09**			**0.1**			**0.15**		
	h	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**
τ=**1**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	0.068	0.025	0.000
	α=**1.01**	0.018	0.008	0.004	0.029	0.012	0.005	0.137	0.085	0.045
	α=**1.05**	0.246	0.196	0.155	0.273	0.220	0.176	0.441	0.373	0.305
τ=**2**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	0.410	0.185	0.014
	α=**1.01**	0.020	0.008	0.004	0.046	0.012	0.005	0.573	0.333	0.137
	α=**1.05**	0.441	0.285	0.188	0.548	0.363	0.233	1.245	0.968	0.720
τ=**3**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	1.084	0.538	0.082
	α=**1.01**	0.020	0.008	0.004	0.047	0.012	0.005	1.346	0.779	0.298
	α=**1.05**	0.561	0.295	0.188	0.790	0.415	0.236	2.411	1.797	1.239
τ=**4**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	2.107	1.119	0.251
	α=**1.01**	0.020	0.008	0.004	0.047	0.012	0.005	2.472	1.463	0.566
	α=**1.05**	0.608	0.295	0.188	0.995	0.419	0.236	3.946	2.876	1.897
τ=**4.5**	α=**1.00**	0.000	0.000	0.000	0.000	0.000	0.000	2.762	1.505	0.387
	α=**1.01**	0.020	0.008	0.004	0.047	0.012	0.005	3.176	1.903	0.755
	α=**1.05**	0.613	0.295	0.188	1.083	0.419	0.236	4.866	3.530	2.290

Table 4.12. The Difference in the Optimal Quantity between $\sigma_p = 1$ and $\sigma_p = 0$

	μ	**0.09**			**0.1**			**0.15**		
	h	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**
τ=**1**	α=**1.00**	0	0	0	0	0	0	2	2	1
	α=**1.01**	1	0	0	1	1	1	1	1	1
	α=**1.05**	1	1	1	1	1	1	1	2	1
τ=**2**	α=**1.00**	0	0	0	0	0	0	2	1	1
	α=**1.01**	0	0	0	0	0	0	1	2	1
	α=**1.05**	1	0	0	1	0	1	1	1	1
τ=**3**	α=**1.00**	0	0	0	0	0	0	1	1	1
	α=**1.01**	0	0	0	0	0	0	1	1	1
	α=**1.05**	0	0	0	1	0	0	1	1	0
τ=**4**	α=**1.00**	0	0	0	0	0	0	1	1	1
	α=**1.01**	0	0	0	0	0	0	0	0	1
	α=**1.05**	0	0	0	1	0	0	1	0	1
τ=**4.5**	α=**1.00**	0	0	0	0	0	0	0	0	0
	α=**1.01**	0	0	0	0	0	0	0	0	0
	α=**1.05**	0	0	0	0	0	0	0	0	0

Table 4.13. The Value of the Volatility Information

	μ	**0.09**			**0.1**			**0.15**		
	h	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**	**0.01**	**0.03**	**0.05**
τ=**1**	α=**1.00**	0	0	0	0	0	0	97.32	144.1	5.207
	α=**1.01**	14.52	0	0	32.2	0.314	1.372	25.55	14.14	7.164
	α=**1.05**	12.77	3.17	11.91	8.822	18	7	9.302	16	6
τ=**2**	α=**1.00**	0	0	0	0	0	0	14.81	7.935	23.73
	α=**1.01**	0	0	0	0	0	0	6.632	10.09	8.037
	α=**1.05**	0.141	0	0	1.264	0	0.253	3.699	2.814	2.372
τ=**3**	α=**1.00**	0	0	0	0	0	0	2.941	2.095	5.249
	α=**1.01**	0	0	0	0	0	0	1.587	0.009	1.223
	α=**1.05**	0	0	0	1.622	0	0	1.163	1.48	0
τ=**4**	α=**1.00**	0	0	0	0	0	0	0.446	0.05	0.113
	α=**1.01**	0	0	0	0	0	0	0	0	0.368
	α=**1.05**	0	0	0	0.178	0	0	0.107	0	0.072
τ=**4.5**	α=**1.00**	0	0	0	0	0	0	0	0	0
	α=**1.01**	0	0	0	0	0	0	0	0	0
	α=**1.05**	0	0	0	0	0	0	0	0	0

5. CONCLUSION

In this thesis study, we studied a model for inventory management under stochastic purchase price and demand. We analyzed single item and two period problem, where the price and demand is independent in the first period and the secondary market demand depends on the price. In the first selling season, sales revenues depend on individual arrival times of demands and also there is a holding cost depends on the individual arrival times of demands as well. Over the second selling season, to maximize the sales revenue one need to optimize the price since the demand depends on the price the firm assigns. We modeled the problem where the secondary market revenue is just a salvage value in a special case and find closed form analytical solution for the optimal inventory quantity and the expected optimal revenues. In a more general problem, we first model the demand of the secondary market and find a solution approach to deal with it. Also, numerical experiments are made to validate the model and help to determine the effect of the individual parameters to the optimal quantity and optimal revenue. Moreover, we observe the value of the information of the volatility of the price in the inventory management context.

There are several avenues for future research. Firstly, correlation between price and demand process can be analyzed. Over the secondary selling season, we assume that there is a linear relationship between the price and the demand. Other type of the relationships, for instance exponential, can be investigated. Secondly, better modeling of the primary market customers can be done. Maybe, the demand process is distributed with normal distribution or linear distribution. Also, the price and the demand can be related in the first selling season as well. Holding cost also can change with time, in contrast to constant holding cost multiplier. Lastly, the opportunity of the more than one ordering can be allowed. It would be more realistic problem of that the firm reorders the commodity if the inventory level reaches zero before the end of first selling season.

REFERENCES

Berling, P., and V. Martínez-de-Albéniz, 2011, "Optimal Inventory Policies when Purchase Price and Demand are Stochastic", *Operations research*, Vol. 59, No. 1, pp. 109-124.

Berling, P., and Z. Xie, 2014, "Approximation Algorithms for Optimal Purchase/Inventory Policy when Purchase Price and Demand are Stochastic", *OR Spectrum*, Vol. 36, No. 4, pp. 1077-1095.

Chen, D., S. Benjaafar, and W. Cooper, 2014, "On the Impact of Input Price Variability and Correlation in Stochastic Inventory Systems".

Çetinkaya, S., and M. Parlar, 2010, "A One-time Excess Inventory Disposal Decision under a Stationary Base-stock Policy", *Stochastic Analysis and Applications*, Vol. 28, No. 3, pp. 540-557.

Fabian, T., J. L. Fisher, M. W. Sasieni, and A. Yardeni, 1959, "Purchasing Raw Material on a Fluctuating Market", *Operations Research*, Vol. 7, No. 1, pp. 107-122.

Gaur, V., N. Osadchiy, S. Seshadri, and M. G. Subrahmanyam, 2015, "Optimal Timing of Inventory Decisions with Price Uncertainty", *SSRN 2677045*.

Gaur, V., and S. Seshadri, 2005, "Hedging Inventory Risk Through Market Instruments", *Manufacturing & Service Operations Management*, Vol. 7, No. 2, pp. 103-120.

Gavirneni, S., 2004, "Periodic Review Inventory Control with Fluctuating Purchasing Costs", *Operations Research Letters*, Vol. 32, No. 4, pp. 374-379.

Goel, A., and G. Gutierrez, 2012, "Integrating Commodity Markets in the Optimal Procurement Policies of a Stochastic Inventory System, *SSRN 930486*.

Haksöz, Ç., and S. Seshadri, 2007, "Supply Chain Operations in the Presence of a Spot Market: a Review with Discussion", *Journal of the Operational Research Society*, Vol. 58, No. 11, pp. 1412-1429.

Kalymon, B., 1971, “Stochastic Prices in a Single Item Inventory Purchasing Model”, *Operations Research*, Vol. 19, No. 6, pp. 1434-1458.

Khouja, M., 1999, “The Single-period (News-vendor) Problem: Literature Review and Suggestions for Future Research”, *Omega*, Vol. 27, No. 5, pp. 537-553.

Khouja, M. J., 2000, “Optimal Ordering, Discounting, and Pricing in the Single-period Problem”, *International Journal of Production Economics*, Vol. 65, No. 2, pp. 201-216.

Petruzzi, N. C., and M. Dada, 1999, “Pricing and the Newsvendor Problem: A Review with Extensions”, *Operations Research*, Vol. 47, No. 2, pp. 183-194.

Scarf, H., 1960, “The Optimality of (S,s) Policies in the Dynamic Inventory Problem”, In: K. J. Arrow, S. Karlin, and S. Suppes (Eds.) *Math. Methods in the Social Sciences*, Stanford University Press.

Secomandi, N., and S. Kekre, 2014, “Optimal Energy Procurement in Spot and Forward Markets”, *Manufacturing & Service Operations Management*, Vol. 16, No. 2, pp. 270-282.

Whitin, T. M., 1955, “Inventory Control and Price Theory”, *Management Science*, Vol. 2, No. 1, pp. 61-68.

Xie, Z., C. S. Park, and L. Zheng, 2013, “Procurement Models under Purchase Price Uncertainty for Chinese Oil Refineries”, *International Journal of Production Research*, Vol. 51, No. 10, pp. 2952-2968.

Young, L., 1978, “Price, Inventory and the Structure of Uncertain Demand”, *New Zealand Operations Research*, Vol. 6, No. 2, pp. 157-177.

Zhang, J., J. Chen, and C. Lee, 2008, “Joint Optimization on Pricing, Promotion and Inventory Control with Stochastic Demand”, *International Journal of Production Economics*, Vol. 116, No. 2, pp. 190-198.

Zhu, X., and S. Çetinkaya, 2015, "A Stochastic Inventory Model for an Immediate Liquidation and Price-promotion Decision under Price-dependent Demand", *International Journal of Production Research*, Vol. 53, No. 12, pp. 3789-3809.

APPENDIX A: LOGNORMAL PARTIAL MOMENT

Secomandi an Kekre (2013) proved the following Lemma A.1 in their study.

Lemma A.1. *Denote by $g(w;\mu,\sigma)$ and $\phi(y;\mu,\sigma)$ the probability density functions of a Log-normal random variable W and a normal random variable Y, respectively, with parameters μ and σ, and $\Phi(.)$ the cumulative distribution function of the standard normal random variable. Let $\gamma \in \Re$ and $q \in \Re_+$. It holds that*

$$E[W^\gamma 1_{W\le q}] = \int_0^q w^\gamma g(w;\mu,\sigma)dw = e^{(\frac{\sigma^2\gamma}{2}+\mu)\gamma} \ \Phi(\frac{\ln q-(\mu+\sigma^2\gamma)}{\sigma}).$$

Let $P(\tau) = P(0)e^{(\mu-\frac{1}{2}\sigma^2)\tau+\sigma W(\tau)}$ where $W(\tau) \sim N(0,\tau)$. Then $P(\tau) \sim LN(\ln P(0)+((\mu-\frac{1}{2}\sigma^2)\tau),\sigma^2\tau)$.

From Lemma A.1, it can be easily shown the following results where $P(\tau)$ denoted by y and $\mu_\tau = \ln P(0) + (\mu - \frac{1}{2}\sigma^2)\tau$.

(i) $\int_{\frac{A}{2B}}^\infty f(y)\,dy = \int_{\frac{A}{2B}}^\infty y^0 f(y)\,dy = 1-\int_0^{\frac{A}{2B}} y^0 f(y)\,dy = 1-\Phi(\frac{\ln(\frac{A}{2B})-In(P(0))-(\mu-\frac{1}{2}\sigma^2)\tau}{\sigma_P\sqrt{\tau}})$.

(ii) $\int_{\frac{A-(Q-i)}{B}}^\infty f(y)\,dy = 1-\int_0^{\frac{A-(Q-i)}{B}} y^0 f(y)\,dy = 1-\Phi(\frac{\ln(\frac{A-(Q-i)}{B})-\mu_\tau}{\sigma_P\sqrt{\tau}})$.

(iii) $\int_0^{\frac{A-(Q-i)}{B}} yf(y)\,dy = e^{(\frac{\sigma_P^2\tau}{2}+\mu_\tau)}\Phi(\frac{\ln(\frac{A-(Q-i)}{B})-\mu_\tau-\sigma_P^2\tau}{\sigma_P\sqrt{\tau}})$.

(iv) $\int_{\frac{A-(Q-i)}{B}}^{\frac{A}{2B}} yf(y)\,dy = \int_0^{\frac{A}{2B}} yf(y)\,dy - \int_0^{\frac{A-(Q-i)}{B}} yf(y)\,dy$
$= e^{(\frac{\sigma_P^2\tau}{2}+\mu_\tau)}\left[\Phi\left(\frac{\ln(\frac{A}{2B})-\mu_\tau-\sigma_P^2\tau}{\sigma_P\sqrt{\tau}}\right) - \Phi\left(\frac{\ln(\frac{A-(Q-i)}{B})-\mu_\tau-\sigma_P^2\tau}{\sigma_P\sqrt{\tau}}\right)\right]$.

(v) $\int_{\frac{A-(Q-i)}{B}}^{\frac{A}{2B}} y^2 f(y)\,dy = \int_0^{\frac{A}{2B}} y^2 f(y)\,dy - \int_0^{\frac{A-(Q-i)}{B}} y^2 f(y)\,dy$
$= e^{2(\sigma_P^2\tau+\mu_\tau)}\left[\Phi\left(\frac{\ln(\frac{A}{2B})-\mu_\tau-2\sigma_P^2\tau}{\sigma_P\sqrt{\tau}}\right) - \Phi\left(\frac{\ln(\frac{A-(Q-i)}{B})-\mu_\tau-2\sigma_P^2\tau}{\sigma_P\sqrt{\tau}}\right)\right]$.